Problem 1 Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}$$

1. (5 points) Show that f is differentiable and strictly increasing for all $x \in \mathbb{R}$.

2. (5 points) The following MATLAB commands have been used to generate the plot in Figure 1.

```
>> k=21;
>> x=single(linspace(-1,1,129)*2^-k);
>> f=@(x)(exp(x)-exp(-x))/2;
>> plot(x,f(x))
```

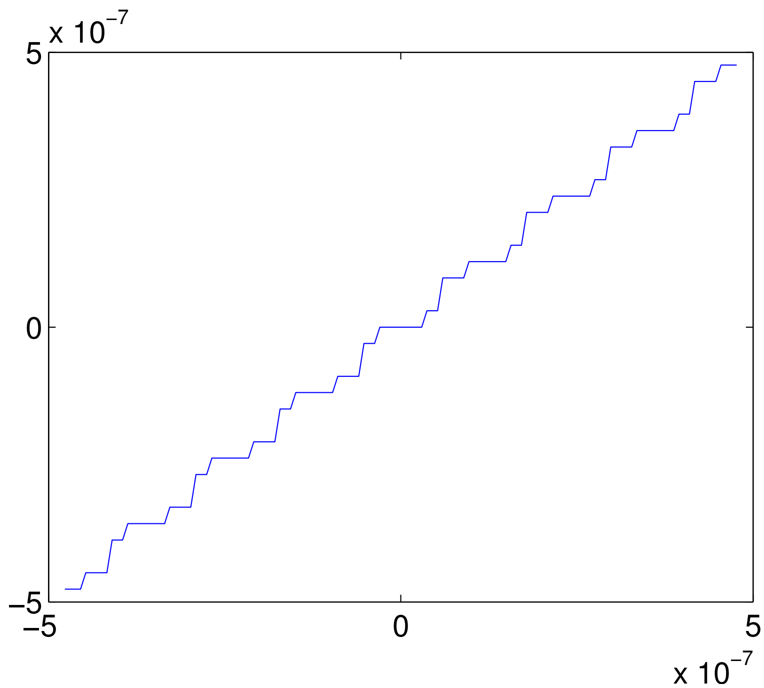

Figure 1: An untrustworthy plot of the function f on an interval around zero.

Explain why this is clearly not an accurate representation of the graph of f on the interval $[-2^{-21}, 2^{-21}]$?

3. (5 points) Consider the nominator of $f(x)$, i.e. the expression

$$N(x) = e^x - e^{-x}.$$

Show that we do not have to worry about catastrophic cancellation when $x > \frac{\log(2)}{2}$.

4. (2 points) Let p_n be the Taylor polynomial for f of order n at the point $x_0 = 0$. Show that

$$p_7(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040}.$$

5. (6 points) Show that

$$\frac{|f(x) - p_7(x)|}{|f(x)|} \leq \frac{|x|^8}{8!}, \quad x > 0.$$

6. (2 points) Show that p_7 approximates f with a relative error τ which is smaller than the single precision round-off error, i.e. $u = 2^{-24}$, on the interval $(0, \log(2)/2]$.

Problem 2 An infinitely differentiable function $f : [0, 1] \to \mathbb{R}$ has been integrated numerically on the interval $[0, 1]$ using the standard trapezoidal rule $T = T(h)$ and stepsizes $h = h(k) = 2^{-k}$ for many different values of k. The results and some auxiliary calculations are given in the table below.

k	Th	(Th-T2h)/3	(T2h-T4h)/(Th-T2h)
24	-0.1056405560	-5.2828e-15	0.0674255692
23	-0.1056405560	-3.5620e-16	35.2987012987
22	-0.1056405560	-1.2573e-14	2.6405445180
21	-0.1056405560	-3.3200e-14	4.6455343458
20	-0.1056405560	-1.5423e-13	3.8538436160
19	-0.1056405560	-5.9439e-13	4.0087321291
18	-0.1056405560	-2.3828e-12	4.0049681024
17	-0.1056405560	-9.5428e-12	3.9991143549
16	-0.1056405559	-3.8163e-11	4.0001060634
15	-0.1056405558	-1.5266e-10	3.9999743333
14	-0.1056405554	-6.1062e-10	4.0000016515
13	-0.1056405535	-2.4425e-09	4.0000000947
12	-0.1056405462	-9.7699e-09	3.9999994938
11	-0.1056405169	-3.9080e-08	3.9999985076
10	-0.1056403996	-1.5632e-07	3.9999939668
9	-0.1056399307	-6.2527e-07	3.9999758652
8	-0.1056380549	-2.5011e-06	3.9999034071
7	-0.1056305516	-1.0004e-05	3.9996127750
6	-0.1056005394	-4.0012e-05	3.9984373960
5	-0.1054805023	-1.5999e-04	3.9935271855
4	-0.1050005413	-6.3891e-04	3.9703436822
3	-0.1030838038	-2.5367e-03	3.8067034455
2	-0.0954736974	-9.6565e-03	1.4391333304
1	-0.0665042792	-1.3897e-02	
0	-0.0248134239		

1. (5 points) Explain, why it is immediately clear that computed values of the tell-tale fraction
$$\frac{T_{2h} - T_{4h}}{T_h - T_{2h}}$$
are completely wrong for $k > 19$.

2. (4 points) Explain, why the expression
$$T_h - T_{2h}$$
cancelled catastrophically for large values of k.

3. (5 points) Explain why the computed approximations of the integral are inaccurate for small values of k.

4. (7 points) Determine the range of k where you are confident that you can trust the error estimates. Remember to justify your choice!

5. (5 points) Determine the smallest value of k from which the value of the integral can be approximated with a *relative* error which is smaller than $\tau = 10^{-7}$.

Problem 3 Let $y \in (0,1)$ and consider the non-linear equation

$$g(y) = 0$$

where

$$g(y) = \frac{\sqrt{1-y^2}}{y} - \tan(y).$$

1. (5 points) Show that this equation has at least one solution on the interval $(0,1)$.
2. (5 points) Show that the solution is unique.
3. (7 points) Write down an iteration which is certain to converge to the solution, provided you begin with a good initial guess.
4. (8 points) The following table displays the results of applying Newton's method to the problem at hand.

n	x(n)	g(x(n))
0	5.000000000000000e-01	1.185748317725087e+00
1	7.003884584051097e-01	1.761416298335665e-01
2	7.389598589869100e-01	5.697906526650476e-04
3	7.390851339148572e-01	-3.182480501351392e-09
4	7.390851332151607e-01	-1.110223024625157e-16
5	7.390851332151607e-01	-1.110223024625157e-16
6	7.390851332151607e-01	-1.110223024625157e-16
7	7.390851332151607e-01	-1.110223024625157e-16
8	7.390851332151607e-01	-1.110223024625157e-16
9	7.390851332151607e-01	-1.110223024625157e-16

In exact arithmetic, Newton's iteration converges quadratically, and the number of correct digits should double for every iteration. Explain why the computed numbers stagnate after $n = 4$.

Problem 4 The "rage" virus has escaped from the laboratory at the heart of the green zone in London and the zombies are attacking the civilian population. Table 1 gives the number of infected during the initial phase.

t (minutes)	infected
0	1
2	6
4	19
6	40
8	69

Table 1: The number of zombies as function of time during the first few minutes after the outbreak.

1. (8 points) Find a polynomial of degree at most 2 which fits the initial data.
2. (7 points) Estimate the rate of infection, i.e. new zombies/minute at $t = 8$ minutes.
3. (10 points) The garrison has the capacity to kill 50 zombies/minute using conventional small arms. Assuming that our model continues to hold, at which time will it be impossible to stabilize the zombie population at a fixed number of individuals.