Problem 1 Consider the function $f : (0, \infty) \to 0$ given by

$$f(x) = \sqrt{9 + x^2} - 3. \tag{1}$$

1. (6 points) Show that f is a strictly positive and strictly increasing function of x.

2. (6 points) The following MATLAB commands

```
>>f=@(x)sqrt(9+x.^2)-3;
>>x=single(linspace(0,1,1025)*2^-8);
>>plot(x,f(x));
```

   have produced the figure given in Figure 1. What evidence do you find to support the statement that this is not a reliable way to compute f.

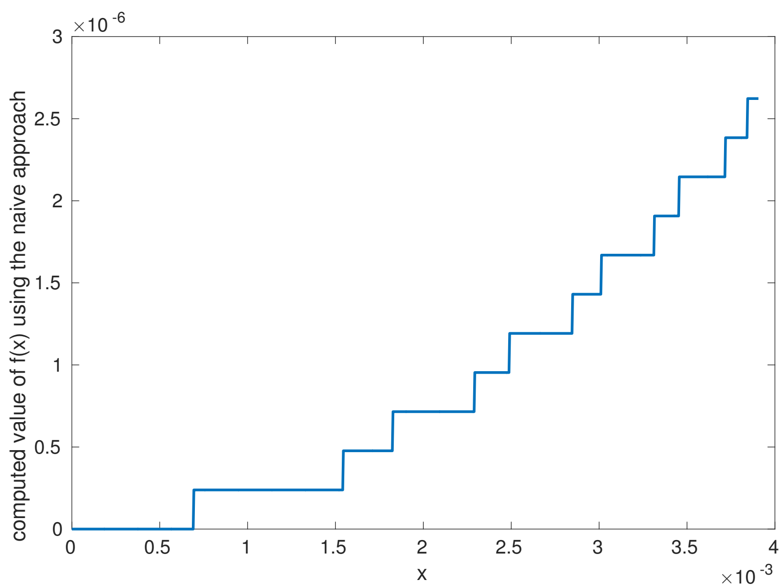

Figure 1: A plot of the computed values of f using the naive implementation based directly on the definition of f.

3. (5 points) Explain why the *computed* value of f equals zero for all $x \leq \frac{1}{2} \times 10^{-3}$ while the true value of f is strictly positive for all $x > 0$.

4. (8 points) Find a reliable way to evaluate f and explain why it can *never* cancel catastrophically.

elevation (degrees)	wind speed (meters/second) -2	-1	0	1	2
0	0	0	0	0	0
10	11424	11438	11453	11467	11481
20	15788	15817	15887	15876	15906
30	17586	17628	17670	17712	17754
40	17708	17759	17810	17861	17913
50	16458	16515	17773	16631	16688
60	13965	14026	14087	14148	14209
70	10293	10354	10415	10476	10538
80	5518	5577	5635	5694	5753
90	-113	-56	-2	56	113

Figure 2: A partial artillery table for a new piece of artillery.

Problem 2 Your brigade has received a new piece of artillery along with the very partial firing table show in Figure 2. It gives the range as a function of the elevation of the gun and the wind speed parallel to the ground. Negative values of the wind indicate head wind (Swedish: "motvind"), positive wind speeds indicate tail wind (Swedish: "medvind").

1. (6 points) Exactly three entries in the column corresponding to $w = 0$ have been corrupted. Find them and explain why they are wrong.
2. (4 points) Assuming the errors are confined to the column $w = 0$, find a way to approximate the correct value of the corrupted data accurately.
3. (5 points) Explain why there is not enough information to estimate how accurate the recovered data is.
4. (5 points) Consider a target located at $d = 6000$. Estimate the elevation θ_{high} for the high trajectory to this target when $w = -2$.
5. (5 points) Estimate the relative error for θ_{high} as accurately as the data will allow.

Problem 3 The trajectory of an artillery shell has been integrated numerically using time steps $h, 2h, 4h, 8h$, and $16h$. Partial results corresponding to time step h along with Richardson's fractions and error estimates are presented in Figure 3a and Figure 3b. Specifically, $x_h(t)$ is the computed approximation of the x coordinate at time t using time step h, several values of Richardson's fraction

$$F_h(t) = \frac{x_{2h}(t) - x_{4h}(t)}{x_h(t) - x_{2h}(t)}, \tag{2}$$

as well as Richardson's error estimate

$$E_h(t) = \frac{x_h(t) - x_{2h}(t)}{2^{p-1}}. \tag{3}$$

We have the same kind of information for the computed values of the y coordinate. Unfortunately, most of the data which describes the experimental setup has been lost after the numbers were produced.

Remark 1 In what follows you are asked to examine the numbers and extract evidence which supports a specific conjecture. Is it *not* enough to simply point to the relevant numbers, it is critical that you explain why the numbers support the given conjecture.

1. (6 points) Let $x(t)$ denote the shell's true x coordinate at time t and let $x_h(t)$ denote the computed approximation using time step h.

   While scanning the data for the x coordinate, what evidence do you find to support the conjecture that we have an asymptotic error expansion of the form

   $$x(t) - x_h(t) = \alpha(t)h^2 + \beta(t)h^3 + O(h^r), \quad 3 < r. \tag{4}$$

2. (6 points) While examining the data for the x coordinate, what evidence do you find to support the conjecture that the function $\beta(t)/\alpha(t)$ does not change sign.

3. (3 points) What evidence do you find to support the conjecture that you have been handed data which describes a *complete* trajectory from the muzzle of the gun to the point of impact?

4. (4 points) What evidence do you find to support the conjecture that the muzzle velocity was greater than 670 m/s?

5. (6 points) What evidence do you find to support the conjecture that the true range is $r = 17461$ m and that this number is accurate to 5 significant figures?

t	x_h(t)	F_16h(t)	F_8h(t)	F_4h(t)	F_2h(t)	F_h(t)	E_h(t)
0.000000e+00	0.00000000e+00	NaN	NaN	NaN	NaN	NaN	0.00000000e+00
5.497990e+00	3.24659494e+03	4.21151059e+00	4.10494790e+00	4.05221775e+00	4.02603872e+00	4.01300110e+00	6.31161104e-04
1.099598e+01	5.82616360e+03	4.21623805e+00	4.10629017e+00	4.05265638e+00	4.02620267e+00	4.01306956e+00	7.55420390e-04
1.649397e+01	7.97981803e+03	4.22733264e+00	4.11095146e+00	4.05477883e+00	4.02721350e+00	4.01356256e+00	7.24402208e-04
2.199196e+01	9.83700048e+03	4.24454536e+00	4.11866844e+00	4.05842372e+00	4.02898368e+00	4.01443475e+00	6.43401519e-04
2.748995e+01	1.14732693e+04	4.26881937e+00	4.12984265e+00	4.06377562e+00	4.03160160e+00	4.01572932e+00	5.48974676e-04
3.298794e+01	1.29345446e+04	4.30232889e+00	4.14552025e+00	4.07134699e+00	4.03532078e+00	4.01757239e+00	4.54824126e-04
3.848593e+01	1.42491089e+04	4.34898884e+00	4.16765184e+00	4.08210942e+00	4.04062588e+00	4.02020588e+00	3.66260487e-04
4.398392e+01	1.54345625e+04	4.41580104e+00	4.19982459e+00	4.09787408e+00	4.04842634e+00	4.02408546e+00	2.85289521e-04
4.948191e+01	1.65023604e+04	4.51632481e+00	4.24922144e+00	4.12232827e+00	4.06058912e+00	4.03015018e+00	2.12439249e-04
5.497990e+01	1.74608397e+04	4.68068524e+00	4.33257063e+00	4.16426748e+00	4.08162148e+00	4.04068146e+00	1.47447296e-04

(a) Data pertaining to the x-coordinate of the shell.

t	y(h)	F_16h(t)	F_8h(t)	F_4h(t)	F_2h(t)	F_h(t)	E_h(t)
0.000000e+00	0.00000000e+00	NaN	NaN	NaN	NaN	NaN	0.00000000e+00
5.497990e+00	1.74000565e+03	4.18677009e+00	4.09294990e+00	4.04631033e+00	4.02310769e+00	4.01154122e+00	4.63609226e-04
1.099598e+01	2.85736608e+03	4.18814477e+00	4.09266937e+00	4.04595034e+00	4.02287545e+00	4.01141238e+00	5.95827378e-04
1.649397e+01	3.51854506e+03	4.19489759e+00	4.09524366e+00	4.04705027e+00	4.02338026e+00	4.01165369e+00	6.23229001e-04
2.199196e+01	3.81248313e+03	4.20542081e+00	4.09975873e+00	4.04913065e+00	4.02437735e+00	4.01214161e+00	6.12096517e-04
2.748995e+01	3.79209546e+03	4.21890015e+00	4.10576474e+00	4.05195763e+00	4.02574780e+00	4.01281619e+00	5.85319972e-04
3.298794e+01	3.49243877e+03	4.23492713e+00	4.11303874e+00	4.05541568e+00	4.02743287e+00	4.01364785e+00	5.51856330e-04
3.848593e+01	2.94013162e+03	4.25334090e+00	4.12149283e+00	4.05945906e+00	4.02940927e+00	4.01462481e+00	5.15596084e-04
4.398392e+01	2.15861212e+03	4.27421489e+00	4.13116312e+00	4.06410550e+00	4.03168577e+00	4.01575143e+00	4.78275188e-04
4.948191e+01	1.17080847e+03	4.29789306e+00	4.14222604e+00	4.06944396e+00	4.03430698e+00	4.01705008e+00	4.40577470e-04
5.497990e+01	1.33293816e-02	4.32499814e+00	4.15500228e+00	4.07563663e+00	4.03735438e+00	4.01856156e+00	4.02731126e-04

(b) Data pertaining to the y-coordinate of the shell.

Figure 3: The x and y coordinates of the shell computed using time step h. Richardson's fraction F_h is given for each value of the time and several different values of the time step. The last column contains Richardson's error estimates E_h.

Problem 4 The Quake's engine hinges (among other) things on the ability to compute reciprocal square roots with an accuracy good enough to fool the human eye. Here we consider the problem of implementing a function which can compute the reciprocal square root y of any floating point number $\alpha > 0$, i.e. $x = \frac{1}{\sqrt{\alpha}}$.

1. (5 points) Explain why this problem is equivalent to solving the non-linear equation
$$f(x) = \frac{1}{x^2} - \alpha = 0, \quad x > 0. \tag{5}$$

2. (5 points) Show that Newton's method for this equation can be implemented without a single division.

3. (8 points) Newton's method hinges on our ability to generate a good initial guess. Use your knowledge of the floating point number system to explain why it suffices to solve this problem for all $\alpha \in [1, 4]$. Obviously, we still need to handle the general case, but you need to explain why the extension from the special case is trivial in comparison with the general case.

4. (7 points) Derive an initial guess x_0 for Newton's method such that
$$\left| x_0 - \frac{1}{\sqrt{\alpha}} \right| < \frac{1}{4} \tag{6}$$
for all $\alpha \in [1, 4]$.