The Joy of Primitive Variable Recovery in Relativistic Magnetohydrodynamics

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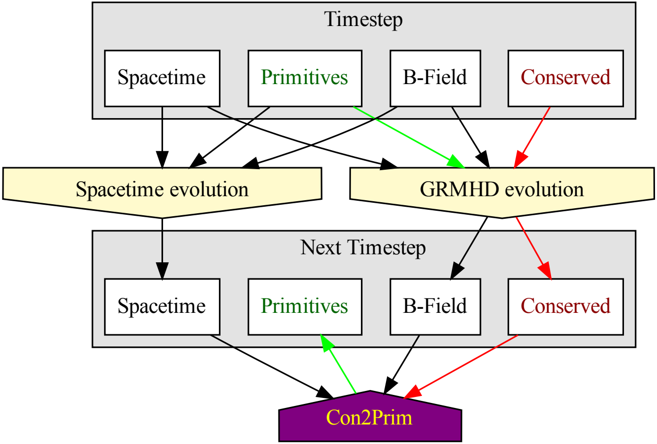
Conserative to Primitive Recovery

Also known as con2prim, C2P



The Situation

Better avoid driving over potholes with GRMHD cars.

The Primitive Variables

Matter related variables that need to be recovered

ρ_E	Fluid contribution to total energy density
ρ	Baryonic mass density
P	Fluid pressure
$\epsilon = \frac{\rho_E}{\rho} - 1$	Specific internal energy
$h = 1 + \epsilon + \frac{P}{\rho}$	Relativistic enthalpy
v^i	3-velocity
W	Lorentz factor
c_s	Adiabatic sound speed

The Magnetic Field

- In the Eularian frame, EM field described by

  B^i Magnetic field
  E^i Electric field

- Important assumption: Ideal MHD

$$E^i = -\epsilon^{ijk} v_j B_k$$

- Role in C2P
  - B^i needed as input
  - E^i provided as result
- B^i can be trivially obtained before C2P
  - Either evolved directly (constrained transport)
  - Or computed from evolved vector potential A^μ

The Conserved Variables

Evolved by GRMHD codes

- Conserved baryonic mass

$$D = \sqrt{\gamma}\rho W$$

- Quasi-conserved momentum density

$$S_i = \sqrt{\gamma}\left[\rho W^2 h v_i + \epsilon_{ijk} E^j B^k\right]$$

- Quasi-conserved energy density

$$\tau = \sqrt{\gamma}\left[\rho W\left(hW - 1\right) - P + \frac{1}{2}\left(E^2 + B^2\right)\right]$$

Matter Composition

- Typically given by the electron fraction Y_e
- Available from evolution scheme before C2P
- Often evolved in terms of conserved tracer DY_e
- Does not need to be recovered by C2P
- Used by C2P only when evaluating EOS

The Equation of State

- Evolved variables insufficient to compute primitives
- Also need EOS
- Can be specified differently
  - Suitable for C2P: $P = P(\rho, \epsilon, Y_e)$
  - Often available only as $P(\rho, T, Y_e)$
- Inversion $T(\rho, \epsilon, Y_e)$ increases computational costs
- Using $P(\rho, T, Y_e)$ directly increases complexity of C2P
- Many C2P schemes require derivatives $\partial_\rho P, \partial_\epsilon P$

The Equation of State

- Different types of EOS in use
  - Include composition Y_e or not
  - Analytic, e.g. ideal gas
  - Hybrid: cold part + analytic thermal contribution
  - Fully tabulated
- Ideally, C2P algorithm should not depend on EOS type

The Equation of State

- EOS have a validity range
  - Physical bounds
  - Technical constraints
- Important: zero-temperature limit $\epsilon \leq \epsilon(\rho, T = 0, Y_e)$
- Ideally,
  - EOS frameworks should provide validity ranges
  - C2P algorithm should use validity ranges

Solution Approaches

- Always cast into a root finding problem
- Different choices for root function
  - Multidimensional?
  - One-dimensional?
  - Which independent variables to use?
- Different root-solving algorithms
  - Newton-Raphson needed for multidimensional formulations
  - Root-bracketing only for one-dimensional formulations
  - Some require derivatives
- Require initial guess or not

Existing Schemes

- Many algorithms in literature, e.g.

Scheme	Dims	Independent variables	EOS form	EOS deriv.	Guess needed
Neilsen	1	DhW	$P(\rho, \epsilon)$	No	No
Newman	1	P	$P(\rho, h)$	No	No
Noble	2	$DhW,\ v^2$	$P(\rho, h)$	Yes	Yes
Siegel	2	$DhW,\ T$	$P(\rho, T)$	Yes	Yes
Duran	3	$W,\ DhW,\ T$	$\epsilon(\rho, T)$	Yes	Yes

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- Meta-scheme: try several schemes until one succeeds

Existing Schemes

- See [Siegel+2018] for a C2P demolition derby
- Found that all schemes can fail

Evolution Errors

- Can give **inaccurate** or **invalid** states
- Specific internal energy
  - Can fall slightly below $T = 0$ value (harmless)
  - Has large error in magnetically dominated regime
  - Has large error in highly relativistic regime
  - Large temperature errors (several MeV)
  - Some schemes also evolve entropy

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- NS surfaces
  - Problematic for evolution schemes
  - Often produces low-density invalid states
- BH center
  - Center extremely problematic for GRMHD
  - Anything can happen..
  - Impact lessened by horizon
  - Alternative: excise interior

Evolution Errors

- C2P needs to detect invalid input
- Ideally, correct harmless problems
- Otherwise, simulations should be aborted
- C2P cannot reduce inaccuracies from evolution

Artificial Atmosphere

- Evolution schemes cannot handle vacuum
- C2P problem also degenerate (velocity undefined)
- Artificial atmosphere: enforce minimum density ρ_{atmo}
- Atmosphere handling typically done during C2P

To Design a C2P Scheme

- Beware: tedious
- Root solvers
  - Need rapid convergence to limit CPU costs
  - Newton-Raphson methods may diverge
  - Using initial guesses makes C2P less predictable
- Root functions
  - May become degenerate for high W or B or ϵ
  - Need to proof uniqueness of solution
  - Avoid bad behavior for invalid input
  - Might even break in Newtonian (!) limit

C2P Headaches

- Failures for high magnetization or Lorentz factors
  $\Rightarrow$ cannot lower atmosphere density
- Inability to distinguish algorithm failure from invalid input
- Reflex of blaming the C2P blinds against evolution problems
- Unpredictable when using initial guesses
- EOS framework
  - Inconsistent treatment of different EOS types
  - Unclear concepts for validity range handling
  - Tabulated EOS often have low quality
  - Slow if based on temperature

RePrimAnd Scheme to the Rescue

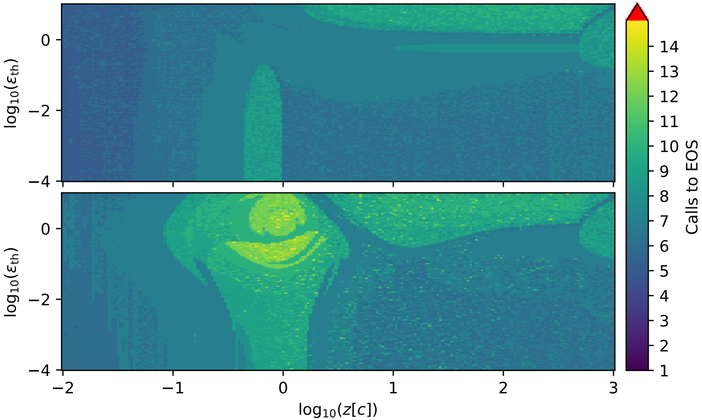
- One-dimensional root function
- Root-solver does not need derivatives
- Linear in Newtonian limit
- Well behaved even for high magnetization
- Tight initial bracket available
- Mathematical proofs for existence + uniqueness of solution
- Reliably identifies invalid input states
- Provides optional corrections
- Error propagation of root solving mapped out

The RePrimAnd Library

- Standalone C++ library for primitive recovery
- Also contains fancy EOS framework
  - Generic interface
  - Open design, can add custom EOS types
  - Universal EOS file format
  - Python bindings for postprocessing
- C2P code completely EOS-agnostic
- Not tied to any evolution code
  - Can be installed in ET via ExternalLibraries thorn

The RePrimAnd Library

- Robustness proven by unit tests for C2P and EOS
- Efficiency proven by benchmarks



$z \equiv Wv$. $B = 10\sqrt{W\rho}$ (bottom), $B = 0$ (top).

The RePrimAnd Library

- Publicly available
  - GitHub: `https://github.com/wokast/RePrimAnd`
  - Zenodo DOI `https://doi.org/10.5281/zenodo.3785074`
- Usable documentation
  `https://www.atlas.aei.uni-hannover.de/holohome/wolfgang.kastaun/doc/reprimand/latest/index.html`
- Scheme described and tested in
  W. Kastaun et al, **Phys. Rev. D 103, 023018 (2021)** (arXiv:2005.01821).
- Further tests within Spritz code (see Jay's talk)
  J. V. Kalinani et al, **arxiv:2107.10620** (2021)
- Already used in
  R. Aguilera-Miret et al, **Phys. Rev. D 102** (2020)

RePrimAnd Todo-List

- Implement fully tabulated EOS
- EOS performance optimizations
- Add TOV solver
- Include the code for non-MHD C2P algorithm
- Thorns for integration into EinsteinToolkit
- RePrimand's EOS as alternative to ET's `EOS_Omni` (?)
- Public EOS collection (?)