MATH1851

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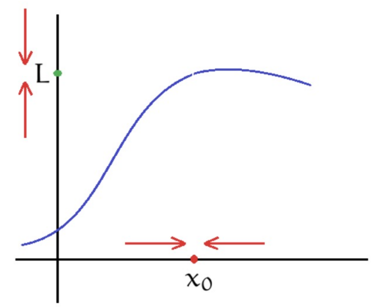
1 Limits and Continuity

1.1 Introduction to the concept of limit

We can conceptualize that the limit of a function $f(x)$ is L as x approaches c, given that we can make $f(x)$ *as close to* L *as we want* for all x sufficiently close to a, from both sides, *without actually letting* x *be* a. We can write this as:

$$\lim_{x \to a} f(x) = L$$



1.2 One-sided limits

There are two sides that x can tend to a number. We can write it as $x \to n^-$ and $x \to n^+$, which represents from the negative (left) / positive (right) side.

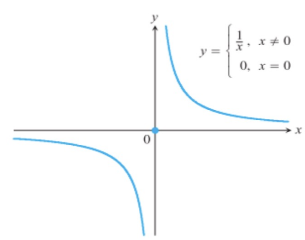
1.3 Existence of limits

Condition for limit to exist

The limit for a function $f(x)$ only exists if and only if:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

For this example, when $x \to 0^-, y \to -\infty$.
Similarly, as $x \to 0^+, y \to +\infty$.
Hence, we can conclude that the limit for this function as $x \to 0$ doesn't exist.



1.4 Continuity

Continuity

A function $f(x)$ is *continuous* at $x = a$ if:

$$\lim_{x \to a} f\left(x\right) = f\left(a\right)$$

Intermediate value theorem

If a function f is continuous on $[a, b]$, there is a number c in $[a, b]$ where $f(c)$ in $[f(a), f(b)]$.

1.5 Computing limits

Using the limit laws

For functions f, g and using $\lim_{x\to a} = \bigcirc$ for simpler notation:

1. $lim_{x\to a}c = c$
2. $\bigcirc(f \pm g) = \bigcirc f \pm \bigcirc g$
3. $\bigcirc(k \cdot f) = k \cdot \bigcirc L$
4. $\bigcirc(f^n) = (\bigcirc f)^n$
5. $\bigcirc(fg) = \bigcirc f \bigcirc g$
6. $\bigcirc(\frac{f}{g}) = \frac{\bigcirc f}{\bigcirc g}$, given that $\bigcirc g \neq 0$. *This strict condition prevents indeterminate forms.*

We can use these laws to break a limit into separate limits, and compute that way. Also note that:

7. $\bigcirc f(g) = f(\bigcirc g)$, given that f is **continuous** at $\bigcirc g$

Limit of a polynomial

For the limit of a polynomial $p(x)$:

$$lim_{x\to a}p(x) = p(a)$$

This can be proven easily with the limit laws above.

Techniques to compute limits

To solve for limits, we have to get the expression to the right form - a polynomial, for us to substitute our limit value into the function.
To do this, often we have to **factorize** or **rationalize**.

Example 1.1 *Indeterminate forms by substitution*

This applies limit law #5. As substituting into the function directly gives $0/0$*, we have to change it into a form such that we could apply the limit laws directly.*

$$\begin{aligned} lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} &= \frac{(x-2)(x+6)}{(x-2)x} \\ &= \frac{x+6}{x} \\ \textit{Substituting 2 gives} &= 4 \end{aligned}$$

The squeeze / sandwich theorem

Suppose $f(x) \leq g(x) \leq h(x)$ in the range $[a, b]$, for c in $[a, b]$:

$$\lim_{x \to c} f \leq \lim_{x \to c} g \leq \lim_{x \to c} h$$

We will make use of the fact that the limits can be equal to solve for the limit of $g(x)$.

Example 1.2 *Squeezing a function*

When we can't seem to factorize a function, we can try squeezing it between two other functions.

$$\lim_{x \to 0} x^2 \cos \frac{1}{x}$$

We know the limits of the function $\cos \frac{1}{x}$*, so we can start from there.*

$$\textit{Given that } x \neq 0, -1 \leq \cos \frac{1}{x} \leq 1$$

$$\textit{Multiplying } x^2 \textit{ on both sides}, -x^2 \leq \cos x^2 \frac{1}{x} \leq x^2$$

$$\textit{As } \lim_{x \to 0} \pm x^2 = 0, \quad \textit{we can conclude that } \lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$$

1.6 Infinite limits

Determining infinite limits

If $f(x)$ gets (negatively) arbitrarily large when x approaches a, we can say:

$$\lim_{x \to a} f(x) = (-)\infty$$

After we know that the limit *may* be infinity, we then have to make sure that *the limit is the same from both sides*, so that the limit actually exists. We can do so by plugging numbers which are approaching the limit from both sides.

Example 1.3 *Infinite limit exists*

$$\lim_{x \to 0} \frac{6}{x^2}$$

$$\textit{Consider both } \lim_{x \to 0^-} \frac{6}{x^2}, \lim_{x \to 0^+} \frac{6}{x^2} :$$

$$\lim_{x \to 0} \frac{6}{x^2} = \infty$$

Example 1.4 *Infinite limit doesn't exist*

$$\lim_{x \to 4} \frac{3}{(4 - x)^3}$$

Checking both sides, we can conclude that the limit doesn't exist, as:

$$\lim_{x \to 4^+} \frac{3}{(4 - x)^3} = -\infty, \quad \lim_{x \to 4^-} \frac{3}{(4 - x)^3} = \infty$$

1.7 Limits at infinity

Infinity operations

Note the following operations:

1. $\infty + k = \infty$
2. For $k < 0,\ k\infty = -\infty$

Determining limits of infinity

It is not hard to see that, for rational numbers n:

$$\lim_{x \to \pm\infty} \frac{k}{x^n} = 0$$

The easiest way to determine the limit would be to *factorize* the function so that we can use the facts above.

Determining limits of infinity of polynomials

Using the above fact, we can produce the fact, for a polynomial $p(x)$ with degree n and largest coefficient a_n:

$$\lim_{x \to \pm\infty} p(x) = a_n x^n$$

Which means we can *only consider the largest term in a polynomial* for limits of infinity.

Example 1.5 *Indeterminate forms by substitution of infinity*

Substituting ∞ into the function gives $\infty - \infty - \infty$, which is indeterminate. Hence, we must factorize it.

$$\begin{aligned} \lim_{x \to \infty} 2x^4 - x^2 - 8x &= \lim_{x \to \infty} [x^4(2 - \frac{1}{x^2} - \frac{8}{x^3})] \\ &= \infty \times 2 \\ &= \infty \end{aligned}$$

Or we can just simply use the theorem above and consider $\lim_{x \to \infty} 2x^4$ only to give ∞.

Example 1.6 *Factor polynomials limit to infinity*

We can simply consider the largest terms on each side and give the final answer easily.

$$\begin{aligned} \lim_{x \to -\infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} &= \lim_{x \to \infty} \frac{\sqrt{3x^2}}{-2x} \\ &= \lim_{x \to \infty} \frac{\sqrt{3}|x|}{-2x} \\ &= \frac{-\infty\sqrt{3}}{-2 \times \infty} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

Note that, as we are considering the negative limit of infinity, we need to add - to the abs sign on line 3.

2 Derivatives

First principle

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

2.1 Differentiation formulas and rules

Basic formulas

- We can differentiate individual items: $(f \pm g)' = f' \pm g'$
- We can factor out a multiplicative constant: $(cf)' = cf'$
- Derivative of a constant is 0: $\frac{d}{dx}k = 0$
- Power rule: $\frac{d}{dx}x^n = nx^{n-1}$

Chain rule

Shorthand: **d**1x2 + **d**2x1

$$(u(v))' = u'(v)v' \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Product rule

Shorthand: **d** from outside to inside

$$(uv)' = uv' + vu' \qquad \text{or} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient rule

Shorthand: move lower **d** upper **-** **d** lower x upper, lower square

$$(\frac{u}{v})' = \frac{vu' - uv'}{v^2} \qquad \text{or} \qquad \frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

	$f(x)$	$f'(x)$
1.	a^x	$\ln a \cdot a^x$
2.	e^{kx}	ke^{kx}
3.	$\ln kx$	x^{-1}
4.	$\sin kx$	$k\cos kx$
5.	$\cos kx$	$-k\sin kx$
6.	$\|x\|$	$\frac{\|x\|}{x}$
7.	$\tan kx$	$k\sec^2 kx$
8.	$\csc x$	$-\csc x\cot x$
9.	$\sec x$	$\sec x\tan x$
10.	$\cot x$	$-\csc^2 x$

Implicit differentiation

Differentiate all xy, add $\frac{dy}{dx}$ behind all differentiations of y.