Problem 1 Consider the function $f : [1, \infty) \to \mathbb{R}$

$$f(x) = \sqrt{x^4 + 1} - \sqrt{x^4 - 1}$$

1. (5 points) Show that f is differentiable and strictly decreasing for all $x > 1$.
2. (5 points) The MATLAB commands

   ```
   >>x=single(linspace(10,100,201));
   >>f=sqrt(x.^4+1)-sqrt(x.^4-1);
   >>plot(x,f)
   ```

   followed by a few purely cosmetic commands have generated the graph displayed in Figure 1. Which features of this graph have nothing to do with reality?

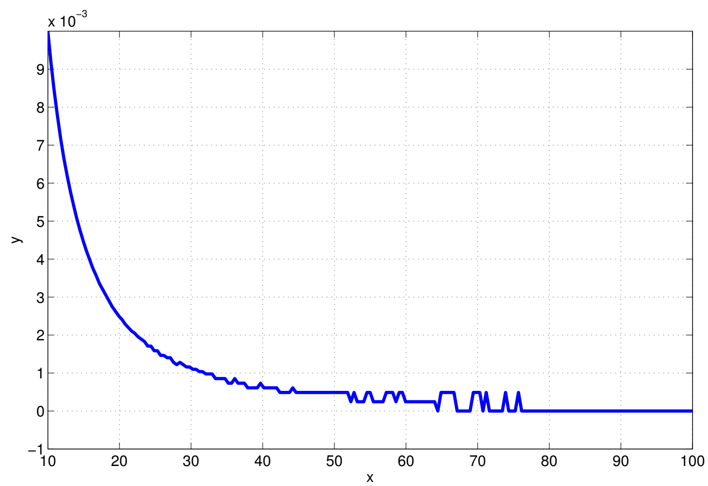

Figure 1: The naive application of MATLAB to the problem of computing f

3. (5 points) Why did the MATLAB commands fail to produce a reliable plot?
4. (5 points) Why is catastrophic cancellation not an issue for the interval

$$1 < x < \sqrt[4]{\frac{5}{3}}.$$

5. (5 points) Find a numerically reliable way to evaluate $f(x)$ for all $x \geq 1$ using MATLAB.

Problem 2 Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = x^3 - x^2 - 4x + 1.$$

A very crude plot of the graph of g can be found in Figure 2.

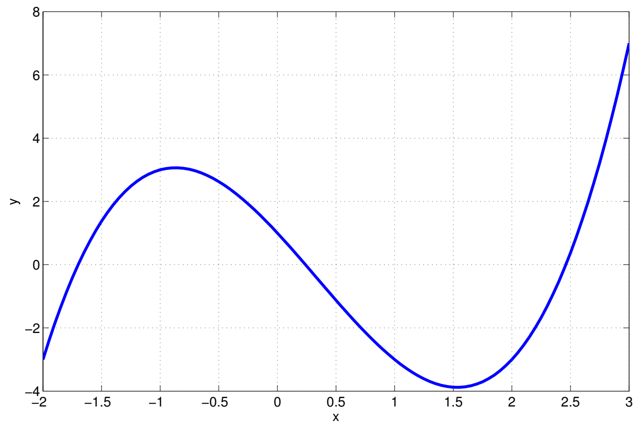

Figure 2: A crude plot of the graph of g.

1. (5 points) Explain why you can be absolutely certain that g has exactly three distinct zeros even though we are not certain that the graph can be trusted.

2. (10 points) Newton's method has been applied to the solution of the equation

$$g(x) = 0 \tag{1}$$

and has produced the results given below

n	x(n)	g(x(n))
0	2.100000000000000e-01	1.251610000000001e-01
1	2.391907083051520e-01	-2.904027288519462e-04
2	2.391232785547966e-01	-1.284441442095385e-09
3	2.391232782565544e-01	1.110223024625157e-16
4	2.391232782565545e-01	0

Explain why you can not trust the computed values of $g(x_3)$ and $g(x_4)$.

3. (10 points) Find an interval of length at most 2×10^{-6} which is certain to contain the smallest positive root of g and determine the root with a relative error which is less than 10^{-6}.

Problem 3 Let $\phi : \mathbb{R} \to \mathbb{R}$ be any function which is infinitely often differentiable.

1. (5 points) Let $x \in \mathbb{R}$ and let $h > 0$. Show that $D_h(x)$ given by

$$D_h(x) = \frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2}$$

satisfies

$$D_h(x) = \phi''(x) + O(h^2).$$

2. (10 points) A specific ϕ has been chosen, together with the point $x_0 = 1$. Then $D_h(x_0)$ has been computed for $h = 2^{-k}x_0$, where $k = 1, 2, \ldots, 20$. The results can be found in the following table.

k	Dh	(Dh-D2h)	(D2h-D4h)/(Dh-D2h)
1	-2.629859253893		
2	-2.394779366424	2.350798874690e-01	
3	-2.314160132340	8.061923408471e-02	2.915928067761
4	-2.292594257216	2.156587512359e-02	3.738277886832
5	-2.287113989657	5.480267559562e-03	3.935186537738
6	-2.285738363786	1.375625870423e-03	3.983835777874
7	-2.285394109742	3.442540441938e-04	3.995961394281
8	-2.285308024504	8.608523785369e-05	3.998990451521
9	-2.285286501836	2.152266824851e-05	3.999747469028
10	-2.285281121149	5.380687071010e-06	3.999985125406
11	-2.285279775970	1.345179043710e-06	3.999978364344
12	-2.285279439762	3.362074494362e-07	4.001038781163
13	-2.285279363394	7.636845111847e-08	4.402439024390
14	-2.285279333591	2.980232238770e-08	2.562500000000
15	-2.285279273987	5.960464477539e-08	0.500000000000
16	-2.285279273987	0.000000000000e+00	Inf
17	-2.285280227661	-9.536743164062e-07	-0.000000000000
18	-2.285278320312	1.907348632812e-06	-0.500000000000
19	-2.285278320312	0.000000000000e+00	Inf
20	-2.285278320312	0.000000000000e+00	NaN

Determine the range of k for which we will be able to trust the corresponding error estimates.

3. (10 points) Find the value of $\phi''(1)$ with a relative error which is at most 10^{-6}.

Problem 4 Consider the problem of computing

$$f(\alpha) = \sqrt[5]{\alpha}$$

using a binary computer.

1. (5 points) Explain carefully why the problem is equivalent to solving the nonlinear equation

$$g(x) = 0, \quad \text{where} \quad g(x) = x^5 - \alpha, \tag{2}$$

   and write down Newton's iteration for equation (2).

2. (5 points) Explain carefully why the problem is essentially solved if we can compute $f(x)$ for all machine numbers $x \in [1, 32]$.

3. (15 points) It is clear that we will need an intelligent way of initializing Newton's iteration, i.e. a function

$$x_0 = x_0(\alpha)$$

   defined for $\alpha \in [1, 32]$. Assuming that we chosen a stepsize $h > 0$ and have defined points

$$t_j = 1 + jh, \quad j = 0, 1, 2, \ldots, N, \quad Nh = 31,$$

   and that we are willing to precompute the values

$$f(t_j), \quad j = 0, 1, 2, \ldots, N,$$

   then we can define x_0 by interpolating f on each subinterval $[t_j, t_{j+1}]$ using the corresponding first order polynomial. Now, what is the smallest value of N for which you will be able to ensure, that

$$|x_0(\alpha) - f(\alpha)| \leq \frac{1}{50}$$

   for all $\alpha \in [1, 32]$?