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Quantum Computing Technical Foundations
September 7, 2022

Assignment 2: Single Qubits

Assignment Due: Thursday 15th September

Solutions can be handwritten on a separate sheet of paper, typed or done on a tablet. You can print this, write the solutions on it, and then scan and upload it.
Send the completed assignment to tclarke@asesoftware.com If you have any questions or difficulties, please do reach out to the same email.

Challenge Questions are Optional

Question 1. ___ does not play dice

A dice has 6 faces numbered 1,2,3,4,5,6. For this question we will count from 1. Rather than a 2-level system like a qubit, this is a 6-level system. We'll call each state by the number on the face. For instance $|3\rangle$ is the state of the dice with face with 3 up.

When we toss the dice, let's say it's in the superposition state similar to $|+\rangle$ for the qubit.

$$|+_6\rangle = \frac{1}{\sqrt{6}}(|1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle)$$

What is the expectation value of the dice?

Hint: rather than computing a 6x6 matrix-vector product, you can do the sum
$\langle N \rangle = \sum_{j=1}^{6} |\langle j|+_6\rangle|^2$

Question 2. Challenge question: Turn the dice

With the coin, turning it over can be modelled with the X-gate. Such that
$\hat{X}|h\rangle = |t\rangle$

For the dice, turning it over corresponds to putting the face at the bottom on top. Since opposite faces of the coin add up to 7, this means

$$|1\rangle \rightarrow |6\rangle$$

$$|2\rangle \rightarrow |5\rangle$$

$$|3\rangle \rightarrow |4\rangle$$

Write down the 6x6 matrix that corresponds to the operator $\hat{F}$ such that:

$$\hat{F}|j\rangle = |k\rangle$$

With j and k being the numbers on opposite faces

Question 3. Challenge question: Superposition: A mater of perspective

The states $|0\rangle$ & $|1\rangle$ are not generally considered superpositions. They correspond to our coin being heads or tails, or the 0 or 1 classical bits. But $|0\rangle$ & $|1\rangle$ are superpositions if you measure in a different basis.

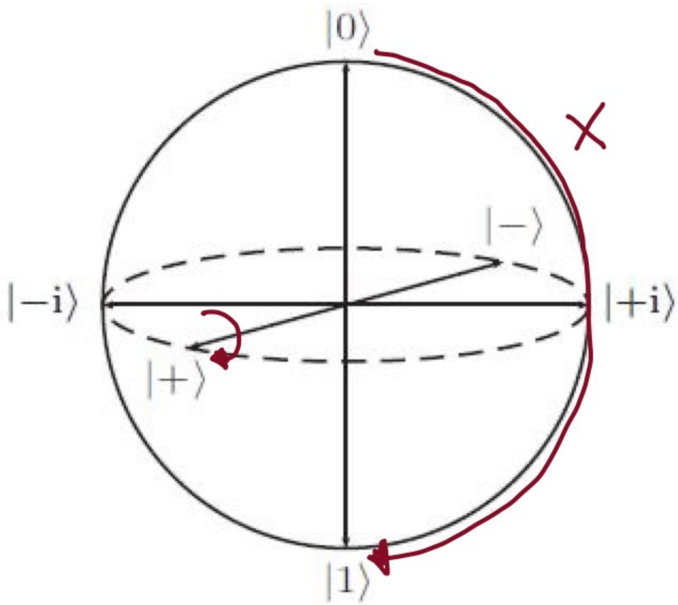
Take the X-basis, (X-axis on the Bloch sphere) given by

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

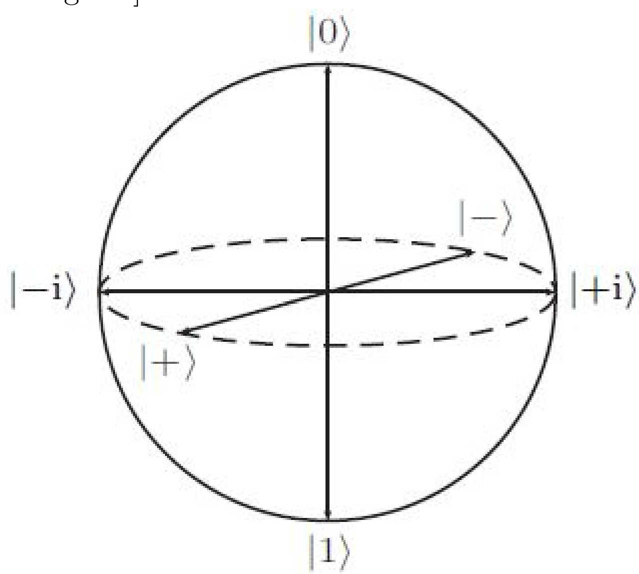
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Express $|0\rangle$ & $|1\rangle$ in terms of $|+\rangle$ & $|-\rangle$. Interpret your result using the Bloch sphere or otherwise

Question 4. On the Bloch Sphere, the X gate can be represented as a rotation of a 180° (π) as shown below .



starting from the top at $|0\rangle$, draw out the path traced by the qubit from the sequence of gates H, Z, H
Where H is the Hadamard gate, Z is the 180° (π) rotation around the Z-axis of the Bloch sphere and X is the not gate
Label each path with the gate. Compare your result to the example of the X gate [You may wish to use this blank diagram]



Question 5. By considering the effect on $|0\rangle$ $|1\rangle$, or by doing matrix multiplication, calculate the following states:

1) $XZ\,|1\rangle$

2) $XH\,|+\rangle$

3) $XYZ\,|0\rangle$

Question 6. Using matrix multiplication, writing out the state vector, or otherwise, show that

$$HZH = X$$

If the previous question is done correctly this should verify your result.

Question 7. All single qubit gates can be described as unitary operators in the following form

$$U(\theta, \varphi, \lambda) = \begin{bmatrix} \cos(\theta/2) & e^{i\lambda}\sin(\theta/2) \\ e^{i\theta}\sin(\theta/2) & e^{i(\varphi+\lambda)}\cos(\theta) \end{bmatrix}$$

By choosing suitable values of $(\theta, \varphi, \lambda)$ show how the following gates can be obtained:

1) Y = $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
2) Z = $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
3) H = $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
4) T = $\begin{bmatrix} 1 & 0 \\ 0 & e^{\pi/4} \end{bmatrix}$

Question 8. Any quantum state can be written as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle$$

$|i\rangle =$

Question 9. Challenge question: Pauli matrices identity

The Pauli matrices satisfy the relations

$$\sigma_j\sigma_k = \delta_{j,k}I + i\epsilon_{jkl}\sigma_l$$

Where σ_j or σ_k can be the gates $\{X, Y, Z\}$. $\delta_{j,k}$ is the Kronecker-Delta $\delta_{j,k} = 1$ if $j = k$ (i.e. the two matrices are the same) or 0 if $j \neq k$. $I$is the identity, ϵ_{jkl} is the Levi-Civita symbol $= 1$ if the pair j, k are in the order X,Y,Z. For example $\sigma_j\sigma_k = XY$ would have $\epsilon_{jkl} = 1$ but $\sigma_j\sigma_k = ZY$ (reverse order) would have $\epsilon_{jkl} = -1$

Show that the gates X, Y satisfy this by computing the matrix products

1) XX
2) XY
3) YX

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