# MATH1851

## Notes for HKU · Spring 2024

**Author:** Jax

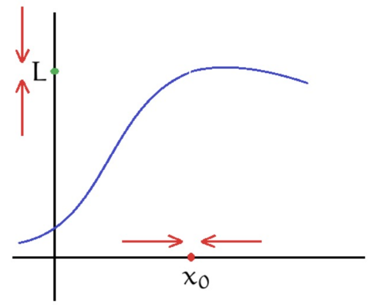
**Contact:** enhanjax@connect.hku.hk

MORE notes on my website!

Credit: Heavily inspired by Paul's Calculus I Notes.

# 1 Limits and Continuity

## 1.1 Introduction to the concept of limit




We can conceptualize that the limit of a function $f(x)$ is $L$ as $x$ approaches $c$, given that we can make $f(x)$ *as close to $L$ as we want* for all $x$ sufficiently close to $a$, from both sides, *without actually letting $x$ be $a$*. We can write this as:

$$\lim_{x \to a} f(x) = L$$

## 1.2 One-sided limits

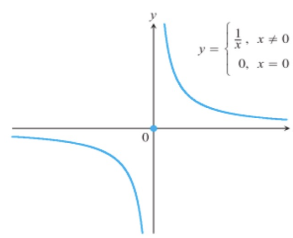
There are two sides that $x$ can tend to a number. We can write it as $x \to n^-$ and $x \to n^+$, which represents from the negative (left) / positive (right) side.

## 1.3 Existence of limits

**Condition for limit to exist**

The limit for a function $f(x)$ only exists if and only if:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$




For this example, when $x \to 0^-, y \to -\infty$.
Similarly, as $x \to 0^+, y \to +\infty$.
Hence, we can conclude that the limit for this function as $x \to 0$ doesn't exist.

## 1.4 Continuity

**Continuity**

A function $f(x)$ is *continuous* at $x = a$ if:

$$\lim_{x \to a} f\left(x\right) = f\left(a\right)$$

**Intermediate value theorem**

If a function $f$ is continuous on $[a,b]$, there is a number $c$ in $[a,b]$ where $f(c)$ in $[f(a), f(b)]$.

## 1.5 Computing limits

**Indeterminate forms**

Indeterminate forms are forms that cannot be solved by simply substituting the value of $x$ into the function. They are:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 1^\infty, \quad \infty^0$$

**Using the limit laws**

For functions $f, g$ and using $\lim_{x \to a} = \bigcirc$ for simpler notation:

1. $lim_{x\to a} c = c$
2. $\bigcirc(f \pm g) = \bigcirc f \pm \bigcirc g$
3. $\bigcirc(k \cdot f) = k \cdot \bigcirc L$
4. $\bigcirc(f^n) = (\bigcirc f)^n$
5. $\bigcirc(fg) = \bigcirc f \bigcirc g$
6. $\bigcirc(\frac{f}{g}) = \frac{\bigcirc f}{\bigcirc g}$, given that $\bigcirc g \neq 0$. *This strict condition prevents indeterminate forms.*

We can use these laws to break a limit into separate limits, and compute that way. Also note that:

7. $\bigcirc f(g) = f(\bigcirc g)$, given that $f$ is **continuous** at $\bigcirc g$

**Limit of a polynomial**

For the limit of a polynomial $p(x)$:

$$lim_{x\to a} p(x) = p(a)$$

This can be proven easily with the limit laws above.

**Techniques to compute limits**

To solve for limits, we have to get the expression to the right form - a polynomial, for us to substitute our limit value into the function.

To do this, often we have to **factorize** or **rationalize**.

**Example 1.1** *Indeterminate forms by substitution*

*This applies limit law #5. As substituting into the function directly gives 0/0, we have to change it into a form such that we could apply the limit laws directly.*

$$\begin{aligned} lim_{x\to 2} \frac{x^2+4x-12}{x^2-2x} &= \frac{(x-2)(x+6)}{(x-2)x} \\ &= \frac{x+6}{x} \\ \textit{Substituting 2 gives} &= 4 \end{aligned}$$

**The squeeze / sandwich theorem**

Suppose $f(x) \leq g(x) \leq h(x)$ in the range $[a, b]$, for $c$ in $[a, b]$:

$$\lim_{x \to c} f \leq \lim_{x \to c} g \leq \lim_{x \to c} h$$

We will make use of the fact that the limits can be equal to solve for the limit of $g(x)$.

**Example 1.2** *Squeezing a function*

*When we can't seem to factorize a function, we can try squeezing it between two other functions.*

$$\lim_{x \to 0} x^2 \cos \frac{1}{x}$$

*We know the limits of the function* $\cos \frac{1}{x}$, *so we can start from there.*

$$\textit{Given that } x \neq 0, -1 \leq \cos \frac{1}{x} \leq 1$$

$$\textit{Multiplying } x^2 \textit{ on both sides}, -x^2 \leq \cos x^2 \frac{1}{x} \leq x^2$$

$$\textit{As } \lim_{x \to 0} \pm x^2 = 0, \quad \textit{we can conclude that } \lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$$

## 1.6 Infinite limits

**Determining infinite limits**

If $f(x)$ gets (negatively) arbitrarily large when $x$ approaches $a$, we can say:

$$\lim_{x \to a} f(x) = (-)\infty$$

After we know that the limit *may* be infinity, we then have to make sure that *the limit is the same from both sides*, so that the limit actually exists. We can do so by plugging numbers which are approaching the limit from both sides.

**Example 1.3** *Infinite limit exists*

$$\lim_{x \to 0} \frac{6}{x^2}$$

$$\textit{Consider both } \lim_{x \to 0^-} \frac{6}{x^2}, \lim_{x \to 0^+} \frac{6}{x^2} :$$

$$\lim_{x \to 0} \frac{6}{x^2} = \infty$$

**Example 1.4** *Infinite limit doesn't exist*

$$\lim_{x \to 4} \frac{3}{(4-x)^3}$$

*Checking both sides, we can conclude that the limit doesn't exist, as:*

$$\lim_{x \to 4^+} \frac{3}{(4-x)^3} = -\infty, \quad \lim_{x \to 4^-} \frac{3}{(4-x)^3} = \infty$$

## 1.7 Limits at infinity

**Infinity operations**

Note the following operations:

1. $\infty + k = \infty$
2. For $k < 0,\ k\infty = -\infty$

**Determining limits of infinity**

It is not hard to see that, for rational numbers $n$:

$$\lim_{x \to \pm\infty} \frac{k}{x^n} = 0$$

The easiest way to determine the limit would be to *factorize* the function so that we can use the facts above.

**Determining limits of infinity of polynomials**

Using the above fact, we can produce the fact, for a polynomial $p(x)$ with degree $n$ and largest coefficient $a_n$:

$$\lim_{x \to \pm\infty} p(x) = a_n x^n$$

Which means we can *only consider the largest term in a polynomial* for limits of infinity.

**Example 1.5** *Indeterminate forms by substitution of infinity*

*Substituting* $\infty$ *into the function gives* $\infty - \infty - \infty$*, which is indeterminate. Hence, we must factorize it.*

$$\begin{aligned} \lim_{x \to \infty} 2x^4 - x^2 - 8x &= \lim_{x \to \infty} [x^4(2 - \frac{1}{x^2} - \frac{8}{x^3})] \\ &= \infty \times 2 \\ &= \infty \end{aligned}$$

*Or we can just simply use the theorem above and consider* $\lim_{x \to \infty} 2x^4$ *only to give* $\infty$*.*

**Example 1.6** *Factor polynomials limit to infinity*

*We can simply consider the largest terms on each side and give the final answer easily.*

$$\begin{aligned} \lim_{x \to -\infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} &= \lim_{x \to \infty} \frac{\sqrt{3x^2}}{-2x} \\ &= \lim_{x \to \infty} \frac{\sqrt{3}|x|}{-2x} \\ &= \frac{-\infty\sqrt{3}}{-2 \times \infty} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

*Note that, as we are considering the negative limit of infinity, we need to add - to the abs sign on line 3.*

## 2 Derivatives

**First principle**

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

### 2.1 Differentiation formulas and rules

**Basic formulas**

- We can differentiate individual items: $(f \pm g)' = f' \pm g'$
- We can factor out a multiplicative constant: $(cf)' = cf'$
- Derivative of a constant is 0: $\frac{d}{dx}k = 0$
- Power rule: $\frac{d}{dx}x^n = nx^{n-1}$

**Chain rule**

Shorthand: **d**1x2 + **d**2x1

$$(u(v))' = u'(v)v' \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

**Product rule**

Shorthand: **d** from outside to inside

$$(uv)' = uv' + vu' \qquad \text{or} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

**Quotient rule**

Shorthand: move lower **d** upper **-** **d** lower x upper, lower square

$$(\frac{u}{v})' = \frac{vu' - uv'}{v^2} \qquad \text{or} \qquad \frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

| | $f(x)$ | $f'(x)$ |
|---|---|---|
| 1. | $a^x$ | $\ln a \cdot a^x$ |
| 2. | $e^{kx}$ | $ke^{kx}$ |
| 3. | $\ln kx$ | $x^{-1}$ |
| 4. | $\sin kx$ | $k\cos kx$ |
| 5. | $\cos kx$ | $-k\sin kx$ |
| 6. | $\lvert x\rvert$ | $\frac{\lvert x\rvert}{x}$ |
| 7. | $\tan kx$ | $k\sec^2 kx$ |
| 8. | $\csc x$ | $-\csc x \cot x$ |
| 9. | $\sec x$ | $\sec x \tan x$ |
| 10. | $\cot x$ | $-\csc^2 x$ |

**Implicit differentiation**

Differentiate all $xy$, add $\frac{dy}{dx}$ behind all differentiations of $y$.

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To find $\frac{dy}{dx}$ for $y^2 = x^2 + \sin(xy)$:

$$
\begin{aligned}
y^2 &= x^2 + \sin(xy) \\
2y\frac{dy}{dx} &= \frac{d}{dx}(\sin(xy)) \\
2y\frac{dy}{dx} &= 2x + \cos(xy)(x\frac{dy}{dx} + y) \\
\text{Then we simply collect terms of } \frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{2x + y\cos(xy)}{2y - x\cos(xy)}
\end{aligned}
$$

## 2.2 Extremum points

**Critical points**

A critical point is a point where $f'(x) = 0$ or $f'(x)$ is undefined, or the end-points of the domain.

**Absolute max/minimum points**

The absolute maximum/minimum points are the points where the function has the largest/smallest value in the entire domain. It can be written as:

$$\max_{x \in D} f(x) \qquad \text{or} \qquad \min_{x \in D} f(x)$$

**Local max/minimum points**

The local maximum/minimum points are the points where the function has the largest/smallest value in a small interval around the point.

**Extreme value theorem**

For $f(x)$ that is *continuous* in $[a, b]$, there must be both a abs maximum and minimum point.

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To find the absolute extremas of $f(x)$ in $[a, b]$, we can:

1. Verify that $f(x)$ is continuous in $[a, b]$
2. Find all critical points in $[a, b]$
3. Evaluate the critical points as well as the end points of the interval

**Rolle's theorem**

For $f(x)$ that is *continuous* in $[a, b]$ and *differentiable* in $(a, b)$, if $f(a) = f(b)$, there must be a critical point $c$ in the interval.

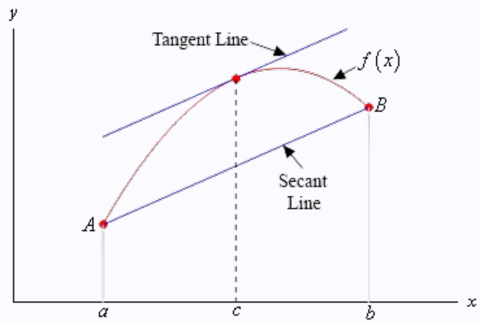
## 2.3 Other cool theorems

**Mean value theorem**

For $f(x)$ that is *continuous* in $[a, b]$ and *differentiable* in $(a, b)$, there must be a point $c$ in $(a, b)$ where:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The theorem tells us that, in describe conditions, there must be a point $c$ where the slope of the *tangent line* is equal to the slope of the line from $a \to b$ (secant line).




Source: https://tutorial.math.lamar.edu/Classes/CalcI/MeanValueTheorem.aspx

**L'Hopital's rule**

For any $a \in [\mathbb{R}, \pm\infty]$, if $\lim_{x \to a}(\frac{f}{g})$ is in indeterminate form after substitution, we can conclude:

$$\lim_{x \to a}(\frac{f}{g}) = \lim_{x \to a}(\frac{f'}{g'})$$

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To find $\lim_{x \to -\infty} xe^x$, we first check if the limit is indeterminate, then we can apply the rule:

$$\begin{aligned}
\lim_{x \to -\infty} xe^x &\implies \infty \times 0 \\
\lim_{x \to -\infty} xe^x &= \lim_{x \to -\infty} \frac{x}{e^{-x}} \\
&= \lim_{x \to -\infty} \frac{1}{-e^{-x}} \text{ (rule applied)} \\
&= 0
\end{aligned}$$

## 2.4 Shape of graph

| graph feature | | f(x) | f'(x) | f''(x) | Notes |
|---|---|---|---|---|---|
| rising (L to R) | | slope > 0 | + | | |
| falling (L to R) | | slope < 0 | - | | |
| extrema | maximum | slope = 0 | = 0 + on L - on R | - at $x_{max}$ | derivative may not exist at a max or min, e.g. or |
| extrema | minimum | slope = 0 | = 0 - on L + on R | + at $x_{min}$ | |
| inflection pt. | | Curvature changes: ↔ | | = 0 potential inflection point | Check f''(x) on either side of a potential inflection point. |
| concave up | | | - + | + | |
| concave down | | | + - | - | |

Source: *https://xaktly.com/CurveSketchSecondDeriv.html*

# 3 Integrals

## 3.1 Area and Estimating with Finite Sums

## 3.2 Definite Integrals

## 3.3 The Fundamental Theorem of Calculus

## 3.4 theorem of Natural Logarithms

## 3.5 Interlude - Hyperbolic Functions

# 4 Parametric Equations

# 5 Polar Coordinates

# 6 Ordinary Differential Equations

## 6.1 1st Order Linear ODEs & Integrating Factors

## 6.2 Bernoulli Equations & Riccati Equations