Problem 1 The infamous conquistador Bernal Diaz del Castillo has been experimenting with Richardson extrapolation in the context of numerical integration. Bernal's results of applying the trapezoidal rule to the function

$$f(x) = x^2 - 1, \quad x \in [0, 1] \tag{1}$$

are given in Figure 1.

1. (5pt) Demonstrate how Bernal could have recovered the correct integral using only the values T_h and T_{2h}.
2. (5pt) Compare the error estimates to the actual error. Find the 6 error error estimates which are worthless.
3. (5pt) Explain how the worthless error estimates could have been identified using Richardson's fractions only!
4. (5pt) Of the 6 worthless error estimates, there are 2 which are bad in a manner which is fundamentally different from the others. Find them and explain why they are particularly misleading.
5. (5pt) The final error estimate is extremely small and is has the correct sign. Why is it still useless?

N	Th	(T2h-T4h)/(Th-T2h)	(Th-T2h)/(2^p-1)	I-Th	Comparison
1	-0.5000000000000000	N/A	N/A	N/A	N/A
2	-0.6250000000000000	N/A	-4.1666666666666664e-02	-4.1666666666666630e-02	14
4	-0.6562500000000000	4.0000000000000000e+00	-1.0416666666666666e-02	-1.0416666666666630e-02	14
8	-0.6640625000000000	4.0000000000000000e+00	-2.6041666666666665e-03	-2.6041666666666297e-03	13
16	-0.6660156250000000	4.0000000000000000e+00	-6.5104166666666663e-04	-6.5104166666662966e-04	12
32	-0.6665039062500000	4.0000000000000000e+00	-1.6276041666666666e-04	-1.6276041666662966e-04	12
64	-0.6666259765625000	4.0000000000000000e+00	-4.0690104166666664e-05	-4.0690104166629659e-05	11
128	-0.6666564941406250	4.0000000000000000e+00	-1.0172526041666666e-05	-1.0172526041629659e-05	11
256	-0.6666641235351562	4.0000000000000000e+00	-2.5431315104166665e-06	-2.5431315103796592e-06	10
512	-0.6666660308837891	4.0000000000000000e+00	-6.3578287760416663e-07	-6.3578287756715923e-07	9
1024	-0.6666665077209473	4.0000000000000000e+00	-1.5894571940104166e-07	-1.5894571936403423e-07	9
2048	-0.6666666269302368	4.0000000000000000e+00	-3.9736429850260414e-08	-3.9736429813252983e-08	8
4096	-0.6666666567325592	4.0000000000000000e+00	-9.9341074625651036e-09	-9.9341074255576700e-09	8
8192	-0.6666666641831398	4.0000000000000000e+00	-2.4835268656412759e-09	-2.4835268286338419e-09	7
16384	-0.6666666660457850	4.0000000000000000e+00	-6.2088171641031898e-10	-6.2088167940288486e-10	6
32768	-0.6666666665114462	4.0000000000000000e+00	-1.5522042910257974e-10	-1.5522039209514560e-10	6
65536	-0.6666666666278616	4.0000000000000000e+00	-3.8805107275644936e-11	-3.8805070268210784e-11	5
131072	-0.6666666666569654	4.0000000000000000e+00	-9.7012768189112340e-12	-9.7012398114770804e-12	5
262144	-0.6666666666696578	2.2930119048660376e+00	-4.2308008948073921e-12	2.9911628729450968e-12	-1
524288	-0.6666666666646417	-2.5303335472875768e+00	1.6720328825196400e-12	-2.0249357746138230e-12	-1
1048576	-0.6666666666660728	-3.5051202482544608e+00	-4.7702582624727563e-13	-5.9385829587199623e-13	0
2097152	-0.6666666666665162	3.2273410115172760e+00	-1.4780769201176250e-13	-1.5043521983670871e-13	1
4194304	-0.6666666666666309	3.8664085188770572e+00	-3.8228679481259555e-14	-3.5749181392930041e-14	0
8388608	-0.6666666666666458	7.7089552238805972e+00	-4.9589961766590323e-15	-2.0872192862952943e-14	-1

Figure 1: The result of applying the trapezoidal rule to the function f. Here $p = 2$ is the order of the trapezoidal rule and I = -2/3 is the exact value of the integral. The last column compares Ricardson's error estimate to the actual error and measure the number of significant figures which they have in common.

Problem 2 Mylady de Winter has taken an interest in the functions $g, h : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = (1-x)^5 \tag{2}$$
$$h(x) = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5 \tag{3}$$

1. (5pt) Show that g and h are equivalent in exact arithmetic.

2. (5pt) Mylady has plotted the graph of h using the MATLAB commands

   ```
   >> x=1+linspace(-1,1,1025)*2^-9;
   >> plot(x,h(x));
   ```

   and her results are given in Figure 2. Explain why this graph signals that h is not a numerically reliable way to evaluate g for certain values of x.

3. (5pt) Suppose Mylady was to use the bisection algorithm to compute the root $x = 1$ for g using h and a randomly selected interval $[a, b]$ containing the root. Barring extreme luck, what kind of relative error should she expect?

4. (10pt) Explain carefully why the expression for h must necessarily experience catastrophic cancellation in the vicinity of $x = 1$.

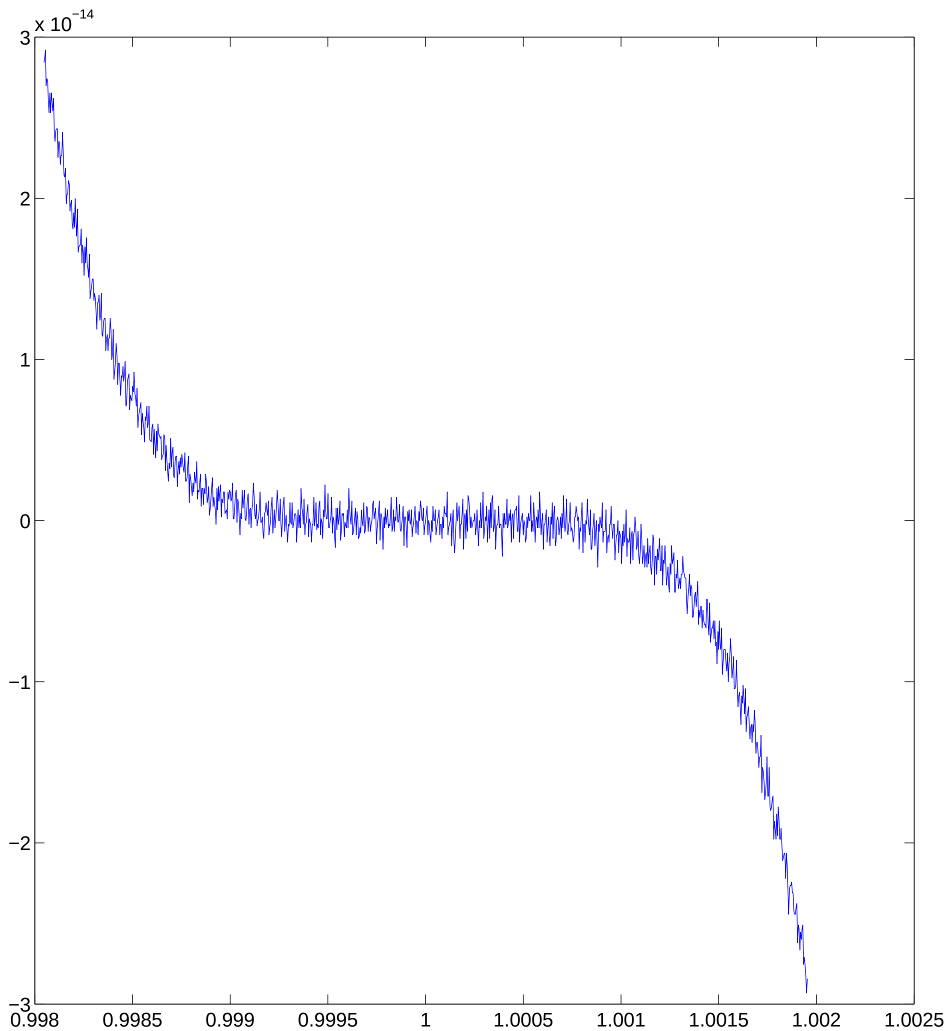

Figure 2: The results of plotting h in `MATLAB`

Problem 3 Hans Diedrich Freiherr von Tiesenhausen is attending the final exam at the U-boat school in Neustadt on the coast of the Baltic Sea. The third problem centers around aiming torpedoes accurately. This is the information which is available

- The submarine is located at origo $(0,0)$
- Target is located at $\mathbf{r} = (r_1, r_2)$
- The targets absolute velocity is $\mathbf{u} = (u_1, u_2)$
- The torpedo travels with speed $w = \|\mathbf{w}\|$ until it runs out of fuel after T seconds.

Here $\|\cdot\|$ denotes the standard Euclidian norm of a vector. The time-to-target is the time between the torpedo is fired and the target is hit.

1. (5pt) Show that the time-to-target τ satisfies the equation

$$t\mathbf{w} = \mathbf{r} + t\mathbf{u} \tag{4}$$

   and show how the firing direction of the torpedo (the vector $\mathbf{w}/\|\mathbf{w}\|$) can be determined from τ and w.

2. (5pt) Explain why τ can be computed by solving the quadratic equation

$$t^2w^2 = \|\mathbf{r}\|^2 + t^2u^2 + 2t\mathbf{r}\cdot\mathbf{u} \tag{5}$$

   where $\mathbf{r}\cdot\mathbf{u}$ is the Euclidian inner product between the vectors $\mathbf{r}$ and $\mathbf{u}$.

3. (5pt) Suppose equation (5) has two real solutions which have been computed accurately. What are the two inequalities which each solution must satisfy in order to have any *practical* value?

4. (10pt) Consider the standard formula for solving (5), i.e.

$$t_\pm = \frac{-B \pm \sqrt{D}}{2A} \tag{6}$$

   where

$$A = \|\mathbf{u}\|^2 - \|\mathbf{w}\|^2 \tag{7}$$
$$B = 2\mathbf{r}\cdot\mathbf{u} \tag{8}$$
$$C = \|\mathbf{r}\|^2 \tag{9}$$
$$D = B^2 - 4AC \tag{10}$$

   Freiherr von Tieshausen believes that catastrophic cancellation is not an issue if the torpedo is much faster than the target. Determine if this is true or not.

   Hint: It may be helpful to recall Cauchy-Schwartz inequality which states

$$|\mathbf{r}\cdot\mathbf{u}| \le \|\mathbf{r}\|\|\mathbf{u}\| \tag{11}$$

Problem 4 Consider the following artillery table for the German 88 mm Flak 36 gun from WWII.

Elevation (degrees)	Range (meters)	Elevation (degrees)	Range (meters)	Elevation (degrees)	Range (meters)
1	1968	31	17752	61	13772
2	3623	32	17819	62	13446
3	5044	33	17869	63	13107
4	6286	34	17905	64	12757
5	7385	35	17925	65	12396
6	8366	36	17930	66	12023
7	9251	37	17922	67	11638
8	10053	38	17898	68	11242
9	10784	39	17861	69	10834
10	11453	40	17810	70	10415
11	12067	41	17745	71	9985
12	12632	42	17667	72	9544
13	13153	43	17576	73	9092
14	13635	44	17471	74	8629
15	14080	45	17353	75	8155
16	14492	46	17223	76	7671
17	14872	47	17079	77	7177
18	15224	48	16923	78	6673
19	15548	49	16754	79	6159
20	15847	50	16573	80	5635
21	16121	51	16379	81	5103
22	16373	52	16173	82	4562
23	16603	53	15955	83	4013
24	16812	54	15724	84	3456
25	17001	55	15481	85	2892
26	17171	56	15226	86	2322
27	17322	57	14959	87	1747
28	17455	58	14680	88	1168
29	17571	59	14390	89	585
30	17670	60	14087	90	0

1. (10pt) It is clear that the maximum range of the gun is less than 18000 m. Interpolate the table using a polynomial of degree 2 and use this polynomial to approximate the maximum range of the gun.
2. (10 pt) Approximate the firing angle for the high trajectory to a target located at a range of 3000 m.
3. (5 pt) Estimate the relative error on your firing angle.