MTH2004

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

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Vector Calculus and Applications

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You have 24 hours to complete this paper from the time of its release
Intended duration 2 hours

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is an **OPEN BOOK** examination

USEFUL FORMULAE

$$
\begin{array}{rlr}
(i) & \vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = \vec{w}\cdot(\vec{u}\times\vec{v}) & (1)\\
(ii) & \vec{u}\times(\vec{v}\times\vec{w}) = \vec{v}(\vec{u}\cdot\vec{w}) - \vec{w}(\vec{u}\cdot\vec{v}) & (2)\\
(iii) & \nabla\cdot(\varphi\vec{v}) = \varphi\nabla\cdot\vec{v} + \vec{v}\cdot\nabla\varphi & (3)\\
(iv) & \nabla\times(\varphi\vec{v}) = \varphi\nabla\times\vec{v} + \nabla\varphi\times\vec{v} & (4)\\
(v) & \nabla\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\nabla\times\vec{u}) - \vec{u}\cdot(\nabla\times\vec{v}) & (5)\\
(vi) & \nabla\times(\vec{u}\times\vec{v}) = \vec{u}(\nabla\cdot\vec{v}) - \vec{v}(\nabla\cdot\vec{u}) + (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v} & (6)\\
(vii) & \vec{u}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{u}) = \nabla(\vec{u}\cdot\vec{v}) - (\vec{u}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{u} & (7)\\
(viii) & \nabla\times(\nabla\times\vec{u}) = \nabla(\nabla\cdot\vec{u}) - \nabla^2\vec{u} & (8)
\end{array}
$$

Formulae in cylindrical polar coordinates (R, φ, z)**:**

$$\vec{r} \equiv x\hat{x} + y\hat{y} + z\hat{z} = R\cos\varphi\,\hat{x} + R\sin\varphi\,\hat{y} + z\,\hat{z}.$$

Also,

$$\hat{R} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y},$$

$$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}.$$

The unit vectors do not change with R or z, but

$$\frac{\partial\hat{R}}{\partial\varphi} = \hat{\varphi}, \quad \frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{R}, \quad \frac{\partial\hat{z}}{\partial\varphi} = 0.$$

The vector differential operators are given by

$$\nabla f = \frac{\partial f}{\partial R}\,\hat{R} + \frac{1}{R}\frac{\partial f}{\partial\varphi}\,\hat{\varphi} + \frac{\partial f}{\partial z}\,\hat{z},$$

$$\nabla\cdot\vec{F} = \frac{1}{R}\frac{\partial}{\partial R}(RF_R) + \frac{1}{R}\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z}\,,$$

$$\nabla\times\vec{F} = \frac{1}{R}\begin{vmatrix} \hat{R} & R\,\hat{\varphi} & \hat{z} \\ \partial/\partial R & \partial/\partial\varphi & \partial/\partial z \\ F_R & RF_\varphi & F_z \end{vmatrix},$$

$$\nabla^2 f = \frac{1}{R}\,\frac{\partial}{\partial R}\left(R\,\frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\,\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}\,.$$

Formulae in spherical polar coordinates (r, θ, φ)**:**

$$\vec{r} \equiv x\hat{x} + y\hat{y} + z\hat{z} = r\sin\theta\cos\varphi\,\hat{x} + r\sin\theta\sin\varphi\,\hat{y} + r\cos\theta\,\hat{z}\ .$$

Also,

$$\hat{\theta} = \cos\theta\cos\varphi\,\hat{x} + \cos\theta\sin\varphi\,\hat{y} - \sin\theta\,\hat{z},$$

$$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}.$$

The unit vectors do not change with r, but

$$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}, \quad \frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}, \quad \frac{\partial\hat{\varphi}}{\partial\theta} = 0,$$

$$\frac{\partial\hat{r}}{\partial\varphi} = \sin\theta\,\hat{\varphi}, \quad \frac{\partial\hat{\theta}}{\partial\varphi} = \cos\theta\,\hat{\varphi}\,, \quad \frac{\partial\hat{\varphi}}{\partial\varphi} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}.$$

The vector differential operators are given by

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\,\hat{\varphi},$$

$$\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}\,,$$

$$\nabla\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\,\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\varphi \\ F_r & rF_\theta & r\sin\theta F_\varphi \end{vmatrix},$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}\,.$$

SECTION A

1. (a) Given the numbers a_{ij} and b_i

$$\begin{aligned} a_{11} &= 2\,, & a_{12} &= 1\,, & a_{13} &= 0\,, \\ a_{21} &= 1\,, & a_{22} &= -1\,, & a_{23} &= 2\,, \\ a_{31} &= 0\,, & a_{32} &= 2\,, & a_{33} &= 2\,, \\ b_1 &= 1\,, & b_2 &= 2\,, & b_3 &= 3\,, \end{aligned}$$

evaluate

(i) a_{ii} ,

(ii) $a_{2j}a_{1i}\delta_{ij}$,

(iii) $\epsilon_{jlm}b_j\delta_{ii}a_{lm}$,

where δ_{ij} is the Kronecker delta and ϵ_{jlm} is the alternating tensor. (6)

(b) Assume a Cartesian coordinate system. Evaluate the following expressions:

(i) $\nabla\phi$,

(ii) $\nabla \cdot \phi\vec{r}$,

(iii) $\nabla \times (f(r)\vec{r})$,

where $\vec{r}$ is the position vector, $r = |\vec{r}|$, $f(r)$ is an arbitrary differentiable scalar function of r and $\phi = xr^2$. (8)

(c) Consider the scalar field $\phi = x^2yz - 4xyz^2$ and the point P $(1, 3, 1)$.

(i) In what direction from the point P is the directional derivative of ϕ a maximum and what is the magnitude of this maximum?

(ii) Calculate the directional derivative of ϕ at the point P in the direction of the vector $2\hat{x} - \hat{y} - 2\hat{z}$. (7)

(d) Sketch the vector field $\vec{F} = (x + y)\,\hat{x} - x\,\hat{y}$. (5)

(e) Calculate the line integral

$$\int_C \vec{F} \cdot \vec{dr}$$

of $\vec{F} = (x^2 + 2y, y)$ from the point $(x, y) = (1, 0)$ to the point $(2, 2)$ along the following paths C in the xy-plane:

(i) a straight line,

(ii) an arc of the parabola $y = x^2 - x$. (14)

(f) An eagle soars on the ascending helical path given in Cartesian coordinates by

$$\vec{r}(t) = (a\cos t, a\sin t, bt)$$

where t is measured in hours, $a = 3$ miles and $b = 4$ miles per hour. What is the distance travelled by the eagle over the time interval $0 \le t \le 2$? (4)

(g) Find the area of the helicoidal surface whose parametric representation is

$$\vec{r}(u,v) = (u\cos v, u\sin v, \frac{u^2}{2}) \quad , 0 \le u \le 1 \quad , 0 \le v \le 2\pi\,.$$

(6)

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SECTION B

2. (a) By considering normals to surfaces, find the angle between the surfaces of the sphere $x^2+y^2+z^2=2$ and the cylinder $x^2+y^2=1$ for all z, at any point where they intersect. (8)

   (b) (i) State the divergence theorem.

   (ii) By evaluating the appropriate surface and volume integrals verify the divergence theorem for the vector field

$$\vec{F} = x\hat{x} + z\hat{z}\,,$$

where the surface is given by $x^2 + y^2 + z^2 = a^2$ and a is a positive constant. [You may find the identity $2\cos^2\phi = 1 + \cos 2\phi$ useful.] (17)

[25]

3. (a) Consider the motion of an incompressible fluid under the action of gravity. Assume that the fluid density ρ is constant. You are given Euler's equation

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right) = -\nabla(p+\Pi)$$

where $\Pi = \rho g z$ is the gravitational potential.

   (i) Show that Euler's equation can be reduced to the Bernoulli equation

$$\frac{\partial\vec{u}}{\partial t} + \nabla H = \vec{u}\times\vec{\zeta}$$

where ζ is the vorticity and H is the Bernoulli function which should be defined. (4)

   (ii) By taking the curl of this equation, show that the vorticity equation may be written as

$$\frac{D\vec{\zeta}}{Dt} = (\vec{\zeta}\cdot\nabla)\vec{u}$$

where the material derivative D/Dt should be defined. (5)

   (iii) A shallow rectangular tank of horizontal cross-sectional area A contains water of depth h. There is a small hole of cross-sectional area a in the base of the tank.

A. Assume that the flow is irrotational and steady. If the downwards flow speed at height h is V and the pressure is the atmospheric pressure p_0, show that

$$\frac{V^2}{2} + gh = \frac{v^2}{2}$$

where v is the speed of the flow out of the hole. (5)

B. Since mass is conserved $va = VA$. Show that

$$\frac{dh}{dt} = -\left[\frac{2gh}{\left(\frac{A^2}{a^2} - 1\right)}\right]^{1/2}$$

(6)

C. Assuming that $A \gg a$, show that the time taken to empty the tank is

$$\left(\frac{A}{a}\right)\left(\frac{2h_0}{g}\right)^{1/2}$$

where h_0 is the height of the fluid at time $t = 0$ (5)

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4. For an electric field $\vec{E}$ defined at position $\vec{r}$, you are given Gauss's law, stating that the total of the electric flux out of a closed surface S is equal to the charge Q enclosed divided by the permittivity ϵ_0 (a constant):

$$\oiint_S \vec{E} \cdot d\vec{S} = Q/\epsilon_0 \, .$$

(a) Consider a sphere of radius R, which carries a volume charge density $\rho = k/r$ where k is a constant and r is the radial coordinate.

(i) Find the total charge Q inside the spherical volume.

(ii) Using Gauss's law, calculate the electrostatic field $\vec{E}(\vec{r})$ and its potential ϕ for points in space where $r \leq R$ and $r > R$, knowing that ϕ tends to zero as $r \to \infty$. Sketch the radial profiles for both expressions $|\vec{E}|$ and ϕ. (21)

(b) Given an electric field

$$\vec{E}(\vec{r}) = \frac{Q}{\pi\epsilon_0 r^2}(1 - \cos(5r))\hat{r}$$

in spherical coordinates and with constant Q, find the charge density $\rho(r)$. (4)

[**25**]