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Quantum Computing Technical Foundations
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Assignment 3: Multiple Qubits

Assignment Due: Friday 23rd September

Solutions can be handwritten on a separate sheet of paper, typed or done on a tablet. You can print this, write the solutions on it, and then scan and upload it.
Send the completed assignment to tclarke@asesoftware.com If you have any questions or difficulties, please do reach out to the same email.
Challenge Questions are optional.

Question 1. Basic binary

Represent the following numbers in binary

1) 4
2) 7
3) 12

Question 2. Back to basics

Represent the following numbers in decimal (or a base of your choice)

1) 00101
2) 1010
3) 110

You can check you answers with python by using bin(integer) or int(0b*binary*) and comparing your answers.

Question 3. Multiple qubits, multiple gates

Apply the following gates to the state $|000\rangle$ and write the resulting statevector. You can use $|+\rangle, |-\rangle, |i\rangle, |-i\rangle$ if it helps simplify your answer.

1) $H_0 H_1 H_2$
2) $X_0 Y_1 Z_2$

Question 4. Entangled or not
For each of the states below, are they entangled or product states?

1) $\frac{1}{\sqrt{2}}(|+\rangle|-\rangle + |-\rangle|+\rangle)$

2) $\frac{1}{\sqrt{2}}(|01\rangle + |00\rangle)$

3) $|GHZ\rangle = \frac{1}{\sqrt{2}}(|00...0\rangle + |11...1\rangle)$

4) $|00\rangle + e^{i\pi/2}|01\rangle + e^{i\pi}|10\rangle + e^{i3\pi/2}|11\rangle$

Question 5. Bell state basis

The 4 Bell States are

$$|\Phi^+\rangle = \frac{1}{2}\left(|00\rangle + |11\rangle\right)$$
$$|\Phi^-\rangle = \frac{1}{2}\left(|00\rangle - |11\rangle\right)$$
$$|\Psi^+\rangle = \frac{1}{2}\left(|01\rangle + |10\rangle\right)$$
$$|\Psi^-\rangle = \frac{1}{2}\left(|01\rangle - |01\rangle\right)$$

As vectors, this looks like

$$|\Phi^+\rangle =; |\Phi^-\rangle =; |\Psi^+\rangle =; |\Psi^+\rangle$$

Verify that they are orthonormal (i.e.$\langle\psi|\varphi\rangle = \delta_{\psi,\varphi}$) by computing

1) $\langle\Phi^+|\Phi^-\rangle$
2) $\langle\Phi^+|\Phi^+\rangle$
3) $\langle\Phi^+|\Psi^+\rangle$

Question 6. Generate the Bell states

In the class, we generated the Bell states by first applying a Hadamard gate to qubit 0 (H_0) and then applying a controlled-not gate with qubit 0 as control and qubit 1 as target ($CNOT_{0,1}$). By changing the states of q_0 q_1 from 0 to 1 we can get all the Bell States.

For example, if we start with both q_0 q_1 in the state $|0\rangle$ we can generate $|\Phi^+\rangle$. (This was the second challenge question in Assignment 1).

Applying the Hadamard gate on q_0 changes it from $|0\rangle$ to the state $|+\rangle$

$$|00\rangle_{1,0} \xrightarrow{H_0} |0+\rangle$$

The controlled-not ($CNOT_{0,1}$), with q_0 as control and q_1 as target then changes q_1 to $|1\rangle$ if q_0 is $|1\rangle$.

We can expand $|0+\rangle$ and consider what happens to $|00\rangle$ and $|01\rangle$ when the CNOT is applied.

$$|0+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

$$|00\rangle \rightarrow |00\rangle$$

$$|01\rangle \rightarrow |11\rangle$$

This gives us

$$|0+\rangle \xrightarrow{CNOT_{0,1}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle$$

By changing the input to $|\psi\rangle$ and applying the same gates, which Bell State is generated by

1) $q_0 = 1$, $q_0 = 1$, $\psi = |01\rangle$
2) $q_0 = 0$, $q_1 = 1$, $\psi = |10\rangle$
3) $q_0 = 1$, $q_1 = 1$, $\psi = |11\rangle$

Question 7. Challenge Question: The no-cloning theorem

One of the most important results in quantum information science is the no-cloning theorem. It says that it's impossible to copy a quantum state with unlimited precision. In this question, we will show why by doing a "proof" (not really a proof) by contradiction.

Let's say we have some machine that can clone a quantum state. So we have two qubits, and we take the state of one of them and copy it onto the other. This looks like:

$$|\psi\rangle_1 |0\rangle_0 \xrightarrow{copy} |\psi\rangle_1 |\psi\rangle_0$$

Our copy machine should be able to copy any state, so let's take a state orthogonal to $|\psi\rangle$ we'll call it $|\varphi\rangle$, such that

$$\langle\psi|\varphi\rangle = 0$$
$$\langle\varphi|\varphi\rangle = \langle\psi|\psi\rangle = 1$$

Our cloning machine should also be able to clone $|\varphi\rangle$

$$|\varphi\rangle_1 |0\rangle_0 \xrightarrow{copy} |\varphi\rangle_1 |\varphi\rangle_0$$

We can say any state of our qubit is a superposition of $|\psi\rangle$ & $|\varphi\rangle$ (this is the same as saying any state is a superposition of $|0\rangle$ & $|1\rangle$) and it should be cloneable.

$$\frac{1}{\sqrt{2}}(|\varphi\rangle + |\psi\rangle)_1 |0\rangle_0 \xrightarrow{copy} \frac{1}{\sqrt{2}}(|\varphi\rangle + |\psi\rangle)_1 \frac{1}{\sqrt{2}}(|\varphi\rangle + |\psi\rangle)_0$$

We'll call this cloned state $|\Psi_1\rangle_{1,0}$

$$|\Psi_1\rangle_{1,0} = \frac{1}{\sqrt{2}}(|\varphi\rangle + |\psi\rangle)_1 \frac{1}{\sqrt{2}}(|\varphi\rangle + |\psi\rangle)_0 \tag{1}$$

(I chose $\frac{1}{\sqrt{2}}$ but could have chosen any superposition)

Any quantum operator must be unitary, this means our copy is a linear operator, let's call the copy operator $\hat{C}$ and apply it to our superposition

$$\hat{C}\frac{1}{\sqrt{2}}(|\varphi\rangle + |\psi\rangle)_1 |0\rangle_0 = \frac{1}{\sqrt{2}}[\hat{C}|\varphi\rangle_1 |0\rangle_0 + \hat{C}|\psi\rangle_1 |0\rangle_0]$$

Applying $\hat{C}$ should copy the state of q_1 leaving us with a cloned state $|\Psi_2\rangle_{1,0}$:

$$|\Psi_2\rangle_{1,0} = \frac{1}{\sqrt{2}}(|\varphi\rangle|\varphi\rangle + |\psi\rangle|\psi\rangle)_{1,0} \tag{2}$$

Comparing equations (1) & (2) they should be the exact same state (if the cloning machine works).

1) From Equation (1), expand the parenthesis to express $|\Psi_1\rangle$ as a sum of products $|a\rangle_1 |b\rangle_0$, where both a and b can be either ψ, φ
2) Calculate the inner product

$$\langle\Psi_1|\Psi_2\rangle_{1,0}$$

Hint:You can then take advantage of the tensor product by using:

$$(\langle a|_1 \langle b|_0)(|c\rangle_0 |d\rangle_1 = \langle a|d\rangle_1 \langle b|c\rangle_0$$

Or equivalently

$$\langle a|_1 \langle b|_0 |c\rangle_0 |d\rangle_1 = \langle a|d\rangle_1 \langle b|c\rangle_0$$

Any result other than 1 suggests they are not the same state.

3) Interpret what your result for (2) means about whether or not it is possible to clone a quantum state.

Question 8. Challenge question: The Quantum Fourier Transform

This question is just here because people kept doing tensor products in the previous assignments. There is a lot of linear algebra here. Much of the excitement around quantum computing is derived from applications of the quantum Fourier transform. For 2 qubits, it can be done in the following steps

- Apply a Hadamard on qubit 0,
- Apply a controlled Z gate with angle $\theta = \pi/4$ with qubit 1 as control, qubit 0 as target,
- Apply a Hadamard on qubit 1,
- Swap the qubits 0 and 1 to preserve their orders

We can represent these steps as matrix multiplication with the last one on the left and the first one on the right as

$$\mathbb{F} = SWAP(H_1 \otimes \mathbb{1}_0)CRz_{1,0}(\theta)(\mathbb{1} \otimes H_0)$$

Where $CRz_{1,0}$ can be writen as

$$CRz_{1,0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{bmatrix} = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & Rz(\theta) \end{bmatrix}$$

(The second one is the block form of the first)

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For 1) - 3) you may wish to block the tensor products so you only need to do 2x2 matrix multiplication

1) Compute $(\mathbb{1} \otimes H_0)$
2) Compute $(H_1 \otimes \mathbb{1}_0)$
3) Compute $(H_1 \otimes \mathbb{1}_0)CRz_{1,0}(\theta)(\mathbb{1} \otimes H_0)$
4) Compute $\mathbb{F}$
5) $\mathbb{F}$ can reveal the periodicity in quantum states. For example, take

$$|\psi\rangle = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Apply $\mathbb{F}|\psi\rangle = |k\rangle$ and you should end up with a number, k. One of the following

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$