Problem 1 Let I denote the closed interval $I = [-1, 1]$. A continuous function

$$f : I \to \mathbb{R}$$

has been carefully sampled on 75 equidistant points spread across I and the results are given in the table below.

n	x(n)	f(x(n))	n	x(n)	f(x(n))	n	x(n)	f(x(n))
1	-1.0000	-1.0290	26	-0.3243	-0.0295	51	0.3514	0.0439
2	-0.9730	-1.0095	27	-0.2973	0.0032	52	0.3784	0.0199
3	-0.9459	-0.9855	28	-0.2703	0.0336	53	0.4054	-0.0050
4	-0.9189	-0.9577	29	-0.2432	0.0616	54	0.4324	-0.0306
5	-0.8919	-0.9262	30	-0.2162	0.0872	55	0.4595	-0.0567
6	-0.8649	-0.8916	31	-0.1892	0.1103	56	0.4865	-0.0829
7	-0.8378	-0.8543	32	-0.1622	0.1308	57	0.5135	-0.1090
8	-0.8108	-0.8145	33	-0.1351	0.1487	58	0.5405	-0.1349
9	-0.7838	-0.7727	34	-0.1081	0.1640	59	0.5676	-0.1600
10	-0.7568	-0.7291	35	-0.0811	0.1765	60	0.5946	-0.1843
11	-0.7297	-0.6842	36	-0.0541	0.1863	61	0.6216	-0.2073
12	-0.7027	-0.6382	37	-0.0270	0.1935	62	0.6486	-0.2288
13	-0.6757	-0.5915	38	0.0000	0.1980	63	0.6757	-0.2483
14	-0.6486	-0.5443	39	0.0270	0.1998	64	0.7027	-0.2656
15	-0.6216	-0.4969	40	0.0541	0.1991	65	0.7297	-0.2803
16	-0.5946	-0.4496	41	0.0811	0.1958	66	0.7568	-0.2920
17	-0.5676	-0.4025	42	0.1081	0.1900	67	0.7838	-0.3004
18	-0.5405	-0.3561	43	0.1351	0.1818	68	0.8108	-0.3049
19	-0.5135	-0.3105	44	0.1622	0.1713	69	0.8378	-0.3053
20	-0.4865	-0.2658	45	0.1892	0.1587	70	0.8649	-0.3011
21	-0.4595	-0.2224	46	0.2162	0.1439	71	0.8919	-0.2918
22	-0.4324	-0.1804	47	0.2432	0.1272	72	0.9189	-0.2771
23	-0.4054	-0.1399	48	0.2703	0.1087	73	0.9459	-0.2563
24	-0.3784	-0.1012	49	0.2973	0.0885	74	0.9730	-0.2291
25	-0.3514	-0.0643	50	0.3243	0.0669	75	1.0000	-0.1950

1. (5 points) Show that the function f has at least two zeros in I.
2. (10 points) Compute each of the zeros with a *relative* error less than $\tau = 0.05$.
3. (10 points) It is known that the function f is also twice differentiable with a second derivative which is continuous and

$$\forall x \in [-1, 1] \; : \; |f''(x)| \leq 10.5.$$

   Compute the value of $f(0.72)$ with an absolute error less than $\nu = 0.05$.

Problem 2 The integral $\int_0^1 f(x)dx$ of a function $f : [0,1] \to \mathbb{R}$ has been computed numerically using Simpson's rule and many different stepsizes $h = \frac{1}{2N}$. The results along with some auxiliary values are given below. It is known that f is infinitely often differentiable.

N	Sh	(Sh-S2h)	(S2h-S4h)/(Sh-S2h)
524288	2.45837007000238e-01	0.0000e+00	Inf
262144	2.45837007000238e-01	1.2212e-15	4.545455e-01
131072	2.45837007000237e-01	5.5511e-16	-3.050000e+00
65536	2.45837007000236e-01	-1.6931e-15	-5.245902e-01
32768	2.45837007000238e-01	8.8818e-16	-7.500000e-01
16384	2.45837007000237e-01	-6.6613e-16	4.166667e-02
8192	2.45837007000238e-01	-2.7756e-17	-9.000000e+00
4096	2.45837007000238e-01	2.4980e-16	6.666667e-01
2048	2.45837007000237e-01	1.6653e-16	2.933333e+01
1024	2.45837007000237e-01	4.8850e-15	1.526136e+01
512	2.45837007000232e-01	7.4551e-14	1.599442e+01
256	2.45837007000158e-01	1.1924e-12	1.600126e+01
128	2.45837006998965e-01	1.9080e-11	1.600174e+01
64	2.45837006979885e-01	3.0531e-10	1.600662e+01
32	2.45837006674572e-01	4.8870e-09	1.602645e+01
16	2.45837001787536e-01	7.8322e-08	1.610547e+01
8	2.45836923465701e-01	1.2614e-06	1.641672e+01
4	2.45835662055614e-01	2.0708e-05	1.758315e+01
2	2.45814953836298e-01	3.6412e-04	0.000000e+00
1	2.45450838083980e-01	0.0000e+00	0.000000e+00

1. (5pt) Explain why the computed value of the fraction $\frac{S_{2h}-S_{4h}}{S_h-S_{2h}}$ will always deviate dramatically from the correct value as h tends to zero.

2. (10pt) Determine the range of N where the *computed* value of the fraction

$$\frac{S_{2h} - S_{4h}}{S_h - S_{2h}}$$

shows the same behavior as if it had been computed without any rounding errors.

3. (10pt) Find the smallest value of N for which you are certain the relative error is less than $\tau = 10^{-11}$.

Problem 3 Consider the function $f : [2, \infty) \to \mathbb{R}$ given by

$$f(x) = \sqrt{x+1} - \sqrt{x-1}.$$

The following MATLAB commands have been used to generate a plot of the graph of $\log_2(f)$ for $x \in [2^{20}, 2^{21}]$:

```
>> f=@(x)sqrt(x+1)-sqrt(x-1);
>> x=single(linspace(2^20,2^21,1025));
>> plot(log2(x),log2(f(x)))
```

The plot is presented in Figure 1.

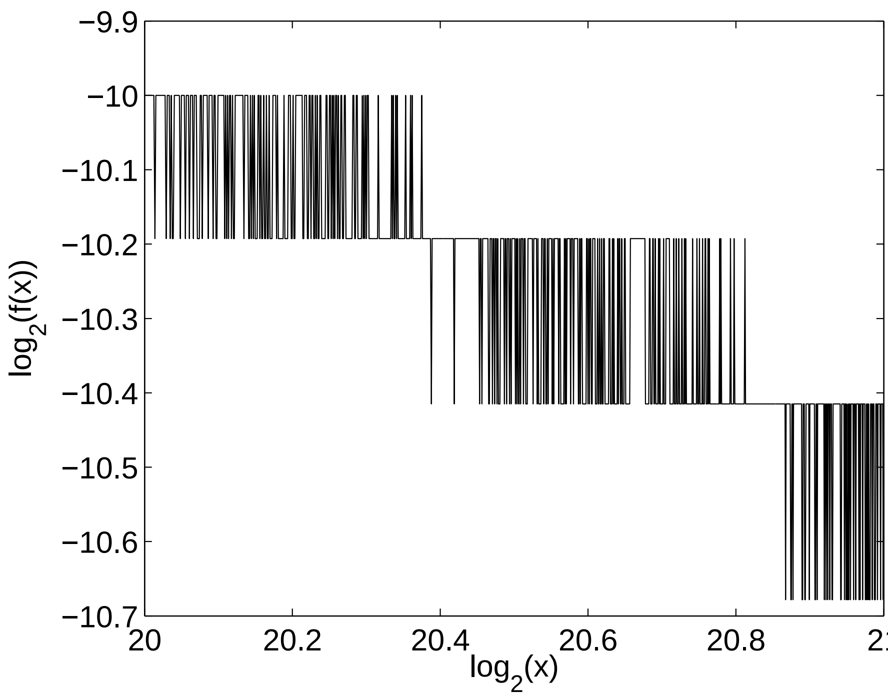

Figure 1: An inferior plot of $\log_2(f(x))$ as a function of $\log_2(x)$.

1. (5pt) List as many differences between this `MATLAB` plot and the true graph of $\log_2(f)$ as you can.
2. (10pt) Show that the condition number of f is given by

$$\kappa_f(x) = \frac{x}{2\sqrt{x+1}\sqrt{x-1}}$$

   and explain why it is at least not theoretically impossible to compute f with a relative error which is less than the unit roundoff error u.
3. (10pt) Find a reliable way of computing f in `MATLAB`.

Problem 4 This problem centers on the rapid calculation of reciprocal values on a binary computer with no hardware division. Let $\alpha \neq 0$ be a machine number. The goal is to compute the value $\frac{1}{\alpha}$ without doing any divisions.

1. (5 points) Explain, how we can easily compute reciprocal values for all non-zero machine numbers, if we can handle all positive machine numbers in the interval $[1, 2)$.

2. (10 points) Find a function $g : \mathbb{R} \to \mathbb{R}$ such that the fixpoint iteration given by
$$x_0 \in \mathbb{R}, \quad \text{and} \quad x_{n+1} = g(x_n), \quad n = 0, 1, 2 \ldots$$
satisfies
$$1 - \alpha x_{n+1} = (1 - \alpha x_n)^3.$$
Moreover, it must be possible to evaluate g without doing any divisions.

3. (10 points) Let $\alpha \in [1, 2)$. Show that if x_0 is chosen such that
$$0 < x_0 < \frac{2}{\alpha}$$
then not only is the sequence $\{x_n\}_{n=0}^{\infty}$ convergent, but
$$x_n \to \frac{1}{\alpha}, \quad n \to \infty, \quad n \in \mathbb{N}.$$