5DV005 Teknisk-vetenskapliga beräkningar

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Instructions: This exam consists of four pages with one major problem per page. The maximum score is 100 points or 25 points per problem. You may use your notes, books as well as a programmable calculator. You may answer the questions in English or Swedish.

Please fill in your name and personal identification number.

NAME :

ID :

Problem 1 Consider the functions $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ given by

$$f(x) = \sqrt{x^2 + 4} - 2,$$
$$g(x) = \frac{x^2}{\sqrt{x^2 + 4} + 2}.$$

1. Show, that
$$f(x) = g(x)$$
for all $x \in \mathbb{R}$.

2. Consider the following MATLAB commands

```
>> x=single(linspace(0,2^(-11),16))';
>> f=sqrt(x.^2+4)-2;
>> g=x.^2./(sqrt(x.^2+4)+2);
```

Explain, why the MATLAB command

```
>>[f g]
```

produces the output

```
ans =

   1.0e-07 *

         0         0
         0    0.0026
         0    0.0106
         0    0.0238
         0    0.0424
         0    0.0662
         0    0.0954
         0    0.1298
         0    0.1695
         0    0.2146
         0    0.2649
         0    0.3205
         0    0.3815
         0    0.4477
         0    0.5192
         0    0.5960
```

despite the fact that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Problem 2 Let $f : (0, \infty) \to \mathbb{R}$ be the logarithm with base e, i.e.

$$f(x) = \log(x)$$

and consider the linear function

$$g(x) = ax + b,$$

where

$$\begin{aligned} a &= \log(2), \\ b &= -\frac{1 + a + \log(a)}{2}. \end{aligned}$$

1. Show that
$$-3 \cdot 10^{-2} \leq f(x) - g(x) \leq 3 \cdot 10^{-2}$$
for all $x \in [1, 2]$.
2. Compute $f(1.5)$ with a relative error of at most 10^{-6}.

Problem 3 Consider the function $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ given by

$$f(n) = \sum_{j=0}^{n} j^2.$$

1. Find a polynomial g of degree at most 3, such that

   $$f(n) = g(n)$$

   for $n = 0, 1, 2, 3$.

2. Show, that

   $$f(n) = g(n)$$

   for all $n \in \mathbb{N}$.

Problem 4 Consider the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = e^x.$$

Let n be a positive integer, let $h = 1/n$ and let $x_j = jh$ for $j = 0, 1, 2, \ldots, n$. The trapezoidal rule is given by

$$T_n(f) = \frac{1}{2}h \left\{ f(x_0) + 2 \left(\sum_{j=1}^{n-1} f(x_j) \right) + f(x_n) \right\}.$$

1. Show, that
$$T_n(f) > \int_0^1 f(x)dx.$$

2. Show, that
$$T_n(f) \to \int_0^1 f(x)dx, \quad n \to \infty, \quad n \in \mathbb{N}, \quad nh = 1.$$