MATH1853
Assignment 1
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Document prepared with LaTeX by myself. Solutions are my own.

Question 1

Given $\lim_{x\to 0}(f+g) = 2$ and $lim_{x\to 0}(2f-g) = -5$. We can find $lim_{x\to 0}(fg)$ by:

$$\begin{aligned}
\lim_{x\to 0}(f+g) &= 2 \\
\lim_{x\to 0}(2f-g) &= -5 \\
\lim_{x\to 0}(f+g) + \lim_{x\to 0}(2f-g) &= 2 + (-5) \\
\lim_{x\to 0}(3f) &= -3 \\
\lim_{x\to 0}(f) &= -1 \\
\therefore \lim_{x\to 0}(g) &= 2 - \lim_{x\to 0}(f) \\
&= 3 \\
\lim_{x\to 0}(fg) &= \lim_{x\to 0}(f) \cdot \lim_{x\to 0}(g) \\
&= (-1)\cdot(3) \\
&= -3
\end{aligned}$$

Question 2

Question 2a

$$\begin{aligned}
\lim_{t\to 2}\left(\frac{2^{2t}+2^t-20}{2^t-4}\right) &= \lim_{t\to 2}\frac{(2^t-4)(2^t+5)}{2^t-4} \\
&= \lim_{t\to 2}(2^t+5) \\
&= 2^2+5 \\
&= 9
\end{aligned}$$

Question 2b

$$\begin{aligned}
\lim_{x\to 0}(\tan(\frac{\pi}{4}\cos(\sin x^{\frac{1}{3}}))) &= \tan(\frac{\pi}{4}\cos(\sin 0^{\frac{1}{3}})) \\
&= \tan(\frac{\pi}{4}\cos(0)) \\
&= 1
\end{aligned}$$

Question 3

To prove $\lim_{x\to 0^+}(\sqrt{x}\cdot(1+\sin^2\frac{2\pi}{x}))=0$:

$$\begin{aligned}
-1 \le \sin(\frac{1}{x}) &\le 1 \\
0 \le \sin^2(\frac{1}{x}) &\le 1 \\
0 \le \sin^2(\frac{2\pi}{x}) &\le 1 \\
1 \le (1+\sin^2(\frac{2\pi}{x})) &\le 2 \\
\sqrt{x} \le \sqrt{x}\cdot(1+\sin^2(\frac{2\pi}{x})) &\le 2\sqrt{x} \\
\lim_{x\to 0^+}\sqrt{x} \le \lim_{x\to 0^+}\sqrt{x}\cdot(1+\sin^2(\frac{2\pi}{x})) &\le \lim_{x\to 0^+}2\sqrt{x} \\
0 \le \lim_{x\to 0^+}\sqrt{x}\cdot(1+\sin^2(\frac{2\pi}{x})) &\le 0 \\
\therefore \lim_{x\to 0^+}\sqrt{x}\cdot(1+\sin^2(\frac{2\pi}{x})) &= 0
\end{aligned}$$

Question 4

Question 4a

We only need to consider the highest degree terms in the numerator and denominator, as x grows the highest degree terms would have the most effect on the limit:

$$\begin{aligned}
\lim_{x\to -\infty}(\frac{4x-3}{\sqrt{25x^2+4x}}) &= \lim_{x\to -\infty}(\frac{4x}{\sqrt{25x^2}}) \\
&= \lim_{x\to -\infty}(\frac{4x}{5|x|}) \\
&= (\frac{4\infty}{-5\infty}) \\
&= -\frac{4}{5}
\end{aligned}$$

Question 4b

Note: The limit of $\lim_{x\to\infty}(\frac{k}{x^n})$ for a constant k must be 0. For this question, we must rationalize the numerator, because we can't combine the x^2 inside and outside the root:

$$
\begin{aligned}
\lim_{x\to\infty}(\sqrt{4x^4+9x}-2x^2) &= \lim_{x\to\infty}(\sqrt{4x^4+9x}-2x^2)\cdot\frac{\sqrt{4x^4+9x}+2x^2}{\sqrt{4x^4+9x}+2x^2} \\
&= \lim_{x\to\infty}(\frac{4x^4+9x-4x^4}{\sqrt{4x^4+9x}+2x^2}) \\
&= \lim_{x\to\infty}(\frac{9x}{\sqrt{4x^4+9x}+2x^2}) \\
&= \lim_{x\to\infty}(\frac{\frac{9}{x}}{\sqrt{\frac{4x^4+9x}{x^4}}+2}) \\
&= \frac{\lim_{x\to\infty}\frac{9}{x}}{\lim_{x\to\infty}\sqrt{\frac{4x^4+9x}{x^4}}+2} \\
&= \frac{0}{\sqrt{4}+2} \\
&= 0
\end{aligned}
$$

Question 5

To find the discontinuous points of $f(x) = \begin{cases} -x+5, & \text{if } x \leq -1 \\ \sin(x^2-1), & \text{if } -1 \leq x \leq 1 \\ \sqrt{x}, & \text{if } x > 1 \end{cases}$:

A function $f(x)$ is *discontinuous* at $x=a$ if: $\lim_{x\to a} f(x) \neq f(a)$, or such that the limit does not exist.

$$
\begin{aligned}
\lim_{x\to -1^-}(-x+5) &= 6 \\
\lim_{x\to -1^+}(\sin(x^2-1)) &= \sin(0) = 0 \\
\because \lim_{x\to -1^-}(-x+5) &\neq \lim_{x\to -1^+}(\sin(x^2-1))
\end{aligned}
$$

$\therefore$ Function has no limit at $x=-1$

$$
\begin{aligned}
\lim_{x\to 1^-}(\sin(x^2-1)) &= \sin(0) = 0 \\
\lim_{x\to 1^+}(\sqrt{x}) &= \sqrt{1} = 1 \\
\because \lim_{x\to 1^-}(\sin(x^2-1)) &\neq \lim_{x\to 1^+}(\sqrt{x})
\end{aligned}
$$

$\therefore$ Function has no limit at $x=1$

$\because$ We know the function is continous on the left and right by their functions in the domain.

$\therefore$ Function is discontinuous at $x=-1$ and $x=1$.

Question 6

Using the IVT to show that $\cos x = x$ has a solution in the interval $[0, 1]$, we have to consider the roots of $f(x) = \cos x - x$:

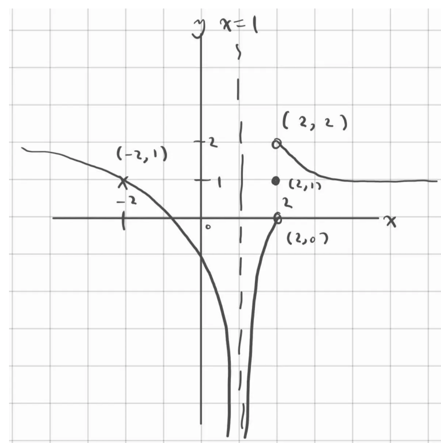
$$\begin{aligned}
f(x) &= \cos x - x \\
f(0) &= \cos 0 - 0 \\
&= 1 \\
f(1) &= \cos 1 - 1 \\
&= 0.5403 - 1 \\
&= -0.4597 \\
&\because 0 \text{ lies in the interval } [f(0), f(1)] \\
&\therefore \text{By the IVT, there is a root in } [0, 1]\text{, so there is a solution to } \cos x = x \text{ in } [0, 1].
\end{aligned}$$

Question 7

Checking out the conditions of the function $f(x)$:

1. $\lim_{x\to -2} f(x) = 1$
2. $\lim_{x\to 1} f(x) = -\infty$
3. $\lim_{x\to 2^-} f(x) = 0$
4. $\lim_{x\to 2^+} f(x) = 2$
5. $f(2) = 1$
6. $\lim_{x\to \infty} f(x) = 1$

Now we sketch the graph, labelling the axis, intercepts and asymptotes:



Question 8

Let $P(n)$ denote the perimeter of a regular n-gon inscribed in a unit circle.

Question 8a

We know that the perimeter of a unit circle is 2π. As the number of sides of the polygon increases, it gets closer and closer to being a circle as it's surface smooths out. Hence, we can say that $P(n) \to 2\pi$ as $n \to \infty$.

Question 8b

We can find the unknown side of a triangle given the two sides and the angle between them using the cosine rule: $c = sqrta^2 + b^2 - 2ab\cos C$.

Consider the n triangles inside a regular n-gon, with two of their sides from the center of the circle to the corners of the polygon. We then can make the following conclusions:

1. $a, b = 1$ as we are dealing with a unit circle.

2. $C = \frac{2\pi}{n}$, as there are n triangles.

We then can write:

$$
\begin{aligned}
P(n) &= n\sqrt{2 - 2\cos(\frac{2\pi}{n})} \\
&= n\sqrt{2 - 2(1 - 2\sin^2(\frac{\pi}{n}))} \\
&= n\sqrt{4\sin^2(\frac{\pi}{n})} \\
&= 2n\sin(\frac{\pi}{n})
\end{aligned}
$$

Question 8c

Question 8a shows us that $\lim_{x\to\infty} P(n) = 2\pi$. Therefore:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n}{\pi}\sin(\frac{\pi}{n}) &= \lim_{n\to\infty} \frac{2n\sin(\frac{\pi}{n})}{2\pi} \\
&= \lim_{n\to\infty} \frac{P(n)}{2\pi} \\
&= \frac{\lim_{n\to\infty} P(n)}{\lim_{n\to\infty} 2\pi} \\
&= \frac{\lim_{n\to\infty} 2\pi}{\lim_{n\to\infty} 2\pi} \\
&= 1
\end{aligned}
$$

Question 8d

Let $\theta = \frac{\pi}{n}$. The limit $n \to \infty$ is equivalent to $\theta \to 0$, as $\frac{\pi}{n} \to 0$ as $n \to \infty$. Therefore:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n}{\pi}\sin(\frac{\pi}{n}) &= \lim_{\theta\to 0} \theta^{-1}\sin(\theta) \\
&= \lim_{\theta\to 0} \frac{\sin\theta}{\theta}
\end{aligned}
$$

Therefore, we can say that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \lim_{n\to\infty} \frac{n}{\pi}\sin(\frac{\pi}{n}) = 1$.