# LOFO 2020

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**1A.** Prove that $(\neg P \vee Q) \Leftrightarrow (P \Rightarrow Q)$ is a tautology.

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ | $p \Rightarrow q$ | $\varphi$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 |

**1B.** Prove that $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$ is a tautology.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p \wedge \neg q$ | $\varphi$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |

**2A.** Is the following Hilbert system complete with regards to $\mathcal{F}_{\{\bot,\neg,\vee,\Rightarrow\}}$?

$$\frac{A \Rightarrow B \qquad A}{B}\ [\textit{Modus Ponens}]$$

$$\frac{}{\bot \Rightarrow A}\ [\bot] \qquad \frac{}{A \Rightarrow B \Rightarrow A}\ [\Rightarrow_1] \qquad \frac{}{(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C}\ [\Rightarrow_2]$$

$$\frac{}{(A \Rightarrow \bot) \Rightarrow \neg A}\ [\neg_1] \qquad \frac{}{A \Rightarrow \neg A \Rightarrow \bot}\ [\neg_2]$$

No, because no rule can handle the $\vee$ symbol.

**2B.** Is the following Hilbert system complete with regards to $\mathcal{F}_{\{\bot,\wedge,\vee,\Rightarrow\}}$?

$$\frac{A \Rightarrow B \qquad A}{B}\,[\textit{Modus Ponens}]$$

$$\frac{}{A \Rightarrow A \vee B}\,[\vee_1] \qquad \frac{}{B \Rightarrow A \vee B}\,[\vee_2] \qquad \frac{}{A \vee B \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C}\,[\vee_3]$$

$$\frac{}{A \wedge B \Rightarrow A}\,[\wedge_1] \qquad \frac{}{A \wedge B \Rightarrow B}\,[\wedge_2] \qquad \frac{}{A \Rightarrow B \Rightarrow A \wedge B}\,[\wedge_3]$$

$$\frac{}{A \Rightarrow B \Rightarrow A}\,[\Rightarrow_1] \qquad \frac{}{(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C}\,[\Rightarrow_2]$$

No, because no rule can handle the $\bot$ symbol.

**3A.** Prove that $\{Q \wedge P, R\} \vdash_{\mathcal{N}} P \wedge (R \wedge Q)$. This can be done using a proof of depth 4.

$$\boxed{\dfrac{\dfrac{Q \wedge P}{P}\,\wedge\text{Elim2} \qquad \dfrac{R \qquad \dfrac{Q \wedge P}{Q}\,\wedge\text{Elim1}}{R \wedge Q}\,\wedge\text{Intro}}{P \wedge (R \wedge Q)}\,\wedge\text{Intro}}$$

**3B.** Prove that $\{P \vee Q\} \vdash_{\mathcal{N}} P \vee (Q \vee R)$. This can be done using a proof of depth 4.

$$\boxed{\dfrac{P \vee Q \qquad \dfrac{[P]}{P \vee (Q \vee R)}\,\vee\text{Intro1} \qquad \dfrac{\dfrac{[Q]}{Q \vee R}\,\vee\text{Intro1}}{P \vee (Q \vee R)}\,\vee\text{Intro2}}{P \vee (Q \vee R)}\,\vee\text{Elim}}$$

**4A.** Prove that $\{\neg P \Rightarrow Q\} \vdash_{\mathcal{N}} P \vee Q$ by filling the blanks of the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg P \Rightarrow Q \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P}\,1}{P \vee Q}\,[\vee_I^l] \qquad \dfrac{}{\neg(P \vee Q)}\,2}{\bot}\,[\neg_E]}{\neg P}\,[\neg_I]^1}{Q}\,[\Rightarrow_E]}{P \vee Q}\,[\vee_I^r] \qquad \dfrac{}{\neg(P \vee Q)}\,2}{\dfrac{\bot}{\neg\neg(P \vee Q)}\,[\neg_I]^2}\,[\neg_E]}{P \vee Q}\,[\neg\neg]$$

**4B.** Prove that $\vdash_{\mathcal{N}} ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ by filling the blanks of the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \Rightarrow Q) \Rightarrow P}^{3} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{P}^{1} \quad \dfrac{}{\neg P}^{2}}{\bot}[\neg_E]}{Q}[\bot_E]}{P \Rightarrow Q}[\Rightarrow_I]^{1}}{P}[\Rightarrow_E] \quad \dfrac{}{\neg P}^{2}}{\bot}[\neg_E]}{\neg\neg P}[\neg_I]^{2}}{\dfrac{}{P}[\neg\neg]}$$

$$\dfrac{P}{((P \Rightarrow Q) \Rightarrow P) \Rightarrow P}[\Rightarrow_I]^{3}$$

**5A.** Define a term Square $\in \Lambda$ such that for any natural integer $n$:

$$\text{Square } \underline{n} \to_\beta^* \underline{n^2}$$

Then guess its type.

Square $= \lambda u \cdot \text{Mult } uu$ is of type $((\sigma \to \sigma) \to (\sigma \to \sigma)) \to ((\sigma \to \sigma) \to (\sigma \to \sigma))$.

**5B.** Define a term Double $\in \Lambda$ such that for any natural integer $n$:

$$\text{Double } \underline{n} \to_\beta^* \underline{2 \times n}$$

Then guess its type.

Double $= \lambda u \cdot \text{Plus } uu$ is of type $((\sigma \to \sigma) \to (\sigma \to \sigma)) \to ((\sigma \to \sigma) \to (\sigma \to \sigma))$.

**6.** Prove that $\Theta = (\lambda xy \cdot y(xxy))(\lambda uv \cdot v(uuv))$ is a fixed-point combinator.

$$\begin{array}{rcl}
\Theta A & = & \theta\theta A \\
& = & ((\lambda xy \cdot y(xxy)\theta)A \\
& \to_\beta & (\lambda y \cdot y(\theta\theta y))A \\
& \to_\beta & A(\theta\theta A) \\
& = & A\Theta A
\end{array}$$

Thus $\Theta A \leftrightarrow_\beta^* A(\Theta A)$.

**7.** Prove that $\vdash KI : \tau \to \sigma \to \sigma$.

$K$ is of primary type $\sigma \to \tau \to \sigma$, thus also of type $(\sigma \to \sigma) \to \tau \to (\sigma \to \sigma)$. $I$ is of type $\sigma \to \sigma$. $KI$ is therefore of type $\tau \to \sigma \to \sigma$ by the application rule.

**8A.** Prove that $\vdash_{\mathcal{NI}} (P \wedge Q) \Rightarrow (Q \wedge P)$.
Then find a term in $\Lambda_{ext}$ of type $\sigma \times \tau \to \tau \times \sigma$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{P \wedge Q}\,{}^{1}}{Q}\,[\wedge_E^r] \qquad \dfrac{\dfrac{}{P \wedge Q}\,{}^{1}}{P}\,[\wedge_E^l]}{Q \wedge P}\,[\wedge_I]}{(P \wedge Q) \Rightarrow (Q \wedge P)}\,[\Rightarrow_I]^1
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{x : \sigma \times \tau}\,{}^{1}}{\Pi_2(x) : \tau} \qquad \dfrac{\dfrac{}{x : \sigma \times \tau}\,{}^{1}}{\Pi_1(x) : \sigma}}{\langle \Pi_2(x), \Pi_1(x) \rangle : \tau \times \sigma}}{\lambda x \cdot \langle \Pi_2(x), \Pi_1(x) \rangle : \sigma \times \tau \to \tau \times \sigma}\,{}^{1}
$$

**8B.** Prove that $\vdash_{\mathcal{NI}} (P \vee Q) \Rightarrow (Q \vee P)$.
Then find a term in $\Lambda_{ext}$ of type $\sigma \cup \tau \to \tau \cup \sigma$.

$$
\dfrac{\dfrac{\dfrac{}{P \vee Q}\,{}^{1} \qquad \dfrac{\dfrac{}{P}\,{}^{2}}{Q \vee P}\,[\vee_I^r] \qquad \dfrac{\dfrac{}{Q}\,{}^{2}}{Q \vee P}\,[\vee_I^l]}{Q \vee P}\,[\vee_E]^2}{(P \vee Q) \Rightarrow (Q \vee P)}\,[\Rightarrow_I]^1
$$

$$
\dfrac{\dfrac{\dfrac{}{x : \sigma \cup \tau}\,{}^{1} \qquad \dfrac{\dfrac{}{y : \sigma}\,{}^{2}}{K_2(y) : \tau \cup \sigma} \qquad \dfrac{\dfrac{}{z : \tau}\,{}^{2}}{K_1(z) : \tau \cup \sigma}}{\oplus(K_2(y), K_1(z), x) : \tau \cup \sigma}\,{}^{2}}{\lambda x \cdot \oplus(K_2(y), K_1(z), x) : (\sigma \cup \tau) \to (\tau \cup \sigma)}\,{}^{1}
$$