Problem 1 Consider the two real functions $f, g : [1, \infty) \to \mathbb{R}$ given by

$$f(x) = \sqrt{1 + \frac{1}{x^2}} - \sqrt{1 - \frac{1}{x^2}} \tag{1}$$

and

$$g(x) = \frac{2}{x^2 \left(\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}} \right)} \tag{2}$$

The functions have been implemented and plotted using the `MATLAB` commands

```
>> f=@(x)sqrt(1+1./x.^2)-sqrt(1-1./x.^2);
>> g=@(x)(2./x.^2)./(sqrt(1+1./x.^2)+sqrt(1-1./x.^2));
>> x=linspace(1,2,1025)*2^24;
>> plot(log2(x),log2(f(x)),log2(x),log2(g(x)));
```

The result is given in Figure 1.

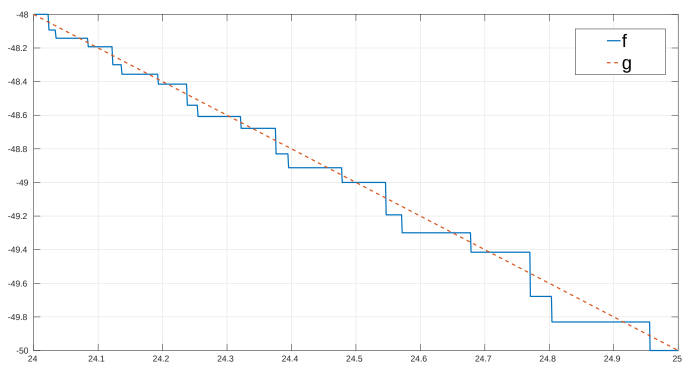

Figure 1: Illustration of the naive implementation of functions f and g.

1. (5 points) Prove that f is a differentiable and strictly decreasing for all $x \in [0, \infty)$.
2. (5 points) Explain why it is immediately clear that the implementation of f is useless.
3. (5 points) Prove that $f(x) = g(x)$ for all $x \in [1, \infty)$.
4. (5 points) Prove that the rendering of g as a straight line is an acceptable representation of the mathematical reality.
5. (5 points) Find the largest value of $b > 1$ such that f can be safely evaluated as stated for all $x \in [1, b)$.

Problem 2 Let T denote the target value

$$T = \int_0^1 \phi(x)dx \tag{3}$$

where $\phi : [0,1] \to \mathbb{R}$ is an unknown function. An unknown method A_h has been applied to approximate T using a variety of stepsize $h = 2^{-N}$. All available results are given in Figure 2. As usual Richardson's fraction F_h is given by

$$F_h = \frac{A_{2h} - A_{4h}}{A_h - A_{2h}} \tag{4}$$

and the error estimates are given by

$$E_h = \frac{A_h - A_{2h}}{2^p - 1}. \tag{5}$$

The correct value of p has been used when computing E_h.

1. (5 points) What evidence do you find to suggest that A_h obeys an asymptotic error expansion of the form

$$T - A_h = \alpha h^p + \beta h^q + O(h^r) \tag{6}$$

2. (5 points) What evidence do you find to suggest $p = 2$ and $q = 4$?

3. (5 points) What evidence do you find to suggest that ϕ is several times differentiable?

4. (10 points) Compute the value of T with a relative error which is less than 10^{-8}.

N	Ah	Richardson's fraction	Error estimate
1	3.4157661036209865e+00	0.0000000000000000e+00	0.0000000000000000e+00
2	3.4095318985182499e+00	0.0000000000000000e+00	-2.0780683675788816e-03
4	3.4080250105398555e+00	4.1371390522202764e+00	-5.0229599279812598e-04
8	3.4076521208305195e+00	4.0411090482426735e+00	-1.2429656977867390e-04
16	3.4075591515592141e+00	4.0108920302388045e+00	-3.0989757101806958e-05
32	3.4075359253067408e+00	4.0027667576706731e+00	-7.7420841577691135e-06
64	3.4075301197516557e+00	4.0006945301165375e+00	-1.9351850283714591e-06
128	3.4075286684259454e+00	4.0001738024770566e+00	-4.8377523675924294e-07
256	3.4075283055984640e+00	4.0000435047169365e+00	-1.2094249379757116e-07
512	3.4075282148918387e+00	4.0000108052182810e+00	-3.0235541773985610e-08
1024	3.4075281922151994e+00	4.0000030158674580e+00	-7.5588797443515432e-09
2048	3.4075281865460343e+00	3.9999962399609896e+00	-1.8897217124447252e-09
4096	3.4075281851287462e+00	4.0000090867815841e+00	-4.7242935489559079e-10
8192	3.4075281847744137e+00	3.9998809353717197e+00	-1.1811085443014235e-10
16384	3.4075281846858325e+00	4.0000902404909082e+00	-2.9527047473720813e-11
32768	3.4075281846637044e+00	4.0031107008107893e+00	-7.3760257161363061e-12
65536	3.4075281846581515e+00	3.9849648112603968e+00	-1.8509638266550610e-12
131072	3.4075281846567593e+00	3.9885167464114835e+00	-4.6407322429331543e-13
262144	3.4075281846564440e+00	4.4154929577464790e+00	-1.0510111299784815e-13
524288	3.4075281846563468e+00	3.2420091324200913e+00	-3.2418512319054571e-14
1048576	3.4075281846563201e+00	3.6499999999999999e+00	-8.8817841970012523e-15
2097152	3.4075281846563081e+00	2.2222222222222223e+00	-3.9968028886505635e-15
4194304	3.4075281846560519e+00	4.6793760831889082e-02	-8.5413158027828714e-14
8388608	3.4075281846563774e+00	-7.8717598908594821e-01	1.0850579694003197e-13

Figure 2: All results available after numerical integration of the unknown function ϕ.

Problem 3 Consider the problem of solving the equation

$$h(x) = 0 \tag{7}$$

where $h : \mathbb{R} \to \mathbb{R}$ continuous function.

1. (5 points) What are steps which must be completed before we even consider feeding the problem to a computer?
2. (5 points) What are the steps which must be completed before we can apply the bisection algorithm to our problem?
3. (5 points) Supppose that h is also differentiable. What are the steps which must be completed before Newton's method can be applied with any hope of success?
4. (10 points) Explain how to convert Newton's method to a reliable algorithm for solving problems of the type given by equation (7).

Problem 4 The trajectory of an artillery shell has been approximated using a variety of different time steps and the MATLAB function `rode`. All available information is given in Figure (3) and Figure (4).

1. (5 points) Estimate the elevation of the gun as accurately as possible.
2. (10 points) Prove that the shell is fired into a powerful headwind (Swedish: motvind)
3. (10 points) Let $(r, 0)$ denote the point of impact. Prove that the shell impacts *behind* the gun with $r < -80$ (meters).

Data related to component 1

time	approximation	F_(16h)	F_(8h)	F_(4h)	F_(2h)	F_(1h)	Error estimate
0.00	0.0000000000000000e+00	NaN	NaN	NaN	NaN	NaN	0.0000000000000000e+00
5.00	2.4462793833263643e+02	4.913949	4.495426	4.248143	4.123263	4.061335	2.6877370037254877e-03
10.00	3.8745475761458499e+02	5.035988	4.521543	4.254514	4.125075	4.061934	3.2986492458159469e-03
15.00	4.6247866536962556e+02	5.160824	4.558467	4.267666	4.130490	4.064373	3.2225182726506318e-03
20.00	4.9088229134264833e+02	5.302231	4.607578	4.287303	4.139181	4.068450	2.9017072577820122e-03
25.00	4.8666969375226660e+02	5.469940	4.671525	4.314315	4.151499	4.074322	2.4994208512794103e-03
30.00	4.5966510877944012e+02	5.675454	4.755330	4.350983	4.168528	4.082514	2.0808957552844731e-03
35.00	4.1712243008002292e+02	5.938627	4.869451	4.402447	4.192784	4.094268	1.6705748998523025e-03
40.00	3.6450998655173413e+02	6.314118	5.044875	4.484464	4.232123	4.113496	1.2636820135677833e-03
45.00	3.0539054853735337e+02	6.995843	5.405980	4.665709	4.322391	4.158481	8.1605280284217463e-04
50.00	2.3996483691705049e+02	8.647269	6.553589	5.358414	4.707553	4.362439	3.1676947414401485e-04

Data related to component 2

time	approximation	F_(16h)	F_(8h)	F_(4h)	F_(2h)	F_(1h)	Error estimate
0.00	0.0000000000000000e+00	NaN	NaN	NaN	NaN	NaN	0.0000000000000000e+00
5.00	3.2258505354684148e+03	4.838000	4.461550	4.232272	4.115587	4.057560	1.7105721701379178e-02
10.00	5.6781778855718458e+03	4.947954	4.482216	4.236026	4.116112	4.057521	2.1756461818768003e-02
15.00	7.5663812566492243e+03	5.057970	4.512172	4.245818	4.119878	4.059142	2.2248593458243704e-02
20.00	9.0165565515187864e+03	5.178551	4.551267	4.260630	4.126207	4.062052	2.1181086139525480e-02
25.00	1.0107486140041645e+04	5.315789	4.600300	4.280446	4.135009	4.066187	1.9491143250585690e-02
30.00	1.0889489228245240e+04	5.476333	4.661564	4.306174	4.146681	4.071732	1.7538275847376401e-02
35.00	1.1394489679007136e+04	5.670142	4.739676	4.339956	4.162247	4.079186	1.5454313487983503e-02
40.00	1.1641573608270419e+04	5.912753	4.842837	4.385855	4.183709	4.089542	1.3279337616647050e-02
45.00	1.1640483068549285e+04	6.220240	4.981462	4.449447	4.213915	4.104234	1.1076520471154557e-02
50.00	1.1396607468404847e+04	6.601068	5.163495	4.535558	4.255454	4.124593	9.0014931332310279e-03

Figure 3: Data related to the position of the shell

```
Data related to component 3

 time |           approximation |   F_(16h) |    F_(8h) |    F_(4h) |    F_(2h) |    F_(1h) |            Error estimate
 0.00 |   6.7981479343173348e+01 |       NaN |       NaN |       NaN |       NaN |       NaN |   0.0000000000000000e+00
 5.00 |   3.9786385635632236e+01 |  4.102738 |  4.190924 |  4.118534 |  4.063679 |  4.032788 |  -1.9170730239418768e-04
10.00 |   2.2198514308314380e+01 |  4.166372 |  4.189418 |  4.111638 |  4.059002 |  4.030179 |  -2.1123581422581120e-04
15.00 |   1.0533006994800211e+01 |  4.203307 |  4.188098 |  4.107374 |  4.056139 |  4.028585 |  -1.9499333841643818e-04
20.00 |   2.5159176479339926e+00 |  4.231212 |  4.188374 |  4.105007 |  4.054440 |  4.027620 |  -1.7310401510606255e-04
25.00 |  -3.0692793365027917e+00 |  4.257033 |  4.190531 |  4.104086 |  4.053561 |  4.027085 |  -1.5270248358278948e-04
30.00 |  -6.9257410120747442e+00 |  4.284209 |  4.194898 |  4.104501 |  4.053391 |  4.026913 |  -1.3494169292845490e-04
35.00 |  -9.5003468286793282e+00 |  4.316791 |  4.203282 |  4.107035 |  4.054285 |  4.027272 |  -1.1879691435545681e-04
40.00 |  -1.1167332717504847e+01 |  4.367368 |  4.223962 |  4.116045 |  4.058451 |  4.029270 |  -1.0060245743481744e-04
45.00 |  -1.2455355314060149e+01 |  4.387418 |  4.250349 |  4.132139 |  4.067130 |  4.033757 |  -8.1454525197699468e-05
50.00 |  -1.3993521646811633e+01 |  4.354240 |  4.249318 |  4.134849 |  4.069189 |  4.034949 |  -7.5690384310433956e-05

Data related to component 4

 time |           approximation |   F_(16h) |    F_(8h) |    F_(4h) |    F_(2h) |    F_(1h) |            Error estimate
 0.00 |   7.7703186451156148e+02 |       NaN |       NaN |       NaN |       NaN |       NaN |   0.0000000000000000e+00
 5.00 |   5.7364661162707205e+02 |  3.872026 |  4.113521 |  4.087056 |  4.049491 |  4.026052 |  -1.0174424587603426e-03
10.00 |   4.3622702105381001e+02 |  3.882022 |  4.094587 |  4.073145 |  4.041665 |  4.021950 |  -1.0753204657589777e-03
15.00 |   3.3475473034462766e+02 |  3.862219 |  4.076060 |  4.062292 |  4.035927 |  4.019015 |  -9.5477161888387252e-04
20.00 |   2.5452370005094750e+02 |  3.830598 |  4.059598 |  4.053900 |  4.031703 |  4.016897 |  -8.1865538306639485e-04
25.00 |   1.8748184462896086e+02 |  3.794151 |  4.045645 |  4.047628 |  4.028711 |  4.015433 |  -7.0188428229774524e-04
30.00 |   1.2878175853367665e+02 |  3.756066 |  4.034426 |  4.043348 |  4.026840 |  4.014559 |  -6.0919778797578295e-04
35.00 |   7.5237930991049083e+01 |  3.718411 |  4.026144 |  4.041042 |  4.026065 |  4.014256 |  -5.3892101691133121e-04
40.00 |   2.4606493123828557e+01 |  3.683342 |  4.020141 |  4.040229 |  4.026129 |  4.014398 |  -4.9237146634576823e-04
45.00 |  -2.4493181244058526e+01 |  3.673342 |  4.017247 |  4.039423 |  4.025889 |  4.014318 |  -4.8342737078262604e-04
50.00 |  -7.2146059208279098e+01 |  3.700208 |  4.020416 |  4.039285 |  4.025472 |  4.014032 |  -4.9502757887391147e-04
```

Figure 4: Data related to the velocity of the shell