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Quantum Computing Technical Foundations
September 9, 2022

Assignment 2: Single Qubits

Assignment Due: Friday 16th September

Solutions can be handwritten on a separate sheet of paper, typed or done on a tablet. You can print this, write the solutions on it, and then scan and upload it.
Send the completed assignment to tclarke@asesoftware.com If you have any questions or difficulties, please do reach out to the same email.
Challenge Questions are Optional

Question 1. ___ does not play dice

A dice has 6 faces numbered 1,2,3,4,5,6. For this question we will count from 1. Rather than a 2-level system like a qubit, this is a 6-level system. We'll call each state by the number on the face. For instance $|3\rangle$ is the state of the dice with face with 3 up.
When we toss the dice, let's say it's in the superposition state similar to $|+\rangle$ for the qubit.

$$|+_6\rangle = \frac{1}{\sqrt{6}}(|1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle)$$

What is the expectation value of the dice?
Hint: rather than computing a 6x6 matrix-vector product, you can do the sum
$\langle N\rangle = \sum_{j=1}^{6} |\langle j|+_6\rangle|^2$

Question 2. Challenge question 1: Turn the dice

With the coin, turning it over can be modelled with the X-gate. Such that
$\hat{X}|h\rangle = |t\rangle$
For the dice, turning it over corresponds to putting the face at the bottom on top. Since opposite faces of the coin add up to 7, this means

$$|1\rangle \rightarrow |6\rangle$$

$$|2\rangle \rightarrow |5\rangle$$

$$|3\rangle \rightarrow |4\rangle$$

And the reverse for $|4\rangle, |5\rangle, |6\rangle$
Write down the 6x6 matrix that corresponds to the operator $\hat{F}$ such that:

$$\hat{F}|j\rangle = |k\rangle$$

With j and k being the numbers on opposite faces

Question 3. Challenge question 2: Superposition: A mater of perspective

The states $|0\rangle$ & $|1\rangle$ are not generally considered superpositions. They correspond to our coin being heads or tails, or the 0 or 1 classical bits. But $|0\rangle$ & $|1\rangle$ are superpositions if you measure in a different basis.

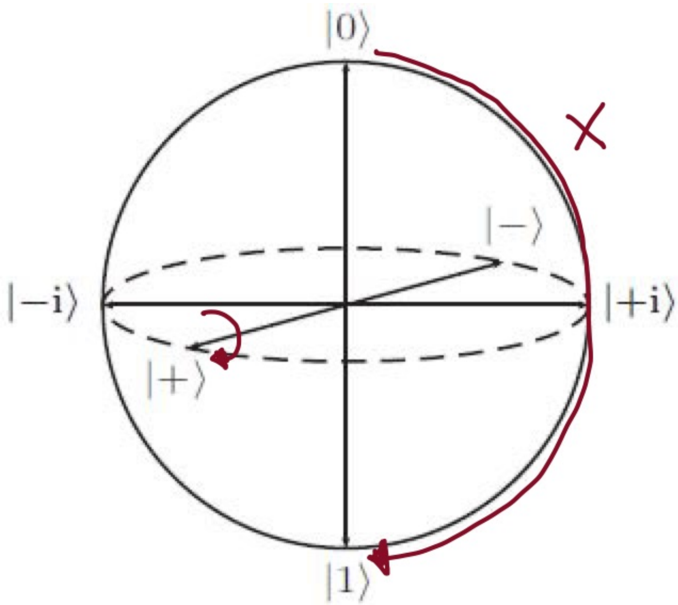
Take the X-basis, (X-axis on the Bloch sphere) given by

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Express $|0\rangle$ & $|1\rangle$ in terms of $|+\rangle$ & $|-\rangle$. Interpret your result using the Bloch sphere or otherwise

Question 4. On the Bloch Sphere, the X gate can be represented as a rotation of a 180° (π) as shown below .

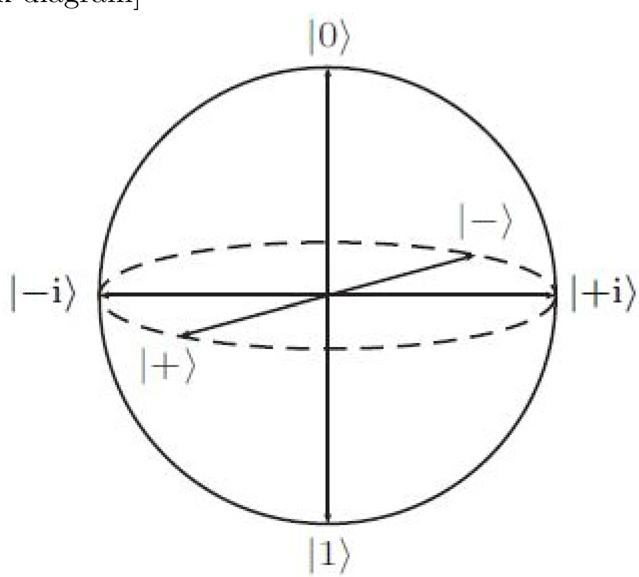


starting from the top at $|0\rangle$, draw out the path traced by the qubit from the sequence of gates H, Z, H

Where H is the Hadamard gate, Z is the 180° (π) rotation around the Z-axis of the Bloch sphere and X is the not gate

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$,

Label each path with the gate. Compare your result to the example of the X gate [You may wish to use this blank diagram]



Question 5. By considering the effect on $|0\rangle$ $|1\rangle$, or by doing matrix multiplication, calculate the following states:
$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

1) $XZ\,|1\rangle$

2) $XH\,|+\rangle$

3) $XYZ\,|0\rangle$

Question 6. Using matrix multiplication, writing out the state vector, or otherwise, show that

$$HZH = X$$

If the previous question is done correctly this should verify your result.

Question 7. All single qubit gates can be described as unitary operators in the following form

$$U(\theta, \varphi, \lambda) = \begin{bmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\varphi}\sin(\theta/2) & e^{i(\varphi+\lambda)}\cos(\theta) \end{bmatrix}$$

By choosing suitable values of $(\theta, \varphi, \lambda)$ show how the following gates can be obtained:

1) Y $= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
2) Z $= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
3) H $= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
4) T $= \begin{bmatrix} 1 & 0 \\ 0 & e^{\pi/4} \end{bmatrix}$

Question 8. Local vs global phase

Any quantum state can be written as

$$|\psi\rangle = \cos{(\theta/2)}\,|0\rangle + \sin{(\theta/2)}e^{i\varphi}\,|1\rangle$$

We can factor out a global phase- the phase common to all the states- and remove it. For instance the global phase $e^{i\Theta}$ can be removed like

$$\begin{aligned} e^{i\Theta}\,|\psi\rangle &= e^{i\Theta}(\cos{(\theta/2)}\,|0\rangle + \sin{(\theta/2)}e^{i\varphi}\,|1\rangle) \\ &= |\psi\rangle \end{aligned}$$

For the following states, factor out the global phase (if there is any) and write down the angles (θ, φ) that describe their position on the Bloch Sphere

1) $|i\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i\,|0\rangle$
2) $|\psi\rangle = \sqrt{\frac{1}{3}}i\,|0\rangle + \frac{2}{3}\,|1\rangle$ Hint: you can multiply everything by a global phase to move the i to the $|1\rangle$

Question 9. Challenge Question 4: Why global phase doesn't matter

In the class, I said we could ignore global phases. In this question we will explore that idea a little.

1) Consider a generic quantum state $|\psi\rangle = \cos(\alpha)\,|0\rangle + \sin(\alpha)\,|1\rangle$ and the same state with a global phase added $|\psi_G\rangle = e^{i\Phi}\cos(\alpha)\,|0\rangle + e^{i\Phi}\sin(\alpha)\,|1\rangle$.

   The probability of measuring $|\psi\rangle$ from $|\psi_G\rangle$ is given by the inner product as $|\,\langle\varphi_G|\varphi\rangle\,|^2$. Calculate this probability.

2) What if we had a relative phase difference instead, a phase shift between $|0\rangle$ & $|1\rangle$. Let's define this state $|\psi_R\rangle = \cos(\alpha)\,|0\rangle + e^{i\varphi}\sin(\alpha)\,|1\rangle$
   Compute the inner product of $|\,\langle\varphi_R|\varphi\rangle\,|^2$

3) We'll choose $\alpha = \pi/4$ such that $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|\psi_R\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}\,|1\rangle)$ What is the value of α such that the inner product from (2) is zero?

Question 10. Challenge question 5: The Pauli matrices

The Pauli matrices satisfy the relations

$$\sigma_j\sigma_k = \delta_{j,k}I + i\epsilon_{jkl}\sigma_l$$

Where σ_j or σ_k can be the gates $\{\sigma_x = X, \sigma_y = Y, \sigma_z = Z\}$. $\delta_{j,k}$ is the Kronecker-Delta $\delta_{j,k} = 1$ if $j = k$ (i.e. the two matrices are the same) or 0 if $j \neq k$. I is the identity matrix , ϵ_{jkl} is the Levi-Civita symbol $= 1$ if the pair j, k are in the order X,Y,Z. For example $\sigma_j\sigma_k = XY$ would have $\epsilon_{jkl} = 1$ but $\sigma_j\sigma_k = ZY$ (reverse order) would have $\epsilon_{jkl} = -1$

Show that the gates X, Y satisfy this by computing the matrix products

1) XX
2) XY
3) YX

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