Correction of the midterm exam S2 2023

Exercise 1: polynomials

Consider the polynomial $P(X) = X^6 - X^5 - 3X^4 + 7X^3 + 14X^2 + 6X$.

1. *Show that -1 is a root of P and find its (exact) order of multiplicity.*

   - $P(-1) = 1 + 1 - 3 - 7 + 14 - 6 = 0$. -1 is a root of P.
   - $P'(X) = 6X^5 - 5X^4 - 12X^3 + 21X^2 + 28X + 6$. Hence $P'(-1) = -6 - 5 + 12 + 21 - 28 + 6 = 0$
   - $P''(X) = 30X^4 - 20X^3 - 36X^2 + 42X + 28$. Hence, $P''(-1) = 30 + 20 - 36 - 42 + 28 = 0$.
   - $P'''(X) = 120X^3 - 60X^2 - 72X + 42$. Hence, $P'''(-1) = -66 \neq 0$.

   Finally, -1 is a root of P of multiplicity 3.

2. *What can we deduce in terms of divisibility?*

   We can deduce that $(X+1)^3 \mid P$ and that $(X+1)^4 \nmid P$.

3. *Using only one euclidean division, factorize P as a product of irreducible polynomials in $\mathbb{R}[X]$.*

   Remind that the irreducible polynomials in $\mathbb{R}[X]$ are only those of degree 1 and of degree 2 with a negative discriminant.

   Let us do the euclidean division of P by $(X+1)^3 = X^3 + 3X^2 + 3X + 1$. We find a zero reminder and the quotient $Q = X^3 - 4X^2 + 6X$.
   Thus, $P = (X+1)^3\left(X^3 - 4X^2 + 6X\right) = (X+1)^3 X\left(X^2 - 4X + 6\right)$. The polynomial $X^2 - 4X + 6$ has a discriminant equal to $-8 < 0$. It is hence irreducible in $\mathbb{R}[X]$. Finally, P can be written as this product of irreducible polynomials in $\mathbb{R}[X]$: $(X+1)^3 X\left(X^2 - 4X + 6\right) = (X+1)(X+1)(X+1)X(X^2 - 4X + 6)$.

Exercise 2: differential equations

The questions of the exercise are independent.

1. *Consider the differential equation $(E_1): \ (x+1)y' - 2y = (x+1)^3\cos(3x)$ on $I =]-1, +\infty[$.*

   (a) *Solve (E_1) on I.*

   - The solutions of the homogeneous equation are the functions of the form

   $$y_0(x) = ke^{-\int \frac{-2}{x+1}\mathrm{d}x} = ke^{2\ln(x+1)} = k(x+1)^2 \text{ with } k \in \mathbb{R}$$

   - Let us find a particular solution of (E_1) as a function of the form $y_p(x) = k(x)(x+1)^2$ (variation of the parameter).

   Then $y_p'(x) = k'(x)(x+1)^2 + 2k(x)(x+1)$. This leads to:

   $$\begin{aligned} y_p(x) \text{ solution of } (E_1) &\iff (x+1)y_p'(x) - 2y_p(x) = (x+1)^3\cos(3x) \\ &\iff (x+1)^3k'(x) + 2k(x)(x+1)^2 - 2k(x)(x+1)^2 = (x+1)^3\cos(3x) \end{aligned}$$

   Hence, $k'(x) = \cos(3x)$. Let us choose, for example $k(x) = \dfrac{1}{3}\sin(3x)$. $y_p(x) = \dfrac{(x+1)^2}{3}\sin(3x)$ is hence a particular solution of (E_1).

   Finally, $S = \left\{ \begin{array}{lcl} I & \longrightarrow & \mathbb{R} \\ x & \longmapsto & k(x+1)^2 + \frac{(x+1)^2}{3}\sin(3x) \end{array} ; k \in \mathbb{R} \right\}$

   (b) *Find the solutions of (E_1) satisfying to: $y(0) = 1$.*

   We have to choose the parameter k such that $y(0) = 1$. Then, $y(0) = k(0+1)^2 + \frac{(0+1)^2}{3}\sin(3 \times 0) = k$. Thus, $k = 1$ and there exists only one function y, solution de (E_1) such that $y(0) = 1$. It is the function $y : x \longmapsto (x+1)^2 + \dfrac{(x+1)^2}{3}\sin(3x)$.

2. *Consider the differential equation* $(E_2):\ y''+4y'+13y=(25x^2+16x+2)e^{2x}$ *on* $J=\mathbb{R}$.

   (a) *Show that* $y_p : x \longmapsto x^2e^{2x}$ *is a particular solution of* (E_2).

   We compute $y_p'(x)=\left(2x+2x^2\right)e^{2x}$ and $y_p''(x)=\left(2+4x+4x+4x^2\right)e^{2x}$. By injecting these expressions in (E_2), we get:

   $$y_p''(x)+4y_p'(x)+13y_p(x)=(2+8x+4x^2+8x+8x^2+13x^2)e^{2x}=(25x^2+16x+2)e^{2x}$$

   y_p is hence a solution of (E_2).

   (b) *Find the solution set of* (E_2).

   - The characteristic equation associated to (E_2) is $(C)\ r^2+4r+13=0$. Its discriminant is $\Delta=-36$. The roots of (C) are hence $r_1=\dfrac{-4+6i}{2}=-2+3i$ and $r_2=-2-3i$.
   - We can hence deduce that $S=\left\{\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & e^{-2x}\left(k_1\cos(3x)+k_2\sin(3x)\right)+(25x^2+16x+2)e^{2x}\end{array}\ ;(k_1,k_2)\in\mathbb{R}^2\right\}$

Exercise 3: local analysis of functions

1. *Let* f *and* g *be two real functions defined on* $\mathbb{R}$. *Let* $a\in\mathbb{R}\cup\{+\infty,-\infty\}$. *Write the mathematical definitions of:* $f(x)\sim g(x)$ *and* $f(x)=o\big(g(x)\big)$ *as* x *approaches* a.

   - $f(x)\sim g(x)\iff f(x)=g(x)(1+\varepsilon(x))$ with $\lim\limits_{x\to a}\varepsilon(x)=0$. We can also write: $f(x)\sim g(x)\iff \lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=1$.
   - $f(x)=o(g(x))\iff f(x)=g(x)\varepsilon(x)$ with $\lim\limits_{x\to a}\varepsilon(x)=0$. We can also write: $f(x)=o(g(x))\iff \lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=0$.

2. *Find simple equivalents (others than the function* f *itself) of* $f(x)=3x^3-2x^2+6x$ *in* $a=0$ *AND in* $a=+\infty$. *Justify your answers.*

   - Note that $\dfrac{3x^3-2x^2+6x}{6x}=\dfrac{x^2}{2}-\dfrac{x}{3}+1$. Hence, $\lim\limits_{x\to 0}\dfrac{f(x)}{6x}=1$. Finally, $f(x)\sim 6x$ as x approaches 0.
   - Note that $\dfrac{3x^3-2x^2+6x}{3x^3}=1-\dfrac{2}{3x^2}+\dfrac{2}{x^2}$. Hence, $\lim\limits_{x\to+\infty}\dfrac{f(x)}{3x^3}=1$. Finally, $f(x)\sim 3x^3$ as x approaches $+\infty$.

3. *Let* h *and* k *be two functions such that, as* x *approaches* 0,

   $$h(x)=1+2x+x^2-3x^3+o(x^3)\quad \textit{and}\quad k(x)=-x+3x^2+o(x^2)$$

   (a) *Find equivalents in* 0, *the simplest as possible, of:* $h(x)$ *(don't justify),* $k(x)$ *(don't justify) and* $xh(x)+k(x)$ *(justify).*

   We have $h(x)\sim 1$ in 0 and $k(x)\sim -x$ in 0. Furthermore, $xh(x)+k(x)=5x^2+o(x^2)$. Hence, $xh(x)+k(x)\sim 5x^2$ in 0.

   (b) *Do you have enough information to write the Taylor expansion of* $h(x)+k(x)$ *at the order* 1*? And at the order* 2*? And at the order* 3*? Write explicitly the Taylor expansion when your answer is positive.*

   We can write TEs at the orders 1 and 2, but we don't have enough information for a TE at the order 3 (because of k). The possible TEs are:

   $$h(x)+k(x)=1+x+o(x)\text{ (order 1) and } h(x)+k(x)=1+x+4x^2+o(x^2)\text{ (order 2)}$$

Exercise 4: Taylor expansions

At each question, write all the basic Taylor expansions that you will use before answering the question.

1. *Find the Taylor expansion in* 0, *at the order* 3, *of* $f(x)=\cos(x)e^{-2x}$.

   $$f(x)=\left(1-\frac{x^2}{2}+o(x^3)\right)\times\left(1-2x+\frac{(-2x)^2}{2}+\frac{(-2x)^3}{3!}+o(x^3)\right)$$

Thus,

$$f(x) = \left(1 - \frac{x^2}{2} + o(x^3)\right) \times \left(1 - 2x + 2x^2 - \frac{4x^3}{3} + o(x^3)\right) = 1 - 2x + 2x^2 - \frac{4x^3}{3} - \frac{x^2}{2} + x^3 + o(x^3)$$

Finally, $f(x) = 1 - 2x + \frac{3x^2}{2} - \frac{x^3}{3} + o(x^3)$.

2. *Find the Taylor expansion in* 0*, at the order* 2*, of* $g(x) = \sqrt{1+x}$*, using one of the five basic Taylor expansions.*

$$g(x) = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2 + o(x^2) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$$

3. *Find the Taylor expansion in* 0*, at the order* 2*, of* $k(x) = \ln\left(1 + \sqrt{1+x}\right)$.

Using previous question:

$$h(x) = \ln\left(2 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)\right) = \ln\left(2\left(1 + \frac{x}{4} - \frac{x^2}{16} + o(x^2)\right)\right) = \ln(2) + \ln\left(1 + \frac{x}{4} - \frac{x^2}{16} + o(x^2)\right)$$

Let $u(x) = \frac{x}{4} - \frac{x^2}{16} + o(x^2)$ (which tends to 0 as x approaches 0). Then $u^2(x) = \frac{x^2}{16} + o(x^2)$. Since, as u approaches 0, $\ln(1+u) = u - \frac{u^2}{2} + o(u^2)$, we get:

$$h(x) = \ln(2) + \frac{x}{4} - \frac{x^2}{16} - \frac{x^2}{32} + o(x^2) = \ln(2) + \frac{x}{4} - \frac{3x^2}{32} + o(x^2)$$

Exercise 5: finding limits

1. *Find* $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2}{\sin(\frac{x}{2})}$.

- Let $N(x) = e^x + e^{-x} - 2$.
Then, $N(x) = \left(1 + x + \frac{x^2}{2} + o(x^2)\right) + \left(1 - x + \frac{(-x)^2}{2} + o(x^2)\right) - 2 = x^2 + o(x^2)$. Thus, $N(x) \sim x^2$ in 0.
- Let $D(x) = \sin\left(\frac{x}{2}\right)$. Using TEs, we get $D(x) \sim \frac{x}{2}$.
- We deduce that $\frac{N(x)}{D(x)} \sim \frac{x^2}{\frac{x}{2}} = 2x$. Since $\lim\limits_{x\to 0} 2x = 0$, we get: $\lim\limits_{x\to 0} \frac{N(x)}{D(x)} = 0$

2. *Find* $\lim\limits_{x\to+\infty} \left(1 + \ln\left(1 + \frac{1}{x}\right)\right)^x$.

$$\left(1 + \ln\left(1 + \frac{1}{x}\right)\right)^x = e^{x\ln\left(1+\ln\left(1+\frac{1}{x}\right)\right)} = e^{x\left(\ln\left(1+\frac{1}{x}+o\left(\frac{1}{x}\right)\right)\right)} = e^{x\left(\frac{1}{x}+o\left(\frac{1}{x}\right)\right)} = e^{1+o(1)}$$

Hence, $\lim\limits_{x\to+\infty} \left(1 + \ln\left(1 + \frac{1}{x}\right)\right)^x = e^1$

Exercise 6: vector spaces 1

1. *For each of these sets, say whether it is a vector space over* $\mathbb{R}$ *or not. Justify your answers rigorously.*

(a) $E = \left\{(x, y) \in \mathbb{R}^2,\, x \leq y\right\}$

$E \subset \mathbb{R}^2$, but E is not a linear subspace of $\mathbb{R}^2$.

For example, consider the vector $u = (1, 2)$. Then $u \in E$ but $-u = (-1, -2) \notin E$. Thus, E is not closed for the scalar multiplication.

E is hence not a vector space over $\mathbb{R}$.

(b) $F = \{(x, y, z) \in \mathbb{R}^3,\ x - y = 0\}$.

By definition, $F \subset \mathbb{R}^3$. Furthermore, $0_{\mathbb{R}^3} = (0, 0, 0) \in F$ because $0 - 0 = 0$.
Let $(u_1 = (x, y, z), u_2 = (x', y', z')) \in F^2$ and $\lambda \in \mathbb{R}$.
Then $\lambda u_1 + u_2 = (\lambda x + x', \lambda y + y', \lambda z + z')$. Note that $(\lambda x + x') - (\lambda y + y') = \lambda (x - y) + (x' - y') = \lambda \times 0 + 0 = 0$ because u_1 and u_2 are elements of F.
This proves that $\lambda u_1 + u_2 \in F$.
Finally, F is a linear subspace of $\mathbb{R}^3$. It is hence a $\mathbb{R}$-vector space.

(c) $G = \{P \in \mathbb{R}[X],\ X \mid P\}$

By definition, $G \subset \mathbb{R}[X]$. Furthermore, $0_{\mathbb{R}[X]} = X \times 0_{\mathbb{R}[X]}$, that is, $X \mid 0_{\mathbb{R}[X]}$. Hence, $0_{\mathbb{R}[X]} \in G$.
Let $(P_1, P_2) \in G^2$ and $\lambda \in \mathbb{R}$. Since $X \mid P_1$ and $X \mid P_2$, $\exists\, (Q_1, Q_2) \in (\mathbb{R}[X])^2$ such that $P_1 = XQ_1$ et $P_2 = XQ_2$. Thus, $\lambda P_1 + P_2 = X (\lambda Q_1 + Q_2)$ which show that $X \mid (\lambda P_1 + P_2)$. Hence, $\lambda P_1 + P_2 \in G$.
Finally, G is a linear subspace of $\mathbb{R}[X]$. It is hence a $\mathbb{R}$-vector space.

2. *In this question, you don't have to justify your answers.*

   Find a linear subspace of E (other than E and $\{0_E\}$) in the following cases:

   (a) $E = \mathbb{R}^4$

   For example: $F = \{(x, y, z, t) \in \mathbb{R}^4,\ x + z + t = 0\}$.

   (b) $E = \mathbb{R}^{\mathbb{R}}$

   For example: $F = \{f \in \mathbb{R}^{\mathbb{R}},\ f(0) = 0\}$

   (c) $E = \{(u_n) \in \mathbb{R}^{\mathbb{N}},\ (u_n) \text{ converges}\}$

   For example: $F = \{(u_n) \in \mathbb{R}^{\mathbb{N}},\ (u_n) \text{ converges to } 0\}$

Exercise 7: vector spaces 2

The questions are independent.

1. *In $\mathbb{R}^3$, consider the linear subspaces*

$$F = \{(x, y, z) \in \mathbb{R}^3,\ x = 0\} \text{ and } G = \{(x, y, z) \in \mathbb{R}^3,\ x = y\}$$

   (a) *Is it true that $F \cap G = \{0_{\mathbb{R}^3}\}$? Justify.*

   The vector $u = (0, 0, 15) \in F \cap G$. Thus, $F \cap G \neq \{0_{\mathbb{R}^3}\}$.

   (b) *Write the definition of the set $F + G$.*

$$F + G = \{u \in E \text{ tel que } \exists\, (u_1, u_2) \in F \times G \text{ tel que } u = u_1 + u_2\}$$

   (c) *Is vector $u = (1, 2, 3)$ an element of $F + G$? Justify.*

   For example, we can write $u = (1, 2, 3) = u_1 + u_2$ with $u_1 = (0, 1, 6)\$ \in F$ and $u_2 = (1, 1, -3) \in G$. Thus, $u \in F + G$.

   (d) *Consider the decomposition that you got at question (c). Is it unique? Justify. Explain why you could know it before doing any computation.*

   No, the decomposition above is not unique. For example, we can also write $u = (1, 2, 3) = v_1 + v_2$ with $v_1 = (0, 1, 0) \in F$ and $v_2 = (1, 1, 3) \in G$. There is an infinite number of possibilities. We could know that from question (a), because $F \cap G \neq \{0_{\mathbb{R}^3}\}$.

2. *Let E be a $\mathbb{R}$-vector space and $\mathcal{F} = (u_1, u_2, \cdots, u_n) \in E^n$ a n-vector family of E. ($n \in \mathbb{N}^*$)*

(a) *Write the mathematical definition of: $\mathcal{F}$ is linearly independent.*

$$\mathcal{F} \text{ linearly independent} \iff \forall\, (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n, (\lambda_1 u_1 + \cdots + \lambda_n u_n = 0_E \implies \lambda_1 = \cdots = \lambda_n = 0)$$

(b) *Write the mathematical definition of: $\mathcal{F}$ is a spanning family of E.*

$$\mathcal{F} \text{ spanning family of } E \iff \forall\, u \in E,\ \exists\, (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \text{ such that } u = \lambda_1 u_1 + \cdots + \lambda_n u_n$$

(c) *In $E = \mathbb{R}^3$, find an example of a 2-vector family which is independent. Then an example of a 3-vector family which is not independent. You don't have to justify.*

- Let $u_1 = (1, 0, 0) \in \mathbb{R}^3$ and $u_2 = (0, 1, 0) \in \mathbb{R}^3$. The family (u_1, u_2) is linearly independent because the two vectors are not colinear.

- Let $\mathbb{R}^3$, $v_1 = (1, 2, 3)$, $v_2 = (4, 5, 6)$ and $v_3 = (5, 7, 9)$. The family (v_1, v_2, v_3) is linearly dependent because $v_3 = v_1 + v_2$.

(d) *In $E = \mathbb{R}^2$, find an example of a spanning family of E. You don't have to justify.*

The family $(e_1 = (1, 0), e_2 = (0, 1))$ is a spanning family of $\mathbb{R}^2$ because, for all $u = (x, y) \in \mathbb{R}^2$, $u = xe_1 + ye_2$.