# Teknisk-vetenskapliga beräkningar
# Fall 2011
# The 2nd Exam

January 9th, 2012

**Instructions:** This exam consists of four major problems. The maximum score is 100 points or 25 points per problem. You are allowed to use anything which is either printed or written on paper prior to the exam. This includes lecture notes, your own notes, your mandatory projects and any textbook that you might care to reference. Moreover, you may use a programmable calculator. While the class was taught in English you may write your answers in Swedish or English.

**Problem 1** The purpose of this problem is to estimate the value of the integral

$$I = \int_0^1 f(x)dx$$

where $f : [0,1] \to \mathbb{R}$ is a real function. Most of the information about $f$ has been lost. However, it is known that

$$f(0) = 0, \quad f(1) = \sin(1)$$

and if

$$x_j = \frac{j}{1000}, \quad j = 0, 1, \dots, 1000$$

then

$$\begin{aligned} A &= \sum_{j=0}^{1000} f(x_j) &=& \quad 1.255022509615711 \times 10^2, \\ B &= \sum_{j=0}^{500} f(x_{2j}) &=& \quad 6.2962086683414704 \times 10^1, \\ C &= \sum_{j=0}^{250} f(x_{4j}) &=& \quad 3.1692597996404810 \times 10^1. \end{aligned}$$

Moreover, it is known that $f$ is infinitely often differentiable in the entire interval $[0,1]$.

1. (10 points) Let $T_h$ denote the result of applying the trapezoidal rule to $f$ with stepsize $h$. Show, that

$$\begin{aligned} T_h &\approx 0.125081515469167 \\ T_{2h} &\approx 0.125082702382022 \\ T_{4h} &\approx 0.125087450016003 \end{aligned}$$

2. (10 points) Explain why Richardson's technique can be used to obtain a reliable error estimate in the case of $h = \frac{1}{1000}$.

3. (5 points) Estimate the integral with an absolute error which is less than $10^{-6}$.

**Problem 2** The purpose of this problem is to approximate the solutions of the nonlinear equation

$$x = f(x) \tag{1}$$

where $f(x) = \tan(x)$.

Let $I_j$ be the intervals given by

$$I_0 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

and

$$I_j = I_0 + j \cdot \pi, \quad j \in \mathbb{Z}.$$

1. (5 points) Show that equation (1) has exactly one solution, say $\xi_j$, in each interval $I_j$ for all $j \in \mathbb{Z}$.

2. (10 points) Estimate the number $\xi_1$ with a relative error which is less than $10^{-6}$. (Hint: The approximation $\xi_1 \approx 4.5$ is not bad, but it is not good enough).

3. (10 points) Find a $j \neq 0$ for which you can estimate the corresponding solution $\xi_j$ with a relative error which is less than $10^{-1000}$.

**Problem 3** Let $A$ be the matrix

$$A = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 4 & 10 & 4 & 0 \\ 0 & 4 & 10 & 4 \\ 0 & 0 & 4 & 10 \end{bmatrix}$$

1. (5 points) Show that

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

   constitute the $LU$ factorization of the matrix $A$.

2. (5 points) Solve the linear system $Ax = f$, where $f = (0, 0, 0, 1)^T$ using the LU factorization.

3. (5 points) Given that $A^{-1}$ is given by

$$A^{-1} = \begin{bmatrix} 42.5 & -21 & 10 & -4 \\ -21 & 10.5 & -5 & 2 \\ 10 & -5 & 2.5 & -1 \\ -4 & 2 & -1 & 0.5 \end{bmatrix}$$

   compute the infinity norm condition number of $A$.

4. (10 points) Let $g \in \mathbb{R}^4$ and let $y$ denote the exact solution of $Ay = g$. Let $\hat{y}$ denote the computed solution using, say, Gaussian elimination with partial pivoting and double precision floating point arithmetic. Explain, why we can not expect the inequality

$$\frac{\|y - \hat{y}\|_\infty}{\|y\|_\infty} \leq 2^{-53}$$

   to hold.

**Problem 4** Let $f : \mathbb{R} \to \mathbb{R}$ be a finitely often differentiable function, such that

$$f(0) = 0, \quad f(1) = -4, \quad \text{and} \quad f(2) = -2$$

1. (5 points) Show that

   $$p(x) = 3x^2 - 7x$$

   is the interpolating polynomial for $f$ corresponding to the nodes $\{0, 1, 2\}$.

2. (5 points) It is known that $|f^{(3)}(y)| \leq 6$ for all $y \in \mathbb{R}$. Compute an approximation for $f(1.5)$ and estimate the absolute error.

3. (5 points) Prove that your approximation for $f(1.5)$ has the correct sign.

4. (5 points) Now, suppose that additional information is made available. Specifically, $f(3) = 12$. Compute the interpolating polynomial for $g$ corresponding to the nodes $\{0, 1, 2, 3\}$.

5. (5 points) Finally, prove that $f'$ has at least one zero.