Miscellaneous

Mag to dB: $M_{dB}(\omega) = 20\log_{10}(|M(j\omega)|)$

dB to Mag: $|M(j\omega)| = 10^{\frac{M_{dB}(\omega)}{20}}$

Pade approximation $[n,n]$:

$$e^{-sh} = \frac{\sum_{i=1}^{n} \frac{(2n-i)!n!}{(2n)!(n-i)!i!}(-sh)^i}{\sum_{i=1}^{n} \frac{(2n-i)!n!}{(2n)!(n-i)!i!}(sh)^i}$$

Signal Properties

Energy: $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

Average Power: $P_f = \lim_{M\to\infty} \frac{1}{M}\int_{-\frac{M}{2}}^{\frac{M}{2}} |x(t)|^2 dt$

Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Spectral coefficient: $a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$

$|a_k|, \angle a_k$: the kth harmonic amplitude and phase

Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Parseval's theorem

Given periodic and piecewise continuous $x(t)$

$$\text{Power}: P_x = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\text{Energy}: E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Frequency based control

Lead controller

$$C_{\text{lead}}(s) = \frac{\sqrt{\alpha}s + \omega_m}{s + \sqrt{\alpha}\omega_m}, \quad \alpha > 1$$

Max phase lead: $\phi_m = \arcsin\left(\frac{\alpha-1}{\alpha+1}\right)$

Lag controller

$$C_{\text{lag}}(s) = \frac{10s + \omega_m}{10s + \frac{\omega_m}{\beta}}, \quad \beta > 1$$

Max phase lag: $\phi_m = -\arcsin\left(\frac{\beta-1}{\beta+1}\right)$

Notch Filter / Skewed Notch Filter

$$F(s) = \frac{s^2 + 2\zeta_n\omega_n s + \omega_n^2}{s^2 + 2\zeta_d\omega_d s + \omega_d^2}, \quad \zeta_n < \zeta_d$$

Notch: $w_n = w_d$, Skewed: $w_n \neq w_d$

Butterworth low-pass filters

A special family of low-pass filters

$$|C_{b,n}(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_b}\right)^{2n}}$$

where n is the order of the filter and ω_b is the bandwidth of the filter.

Robust stability

Multiplicative uncertainty

$$\Pi = \left\{P(s) : \left|\frac{P(j\omega)}{P_n(j\omega)} - 1\right| < \ell_m(\omega)\right\}$$

If conditions of theorem are met, then the controller $C(s)$ robustly stabilizes the system iff it stabilizes the nominal process and:

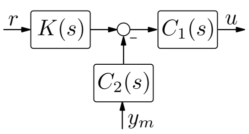
$$|T_n(j\omega)| < \frac{1}{|\ell_m(\omega)|}, \quad \forall\omega$$

2DoF control

In case of **single prefilter**, we choose:

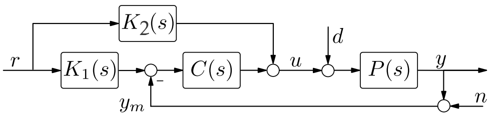
$$K(s) = \frac{T_{\text{ref}}(s)}{T(s)}$$

it's possible to split the controller: $C(s) = C_1(s)C_2(s)$ and to move one of them to the feedback inorder to control the complementary sensitivity function zeros:

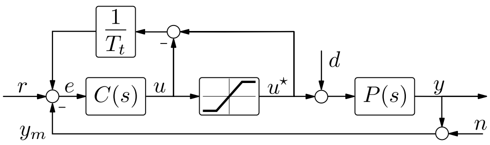


In case of a **double prefilter**, choose:

$$K_1(s) = T_{\text{ref}}(s) \quad K_2(s) = \frac{T_{\text{ref}}(s)}{P(s)}$$



Integrator Anti-windup



Pole placement controller

The plant P and controller C are given by:

$$P(s) = \frac{a(s)}{b(s)} = \frac{\sum_{i=0}^{m} a_i s^i}{s^n + \sum_{i=0}^{n-1} b_i s^i}$$

$$C(s) = \frac{x(s)}{y(s)} = \frac{\sum_{i=0}^{n-1} x_i s^i}{s^{p-n} + \sum_{i=0}^{p-n-1} y_i s^i}$$

The closed loop characteristic polynomial:

$$\Delta_{\text{cl}} = D_0 + N_0 = b(s)\cdot y(s) + a(s)\cdot x(s)$$

We require $m + n \leq p \leq 2n$ and $p > n$.

Case	$C(s)$
$p = 2n$	Strictly proper
$p = 2n-1$	Proper
$m+n \leq p < 2n-1$	Improper

Internal model principle

Necessary condition for disturbance attenuation in steady-state and tracking with zero steady-state error:

$$\text{d(t)}: \quad C(s) = \frac{\overline{C}(s)}{\Gamma(s)} \qquad D(s) = \frac{N(s)}{\Gamma(s)}$$

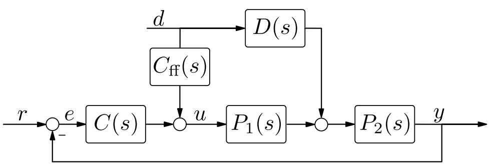
$$\text{r(t)}: \quad L(s) = \frac{\overline{L}(s)}{\Gamma_r(s)} \qquad R(s) = \frac{N_r(s)}{\Gamma_r(s)}$$

where $\Gamma(s)$ and $\Gamma_r(s)$ are the generating polynomials

Feed forward control

Ideal feed forward controller:

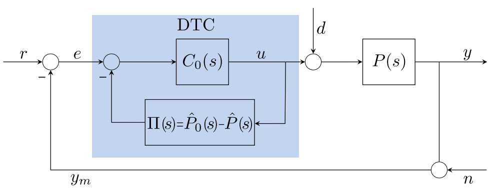
$$C_{\text{ff}} = -\frac{D(s)}{P_1(s)}$$



Dead time compensator

Smith controller (For stable processes)

System transfer function with dead time $P = P_0 e^{-\theta s}$



If $\hat{P}_0(s) = P_0(s)$ and $\hat{\theta} = \theta$:

$$T_{\text{ideal}} = \frac{C_0 P}{1 + C_0 P_0} = \frac{C_0 P_0 e^{-\theta s}}{1 + C_0 P_0} = Q(s)e^{-\theta s}$$

Multiplicative uncertainty

$$L(s) = \frac{P(s) - \hat{P}(s)}{\hat{P}(s)} = \frac{P_0(s)}{\hat{P}_0(s)}e^{-\Delta\theta s} - 1$$

Assumptions:

1. $Q(s)$ is asymptotically stable
2. $\hat{P}_0(s) = P_0(s)$

(a) Assuming (1) then the necessary condition for practical stability: $\lim_{\omega\to\infty}|Q(j\omega)| < \frac{1}{2}$ If $\hat{P}_0(s) = P_0(s)$ but $\hat{\theta} \neq \theta$ the condition is also sufficient

(b) Assuming (1),(2) then the closed loop is asymptotically stable for every $\Delta\theta$ if $|Q(j\omega)| < \frac{1}{2}$, $\forall\omega > 0$

(c) Assuming (1), (2) then there exists a positive $(\Delta\theta_m)$ which guarantees asymptotic stability for all $|\Delta\theta| < (\Delta\theta_m)$ if: (i) $|Q(j\omega)| < 1, \forall\omega > 0$ and (ii) $\lim_{\omega\to\infty}|Q(j\omega)| < \frac{1}{2}$

(d) Assuming (1) then the closed loop is robustly stable if: $|Q(j\omega)\ell_m(\omega)| < 1, \forall\omega > 0$

(e) Assuming (1) and condition (a) and if $|Q(j\omega)| < 1, \forall\omega > 0$, the closed loop $GM \geq 2$, and $PM > 60\circ$

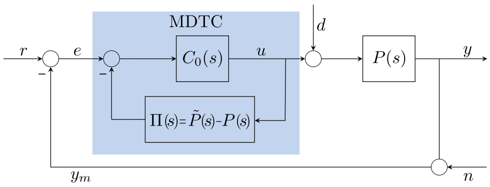
(f) Assuming (1) and condition (a), $P(s)$ is stable without a zero at origin, $C_0(s)$ includes an integrator then the DTC guarantees zero steady state error for a step change in reference and disturbance.

Inversion based primary controller design

For a stable and minimum phase $\hat{P}_0(s)$, choose $C_0(s) = \frac{k}{s}\hat{P}_0^{-1}(s)$ which leads to $Q(s) = \frac{k}{s+k}$ which (1) guarantees stability for all $k > 0$ in the ideal case (2) satisfies Theorems 1, 2b, 4, 5 ,and (3) has a single tuning parameter, k - as a rule of thumb $k \in \left(\frac{1}{\hat{\theta}}, \frac{1.5}{\hat{\theta}}\right)$.

Modified DTC

Modified DTC-holds for unstable processes.



$$\hat{\Pi}(s) = \tilde{P}(s) - P_0(s)e^{-\theta s}$$

$$\Pi(s) = \tilde{P}(s) - P(s) = \boldsymbol{C}e^{-\boldsymbol{A}\theta}\cdot\int_{-\theta}^{0} e^{(s\boldsymbol{I}-\boldsymbol{A})\tau}d\tau\cdot\boldsymbol{B}$$

where,

$P_0(s) = \boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}$ and $\tilde{P}(s) = \boldsymbol{C}e^{-\boldsymbol{A}\theta}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}$

The choice of $\tilde{P}(s)$

A possible choice of $\tilde{P}(s)$ in order to guarantee stability of $\Pi(s)$ is $D_{\tilde{P}}(s) = D_{P_0}(s)$, and the numerator is chosen such that it cancels all undesirable poles (or unstable poles) of $\tilde{P}_0(s)$ inside $\Pi(s)$, the number of required parameters for $N_{\tilde{P}}(s)$ is equal to the number of constraints.

Integral action in MDTC

Requires an integrator in $C_0(s)$ and $\Pi(0) = 0$.

Strong stabilization

The process $P(s)$ is strongly stabilizable if it is possible to stabilize it with a stable controller. A rational process $P(s)$ is strongly stabilizable $\iff$ between every pair of real zeros in RHP (including ∞) there is an even number of poles.

★ The number of zeros at $+\infty$ is equal to the process' pole excess.

Bode's sensitivity integral

Lemma 1. Let $L(s) = e^{-\tau s}L_0$ be the open-loop transfer function with $\tau \geq 0$ where $L_0(s)$ is stable and rational having poles excess $n_r \geq 1$ and define $\kappa \triangleq \lim_{s\to\infty} sL_0(s)$, then

$$BI = \begin{cases} 0 & \tau > 0 \\ -\kappa\frac{\pi}{2} & \tau = 0 \end{cases}$$

where $BI = \int_0^{\infty}\ln(|S(j\omega)|)d\omega$ and where $S(s)$, the sensitivity function, is assumed stable.

Lemma 2. Consider a feedback control loop with open loop transfer function as in Lemma 1 and having unstable poles $\{p_i\}_{i=1}^{N}$, $\tau \geq 0$, having poles excess $n_r \geq 1$ and define $\kappa \triangleq \lim_{s\to\infty} sL_0(s)$, then

$$BI = \begin{cases} \pi\sum_{i=1}^{N}\text{Re}\{p_i\} & n_r > 1 \\ -\kappa\frac{\pi}{2} + \pi\sum_{i=1}^{N}\text{Re}\{p_i\} & n_r = 1 \end{cases}$$

where $BI = \int_0^{\infty}\ln(|S(j\omega)|)d\omega$ and $S(s)$, the sensitivity function, is assumed stable.

Control in state space

State space modeling

$$P(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right] \quad : \quad \begin{cases}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)+Du(t)\end{cases}$$

where $x\in\mathbb{R}^n, u\in\mathbb{R}^m, y\in\mathbb{R}^r$.

$$[\dot{x}(t)]_{n\times 1}=[A]_{n\times n}[x(t)]_{n\times 1}+[B]_{n\times m}[u(t)]_{m\times 1}$$
$$[y(t)]_{r\times 1}=[C]_{r\times n}[x(t)]_{n\times 1}+[D]_{r\times m}[u(t)]_{m\times 1}$$

Canonical realization

$$P(s)=\frac{b_n s^n+b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}, \quad a_n=1$$

Companion form

$$\left[\begin{array}{cccc|c} 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ -a_0 & -a_1 & \cdots & -a_{n-1} & 1\\ \hline b_0-b_na_0 & b_1-b_na_n & \cdots & b_{n-1}-b_na_{n-1} & b_n\end{array}\right]$$

Observer form

$$\left[\begin{array}{cccc|c} -a_{n-1} & 1 & \cdots & 0 & b_{n-1}-b_na_{n-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -a_1 & 0 & \cdots & 1 & b_1-b_na_1\\ -a_0 & 0 & \cdots & 0 & b_0-b_na_0\\ \hline 1 & 0 & \cdots & 0 & b_n\end{array}\right]$$

State equation solution

$$x(t)=e^{At}x_0+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Transformed solution

$$x(s)=(sI-A)^{-1}x_0+(sI-A)^{-1}Bu(s)$$
$$y(s)=C(sI-A)^{-1}x_0+\left(C(sI-A)^{-1}B+D\right)u(s)$$

Matrix similarity transformation

$$\tilde{A}=T^{-1}AT,\quad \tilde{B}=T^{-1}B,\quad \tilde{C}=CT,\quad \tilde{D}=D$$

Controllability

Controllability matrix

$$\mathcal{C}\triangleq\begin{bmatrix}B & AB & \ldots & A^{n-1}B\end{bmatrix}$$

Given two minimal realizations with controllability matrices C_1, C_2, then the similarity transformation connecting between them is: $x_1=Tx_2, A_2=T^{-1}A_1T, B_2=T^{-1}B_1$, given by $T=C_1C_2^{-1}$.

PBH controllability test

The matrix $\begin{bmatrix}A-sI & B\end{bmatrix}$ has full rank $\forall s\in\mathbb{C}$

Controllability Gramian

$$W_C(t)\triangleq\int_0^t e^{A\tau}BB'e^{A'\tau}d\tau$$

The control input

$$u(t)=B^\top e^{A^\top(t_1-t)}W_C^{-1}(t_1)\left[x_1-e^{A^\top t_1}x_0\right]$$

transfers the system from $x(0)=x_0$ to $x(t_1)=x_{t_1}$, and does it with minimum control energy.

Observability

Observability matrix

$$\mathcal{O}\triangleq\begin{bmatrix}C & CA & \cdots & CA^{n-1}\end{bmatrix}^\top$$

PBH observability test

The matrix $\begin{bmatrix}A-sI\\ C\end{bmatrix}$ has full rank $\forall s\in\mathbb{C}$

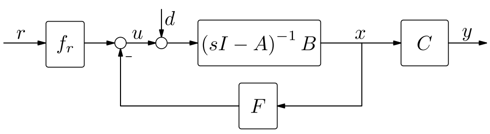
Observability Gramian

$$W_O(t)\triangleq\int_0^t e^{A\tau}CC'e^{A'\tau}d\tau$$

Stabilizability and Detectability

The pair (A,B) is **stabilizable** if all uncontrollable modes are stable.
The pair (C,A) is **detectable** if all unobservable modes are stable.

State Feedback

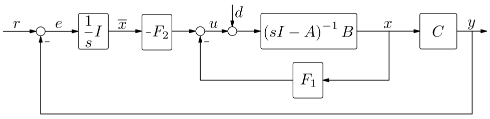


$$u(t)=f_r r(t)-Fx(t)$$

Ackermann's formula

$$F=\begin{bmatrix}0 & \cdots & 1\end{bmatrix}\mathcal{C}^{-1}\Delta_{cl}(A)$$

Adding an integrator to state feedback



Let $\tilde{x}(t)\triangleq\begin{bmatrix}x(t)\\ \bar{x}(t)\end{bmatrix}\in\mathbb{R}^{n+r}$, where $\bar{x}(t)\triangleq\int_0^t e(\tau)d\tau\in\mathbb{R}^r$
Control law: $u(t)=-\begin{bmatrix}F_1 & F_2\end{bmatrix}\tilde{x}(t)$

$$\dot{\tilde{x}}(t)=\begin{bmatrix}A-BF_1 & -BF_2\\ -C & 0\end{bmatrix}\tilde{x}(t)+\begin{bmatrix}0\\ I\end{bmatrix}r(t)+\begin{bmatrix}B\\ 0\end{bmatrix}d(t)$$
$$y(t)=\begin{bmatrix}C & 0\end{bmatrix}\tilde{x}(t)$$

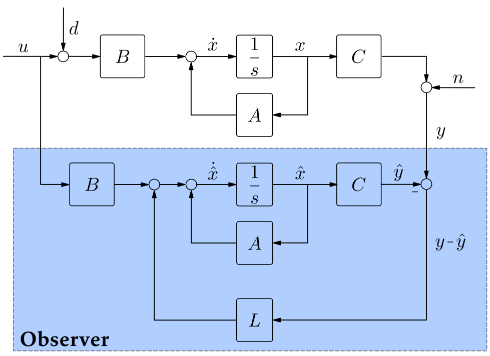
Closed-loop transfer functions

$$T(s)=\begin{bmatrix}C & 0\end{bmatrix}\left(sI-\tilde{A}\right)^{-1}\begin{bmatrix}0\\ I\end{bmatrix}$$
$$S_d(s)=\begin{bmatrix}C & 0\end{bmatrix}\left(sI-\tilde{A}\right)^{-1}\begin{bmatrix}B\\ 0\end{bmatrix}$$

Prototype table

$\mathcal{O}$	Transfer function poles
	Bessel
1	$\frac{s}{\omega_0}+1$
2	$\frac{s}{\omega_0}+0.866\pm 0.5j$
3	$(\frac{s}{\omega_0}+0.942)(\frac{s}{\omega_0}+0.745\pm 0.711j)$
4	$(\frac{s}{\omega_0}+0.657\pm 0.830j)(\frac{s}{\omega_0}+0.904\pm 0.271j)$
	ITAE
1	$\frac{s}{\omega_0}+1$
2	$\frac{s}{\omega_0}+0.707\pm 0.707j$
3	$(\frac{s}{\omega_0}+0.708)(\frac{s}{\omega_0}+0.521\pm 1.068j)$
4	$(\frac{s}{\omega_0}+0.424\pm 1.263j)(\frac{s}{\omega_0}+0.626\pm 0.414j)$

Asymptotic Observer

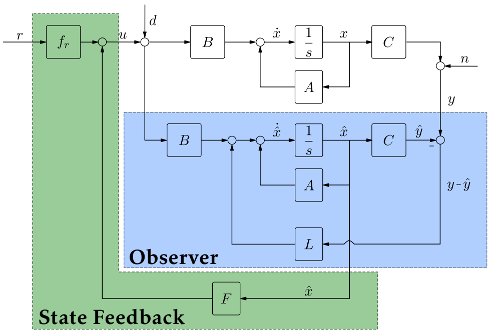


$$\dot{\hat{x}}(t)=A\hat{x}(t)+Bu(t)+L(y(t)-C\hat{x}(t)) \quad ,\hat{x}(0)=\hat{x}_0$$
$$\dot{e}(t)=(A-LC)e(t)+Bd(t)-Ln(t) \quad ,e(0)=x_0-\hat{x}_0$$
$$\Delta_{ob}(s)=\det(sI-A+LC)$$

Ackermann's formula

$$L=\Delta_{ob}(A)\mathcal{O}^{-1}\begin{bmatrix}0\\ \vdots\\ 1\end{bmatrix}$$

Observer based state feedback



$$u(t)=f_r r(t)-F\hat{x}(t)$$

Closed loop system equations

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix}=\begin{bmatrix}A & -BF\\ LC & A-BF-LC\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}+\begin{bmatrix}B\\ B\end{bmatrix}f_r r(t)$$
$$+\begin{bmatrix}B\\ 0\end{bmatrix}d(t)+\begin{bmatrix}0\\ L\end{bmatrix}n(t)$$
$$y(t)=\begin{bmatrix}C & 0\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}$$

Closed loop transfer function

$$y(s)=C(sI-(A-BF))^{-1}Bf_r r(s)$$

Controller transfer function

$$\dot{\hat{x}}(t)=(A-LC)\hat{x}(t)+Bu(t)+Ly(t)+Ln(t)$$
$$u(s)=C_r\cdot r(s)-C_y\cdot(y(s)+n(s))$$

where $C_r=(I-F\phi^{-1}(s)B)f_r$, $C_y=F\phi^{-1}(s)L$, and $\phi(s)=(sI-A+BF+LC)$

Disturbance generator

$$\dot{x}_d(t)=A_dx_d(t)$$
$$d(t)=C_dx_d(t)$$

The augmented realization

$$\begin{bmatrix}\dot{x}(t)\\ \dot{x}_d(t)\end{bmatrix}=\left[\begin{array}{c|c}A & BC_d\\ \hline 0 & A_d\end{array}\right]\begin{bmatrix}x(t)\\ x_d(t)\end{bmatrix}+\left[\begin{array}{c}B\\ \hline 0\end{array}\right]u(t)$$
$$y(t)=\left[\begin{array}{c|c}C & 0\end{array}\right]\begin{bmatrix}x(t)\\ x_d(t)\end{bmatrix}$$

Integrated observer

$$\begin{bmatrix}\dot{\hat{x}}(t)\\ \dot{\hat{x}}_d(t)\end{bmatrix}=\left[\begin{array}{c|c}A-L_1C & BC_d\\ \hline -L_2C & A_d\end{array}\right]\begin{bmatrix}\hat{x}(t)\\ \hat{x}_d(t)\end{bmatrix}$$
$$+\left[\begin{array}{c}B\\ \hline 0\end{array}\right]u(t)+\left[\begin{array}{c}L_1\\ \hline L_2\end{array}\right]y(t)$$

where $\begin{bmatrix}L_1 & L_2\end{bmatrix}=L$.

Internal model control (IMC) in state space

$$u(t)=f_rr(t)-F\hat{x}(t)-\hat{d}(t)=f_rr(t)-\begin{bmatrix}F & C_d\end{bmatrix}\hat{\eta}$$

LQR control

Finds the control signal which stabilizes the system and minimizes the cost function:

$$\mathcal{J}=\int_0^\infty\left(x^\top(t)C_z^\top C_zx(t)+\rho u^2(t)\right)dt$$

Optimal control law

A static state feedback $u(t)=-Fx(t)$ with

$$F=\frac{1}{\rho}B^\top P$$

where P is a stabilizing solution of the the following ARE:

Continuous algebraic Riccati equation

$$A^\top P+PA+C_z^\top C_z-\frac{1}{\rho}PBB^\top P=0$$

Cross-term in LQR cost function

$$\mathcal{J}=\int_0^\infty\left(x^\top(t)C_z^\top C_zx(t)+2x^\top(t)Su(t)+\rho u^2(t)\right)dt$$

To solve it we define:

$$\tilde{z}(t)=\tilde{C}_z^\top x(t),\quad \tilde{u}(t)=u(t)+\frac{1}{\rho}S^\top x(t)$$

satisfying the cross-term weight:

$$\tilde{C}_z^\top\tilde{C}_z=C_z^\top C_z-\frac{1}{\rho}S^\top S\geq 0$$

and the criterion becomes:

$$\mathcal{J}=\int_0^\infty\left(x^\top(t)\tilde{C}_z^\top\tilde{C}_zx(t)+\rho\tilde{u}^2(t)\right)dt$$

the optimal feedback gain is:

$$F=\frac{1}{\rho}\left(B^\top P+S^\top\right)$$

where P is the solution of the ARE:

$$A^\top P+PA+C_z^\top C_z-\frac{1}{\rho}(PB+S)\left(B^\top P+S^\top\right)=0$$

Cost function with exponential decay constraint

$$\mathcal{J}=\int_0^\infty e^{2\alpha t}\left(x^\top(t)C_z^\top C_zx(t)+\rho u^2(t)\right)dt$$

To solve it we define:

$$x_\alpha(t)\triangleq e^{\alpha t}x(t)\quad ,\quad u_\alpha(t)\triangleq e^{\alpha t}u(t)$$

and the criterion becomes:

$$\mathcal{J}_\alpha=\int_0^\infty\left(x_\alpha^\top(t)C_z^\top C_zx_\alpha(t)+\rho u_\alpha^2(t)\right)dt$$

the optimal feedback gain is:

$$F=\frac{1}{\rho}B^\top P_\alpha$$

where P_α is the solution of the ARE

$$(\alpha I+A)^\top P_\alpha+P_\alpha(\alpha I+A)+C_z^\top C_z-\frac{1}{\rho}P_\alpha BB^\top P_\alpha=0$$

Return difference equality

$$1+\frac{1}{\rho}P_z^\top(-s)P_z(s)=(1+L(-s))(1+L(s))$$

where

$$L(s)\triangleq F(sI-A)^{-1}B$$
$$u(s)\mapsto C_zx(s):\ P_z(s)\triangleq C_z(sI-A)^{-1}B$$

Optimal closed loop poles

The optimal control law satisfies $|1+L(j\omega)| \geq 1, \forall\omega$ which implies that the LQR control law guarantees

$$GM \in \left(\frac{1}{2}, \infty\right) \quad , \quad PM \geq 60°$$

Location of Closed Loop Poles

Define $P_z(s) = \frac{N_z(s)}{\Delta_{ol}(s)}$, then the closed-loop poles are the stable roots of

$$\Delta_{ol}(-s)\Delta_{ol}(s) + \frac{1}{\rho} N_z(-s) N_z(s) = 0$$

Symmetric Root Locus (SRL)

The R.L. problem has the form $kG_0(s) = -1$ where $G_0(s) = P_z(-s)P_z(s)$ and $k = \frac{1}{\rho} \geq 0$.

- SRL has $2n$ symmetrical branches (about the imaginary as well as the real axis).
- The pole excess of $G_0(s)$ is $2(n-m)$ (even).
- Asymptotes' center of gravity always at the origin.
- No branch can intersect or be on the imaginary axis

$(n-m)$ is odd	$(n-m)$ is even
$K < 0$	$K \geq 0$
$\gamma = \frac{180}{n-m} \cdot l$	$\gamma = \frac{90}{n-m} \cdot (2l+1)$

where $l = 0, 1, \dots, 2(n-m)-1$

Linear quadratic estimator (LQE)

Let:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) + \boldsymbol{B}_w w(t) \quad , \boldsymbol{x}(0) = 0 \\ y(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \sqrt{\sigma} n(t) \end{aligned}$$

where $w(t)$ and $n(t)$ are white Gaussian noises with unit intensities. We minimize:

$$\mathcal{J} = \int_0^\infty \|\boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)\|_2 \, dt$$

Kalman filter

$$\dot{\hat{\boldsymbol{x}}}(t) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\hat{\boldsymbol{x}}(t) + \boldsymbol{B}u(t) + \boldsymbol{L}(y(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t))$$

where $\boldsymbol{L} \triangleq \frac{1}{\sigma}\boldsymbol{Q}\boldsymbol{C}^\top$ and $\boldsymbol{Q} = \boldsymbol{Q}^\top \geq 0$ is the unique stabilizing solution of the CARE

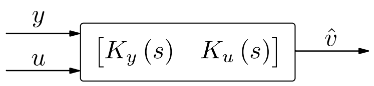
$$\boldsymbol{Q}\boldsymbol{A}^\top + \boldsymbol{A}\boldsymbol{Q} + \boldsymbol{B}_w\boldsymbol{B}_w^\top - \frac{1}{\sigma}\boldsymbol{Q}\boldsymbol{C}^\top\boldsymbol{C}\boldsymbol{Q} = 0$$

For a partial observation of the state vector

Define $\boldsymbol{v}\hat{(}t) = \boldsymbol{C}_v\boldsymbol{x}\hat{(}t)$. We with to minimize:

$$\mathcal{J} = \int_0^\infty \|\boldsymbol{C}_v\boldsymbol{x}(t) - \hat{\boldsymbol{v}}(t)\|_2 \, dt$$

Kalman filter (partial observer) transfer function:



Denoting $\boldsymbol{A}_L \triangleq \boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ we have

$$y \to \hat{\boldsymbol{v}} : K_y(s) = \boldsymbol{C}_v(s\boldsymbol{I} - \boldsymbol{A}_L)^{-1}\boldsymbol{L}$$

$$u \to \hat{\boldsymbol{v}} : K_u(s) = \boldsymbol{C}_v(s\boldsymbol{I} - \boldsymbol{A}_L)^{-1}\boldsymbol{B}$$

and

$$\hat{\boldsymbol{v}}(s) = K_u(s)u(s) + K_y(s)y(s)$$

Filtering error:

Let $\varepsilon(t) \triangleq x(t) - \hat{x}(t)$

$$\dot{\varepsilon} = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\hat{\varepsilon} + \boldsymbol{B}_w w - \sqrt{\sigma}\boldsymbol{L}n$$

Measurement error:

Let $y_\varepsilon = y - \boldsymbol{C}\hat{x} = \boldsymbol{C}\varepsilon(t) + \sqrt{\sigma}n$

$$\frac{y_\varepsilon}{w} = S_w(s) = \boldsymbol{C}(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})^{-1})\boldsymbol{B}_w$$

$$\frac{y_\varepsilon}{n} = S_n(s) = \sqrt{\sigma}(\boldsymbol{I} - \boldsymbol{C}(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}))^{-1}\boldsymbol{L})$$

Resulting relation:

$$\sigma = |S_w(j\omega)|^2 + |S_n(j\omega)|^2$$

Stability margins of the optimal filter:

$$\frac{1}{2} \leq GM < \infty, \quad PM \geq 60°$$

Signal to noise ratio:

$$SNR(\omega) = \frac{|Y_s(j\omega)|}{|Y_n(j\omega)|}$$

where

$$Y_s(s) \triangleq \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_w, \quad Y_n(s) \triangleq \sqrt{\sigma}.$$

Extreme cases

If no process noise is present, meaning $\boldsymbol{B}_w = 0$, the filter leaves the stable modes in their place, and reflects with respect to the imaginary axis the unstable modes. If measurement noise is infinitesimal, meaning $\sigma \to 0$, we define:

$$P_\omega(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}_w = \frac{N_\omega(s)}{\Delta_{\text{ol}}(s)}$$

and:

- m eigenvalues of $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ are approaching to m stable roots of $\frac{N_\omega(-s)}{N_\omega(s)}$
- $n-m$ eigenvalues of $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ are approaching to $n-m$ stable roots of $s^{2(n-m)} + \frac{(-1)^{n-m}}{\sigma} = 0$

Coloring Process Disturbances

Let $d(t)$ be the colored disturbance generated form the **shaping filter** $W_d(s)$, with the following minimal realization

$$\begin{aligned} \dot{\boldsymbol{x}}_d(t) &= \boldsymbol{A}_d\boldsymbol{x}_d(t) + \boldsymbol{B}_d\tilde{w}(t) \\ d(t) &= \boldsymbol{C}_d\boldsymbol{x}_d(t) \end{aligned}$$

where $\tilde{w}(t)$ is a white Gaussian noise. The augmented realization of the plant and the filter is

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{x}}_d(t) \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B}_w\boldsymbol{C}_d \\ \hline \boldsymbol{0} & \boldsymbol{A}_d \end{array}\right] \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_d(t) \end{bmatrix} + \left[\begin{array}{c} \boldsymbol{B} \\ \hline \boldsymbol{0} \end{array}\right] u(t) + \left[\begin{array}{c} \boldsymbol{0} \\ \hline \boldsymbol{B}_d \end{array}\right] w(t)$$

$$y(t) = \left[\begin{array}{c|c} \boldsymbol{C} & \boldsymbol{0} \end{array}\right] \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_d(t) \end{bmatrix} + \sqrt{\sigma}n(t)$$

Given the augmented realization of the plant and shaping filter, the optimal observer is

$$\begin{bmatrix} \dot{\hat{\boldsymbol{x}}}(t) \\ \dot{\hat{\boldsymbol{x}}}_d(t) \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{A} - \boldsymbol{L}_1\boldsymbol{C} & \boldsymbol{B}_w\boldsymbol{C}_d \\ \hline -\boldsymbol{L}_2\boldsymbol{C} & \boldsymbol{A}_d \end{array}\right] \begin{bmatrix} \hat{\boldsymbol{x}}(t) \\ \hat{\boldsymbol{x}}_d(t) \end{bmatrix} + \left[\begin{array}{c} \boldsymbol{B} \\ \hline \boldsymbol{0} \end{array}\right] u(t) + \left[\begin{array}{c} \boldsymbol{L}_1 \\ \hline \boldsymbol{L}_2 \end{array}\right] y(t)$$

where $[\boldsymbol{L}_1 \quad \boldsymbol{L}_2] = \boldsymbol{L}$.

Coloring measurement noise

Let $n(t)$ be the colored disturbance generated form the shaping filter $W_n(s)$, with the following minimal realization (for simplicity we assume $\boldsymbol{D}_n = 1$)

$$\begin{aligned} \dot{\boldsymbol{x}}_n(t) &= \boldsymbol{A}_n\boldsymbol{x}_n(t) + \boldsymbol{B}_n\tilde{n}(t) \\ n(t) &= \boldsymbol{C}_n\boldsymbol{x}_n(t) + \tilde{n}(t) \end{aligned}$$

where $\tilde{n}(t)$ is a white Gaussian noise. The optimal observer is

$$\begin{bmatrix} \dot{\hat{\boldsymbol{x}}}(t) \\ \dot{\hat{\boldsymbol{x}}}_n(t) \end{bmatrix} = \left(\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{0} \\ \hline -\frac{1}{\sqrt{\sigma}}\boldsymbol{B}_n\boldsymbol{C} & \boldsymbol{A}_n - \boldsymbol{B}_n\boldsymbol{C}_n \end{array}\right] - \left[\begin{array}{c} \boldsymbol{L}_1 \\ \hline \boldsymbol{L}_2 \end{array}\right] \left[\begin{array}{c|c} \boldsymbol{C} & \sqrt{\sigma}\boldsymbol{C}_n \end{array}\right]\right) \begin{bmatrix} \hat{\boldsymbol{x}}(t) \\ \hat{\boldsymbol{x}}_n(t) \end{bmatrix} + \left[\begin{array}{c} \boldsymbol{B} \\ \hline \boldsymbol{0} \end{array}\right] u(t) + \left[\begin{array}{c} \boldsymbol{L}_1 \\ \hline \boldsymbol{L}_2 + \frac{1}{\sqrt{\sigma}}\boldsymbol{B}_n \end{array}\right] y(t)$$

where $[\boldsymbol{L}_1 \quad \boldsymbol{L}_2] = \boldsymbol{L}$.

Shaping filter with a dominant frequency ω_0

$$\phi_n(\omega) = \frac{\omega^4 + \omega_0^4}{(\omega^2 - \omega_0^2)^2} \to W_n(s) = \frac{s^2 + \sqrt{2}\omega_0 s + \omega_0^2}{s^2 + \omega_0^2}$$

$$\dot{\boldsymbol{x}}_n(t) = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix} \boldsymbol{x}_n(t) + \begin{bmatrix} \sqrt{2}\omega_0 \\ 0 \end{bmatrix} \tilde{n}(t),$$

$$n(t) = [1 \quad 0]\boldsymbol{x}_n(t) + \tilde{n}(t)$$

LQG control

The solution to the LQG problem is the combination of the LQR controller minimizing

$$\mathcal{J}_{LQR} = \int_0^\infty (x^\top(t)\boldsymbol{X}x(t) + u^\top(t)\boldsymbol{R}u(t))dt$$

and a Kalman filter. The LQG controller:

$$K_{LQG} = \left[\begin{array}{c|c} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{F} - \boldsymbol{L}\boldsymbol{C} & \boldsymbol{L} \\ \hline -\boldsymbol{F} & 0 \end{array}\right]$$

where: $\boldsymbol{F} = \boldsymbol{R}^{-1}\boldsymbol{B}^\top\boldsymbol{P}, \quad \boldsymbol{L} = \boldsymbol{Q}\boldsymbol{C}^\top\boldsymbol{N}^{-1}$ are solutions of the Riccati equations:

$$\boldsymbol{A}^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{X} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top\boldsymbol{P} = 0$$

$$\boldsymbol{Q}\boldsymbol{A}^\top + \boldsymbol{A}\boldsymbol{Q} + \boldsymbol{W} - \boldsymbol{Q}\boldsymbol{C}^\top\boldsymbol{N}^{-1}\boldsymbol{C}\boldsymbol{Q} = 0$$

Disturbance generators

Step + Ramp

$$d(t) = (d_0 + d_1 t) \cdot \mathbb{1}(t) \to D(s) = \frac{d_0 s + d_1}{s^2}$$

$$\dot{\boldsymbol{x}}_d(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}_d(t), \quad \boldsymbol{x}_d(0) = \begin{bmatrix} d_0 \\ d_r \end{bmatrix}$$

$$d(t) = [1 \quad 0]\boldsymbol{x}_d(t)$$

Harmonic signal

$$d(t) = a\sin(\omega t + \varphi) \to D(s) = \frac{a\sin(\varphi)s + a\omega\cos(\varphi)}{s^2 + \omega^2}$$

$$\dot{\boldsymbol{x}}_d(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \boldsymbol{x}_d(t), \quad \boldsymbol{x}_d(0) = \begin{bmatrix} a\sin(\phi) \\ a\cos(\phi) \end{bmatrix}$$

$$d(t) = [1 \quad 0]\boldsymbol{x}_d(t)$$

Sampled data system

Spectrum of sampled signal

$$X^{\star}(\omega) \triangleq \mathcal{F}\left\{x^{\star}(t)\right\} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega + k\omega_s)$$

Sampling theorem (Shannon-Nyquist)

A continuous signal, $x(t)$, can be uniquely reconstructed from its samples, $x(kT_s)$, if: $\forall|\omega| > \omega_b$: $|X(j\omega)| = 0$, and $\omega_s > 2\omega_b$.

Nyquist frequency

The highest frequency in a band-limited signal that can be reconstructed (in an unbiased way) from its samples:

$$\omega_N \triangleq \frac{\omega_s}{2} = \frac{\pi}{T_s}$$

Ideal filter (the sinc interpolator)

$$g_I(t) = \frac{1}{T_s}\mathrm{sinc}\left(\frac{\omega_s}{2}t\right) \to G_I(\omega) = \begin{cases}1 & |\omega| \le \frac{\omega_s}{2}\\ 0 & \text{otherwise}\end{cases}$$

which is non-causal. With frequency response:

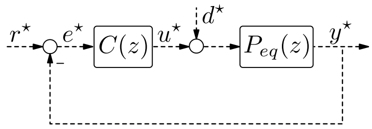
$$G(j\omega) = \begin{cases}1, & -\frac{\omega_s}{2} \le \omega \le \frac{\omega_s}{2}\\ 0, & \text{elsewhere}\end{cases}$$

Given the sampled signal, $x(kT_s)$, the ideal reconstructed signal is given by

$$x^{\star}_{\text{ideal}}(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\,\mathrm{sinc}\left(\frac{\omega_s}{2}(t - kT_s)\right)$$

And if the conditions of the Shannon-Nyquist Sampling Theorem are satisfied, then $x^{\star}_{\text{ideal}}(t) = x(t)$.

ZOH Discrete Equivalent Plant (DEP)



The discrete model of the continuous plant $P(s)$ $y^{\star}_k \mapsto u^{\star}_k$:

$$\boldsymbol{P}_{eq} : \left[\begin{array}{c|c}\boldsymbol{A}_s & \boldsymbol{B}_s\\ \hline \boldsymbol{C} & D\end{array}\right]$$

where

$$\boldsymbol{A}_s = e^{\boldsymbol{A}T_s} \quad , \quad \boldsymbol{B}_s = \int_0^{T_s} e^{\boldsymbol{A}\tau}\boldsymbol{B}\,d\tau$$

The DEP transfer function is:

$$P_{eq}(z) = \left(1 - z^{-1}\right)\mathbb{Z}\left\{\frac{P(s)}{s}\right\}$$

Triangular hold DEP

Given $P(s)$ the triangular discrete equivalent plant is given by

$$P_{\text{Tri}}(z) = \frac{(z-1)^2}{T_s z}\mathcal{Z}\left\{\frac{P(s)}{s^2}\right\}$$

which is causal.

Pathological sampling

A sampling frequency ω_s is called pathological with respect to $P(s)$ if complex poles of $P(s)$ satisfy

$$\mathfrak{Re}\{p_i\} = \mathfrak{Re}\left\{p_j\right\} \quad \text{and} \quad \mathfrak{Im}\left\{p_i - p_j\right\} = m\omega_s$$

where p_i, p_j are poles of $P(s)$, ω_s is the sampling frequency, and $m = 1, 2 \ldots$.

Residue theorem

Assuming strictly proper $F(s)$

$$F(z) = \sum_{\text{poles of } F(s)} R_i\left\{F(s)\frac{1}{1 - e^{-T_s}z^{-1}}\right\}$$

For a single pole

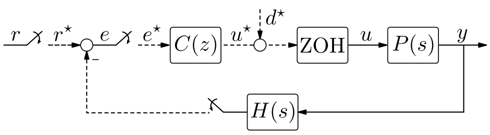
$$R_i\{F(s)\} = \left.(s - s_i)F(s)\right|_{s=s_i}$$

For a pole of q_i multiplicity

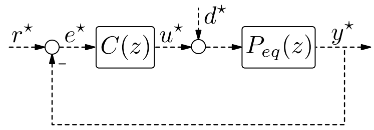
$$R_i = \frac{1}{(q_i - 1)!}\frac{d^{q_i-1}}{ds^{q_i-1}}\left.\left((s - s_i)^{q_i}F(s)\right)\right|_{s=s_i}$$

Stability of Closed-Loop Sampled Data System

It can be shown that the following Sampled Data (SD) system:



is equivalent at the sampling instants to the following discrete system:



The closed-loop transfer functions are"

$$y(z) = \frac{C(z)P_{eq}(z)}{1 + C(z)[HPh](z)}r(z) + \frac{P_{eq}(z)}{1 + C(z)[HPh](z)}d(z)$$

where

$$[HPh](z) = (1 - z^{-1})\mathbb{Z}\left\{\frac{P(s)H(s)}{s}\right\}$$

Denote

$$C(z) = \frac{N_c(z)}{D_c(z)}, \quad P_{eq}(z) = \frac{N_p(z)}{D_p(z)}$$

then the open loop of the discrete system is

$$L(z) = C(z)P_{eq}(z) = \frac{N_c(z)N_p(z)}{D_c(z)D_p(z)} = \frac{N_o(z)}{D_o(z)}$$

and the characteristic polynomial of the discrete closed loop system is:

$$\Delta_{cl}(z) = D_o(z) + N_o(z).$$

Theorem:

1. Let $L(z)$ be minimal, then the discrete system is stable iff all roots of $\Delta_{cl}(z)$ are in the Open Unit Disk (OUD).
2. Stability of the discrete system implies the stability of the sampled data system.

Discretization of continuous design

Forward rectangular rule (FRR)

$$s \longleftarrow \frac{z-1}{T_s}$$

Backward rectangular rule (BRR)

$$s \longleftarrow \frac{1 - z^{-1}}{T_s}$$

Trapezoid/Tustin's rule (TR) / Bilinear rule

$$s \longleftarrow \frac{2}{T_s}\cdot\frac{z-1}{z+1}$$

Tustin with frequency prewarping

Ensures a match between the continuous and discrete responses at ω_0

$$s \longleftarrow \frac{\omega_0}{\tan\left(\frac{\omega_0 T_s}{2}\right)}\cdot\frac{z-1}{z+1}$$

Impulse invariance (Z transform)

$$C_D(z) = \mathbb{Z}\{C_D(kT_s)\} = T_s\mathbb{Z}\{C(t = kT_s)\} = T_s C(z)$$

Step invariance

$$C_D(z) = \left(1 - z^{-1}\right)\mathbb{Z}\left\{\frac{C(s)}{s}\right\}$$

Zero-Pole matching

	$C_D(z)$	$C(s)$
Pole	$e^{p_i T}$	p_i
Finite zero	$e^{z_i T}$	z_i
Infinite zero	$z = -1$	z_i
LPF gain	$C_D(z = 1)$	$C(s = 0)$
HPF gain	$C_D(z = -1)$	$C(s = \infty)$

Frequency response for discrete systems

For input $u_k = a\sin(\omega kT)$:

$$Y_{ss}(kT) = a\left|P(e^{j\omega T})\right|\sin\left(k\omega T + \angle P(e^{j\omega T})\right)$$

Direct discrete controller design - minimal prototype

zero tracking error in minimum steps at sampling instants

$$P_{eq}(z) = \frac{A(z)}{B(z)}, \quad C(z) = \frac{P_{eq}^{-1}(z)}{z^{n-m} - 1}$$

where P_{eq} is stable and MP

Deadbeat control

Zero tracking error at sampling instants after n steps and control reaches steady state in n steps.

$$P(z) = \frac{A(z)}{B(z)}, \quad C(z) = \frac{\alpha B(z)}{z^n - \alpha A(z)}, \quad \alpha = \frac{1}{A(1)}$$

where $P(z)$ is stable and MP.

$\star$ Equivalent to pole placement with $\Delta_{cl}(z) = z^n$

Pole placement controller

Same formulas and conditions as in continuous case

Discrete systems in state space

$$\begin{cases} x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}u_k\\ y_k = \boldsymbol{C}x_k + \boldsymbol{D}u_k\end{cases}, x[k = 0] = x_0$$

$$x_n = A^n x_0 + \sum_{i=0}^{n-1} A^{n-i-1}Bu_i$$

transfer function of the discrete system:

$$\boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

Controllability matrix

$$\mathcal{C} \triangleq \begin{bmatrix}\boldsymbol{B} & \boldsymbol{AB} & \ldots & \boldsymbol{A}^{n-1}\boldsymbol{B}\end{bmatrix}$$

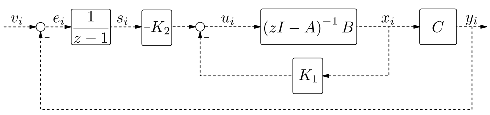
Observability matrix

$$\mathcal{O} \triangleq \begin{bmatrix}\boldsymbol{C} & \boldsymbol{CA} & \cdots & \boldsymbol{CA}^{n-1}\end{bmatrix}^{\top}$$

Ackermann's formula for state feedback

$$\boldsymbol{K} = [0 \quad \cdots \quad 1]\mathcal{C}^{-1}\Delta_{cl}(\boldsymbol{A})$$

State feedback with integration on error



Control law: $u_i = -\begin{bmatrix}\boldsymbol{K_1} & \boldsymbol{K_2}\end{bmatrix}\begin{bmatrix}x_i\\ s_i\end{bmatrix}$

$$\begin{bmatrix}x_{i+1}\\ s_{i+1}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ -\boldsymbol{C} & \boldsymbol{I}\end{bmatrix}\begin{bmatrix}x_i\\ s_i\end{bmatrix} + \begin{bmatrix}\boldsymbol{B}\\ \boldsymbol{0}\end{bmatrix}u_i + \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix}v_i$$

$$y_i = \begin{bmatrix}\boldsymbol{C} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}x_i\\ s_i\end{bmatrix}$$

Ackermann's formula for observer

$$\boldsymbol{L} = \Delta_{ob}(\boldsymbol{A})\mathcal{O}^{-1}[0 \quad \cdots \quad 0 \quad 1]^{\top}$$

Discrete Dead time (delay)

Transfer function - z^{-l} Frequency response $z^{-j\omega Tl}$

$$\left|z^{-j\omega Tl}\right| = 1, \quad \forall\omega T \in [0, \pi]$$

$$\angle z^{-j\omega T} = -\omega Tl$$

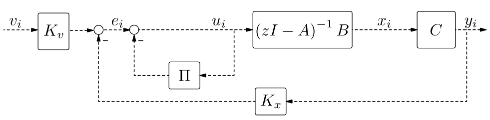
Discrete DT is finite dimensional ($-l$ poles at origin)

Discrete system with an input delay of l sampling intervals

$$x_{i+1} = \boldsymbol{A}x_i + \boldsymbol{B}u_{i-l}$$

$$\underbrace{\begin{bmatrix}x_{i+1}\\ u_{i-l+1}\\ u_{i-l+2}\\ \vdots\\ u_i\end{bmatrix}}_{\bar{\bar{x}}_{i+1}} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B} & \boldsymbol{0} & \cdots & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \cdots & \boldsymbol{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{I}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0}\end{bmatrix}\underbrace{\begin{bmatrix}x_i\\ u_{i-l}\\ u_{i-l+1}\\ \vdots\\ u_{i-1}\end{bmatrix}}_{\bar{\bar{x}}_i} + \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\\ \vdots\\ \boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix}u_i$$

State feedback for system with dead time



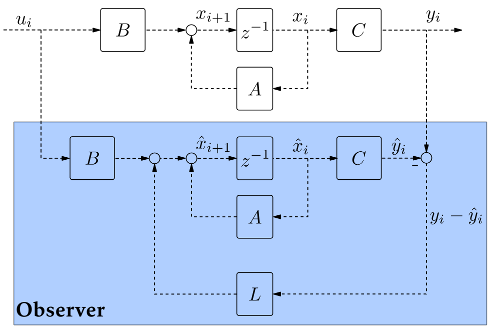
$$u_i = k_v v_i - (\boldsymbol{K_x} \cdot \boldsymbol{K_{u,1}} \cdot \boldsymbol{K_{u,2}} \cdot \ldots \cdot \boldsymbol{K_{u,l}})\bar{\bar{x}}_i$$

$$= k_v v_i - \boldsymbol{K_x}x_i - \sum_{k=1}^{l}\boldsymbol{K_{u,l+1-k}}u_{i-k}$$

$$u(z) = k_v v(z) - \boldsymbol{K_x}x(z)\underbrace{\left(\sum_{k=1}^{l}\boldsymbol{K_{u,l+1+k}}z^{-k}\right)}_{\Pi(z)}u(z)$$

Discrete observer

Regular full order observer system

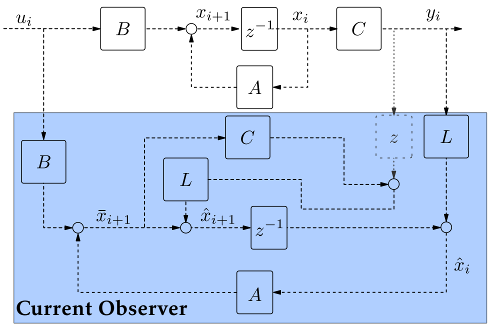


$\boldsymbol{L}$ can be obtained using Ackermann's formula:

$$\boldsymbol{L} = \Delta_{ob}(\boldsymbol{A})\mathcal{O}^{-1}\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}^\top$$

where: $\Delta_{\text{ob}}(z) = \det(z\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C})$

Current observer



$\boldsymbol{L}$ can be obtained using Ackermann's formula:

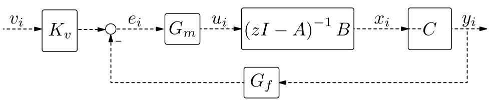
$$\boldsymbol{L} = \Delta_{ob}(\boldsymbol{A})\mathcal{O}^{-1}_{\boldsymbol{CA},\boldsymbol{A}}\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}^\top$$

where: $\Delta_{\text{ob}}(z) = \det(z\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C}\boldsymbol{A})$

Theorem:

1. If $(\boldsymbol{C}, \boldsymbol{A})$ is observable and if $\det(\boldsymbol{A}) \neq 0$ then the pair $(\boldsymbol{CA}, \boldsymbol{A})$ is observable.
2. If $\det(\boldsymbol{A}) = 0$ and $\boldsymbol{A}$ has q poles at the origin and $(\boldsymbol{C}, \boldsymbol{A})$ observable then $(\boldsymbol{CA}, \boldsymbol{A})$ is not observable and only $n - q$ poles (those not at the origin) can be placed arbitrarily.

Discrete state feedback + observer



	Current	Regular
G_m	$\left[\boldsymbol{I} + \boldsymbol{K}\boldsymbol{\Phi}_0^{-1}(\boldsymbol{I} - \boldsymbol{L}_C\boldsymbol{C})\boldsymbol{B}\right]^{-1}$	$\left[\boldsymbol{I} + \boldsymbol{K}\boldsymbol{\Phi}_0^{-1}\boldsymbol{B}\right]^{-1}$
G_f	$z\boldsymbol{K}\boldsymbol{\Phi}_0^{-1}\boldsymbol{L}_C$	$\boldsymbol{K}\boldsymbol{\Phi}_0^{-1}\boldsymbol{L}$
$\boldsymbol{\Phi}_0$	$z\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{L}_C\boldsymbol{C}\boldsymbol{A}$	$z\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C}$

Discrete LQR

$$\mathcal{J} = \sum_{i=1}^{\infty}\begin{bmatrix} \boldsymbol{x}_i^\top & u_i^\top \end{bmatrix}\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^\top & \boldsymbol{R} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_i \\ u_i \end{bmatrix}$$

The optimal state feedback $u_i = -\boldsymbol{K}x_i$

$$\boldsymbol{K} = \left(\boldsymbol{R} + \boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{B}\right)^{-1}\left(\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}^\top\right)$$

Discrete Riccati equation

$$\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} = \boldsymbol{P} + (\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{B} + \boldsymbol{S})(\boldsymbol{R} + \boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{B})^{-1}(\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}^\top)$$

open loop of optimal state feedback

$$L(z) = \boldsymbol{K}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$$

Sampled-Data LQR

finding the discrete feedback which internally stabilizes the closed loop and minimizes the continuous cost function:

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} \boldsymbol{x}^\top(t) & u^\top(t) \end{bmatrix}\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(t) \\ u(t) \end{bmatrix} dt$$

Find stabilizing piecewise-constant control signal

$$u(t) = u_i = -\boldsymbol{K}\boldsymbol{x}_i \quad , t \in [iT, (i+1)T)$$

Equivalent discrete criterion

$$\mathcal{J} = \sum_{i=0}^{\infty}\begin{bmatrix} \boldsymbol{x}_{iT}^\top & u_{iT}^\top \end{bmatrix}\begin{bmatrix} \boldsymbol{Q}_d & \boldsymbol{S}_d \\ \boldsymbol{S}_d^\top & \boldsymbol{R}_d \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_{iT} \\ u_{iT} \end{bmatrix}$$

where:

$$\boldsymbol{Q}_d = \int_0^T e^{\boldsymbol{A}^\top\tau}\boldsymbol{Q}e^{\boldsymbol{A}\tau}d\tau$$

$$\boldsymbol{S}_d = \int_0^T e^{\boldsymbol{A}^\top\tau}\boldsymbol{Q}\boldsymbol{B}_s(\tau)d\tau$$

$$\boldsymbol{R}_d = \int_0^T (\boldsymbol{B}_s^\top(\tau)\boldsymbol{Q}\boldsymbol{B}_s(\tau) + \boldsymbol{R})d\tau$$

and $\boldsymbol{B}_s(v) = \int_0^v e^{\boldsymbol{A}\sigma}d\sigma\boldsymbol{B}$

Aliasing (frequency folding)

The frequency ω_0 is called the alias of $\omega_0 \pm n\omega_s$, $\forall n \in \mathbb{Z}$. The fundamental alias for $\omega_1 > \omega_N$ is given by

$$\omega = |(\omega_1 + \omega_N)\text{mod}(\omega_s) - \omega_N|$$