Problem 1 The trajectory of an artillery shell has be carefully computed and some of the results are given in Table 1.

t	x(t)	y(t)	x'(t)	y'(t)
s	m	m	m/s	m/s
0	0	0	5.5154e+02	5.5154e+02
5	2.4423e+03	2.3287e+03	4.3549e+02	3.9151e+02
10	4.4263e+03	3.9967e+03	3.6321e+02	2.8147e+02
15	6.1129e+03	5.1866e+03	3.1433e+02	1.9775e+02
20	7.5925e+03	5.9994e+03	2.7928e+02	1.2931e+02
25	8.9200e+03	6.4952e+03	2.5288e+02	7.0279e+01
30	1.0130e+04	6.7123e+03	2.3199e+02	1.7390e+01
35	1.1246e+04	6.6763e+03	2.1452e+02	-3.1169e+01
40	1.2279e+04	6.4064e+03	1.9904e+02	-7.6230e+01
45	1.3238e+04	5.9195e+03	1.8458e+02	-1.1798e+02
50	1.4125e+04	5.2324e+03	1.7062e+02	-1.5625e+02
55	1.4944e+04	4.3632e+03	1.5692e+02	-1.9079e+02
60	1.5695e+04	3.3310e+03	1.4350e+02	-2.2142e+02
65	1.6380e+04	2.1555e+03	1.3046e+02	-2.4812e+02
70	1.7001e+04	8.5618e+02	1.1795e+02	-2.7100e+02
T	1.7353e+04	0	1.1055e+02	-2.8335e+02

Figure 1: The trajectory of a shell fired at time $t = 0$ until the time of impact $t = T = 70.3088$ s.

1. (5 points) Which property about the trajectory must either be proved or assumed before it follows with certainty that the shell reaches its heighest point some time between 30 and 35 seconds after being fired?

2. (5 points) Compute the total work done by the friction between the shell and the surrounding atmosphere from time $t = 0$ until the time of impact $t = T$ *relative* to the initial kinetic energy of the shell.

3. (15 points) The shell is at height $y = 6000$ meters twice during its flight. Pick one of the to points and compute the corresponding value of the x coordinate as accurately as you can. Estimate the relative error on you

Problem 2 An unknown function $f : [0, 1] \to \mathbb{R}$ has been integrated numerically using the trapezoid rule T_h where $h = 1/N$ and $N = 2^k$. The computed results are given as Table 2. The table also contains Richardson's fractions $F_h = \frac{T_{2h}-T_{4h}}{T_h-T_{2h}}$ and his error estimates $E_h = \frac{T_h-T_{2h}}{3}$.

k	Th	fraction Fh	Eh
0	1.1436776435894214	0.00000000e+00	0.0000e+00
1	0.7757889068338907	0.00000000e+00	-1.2263e-01
2	0.6901947335054285	4.29805818e+00	-2.8531e-02
3	0.6695218208542304	4.14040221e+00	-6.8910e-03
4	0.6644028353205115	4.03847843e+00	-1.7063e-03
5	0.6631262165469027	4.00979967e+00	-4.2554e-04
6	0.6628072580636909	4.00246064e+00	-1.0632e-04
7	0.6627275307173217	4.00061582e+00	-2.6576e-05
8	0.6627075996480573	4.00015400e+00	-6.6437e-06
9	0.6627026169287019	4.00003850e+00	-1.6609e-06
10	0.6627013712518608	4.00000963e+00	-4.1523e-07
11	0.6627010598328377	4.00000240e+00	-1.0381e-07
12	0.6627009819780926	4.00000055e+00	-2.5952e-08
13	0.6627009625144066	4.00000006e+00	-6.4879e-09
14	0.6627009576484860	4.00000078e+00	-1.6220e-09
15	0.6627009564320073	4.00000475e+00	-4.0549e-10
16	0.6627009561278847	3.99996130e+00	-1.0137e-10
17	0.6627009560518562	4.00011244e+00	-2.5343e-11
18	0.6627009560328556	4.00137897e+00	-6.3335e-12
19	0.6627009560280878	3.98514379e+00	-1.5893e-12
20	0.6627009560268888	3.97675711e+00	-3.9964e-13
21	0.6627009560266003	4.15505964e+00	-9.6182e-14
22	0.6627009560265480	5.51804671e+00	-1.7431e-14
23	0.6627009560264855	8.36589698e-01	-2.0835e-14

Figure 2: The results obtained by integrating an unknown function numerically using $N = 2^k$ subintervals along with some auxiliary numbers explained in the main text.

1. (5 points) The function f is costly to compute and it requires a full CPU second to evaluate a f at a single point x. Assuming that the values are recycled how many CPU days would it require to compute the above table from scratch?

2. (10 points) Consider the value of Richardson's fraction corresponding to $k = 14$, i.e. $F_h = 4.00000078$. What conclusions can be drawn about this value which would be impossible to reach if you did not have the rest of the table available?

3. (10 points) Estimate the integral with a a relative error which is less than $\tau = 10^{-8}$ and explain why your error estimate is realiable.

Problem 3 Consider the problem of computing the function

$$f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2} \tag{1}$$

from scratch using floating point arithmetic.

1. (5 point) Compute the condition number $\kappa_f(x)$ of f for all $x \neq 0$.
2. (10 point) Explain why it is theoretically possible to compute $f(x)$ with a relative error which is less than $2u$ where u is the unit round off error for all x in the interval $[-1, 1]$.
3. (10 point) Explain why the definition of f is unsuitable for direct numerical computation for x sufficiently close to 0 and explain in detail how to ensure that the relative error is less than a given tolerance τ.

Problem 4 Consider the problem of computing the reciprocal square root of a postive real number α, i.e. $\alpha \to 1/\sqrt{\alpha}$.

1. (5 points) Show that $x = 1/\sqrt{\alpha}$ if and only if x is the solution of the equation
$$f(x) = 0, \tag{2}$$
where $f(x) = \frac{1}{x^2} - \alpha$.

2. (10 points) Of all the basic arithmetic operations divisions has always been the slowest in terms of CPU cycles. Show that Newton's method can be applied to computing the reciprocal square root of α without doing any divisions.

3. (10 points) The proper use of Newton's method requires the construction of a good initial guess. This can be very difficult if α is an arbitrary positive number in the representable range. However, any positive floating point number α admits a binary representation of the form
$$\alpha = (1.f_1 f_2 f_3 \ldots . f_k)_2 \times 2^m, \quad f_k \in \{0, 1\} \tag{3}$$
where $k > 0$ and m are integers. Explain why it suffices to construct a good initial guess for $\alpha \in [1, 4]$ and why $x_0(\alpha) = \frac{3}{4}$ is not a bad initial guess for $1/\sqrt{\alpha}$ for all $\alpha \in [1, 4]$.