# Teknisk-vetenskapliga beräkningar
# Exam

April 12th, 2012

**Instructions:** This exam consists of four major problems. The maximum score is 100 points or 25 points per problem. You are allowed to use anything which is either printed or written on paper prior to the exam. This includes lecture notes, your own notes, your mandatory projects and any textbook that you might care to reference. Moreover, you may use a programmable calculator. While the class was taught in English you may write your answers in Swedish or English.

**Problem 1** This problem concerns the computation of $s^{1/3}$ where $s$ is a real number.

1. (5pt) Explain why it is enough to solve the equation $f(x) = 0$ where

$$f(x) = s - x^3$$

   and write down Newton's iteration for this problem.

2. (10pt) Let $s \in [1,8]$. Show that the initial guess

$$x_0 = x_0(s) = as + b, \quad a = \frac{1}{7}, \quad b = \frac{3}{7} + \frac{1}{3}\sqrt{\frac{7}{3}}$$

   satisfies

$$|s^{1/3} - x_0| \leq \frac{1}{3}\sqrt{\frac{7}{3}} - \frac{3}{7} \approx 0.0806.$$

3. (10pt) Compute $(12)^{1/3}$ with a relative error of at most $10^{-6}$.

**Problem 2** This problem concerns the fundamental differences between real arithmetic and floating point arithmetic. You will be investigating the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{\sinh(x)-x}{x^3}, & x \neq 0, \\ \frac{1}{6}, & x = 0. \end{cases}$$

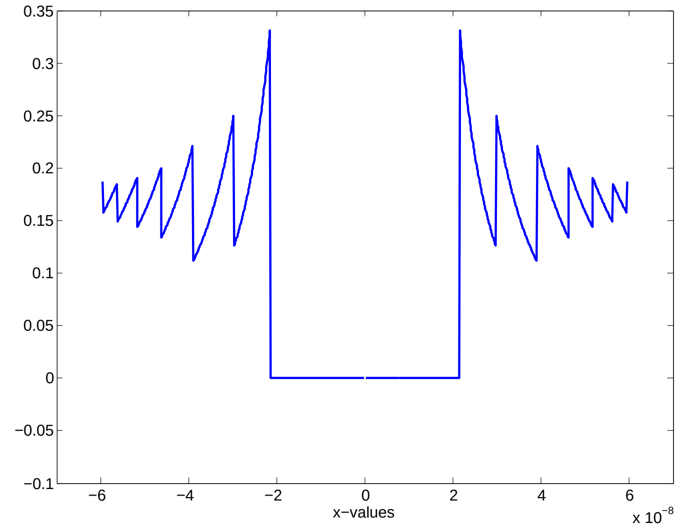
where $\sinh(x)$ is the hyperbolic sinus function given by

$$\sinh(x) = \frac{\exp(x) - \exp(-x)}{2}.$$

1. (5pt) Show that $f$ is continuous for all $x \in \mathbb{R}$, including the special case of $x = 0$.

2. (10pt) The following plot was generated with the MATLAB command

```
>> k=-24; x=linspace(-2^-k,2^-k,1025); f=(sinh(x)-x)./x.^3; plot(x,f)
```




Explain the presence of the violent oscillations as well as the strange plateau in the interval $[-2,2] \times 10^{-8}$.

3. (10pt) Show that the correct double precision representation of $f$ satisfies

$$\text{fl}(f(x)) = \text{fl}\left(\frac{1}{6}\right)$$

for all $[-1,1] \times 10^{-8}$.

**Problem 3** This problem concerns the numerical computation of the integral

$$I = \int_0^1 \exp(-x^2)dx.$$

In the table below the column `Sh` contains the Simpson sum $S_h$ corresponding to the stepsize $h = \frac{1}{2N}$ and the column `fraction` contains the values of the usual fraction $\nu$ given by

$$\nu = \frac{S_{2h} - S_{4h}}{S_h - S_{2h}}$$

Here is the table of computed values

| N | Sh | (Sh-S2h)/15 | fraction |
|---|---|---|---|
| 524288 | 0.7468241328124283 | 5.92e-17 | -1.00000000 |
| 262144 | 0.7468241328124274 | -5.92e-17 | -1.25000000 |
| 131072 | 0.7468241328124283 | 7.40e-17 | -0.50000000 |
| 65536 | 0.7468241328124272 | -3.70e-17 | -0.80000000 |
| 32768 | 0.7468241328124278 | 2.96e-17 | -2.00000000 |
| 16384 | 0.7468241328124273 | -5.92e-17 | -0.62500000 |
| 8192 | 0.7468241328124282 | 3.70e-17 | -0.20000000 |
| 4096 | 0.7468241328124277 | -7.40e-18 | -7.00000000 |
| 2048 | 0.7468241328124278 | 5.18e-17 | -10.42857143 |
| 1024 | 0.7468241328124270 | -5.40e-16 | 13.71232877 |
| 512 | 0.7468241328124351 | -7.41e-15 | 16.05494505 |
| 256 | 0.7468241328125462 | -1.19e-13 | 16.00161782 |
| 128 | 0.7468241328143305 | -1.90e-12 | 15.99929616 |
| 64 | 0.7468241328428812 | -3.05e-11 | 15.99704551 |
| 32 | 0.7468241332996726 | -4.87e-10 | 15.98792233 |
| 16 | 0.7468241406069851 | -7.79e-09 | 15.94720317 |
| 8 | 0.7468242574357303 | -1.24e-07 | 15.70468214 |
| 4 | 0.7468261205274666 | -1.95e-06 | 11.10927205 |
| 2 | 0.7468553797909874 | -2.17e-05 | 0.00000000 |
| 1 | 0.7471804289095103 | 0.00e+00 | 0.00000000 |

1. (5pt) Explain why the fraction $\nu$ misbehaves for large values of $N$.
2. (8pt) Determine the range of values of $N$ for which you can trust the error estimate, but remember to justify your choice.
3. (12pt) Compute the integral $I$ with a relative error which is less than $10^{-12}$.

**Problem 4** Let $A$ be the matrix given by

$$A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

1. (8pt) Compute an $LU$ factorization of $A$.

2. (7pt) Given that $A^{-1}$ satisfies

$$A^{-1} = \frac{1}{256}\begin{bmatrix} 85 & -42 & 20 & -8 \\ -42 & 84 & -40 & 16 \\ 20 & -40 & 80 & -32 \\ -8 & 16 & -32 & 64 \end{bmatrix}$$

compute the condition number of $A$ with respect to the infinity norm.

3. (10pt) Find the floating point representation of the solution of the linear system $Ax = f$, where $f = (1,1,1,1)^T$.