Problem 1 Let I be an interval and let $h : I \to \mathbb{R}$ be a differentiable function, such that the equation

$$h(x) = 0 \tag{1}$$

has exactly one solution $r \neq 0$. Moreover, $h'(r) \neq 0$.

1. (5 points) Give a real life example of a function h where a small residual is much more important than a small error.
2. (10 points) Explain, why it is theoretically impossible to compute $h(x)$ with a small relative error using floating point arithmetic, when we are very close to the true root r.
3. (10 points) Explain, why it is possible for a real implementation of the bisection algorithm to return an interval $[a_n, b_n]$ which does not contain r, even though the initial interval $[a_0, b_0]$ contains r.

Problem 2 Consider the problem of two dimensional projectile motion. Figure 1 contains a partial artillery table for a particular gun fired under standard conditions, i.e. constant gravity, constant temperature, homogene atmosphere, and no wind.

Elevation theta (degrees)	Range r (meters)	Flight time tau (seconds)	Derivative dr/dtheta (meters/radian)
4.00	6286	10.15	66754
8.00	10053	18.92	43813
12.00	12632	26.75	31091
16.00	14492	33.88	22668
20.00	15847	40.46	16416
24.00	16812	46.59	11395
28.00	17455	52.32	7128
32.00	17819	57.71	3349
36.00	17931	62.77	-99
40.00	17810	67.54	-3317
44.00	17471	72.01	-6372
48.00	16923	76.20	-9310
52.00	16173	80.09	-12162
56.00	15226	83.68	-14951
60.00	14087	86.95	-17691
64.00	12757	89.88	-20385
68.00	11242	92.46	-23026
72.00	9544	94.66	-25591
76.00	7671	96.46	-28031
80.00	5635	97.83	-30254
84.00	3456	98.76	-32104
88.00	1168	99.23	-33313

Figure 1: A partial ballistics table for a gun fired under standard conditions.

1. (5 points) Explain, why you can be virtually certain that the maximal range of the gun corresponds to an elevation between 32 and 36 degrees.
2. (5 points) Estimate one of two possible flight times to a target $r = 8000$ meters. Your relative error must less than 2 procent.
3. (10 points) Use Newton's method to compute the elevation corresponding to both the high and the low trajectory to a target at $r = 10000$ meters.

   Hint: Do not forget to convert between degrees and radians as needed!
4. (5 points) Explain why it is theoretically possible to have two rounds hit the same target at $r = 10000$ meters *simultanously* using a *single* gun provided the crew can reload in less than 60 seconds.

Problem 3 Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{x-\cos(x)\sin(x)}{x^3}, & x \neq 0 \\ \frac{2}{3}, & x = 0. \end{cases} \tag{2}$$

1. (5 point) Show that f is differentiable for $x \neq 0$.
2. (5 point) Show that $f(x) \to f(0)$ for $x \to 0$, $x \neq 0$.
3. (5 point) The following MATLAB commands

```
>> f=@(x)(x-cos(x).*sin(x))./x.^3;
>> x=linspace(-1,1,1025)*2^-23;
>> plot(x,f(x));
```

have been used to generate the graph given in Figure 2. Explain, why you can immediately conclude that this graph is not an accurate representation of the mathematical reality and we cannot rely on the above implementation of f.

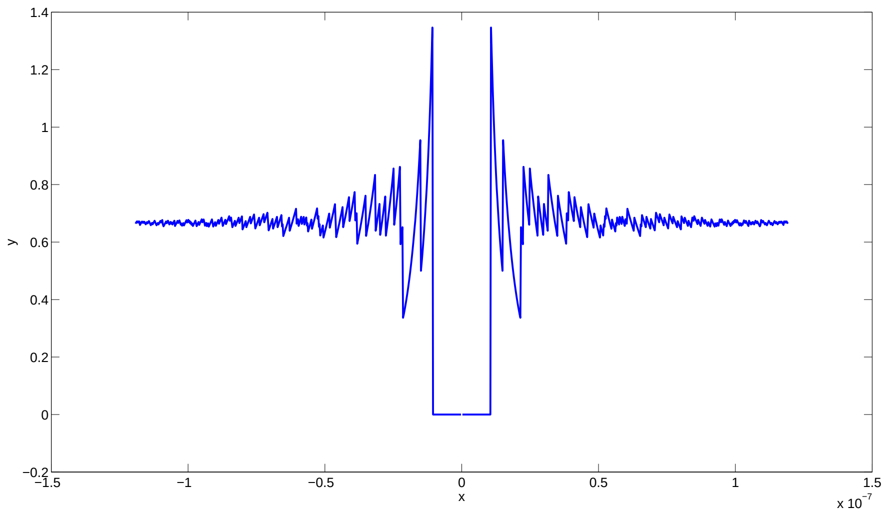

Figure 2: A plot of the function $x \to f(x)$ using a naive MATLAB implementation.

4. (5 point) Show that f can be computed using the *alternating* series

$$f(x) = 4\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+3)!}(2x)^{2j}. \tag{3}$$

Hint: You are free to exploit the identity

$$2\sin(x)\cos(x) = \sin(2x), \tag{4}$$

as well as the Taylor series expansion for $x \to \sin(x)$ at $x = 0$, i.e.

$$\sin(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}x^{2j+1}. \tag{5}$$

5. (5 point) Let $T_n(x)$ denote the polynomial

$$T_n(x) = 4 \sum_{j=0}^{n} \frac{(-1)^j}{(2j+3)!} (2x)^{2j}, \tag{6}$$

and consider $T_n(x)$ as an approximation of $f(x)$. Show that

$$\frac{|f(x) - T_0(x)|}{|f(x)|} \leq 10^{-14}, \tag{7}$$

for all $x \in [-2^{-23}, 2^{-23}]$.

Hint: Alternating series converge when the absolute value of the terms are decreasing monotonically. The truncation error is closely related to the absolute value of the first term which is neglected.

Problem 4 An unknown function $g : \mathbb{R} \to \mathbb{R}$ has been differentiated numerically using an unknown numerical method $D = D(h)$ at the point $x = 2$ and step sizes

$$h = h(k) = h_0 \times 2^{-k}, \quad k = 0, 1, 2, \dots 19, \quad h_0 = \frac{3}{2}. \tag{8}$$

It is known that g is infinitely often differentiable at every point $x \in \mathbb{R}$. Figure 3 contains the computed approximations of $g'(2)$ as well as some auxiliary numbers.

k	Dh	Fractions	Error estimate
		(D2h-D4h)/(Dh-D2h)	(Dh-D2h)/(2^p-1)
0	-1.8671501966806722e-01	0.0000000000000000e+00	0.000000e+00
1	-2.2506471579681531e-01	0.0000000000000000e+00	-1.278323e-02
2	-2.3563099682161945e-01	3.6294412422613886e+00	-3.522094e-03
3	-2.3833698477779799e-01	3.9047775510893512e+00	-9.019960e-04
4	-2.3901755946574385e-01	3.9760337904218725e+00	-2.268582e-04
5	-2.3918795880368884e-01	3.9939984283598209e+00	-5.679978e-05
6	-2.3923057462996411e-01	3.9984989812080878e+00	-1.420528e-05
7	-2.3924122958621533e-01	3.9996247070843687e+00	-3.551652e-06
8	-2.3924389338776564e-01	3.9999061679227328e+00	-8.879339e-07
9	-2.3924455934204994e-01	3.9999765946503314e+00	-2.219848e-07
10	-2.3924472583085313e-01	3.9999944233381335e+00	-5.549627e-08
11	-2.3924476745310130e-01	3.9999954467843857e+00	-1.387408e-08
12	-2.3924477785870599e-01	3.9999836090244298e+00	-3.468535e-09
13	-2.3924478046001241e-01	4.0001456969398443e+00	-8.671021e-10
14	-2.3924478111014955e-01	4.0011657017377171e+00	-2.167124e-10
15	-2.3924478127264592e-01	4.0009331205529239e+00	-5.416546e-11
16	-2.3924478131448268e-01	3.8840577498726221e+00	-1.394559e-11
17	-2.3924478132539662e-01	3.8333333333333335e+00	-3.637979e-12
18	-2.3924478132297131e-01	-4.5000171662031789e+00	8.084367e-13
19	-2.3924478132782193e-01	-5.0000000000000000e-01	-1.616873e-12

Figure 3: The results obtained by differentiating the function $x \to g(x)$ numerically using the method $D = D(h)$ at the point $x = 2$.

1. (5 points) Explain, why it is likely the numerical method D has order $p = 2$.
2. (5 points) Explain, why Richardson's fractions deviate from 4 for small values of k.
3. (5 points) Explain, why the computed value of Richardson's fraction deviates from 4 for large values of k.
4. (5 points) Find the range of k where the computed value of Richardson's fraction behaves as if it had been computed in exact arithmetic.
5. (5 points) Explain, why the number of correct significant figures of the computed error estimate is in all likelihood maximal at $k = 11$.