Teknisk-vetenskapliga beräkningar
Fall 2011
Final exam

November 7th, 2011

Instructions: This exam consists of four major problems. The maximum score is 100 points or 25 points per problem. You are allowed to use anything which is either printed or written on paper prior to the exam. This includes lecture notes, your own notes, your mandatory projects and any textbook that you might care to reference. Moreover, you may use a programmable calculator. While the class was taught in English you may write your answers in Swedish or English.

Problem 1 The purpose of this problem to show that the equation

$$x = g(x)$$

where $g : \mathbb{R} \to \mathbb{R}$ is given by

$$g(x) = \cos(\sin(x))$$

has a unique solution $\xi \in \mathbb{R}$. Moreover, the goal is to compute ξ with a relative error of at most 10^{-6}.

To this end, let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x - \cos(\sin(x)).$$

1. Show that f is strictly increasing.
2. Show that $f(x) \to -\infty$ for $x \to -\infty$ and $f(x) \to \infty$ for $x \to \infty$.
3. Show that f has exactly one zero ξ in $\mathbb{R}$.

The following table contains the results of applying 20 steps of the standard fixed point iteration for g using $x_1 = \frac{1}{2}$ as the initial guess. Moreover, the table contains some auxiliary information which may aid your computations.

n	x(n)	f(x(n))	x(n)-x(n-1)
1	0.5000000000000000	-3.87260051e-01	N/A
2	0.8872600507176527	1.73079954e-01	3.87260051e-01
3	0.7141800964160075	-7.88689935e-02	-1.73079954e-01
4	0.7930490899221964	3.63165874e-02	7.88689935e-02
5	0.7567325025302492	-1.67053697e-02	-3.63165874e-02
6	0.7734378722139057	7.69401878e-03	1.67053697e-02
7	0.7657438534347033	-3.54218258e-03	-7.69401878e-03
8	0.7692860360180922	1.63112223e-03	3.54218258e-03
9	0.7676549137834672	-7.51035113e-04	-1.63112223e-03
10	0.7684059488966346	3.45823038e-04	7.51035113e-04
11	0.7680601258582556	-1.59235011e-04	-3.45823038e-04
12	0.7682193608696493	7.33208342e-05	1.59235011e-04
13	0.7681460400354824	-3.37609227e-05	-7.33208342e-05
14	0.7681798009581497	1.55454073e-05	3.37609227e-05
15	0.7681642555508844	-7.15796373e-06	-1.55454073e-05
16	0.7681714135146118	3.29592310e-06	7.15796373e-06
17	0.7681681175915115	-1.51762531e-06	-3.29592310e-06
18	0.7681696352168211	6.98798704e-07	1.51762531e-06
19	0.7681689364181168	-3.21765594e-07	-6.98798704e-07
20	0.7681692581837104	1.48158688e-07	3.21765594e-07

4. Find an interval $[a, b]$ such that not only is $\xi \in [a, b]$, but the midpoint $m = \frac{a+b}{2}$ satisfies
$$\frac{|\xi - m|}{|\xi|} \leq 10^{-6}.$$

Problem 2 This problem deals with some of the differences between exact arithmetic and floating point arithmetic.

Consider the function

$$f(x) = \frac{x - \tan(x)}{x^3}, \quad x \neq 0$$

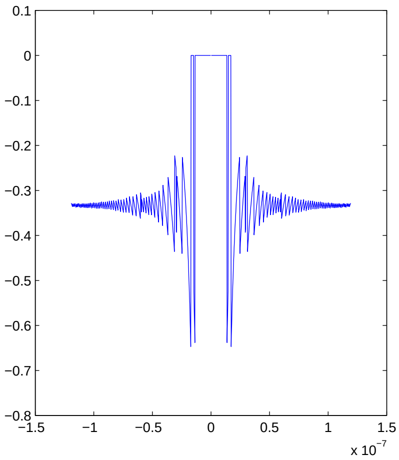
as well as MATLAB function `f.m` given by

```
function y=f(x);
    y=(x-tan(x))./x.^3;
```

The command

```
>>plot(x,f(x));
```

has produced the following plot



1. Show that in exact arithmetic
$$f(x) \to -\frac{1}{3}, \quad x \to 0, \quad x \neq 0.$$
2. Why can you with absolute certainty conclude that the plot is not an accurate representation of the graph of f?
3. Explain the presence of the violent oscillations.
4. Explain the presence of an interval where the computed value of f is zero.
5. Write a MATLAB function `good.m` which will return an accurate approximation of $f(x)$ for all x which satisfy $|x| < 2^{-26}$.

Remark: The following series expansion can be used without proof

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^7).$$

Problem 3 An unknown function $f : [0, 1] \to \mathbb{R}$ has been integrated numerically using Simpson's method and various step-sizes h given by

$$2Nh = 1$$

for several different integers N. These calculations have produced a sequence of approximations S_h for the integral

$$I = \int_0^1 f(x)dx.$$

The values of S_h together some auxiliary information is displayed in the following table.

N	Sh	(Sh-S2h)	(S2h-S4h)/(Sh-S2h)
131072	0.384615384615386358	-3.9968e-15	-1.138888888888888840
65536	0.384615384615390354	4.5519e-15	-5.365853658536585691
32768	0.384615384615385802	-2.4425e-14	4.638636363636363669
16384	0.384615384615410227	-1.1330e-13	6.255756981871631872
8192	0.384615384615523526	-7.0877e-13	6.060228696741854826
4096	0.384615384616232292	-4.2953e-12	6.061452369567184029
2048	0.384615384620527578	-2.6036e-11	6.063012641332830022
1024	0.384615384646563252	-1.5785e-10	6.062730513114103381
512	0.384615384804417870	-9.5703e-10	6.062566124062662709
256	0.384615385761447881	-5.8021e-09	6.062068714123094004
128	0.384615391563505604	-3.5172e-08	6.060761569428588125
64	0.384615426735978205	-2.1317e-07	6.057312575148287692
32	0.384615639907948448	-1.2912e-06	6.048222653848895014
16	0.384616931157204467	-7.8098e-06	6.024397728481678982
8	0.384624740920206487	-4.7049e-05	5.963331393888615217
4	0.384671790038695838	-2.8057e-04	5.817782605448264199
2	0.384952359524038168	-1.6323e-03	N/A
1	0.386584651795482348	N/A	N/A

Caution: Note that the third column contains the number $S_h - S_{2h}$ rather than the numbers $(S_h - S_{2h})/(2^p - 1)$ for some appropriate power of p!

1. Determine the order p of the Simpson method when applied to the unknown function f. (Hint: It is not an integer in this case.)

2. Determine the range of N for which you will be able to compute reliable error estimates.

3. Explain, why the computed values of the tell-tale fraction

$$\frac{S_{2h} - S_{4h}}{S_h - S_{2h}}$$

are clearly wrong for the two largest values of N.

4. Find the value of N which corresponds to the best approximation S_h of I and estimate the relative error, i.e

$$\frac{|I - S_h|}{|I|}$$

as accurately as possible.

5. Explain why it is unlikely in the extreme that f is infinitely often differentiable on the entire interval $[0, 1]$?

Problem 4 The function $f : \mathbb{R} \to \mathbb{R}$ satisfies

x	0	1	2	3
f(x)	3	1	-2	-3

1. Compute the third order interpolating polynomial for f.

It is also known that f is strictly decreasing, so the inverse function $g = f^{-1}$ exists.

2. Show that the third order interpolating polynomial for g is

$$q(y) = (3-y)\left(\frac{1}{2} + \frac{1}{30}(y-1) + \frac{1}{30}(y-1)(y+2)\right)$$

Finally, it is known that g is four times differentiable and $g^{(4)}$ is continuous, and

$$|g^{(4)}(y)| \leq \frac{1}{30} \times 10^{-3}, \quad y \in \mathbb{R}$$

3. Solve the equation $f(x) = 0$ with an absolute error which is less than 10^{-3}.