Problem 1 Consider the two real functions $f, g : [1, \infty) \to \mathbb{R}$ given by

$$f(x) = \sqrt{1 + \frac{1}{x^2}} - \sqrt{1 - \frac{1}{x^2}} \tag{1}$$

and

$$g(x) = \frac{2}{x^2 \left(\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}} \right)} \tag{2}$$

The functions have been implemented and plotted using the `MATLAB` commands

```
>> f=@(x)sqrt(1+1./x.^2)-sqrt(1-1./x.^2);
>> g=@(x)(2./x.^2)./(sqrt(1+1./x.^2)+sqrt(1-1./x.^2));
>> x=linspace(1,2,1025)*2^24;
>> plot(log2(x),log2(f(x)),log2(x),log2(g(x)));
```

The result is given in Figure 1.

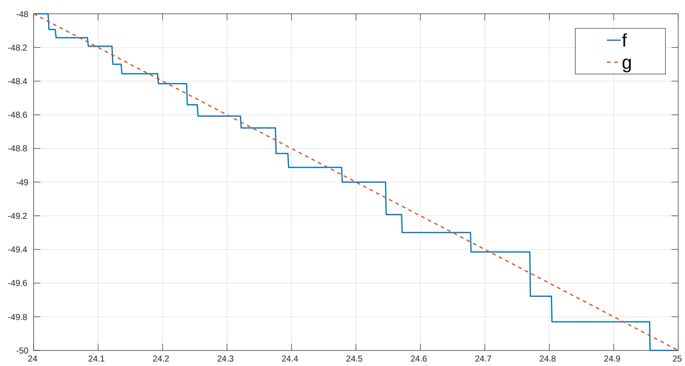

Figure 1: Illustration of the naive implementation of functions f and g.

1. (5 points) Prove that f is a differentiable and strictly decreasing for all $x \in [0, \infty)$.

   Solution

   Remark 1 The was a typographical error in this question. The question does not make sense unless $x \in (1, \infty)$.

   For $x \in (1, \infty)$ we see that f is built from differentiable functions using elementary arithmetical operations and function compositions. Hence it is itself differentiable. The omission of the point $x = 1$ is critical, because

$x \to \sqrt{1 - \frac{1}{x^2}}$ is not differentiable at this point. For $x > 1$ we find

$$\begin{aligned} f'(x) &= -(1 + x^{-2})^{-\frac{1}{2}} x^{-3} - (1 - x^{-2})^{-\frac{1}{2}} x^{-3} \\ &= -x^{-3} \left((1 + x^{-2})^{-\frac{1}{2}} + (1 - x^{-2})^{-\frac{1}{2}} \right) < 0. \end{aligned}$$

Hence, it follows that f is strictly decreasing for $x > 1$.

2. (5 points) Explain why it is immediately clear that the implementation of f is useless.

   Solution We are not given a plot of f, but rather a plot of $\log_2(f(x)$ as a function of $\log_2(x)$. This is not an obstacle, because $\log_2$ is a strictly increasing function. We deduce from the plot that the implemetation of f does not yield a function which is strictly decreasing. Instead we receive the inaccurate impression that $f'(x) = 0$ has infinitely many solutions. As these observations are in direct opposition to the mathematical reality, it is clear that the naive implementation of f is useless.

3. (5 points) Prove that $f(x) = g(x)$ for all $x \in [1, \infty)$.

   Solution We have

$$f(x) = \sqrt{1 + \frac{1}{x^2}} - \sqrt{1 - \frac{1}{x^2}} \tag{3}$$

$$= \left(\sqrt{1 + \frac{1}{x^2}} - \sqrt{1 - \frac{1}{x^2}} \right) \left(\frac{\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}}}{\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}}} \right) \tag{4}$$

$$= \frac{\left(1 + \frac{1}{x^2}\right) - \left(1 - \frac{1}{x^2}\right)}{\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}}} = \frac{\frac{2}{x^2}}{\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}}} \tag{5}$$

$$= g(x) \tag{6}$$

4. (5 points) Prove that the rendering of g as a straight line is an acceptable representation of the mathematical reality.

   Solution It is critical to observe that the rendering of g is given only for *large* values of x. In this case, we have

$$g(x) \approx \frac{1}{x^2} = x^{-2} \tag{7}$$

   hence

$$\log_2(x) \approx -2 \log_2(x) \tag{8}$$

   and it is clear that the rendering of g as a straight line in a log-log plot is an acceptable representation of the mathematical reality for large values of x.

5. (5 points) Find the largest value of $b > 1$ such that f can be safely evaluated as stated for all $x \in [1, b)$.

Hint: When is catastropic cancelation not an issue in the subtraction $d = a - b$?

Problem 2 Let T denote the target value

$$T = \int_0^1 \phi(x)dx \tag{9}$$

where $\phi : [0,1] \rightarrow \mathbb{R}$ is an unknown function. An unknown method A_h has been applied to approximate T using a variety of stepsize $h = 2^{-N}$. All available results are given in Figure 2. As usual Richardson's fraction F_h is given by

$$F_h = \frac{A_{2h} - A_{4h}}{A_h - A_{2h}} \tag{10}$$

and the error estimates are given by

$$E_h = \frac{A_h - A_{2h}}{2^p - 1}. \tag{11}$$

The correct value of p has been used when computing E_h.

1. (5 points) What evidence do you find to suggest that A_h obeys an asymptotic error expansion of the form

$$T - A_h = \alpha h^p + \beta h^q + O(h^r) \tag{12}$$

   Solution We observe that the computed values of Richardson's fraction is converging toward $2^p = 4$. The convergence is monotone from above for rows 3 through 11. In row 12 the computed values of Richardson's fraction has executed an illegal jump to the other side of 4, evidence of subtractive cancellation. This trend is only strengthend as the number of intervals is increased even futher.

2. (5 points) What evidence do you find to suggest $p = 2$ and $q = 4$?

   Solution It is likely that $p = 2$ since the computed value of Richardson's fraction are converging towards $4 = 2^2$ as h tends to zero, at least until the point where subtractive cancellation becomes an issue. The deviation away from 4 decreases as h^2 indicating that $q - p = 2$ or equivalently that $q = 4$.

3. (5 points) What evidence do you find to suggest that ϕ is several times differentiable?

   Solution If ϕ was not several times differentiable, then the computed values of Richardson's fraction would not be converging towards 4. In the past we have Richadson's fraction converge to an non-integer power of 2 when the given input function had even a single point where it was not differentiable.

4. (10 points) Compute the value of T with a relative error which is less than 10^{-8}.

   Solution This is question of finding the right error estimate and arguing why it is trustworthy. It is clear from the table that the target value satisfies $T > 3.4$. The error estimate on row 10 is trustworthy because it is inside the range of values for which the computed values of Richardson's fraction are behaving as if they had been computed in exact arithmetic. We find that the relative error estimate for this row is

$$\frac{|T - A_h|}{|T|} \lesssim \frac{E_h}{|T|} \leq \frac{3.03 \times 10^{-8}}{3.03} = 10^{-8} \tag{13}$$

   It follows that the corresponding value of $A_h = 3.4075282148918387$ is acceptable.

N	Ah	Richardson's fraction	Error estimate
1	3.4157661036209865e+00	0.0000000000000000e+00	0.0000000000000000e+00
2	3.4095318985182499e+00	0.0000000000000000e+00	-2.0780683675788816e-03
4	3.4080250105398555e+00	4.1371390522202764e+00	-5.0229599279812598e-04
8	3.4076521208305195e+00	4.0411090482426735e+00	-1.2429656977867390e-04
16	3.4075591515592141e+00	4.0108920302388045e+00	-3.0989757101806958e-05
32	3.4075359253067408e+00	4.0027667576706731e+00	-7.7420841577691135e-06
64	3.4075301197516557e+00	4.0006945301165375e+00	-1.9351850283714591e-06
128	3.4075286684259454e+00	4.0001738024770566e+00	-4.8377523675924294e-07
256	3.4075283055984640e+00	4.0000435047169365e+00	-1.2094249379757116e-07
512	3.4075282148918387e+00	4.0000108052182810e+00	-3.0235541773985610e-08
1024	3.4075281922151994e+00	4.0000030158674580e+00	-7.5588797443515432e-09
2048	3.4075281865460343e+00	3.9999962399609896e+00	-1.8897217124447252e-09
4096	3.4075281851287462e+00	4.0000090867815841e+00	-4.7242935489559079e-10
8192	3.4075281847744137e+00	3.9998809353717197e+00	-1.1811085443014235e-10
16384	3.4075281846858325e+00	4.0000902404909082e+00	-2.9527047473720813e-11
32768	3.4075281846637044e+00	4.0031107008107893e+00	-7.3760257161363061e-12
65536	3.4075281846581515e+00	3.9849648112603968e+00	-1.8509638266550610e-12
131072	3.4075281846567593e+00	3.9885167464114835e+00	-4.6407322429331543e-13
262144	3.4075281846564440e+00	4.4154929577464790e+00	-1.0510111299784815e-13
524288	3.4075281846563468e+00	3.2420091324200913e+00	-3.2418512319054571e-14
1048576	3.4075281846563201e+00	3.6499999999999999e+00	-8.8817841970012523e-15
2097152	3.4075281846563081e+00	2.2222222222222223e+00	-3.9968028886505635e-15
4194304	3.4075281846560519e+00	4.6793760831889082e-02	-8.5413158027828714e-14
8388608	3.4075281846563774e+00	-7.8717598908594821e-01	1.0850579694003197e-13

Figure 2: All results available after numerical integration of the unknown function ϕ.

Problem 3 Consider the problem of solving the equation

$$h(x) = 0 \tag{14}$$

where $h : \mathbb{R} \to \mathbb{R}$ continuous function.

1. (5 points) What are steps which must be completed before we even consider feeding the problem to a computer?

   Solution It is pointless to apply a computer to a problem which has no solution at all. Before feeding the problem to a computer, we must therefore detemine that equation (14) does in fact have at least one solution.

2. (5 points) What are the steps which must be completed before we can apply the bisection algorithm to our problem?

   Solution We need to construct a bracket around each solution we desire. A bracket around a root r is an interval (a, b) containing r, such that g is continuous on $[a, b]$ and such that $g(a)$ and $g(b)$ have distinct signs. If r is the only root in the interval (a, b), then the bisection algorithm will converge to r, at least in exact arithmetic.

3. (5 points) Supppose that h is also differentiable. What are the steps which must be completed before Newton's method can be applied with any hope of success?

   Solution The succes of Newton's method hinges on our ability to succesfully choose a good initial guess, i.e. find a good approximation of the desired root. If a bad initial guess is chosen, then Newton's method will either converge exceedingly slowly or not at all.

4. (10 points) Explain how to convert Newton's method to a reliable algorithm for solving problems of the type given by equation (14).

   Solution The key is to maintain a bracket around the root at all times. Starting from a large bracket, several steps of the bisection algorithm is first applied to reduce the size of the bracket. Given any small bracket (a, b), we compute the value of f at the midpoint $c = a + \frac{b-a}{2}$. If $|f(c)|$ is sufficient small, then we exit. Otherwise we use $x_0 = c$ as the starting point for a single step of Newton's method, i.e $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. If x_1 falls outside (a, b) it is discarded. Otherwise we compute $f(x_1)$. If $|f(x_1)|$ is sufficiently small we exit. Otherwise we select the smallest available bracket. We know the values for $f(a)$, $f(x_0)$, $f(x_1)$ and $f(b)$, so there at precisely 5 intervals to examine.

Problem 4 The trajectory of an artillery shell has been approximated using a variety of different time steps and the `MATLAB` function `rode`. All available information is given in Figure (3) and Figure (4).

1. (5 points) Estimate the elevation of the gun as accurately as possible.

   Solution We can do considerably better than just estimating the elevation of the gun. The velocity of the shell is given by $v = v_0(\cos(\theta), \sin(theta))$, hence $\tan(\theta) = \frac{v_y}{v_x}$ and the two components are given in the third (v_x) and fourth (v_y) part of the table. We have

   $$\tan(\theta) = \frac{7.7703186451156148 \times 10^2}{6.7981479343173348 \times 10^1} \Rightarrow \theta = 85 degrees. \tag{15}$$

2. (10 points) Prove that the shell is fired into a powerful headwind (Swedish: motvind)

   Solution We need to not only find the right values, but prove that they are trustworthy. Let $v_x(t)$ denote the velocity of the shell parallel to the ground at time t. In our case $v_x(0) > 0$. When firing in vacuum v_x is a constant, i.e. $v_x(t) = v_x(0)$. When firing into an atmosphere with no wind, $v_x(t)$ will decrease, but it will never become zero. If there is a strong headwind, then is possible for v_x to become negative. In the third part of the table, we see v_x change sign from positive to negative as we move from $t = 20$ to $t = 25$. The critital point is to argue why we can trust the change in sign. For both values of t we can trust the error estimates, because the computed value of Richardson's fraction exhibits monotone convergence toward 4. In fact, we can not only assertain that the order of the method is $p = 2$, but it is clear that the secondary term in the AEX has order $q = 3$. Thus we have every reason to trust the two error estimates. They are both tiny, orders of magnitude smaller than the computed values of v_x. Hence, there is no doubt that the shell stops moving forward and starts to move backward in the direction of the gun. Since $v_x(0) \approx 70$ meters per second, the headwind has to be substantial.

3. (10 points) Let $(r, 0)$ denote the point of impact. Prove that the shell impacts *behind* the gun with $r < -80$ (meters).

Data related to component 1

time	approximation	F_(16h)	F_(8h)	F_(4h)	F_(2h)	F_(1h)	Error estimate
0.00	0.0000000000000000e+00	NaN	NaN	NaN	NaN	NaN	0.0000000000000000e+00
5.00	2.4462793833263643e+02	4.913949	4.495426	4.248143	4.123263	4.061335	2.6877370037254877e-03
10.00	3.8745475761458499e+02	5.035988	4.521543	4.254514	4.125075	4.061934	3.2986492458159469e-03
15.00	4.6247866536962556e+02	5.160824	4.558467	4.267666	4.130490	4.064373	3.2225182726506318e-03
20.00	4.9088229134264833e+02	5.302231	4.607578	4.287303	4.139181	4.068450	2.9017072577820122e-03
25.00	4.8666969375226660e+02	5.469940	4.671525	4.314315	4.151499	4.074322	2.4994208512794103e-03
30.00	4.5966510877944012e+02	5.675454	4.755330	4.350983	4.168528	4.082514	2.0808957552844731e-03
35.00	4.1712243008002292e+02	5.938627	4.869451	4.402447	4.192784	4.094268	1.6705748998523025e-03
40.00	3.6450998655173413e+02	6.314118	5.044875	4.484464	4.232123	4.113496	1.2636820135677833e-03
45.00	3.0539054853735337e+02	6.995843	5.405980	4.665709	4.322391	4.158481	8.1605280284217463e-04
50.00	2.3996483691705049e+02	8.647269	6.553589	5.358414	4.707553	4.362439	3.1676947414401485e-04

Data related to component 2

time	approximation	F_(16h)	F_(8h)	F_(4h)	F_(2h)	F_(1h)	Error estimate
0.00	0.0000000000000000e+00	NaN	NaN	NaN	NaN	NaN	0.0000000000000000e+00
5.00	3.2258505354684148e+03	4.838000	4.461550	4.232272	4.115587	4.057560	1.7105721701379178e-02
10.00	5.6781778855718458e+03	4.947954	4.482216	4.236026	4.116112	4.057521	2.1756461818768003e-02
15.00	7.5663812566492243e+03	5.057970	4.512172	4.245818	4.119878	4.059142	2.2248593458243704e-02
20.00	9.0165565515187864e+03	5.178551	4.551267	4.260630	4.126207	4.062052	2.1181086139525480e-02
25.00	1.0107486140041645e+04	5.315789	4.600300	4.280446	4.135009	4.066187	1.9491143250585690e-02
30.00	1.0889489228245240e+04	5.476333	4.661564	4.306174	4.146681	4.071732	1.7538275847376401e-02
35.00	1.1394489679007136e+04	5.670142	4.739676	4.339956	4.162247	4.079186	1.5454313487983503e-02
40.00	1.1641573608270419e+04	5.912753	4.842837	4.385855	4.183709	4.089542	1.3279337616647050e-02
45.00	1.1640483068549285e+04	6.220240	4.981462	4.449447	4.213915	4.104234	1.1076520471154557e-02
50.00	1.1396607468404847e+04	6.601068	5.163495	4.535558	4.255454	4.124593	9.0014931332310279e-03

Figure 3: Data related to the position of the shell

Data related to component 3

time	approximation	F_(16h)	F_(8h)	F_(4h)	F_(2h)	F_(1h)	Error estimate
0.00	6.7981479343173348e+01	NaN	NaN	NaN	NaN	NaN	0.0000000000000000e+00
5.00	3.9786385635632236e+01	4.102738	4.190924	4.118534	4.063679	4.032788	-1.9170730239418768e-04
10.00	2.2198514308314380e+01	4.166372	4.189418	4.111638	4.059002	4.030179	-2.1123581422581120e-04
15.00	1.0533006994800211e+01	4.203307	4.188098	4.107374	4.056139	4.028585	-1.9499333841643818e-04
20.00	2.5159176479339926e+00	4.231212	4.188374	4.105007	4.054440	4.027620	-1.7310401510606255e-04
25.00	-3.0692793365027917e+00	4.257033	4.190531	4.104086	4.053561	4.027085	-1.5270248358278948e-04
30.00	-6.9257410120747442e+00	4.284209	4.194898	4.104501	4.053391	4.026913	-1.3494169292845490e-04
35.00	-9.5003468286793282e+00	4.316791	4.203282	4.107035	4.054285	4.027272	-1.1879691435545681e-04
40.00	-1.1167332717504847e+01	4.367368	4.223962	4.116045	4.058451	4.029270	-1.0060245743481744e-04
45.00	-1.2455355314060149e+01	4.387418	4.250349	4.132139	4.067130	4.033757	-8.1454525197699468e-05
50.00	-1.3993521646811633e+01	4.354240	4.249318	4.134849	4.069189	4.034949	-7.5690384310433956e-05

Data related to component 4

time	approximation	F_(16h)	F_(8h)	F_(4h)	F_(2h)	F_(1h)	Error estimate
0.00	7.7703186451156148e+02	NaN	NaN	NaN	NaN	NaN	0.0000000000000000e+00
5.00	5.7364661162707205e+02	3.872026	4.113521	4.087056	4.049491	4.026052	-1.0174424587603426e-03
10.00	4.3622702105381001e+02	3.882022	4.094587	4.073145	4.041665	4.021950	-1.0753204657589777e-03
15.00	3.3475473034462766e+02	3.862219	4.076060	4.062292	4.035927	4.019015	-9.5477161888387252e-04
20.00	2.5452370005094750e+02	3.830598	4.059598	4.053900	4.031703	4.016897	-8.1865538306639485e-04
25.00	1.8748184462896086e+02	3.794151	4.045645	4.047628	4.028711	4.015433	-7.0188428229774524e-04
30.00	1.2878175853367665e+02	3.756066	4.034426	4.043348	4.026840	4.014559	-6.0919778797578295e-04
35.00	7.5237930991049083e+01	3.718411	4.026144	4.041042	4.026065	4.014256	-5.3892101691133121e-04
40.00	2.4606493123828557e+01	3.683342	4.020141	4.040229	4.026129	4.014398	-4.9237146634576823e-04
45.00	-2.4493181244058526e+01	3.673342	4.017247	4.039423	4.025889	4.014318	-4.8342737078262604e-04
50.00	-7.2146059208279098e+01	3.700208	4.020416	4.039285	4.025472	4.014032	-4.9502757887391147e-04

Figure 4: Data related to the velocity of the shell