Problem 1 Consider the problem of computing the function $f : (0, \infty) \to \mathbb{R}$ given by

$$f(x) = \frac{\sqrt{1 + \sin(x)^2} - 1}{x}, \quad x > 0 \tag{1}$$

1. Show that $f(x) > 0$ for all $x > 0$.
2. Show that $f(x) \to 0$ for $x \to 0_+$.
3. Figure 1 shows the results of the MATLAB commands

```
a=0; b=2e-7;
s=linspace(a,b,1025);
f=@(x)(sqrt(1+sin(x).^2)-1)./x;
plot(s,f(s),'LineWidth',2); grid on; grid minor;
xlabel('x'); ylabel('y=f(x)');
```

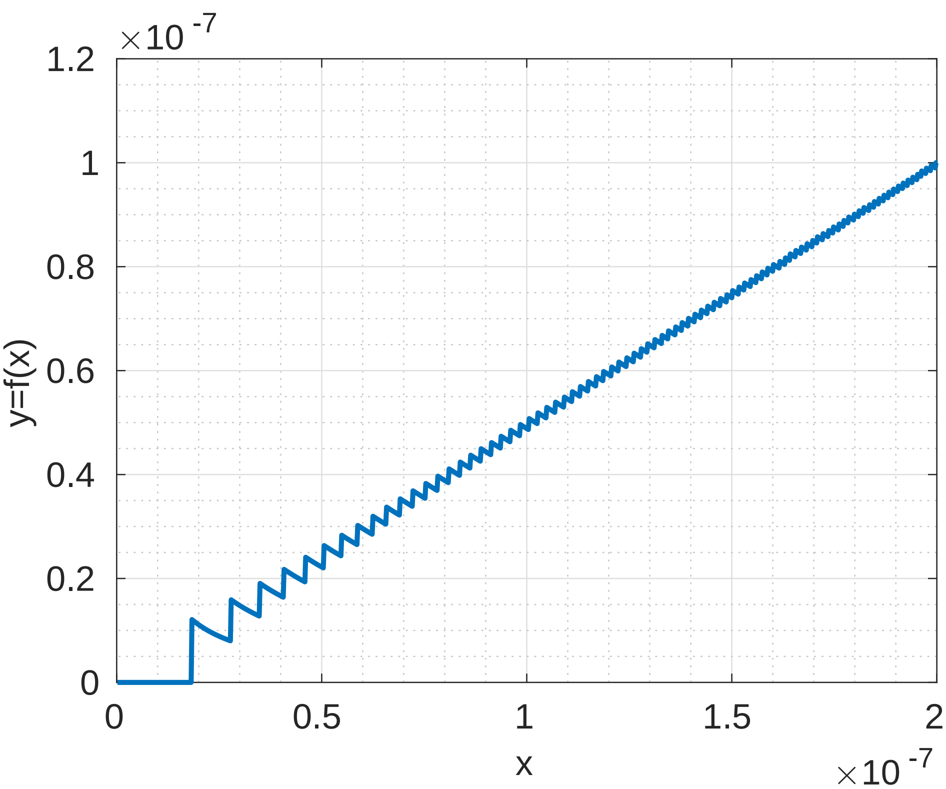

Figure 1: The result of a naive computation of $f$using MATLAB

Why is it immediately clear that this is not a numerically reliable way of computing f? Give as many reasons as you can!

4. Explain why $\psi(x) = \frac{x}{2}$ is a good approximation of f for $x \in [0, 2 \times 10^{-7}]$?
5. What is the largest relative error associated with the naive approximation of $f(x)$, when $x \in [0, 2 \times 10^{-7}]$?

Problem 2 Consider the problem of solving a non-linear equation

$$g(x) = 0 \tag{2}$$

where $g : \mathbb{R} \to \mathbb{R}$.

1. Let $a \neq b$ and suppose $g(a)g(b) < 0$. Which property of g will allow you to draw the nontrivial conclusion that g has at least one root between a and b?

Now consider an iterative method for solving our non-linear equation (2). Let x_j denote the exact value of the jth approximation, and let $\hat{x}_j$ denote the computed value of x_j.

2. Assume that the *computed* residual $\hat{g}(\hat{x}_j)$ is exactly equal to 0 for some value of j. Explain, why this should not be taken as evidence that $\hat{x}_j$ is a root of g.

3. Give an example of an iterative method, a function g and an initial guess x_0 for a root r for which $|g(x_j)| < \epsilon$, where ϵ is a tiny number, but x_j is far from any root of g for all large values of j.

4. What are the advantage(s)/disadvantage(s) of the bisection method?

5. Describe an iterative method which is both fast and robust.

Problem 3 The integral $I = \int_0^1 \phi(x)dx$ of an unknown function $\phi : [0,1] \to \mathbb{R}$ has been computed numerically using the trapezoidal rule and subjected to Richardson's techniques. Figure 3 contains all the available data. The number N is the number of sub-intervals, A_h is the computed approximation corresponding to the step size $h = 1/N$. Richardson's fraction is the number

$$F_h = \frac{A_{2h} - A_{4h}}{A_h - A_{2h}} \tag{3}$$

As there is some doubt about the order p of the method, the numbers $A_h - A_{2h}$ has been provided instead of the usual error estimates given by

$$E_h = \frac{A_h - A_{2h}}{2^p - 1}. \tag{4}$$

1. What evidence do you find to support the conjecture that ϕ is not infinitely often differentiable?

2. What is the order p of the dominant term of the asymptotic error expansion for the trapezoidal method when applied to ϕ?

3. What evidence do you find to support the conjecture that the approximations continue to improve as long as $N \leq 2097152$?

4. What are all the consequences of increasing the number of sub-intervals beyond this point?

5. What is the smallest value of N which will allow you to compute the integral I with a relative error less than $\tau = 10^{-6}$?

N	Approximation A(h)	Richardson's fraction	A(h)-A(2h)
1	1.3591409142295228e+00	0.0000000000000000e+00	0.0000000000000000e+00
2	1.2624814525140426e+00	0.0000000000000000e+00	-9.6659461715480122e-02
4	1.2500878518926526e+00	7.7991428535026399e+00	-1.2393600621390055e-02
8	1.2516121251639316e+00	-8.1308259187609906e+00	1.5242732712790197e-03
16	1.2536843987528026e+00	7.3555599968317309e-01	2.0722735888709654e-03
32	1.2548090999163359e+00	1.8425103983717011e+00	1.1247011635333592e-03
64	1.2553062339580381e+00	2.2623700434644585e+00	4.9713404170215192e-04
128	1.2555071315593258e+00	2.4745643477851758e+00	2.0089760128771950e-04
256	1.2555844894995216e+00	2.5969874686322156e+00	7.7357940195810215e-05
512	1.2556134303720221e+00	2.6729650322218466e+00	2.8940872500493597e-05
1024	1.2556240616527965e+00	2.7222376226031502e+00	1.0631280774386909e-05
2048	1.2556279204209184e+00	2.7550970772221004e+00	3.8587681219226511e-06
4096	1.2556293097578939e+00	2.7774169910282742e+00	1.3893369754658380e-06
8192	1.2556298072348882e+00	2.7927662816183387e+00	4.9747699426561098e-07
16384	1.2556299846890373e+00	2.8034114538352468e+00	1.7745414915282254e-07
32768	1.2556300478211850e+00	2.8108365631201058e+00	6.3132147731792543e-08
65536	1.2556300702399685e+00	2.8160380665719309e+00	2.2418783496291894e-08
131072	1.2556300781907634e+00	2.8196908291928016e+00	7.9507949113377663e-09
262144	1.2556300810079308e+00	2.8222657072051205e+00	2.8171673882582127e-09
524288	1.2556300820055022e+00	2.8240259338450122e+00	9.9757135885170101e-10
1048576	1.2556300823585662e+00	2.8254687404681209e+00	3.5306402246249036e-10
2097152	1.2556300824834798e+00	2.8264650411244290e+00	1.2491363499123054e-10
4194304	1.2556300825276043e+00	2.8309371524615159e+00	4.4124481846097297e-11
8388608	1.2556300825432214e+00	2.8254020161232991e+00	1.5617063198192227e-11

Figure 2: The results of integrating ϕ numerically using the trapezoidal rule and a selection of step-sizes

Problem 4

An artillery trajectory has been computed. The script used to compute all the numbers is given in Figure 3. A plot of the computed trajectory is given in Figure 4. Auxiliary data computed by the script is given in Figure 5 and Figure 6.

1. What evidence do you find to support the conjecture that this is a flat trajectory?
2. What evidence do you find to support the conjecture that air resistance was included in the calculation?
3. What evidence do you find to support the conjecture that the shell impacted the ground with a speed which was less than 297 meters/s.
4. What evidence do you find to support the conjecture that the shell did not impact the ground with an angle of 45 degrees?
5. Why is it good physics that the speed of the shell at the point of impact is smaller than the muzzle velocity?

```
% Load preprogrammed drag coefficients
load shells
% Define the shell and enviroment
param.mass=10;
param.cali=0.088;
param.drag=@(x)mcg7(x);
param.atmo=@(x)atmosisa(x);
param.grav=@(x)9.82;
% Define shot
v0=830; theta=20*pi/180;

% Various other stuff.
method='rk2'; dt=1; maxstep=200;
[r, flag, t, tra]=range_rkx(param,v0,theta,method,dt,maxstep);
% Generate plot.
plot(tra(1,:),tra(2,:),'k-','LineWidth',4); grid on; grid minor;
xlabel('x (meters)'); ylabel('y (meters)');

% Allocate space for data collection
data=zeros(4,5);
% Data collection
for k=1:5
  [r, flag, t, tra]=range_rkx(param,v0,theta,method,dt,maxstep);
  data(:,k)=tra(:,end);
  dt=dt/2; maxstep=maxstep*2;
end

% Compute speed of impact
v=sqrt(data(3,:).^2+data(4,:).^2);
% Compute some angle
angle=atan(data(3,:)./data(4,:));

% Run Richardson's scheme, generates the first table
richardson(v,2);
% Run Richardson's scheme, generates the second table
richardson(angle,2);
```

Figure 3: The script used to compute the trajectory and all auxiliary numbers.

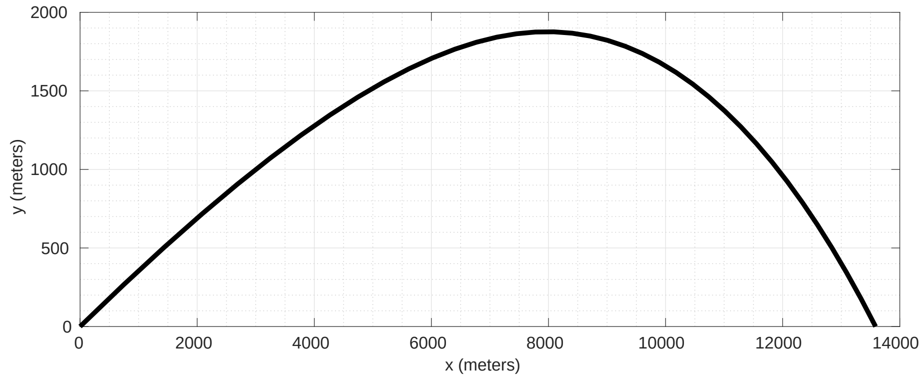

Figure 4: The computed trajectory of an artillery shell.

k	approximation	fraction	error estimate
1	2.961508712574e+02	0.00000000	0.000000000000e+00
2	2.961018547930e+02	0.00000000	-1.633882148523e-02
3	2.960888812097e+02	3.77817473	-4.324527759119e-03
4	2.960859000494e+02	4.35185705	-9.937200846745e-04
5	2.960851842022e+02	4.16452023	-2.386157417125e-04

Figure 5: The first table generated by the script

k	approximation	fraction	error estimate
1	-9.374500477428e-01	0.00000000	0.000000000000e+00
2	-9.368320178834e-01	0.00000000	2.060099531333e-04
3	-9.366844734718e-01	4.18877172	4.918147054445e-05
4	-9.366488584576e-01	4.14275875	1.187167139971e-05
5	-9.366401377402e-01	4.08395461	2.906905816465e-06

Figure 6: The second table generated by the script