Problem 1 Consider the problem of aiming a piece of artillery. The muzzle of the gun is located at $(0,0)$, the target is located at $(d,0)$ and the objective is to solve the equation

$$r(\theta) = d, \tag{1}$$

where $r(\theta)$ is the range of the gun when fired using the elevation θ. There is a single elevation ψ for which

$$r'(\psi) = 0. \tag{2}$$

The value of ψ can be treated as a known value. Let $r_{\max}$ denote the maximum range of the gun.

1. (5 points) Let $0 < d < r_{\max}$ denote the location of a target which is within range of the gun. Find a set of elevations $a < b < c$ such that

   $$a < \theta_{\text{low}} < b < \theta_{\text{high}} < c \tag{3}$$

   where θ_{low} and θ_{high} are the elevations for the low and the high trajectory from the gun to the target.

2. (10 points) Let a and b denote a pair of elevations, such that

   $$0 < a < b < \frac{\pi}{2} \tag{4}$$

   and

   $$r(a) < d < r(b) \tag{5}$$

   By continuity there is an elevation $\theta \in (a,b)$ such that

   $$r(\theta) = d. \tag{6}$$

   Show how to estimate both θ and the relative error in terms of a and b.

3. (5 points) Let $0 < \bar{\theta} < \frac{\pi}{2}$ denote a good approximation of a solution θ of equation (1). Explain how to estimate the relative error regardless of the algorithm used to compute $\bar{\theta}$. Remember to explain when your estimate is reliable.

4. (5 points) In exact arithmetic we can always solve equation (1) exactly. In practice there is a hard limit for how accurately the equation can be solved, regardless of which algorithm is used. Explain why this is the case.

Problem 2 Consider the problem of aiming a piece of artillery. The muzzle is located at $(0,0)$ and the gun is being fired into a constant wind which blows parallel to the ground, i.e.

$$\mathbf{w} = -(w, 0), \quad w > 0. \tag{7}$$

Careful computation has established that the range of the gun is affected by the wind as follows

Elevation (degrees)	w=0 m/s Range (meters)	w=1 m/s Range (meters)	w=3 m/s Range (meters)
0	0	0	0
10	11453	11438	11410
20	15847	15817	15758
30	17670	17628	17544
40	17810	17759	17657
50	16573	16515	16400
60	14087	14026	13904
70	10415	10354	10232
80	5635	5577	5459
90	0	-56	-169

Unfortunately, the data corresponding to $w = 2$ m/s has been lost.

1. (5 points) As far as possible, do a basic sanity check of the numbers and explain why they are behaving as we have a right to expect.
2. (5 points) Find a polynomial of degree 2 which can be used to approximate the range $r_{40,2}$ of the gun when the elevation is 40 degrees and the wind is $w = 2$ m/s.
3. (5 points) Given the above information, what is the best error bound that you have for the range $r_{40,2}$ and what additional information do you need in order to compute a better bound?
4. (5 points) Give a physical reason why the case of $w = 1000$ m/s is absurd and irrelevant.
5. (5 points) Give a purely mathematical reason why the case of $w = 1000$ m/s can not be investigated using the data given.

Problem 3 The trajectory of a shell has been been computed numerically using one of the standard Runge-Kutta methods and time steps $2^j h$, where $h > 0$ and $j = 0, 1, 2, 3, 4, 5, 6$. The computed x coordinates corresponding to the smallest time step, as well as several auxiliary numbers are given in Figure 1. These numbers include Richardson's fractions

$$F_h = \frac{x_{2h} - x_{4h}}{x_h - x_{2h}}, \tag{8}$$

and his error estimates

$$E_h = \frac{x_h - x_{2h}}{2^p - 1}. \tag{9}$$

Consider the hypothesis that there exists functions α and β such that

$$x(t) - x_h(t) = \alpha(t)h^p + \beta(t)h^q + O(h^r), \quad 0 < p < q < r, \tag{10}$$

where $x(t)$ is the x coordinate of the shell at time t and $x_h(t)$ is the computed value of the x coordinate of the shell at time t based on the time step.

1. (5 points) Scanning the rows of Figure 1 what evidence can you find to support this hypothesis?
2. (5 points) What evidence do you find to support the idea that $p = 2$?
3. (5 points) What evidence do you find to support the idea that $q = 3$?
4. (5 points) What evidence do you find to support the idea that $\frac{\beta(t)}{\alpha(t)}$ changes sign in the interval from 60 to 70 seconds?
5. (5 points) Given that the kill radius of the shell is 15 m what evidence do you find to support the idea that the time step h is far smaller than absolutely necessary in order to compute an adequate firing solution?

t (second	x(h) (meters)	F(16h)	F(8h)	F(4h)	F(2h)	F(1h)	E(h) (meters)
0.000000e+00	0.00000000e+00	NaN	NaN	NaN	NaN	NaN	0.00000000e+00
1.000000e+01	4.42636428e+03	4.40208302e+00	4.19652547e+00	4.09686835e+00	4.04805949e+00	4.02393324e+00	2.02982097e-03
2.000000e+01	7.59249294e+03	4.45743911e+00	4.21826652e+00	4.10638756e+00	4.05249823e+00	4.02607450e+00	1.81580364e-03
3.000000e+01	1.01303686e+04	4.55971288e+00	4.26316429e+00	4.12731515e+00	4.06258854e+00	4.03102724e+00	1.33200320e-03
4.000000e+01	1.22788706e+04	4.75684747e+00	4.35446129e+00	4.17104781e+00	4.08396761e+00	4.04159417e+00	8.52917015e-04
5.000000e+01	1.41255017e+04	5.22603846e+00	4.58941697e+00	4.28826081e+00	4.14246662e+00	4.07081183e+00	4.30192140e-04
6.000000e+01	1.56951869e+04	7.26961082e+00	5.99834507e+00	5.15708498e+00	4.63447539e+00	4.33433302e+00	7.88617763e-05
7.000000e+01	1.70007109e+04	-1.55835424e+00	2.57912803e+00	3.45735493e+00	3.76068409e+00	3.88742139e+00	-2.05576781e-04
8.000000e+01	1.80630173e+04	2.41140693e+00	3.46805668e+00	3.78135133e+00	3.90087282e+00	3.95280986e+00	-4.37688787e-04
9.000000e+01	1.89121966e+04	3.03968536e+00	3.66792881e+00	3.86268274e+00	3.93770491e+00	3.97035220e+00	-6.31441520e-04
1.000000e+02	1.95822498e+04	3.29433745e+00	3.75543545e+00	3.89934326e+00	3.95452039e+00	3.97840897e+00	-7.96168201e-04

Figure 1: The partial results of tracking an artillery shell numerically for $T = 100$ seconds using a standard Runge-Kutta method and time steps $2^j h$, where the smallest time step is $h = 2^{-5}$ seconds. The x-coordinate of the shell is provided every 10 seconds. Richardson's fractions and his standard error estimate has been computed for each of these times.

Problem 4 Consider the problem of computing the side a in an arbitrary triangle ABC, see figure 2. In exact arithmetic we have the formula

$$a^2 = b^2 + c^2 - 2bc\cos(A) \tag{11}$$

where a, b, and c are sides and A is the angle opposing the side A.

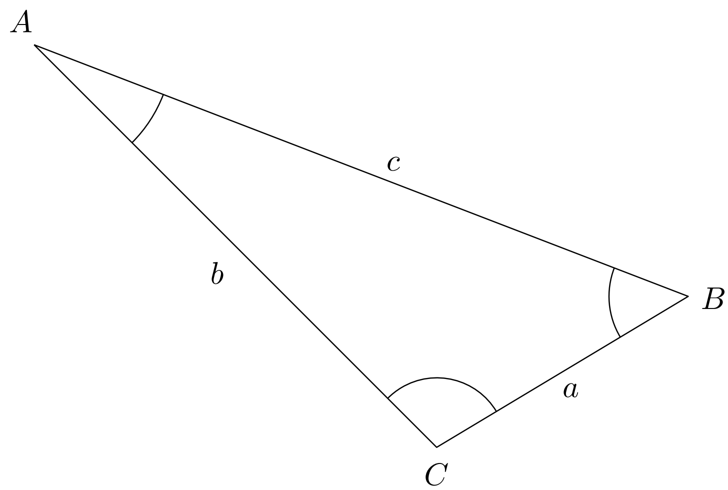

Figure 2: A general triangle ABC with sides a, b, and c.

1. (5 points) Characterize the set of triangles for which this formula is likely to fail miserably when the computations are carried out in floating point arithmetic.
2. (10 points) Show that the formula

   $$a^2 = (b-c)^2 + 4bc\sin^2\left(\frac{A}{2}\right). \tag{12}$$

   is mathematically equivalent to formula (11).

   Hint: Trigonometric identities are a pain to memorize. Fortunately, the problem can be completed using the definitions

   $$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \tag{13}$$

   where i is the imaginary unit, $i^2 = -1$.
3. (10 points) Why will equation (12) always provide sensible results even in floating point arithmetic?

   Hint: The real goal is to compute $a > 0$ rather than a^2.