Problem 1 Consider the quadratic equation

$$2^{-13}x^2 + x - 2^{-13} = 0. \tag{1}$$

Let x_1 and x_2 denote the two roots of this equation.

1. (5 points) Show that if we use the usual solution formula and single precision arithmetic, then we obtain the following approximations of the two roots

$$\hat{x}_1 = 0, \tag{2}$$
$$\hat{x}_2 = -2^{13}. \tag{3}$$

2. (10 points) Compute the absolute and relative error for each root.

3. (10 points) Show how a single precision computer can nevertheless obtain an accurate value for x_1.

   Hint The product $x_1 x_2$ can be read off from equation (1).

Problem 2 Consider the function

$$f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

1. (5 points) Show that the condition number of f at the point $x \neq 0$ is given by

$$\kappa_f(x) = \frac{x\cosh(x)}{\sinh(x)} > 0.$$

2. (10 points) Establish the inequality

$$0 < \kappa_f(x) < 2,$$

for all x such that $0 < |x| < 1$.

3. (10 points) Explain, why it is **not** theoretically impossible to compute $f(x)$ with a relative error τ smaller than 2^{-53}, if you use double precision floating point arithmetic and $\frac{1}{2} < |x| < 1$.

Problem 3 Consider the function

$$g(x) = x^3, \quad x \in [0,1] \tag{4}$$

as well as the corresponding trapezoidal rule T_h with stepsize h, i.e.

$$T_h = \sum_{j=0}^{n-1} \frac{1}{2} h \left[g(x_j) + g(x_{j+1}) \right], \quad x_j = jh, \quad nh = 1. \tag{5}$$

Let I denote the integral

$$I = \int_0^1 g(x) dx.$$

1. (10 points) Show that

$$I - T_h = -\frac{1}{4} h^2. \tag{6}$$

   Hint You are free to use the identity

$$\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2} \right)^2 .$$

   without any proof.

2. (5 points) Show that the fraction F_h given by

$$F_h = \frac{T_{2h} - T_{4h}}{T_h - T_{2h}} \tag{7}$$

   satisfies

$$F_h = 4. \tag{8}$$

3. (10 points) Now suppose that we try to compute F_h using the definition (7) and floating point arithmetic. Explain, why the computed values of F_h will most likely deviate from 4 when h is sufficiently small.

Problem 4 Starfleet Command requires the solution of a particular nonlinear equation

$$f(x) = 0, \tag{9}$$

and your predecessor managed to accumulate a lot of information about f before his unfortunate demise in a transporter accident. Specifically, it is known that there is a unique solution x^* of equation (9), f is two times differentiable, f'' is continuous, $f''(x^*) = 2$ and that the following function values are accurate to 5 significant figures

x	f(x)	f'(x)	f''(x)
1.2500e+00	-1.1158e-01	1.6019e+00	1.8884e+00
1.2600e+00	-9.5462e-02	1.6209e+00	1.9045e+00
1.2700e+00	-7.9158e-02	1.6400e+00	1.9208e+00
1.2800e+00	-6.2661e-02	1.6593e+00	1.9373e+00
1.2900e+00	-4.5971e-02	1.6788e+00	1.9540e+00
1.3000e+00	-2.9086e-02	1.6984e+00	1.9709e+00
1.3100e+00	-1.2003e-02	1.7182e+00	1.9880e+00
1.3200e+00	5.2783e-03	1.7381e+00	2.0053e+00
1.3300e+00	2.2760e-02	1.7583e+00	2.0228e+00
1.3400e+00	4.0445e-02	1.7786e+00	2.0404e+00
1.3500e+00	5.8333e-02	1.7991e+00	2.0583e+00

1. (5 points) Explain why we can be absolutely certain that x^* satisfies

$$1.31 < x^* < 1.32.$$

2. (10 points) Consider the application of Newton's method to our situation. Show that exists ξ_n between x_n and x^*, such that Newton's method satisfies

$$x^* - x_{n+1} = -\frac{1}{2}\frac{f''(\xi_n)}{f'(x_n)}(x^* - x_n)^2.$$

3. (10 points) Find an initial guess x_0 such that

$$x_0 < x^* < x_1, \quad |x^* - x_0| < 10^{-2},$$

and where you are reasonably sure that

$$|x^* - x_1| < 2 \times 10^{-4}.$$

where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

is the result of doing one step of Newton's method.