Eldiias Dzhamankulov

Monte-Carlo Methods and Malliavin Calculus

Project B

1 Part A. An integration by part formula

We consider $X \in \mathbb{D}, \phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R}), Y \in L^2(\Omega)$ and $h \in L^2([0,T] \times \Omega)$ such that

$$\int_0^T h(t) D_t X dt > 0$$

almost surely and

$$u(t) = \frac{h(t) Y_t}{\int_0^T h(t) D_t X dt} \in dom(\delta)$$

a. Prove that

$$E[< DZ, u >_{L^2([0,T])}] = E[\Phi'(X)Y] \tag{1}$$

where $Z = \Phi(X)$.

Solution

$$E[< DZ, u >_{L^2([0,T])}] = E\Big[\int_0^T D_t \Phi(X) u_t dt\Big] = E\Big[\int_0^T \Phi^{'}(X) D_t X u_t dt\Big] = E\Big[\int_0^T \frac{\Phi^{'}(X) D_t X h(t) Y}{\int_0^T D_t X h(t) dt} dt\Big]$$

Here the first equality comes from the definition, second follows from Chain Rule. As long as $\int_0^T D_t X h(t) dt$ doesn't depend on t (constant), we get the following:

$$E\Big[\int_0^T \frac{\Phi^{'}(X) D_t X h(t) Y}{\int_0^T D_t X h(t) dt} dt\Big] = E\Big[\frac{\int_0^T \Phi^{'}(X) D_t X h(t) Y dt}{\int_0^T D_t X h(t) dt}\Big].$$

Similarly $\Phi^{'}(X)$ and Y does not depend on t, such that

$$E\Big[\frac{\int_0^T \Phi'(X) D_t X h(t) Y dt}{\int_0^T D_t X h(t) dt}\Big] = E\Big[\Phi'(X) Y \frac{\cancel{\int_0^T D_t X h(t) dt}}{\cancel{\int_0^T D_t X h(t) dt}}\Big] = E[\Phi'(X)Y].$$

b. Deduce that

$$E[\Phi(X)\delta(u)] = E[\Phi'(X)Y] \tag{2}$$

Solution
From Skorokhod integral we get the following:

- $\delta(u)$ - adjoint operator for D
- $dan(\delta) \subset L^2(\Omega \times [0,T])$ with values in $L^2(\mu)$ s.t. $\delta(u)$ is a unique in $L^2(\mu)$, therefore $\forall F \in \mathbb{D}$,

$$E[F\delta(u)] = E[< DZ, u >]$$

For our case we have got : $X \in \mathbb{D}, \Phi \in C^1 \cap Lip(\mathbb{R},\mathbb{R}) \subset \mathcal{S}-$ set of elementary random variables.
So $\Phi \in \mathcal{S}$, therefore $\Phi \in \mathbb{D}$ as $\mathcal{S} \subset \mathbb{D}$ in $\|\cdot\|_{\mathbb{D}}$
Hence from previous question we get

$$E[< DZ, u >_{L^2([0,T])}] = E[\Phi'(X)Y]$$

but

$$E[< DZ, u >_{L^2[0,T]}] = E[\Phi\delta(u)] \to E[\Phi\delta(u)] = E[\Phi'(X)Y]$$

2 Part B. Delta for asian options

We consider a risky asset whose dynamic on $[0,T]$ is given by the following equation

$$dX_t^x = rX_t^x + \sigma X_t^x dB_t; X_0^x = x \tag{3}$$

where $r \leq 0$ is risk free rate, $x \leq 0$ the initial condition, $\sigma \leq 0$ and $(B_t)_{t\in([0,T])}$ the standard Brownian motion defined in the course.

Let $F = \Phi(\int_0^T X_s^x ds)$ where $\Phi \in C^1 \cap Lip(\mathbb{R},\mathbb{R})$. We consider:

$$P(x) = e^{-rT} E\Big[\Phi\Big(\int\limits_0^T X_s^x ds\Big)\Big]$$

a) Prove that $P \in C^1$ and that:

$$\Delta(x) = \frac{dP(x)}{dx} = \frac{e^{-rT}}{x} E\Big[\Phi'\Big(\int\limits_0^T X_s^x ds\Big)\int\limits_0^T X_s^x ds\Big]; \forall \Phi \in C^1 \cap Lip(\mathbb{R},\mathbb{R}). \tag{4}$$

Solution

From dynamic (3) we get:

$$X_t^x = xe^{(r-\frac{\sigma^2}{2})t+\sigma B_t} \forall t \in [0,T],$$

therefore $X_t^x \in C^1$ and also

$$\frac{dX_t^x}{dx} = e^{(r-\frac{\sigma^2}{2})t+\sigma B_t} = \frac{X_t^x}{x},$$

so:

$$\frac{d(\int_0^T X_t^x dt)}{dx} = \frac{\int_0^T X_s^x ds}{x}$$

Since $\Phi \in C^1 \cap Lip(\mathbb{R},\mathbb{R})$, there is a sequence $\Phi_n \in C_k^1(\mathbb{R},\mathbb{R})$ that converge to Φ in L^2, therefore by Lebesgue's differentiation theorem, we have $P \in C^1$.

Also according to Chain Rule we have:

$$\Delta(x) = \frac{dP(x))}{dx} = e^{-rT}\frac{d}{dx}E\Big[\Phi\Big(\int\limits_0^T X_s^x ds\Big)\Big] = e^{-rT}E\Big[\frac{d\Phi}{dx}\Big(\int\limits_0^T X_s^x ds\Big)\Big]$$

$$= e^{-rT}E\Big[\frac{d\Phi}{dx}\Big(\int\limits_0^T X_s^x ds\Big)\frac{d(\int\limits_0^T X_s^x ds)}{dx}\Big] = e^{-rT}E\Big[\Phi'\Big(\int\limits_0^T X_s^x ds\Big)\frac{\int\limits_0^T X_s^x ds}{x}\Big]$$

$$= \frac{e^{-rT}}{x}E\Big[\Phi'\Big(\int\limits_0^T X_s^x ds\Big)\int\limits_0^T X_s^x ds\Big]$$

b) Taking h=1 and supposing that we may apply the results of part A, show that

$$\Delta(x) = \frac{dP(x))}{dx} = \frac{e^{-rT}}{x} E\Big[\Phi\Big(\int_0^T X_s^x ds\Big)\Pi_1\Big]; \forall \Phi \in C^1 \cap Lip(\mathbb{R},\mathbb{R}) \tag{5}$$

where

$$\Pi_1 = \frac{\int_0^T X_s^x ds}{\int_0^T sX_s^x ds}\left[\frac{B_T}{\sigma} + \frac{\int_0^T s^2 X_s^x ds}{x\int_0^T sX_s^x ds}\right]$$

Solution

From (2), we have:

$$E[\Phi(X)\delta(u)] = E[\Phi'(X)Y]$$

Previously we proved (4), so, here we get the following:

$$\Delta(x) = \frac{e^{-rT}}{x} E\Big[\Phi'\Big(\int_0^T X_s^x ds\Big)\int_0^T X_s^x ds\Big] = \frac{e^{-rT}}{x} E\Big[\Phi\Big(\int_0^T X_s^x ds\Big)\delta(u)\Big]$$

where

$$u(t) = \frac{Y_t}{\int_0^T D_t X dt} = \frac{\int_0^T X_u^x du}{\int_0^T D_u(\int_0^T X_s^x ds)du} \tag{6}$$

Here, $D_u(\int_0^T X_s^x ds)$ can be expressed as:

$$D_u(\int_0^T X_s^x ds) = (\int_0^T D_u X_s^x ds) = (\int_0^T \sigma X_s^x D_u W_s ds) = \sigma(\int_0^T X_s^x \mathbb{1}_{u\le s} ds) = \sigma\int_u^T X_s^x ds \tag{7}$$

Then, combining (6) and (7) we get the following:

$$u(t) = \frac{\int_0^T X_u^x du}{\int_0^T \sigma \int_u^T X_s^x ds du}$$

So, Δ is equal to:

$$\Delta(x) = \frac{e^{-rT}}{x} E\left[\Phi\Big(\int_0^T X_s^x ds\Big)\delta\left(\frac{\int_0^T X_u^x du}{\int_0^T \sigma \int_u^T X_s^x ds du}\right)\right] \tag{8}$$

$$\int_0^T \sigma \int_u^T X_s^x ds du = \sigma \int_0^T X_s^x (\int_0^s du) ds = \sigma \int_0^T X_s^x s ds$$

Hence,

$$\Delta(x) = \frac{e^{-rT}}{x} E\left[\Phi\Big(\int_0^T X_s^x ds\Big)\delta\left(\frac{\int_0^T X_u^x du}{\sigma \int_0^T X_s^x s ds}\right)\right] \tag{9}$$

To prove (5) we need to prove the following:

$$\delta\left(\frac{\int_0^T X_u^x du}{\sigma\int\limits_0^T X_s^x s ds}\right) = \frac{\int_0^T X_s^x ds}{\int_0^T sX_s^x ds}\left[\frac{B_T}{\sigma} + \frac{\int_0^T s^2 X_s^x ds}{x\int_0^T sX_s^x ds}\right]$$

We can get this relation from the proposition (7) in lecture notes, which says the following:
"Let $u \in dom(\delta)$ and $F \in \mathbb{D}$ such that $F \times u \in dom(\delta)$ then $\delta(Fu) = F\delta(u) - \int\limits_0^T D_t F u_t dt$"
More over, we can say that:

$$\delta(Fu) = F\int\limits_0^T u_t dB_t - \int\limits_0^T D_t F u_t dt. \tag{10}$$

Let $F = \frac{\int^T X_t^x dt}{\int^T sX_s^x dt}$ and $u = \frac{1}{\sigma}$:

$$\delta(Fu) = \frac{\int\limits_0^T X_t^x dt}{\int\limits_0^T sX_s^x ds}\int\limits_0^T \frac{1}{\sigma} dB_t - \int\limits_0^T D_t\left(\frac{\int\limits_0^T X_t^x dt}{\int\limits_0^T sX_s^x ds}\right)\frac{1}{\sigma}dt = \frac{\int\limits_0^T X_t^x dt}{\int\limits_0^T sX_s^x ds}\frac{B_T}{\sigma} - \int\limits_0^T D_t\left(\frac{\int\limits_0^T X_t^x dt}{\int\limits_0^T sX_s^x ds}\right)\frac{1}{\sigma}dt \tag{11}$$

Here we have the following:

$$D_t\left(\frac{\int\limits_0^T X_t^x dt}{\int\limits_0^T sX_s^x ds}\right) = \frac{(D_t\int\limits_0^T X_t^x dt)\int\limits_0^T sX_s^x ds - (D_t\int\limits_0^T sX_s^x ds)\int\limits_0^T X_t^x dt}{(\int\limits_0^T sX_s^x ds)^2} = -\frac{(D_t\int\limits_0^T sX_s^x ds)\int\limits_0^T X_t^x dt}{(\int\limits_0^T sX_s^x ds)^2}$$

So, we get the following:

$$\begin{aligned}
\delta(Fu) &= \frac{\int\limits_0^T X_t^x dt}{\int\limits_0^T sX_s^x ds}\frac{B_T}{\sigma} + \int\limits_0^T \frac{(D_t\int\limits_0^T sX_s^x ds)\int\limits_0^T X_t^x dt}{(\int\limits_0^T sX_s^x ds)^2}\frac{1}{\sigma}dt = \frac{\int\limits_0^T X_t^x dt}{\int\limits_0^T sX_s^x ds}\frac{B_T}{\sigma} + \frac{\int\limits_0^T(D_t\int\limits_0^T sX_s^x ds)\frac{1}{\sigma}dt\int\limits_0^T X_t^x dt}{(\int\limits_0^T sX_s^x ds)^2} \\
&= \frac{\int\limits_0^T X_t^x dt}{\int\limits_0^T sX_s^x ds}\frac{B_T}{\sigma} + \frac{\frac{\int\limits_0^T sX_s^x ds}{x}(\int\limits_0^T \sigma dB_t)\frac{1}{\sigma}\int\limits_0^T X_t^x dt}{(\int\limits_0^T sX_s^x ds)^2} = \frac{\int\limits_0^T X_t^x dt}{\int\limits_0^T sX_s^x ds}\frac{B_T}{\sigma} + \frac{\int\limits_0^T s^2 X_s^x ds\int\limits_0^T X_t^x dt}{x(\int\limits_0^T sX_s^x ds)^2} \\
&= \frac{\int\limits_0^T X_t^x dt}{\int\limits_0^T sX_s^x ds}\left[\frac{B_T}{\sigma} + \frac{\int\limits_0^T s^2 X_s^x ds}{x(\int\limits_0^T sX_s^x ds)}\right]
\end{aligned} \tag{12}$$

c) Taking h=X_t^x and supposing that we may apply the results of part A, show that

$$\Delta(x) = \frac{dP(x))}{dx} = \frac{e^{-rT}}{x}E\Big[\Phi\Big(\int\limits_0^T X_s^x ds\Big)\Pi_2\Big]; \forall\Phi \in C^1 \cap Lip(\mathbb{R},\mathbb{R}) \tag{13}$$

where

$$\Pi_2 = \frac{2}{\sigma^2}\Big[\frac{X_T^x - x}{\int_0^T X_s^x ds} - r\Big] + 1$$

Solution

From (2), we have:

$$E[\Phi(X)\delta(u)] = E[\Phi'(X)Y]$$

Previously we proved (4), so, here we get the following:

$$\Delta(x) = \frac{e^{-rT}}{x} E\Big[\Phi'\Big(\int_0^T X_s^x ds\Big) \int_0^T X_s^x ds\Big] = \frac{e^{-rT}}{x} E\Big[\Phi\Big(\int_0^T X_s^x ds\Big)\delta(u)\Big]$$

where

$$u(t) = \frac{Y_t X_t^x}{\int_0^T X_t^x D_t X dt} = \frac{\int_0^T X_u^x du X_t^x}{\int_0^T X_u^x D_u(\int_0^T X_s^x ds) du} \tag{14}$$

Here, $D_u(\int_0^T X_s^x ds)$ can be expressed as:

$$D_u(\int_0^T X_s^x ds) = (\int_0^T D_u X_s^x ds) = (\int_0^T \sigma X_s^x D_u W_s ds) = \sigma(\int_0^T X_s^x \mathbb{1}_{u \le s} ds) = \sigma \int_u^T X_s^x ds \tag{15}$$

Then, combining (14) and (15) we get the following:

$$u(t) = \frac{\int_0^T X_u^x du X_t^x}{\int_0^T \sigma \int_u^T X_s^x ds X_u^x du}$$

So, Δ is equal to:

$$\Delta(x) = \frac{e^{-rT}}{x} E\Big[\Phi\Big(\int_0^T X_s^x ds\Big)\delta\Big(\frac{\int_0^T X_u^x du X_t^x}{\int_0^T \sigma \int_u^T X_s^x ds X_u^x du}\Big)\Big] \tag{16}$$

$$\int_0^T \sigma \int_u^T X_s^x ds X_u^x du = \sigma \int_0^T X_s^x (\int_0^s X_u^x du) ds = \sigma \int_0^T \frac{1}{2} d(\int_0^T X_u^x du)^2 ds = \frac{\sigma}{2}(\int_0^T X_u^x du)^2$$

Hence,

$$\Delta(x) = \frac{e^{-rT}}{x} E\Big[\Phi\Big(\int_0^T X_s^x ds\Big)\delta\Big(\frac{\int_0^T X_u^x du X_t^x}{\frac{\sigma}{2}(\int_0^T X_u^x du)^2}\Big)\Big] = \frac{e^{-rT}}{x} E\Big[\Phi\Big(\int_0^T X_s^x ds\Big)\frac{2}{\sigma}\delta\Big(\frac{X_t^x}{(\int_0^T X_u^x du)}\Big)\Big] \tag{17}$$

To prove (13) we need to prove the following:

$$\delta\Big(\frac{X_t^x}{\int_0^T X_u^x du}\Big) = \frac{2}{\sigma^2}\Big[\frac{X_T^x - x}{\int_0^T X_s^x ds} - r\Big] + 1$$

Similarly, we can get this relation from the proposition (7) in lecture notes.

Using (10) with $F = \frac{1}{\int^T X_u^x du}$, so that $D_u F = -\frac{\sigma \int_u^T X_s^x ds}{(\int_0^T X_s^x ds)^2}$ and $u = X_t^x$ after a few computations we get the following:

$$\delta(\frac{X_t^x}{\int_0^T X_u^x du}) = \frac{\int_0^T X_t^x dB_t}{\int_0^T X_t^x dt} + \frac{\sigma}{2} = \frac{X_T^x - x}{\sigma \int_0^T X_t^x dt} - \frac{2r}{\sigma^2} + 1$$

So, we get the following:

$$\Delta = \frac{e^{-rT}}{x} E\Big[\Phi(\int_0^T X_t^x dt)\Big[\frac{2}{\sigma^2}\Big(\frac{X_T^x - x}{\int_0^T X_t^x dt} - r\Big) + 1\Big]\Big]$$

d) Let $\mathbb{W}$ be the set of all the random variables Π in $L^2(\Omega)$ such that

$$\Delta(x) = e^{-rT} E\Big[\Phi(\int_0^T X_s^x ds)\Pi\Big]; \forall \Phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R}). \tag{18}$$

Prove that the weight $\Pi_0 \in \mathbb{W}$ minimizing the variance of $\Phi\big(\int_0^T X_s^x ds\big)\Pi$ is given by

$$\Pi_0 = E\Big[\Pi|\sigma(\int_0^T X_s^x ds)\Big]$$

where Π is an arbitrary element of $\mathbb{W}$.

Solution

Let denote $\epsilon = \Pi - \Pi_0$ and $\mathcal{F} = \sigma(\int_0^T X_s^x ds)$. We have $E[\epsilon|\mathcal{F}] = E[\Pi|\mathcal{F}] - \Pi_0 =$, so ϵ is independent from $\mathcal{F}$. Moreover, $E[\epsilon] = E[\Pi] - E[E[\Pi|\mathcal{F}]] = 0$. We have $\Pi = \Pi_0 + \epsilon$, so $\Pi^2 = \Pi_0^2 + 2\Pi_0\epsilon + \epsilon^2$. Then

$$E[\Phi^2\Big(\int_0^T X_s^x ds\Big)(\Pi^2 - \Pi_0^2)] = E[\Phi^2\Big(\int_0^T X_s^x ds\Big)2\Pi_0\epsilon + \Phi^2\Big(\int_0^T X_s^x ds\Big)\epsilon^2] = E[\Phi^2\Big(\int_0^T X_s^x ds\Big)2\Pi_0\epsilon] + E[\Phi^2\Big(\int_0^T X_s^x ds\Big)\epsilon^2].$$

ϵ is independent from $\mathcal{F}$ and ϵ is independent from $\Phi^2\Big(\int_0^T X_s^x ds\Big)2\Pi_0$, as the latter is $\mathcal{F}$-measurable.

$E[\Phi^2\Big(\int_0^T X_s^x ds\Big)2\Pi_0\epsilon] = E[\Phi^2\Big(\int_0^T X_s^x ds\Big)2\Pi_0]E[\epsilon] = 0$ as $E[\epsilon] = 0$.

Similarly we have $E[\Phi^2\Big(\int_0^T X_s^x ds\Big)\epsilon^2] = E[\Phi^2\Big(\int_0^T X_s^x ds\Big)]E[\epsilon^2]$, therefore

$$E[\Phi^2\Big(\int_0^T X_s^x ds\Big)(\Pi^2 - \Pi_0^2)] = E[\Phi^2\Big(\int_0^T X_s^x ds\Big)]E[\epsilon^2]$$

We have both $E[\epsilon^2] \geq 0$ and $E[\Phi^2\Big(\int_0^T X_s^x ds\Big)] \geq 0$ (expectation of positive random variable), so:

$$E[\Phi\Big(\int_0^T X_s^x ds\Big)^2(\Pi^2 - \Pi_0^2)] \geq 0$$

, therefore

$$[E[\Phi\Big(\int_0^T X_s^x ds\Big)^2\Pi^2] \geq [E[\Phi\Big(\int_0^T X_s^x ds\Big)^2\Pi_0^2$$

So, $E[\Phi\Big(\int_0^T X_s^x ds\Big)^2\Pi^2] - E[\Phi\Big(\int_0^T X_s^x ds\Big)\Pi^2 \geq E[\Phi\Big(\int_0^T X_s^x ds\Big)^2\Pi_0^2] - E[\Phi\Big(\int_0^T X_s^x ds\Big)\Pi]^2$

Or $E[\Phi(\int_0^T X_s^x ds)\Pi] = E[\Phi(\int_0^T X_s^x ds)\Pi_0]$ because $\Pi_0 = E[\Pi|\sigma(\int_0^T X_s^x ds)]$.

Hence, $E[\Phi(\int_0^T X_s^x ds)^2\Pi^2] - E[\Phi(\int_0^T X_s^x ds)\Pi]^2 \geq E[\Phi(\int_0^T X_s^x ds)^2\Pi_0^2] - E[\Phi(\int_0^T X_s^x ds)\Pi_0]^2$.

Which is equivalent to $Var(\Phi(\int_0^T X_s^x ds)\Pi) \geq Var(\Phi(\int_0^T X_s^x ds)\Pi_0), \forall \Pi$.
So, $\Pi_0 = \underset{x}{\operatorname{argmin}} Var(\Phi(\int_0^T X_s^x ds)\Pi)$.

f) For $\Pi \in L^2$ prove that

$$g(x) = e^{-rT}E[\Phi(\int_0^T X_s^x ds)\Pi]; \forall \Phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R})$$

$$\Updownarrow$$

$g(x) = e^{-rT}E[\Phi(\int_0^T X_s^x ds)\Pi]; \forall \Phi \in C(\mathbb{R}, \mathbb{R})$ with linear growth

Proof for $\Downarrow$

Φ is C^1 so it is C. $\forall x, |\Phi(x) - \Phi(0)| \leq k|x|$ as Φ is Lipschitz

Using triangle inequality $|\Phi(x)| \leq k|x| + |\Phi(0)|$, as $|\Phi(0)|$ is a constant, we can denote it as function with linear growth.

So we conclude that $g(x) = e^{-rT}E[\Phi(X_T^x)\Pi]; \forall \Phi \in C(\mathbb{R}, \mathbb{R})$ with linear growth

Proof for $\Uparrow$

Since we consider function g with $\forall \Phi \in C(\mathbb{R}, \mathbb{R})$ with linear growth, then we can consider as a polynomial function of order 1. Therefore, $Var(\Phi) < \infty$. We then apply Lemma 7 which states that the relation $\Delta_0(x) = E[f(X_T^x)\Pi]$ (and similarly $\Gamma_0(x) = E[f(X_T^x)\Pi]$) is true for all $f(X_T^x) \in L^2 \iff$ it is true for all $f \in C_K^\infty$. So, $\Phi \in C^1 \Rightarrow \Phi \in Lip(\mathbb{R}, \mathbb{R}) \Rightarrow \Phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R})$

3 Part B. Numerical Part

For the third part we used python environment. This part is written in the following structure:

- Task
- Python code
- Description and conclusions

a. Using Monte-Carlo Methods compute the prices of asian options with payoff $(\int_0^T X_s^x ds - K_1)_+$ and $\mathbb{1}_{K_1 < \int_0^T X_s^x ds < K_2}$ where K_1=100 and K_2=110. Precise the empirical variance of each estimator and the corresponding confidence intervals. Study empirically the convergence of each estimators when the number of simulations increase (for example: from N=1000 to N = 51000 by 2000).

```
import numpy as np
from random import gauss
from math import exp, sqrt, log
import matplotlib.pyplot as plt
from scipy.stats import norm
from mpl_toolkits.mplot3d import axes3d, Axes3D
```

Above you can find all the libraries that were used for the program.

```
T=1
r=0.03
sigma=0.2
x0=100
K1=100
K2=110
m=50
n=10000
```

Above - all the constant input variables and their values (except m and n, they will be changed).

```
def X(x0,t,b,r,s):
    y=x0*exp(((r-0.5*s**2)*t)+s*b)
    return (y)

def dig(K1,K2,x):
    return int(K1<x<K2)
```

Here we denote functions, that count the price of asian and digital options at the moment t.

```
def prix(n,m,x,T,k1,k2,r,s):
    y=0
    pr_call=0
    var_call=0
    pr_dig=0
    var_dig=0
    for i in range(n):
        b=0
        for j in range(m):
            db=gauss(0.0,sqrt(T/m))
            b += db
            x_s=X(x,T*j/m,b,r,s)
            y +=x_s
        y *= T/m
        pr_call += max(y-k1,0)
        var_call += max(y-k1,0)*max(y-k1,0)

        pr_dig += dig(k1,k2,y)
        var_dig += dig(k1,k2,y)*dig(k1,k2,y)
    var_call = ((var_call/n - (pr_call/n)**2))*exp(-2*r*T)/n
    pr_call *= exp(-r*T)/n
    var_dig = ((var_dig/n - (pr_dig/n)**2))*exp(-2*r*T)/n
    pr_dig *=  exp(-r*T)/n
    res=[pr_call,pr_dig, var_call, var_dig]
    return res
```

Above is the function that counts the value of both options. As the output we get the expected value and empirical variance. Under the inputs above we obtain the following result:

- Expected value of asian call 0.65
- Expected value of digital call 0.31
- Variance of asian call 7.1e-05
- Variance of digital call 2.05e-05

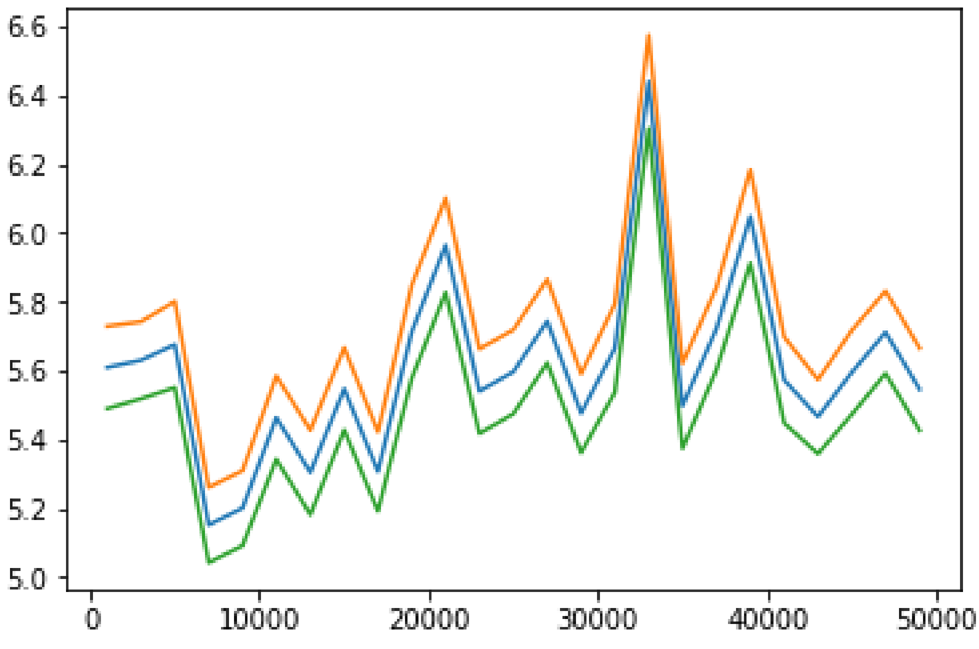
```
#study the convergence with N
m=200  #on fixe M
n=1000
N=51000
step=2000
fin=int((N-n)/step+1)
Res=[]
price_call=[]
price_dig=[]
var1=[]
var2=[]
```

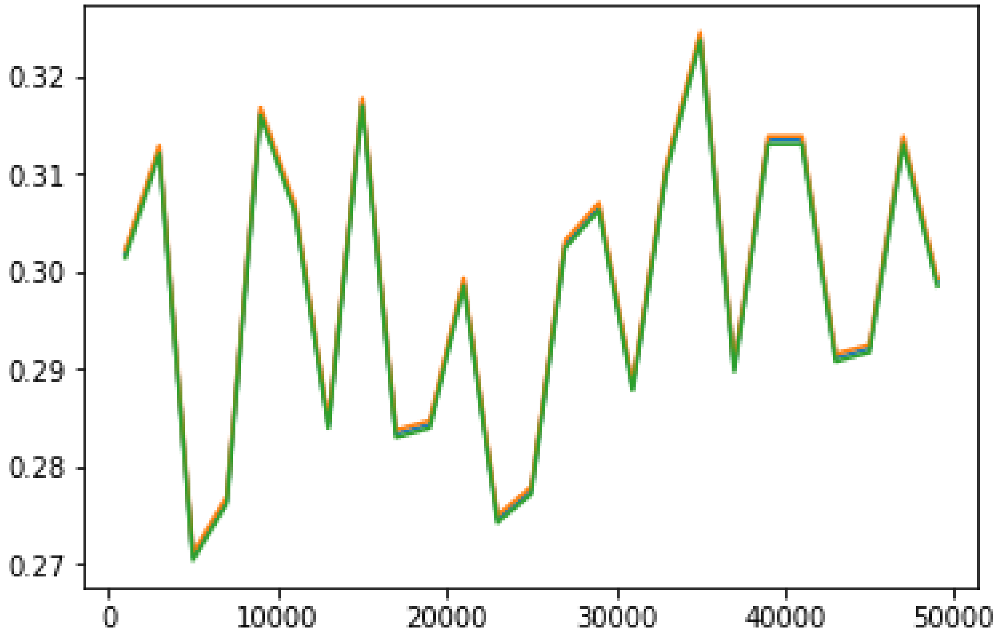
```
kg=[]
up_bound_call=[]
low_bound_call=[]
up_bound_dig=[]
low_bound_dig=[]
for i in range(n,N,step):
    kg.append(i)
    Res=prix(n,m,x0,T,K1,K2,r,sigma)
    price_call.append(Res[0])
    price_dig.append(Res[1])
    up_bound_call.append(Res[0]+1.96*Res[2])
    low_bound_call.append(Res[0]-1.96*Res[2])
    up_bound_dig.append(Res[1]+1.96*Res[3])
    low_bound_dig.append(Res[1]-1.96*Res[3])
plt.figure(1)
plt.plot(kg,price_call)
plt.plot(kg,up_bound_call)
plt.plot(kg,low_bound_call)
plt.show

plt.figure(2)
plt.plot(kg,price_dig)
plt.plot(kg,up_bound_dig)
plt.plot(kg,low_bound_dig)

plt.show
```

Here we study the convergence with number of iterations.





Here we can see that both options don't converge to some value.That is very unpleasant result which tells us that no matter how many iterations we use, we still can be quite far from true value. The result stays the same - nothing converges - both after changing the parameters m and n.

b) Compute $\Delta(x)$ for the two options using the finite difference method. Precise the empirical variance of each estimator and the corresponding confidence intervals. What happens when you make ϵ varies?

```
#calculation of delta

def delta(n,m,x,T,k1,k2,r,s,eps):
    y1=0
    y2=0
    delta_call=0
    var_delta_call=0
    delta_dig=0
    var_delta_dig=0
    for i in range(n):
        b=0
        for j in range(m):
            db=gauss(0.0,sqrt(T/m))
            b += db
            x_s_1=X(x+eps,T*j/m,b,r,s)
            x_s_2=X(x-eps,T*j/m,b,r,s)
            y1 +=x_s_1
            y2 +=x_s_2
        y1 *= T/m
        y2 *= T/m

        delta_call += max(y1-k1,0)-max(y2-k1,0)
        var_delta_call += (max(y1-k1,0)-max(y2-k1,0))*(max(y1-k1,0)-max(y2-k1,0))

        delta_dig += dig(k1,k2,y1)-dig(k1,k2,y2)
        var_delta_dig += (dig(k1,k2,y1)-dig(k1,k2,y2))*(dig(k1,k2,y1)-dig(k1,k2,y2))
    var_delta_call = ((var_delta_call/n - (delta_call/n)**2))*exp(-2*r*T)/(4*n*eps**2)
    delta_call *= exp(-r*T)/(2*n*eps)
    var_delta_dig = ((var_delta_dig/n - (delta_dig/n)**2))*exp(-2*r*T)/(4*n*eps**2)
    delta_dig *=  exp(-r*T)/(2*n*eps)
```

```
    res=[delta_call,delta_dig, var_delta_call, var_delta_dig]
    return res

#example
n=10000
m=200
eps=0.1

DRes=delta(n,m,x0,T,K1,K2,r,sigma,eps)
```

As the output we get the expected value and empirical variance computed under $\epsilon = 0.1$. Under the inputs above we obtain the following result:

- Expected delta of asian call 0.58
- Expected delta of digital call 4.85e-04
- Variance of delta of asian call 2.89e-05
- Variance of delta of digital call 2.99e-05

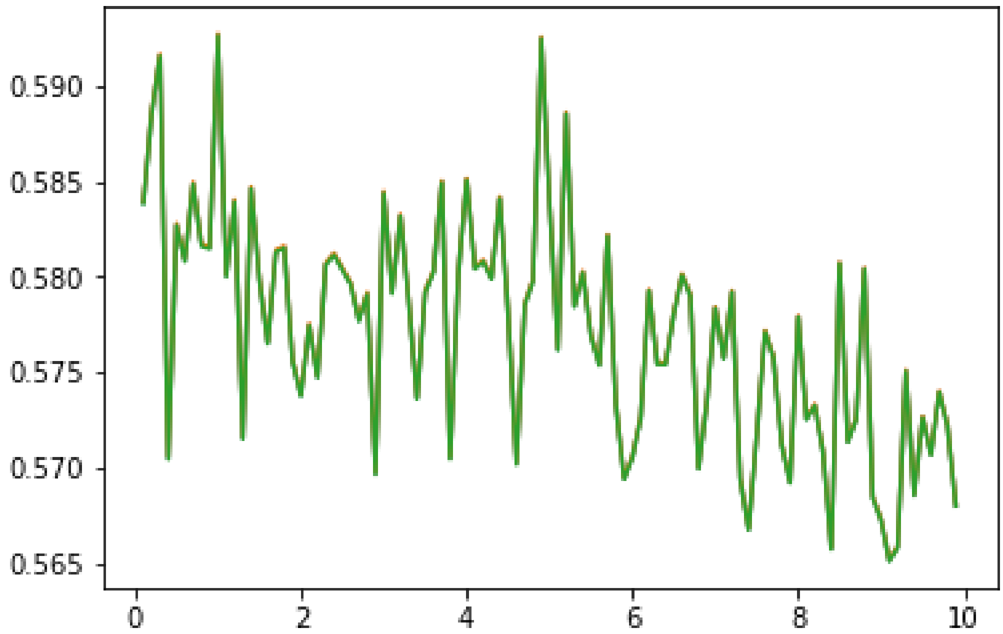
We can see that result differs a bit. By this moment we cannot say whether the difference is significant and if so, which one is better.

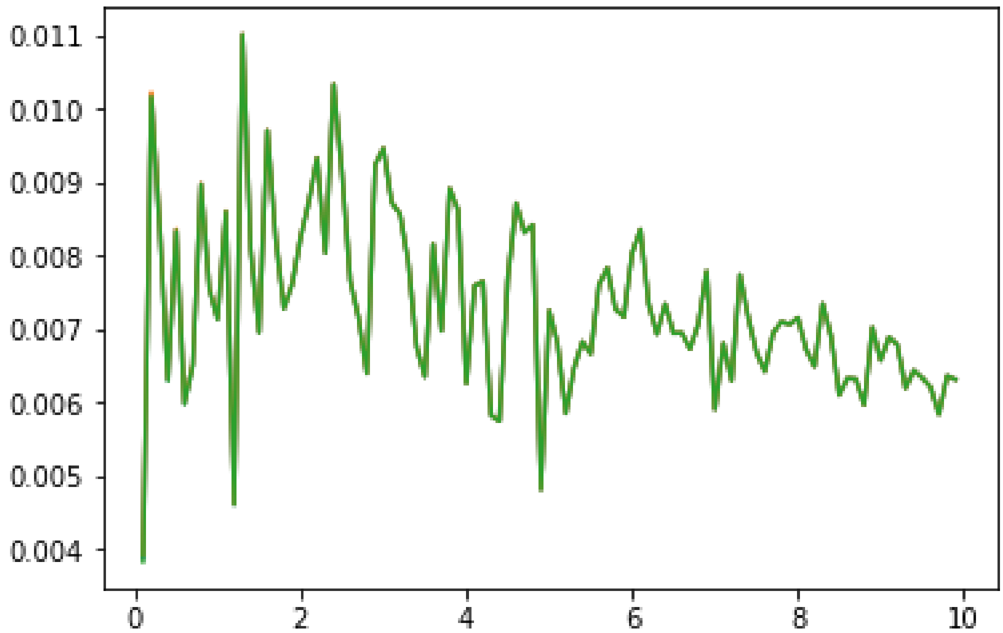
Then let's check convergence under ϵ.

```
#study the convergence with eps
delta_call=[]
delta_dig=[]
up_bound_call=[]
low_bound_call=[]
up_bound_dig=[]
low_bound_dig=[]
kg=[]
for e in range(1,100):
    kg.append(e/10)
    Res=delta(n,m,x0,T,K1,K2,r,sigma,e/10)
    delta_call.append(Res[0])
    delta_dig.append(Res[1])
    up_bound_call.append(Res[0]+1.96*Res[2])
    low_bound_call.append(Res[0]-1.96*Res[2])
    up_bound_dig.append(Res[1]+1.96*Res[3])
    low_bound_dig.append(Res[1]-1.96*Res[3])

plt.figure(1)
plt.plot(kg,delta_call)
plt.plot(kg,up_bound_call)
plt.plot(kg,low_bound_call)
plt.show

plt.figure(2)
plt.plot(kg,delta_dig)
plt.plot(kg,up_bound_dig)
plt.plot(kg,low_bound_dig)
```





Here we can see even using finite difference method doesn't solve the problem of not convergence. But, it's easy to see that the higher the value of ϵ, the smaller value of options.

c) Compute $\Delta(x)$ for the two options using the two methods that may be deduced from part B. Precise the empirical variance of each estimator and the corresponding confidence intervals.

```
def Pi1(n,m,x,T,k1,k2,r,s):
    y=0
    y1=0
    y2=0
    pr_call=0
    var_call=0
    pr_dig=0
```

```
    var_dig=0
    delta1_call=0
    var_delta1_call=0
    delta1_dig=0
    var_delta1_dig=0
    for i in range(n):
        b=0
        for j in range(m):
            db=gauss(0.0,sqrt(T/m))
            b += db
            x_s=X(x,T*j/m,b,r,s)
            y +=x_s
            y1 +=x_s*j
            y2 +=x_s*j*j
        y *= T/m
        y1 *= T/m
        y2 *= T/m
        delta1_call += max(y-k1,0)
        var_delta1_call += max(y-k1,0)*max(y-k1,0)

        delta1_dig += dig(k1,k2,y)
        var_delta1_dig += dig(k1,k2,y)*dig(k1,k2,y)

    Pi1=y/y1*(b/s+y2/(x*y1))

    var_delta1_call = ((var_delta1_call/n - (delta1_call/n)**2))*exp(-2*r*T)/n
    delta1_call *= exp(-r*T)/n*Pi1
    var_delta1_dig = ((var_delta1_dig/n - (delta1_dig/n)**2))*exp(-2*r*T)/n
    delta1_dig *=  exp(-r*T)/n*Pi1
    res=[delta1_call,delta1_dig, var_delta1_call, var_delta1_dig]
    return res

Pi1(n,m,x0,T,K1,K2,r,sigma)

def Pi2(n,m,x,T,k1,k2,r,s):
    y=0
    y1=0
    y2=0
    delta1_call=0
    var_delta1_call=0
    delta1_dig=0
    var_delta1_dig=0
    for i in range(n):
        b=0
        for j in range(m):
            db=gauss(0.0,sqrt(T/m))
            b += db
            x_s=X(x,T*j/m,b,r,s)
            y +=x_s
        y1 += x_s
        y *= T/m
        delta1_call += max(y-k1,0)
        var_delta1_call += max(y-k1,0)*max(y-k1,0)

        delta1_dig += dig(k1,k2,y)
        var_delta1_dig += dig(k1,k2,y)*dig(k1,k2,y)
```

```
    y1 *= 1/n
    Pi2=(2/(s*s)*((y1-x)/y -r))+1

    var_delta1_call = ((var_delta1_call/n - (delta1_call/n)**2))*exp(-2*r*T)/n
    delta1_call *= exp(-r*T)/n*Pi2
    var_delta1_dig = ((var_delta1_dig/n - (delta1_dig/n)**2))*exp(-2*r*T)/n
    delta1_dig *=  exp(-r*T)/n*Pi2
    res=[delta1_call,delta1_dig, var_delta1_call, var_delta1_dig]
    return res

Pi2(n,m,x0,T,K1,K2,r,sigma)
```

As the output we get the expected value and empirical variance computed under $\epsilon = 0.1$. Under the inputs above we obtain the following result:

- Expected delta of asian call using first Pi 0.305
- Expected delta of digital call using first Pi 0.142
- Variance of delta of asian call using first Pi 7.34e-06
- Variance of delta of digital call using first Pi 2.01e-05
- Expected delta of asian call using second Pi 0.598
- Expected delta of digital call using second Pi 0.283
- Variance of delta of asian call using second Pi 7.31e-06
- Variance of delta of digital call using second Pi 2.04e-05

Here we get quite similar result. The expected values are almost the same, confidence intervals coincides. So, we can say, that all methods work fine. But the options provided do not converge. Hence, it is very difficult to make a correct decision. We can see that finite difference method gives higher variances. That proves the proposition, that Π gives the lowest variances. Then, comparing those two methods, we see, that for asian option we have lower variance when using second Π and for digital - first Π.