Problem 1 This problem centers around the computation of $\sin(x)$ for $x \in \mathbb{R}$. Let $p(x) = x - \frac{1}{6}x^3$.

1. (5 points) Show that the error $e(x) = \sin(x) - p(x)$ satisfies
$$|e(x)| \leq \frac{1}{5!}|x|^5$$
for all $x \in \mathbb{R}$.

2. (5 points) Let $\delta_0 = \sqrt{6}$ and let $\delta \in (0, \delta_0)$. Show that the relative error $r(x) = e(x)/\sin(x)$ satisfies
$$|r(x)| \leq \frac{1}{5!}\left(\frac{1}{1 - \frac{1}{6}\delta^2}\right)\delta^4$$
for all $x \in \mathbb{R}$ such that $0 < |x| \leq \delta$.

   Hint You can use without proof that
$$1 - \frac{1}{6}x^2 \leq \frac{\sin(x)}{x} \leq 1$$
for all $x \in \mathbb{R}$.

3. (5 points) Determine δ such that
$$|r(x)| \leq u$$
where $u = 2^{-53}$ is the double precision unit roundoff.

4. (10 points) It is known that $\sin(3x)$ can be computed in terms of $\sin(x)$ using the formula
$$\sin(3x) = 3\sin(x) - 4\sin(x)^3.$$
Give an algorithm which combines this identity with the good properties of p and allows you to compute $\sin(x)$ for all $x \in [-\pi, \pi]$.

Problem 2 The definite integral $T = \int_a^b f(x)dx$ has been approximated using the composite trapezoidal rule A_h and stepsizes $h_k = 2^{-k}h_0$. Figure 1 lists all the numbers which have been generated. A_h is the approximation which corresponds to the stepsize h, F_h is Richardson's fraction and Est_h is Richardson's error estimate.

k	A_h	F_h	Est_h
0	1.8393972059e-01	0.0000000000e+00	0.0000000000e+00
1	1.6778619276e-01	0.0000000000e+00	-5.3845092763e-03
2	1.6248840509e-01	3.0491082040e+00	-1.7659292213e-03
3	1.6107989608e-01	3.7612735260e+00	-4.6950300451e-04
4	1.6072242724e-01	3.9402287492e+00	-1.1915628112e-04
5	1.6063272479e-01	3.9850511717e+00	-2.9900815819e-05
6	1.6061027820e-01	3.9962624105e+00	-7.4821952984e-06
7	1.6060466525e-01	3.9990655786e+00	-1.8709858969e-06
8	1.6060326192e-01	3.9997663932e+00	-4.6777379302e-07
9	1.6060291109e-01	3.9999415985e+00	-1.1694515570e-07
10	1.6060282338e-01	3.9999853886e+00	-2.9236395721e-08
11	1.6060280145e-01	3.9999963963e+00	-7.3091055153e-09
12	1.6060279597e-01	3.9999990937e+00	-1.8272767928e-09
13	1.6060279460e-01	3.9999985013e+00	-4.5681936937e-10
14	1.6060279426e-01	4.0000020253e+00	-1.1420478452e-10
15	1.6060279417e-01	3.9999996760e+00	-2.8551198443e-11
16	1.6060279415e-01	4.0002125845e+00	-7.1374202844e-12
17	1.6060279414e-01	4.0004252142e+00	-1.7841654080e-12
18	1.6060279414e-01	3.9925467382e+00	-4.4687401927e-13
19	1.6060279414e-01	4.0147119940e+00	-1.1130911008e-13
20	1.6060279414e-01	4.3029327611e+00	-2.5868196474e-14
21	1.6060279414e-01	2.7573964497e+00	-9.3813845581e-15
22	1.6060279414e-01	2.7780821918e+00	-3.3769283666e-15
23	1.6060279414e-01	2.4013157895e+00	-1.4062824979e-15

Figure 1: Summary of all available data from the approximation of T

.

1. (5 points) Assuming each evaluation of the function f requires 1 CPU second, what is the total CPU time necessary to generate this table?
2. (10 points) What evidence do you find to support the hypothesis that the error $E_h = T - A_h$ satisfies an asymptotic error expansion of the form
$$E_h = \alpha h^2 + \beta h^4 + O(h^r), \quad h \to 0, \quad h > 0, \quad 4 < r.$$
3. (5 points) Show taht $T < A_h$ for $k = 11$.
4. (5 points) Show that $A_h + \alpha h^2 < T$ for $k = 11$.

iter	approximation	residual
1	1.410714285714286e+00	-9.885204081632404e-03
2	1.414110429447853e+00	-2.916933268093391e-04
3	1.414213690129920e+00	3.613508865463189e-07
4	1.414213562368437e+00	-1.317590481164643e-11
5	1.414213562373095e+00	4.440892098500626e-16

Figure 2: Table 1

iter	approximation	residual
1	1.300000000000000e+00	-3.099999999999998e-01
2	1.419230769230769e+00	1.421597633136118e-02
3	1.414222430685845e+00	2.508345498064557e-05
4	1.414213562400901e+00	7.864642270760669e-11
5	1.414213562373095e+00	-4.440892098500626e-16

Figure 3: Table 2

Problem 3 A student has been solving the nonlinear equation $f(x) = x^2 - 2 = 0$ using three different iterative methods. The results are given in Figures 2, 3, 4.

1. (10 points) It is known that the student used the bisection method, the secant method and Newton's method to generate the data. Identify the method used to generate each of the three tables.
2. (5 points) What is the one advantage of bisection method which has been completely suppressed in this problem?
3. (5 points) Let x denote any good approximation of $\sqrt{2}$. Show that the relative error can be estimated as follows
$$\frac{\sqrt{2}-x}{\sqrt{2}} \approx \frac{2-x^2}{4}.$$
4. (5 points) Use the available information given find $\sqrt{2}$ with a relative error which is less than 10^{-10}.

iter	approximation	residual
1	1.400000000000000e+00	-4.000000000000026e-02
2	1.450000000000000e+00	1.025000000000000e-01
3	1.425000000000000e+00	3.062499999999968e-02
4	1.412500000000000e+00	-4.843750000000480e-03
5	1.418750000000000e+00	1.285156249999941e-02
6	1.415625000000000e+00	3.994140624999698e-03
7	1.414062500000000e+00	-4.272460937500000e-04
8	1.414843750000000e+00	1.782836914062447e-03
9	1.414453125000000e+00	6.776428222656783e-04
10	1.414257812500000e+00	1.251602172853694e-04
11	1.414160156250000e+00	-1.510524749759323e-04
12	1.414208984375000e+00	-1.294851303068612e-05
13	1.414233398437500e+00	5.610525608101824e-05
14	1.414221191406250e+00	2.157822251369623e-05
15	1.414215087890625e+00	4.314817488193512e-06
16	1.414212036132813e+00	-4.316857084463166e-06
17	1.414213562011719e+00	-1.022126827621150e-09
18	1.414214324951172e+00	2.156897098704036e-06
19	1.414213943481445e+00	1.077937340276947e-06
20	1.414213752746582e+00	5.384575700873029e-07
21	1.414213657379150e+00	2.687177129701013e-07
22	1.414213609695435e+00	1.338477906287494e-07
23	1.414213585853576e+00	6.641283123443031e-08
24	1.414213573932648e+00	3.269535220340458e-08
25	1.414213567972183e+00	1.583661290993632e-08
26	1.414213564991951e+00	7.407243263202190e-09
27	1.414213563501835e+00	3.192557773701310e-09
28	1.414213562756777e+00	1.085215473040080e-09
29	1.414213562384248e+00	3.154454475406965e-11
30	1.414213562197983e+00	-4.952911414335404e-10
31	1.414213562291116e+00	-2.318729652728280e-10
32	1.414213562337682e+00	-1.001640992370767e-10
33	1.414213562360965e+00	-3.431011030841091e-11
34	1.414213562372606e+00	-1.382671754868170e-12
35	1.414213562378427e+00	1.508126956650813e-11
36	1.414213562375517e+00	6.849187883517516e-12
37	1.414213562374061e+00	2.732924997417285e-12
38	1.414213562373334e+00	6.754596881819452e-13
39	1.414213562372970e+00	-3.532729664357248e-13
40	1.414213562373152e+00	1.612043831755727e-13
41	1.414213562373061e+00	-9.592326932761353e-14
42	1.414213562373106e+00	3.241851231905457e-14

Figure 4: Table 3

Problem 4 This problem concerns the computation of the number π from scratch. Consider the function $g(x) = \pi + \sin(x)$ and the iteration given by $x_{n+1} = g(x_n)$, where $x_0 \in \mathbb{R}$.

1. (5 points) Show that π is a fixed point for g.
2. (10 points) Show that the iteration satisfies

$$|\pi - x_{n+1}| \leq \frac{1}{6}|\pi - x_0|^3.$$

3. (10 points) Show that the iteration converges to π if x_0 is chosen such that

$$\frac{1}{6}|\pi - x_0|^2 < 1.$$