Problem 1 Let $f : \mathbb{R} \to \mathbb{R}$ denote the hyperbolic sine function given by

$$f(x) = \frac{e^x - e^{-x}}{2}. \tag{1}$$

In this problem you will consider the practical problem of computing f reliably on a binary computer using floating point arithmetic.

1. (5 points) Show that f is differentiable and strictly increasing.

2. (5 points) Figure 1 shows the graph generated by the `MATLAB` commands

```
>> f=@(x)(exp(x)-exp(-x))/2;
>> x=single(linspace(-1,1,1025)*2^(-21));
>> plot(x,f(x));
```

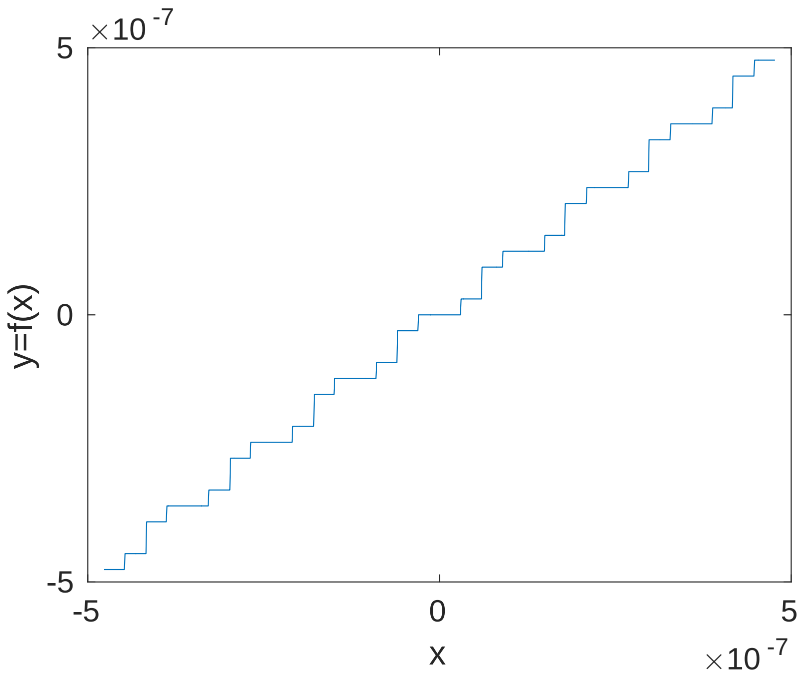

Figure 1: A naive attempt to construct the graph of f.

Why does Figure 1 immediately reveal that f has been computed in an unreliable manner?

Hint: There is more than one reason.

3. (5 points) Show that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{2}$$

4. (10 points) Find a numerically reliable way to evaluate f accurately on the interval $(-2^{-21}, 2^{-21})$ using single precision arithmetic and explain why your approach has no risk of catastrophic cancellation.

Remark 1 Recall that the single precision round off error is $u = 2^{-24}$.

Problem 2 Let $g : [0,1] \to \mathbb{R}$ be given by

$$g(x) = e^x \sin(x) \tag{3}$$

and consider the problem of computing the integral

$$T = \int_0^1 g(x)dx, \qquad \text{T : target}, \tag{4}$$

using the trapezoidal rule with uniform step size. Richardson's technique has been applied to the problem and Figure 2 contains all the results.

k	Th	Fh	Eh
0	1.1436776435894214	0.00000000e+00	0.0000000000000000e+00
1	0.9670583634015181	0.00000000e+00	-5.8873093395967767e-02
2	0.9237047412420658	4.07392212e+00	-1.4451207386484088e-02
3	0.9129205113631961	4.02009440e+00	-3.5947432929565779e-03
4	0.9102279021357744	4.00512253e+00	-8.9753640914054988e-04
5	0.9095549663092243	4.00128678e+00	-2.2431194218338243e-04
6	0.9093867458976557	4.00032208e+00	-5.6073470522869741e-05
7	0.9093446916415655	4.00008054e+00	-1.4018085363387556e-05
8	0.9093341781304715	4.00002014e+00	-3.5045036980152489e-06
9	0.9093315497560062	4.00000503e+00	-8.7612482176554118e-07
10	0.9093308926625975	4.00000126e+00	-2.1903113622823156e-07
11	0.9093307283892565	4.00000027e+00	-5.4757780310055182e-08
12	0.9093306873209228	4.00000015e+00	-1.3689444577913434e-08
13	0.9093306770538404	4.00000040e+00	-3.4223608021595928e-09
14	0.9093306744870665	3.99999485e+00	-8.5559130151106422e-10
15	0.9093306738453759	4.00001799e+00	-2.1389686318447806e-10
16	0.9093306736849540	4.00001869e+00	-5.3473965995938975e-11
17	0.9093306736448422	3.99936894e+00	-1.3370600922731532e-11
18	0.9093306736348192	4.00198274e+00	-3.3409941480044836e-12
19	0.9093306736323076	3.99058480e+00	-8.3721918286983055e-13
20	0.9093306736316904	4.06961684e+00	-2.0572432646304151e-13
21	0.9093306736315040	3.31089934e+00	-6.2135481944854590e-14
22	0.9093306736314545	3.76457399e+00	-1.6505315632760660e-14
23	0.9093306736314497	1.03720930e+01	-1.5913196686293911e-15

Figure 2: The results of integrating g numerically using the trapezoidal rule and many different values of the step size.

- The first column gives $k = \log_2(N)$ where N is the number of sub-intervals.
- Row k contains information pertaining to the step-size $h = 2^{-N}$.
- The second column contains the trapezoidal sums T_h.
- Richardson's fraction

  $$F_h = \frac{T_{2h} - T_{4h}}{T_h - T_{2h}} \tag{5}$$

  is the third column.
- Richardson's error estimate

  $$E_h = \frac{T_h - T_{2h}}{3} \tag{6}$$

  is in the fourth and final column.

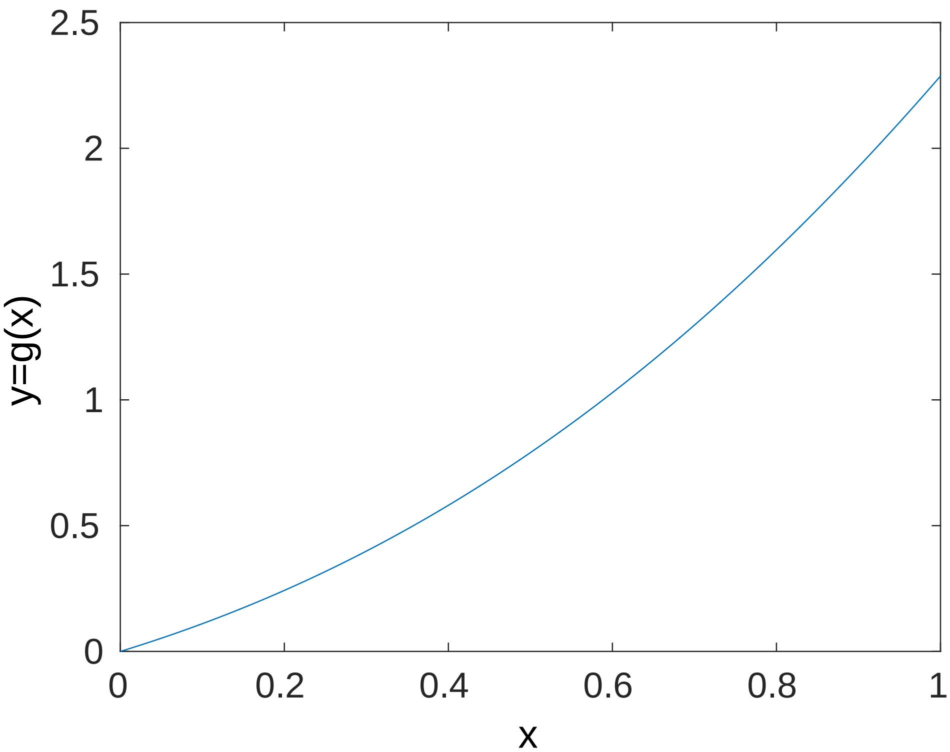

Figure 3: A reliable plot of the graph of g.

A reliable plot of the graph of g is given in Figure 3.

1. (5 points) While some of the error estimates are worthless they all have the same sign. Explain why the error $T - T_h$ will be negative if T_h is computed without any rounding errors at all.

   Hint: Take a good look at the graph of g and recall the geometric interpretation of the trapezoidal rule. Supplement with a short analysis of g and its derivatives as necessary.

2. (5 points) Find the first value of N where the computed value of Richardson's fraction deviates from the expected behavior.

3. (5 points) Explain why some people might believe that $N = 14$ is the correct answer to the previous question.

4. (10 points) Compute the value of T with a relative error which is less than $\tau = 10^{-7}$. Explain why you are confident that your estimate for the relative error is reliable.

Problem 3 Consider the general problem of solving a nonlinear equation

$$h(x) = 0 \tag{7}$$

using an iterative method of your choice. Here $h : \mathbb{R} \to \mathbb{R}$ is infinitely often differentiable and has exactly one root ξ of multiplicity 1, i.e. $h(\xi) = 0$ and $h'(\xi) \neq 0$.

1. (5 points) Explain, how $h(x_k)$ may be extremely small, while x_k is nowhere near ξ.
2. (5 points) Explain, how the difference between successive approximation $|x_{k+1} - x_k|$ can be extremely small, while x_k is nowhere near ξ.
3. (7 points) Explain the value of computing brackets (a_k, b_k) such that $h(a_k)h(b_k) < 0$ and $|b_k - a_k| \to 0$ for $k \to \infty$.
4. (8 points) Suppose that all calculations are done in floating point arithmetic. Explain, why the happy occurrence that $h(x_k)$ is evaluated as 0, should not be taken as evidence that x_k is exactly equal to ξ.

Problem 4 A trajectory for an artillery shell has been carefully computed using an unknown method and different values of the time step h. Richardson's techniques have been applied and all relevant information about the trajectory is given in Figures 4, 5, 6, and 7. As usual, the shell starts at $(0, 0)$ and is fired in the direction of the positive x axis. As usual, gravity acts parallel to the y axis and pulls the shell downward.

1. (5 points) What evidence do you find to support the conjecture that the shell is moving through an atmosphere?

2. (5 points) What evidence do you find to support the conjecture that the trajectory has been computed using a method for which the global error is $O(h^3)$ where h is the time step size

3. (5 points) What evidence do you find to support the conjecture that the shell is aimed at a target located at $(7000, 0)$ and that a real shell would destroy an (unarmored) target located there?

4. (10 points) Why can we say with certainty that the shell reaches the highest point of its trajectory roughly 6 seconds after leaving the muzzle of the gun?

   Remark 2 Notice that this question is valued at 10 points. To get full credit you will have to argue why every single number you reference is trustworthy to the extent which you require for your arguments to be valid.

t	x_h(t)	F_4h(t)	F_2h(t)	F_h(t)	E_h(t)
0.000000e+00	0.00000000e+00	NaN	NaN	NaN	0.00000000e+00
1.162263e+00	8.76094533e+02	8.34392699e+00	8.16877044e+00	8.08378819e+00	-8.24670813e-06
2.324526e+00	1.70144667e+03	8.32016787e+00	8.15807091e+00	8.07870182e+00	-1.42605950e-05
3.486790e+00	2.48165346e+03	8.29797811e+00	8.14808252e+00	8.07395649e+00	-1.86593398e-05
4.649053e+00	3.22142548e+03	8.27706548e+00	8.13867842e+00	8.06949250e+00	-2.18736727e-05
5.811316e+00	3.92476296e+03	8.25720131e+00	8.12975853e+00	8.06526258e+00	-2.42092704e-05
6.973579e+00	4.59509018e+03	8.23820452e+00	8.12124299e+00	8.06122890e+00	-2.58864566e-05
8.135843e+00	5.23535942e+03	8.21993010e+00	8.11306746e+00	8.05736095e+00	-2.70662404e-05
9.298106e+00	5.84813244e+03	8.20226084e+00	8.10517957e+00	8.05363387e+00	-2.78677396e-05
1.046037e+01	6.43564506e+03	8.18510112e+00	8.09753644e+00	8.05002698e+00	-2.83800520e-05
1.162263e+01	6.99985882e+03	8.16837234e+00	8.09010256e+00	8.04652339e+00	-2.86704747e-05

Figure 4: Data related to the x-coordinate of the shell.

t	y_h(t)	F_4h(t)	F_h(t)	F_h(t)	E_h(t)
0.000000e+00	0.00000000e+00	NaN	NaN	NaN	0.00000000e+00
1.162263e+00	6.45662404e+01	8.11995685e+00	8.06265232e+00	8.03218926e+00	-1.15052119e-06
2.324526e+00	1.12507680e+02	8.09089311e+00	8.04971071e+00	8.02606818e+00	-2.03242877e-06
3.486790e+00	1.44920697e+02	8.06622164e+00	8.03877559e+00	8.02090897e+00	-2.72984544e-06
4.649053e+00	1.62729111e+02	8.04544653e+00	8.02961870e+00	8.01660138e+00	-3.29879992e-06
5.811316e+00	1.66718926e+02	8.02807993e+00	8.02201467e+00	8.01303665e+00	-3.77712767e-06
6.973579e+00	1.57564891e+02	8.01365113e+00	8.01574608e+00	8.01010995e+00	-4.19067116e-06
8.135843e+00	1.35851114e+02	8.00171668e+00	8.01060850e+00	8.00772285e+00	-4.55724428e-06
9.298106e+00	1.02087271e+02	7.99186872e+00	8.00641434e+00	8.00578521e+00	-4.88921389e-06
1.046037e+01	5.67215101e+01	7.98374043e+00	8.00299537e+00	8.00421635e+00	-5.19520874e-06
1.162263e+01	1.50839035e-01	7.97700855e+00	8.00020370e+00	8.00294539e+00	-5.48126714e-06

Figure 5: Data related to the y coordinate of the shell.

t	x'_h(t)	F_4h(t)	F_2h(t)	F_h(t)	E_h(t)
0.000000e+00	7.77446486e+02	NaN	NaN	NaN	0.00000000e+00
1.162263e+00	7.31078833e+02	8.77590525e+00	8.36025272e+00	8.17351231e+00	4.19329581e-07
2.324526e+00	6.89970363e+02	8.75286071e+00	8.35019552e+00	8.16882549e+00	6.82624692e-07
3.486790e+00	6.53267311e+02	8.73309525e+00	8.34152540e+00	8.16477453e+00	8.43479565e-07
4.649053e+00	6.20289724e+02	8.71604866e+00	8.33401423e+00	8.16125665e+00	9.36351528e-07
5.811316e+00	5.90489303e+02	8.70128355e+00	8.32748228e+00	8.15819087e+00	9.83798584e-07
6.973579e+00	5.63418941e+02	8.68845447e+00	8.32178668e+00	8.15551248e+00	1.00081744e-06
8.135843e+00	5.38710360e+02	8.67728579e+00	8.31681257e+00	8.15316947e+00	9.97507479e-07
9.298106e+00	5.16057426e+02	8.66755562e+00	8.31246696e+00	8.15111941e+00	9.80742470e-07
1.046037e+01	4.95203545e+02	8.65908386e+00	8.30867401e+00	8.14932807e+00	9.55239898e-07
1.162263e+01	4.75932004e+02	8.65172327e+00	8.30537140e+00	8.14776633e+00	9.24257214e-07

Figure 6: Data related to the x component of the velocity of the shell.

t	y'_h(t)	F_4h(t)	F_2h(t)	F_h(t)	E_h(t)
0.000000e+00	6.30631576e+01	NaN	NaN	NaN	0.00000000e+00
1.162263e+00	4.82288764e+01	7.33697610e+00	7.77759697e+00	7.91241324e+00	-2.31963751e-08
2.324526e+00	3.44243859e+01	7.63232155e+00	7.88800270e+00	7.95992838e+00	-5.02648995e-08
3.486790e+00	2.14832854e+01	7.79019506e+00	7.94811684e+00	7.98597643e+00	-7.72244039e-08
4.649053e+00	9.27342264e+00	7.88408274e+00	7.98411036e+00	8.00159476e+00	-1.02075057e-07
5.811316e+00	-2.31136169e+00	7.94329575e+00	8.00679145e+00	8.01141103e+00	-1.23911525e-07
6.973579e+00	-1.33571946e+01	7.98174133e+00	8.02140215e+00	8.01769147e+00	-1.42429599e-07
8.135843e+00	-2.39345479e+01	8.00681510e+00	8.03077239e+00	8.02166766e+00	-1.57648342e-07
9.298106e+00	-3.41015013e+01	8.02277929e+00	8.03655084e+00	8.02406057e+00	-1.69752605e-07
1.046037e+01	-4.39062095e+01	8.03223970e+00	8.03975357e+00	8.02531594e+00	-1.79003860e-07
1.162263e+01	-5.33887916e+01	8.03686120e+00	8.04103368e+00	8.02572120e+00	-1.85690237e-07

Figure 7: Data related to the y component of the velocity of the shell.