Problem 1 Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \cosh(x) - \cos(x).$$

1. (5 points) The following MATLAB commands have been issued

```
>>f=@(x)cosh(x)-cos(x);
>>x=linspace(-1,1,1024)*2^(-10); x=single(x);
>>plot(x,f(x))
```

   and the result is given as Figure 1. List as many differences between this plot and the correct graph of f as you can.

2. (10 points) Prove that the naive expression for f used in the previous question cannot cancel catastrophically when

$$x > \cosh^{-1}(2) = \log(2 + \sqrt{3}).$$

3. (10 points) Find a polynomial p for which you are certain the following two properties are true

   - For all x we have $f(x) - p(x) = O(x^{10})$.
   - Catastrophic cancellation cannot occur when evaluating p for small values of x.

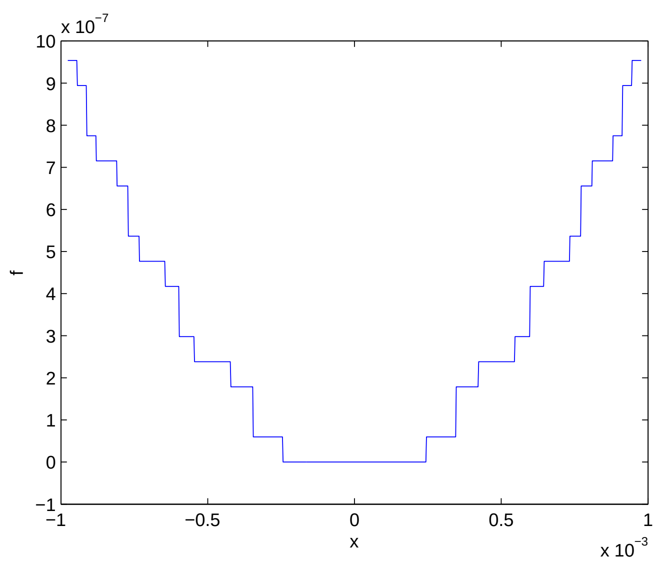

Figure 1: A plot of f as a function of x using a bad MATLAB implementation.

Problem 2 Consider the function $g : [-1, 1] \to \mathbb{R}$ given by

$$g(x) = (1 - |x|^3)^{\frac{1}{3}}.$$

1. (15 points) Simpson's rule has been used to generate many different approximations of the two integrals

$$I_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x)dx \quad \text{and} \quad I_2 = \int_{-1}^{1} g(x)dx$$

and all the results are given in Figure 2 and Figure 3. Unfortunately, we have forgotten which figure belongs to which integral!

N	Sh	(Sh-S2h)	(S2h-S4h)/(Sh-S2h)
131072	9.89322425564950e-01	-9.4369e-15	-6.117647e-01
65536	9.89322425564960e-01	5.7732e-15	9.230769e-01
32768	9.89322425564954e-01	5.3291e-15	8.958333e-01
16384	9.89322425564949e-01	4.7740e-15	-1.255814e+00
8192	9.89322425564944e-01	-5.9952e-15	-3.703704e-02
4096	9.89322425564950e-01	2.2204e-16	9.000000e+00
2048	9.89322425564950e-01	1.9984e-15	1.083333e+01
1024	9.89322425564948e-01	2.1649e-14	1.598974e+01
512	9.89322425564926e-01	3.4617e-13	1.599487e+01
256	9.89322425564580e-01	5.5369e-12	1.599390e+01
128	9.89322425559043e-01	8.8557e-11	1.597706e+01
64	9.89322425470486e-01	1.4149e-09	1.590904e+01
32	9.89322424055610e-01	2.2509e-08	1.564736e+01
16	9.89322401546288e-01	3.5221e-07	1.472995e+01
8	9.89322049334740e-01	5.1881e-06	1.215511e+01
4	9.89316861277389e-01	6.3061e-05	5.970799e+01
2	9.89253799880431e-01	3.7653e-03	0.000000e+00
1	9.85488530462065e-01	0.0000e+00	0.000000e+00

Figure 2: The first set of calculations, $2Nh$ equals the length of the relevant interval.

Examine the numbers carefully and give as many reasons as you can as to why Figure 3 contains the results obtained by applying Simpson's rule to the integral I_2.

Hint There are at least two reasons which are very different in nature.

2. (10 points) Compute the value of I_1 with a relative error which is less than $\tau = 10^{-9}$. Remember to explain why you can trust your error estimate!

Hint Remember to determine and use the correct order p.

N	Sh	(Sh-S2h)	(S2h-S4h)/(Sh-S2h)
131072	1.76663869100275e+00	9.0100e-08	2.519844e+00
65536	1.76663860090236e+00	2.2704e-07	2.519845e+00
32768	1.76663837386342e+00	5.7210e-07	2.519849e+00
16384	1.76663780176041e+00	1.4416e-06	2.519856e+00
8192	1.76663636014718e+00	3.6327e-06	2.519870e+00
4096	1.76663272748955e+00	9.1538e-06	2.519897e+00
2048	1.76662357366548e+00	2.3067e-05	2.519953e+00
1024	1.76660050696801e+00	5.8127e-05	2.520063e+00
512	1.76654237998170e+00	1.4648e-04	2.520284e+00
256	1.76639589630144e+00	3.6918e-04	2.520724e+00
128	1.76602671582609e+00	9.3060e-04	2.521593e+00
64	1.76509611380473e+00	2.3466e-03	2.523265e+00
32	1.76274951440854e+00	5.9211e-03	2.526195e+00
16	1.75682842297277e+00	1.4958e-02	2.529481e+00
8	1.74187058957818e+00	3.7836e-02	2.521815e+00
4	1.70403504038579e+00	9.5414e-02	2.885182e+00
2	1.60862078851493e+00	2.7529e-01	0.000000e+00
1	1.33333333333333e+00	0.0000e+00	0.000000e+00

Figure 3: The second set of calculations, $2Nh$ equals the length of the relevant interval.

Problem 3 Consider the function $h : [0, \infty) \to \mathbb{R}$ given by

$$h(x) = x^2 e^{-x} - \frac{1}{4}x.$$

1. (5 points) Explain why h has at least 3 distinct zeros on the interval $[0, \infty)$. You may rely on the following table of values of h.

x	h(x)
0.2200	-0.0162
0.4400	0.0147
0.6600	0.0601
0.8800	0.1012
1.1000	0.1278
1.3200	0.1355
1.5400	0.1234
1.7600	0.0929
1.9800	0.0463
2.2000	-0.0137

2. (10 points) Newton's method has been used to compute a sequence of 10 approximations of a zero for h and these are the results

n	x(n)	h(x(n))
0	4.000000000000000e-01	7.251207365702311e-03
1	3.594915545716667e-01	3.365926732381980e-04
2	3.574094923949957e-01	1.050048316519892e-06
3	3.574029562463180e-01	1.043082287210950e-11
4	3.574029561813888e-01	-1.387778780781446e-17
5	3.574029561813890e-01	1.387778780781446e-17
6	3.574029561813888e-01	-1.387778780781446e-17
7	3.574029561813890e-01	1.387778780781446e-17
8	3.574029561813888e-01	-1.387778780781446e-17
9	3.574029561813890e-01	1.387778780781446e-17

Find a nonzero zero of h such that the relative error is less than $\tau = 10^{-6}$.

3. (10 points) When solving a general non-linear equation $h(x) = 0$ using Newton's method we iterate until $|h(x_n)| \leq \tau$ or we have completed `maxit` iterations. The parameters τ and `maxit` are specified by the user. Explain why it is usually pointless to use $\tau = 0$, despite the fact that we want to compute x such that $h(x) = 0$.

Problem 4 Consider a ball which is being shot straight into the air at time $t = 0$. Let $y(t)$ denote the height of the ball above sea level at time t and let $v(t)$ denote the velocity. The ball is subject to gravity and air resistance and the motion of the ball is governed by the initial value problem

$$\begin{pmatrix} y'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} f_1(y(t), v(t)) \\ f_2(y(t), v(t)) \end{pmatrix}, \quad \begin{pmatrix} y(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix},$$

where the two functions f_1 and f_2 are given by

$$f_1(y, v) = v, \quad f_2(y, v) = -g - \operatorname{sign}(v) \frac{k}{m} v^2.$$

Here g is the constant of gravity, m is the mass of the ball and k is an aerodynamic constant.

1. (5 points) Let t^* denote the time when the ball reaches its maximum height above sea level. Show that $v(t^*) = 0$.

2. (10 points) The trajectory of the ball has been approximated using Euler's explicit method and time step $h_1 = 0.1$. The results are given in Figure 4. Compute an approximation $t_1 \approx t^*$ using this data.

   Hint The fifth column is there to simplify your life. It contains the relevant values of f_2, i.e.
   $$f_2(n) = f_2(y_n, v_n).$$

3. (10 points) An even cruder approximation of the trajectory has also been computed using Euler's method and time step $h_2 = 2h_1 = 0.2$. The results are given in Figure 5. Use this data to estimate the accuracy of your approximation t_1 of t^*.

n	t(n)	y(n)	v(n)	f2(n)
0	0	0	10.0000	-19.8200
1	0.1000	1.0000	8.0180	-16.2488
2	0.2000	1.8018	6.3931	-13.9072
3	0.3000	2.4411	5.0024	-12.3224
4	0.4000	2.9414	3.7702	-11.2414
5	0.5000	3.3184	2.6460	-10.5201
6	0.6000	3.5830	1.5940	-10.0741
7	0.7000	3.7424	0.5866	-9.8544
8	0.8000	3.8010	-0.3988	-9.8041
9	0.9000	3.7611	-1.3793	-9.6298
10	1.0000	3.6232	-2.3422	-9.2714
11	1.1000	3.3890	-3.2694	-8.7511
12	1.2000	3.0621	-4.1445	-8.1023
13	1.3000	2.6476	-4.9547	-7.3651
14	1.4000	2.1521	-5.6912	-6.5810
15	1.5000	1.5830	-6.3493	-5.7886
16	1.6000	0.9481	-6.9282	-5.0200
17	1.7000	0.2553	-7.4302	-4.2992
18	1.8000	-0.4878	-7.8601	-3.6419
19	1.9000	-1.2738	-8.2243	-3.0561
20	2.0000	-2.0962	-8.5299	-2.5441

Figure 4: The approximate trajectory of the ball computed using Euler's explicit method and time step $h_1 = 0.1$.

n	t(n)	y(n)	v(n)	f2(n)
0	0	0	10.0000	-19.8200
1	0.2000	2.0000	6.0360	-13.4633
2	0.4000	3.2072	3.3433	-10.9378
3	0.6000	3.8759	1.1558	-9.9536
4	0.8000	4.1070	-0.8349	-9.7503
5	1.0000	3.9400	-2.7850	-9.0444
6	1.2000	3.3830	-4.5939	-7.7096
7	1.4000	2.4643	-6.1358	-6.0552
8	1.6000	1.2371	-7.3468	-4.4224
9	1.8000	-0.2323	-8.2313	-3.0445
10	2.0000	-1.8785	-8.8402	-2.0050

Figure 5: The approximate trajectory of the ball computed using Euler's explicit method and time step $h_2 = 0.2$.