LOFO 2020

Adrien Pommellet

November 9, 2020

1A. Prove that $(\neg P \vee Q) \Leftrightarrow (P \Rightarrow Q)$ is a tautology.

1B. Prove that $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$ is a tautology.

2A. Is the following Hilbert system complete with regards to $\mathcal{F}_{\{\perp,\neg,\vee,\Rightarrow\}}$?

$$\frac{A \Rightarrow B \qquad A}{B}\,[\textit{Modus Ponens}]$$

$$\frac{}{\perp \Rightarrow A}\,[\perp] \qquad \frac{}{A \Rightarrow B \Rightarrow A}\,[\Rightarrow_1] \qquad \frac{}{(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C}\,[\Rightarrow_2]$$

$$\frac{}{(A \Rightarrow \perp) \Rightarrow \neg A}\,[\neg_1] \qquad \frac{}{A \Rightarrow \neg A \Rightarrow \perp}\,[\neg_2]$$

2B. Is the following Hilbert system complete with regards to $\mathcal{F}_{\{\perp,\wedge,\vee,\Rightarrow\}}$?

$$\frac{A \Rightarrow B \qquad A}{B}\,[\textit{Modus Ponens}]$$

$$\frac{}{A \Rightarrow A \vee B}\,[\vee_1] \qquad \frac{}{B \Rightarrow A \vee B}\,[\vee_2] \qquad \frac{}{A \vee B \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C}\,[\vee_3]$$

$$\frac{}{A \wedge B \Rightarrow A}\,[\wedge_1] \qquad \frac{}{A \wedge B \Rightarrow B}\,[\wedge_2] \qquad \frac{}{A \Rightarrow B \Rightarrow A \wedge B}\,[\wedge_3]$$

$$\frac{}{A \Rightarrow B \Rightarrow A}\,[\Rightarrow_1] \qquad \frac{}{(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C}\,[\Rightarrow_2]$$

3A. Prove that $\{Q \wedge P, R\} \vdash_{\mathcal{N}} P \wedge (R \wedge Q)$. This can be done using a proof of depth 4.

3B. Prove that $\{P \vee Q\} \vdash_{\mathcal{N}} P \vee (Q \vee R)$. This can be done using a proof of depth 4.

4A. Prove that $\{\neg P \Rightarrow Q\} \vdash_{\mathcal{N}} P \vee Q$ by filling the blanks of the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P \Rightarrow Q \qquad \dfrac{\dfrac{\dfrac{\overline{P}}{\qquad}\,[\quad] \qquad \overline{\neg(P \vee Q)}}{\qquad}\,[\quad]}{\qquad}\,[\neg_I]}{\qquad}\,[\quad]}{\qquad}\,[\quad] \qquad \overline{\neg(P \vee Q)}}{\bot}\,[\quad]}{\qquad}\,[\neg_I]}{\qquad}\,[\quad]$$

4B. Prove that $\vdash_{\mathcal{N}} ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ by filling the blanks of the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{(P \Rightarrow Q) \Rightarrow P} \qquad \dfrac{\dfrac{\dfrac{\overline{P} \qquad \overline{\neg P}}{\qquad}\,[\neg_E]}{\qquad}\,[\bot_E]}{\qquad}\,[\quad]}{\qquad}\,[\quad] \qquad \overline{\neg P}}{\bot}\,[\quad]}{\qquad}\,[\neg_I]}{\qquad}\,[\quad]$$

$$\dfrac{\quad}{\qquad}\,[\quad]$$

5A. Define a term Square $\in \Lambda$ such that for any natural integer n:

$$\text{Square } \underline{n} \rightarrow_\beta^* \underline{n^2}$$

Then guess its type.

5B. Define a term Double $\in \Lambda$ such that for any natural integer n:

$$\text{Double } \underline{n} \rightarrow_\beta^* \underline{2 \times n}$$

Then guess its type.

6. Prove that $\Theta = (\lambda xy \cdot y(xxy))(\lambda uv \cdot v(uuv))$ is a fixed-point combinator.

7. Prove that $\vdash KI : \tau \rightarrow \sigma \rightarrow \sigma$.

8A. Prove that $\vdash_{\mathcal{NI}} (P \wedge Q) \Rightarrow (Q \wedge P)$.
Then find a term in Λ_{ext} of type $\sigma \times \tau \rightarrow \tau \times \sigma$.

8B. Prove that $\vdash_{\mathcal{NI}} (P \vee Q) \Rightarrow (Q \vee P)$.
Then find a term in Λ_{ext} of type $\sigma \cup \tau \rightarrow \tau \cup \sigma$.