首先，INS 运动模型：

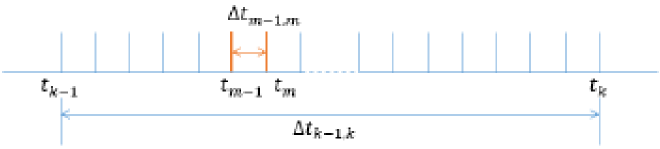
$$\dot{r}^w_{wb} = v^w_{wb} \tag{1}$$
$$\dot{v}^w_{wb} = C^w_b f^b_{ib} + g^w - 2\omega^{w\times}_{ie} v^w_{wb} \tag{2}$$
$$\dot{C}^w_b = C^w_b \omega^{b\times}_{wb}, \omega^b_{wb} = \omega^b_{ib} - C^b_w \omega^w_{ie} \tag{3}$$
$$\omega^w_{ie} = C^w_e \omega^e_{ie} \tag{4}$$

式中，w 为自定义的全局坐标系，b 为 INS 载体坐标系（这里暂时不考虑 IMU 和载体之间安装角的影响，即认为载体坐标系与 IMU 坐标系是一致的）,e 为 ECEF，f^b_{ib}为 IMU 加速度计测量值，ω^b_{ib}为 IMU 陀螺仪测量值，C^w_b为载体坐标系 b 在自定义全局坐标系下的姿态四元数，C^w_b为载体坐标系 b 在自定义世界坐标系下的姿态 DCM，C^e_w为自定义全局坐标系相对于 ECEF 的姿态 DCM，ω^e_{ie}为地球自转角速度；⊗为姿态四元数运算；

将自定义全局坐标系 w 设置为 ECEF，则式(1)-(4)转换为如下形式：

$$\dot{r}^e_{eb} = v^e_{eb} \tag{5}$$
$$\dot{v}^e_{eb} = C^e_b f^b_{ib} + g^e - 2\omega^{e\times}_{ie} v^e_{eb} \tag{6}$$
$$\dot{C}^e_b = C^w_b \omega^{b\times}_{eb}, \omega^b_{eb} = \omega^b_{ib} - C^b_e \omega^e_{ie} \tag{7}$$
$$\dot{C}^e_b = C^e_b \omega^{b\times}_{ib} - \omega^{e\times}_{ie} C^e_b$$

假设有如下 IMU 测量时间段：



对式(1)在$[t_{k-1}, t_k]$进行积分，则有

$$\begin{aligned} v^e_{eb,k} - v^e_{eb,k-1} &= \int_{t_{k-1}}^{t_k} \dot{v}^e_{eb_t} dt = \int_{t_{k-1}}^{t_k} \left[C^e_{b_t} f^{b_t}_{ib_t} + g^e_t - 2\omega^{e\times}_{ie} v^e_{eb_t} \right] dt \\ &= \int_{t_{k-1}}^{t_k} \left[C^e_{b_k} C^{b_k}_{b_t} f^{b_t}_{ib_t} + g^e_t - 2\omega^{e\times}_{ie} v^e_{eb_t} \right] dt \\ &= C^e_{b_{k-1}} \int_{t_{k-1}}^{t_k} C^{b_{k-1}}_{b_t} f^{b_t}_{ib_t} dt + \int_{t_{k-1}}^{t_k} g^e_t dt - 2\omega^{e\times}_{ie} \int_{t_{k-1}}^{t_k} v^e_{eb_t} dt \\ &\approx C^e_{b_{k-1}} \int_{t_{k-1}}^{t_k} C^{b_{k-1}}_{b_t} f^{b_t}_{ib_t} dt + g^e_{k-1} \Delta t_{k-1,k} - 2\omega^{e\times}_{ie} \left[r^e_{eb,k} - r^e_{eb,k-1} \right] \end{aligned} \tag{8}$$

令$\beta^k_{k-1} = \int_{t_{k-1}}^{t_k} C^{b_{k-1}}_{b_t} f^{b_t}_{ib_t} dt$，则有

$$v^e_{eb,k} - v^e_{eb,k-1} \approx C^e_{b_{k-1}} \beta^k_{k-1} + g^e_{k-1} \Delta t_{k-1,k} - 2\omega^{e\times}_{ie} \left[r^e_{eb,k} - r^e_{eb,k-1} \right] \tag{9}$$

式中，β^k_{k-1}表示速度的 IMU 预积分项；

对式(5)在$[t_{k-1}, t_k]$进行积分，则有

$$
\begin{aligned}
r_{eb,k}^{e} - r_{eb,k-1}^{e} &= \int_{t_{k-1}}^{t_k} v_{eb_t}^{e} dt = \int_{t_{k-1}}^{t_k} \left[v_{eb,k-1}^{e} + \int_{t_{k-1}}^{t} \dot{v}_{eb_t}^{e} dt \right] dt \\
&\approx \int_{t_{k-1}}^{t_k} \Bigg[v_{eb,k-1}^{e} + \mathrm{C}_{b_{k-1}}^{e} \int_{t_{k-1}}^{t_k} \mathrm{C}_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} dt + g_t^e \Delta t_{k-1,t} \\
&\quad - 2\omega_{ie}^{e\,\times} \left[r_{eb,t}^{e} - r_{eb,k-1}^{e} \right] \Bigg] dt \\
&= v_{eb,k-1}^{e} \Delta t_{k-1,k} + \mathrm{C}_{b_{k-1}}^{e} \iint_{t_{k-1}}^{t_k} \mathrm{C}_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} dt + \int_{t_{k-1}}^{t_k} g_t^e \Delta t_{k-1,t} dt \\
&\quad - 2\omega_{ie}^{e\,\times} \int_{t_{k-1}}^{t_k} \left[r_{eb,t}^{e} - r_{eb,k-1}^{e} \right] dt \\
&\approx v_{eb,k-1}^{e} \Delta t_{k-1,k} + \mathrm{C}_{b_{k-1}}^{e} \iint_{t_{k-1}}^{t_k} \mathrm{C}_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} dt + \frac{1}{2} g_{k-1}^{e} \Delta t_{k-1,k}^{2} \\
&\quad - 2\omega_{ie}^{e\,\times} \sum_{i=1}^{N} \left[r_{eb,i}^{e} - r_{eb,k-1}^{e} \right] \Delta t_{i-1,i}
\end{aligned}
\tag{10}
$$

令$\alpha_{k-1}^{k} = \iint_{t_{k-1}}^{t_k} \mathrm{C}_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} dt$，则有

$$
\begin{aligned}
r_{eb,k}^{e} - r_{eb,k-1}^{e} \approx v_{eb,k-1}^{e} \Delta t_{k-1,k} + \mathrm{C}_{b_{k-1}}^{e} \alpha_{k-1}^{k} + \frac{1}{2} g_{k-1}^{e} \Delta t_{k-1,k}^{2} \\
- 2\omega_{ie}^{e\,\times} \sum_{i=1}^{N} \left[r_{eb,i}^{e} - r_{eb,k-1}^{e} \right] \Delta t_{i-1,i}
\end{aligned}
\tag{11}
$$

式中，α_{k-1}^{k}表示位置的 IMU 预积分项；
考虑到

$$
\dot{x} = \mathrm{F} x
$$

$$
x(t+\tau) = \exp\left(\int_{t}^{t+\tau} \mathrm{F}(t) dt \right) x(t)
$$

对式(7) 在$[t_{k-1}, t_k]$进行积分，则有

$$
\begin{aligned}
\mathrm{C}_{b_k}^{e} = \mathrm{C}_{b_{k-1}}^{e} \exp\left(\int_{t_{k-1}}^{t_k} \left(\omega_{eb_t}^{b_t\,\times} \right) dt \right) = \mathrm{C}_{b_{k-1}}^{e} \cdot \prod_{i=1}^{N} \exp\left(\omega_{eb_i}^{b_i\,\times} \right) \Delta t_{i-1,i} \\
\mathrm{C}_{e}^{b_{k-1}} \mathrm{C}_{b_k}^{e} = \mathrm{C}_{b_k}^{b_{k-1}} = \prod_{i=1}^{N} \exp\left(\omega_{eb_i}^{b_i\,\times} \right) \Delta t_{i-1,i} \\
\omega_{eb_i}^{b_i} = \omega_{ib_i}^{b_i} - \mathrm{C}_{e}^{b_i} \omega_{ie}^{e} = \omega_{ib_i}^{b_i} - \mathrm{C}_{b_{k-1}}^{b_i} \mathrm{C}_{e}^{b_{k-1}} \omega_{ie}^{e}
\end{aligned}
\tag{12}
$$

令$\gamma_{k-1}^{k} = \exp\left(\int_{t_{k-1}}^{t_k} \omega_{eb_t}^{b_t\,\times} dt \right)$，其表示姿态的 IMU 预积分项，$\gamma_{k-1}^{k}$考虑了地球自转的影响，即$\mathrm{C}_{e}^{b_i} \omega_{ie}^{e}$；进一步有，$\gamma_{k-1}^{k} = \mathrm{C}_{b_k}^{b_{k-1}}$；

在$[t_{k-1}, t_k]$时间内，IMU 预积分项α_{k-1}^{k}、β_{k-1}^{k}和γ_{k-1}^{k}的离散迭代计算如下（简化）：

$$
\gamma_{k-1}^{i} \approx \gamma_{k-1}^{i-1} \exp\left(\omega_{eb_i}^{b_i\,\times} \Delta t_{i-1,i} \right) \approx \gamma_{k-1}^{i-1} \left(\mathrm{I}_3 + \omega_{eb_i}^{b_i\,\times} \Delta t_{i-1,i} \right)
\tag{13}
$$

$$\beta_{k-1}^{i} \approx \int_{t_{k-1}}^{t_i} C_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} dt = \beta_{k-1}^{i-1} + \int_{t_{i-1}}^{t_i} C_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} dt \approx \beta_{k-1}^{i-1} + C_{b_{i-1}}^{b_{k-1}} f_{ib_i}^{b_i} \Delta t_{i-1,i} \tag{14}$$

$$\alpha_{k-1}^{i} \approx \iint_{t_{k-1}}^{t_k} C_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} dt = \alpha_{k-1}^{i-1} + \beta_{k-1}^{i-1} \Delta t_{i-1,i} + \frac{1}{2} C_{b_{i-1}}^{b_{k-1}} f_{ib_i}^{b_i} \Delta t_{i-1,i}^{2} \tag{15}$$

以下推导 IMU 预积分项α_{k-1}^{k}、β_{k-1}^{k}和γ_{k-1}^{k}离散误差系统方程，用于协方差和测量雅克布矩阵的更新：

假设参数$C_{b_{k-1}}^{b_t}$的真实值为$C_{b_{k-1}}^{b_t}$，估计值为$\hat{C}_{b_{k-1}}^{b_t}$，则有

$$C_{b_t}^{b_{k-1}} \approx \hat{C}_{b_t}^{b_{k-1}} \left(I_3 + \delta\theta_{b_i}^{b_{k-1}\times} \right) \tag{16}$$

将式(16)代入式(13)，可得

$$\hat{C}_{b_t}^{b_{k-1}} \left(I_3 + \delta\theta_{b_i}^{b_{k-1}\times} \right) \approx C_{b_{t-1}}^{b_{k-1}} \left(I_3 + \delta\theta_{b_{i-1}}^{b_{k-1}\times} \right) \left(I_3 + \delta\theta_{b_i}^{b_{i-1}\times} \right) \tag{17}$$

考虑到

$$b_{g,t_i} = \hat{b}_{g,t_i} + \delta b_{g,t_i} + \eta_{b_{g,t_i}}$$
$$\omega_{ib_i}^{b_i} = \tilde{\omega}_{ib_i}^{b_i} - b_{g,t_i}$$

$$\delta\theta_{b_i}^{b_{i-1}} \approx \omega_{eb_i}^{b_i} \Delta t_{i-1,i} = \left[\omega_{ib_i}^{b_i} - C_e^{b_i} \omega_{ie}^{e} \right] \Delta t_{i-1,i} = \left[\underbrace{\tilde{\omega}_{ib_i}^{b_i} - \hat{b}_{g,t_i} + \hat{C}_e^{b_i} \omega_{ie}^{e}}_{\hat{\omega}_{eb_i}^{b_i}} - \delta b_{g,t_i} - \eta_{b_{g,t_i}} \right] \Delta t_{i-1,i}$$
$$= \left[\hat{\omega}_{eb_i}^{b_i} - \delta b_{g,t_i} - \eta_{b_{g,t_i}} \right] \Delta t_{i-1,i}$$

将上述式子代入式(17)中并展开，可得

$$\delta\theta_{b_i}^{b_{k-1}} \approx \delta\theta_{b_{i-1}}^{b_{k-1}} - \left[\hat{\omega}_{eb_i}^{b_i} \right]^{\times} \Delta t_{i-1,i} \delta\theta_{b_{i-1}}^{b_{k-1}} - \Delta t_{i-1,i} \delta b_{g,t_i} - \Delta t_{i-1,i} \eta_{b_{g,t_i}} \tag{18}$$

对于 IMU 预积分β_{k-1}^{i}，则有

$$\beta_{k-1}^{i} = \hat{\beta}_{k-1}^{i} + \delta\beta_{k-1}^{i}$$
$$\beta_{k-1}^{i-1} = \hat{\beta}_{k-1}^{i-1} + \delta\beta_{k-1}^{i-1}$$
$$f_{ib_i}^{b_i} = \tilde{f}_{ib_i}^{b_i} - \underbrace{\left[\hat{b}_{a,t_i} + \delta b_{a,t_i} + \eta_{b_{a,t_i}} \right]}_{b_{a,t_i}}$$

$$\hat{\beta}_{k-1}^{i} + \delta\beta_{k-1}^{i} \approx \hat{\beta}_{k-1}^{i-1} + \delta\beta_{k-1}^{i-1} + \hat{C}_{b_{i-1}}^{b_{k-1}} \left[I + \delta\theta_{b_{i-1}}^{b_{k-1}\times} \right] f_{ib_i}^{b_i} \Delta t_{i-1,i} \tag{19}$$

$$\hat{\beta}_{k-1}^{i} + \delta\beta_{k-1}^{i} \approx \delta\beta_{k-1}^{i-1} + \underbrace{\hat{\beta}_{k-1}^{i-1} + \hat{C}_{b_{i-1}}^{b_{k-1}} f_{ib_i}^{b_i} \Delta t_{i-1,i}}_{=\hat{\beta}_{k-1}^{i}} + \hat{C}_{b_{i-1}}^{b_{k-1}} \delta\theta_{b_{i-1}}^{b_{k-1}\times} f_{ib_i}^{b_i} \Delta t_{i-1,i}$$

$$\delta\beta_{k-1}^{i} \approx \delta\beta_{k-1}^{i-1} - \hat{C}_{b_{i-1}}^{b_{k-1}} \hat{f}_{ib_i}^{b_i\times} \Delta t_{i-1,i} \delta\theta_{b_{i-1}}^{b_{k-1}} - \hat{C}_{b_{i-1}}^{b_{k-1}} \Delta t_{i-1,i} \delta b_{a,t_i} - \hat{C}_{b_{i-1}}^{b_{k-1}} \Delta t_{i-1,i} \eta_{b_{a,t_i}} \tag{20}$$

考虑到式(15)，则有

$$\delta\alpha_{k-1}^{i} \approx \delta\alpha_{k-1}^{i-1} + \delta\beta_{k-1}^{i-1} \Delta t_{i-1,i} \tag{21}$$

$\eta_{b_{g,t_i}}$和$\eta_{b_{a,t_i}}$建模为高斯随机噪声，$\delta b_{g,t_i}$和$\delta b_{a,t_i}$建模为一阶马尔科夫模型，即

$$\dot{b}_{g,t_i} = -\frac{1}{\tau_g} b_{g,t_i} + n_{b_{g,t_i}} \tag{22}$$

$$\dot{b}_{a,t_i} = -\frac{1}{\tau_a} b_{a,t_i} + n_{b_{a,t_i}} \tag{23}$$

式中，τ_g和τ_a为马尔科夫过程相关时间；其离散化形式为

$$b_{g,t_i} = e^{-\frac{\Delta t_{i-1,i}}{\tau_g}} b_{g,t_{i-1}} + n_{b_{g,t_i}} \tag{24}$$

$$b_{a,t_i} = e^{-\frac{\Delta t_{i-1,i}}{\tau_a}} b_{a,t_{i-1}} + n_{b_{g,t_i}} \tag{25}$$

结合式(18)、(20)、(21)、(24）和（25）可得：

$$\begin{bmatrix} \delta\alpha_{k-1}^{i} \\ \delta\beta_{k-1}^{i} \\ \delta\theta_{b_i}^{b_{k-1}} \\ \delta b_{g,t_i} \\ \delta b_{a,t_i} \end{bmatrix} = \begin{bmatrix} \mathrm{I}_3 & \Delta t_{i-1,i}\cdot \mathrm{I}_3 & 0 & 0 & 0 \\ 0 & \mathrm{I}_3 & -\hat{\mathrm{C}}_{b_{i-1}}^{b_{k-1}} \tilde{f}_{ib_i}^{b_i\times} \Delta t_{i-1,i} & 0 & -\hat{\mathrm{C}}_{b_{i-1}}^{b_{k-1}} \Delta t_{i-1,i} \\ 0 & 0 & \mathrm{I}_3 - \left[\hat{\omega}_{eb_i}^{b_i}\right]^{\times} \Delta t_{i-1,i} & -\Delta t_{i-1,i}\cdot \mathrm{I}_3 & 0 \\ 0 & 0 & 0 & e^{-\frac{\Delta t_{i-1,i}}{\tau_g}} & 0 \\ 0 & 0 & 0 & 0 & e^{-\frac{\Delta t_{i-1,i}}{\tau_a}} \end{bmatrix} \begin{bmatrix} \delta\alpha_{k-1}^{i-1} \\ \delta\beta_{k-1}^{i-1} \\ \delta\theta_{b_{i-1}}^{b_{k-1}} \\ \delta b_{g,t_{i-1}} \\ \delta b_{a,t_{i-1}} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\hat{\mathrm{C}}_{b_{i-1}}^{b_{k-1}} \Delta t_{i-1,i} & 0 & 0 \\ -\Delta t_{i-1,i}\cdot \mathrm{I}_3 & 0 & 0 & 0 \\ 0 & 0 & \Delta t_{i-1,i}\cdot \mathrm{I}_3 & 0 \\ 0 & 0 & 0 & \Delta t_{i-1,i}\cdot \mathrm{I}_3 \end{bmatrix} \begin{bmatrix} \eta_{b_{g,t_i}} \\ \eta_{b_{a,t_i}} \\ n_{b_{g,t_i}} \\ n_{b_{a,t_i}} \end{bmatrix} \tag{26}$$

$$\delta \mathrm{z}_{k-1}^{i} = \Phi_{i-1}^{i} \cdot \delta \mathrm{z}_{k-1}^{i-1} + \mathrm{G}_{i-1}^{i} \cdot \mathrm{u}_{t_i} \tag{27}$$

$$\Phi_{i-1}^{i} = \begin{bmatrix} 0 & \Delta t_{i-1,i}\cdot \mathrm{I}_3 & 0 & 0 & 0 \\ 0 & 0 & -\hat{\mathrm{C}}_{b_{i-1}}^{b_{k-1}} \tilde{f}_{ib_i}^{b_i\times} \Delta t_{i-1,i} & 0 & -\hat{\mathrm{C}}_{b_{i-1}}^{b_{k-1}} \Delta t_{i-1,i} \\ 0 & 0 & -\left[\hat{\omega}_{eb_i}^{b_i}\right]^{\times} \Delta t_{i-1,i} & -\Delta t_{i-1,i}\cdot \mathrm{I}_3 & 0 \\ 0 & 0 & 0 & e^{\frac{\Delta t_{i-1,i}}{\tau_g}} & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{\Delta t_{i-1,i}}{\tau_a}} \end{bmatrix} \tag{28}$$

$$\mathrm{G}_{i-1}^{i} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\hat{\mathrm{C}}_{b_{i-1}}^{b_{k-1}} \Delta t_{i-1,i} & 0 & 0 \\ -\Delta t_{i-1,i}\cdot \mathrm{I}_3 & 0 & 0 & 0 \\ 0 & 0 & \Delta t_{i-1,i}\cdot \mathrm{I}_3 & 0 \\ 0 & 0 & 0 & \Delta t_{i-1,i}\cdot \mathrm{I}_3 \end{bmatrix} \tag{29}$$

$$\delta \mathrm{z}_{k-1}^{i} = \begin{bmatrix} \delta\alpha_{k-1}^{i} \\ \delta\beta_{k-1}^{i} \\ \delta\theta_{b_i}^{b_{k-1}} \\ \delta b_{g,t_i} \\ \delta b_{a,t_i} \end{bmatrix}, \delta \mathrm{z}_{k-1}^{i-1} = \begin{bmatrix} \delta\alpha_{k-1}^{i-1} \\ \delta\beta_{k-1}^{i-1} \\ \delta\theta_{b_{i-1}}^{b_{k-1}} \\ \delta b_{g,t_{i-1}} \\ \delta b_{a,t_{i-1}} \end{bmatrix}, \mathrm{u}_{t_i} = \begin{bmatrix} \eta_{b_{g,t_i}} \\ \eta_{b_{a,t_i}} \\ n_{b_{g,t_i}} \\ n_{b_{a,t_i}} \end{bmatrix} \tag{30}$$

IMU 预积分项α_{k-1}^{k}、β_{k-1}^{k}和γ_{k-1}^{k}的协方差矩阵更新：

$$\mathrm{P}_{k-1}^{i} = \Phi_{i-1}^{i} \mathrm{P}_{k-1}^{i-1} \Phi_{i-1}^{iT} + \mathrm{G}_{i-1}^{i} \mathrm{Q}_{t_i} \mathrm{G}_{i-1}^{iT}$$

$$\mathrm{P}_{k-1}^{i} = \Phi_{i-1}^{i} \mathrm{P}_{k-1}^{i-1} \Phi_{i-1}^{iT} + \frac{1}{2}\left(\Phi_{i-1}^{i} \mathrm{G}_{i-1}^{i} \mathrm{Q}_{t_i} \mathrm{G}_{i-1}^{iT} + \Phi_{i-1}^{iT} \mathrm{G}_{i-1}^{iT} \mathrm{Q}_{t_i} \mathrm{G}_{i-1}^{i}\right)$$

Q_{t_i}表示系统噪声矩阵；

$$Q_{t_i} = \mathrm{cov}\begin{bmatrix} \eta_{b_{g,t_i}} \\ \eta_{b_{a,t_i}} \\ n_{b_{g,t_i}} \\ n_{b_{a,t_i}} \end{bmatrix} \tag{31}$$

假设t_k和t_{k-1}时刻，状态参数为

$$\chi_k = \begin{bmatrix} r^e_{eb,k} \\ v^e_{eb,k} \\ C^e_{b_k} \\ b_{g,k} \\ b_{a,k} \end{bmatrix}, \chi_{k-1} = \begin{bmatrix} r^e_{eb,k-1} \\ v^e_{eb,k-1} \\ C^e_{b_{k-1}} \\ b_{g,k-1} \\ b_{a,k-1} \end{bmatrix}$$

t_k和t_{k-1}时刻的 IMU 预积分观测值为

$$\alpha^k_{k-1}, \beta^k_{k-1}, \theta^{k-1}_k$$

$$\beta^k_{k-1} \approx C^{b_{k-1}}_e \left[v^e_{eb,k} - v^e_{eb,k-1} - \left[g^e_{k-1}\Delta t_{k-1,k} - 2\omega^{e\ \times}_{ie}[r^e_{eb,k} - r^e_{eb,k-1}] \right] \right] \tag{32}$$

$$\alpha^k_{k-1} \approx C^{b_{k-1}}_e \left[r^e_{eb,k} - r^e_{eb,k-1} - v^e_{eb,k-1}\Delta t_{k-1,k} - \frac{1}{2} g^e_{k-1}\Delta t^2_{k-1,k} + 2\omega^{e\ \times}_{ie} \sum_{i=1}^{N} [r^e_{eb,i} - r^e_{eb,k-1}]\Delta t_{i-1,i} \right] \tag{33}$$

$$\theta^{k-1}_k = \mathrm{rotv}(C^{b_{k-1}}_e C^e_{b_k}) = \mathrm{rotv}\left(\exp\left(\int_{t_{k-1}}^{t_k} \left(\omega^{b_t\ \times}_{eb_t} \right) dt \right) \right) \tag{34}$$

式中，rotv(·)表示旋转向量；

速度预积分残差为

$$\epsilon^{k,\beta}_{k-1} = \beta^k_{k-1} - \left[C^{b_{k-1}}_e \left[v^e_{eb,k} - v^e_{eb,k-1} - \left[g^e_{k-1}\Delta t_{k-1,k} - 2\omega^{e\ \times}_{ie}[r^e_{eb,k} - r^e_{eb,k-1}] \right] \right] \right] \tag{36}$$

$$\epsilon^{k,\beta}_{k-1} = \hat{\beta}^k_{k-1} + \delta\beta^k_{k-1} - \left[I_3 + \delta\theta^{b_{k-1}\ \times}_e \right] \hat{C}^{b_{k-1}}_e \left[v^e_{eb,k} - v^e_{eb,k-1} - \left[g^e_{k-1}\Delta t_{k-1,k} - 2\omega^{e\ \times}_{ie}[r^e_{eb,k} - r^e_{eb,k-1}] \right] \right] \tag{37}$$

$$\frac{\partial \epsilon^{k,\beta}_{k-1}}{\partial v^e_{eb,k}} = -\hat{C}^{b_{k-1}}_e \tag{38}$$

$$\frac{\partial \epsilon^{k,\beta}_{k-1}}{\partial v^e_{eb,k-1}} = \hat{C}^{b_{k-1}}_e \tag{39}$$

$$\frac{\partial \epsilon^{k,\beta}_{k-1}}{\partial r^e_{eb,k}} = -2\hat{C}^{b_{k-1}}_e \omega^{e\ \times}_{ie} \tag{40}$$

$$\frac{\partial \epsilon^{k,\beta}_{k-1}}{\partial r^e_{eb,k-1}} = 2\hat{C}^{b_{k-1}}_e \omega^{e\ \times}_{ie} \tag{41}$$

$$\frac{\partial \epsilon_{k-1}^{k,\beta}}{\partial \delta\theta_e^{b_{k-1}}} = \left[\hat{C}_e^{b_{k-1}}\left[v_{eb,k}^e - v_{eb,k-1}^e - \left[g_{k-1}^e \Delta t_{k-1,k} - 2\omega_{ie}^{e\times}\left[r_{eb,k}^e - r_{eb,k-1}^e\right]\right]\right]\right]^{\times} \tag{42}$$

$$\frac{\partial \delta z_{k-1}^{k}}{\partial \delta z_{k-1}^{k-1T}} = \begin{bmatrix} \frac{\partial \delta\alpha_{k-1}^{k}}{\partial \delta\alpha_{k-1}^{k-1}} & \frac{\partial \delta\alpha_{k-1}^{k}}{\partial \delta\beta_{k-1}^{k-1}} & \frac{\partial \delta\alpha_{k-1}^{k}}{\partial \delta\theta_{b_{k-1}}^{b_{k-1}}} & \frac{\partial \delta\alpha_{k-1}^{k}}{\partial \delta b_{g,t_{k-1}}} & \frac{\partial \delta\alpha_{k-1}^{k}}{\partial \delta b_{a,t_{k-1}}} \\ \frac{\partial \delta\beta_{k-1}^{k}}{\partial \delta\alpha_{k-1}^{k}} & \frac{\partial \delta\beta_{k-1}^{k}}{\partial \delta\beta_{k-1}^{k-1}} & \frac{\partial \delta\beta_{k-1}^{k}}{\partial \delta\theta_{b_{k-1}}^{b_{k-1}}} & \frac{\partial \delta\beta_{k-1}^{k}}{\partial \delta b_{g,t_{k-1}}} & \frac{\partial \delta\beta_{k-1}^{k}}{\partial \delta b_{a,t_{k-1}}} \\ \frac{\partial \delta\theta_{b_k}^{b_{k-1}}}{\partial \delta\alpha_{k-1}^{k}} & \frac{\partial \delta\theta_{b_k}^{b_{k-1}}}{\partial \delta\beta_{k-1}^{k-1}} & \frac{\partial \delta\theta_{b_k}^{b_{k-1}}}{\partial \delta\theta_{b_{k-1}}^{b_{k-1}}} & \frac{\partial \delta\theta_{b_k}^{b_{k-1}}}{\partial \delta b_{g,t_{k-1}}} & \frac{\partial \delta\theta_{b_k}^{b_{k-1}}}{\partial \delta b_{a,t_{k-1}}} \\ \frac{\partial \delta b_{g,t_k}}{\partial \delta\alpha_{k-1}^{k}} & \frac{\partial \delta b_{g,t_k}}{\partial \delta\beta_{k-1}^{k-1}} & \frac{\partial \delta b_{g,t_k}}{\partial \delta\theta_{b_{k-1}}^{b_{k-1}}} & \frac{\partial \delta b_{g,t_k}}{\partial \delta b_{g,t_{k-1}}} & \frac{\partial \delta b_{g,t_k}}{\partial \delta b_{a,t_{k-1}}} \\ \frac{\partial \delta b_{a,t_k}}{\partial \delta\alpha_{k-1}^{k}} & \frac{\partial \delta b_{a,t_k}}{\partial \delta\beta_{k-1}^{k-1}} & \frac{\partial \delta b_{a,t_k}}{\partial \delta\theta_{b_{k-1}}^{b_{k-1}}} & \frac{\partial \delta b_{a,t_k}}{\partial \delta b_{g,t_{k-1}}} & \frac{\partial \delta b_{a,t_k}}{\partial \delta b_{a,t_{k-1}}} \end{bmatrix} \tag{43}$$

$$\frac{\partial \epsilon_{k-1}^{k,\beta}}{\partial \delta b_{g,t_{k-1}}} = \frac{\partial \delta\beta_{k-1}^{k}}{\partial \delta b_{g,t_{k-1}}} = \left[\Phi_{k-1}^{k}\right]_{2,4} = \left[\Phi_{k-1}^{1}\Phi_1^2\Phi_2^3 \cdot \cdots \cdot \Phi_{k-1}^{N}\right]_{2,4} \tag{44}$$

$$\frac{\partial \epsilon_{k-1}^{k,\beta}}{\partial \delta b_{a,t_{k-1}}} = \frac{\partial \delta\beta_{k-1}^{k}}{\partial \delta b_{a,t_{k-1}}} = \left[\Phi_{k-1}^{k}\right]_{2,5} = \left[\Phi_{k-1}^{1}\Phi_1^2\Phi_2^3 \cdot \cdots \cdot \Phi_{k-1}^{N}\right]_{2,5} \tag{45}$$

姿态预积分残差为

$$\epsilon_{k-1}^{k,\theta} = \theta_k^{k-1} - \left[\mathrm{rotv}[C_e^{b_{k-1}}C_{b_k}^e]\right] = \hat{\theta}_k^{k-1} + \delta\theta_{b_k}^{b_{k-1}} - \left[\mathrm{rotv}[C_e^{b_{k-1}}C_{b_k}^e]\right]$$

$$\frac{\partial \epsilon_{k-1}^{k,\theta}}{\partial \delta\theta_e^{b_{k-1}}} = \frac{\partial \mathrm{rotv}[C_e^{b_{k-1}}C_{b_k}^e]}{\partial \delta\theta_e^{b_{k-1}}} = \frac{\partial \mathrm{rotv}\left[\exp\left(\delta\theta_e^{b_{k-1}\times}\right)\hat{C}_e^{b_{k-1}}C_{b_k}^e\right]}{\partial \delta\theta_e^{b_{k-1}}}$$

$$= \frac{\partial\left[J_l^{-1}\left(\mathrm{rotv}(\hat{C}_e^{b_{k-1}}C_{b_k}^e)\right)\delta\theta_e^{b_{k-1}} + \mathrm{rotv}(\hat{C}_e^{b_{k-1}}C_{b_k}^e)\right]}{\partial \delta\theta_e^{b_{k-1}}} = J_l^{-1}\left(\mathrm{rotv}(\hat{C}_e^{b_{k-1}}C_{b_k}^e)\right) \tag{46}$$

式中，exp(·)表示旋转向量到 DCM 的映射矩阵（SO3 李群与李代数），$J_l^{-1}(\cdot)$表示 SO3 左雅可比矩阵：

$$J_l^{-1}(\boldsymbol{\phi}) = \frac{\phi}{2}\cot\frac{\phi}{2}\begin{bmatrix}1\\1\\1\end{bmatrix} + \left(1 - \frac{\phi}{2}\cot\frac{\phi}{2}\right)aa^T + \frac{\phi}{2}a^{\times}, \boldsymbol{\phi} = a\phi \tag{47}$$

$$C_{b_k}^e = \hat{C}_{b_k}^e \exp\left(-\delta\theta_e^{b_k\times}\right)$$

$$\frac{\partial \epsilon_{k-1}^{k,\theta}}{\partial \delta\theta_e^{b_k}} = \frac{\partial \mathrm{rotv}[C_e^{b_{k-1}}C_{b_k}^e]}{\partial \delta\theta_e^{b_k}} = \frac{\partial \mathrm{rotv}\left[C_e^{b_{k-1}}\hat{C}_{b_k}^e \exp\left(-\delta\theta_e^{b_k\times}\right)\right]}{\partial \delta\theta_e^{b_k}}$$

$$= \frac{\partial\left[-J_r^{-1}\left(\mathrm{rotv}(C_e^{b_{k-1}}\hat{C}_{b_k}^e)\right)\delta\theta_e^{b_k} + \mathrm{rotv}(C_e^{b_{k-1}}\hat{C}_{b_k}^e)\right]}{\partial \delta\theta_e^{b_k}} = -J_r^{-1}\left(\mathrm{rotv}(C_e^{b_{k-1}}\hat{C}_{b_k}^e)\right) \tag{48}$$

$J_r^{-1}(\cdot)$表示 SO3 右雅可比矩阵：

$$J_r^{-1}(\boldsymbol{\phi})=\frac{\phi}{2}\cot\frac{\phi}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}+\left(1-\frac{\phi}{2}\cot\frac{\phi}{2}\right)aa^T-\frac{\phi}{2}a^{\times},\boldsymbol{\phi}=a\phi \tag{49}$$

$$\frac{\partial\epsilon_{k-1}^{k,\theta}}{\partial\delta b_{g,t_{k-1}}}=\frac{\partial\delta\theta_{k-1}^{k}}{\partial\delta b_{g,t_{k-1}}}=\left[\Phi_{k-1}^{k}\right]_{3,4}=\left[\Phi_{k-1}^{1}\Phi_{1}^{2}\Phi_{2}^{3}\cdots\cdots\Phi_{k-1}^{N}\right]_{3,4} \tag{50}$$

$$\frac{\partial\epsilon_{k-1}^{k,\theta}}{\partial\delta b_{a,t_{k-1}}}=\frac{\partial\delta\theta_{k-1}^{k}}{\partial\delta b_{a,t_{k-1}}}=\left[\Phi_{k-1}^{k}\right]_{3,5}=\left[\Phi_{k-1}^{1}\Phi_{1}^{2}\Phi_{2}^{3}\cdots\cdots\Phi_{k-1}^{N}\right]_{3,5} \tag{51}$$

位置预积分残差为

$$\begin{aligned}\epsilon_{k-1}^{k,\alpha}=\alpha_{k-1}^{k}-&\left[C_e^{b_{k-1}}\left[r_{eb,k}^e-r_{eb,k-1}^e-v_{eb,k-1}^e\Delta t_{k-1,k}-\frac{1}{2}g_{k-1}^e\Delta t_{k-1,k}^2\right.\right.\\&\left.\left.+2\omega_{ie}^{e\times}\sum_{i=1}^{N}\left[r_{eb,i}^e-r_{eb,k-1}^e\right]\Delta t_{i-1,i}\right]\right]\\&=\hat{\alpha}_{k-1}^{k}+\delta\alpha_{k-1}^{k}\\&-\left[C_e^{b_{k-1}}\left[r_{eb,k}^e-r_{eb,k-1}^e-v_{eb,k-1}^e\Delta t_{k-1,k}-\frac{1}{2}g_{k-1}^e\Delta t_{k-1,k}^2\right.\right.\\&\left.\left.+2\omega_{ie}^{e\times}\sum_{i=1}^{N}\left[r_{eb,i}^e-r_{eb,k-1}^e\right]\Delta t_{i-1,i}\right]\right]\end{aligned} \tag{52}$$

$$\frac{\partial\epsilon_{k-1}^{k,\alpha}}{\partial r_{eb,k}^e}=-\hat{C}_e^{b_{k-1}} \tag{53}$$

$$\frac{\partial\epsilon_{k-1}^{k,\alpha}}{\partial r_{eb,k-1}^e}=\hat{C}_e^{b_{k-1}}+2N\omega_{ie}^{e\times}\Delta t_{i-1,i} \tag{54}$$

$$\frac{\partial\epsilon_{k-1}^{k,\alpha}}{\partial v_{eb,k-1}^e}=\hat{C}_e^{b_{k-1}}\Delta t_{k-1,k} \tag{55}$$

$$\begin{aligned}\frac{\partial\epsilon_{k-1}^{k,\alpha}}{\partial\delta\theta_e^{b_{k-1}}}=&\left[\hat{C}_e^{b_{k-1}}\left[r_{eb,k}^e-r_{eb,k-1}^e-v_{eb,k-1}^e\Delta t_{k-1,k}-\frac{1}{2}g_{k-1}^e\Delta t_{k-1,k}^2\right.\right.\\&\left.\left.+2\omega_{ie}^{e\times}\sum_{i=1}^{N}\left[r_{eb,i}^e-r_{eb,k-1}^e\right]\Delta t_{i-1,i}\right]\right]^{\times}\end{aligned} \tag{56}$$

$$\frac{\partial\epsilon_{k-1}^{k,\alpha}}{\partial\delta b_{g,t_{k-1}}}=\frac{\partial\delta\alpha_{k-1}^{k}}{\partial\delta b_{g,t_{k-1}}}=\left[\Phi_{k-1}^{k}\right]_{1,4}=\left[\Phi_{k-1}^{1}\Phi_{1}^{2}\Phi_{2}^{3}\cdots\cdots\Phi_{k-1}^{N}\right]_{1,4} \tag{57}$$

$$\frac{\partial\epsilon_{k-1}^{k,\alpha}}{\partial\delta b_{a,t_{k-1}}}=\frac{\partial\delta\alpha_{k-1}^{k}}{\partial\delta b_{a,t_{k-1}}}=\left[\Phi_{k-1}^{k}\right]_{1,5}=\left[\Phi_{k-1}^{1}\Phi_{1}^{2}\Phi_{2}^{3}\cdots\cdots\Phi_{k-1}^{N}\right]_{1,5} \tag{58}$$

对 IMU 加速度计偏差b_a和陀螺仪偏差b_g构建残差：

$$\epsilon_{k-1}^{k,b_g}=b_{g,t_k}-b_{g,t_{k-1}} \tag{59}$$

$$\epsilon_{k-1}^{k,b_a} = b_{a,t_k} - b_{a,t_{k-1}} \tag{60}$$

$$\frac{\partial \epsilon_{k-1}^{k,b_g}}{\partial b_{g,t_{k-1}}} = -\mathrm{I}_3 \tag{61}$$

$$\frac{\partial \epsilon_{k-1}^{k,b_g}}{\partial b_{g,t_k}} = \mathrm{I}_3 \tag{62}$$

$$\frac{\partial \epsilon_{k-1}^{k,b_g}}{\partial b_{a,t_{k-1}}} = -\mathrm{I}_3 \tag{63}$$

$$\frac{\partial \epsilon_{k-1}^{k,b_g}}{\partial b_{a,t_k}} = \mathrm{I}_3 \tag{64}$$

假设在t_k时刻有 GNSS 位置观测值$r_{ea,k}^e$，a 表示 GNSS 天线中心，有

$$\tilde{r}_{eb,k}^e = r_{ea,k}^e + \mathrm{C}_{b_k}^e l_{ab}^b \tag{65}$$

构建 GNSS 位置观测残差：

$$\epsilon_k^r = \tilde{r}_{eb,k}^e - r_{eb,k}^e = r_{ea,k}^e + \mathrm{C}_{b_k}^e l_{ab}^b - r_{eb,k}^e \approx r_{ea,k}^e + \hat{\mathrm{C}}_{b_k}^e \left(\mathrm{I}_3 - \delta\theta_e^{b_k \times}\right) l_{ab}^b - r_{eb,k}^e \tag{66}$$

$$\frac{\partial \epsilon_k^r}{\partial r_{eb,k}^e} = -\mathrm{I}_3, \frac{\partial \epsilon_k^r}{\partial \delta\theta_e^{b_k}} = \hat{\mathrm{C}}_{b_k}^e l_{ab}^{b\,\times} \tag{67}$$

假设在t_k时刻有 GNSS 伪距观测值$\begin{bmatrix} \tilde{\rho}_{a,k}^1 \\ \tilde{\rho}_{a,k}^2 \\ \tilde{\rho}_{a,k}^3 \\ \vdots \\ \tilde{\rho}_{a,k}^n \end{bmatrix}$，构建 GNSS 伪距观测残差：

$$\begin{bmatrix} \epsilon_{\rho,k}^1 \\ \epsilon_{\rho,k}^2 \\ \epsilon_{\rho,k}^3 \\ \vdots \\ \epsilon_{\rho,k}^n \end{bmatrix} = \begin{bmatrix} \tilde{\rho}_{a,k}^1 \\ \tilde{\rho}_{a,k}^2 \\ \tilde{\rho}_{a,k}^3 \\ \vdots \\ \tilde{\rho}_{a,k}^n \end{bmatrix} - \begin{bmatrix} \|r^1 - \hat{r}_{ea,k}^e\| + c(dt^1 - dt_{a,k}) + \mathrm{I}_{a,k}^1 + \mathrm{T}_{a,k}^1 \\ \|r^2 - \hat{r}_{ea,k}^e\| + c(dt^2 - dt_{a,k}) + \mathrm{I}_{a,k}^2 + \mathrm{T}_{a,k}^2 \\ \|r^3 - \hat{r}_{ea,k}^e\| + c(dt^2 - dt_{a,k}) + \mathrm{I}_{a,k}^3 + \mathrm{T}_{a,k}^3 \\ \vdots \\ \|r^n - \hat{r}_{ea,k}^e\| + c(dt^n - dt_{a,k}) + \mathrm{I}_{a,k}^n + \mathrm{T}_{a,k}^n \end{bmatrix} \tag{68}$$

$$\frac{\partial \begin{bmatrix} \epsilon_{\rho,k}^1 \\ \epsilon_{\rho,k}^2 \\ \epsilon_{\rho,k}^3 \\ \vdots \\ \epsilon_{\rho,k}^n \end{bmatrix}}{\partial r_{eb,k}^e} = \frac{\partial \begin{bmatrix} \epsilon_{\rho,k}^1 \\ \epsilon_{\rho,k}^2 \\ \epsilon_{\rho,k}^3 \\ \vdots \\ \epsilon_{\rho,k}^n \end{bmatrix}}{\partial r_{ea,k}^e} \cdot \frac{\partial r_{ea,k}^e}{\partial r_{eb,k}^e} = \begin{bmatrix} e_k^{1T} \\ e_k^{2T} \\ e_k^{3T} \\ \vdots \\ e_k^{nT} \end{bmatrix} \tag{69}$$

$$\frac{\partial\begin{bmatrix}\epsilon_{\rho,k}^{1}\\ \epsilon_{\rho,k}^{2}\\ \epsilon_{\rho,k}^{3}\\ \vdots\\ \epsilon_{\rho,k}^{n}\end{bmatrix}}{\partial\delta\theta_{e}^{b_k}}=\frac{\partial\begin{bmatrix}\epsilon_{\rho,k}^{1}\\ \epsilon_{\rho,k}^{2}\\ \epsilon_{\rho,k}^{3}\\ \vdots\\ \epsilon_{\rho,k}^{n}\end{bmatrix}}{\partial r_{ea,k}^{e}}\frac{\partial r_{ea,k}^{e}}{\partial\delta\theta_{e}^{b_k}}=\begin{bmatrix}-e_k^{1T}\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b}{}^{\times}\\ -e_k^{2T}\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b}{}^{\times}\\ -e_k^{3T}\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b}{}^{\times}\\ \vdots\\ -e_k^{nT}\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b}{}^{\times}\end{bmatrix}\tag{70}$$

假设在t_k时刻有 GNSS 载波相位观测值$\begin{bmatrix}\tilde{\varphi}_{a,k}^{1}\\ \tilde{\varphi}_{a,k}^{2}\\ \tilde{\varphi}_{a,k}^{3}\\ \vdots\\ \tilde{\varphi}_{a,k}^{n}\end{bmatrix}$，构建 GNSS 载波相位/伪距双差观测残差：

$$\begin{bmatrix}\epsilon_{\varphi,k}^{12}\\ \epsilon_{\varphi,k}^{13}\\ \epsilon_{\varphi,k}^{14}\\ \vdots\\ \epsilon_{\varphi,k}^{1m}\end{bmatrix}=\begin{bmatrix}\tilde{\varphi}_{a,k}^{12}\\ \tilde{\varphi}_{a,k}^{13}\\ \tilde{\varphi}_{a,k}^{14}\\ \vdots\\ \tilde{\varphi}_{a,k}^{1m}\end{bmatrix}-\begin{bmatrix}r_{ab}^{12}-\mathrm{I}_{ab,k}^{12}+\mathrm{T}_{ab,k}^{12}+\lambda^1 B_{ab}^{1}-\lambda^2 B_{ab}^{2}\\ r_{ab}^{13}-\mathrm{I}_{ab,k}^{13}+\mathrm{T}_{ab,k}^{13}+\lambda^1 B_{ab}^{1}-\lambda^3 B_{ab}^{3}\\ r_{ab}^{14}-\mathrm{I}_{ab,k}^{14}+\mathrm{T}_{ab,k}^{14}+\lambda^1 B_{ab}^{1}-\lambda^4 B_{ab}^{4}\\ \vdots\\ r_{ab}^{1m}-\mathrm{I}_{ab,k}^{1m}+\mathrm{T}_{ab,k}^{1m}+\lambda^1 B_{ab}^{1}-\lambda^m B_{ab}^{m}\end{bmatrix}\tag{71}$$

$$\begin{bmatrix}\epsilon_{\rho,k}^{12}\\ \epsilon_{\rho,k}^{13}\\ \epsilon_{\rho,k}^{14}\\ \vdots\\ \epsilon_{\rho,k}^{1m}\end{bmatrix}=\begin{bmatrix}\tilde{\rho}_{a,k}^{12}\\ \tilde{\rho}_{a,k}^{13}\\ \tilde{\rho}_{a,k}^{14}\\ \vdots\\ \tilde{\rho}_{a,k}^{1m}\end{bmatrix}-\begin{bmatrix}r_{ab}^{12}+\mathrm{I}_{ab,k}^{12}+\mathrm{T}_{ab,k}^{12}\\ r_{ab}^{13}+\mathrm{I}_{ab,k}^{13}+\mathrm{T}_{ab,k}^{13}\\ r_{ab}^{14}+\mathrm{I}_{ab,k}^{14}+\mathrm{T}_{ab,k}^{14}\\ \vdots\\ r_{ab}^{1m}+\mathrm{I}_{ab,k}^{1m}+\mathrm{T}_{ab,k}^{1m}\end{bmatrix}\tag{72}$$

$$\frac{\partial\begin{bmatrix}\epsilon_{\varphi,k}^{12}\\ \epsilon_{\varphi,k}^{13}\\ \epsilon_{\varphi,k}^{14}\\ \vdots\\ \epsilon_{\varphi,k}^{1m}\end{bmatrix}}{\partial r_{eb,k}^{e}}=\frac{\partial\begin{bmatrix}\epsilon_{\varphi,k}^{12}\\ \epsilon_{\varphi,k}^{13}\\ \epsilon_{\varphi,k}^{14}\\ \vdots\\ \epsilon_{\varphi,k}^{1m}\end{bmatrix}}{\partial r_{ea,k}^{e}}\cdot\frac{\partial r_{ea,k}^{e}}{\partial r_{eb,k}^{e}}=\begin{bmatrix}e_k^{1T}-e_k^{2T}\\ e_k^{1T}-e_k^{3T}\\ e_k^{1T}-e_k^{4T}\\ \vdots\\ e_k^{1T}-e_k^{nT}\end{bmatrix}\tag{73}$$

$$\frac{\partial\begin{bmatrix}\epsilon_{\varphi,k}^{12}\\ \epsilon_{\varphi,k}^{13}\\ \epsilon_{\varphi,k}^{14}\\ \vdots\\ \epsilon_{\varphi,k}^{1m}\end{bmatrix}}{\partial[B_{ab}^{1}\quad B_{ab}^{2}\quad B_{ab}^{3}\quad B_{ab}^{4}\quad\cdots\quad B_{ab}^{m}]}=\begin{bmatrix}\lambda^1 & -\lambda^2 & 0 & 0 & \cdots & 0\\ \lambda^1 & 0 & -\lambda^3 & 0 & \cdots & 0\\ \lambda^1 & 0 & 0 & -\lambda^4 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \lambda^1 & 0 & 0 & 0 & 0 & \lambda^m\end{bmatrix}\tag{74}$$

$$\frac{\partial\begin{bmatrix}\epsilon_{\rho,k}^{12}\\ \epsilon_{\rho,k}^{13}\\ \epsilon_{\rho,k}^{14}\\ \vdots\\ \epsilon_{\rho,k}^{1m}\end{bmatrix}}{\partial r_{eb,k}^{e}}=\frac{\partial\begin{bmatrix}\epsilon_{\rho,k}^{12}\\ \epsilon_{\rho,k}^{13}\\ \epsilon_{\rho,k}^{14}\\ \vdots\\ \epsilon_{\rho,k}^{1m}\end{bmatrix}}{\partial r_{ea,k}^{e}}\cdot\frac{\partial r_{ea,k}^{e}}{\partial r_{eb,k}^{e}}=\begin{bmatrix}e_k^{1T}-e_k^{2T}\\ e_k^{1T}-e_k^{3T}\\ e_k^{1T}-e_k^{4T}\\ \vdots\\ e_k^{1T}-e_k^{nT}\end{bmatrix} \tag{75}$$

$$\frac{\partial\begin{bmatrix}\epsilon_{\varphi,k}^{12}\\ \epsilon_{\varphi,k}^{13}\\ \epsilon_{\varphi,k}^{14}\\ \vdots\\ \epsilon_{\varphi,k}^{1m}\end{bmatrix}}{\partial\delta\theta_{e}^{b_k}}=\frac{\partial\begin{bmatrix}\epsilon_{\varphi,k}^{12}\\ \epsilon_{\varphi,k}^{13}\\ \epsilon_{\varphi,k}^{14}\\ \vdots\\ \epsilon_{\varphi,k}^{1m}\end{bmatrix}}{\partial r_{ea,k}^{e}}\frac{\partial r_{ea,k}^{e}}{\partial\delta\theta_{e}^{b_k}}=\begin{bmatrix}-(e_k^{1T}-e_k^{2T})\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b\times}\\ -(e_k^{1T}-e_k^{3T})\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b\times}\\ -(e_k^{1T}-e_k^{4T})\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b\times}\\ \vdots\\ -(e_k^{1T}-e_k^{nT})\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b\times}\end{bmatrix} \tag{76}$$

$$\frac{\partial\begin{bmatrix}\epsilon_{\rho,k}^{12}\\ \epsilon_{\rho,k}^{13}\\ \epsilon_{\rho,k}^{14}\\ \vdots\\ \epsilon_{\rho,k}^{1m}\end{bmatrix}}{\partial\delta\theta_{e}^{b_k}}=\frac{\partial\begin{bmatrix}\epsilon_{\rho,k}^{12}\\ \epsilon_{\rho,k}^{13}\\ \epsilon_{\rho,k}^{14}\\ \vdots\\ \epsilon_{\rho,k}^{1m}\end{bmatrix}}{\partial r_{ea,k}^{e}}\frac{\partial r_{ea,k}^{e}}{\partial\delta\theta_{e}^{b_k}}=\begin{bmatrix}-(e_k^{1T}-e_k^{2T})\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b\times}\\ -(e_k^{1T}-e_k^{3T})\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b\times}\\ -(e_k^{1T}-e_k^{4T})\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b\times}\\ \vdots\\ -(e_k^{1T}-e_k^{nT})\cdot\hat{C}_{b_k}^{e}\cdot l_{ab}^{b\times}\end{bmatrix} \tag{77}$$

INS 载体坐标系 b 与导航坐标系 n 的转换, 可以通过三个轴旋转角度 roll=α, pitch=β, yaw=φ 计算得到:

绕 z 轴旋转φ，旋转矩阵为

$$C_z(\varphi)=\begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix} \tag{78}$$

绕 y 轴旋转β，旋转矩阵为

$$C_y(\beta)=\begin{bmatrix}\cos\beta & 0 & -\sin\beta\\ 0 & 1 & 0\\ \sin\beta & 0 & \cos\beta\end{bmatrix} \tag{79}$$

绕 x 轴旋转α，旋转矩阵为

$$C_x(\alpha)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix} \tag{80}$$

$$C_b^n=C_z(\varphi)C_y(\beta)C_x(\alpha)=\begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\beta & 0 & -\sin\beta\\ 0 & 1 & 0\\ \sin\beta & 0 & \cos\beta\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix} \tag{81}$$

$$C_e^n=C_y\left(-\phi-\frac{\pi}{2}\right)C_z(\lambda)$$ (82)、

上式中，ϕ表示纬度，λ表示经度;

对 IMU 预积分α_{k-1}^{k}、β_{k-1}^{k}和γ_{k-1}^{k}的离散迭代计算进行精度优化：

$$\gamma_{k-1}^{i} = \gamma_{k-1}^{i-1} C_{b_i}^{b_{i-1}} = \gamma_{k-1}^{i-1} \ \exp\left(\left(\int_{t_{i-1}}^{t_i} \dot{\rho}_{i-1}^{i} \, dt\right)^{\times}\right) \tag{83}$$

式中，ρ_{i-1}^{i}表示时刻 i-1 到时刻 i 载体坐标系 b 的旋转向量，$\dot{\rho}_{i-1}^{i}$为其关于时间的导数；

$$\begin{aligned}\dot{\rho}_{i}^{i-1} &= \omega_{eb}^{b} + \frac{1}{2}\rho_{i}^{i-1} \times \omega_{eb}^{b} + \frac{1}{\|\rho_{i}^{i-1}\| \cdot \rho_{i}^{i-1}}\left(1 - \frac{\|\rho_{i}^{i-1}\| \sin(\|\rho_{i}^{i-1}\|)}{2\left(1-\cos(\|\rho_{i}^{i-1}\|)\right)}\right) \rho_{i}^{i-1\times} \rho_{i}^{i-1\times} \omega_{eb}^{b} \\ &\approx \omega_{eb}^{b} + \frac{1}{2}\rho_{i}^{i-1} \times \omega_{eb}^{b} + \frac{1}{12}\rho_{i}^{i-1\times} \rho_{i}^{i-1\times} \omega_{eb}^{b} \approx \omega_{eb}^{b} + \frac{1}{2}\rho_{i}^{i-1} \times \omega_{eb}^{b} \\ &\approx \omega_{eb}^{b} + \frac{1}{2}\int_{t_{i-1}}^{t_i} \omega_{eb}^{b} \, dt \times \omega_{eb}^{b}\end{aligned} \tag{84}$$

假设在时刻 i-1 和时刻 i ω_{eb}^{b}是线性变换的，则有

$$\begin{aligned}\rho_{i}^{i-1} &\approx \int_{t_{i-1}}^{t_i}\left(\omega_{eb_t}^{b_t} + \frac{1}{2}\int_{t_{i-1}}^{t_i} \omega_{eb_t}^{b_t} \, dt \times \omega_{eb_t}^{b_t}\right) dt \approx \int_{t_{i-1}}^{t_i} \omega_{eb_t}^{b_t} \, dt + \frac{1}{12}\int_{t_{i-2}}^{t_{i-1}} \omega_{eb_t}^{b_t} \, dt \times \int_{t_{i-1}}^{t_i} \omega_{eb_t}^{b_t} \, dt \\ &\approx \omega_{eb_i}^{b_i} \Delta t_{i-1,i} + \frac{1}{12}\omega_{eb_{i-1}}^{b_{i-1}} \Delta t_{i-2,i-1} \times \omega_{eb_i}^{b_i} \Delta t_{i-1,i}\end{aligned} \tag{85}$$

$$\beta_{k-1}^{i} = \int_{t_{k-1}}^{t_k} C_{b_t}^{b_{t_{k-1}}} f_{ib_t}^{b_t} dt = \beta_{k-1}^{i-1} + \underbrace{C_{b_{t_{i-1}}}^{b_{t_{k-1}}} \int_{t_{i-1}}^{t_i} C_{b_t}^{b_{t_{i-1}}} f_{ib_t}^{b_t} dt}_{\xi(t_i)} \tag{86}$$

$$\beta_{k-1}^{i} = \beta_{k-1}^{i-1} + \gamma_{k-1}^{i-1} \int_{t_{i-1}}^{t_i} C_{b_t}^{b_{t_{i-1}}} f_{ib_t}^{b_t} dt \tag{87}$$

$$\begin{aligned}\int_{t_{i-1}}^{t_i} C_{b_t}^{b_{t_{i-1}}} f_{ib_t}^{b_t} dt &\approx \int_{t_{i-1}}^{t_i} f_{ib_t}^{b_t} \, dt + \frac{1}{2}\int_{t_{i-1}}^{t_i} \omega_{eb_t}^{b_t} dt \times \int_{t_{i-1}}^{t_i} f_{ib_t}^{b_t} \, dt \\ &+ \frac{1}{12}\left(\int_{t_{i-2}}^{t_{i-1}} \omega_{eb_t}^{b_t} dt \times \int_{t_{i-1}}^{t_i} f_{ib_t}^{b_t} dt + \int_{t_{i-2}}^{t_{i-1}} f_{ib_t}^{b_t} dt \times \int_{t_{i-1}}^{t_i} \omega_{eb_t}^{b_t} dt\right) \\ &\approx f_{ib_i}^{b_i} \Delta t_{i-1,i} + \frac{1}{2} f_{ib_i}^{b_i} \Delta t_{i-1,i} \times \omega_{eb_i}^{b_i} \Delta t_{i-1,i} \\ &+ \frac{1}{12}\left(\omega_{eb_{i-1}}^{b_{i-1}} \Delta t_{i-2,i-1} \times f_{ib_i}^{b_i} \Delta t_{i-1,i} + f_{ib_{i-1}}^{b_{i-1}} \Delta t_{i-2,i-1} \times \omega_{eb_i}^{b_i} \Delta t_{i-1,i}\right)\end{aligned} \tag{88}$$

$$
\begin{aligned}
\alpha_{k-1}^{i} &= \iint_{t_{k-1}}^{t_i} C_{b_t}^{b_{t_{k-1}}} f_{ib_t}^{b_t} dt = \int_{t_{k-1}}^{t_i} \beta_{k-1}^{t} dt = \int_{t_{k-1}}^{t_{i-1}} \beta_{k-1}^{t} dt + \int_{t_{i-1}}^{t_i} \beta_{k-1}^{t} dt = \alpha_{k-1}^{i-1} + \int_{t_{i-1}}^{t_i} \beta_{k-1}^{t} dt \\
&= \alpha_{k-1}^{i-1} + \int_{t_{i-1}}^{t_i} \left(\beta_{k-1}^{i-1} + \int_{t_{i-1}}^{t} C_{b_\tau}^{b_{t_{k-1}}} f_{ib_\tau}^{b_\tau} d\tau \right) dt \\
&\approx \alpha_{k-1}^{i-1} + \beta_{k-1}^{i-1} \Delta t_{i-1,i} + \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} C_{b_\tau}^{b_{t_{k-1}}} f_{ib_\tau}^{b_\tau} d\tau\, dt \\
&\approx \alpha_{k-1}^{i-1} + \beta_{k-1}^{i-1} \Delta t_{i-1,i} + \int_{t_{i-1}}^{t_i} \xi(t) dt \\
&\approx \alpha_{k-1}^{i-1} + \beta_{k-1}^{i-1} \Delta t_{i-1,i} + \frac{\xi(t_i) + \xi(t_{i-1})}{2} \Delta t_{i-1,i} \\
&= \alpha_{k-1}^{i-1} + \beta_{k-1}^{i-1} \Delta t_{i-1,i} + \frac{\xi(t_i)}{2} \Delta t_{i-1,i}
\end{aligned}
\tag{89}
$$

当t_{k-1}时刻的零偏更新时，IMU 预积分同样需要更新：

$$
\hat{\alpha}_{k-1}^{k} = \alpha_{k-1}^{k} + \delta\alpha_{k-1}^{k} = \alpha_{k-1}^{k} + \frac{\partial \delta\alpha_{k-1}^{k}}{\partial \delta b_{g,t_{k-1}}} \delta b_{g,t_{k-1}} + \frac{\partial \delta\alpha_{k-1}^{k}}{\partial \delta b_{a,t_{k-1}}} \delta b_{a,t_{k-1}} \tag{90}
$$

$$
\hat{\beta}_{k-1}^{k} = \beta_{k-1}^{k} + \delta\beta_{k-1}^{k} = \beta_{k-1}^{k} + \frac{\partial \delta\beta_{k-1}^{k}}{\partial \delta b_{g,t_{k-1}}} \delta b_{g,t_{k-1}} + \frac{\partial \delta\beta_{k-1}^{k}}{\partial \delta b_{a,t_{k-1}}} \delta b_{a,t_{k-1}} \tag{91}
$$

$$
\hat{\theta}_{k}^{k-1} = \theta_{k}^{k-1} + \delta\theta_{b_k}^{b_{k-1}} = \theta_{k}^{k-1} + \frac{\partial \delta\theta_{b_k}^{b_{k-1}}}{\partial \delta b_{g,t_{k-1}}} \delta b_{g,t_{k-1}} \tag{92}
$$

假设$q_e^{b_k} \triangleq C_e^{b_k}$，$q_e^{b_{k-1}} \triangleq C_e^{b_{k-1}}$，则有

$$
q_e^{b_{k-1}} = \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta_e^{b_{k-1}} \end{bmatrix} \otimes \hat{q}_e^{b_{k-1}} \tag{93}
$$

$$
q_e^{b_k} = \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta_e^{b_k} \end{bmatrix} \otimes \hat{q}_e^{b_k}, q_{b_k}^{e} = \hat{q}_{b_k}^{e} \otimes \begin{bmatrix} 1 \\ -\frac{1}{2}\delta\theta_e^{b_k} \end{bmatrix} \tag{94}
$$

$$
\epsilon_{k-1}^{k,\theta} = \theta_k^{k-1} - \left[\mathrm{rotv}[q_e^{b_{k-1}} \otimes q_{b_k}^{e}]\right] = \hat{\theta}_k^{k-1} + \delta\theta_{b_k}^{b_{k-1}} - \left[\mathrm{rotv}[C_e^{b_{k-1}} C_{b_k}^{e}]\right] \tag{95}
$$

$$
\begin{aligned}
\frac{\partial \epsilon_{k-1}^{k,\theta}}{\partial \delta\theta_e^{b_{k-1}}} &= \frac{\partial \mathrm{rotv}[q_e^{b_{k-1}} \otimes q_{b_k}^{e}]}{\partial \delta\theta_e^{b_{k-1}}} = \frac{\partial \mathrm{rotv}\left[\begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta_e^{b_{k-1}} \end{bmatrix} \otimes \hat{q}_e^{b_{k-1}} \otimes q_{b_k}^{e}\right]}{\partial \delta\theta_e^{b_{k-1}}} \\
&= \frac{\partial \mathrm{rotv}\left[\begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta_e^{b_{k-1}} \end{bmatrix} \otimes \hat{q}_{b_k}^{b_{k-1}}\right]}{\partial \delta\theta_e^{b_{k-1}}} = \frac{\partial \mathrm{rotv}\left[\Omega_R\left(\hat{q}_{b_k}^{b_{k-1}}\right) \cdot \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta_e^{b_{k-1}} \end{bmatrix}\right]}{\partial \delta\theta_e^{b_{k-1}}} \\
&= \left[\frac{1}{2}\Omega_R\left(\hat{q}_{b_k}^{b_{k-1}}\right)\right]_{(2,2):(4,4)}
\end{aligned}
\tag{96}
$$

$$\frac{\partial \epsilon_{k-1}^{k,\theta}}{\partial \delta\theta_e^{b_k}} = \frac{\partial \mathrm{rotv}[\mathrm{q}_e^{b_{k-1}} \otimes \mathrm{q}_{b_k}^e]}{\partial \delta\theta_e^{b_k}} = \frac{\partial \mathrm{rotv}\left[\mathrm{q}_e^{b_{k-1}} \otimes \hat{\mathrm{q}}_{b_k}^e \otimes \begin{bmatrix} 1 \\ -\frac{1}{2}\delta\theta_e^{b_k} \end{bmatrix}\right]}{\partial \delta\theta_e^{b_k}}$$

$$= \frac{\partial \mathrm{rotv}\left[\Omega_L\left(\hat{\mathrm{q}}_{b_k}^{b_{k-1}}\right)\begin{bmatrix} 1 \\ -\frac{1}{2}\delta\theta_e^{b_k} \end{bmatrix}\right]}{\partial \delta\theta_e^{b_k}} = -\left[\frac{1}{2}\Omega_L\left(\hat{\mathrm{q}}_{b_k}^{b_{k-1}}\right)\right]_{(2,2):(4,4)} \tag{97}$$

假设优化滑动窗口大小为 N，在t_k时刻的滑动窗口为

$$\{\mathrm{t}_{k-N}, \dots, \mathrm{t}_{k-3}, \mathrm{t}_{k-2}, \mathrm{t}_{k-1}, \mathrm{t}_k\} \tag{98}$$

对应的状态参数为

$$\chi_{k-N,k} = \{\chi_{k-N}, \cdots, \chi_{k-3}, \chi_{k-2}, \chi_{k-1}, \chi_k\} \tag{99}$$

在t_k时刻利用观测信息对滑动窗口内的参数进行优化，可得

$$\Lambda \cdot \chi_{k-N,k} = \epsilon \tag{100}$$

$$\chi_{k-N,k} = \begin{bmatrix} \chi_{k-N} \\ \vdots \\ \chi_{k-3} \\ \chi_{k-2} \\ \chi_{k-1} \\ \chi_k \end{bmatrix} = \begin{bmatrix} \chi_{k-N} \\ \chi_{k-N+1,k} \end{bmatrix} \tag{101}$$

$$\Lambda = \begin{bmatrix} \Lambda_{k-N} & \Lambda_a \\ \Lambda_a^T & \Lambda_{\chi_{k-N+1,k}} \end{bmatrix} \tag{102}$$

$$\epsilon = \begin{bmatrix} \epsilon_{k-N} \\ \epsilon_{k-N+1,k} \end{bmatrix} \tag{103}$$

$$\begin{bmatrix} \Lambda_{k-N} & \Lambda_a \\ \Lambda_a^T & \Lambda_{\chi_{k-N+1,k}} \end{bmatrix} \begin{bmatrix} \chi_{k-N} \\ \chi_{k-N+1,k} \end{bmatrix} = \begin{bmatrix} \epsilon_{k-N} \\ \epsilon_{k-N+1,k} \end{bmatrix} \tag{104}$$

对上式进行消元可得

$$\underbrace{\left(\Lambda_{\chi_{k-N+1,k}} - \Lambda_a^T \Lambda_{k-N}^{-1} \Lambda_a\right)}_{\hat{\Lambda}_{\chi_{k-N+1,k}}} \cdot \chi_{k-N+1,k} = \underbrace{\epsilon_{k-N+1,k} - \Lambda_a^T \Lambda_{k-N}^{-1} \epsilon_{k-N}}_{\hat{\epsilon}_{k-N+1,k}} \tag{105}$$

$$\hat{\Lambda}_{\chi_{k-N+1,k}} \cdot \chi_{k-N+1,k} = \hat{\epsilon}_{k-N+1,k} \tag{106}$$

在t_{k+1}时刻，消去t_{k-N}时刻的状态参数χ_{k-N}，则滑动窗口为

$$\{\mathrm{t}_{k-N+1}, \dots, \mathrm{t}_{k-3}, \mathrm{t}_{k-2}, \mathrm{t}_{k-1}, \mathrm{t}_{k+1}\} \tag{107}$$

对应的状态参数为

$$\chi_{k-N+1,k+1} = \{\chi_{k-N+1}, \cdots, \chi_{k-3}, \chi_{k-2}, \chi_{k-1}, \chi_{k+1}\} \tag{108}$$

利用该时刻的观测信息对滑动窗口内的参数进行优化，可得

$$\Lambda_{k+1} \cdot \chi_{k-N+1,k+1} = \epsilon_{k+1} \tag{109}$$

结合式(107)和式(110)，可得如下滑动窗口边缘化的优化方程：

$$\begin{bmatrix} \hat{\Lambda}_{\chi_{k-N+1,k}} & 0 \\ & \Lambda_{k+1} \end{bmatrix} \begin{bmatrix} \chi_{k-N+1,k} \\ \chi_{k+1} \end{bmatrix} = \begin{bmatrix} \hat{\epsilon}_{k-N+1,k} \\ \epsilon_{k+1} \end{bmatrix} \tag{110}$$

假设在t_i时刻有一矢量r，将其表示在坐标系$\mathcal{F}^i$，即

$$r = \underbrace{[i_1 \quad i_2 \quad i_3]}_{\mathcal{F}^{iT}} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{111}$$

在t_j时刻由旋转矢量变换至r'，将其表示在坐标系$\mathcal{F}^j$，即

$$r' = \underbrace{[j_1 \quad j_2 \quad j_3]}_{\mathcal{F}^{jT}} \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} \tag{112}$$

$$r' = \mathrm{D}(\rho_j^i) \cdot r \tag{113}$$

$$\mathrm{D}(\rho_j^i) = \mathrm{I}_3 + \frac{\sin(\|\rho_j^i\|)}{\|\rho_j^i\|}\rho_j^{i\times} + \frac{1-\cos(\|\rho_j^i\|)}{\|\rho_j^i\|^2}\rho_j^{i\times}\rho_j^{i\times} \tag{114}$$

$$[i_1 \quad i_2 \quad i_3]\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = [j_1 \quad j_2 \quad j_3]\begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} \Rightarrow \mathcal{F}^{iT}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \mathcal{F}^{jT}\begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} \Rightarrow \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \mathcal{F}^{i}\mathcal{F}^{jT}\begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} \tag{115}$$

$$\mathrm{D}(\rho_j^i) = \mathcal{F}^i\mathcal{F}^{jT} \triangleq \mathrm{C}_j^i \tag{116}$$

式中，$\mathrm{D}(\rho_j^i)$为罗德里格旋转公式，C_j^i表示t_j时刻到t_i时刻的坐标旋转矩阵；

$$\mathrm{C}_{b_{t_i}}^{e} = \mathrm{C}_{b_l(t_i)}^{e_l(t_i)} = \mathrm{C}_{e_l(t_{i-1})}^{e_l(t_i)}\mathrm{C}_{b_l(t_{i-1})}^{e_l(t_{i-1})}\mathrm{C}_{b_l(t_i)}^{b_l(t_{i-1})} = \mathrm{C}_{e_l(t_{i-1})}^{e_l(t_i)}\mathrm{C}_{b_{t_{i-1}}}^{e}\mathrm{C}_{b_l(t_i)}^{b_l(t_{i-1})} \tag{117}$$

$$\mathrm{C}_{b_l(t_{i-1})}^{e_l(t_{i-1})} = \mathrm{C}_{b_{t_{i-1}}}^{e} \tag{118}$$

$$\mathrm{C}_{e_l(t_{i-1})}^{e_l(t_i)} = \mathrm{D}\left(\rho_{e_l(t_{i-1})}^{e_l(t_i)}\right) = \mathrm{D}\left(\begin{bmatrix} 0 \\ 0 \\ -\omega_{ie}^{e}\Delta t_{i-1,i} \end{bmatrix}\right) \tag{119}$$

$$\mathrm{C}_{b_l(t_i)}^{b_l(t_{i-1})} = \mathrm{D}\left(\rho_{b_l(t_i)}^{b_l(t_{i-1})}\right) = \mathrm{D}\left(\int_{t_{i-1}}^{t_i} \omega_{ib_t}^{b_t}\, dt\right) \tag{120}$$

$$\mathrm{C}_{b_t}^{b_{t_{k-1}}} = \mathrm{C}_{e}^{b_{t_{k-1}}}\mathrm{C}_{b_t}^{e} = \mathrm{C}_{e}^{b_{t_{k-1}}}\mathrm{C}_{e_l(t_{k-1})}^{e_l(t)}\mathrm{C}_{b_{t_{k-1}}}^{e}\mathrm{C}_{b_l(t)}^{b_l(t_{k-1})} \tag{121}$$

假设里程计坐标系为 v，在t_{k-1}和t_k时刻之间里程计的测量值为$\mathrm{s}_{v_k}^{v_{k-1}}$，在$t_{k-1}$和$t_k$时刻 b 系相对于$t_{k-1}$时刻的位置增量可表示为

$$\mathrm{s}_{v_k}^{v_{k-1}} = \int_{t_{k-1}}^{t_k} v_{ev_t}^{v_{k-1}}\, dt, \mathrm{s}_{b_k}^{b_{k-1}} = \int_{t_{k-1}}^{t_k} v_{eb_t}^{b_{k-1}} dt \tag{122}$$

$$v_{ev_t}^{e} = v_{eb_t}^{e} + \mathrm{C}_{b_t}^{e}\omega_{eb_t}^{b_t\times}\mathrm{l}_{bv}^{b} \tag{123}$$

式中，l_{bv}^{b}表示里程计坐标系为 v 与 b 系的杆臂关系，结合上述两式可得：

$$\begin{aligned}
\mathrm{s}_{b_k}^{b_{k-1}} &= \int_{t_{k-1}}^{t_k} \mathrm{C}_{e}^{b_{k-1}} v_{eb_t}^{e} dt = \int_{t_{k-1}}^{t_k} \mathrm{C}_{e}^{b_{k-1}}\left(v_{ev_t}^{e} - \mathrm{C}_{b_t}^{e}\omega_{eb_t}^{b_t\times}\mathrm{l}_{bv}^{b}\right) dt \\
&= \int_{t_{k-1}}^{t_k} \mathrm{C}_{b_t}^{b_{k-1}}\mathrm{C}_{v}^{b} v_{ev_t}^{v_t} dt - \mathrm{C}_{e}^{b_{k-1}}\int_{t_{k-1}}^{t_k} \mathrm{C}_{b_t}^{e}\omega_{eb_t}^{b_t\times}\mathrm{l}_{bv}^{b} dt \\
&\approx \int_{t_{k-1}}^{t_k} \mathrm{C}_{b_{k-1}}^{b_{k-1}}\mathrm{C}_{v}^{b} v_{ev_t}^{v_{k-1}} dt - \mathrm{C}_{e}^{b_{k-1}}\int_{t_{k-1}}^{t_k} \mathrm{C}_{b_t}^{e}\omega_{eb_t}^{b_t\times}\mathrm{l}_{bv}^{b} dt \\
&= \mathrm{C}_{v}^{b}\int_{t_{k-1}}^{t_k} v_{ev_t}^{v_{k-1}} dt - \mathrm{C}_{e}^{b_{k-1}}\int_{t_{k-1}}^{t_k} \dot{\mathrm{C}}_{b_t}^{e}\mathrm{l}_{bv}^{b} dt = \mathrm{C}_{v}^{b}\mathrm{s}_{v_k}^{v_{k-1}} - \mathrm{C}_{e}^{b_{k-1}}(\mathrm{C}_{b_k}^{e} - \mathrm{C}_{b_{k-1}}^{e})\mathrm{l}_{bv}^{b} \\
&= \mathrm{C}_{v}^{b}\mathrm{s}_{v_k}^{v_{k-1}} - \mathrm{C}_{b_k}^{b_{k-1}}\mathrm{l}_{bv}^{b} + \mathrm{l}_{bv}^{b}
\end{aligned}$$

(124)

$$r_{eb_k}^{e} = r_{eb_{k-1}}^{e} + \mathrm{C}_{b_{k-1}}^{e}\mathrm{s}_{b_k}^{b_{k-1}} \approx r_{eb_{k-1}}^{e} + \mathrm{C}_{b_{k-1}}^{e}\left(\mathrm{C}_{v}^{b}\mathrm{s}_{v_k}^{v_{k-1}} - \mathrm{C}_{b_k}^{b_{k-1}}\mathrm{l}_{bv}^{b} + \mathrm{l}_{bv}^{b}\right) \tag{125}$$

$$\dot{\mathrm{s}}_{b_k}^{b_{k-1}} = v_{eb_t}^{b_{k-1}} = \mathrm{C}_{b_t}^{b_{k-1}}\mathrm{C}_{v}^{b}v_{ev_t}^{v_t} - \mathrm{C}_{b_t}^{b_{k-1}}{\omega_{eb_t}^{b_t}}^{\times}\mathrm{l}_{bv}^{b} \tag{126}$$

考虑到里程计比例因子则有

$$\begin{aligned}\dot{\mathrm{s}}_{b_k}^{b_{k-1}} &= \mathrm{C}_{b_t}^{b_{k-1}}\mathrm{C}_{v}^{b}v_{ev_t}^{v_t}(1+s_o) - \mathrm{C}_{b_t}^{b_{k-1}}{\omega_{eb_t}^{b_t}}^{\times}\mathrm{l}_{bv}^{b} \\ &= \mathrm{C}_{b_t}^{b_{k-1}}\mathrm{C}_{v}^{b}v_{ev_t}^{v_t}(1+s_o) - \mathrm{C}_{b_t}^{b_{k-1}}\left(\omega_{ib_t}^{b_t} - \mathrm{C}_{e}^{b_t}\omega_{ie}^{e}\right)^{\times}\mathrm{l}_{bv}^{b}\end{aligned} \tag{126}$$

假设有

$$\mathrm{s}_{b_k}^{b_{k-1}} = \hat{\mathrm{s}}_{b_k}^{b_{k-1}} + \delta\mathrm{s}_{b_k}^{b_{k-1}} \tag{127}$$

$$v_{ev_t}^{v_t} = \hat{v}_{ev_t}^{v_t} + \delta v_{ev_t}^{v_t} \tag{128}$$

$$s_o = \hat{s}_o + \delta s_o \tag{129}$$

$$\mathrm{C}_{b_t}^{b_{k-1}} = \hat{\mathrm{C}}_{b_t}^{b_{k-1}}\left(\mathrm{I}_3 + {\delta\theta_{b_t}^{b_{k-1}}}^{\times}\right) \tag{130}$$

$$b_{g,t} = \hat{b}_{g,t_i} + \delta b_{g,t} + \eta_{b_{g,t}} \tag{131}$$

$$\omega_{ib_t}^{b_t} = \tilde{\omega}_{ib_t}^{b_t} - b_{g,t} \tag{132}$$

$$\mathrm{C}_{b_t}^{b_{k-1}}\left(\mathrm{C}_{e}^{b_t}\omega_{ie}^{e}\right)^{\times}\mathrm{l}_{bv}^{b} = \mathrm{C}_{b_t}^{b_{k-1}}\mathrm{C}_{e}^{b_t}{\omega_{ie}^{e}}^{\times}\mathrm{C}_{b_t}^{e}\mathrm{l}_{bv}^{b} = \mathrm{C}_{e}^{b_{k-1}}{\omega_{ie}^{e}}^{\times}\mathrm{C}_{b_t}^{e}\mathrm{l}_{bv}^{b} \tag{133}$$

$$\mathrm{C}_{e}^{b_{k-1}} = \left(\mathrm{I}_3 + \delta\theta_{e}^{b_{k-1}}\right)^{\times}\hat{\mathrm{C}}_{e}^{b_{k-1}} \tag{134}$$

结合上述式子可得

$$\begin{aligned}\delta\dot{\mathrm{s}}_{b_k}^{b_{k-1}} = {} & \hat{\mathrm{C}}_{b_t}^{b_{k-1}}\mathrm{C}_{v}^{b}\hat{v}_{ev_t}^{v_t}(1+\hat{s}_o)\delta v_{ev_t}^{v_t} - \hat{\mathrm{C}}_{b_t}^{b_{k-1}}{\mathrm{l}_{bv}^{b}}^{\times}\left(\delta b_{g,t} + \eta_{b_{g,t}}\right) - \underbrace{\left(\hat{\mathrm{C}}_{e}^{b_{k-1}}{\omega_{ie}^{e}}^{\times}\mathrm{C}_{b_t}^{e}\mathrm{l}_{bv}^{b}\right)^{\times}\delta\theta_{e}^{b_{k-1}}}_{\approx\left(\left(\hat{\mathrm{C}}_{e}^{b_{k-1}}\omega_{ie}^{e}\right)^{\times}\mathrm{l}_{bv}^{b}\right)^{\times}\delta\theta_{e}^{b_{k-1}}} \\ & + \hat{\mathrm{C}}_{b_t}^{b_{k-1}}\mathrm{C}_{v}^{b}\hat{v}_{ev_t}^{v_t}\delta s_o - \hat{\mathrm{C}}_{b_t}^{b_{k-1}}\left(\mathrm{C}_{v}^{b}\hat{v}_{ev_t}^{v_t}(1+\hat{s}_o) - {\tilde{\omega}_{eb_t}^{b_t}}^{\times}\mathrm{l}_{bv}^{b}\right)^{\times}\delta\theta_{b_t}^{b_{k-1}} \\ & \approx \hat{\mathrm{C}}_{b_t}^{b_{k-1}}\mathrm{C}_{v}^{b}\hat{v}_{ev_t}^{v_t}(1+\hat{s}_o)\delta v_{ev_t}^{v_t} - \hat{\mathrm{C}}_{b_t}^{b_{k-1}}{\mathrm{l}_{bv}^{b}}^{\times}\left(\delta b_{g,t} + \eta_{b_{g,t}}\right) - \underbrace{\left({\tilde{\omega}_{ie}^{b_{k-1}}}^{\times}\mathrm{l}_{bv}^{b}\right)^{\times}}_{\approx 0}\delta\theta_{e}^{b_{k-1}} \\ & + \hat{\mathrm{C}}_{b_t}^{b_{k-1}}\mathrm{C}_{v}^{b}\hat{v}_{ev_t}^{v_t}\delta s_o - \hat{\mathrm{C}}_{b_t}^{b_{k-1}}\left(\mathrm{C}_{v}^{b}\hat{v}_{ev_t}^{v_t}(1+\hat{s}_o) - {\tilde{\omega}_{eb_t}^{b_t}}^{\times}\mathrm{l}_{bv}^{b}\right)^{\times}\delta\theta_{b_t}^{b_{k-1}} \\ & \approx \hat{\mathrm{C}}_{b_t}^{b_{k-1}}\mathrm{C}_{v}^{b}\hat{v}_{ev_t}^{v_t}(1+\hat{s}_o)\delta v_{ev_t}^{v_t} - \hat{\mathrm{C}}_{b_t}^{b_{k-1}}{\mathrm{l}_{bv}^{b}}^{\times}\left(\delta b_{g,t} + \eta_{b_{g,t}}\right) + \hat{\mathrm{C}}_{b_t}^{b_{k-1}}\mathrm{C}_{v}^{b}\hat{v}_{ev_t}^{v_t}\delta s_o \\ & - \hat{\mathrm{C}}_{b_t}^{b_{k-1}}\left(\mathrm{C}_{v}^{b}\hat{v}_{ev_t}^{v_t}(1+\hat{s}_o) - {\tilde{\omega}_{eb_t}^{b_t}}^{\times}\mathrm{l}_{bv}^{b}\right)^{\times}\delta\theta_{b_t}^{b_{k-1}}\end{aligned} \tag{135}$$

INS 运动模型:

$$C_{b_k}^{e} = C_{e_I(t_{k-1})}^{e_I(t_k)} C_{b_{k-1}}^{e} C_{b_I(t_k)}^{b_I(t_{k-1})} \tag{136}$$

$$C_{b_t}^{b_{k-1}} = C_e^{b_{k-1}} C_{b_t}^{e} = C_e^{b_{k-1}} C_{e_I(t_{k-1})}^{e_I(t)} C_{b_{k-1}}^{e} C_{b_I(t)}^{b_I(t_{k-1})} \tag{137}$$

$$v_{eb,k}^{e} - v_{eb,k-1}^{e} \approx C_{b_{k-1}}^{e} \beta_{k-1}^{k} + g_{k-1}^{e} \Delta t_{k-1,k} - 2\omega_{ie}^{e\,\times} [r_{eb,k}^{e} - r_{eb,k-1}^{e}] \tag{138}$$

$$\begin{aligned}\beta_{k-1}^{k} &= \int_{t_{k-1}}^{t_k} C_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} dt = C_{b_{k-1}}^{e} \int_{t_{k-1}}^{t_k} C_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} dt = \int_{t_{k-1}}^{t_k} C_{e_I(t_{k-1})}^{e_I(t)} C_{b_{k-1}}^{e} C_{b_I(t)}^{b_I(t_{k-1})} f_{ib_t}^{b_t} dt \\ &\approx \frac{1}{2}\left(I_3 + C_{e_I(t_{k-1})}^{e_I(t_k)}\right) C_{b_{k-1}}^{e} \int_{t_{k-1}}^{t_k} C_{b_I(t)}^{b_I(t_{k-1})} f_{ib_t}^{b_t} dt\end{aligned} \tag{139}$$

令$\gamma_{I,k-1}^{k} = \exp\left(\int_{t_{k-1}}^{t_k} \omega_{ib_t}^{b_t\,\times} dt\right) = C_{b_I(t_k)}^{b_I(t_{k-1})}$，其表示 IMU 角速度测量值的预积分项;

$$C_{b_k}^{e} = C_{e_I(t_{k-1})}^{e_I(t_k)} C_{b_{k-1}}^{e} \gamma_{I,k-1}^{k} \tag{140}$$

$$\begin{aligned}r_{eb,k}^{e} - r_{eb,k-1}^{e} &\approx v_{eb,k-1}^{e} \Delta t_{k-1,k} + C_{b_{k-1}}^{e} \alpha_{k-1}^{k} + \frac{1}{2} g_{k-1}^{e} \Delta t_{k-1,k}^{2} \\ &- 2\omega_{ie}^{e\,\times} \sum_{i=1}^{N} [r_{eb,i}^{e} - r_{eb,k-1}^{e}] \Delta t_{i-1,i}\end{aligned} \tag{141}$$

式中，α_{k-1}^{k}表示位置的 IMU 预积分项;

$$\alpha_{k-1}^{k} = \iint_{t_{k-1}}^{t_k} C_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} dt = \int_{t_{k-1}}^{t_k} \beta_{k-1}^{t} dt = \sum_{i=1}^{N} \frac{1}{2} (\beta_{k-1}^{i} - \beta_{k-1}^{i-1}) \Delta t_{i-1,i} \tag{142}$$

$$\dot{\gamma}_{I,k-1}^{t} = \dot{C}_{b_I(t)}^{b_I(t_{k-1})} = C_{b_I(t)}^{b_I(t_{k-1})} \omega_{ib_t}^{b_t} \tag{143}$$

$$\dot{\beta}_{k-1}^{t} = C_{b_t}^{b_{k-1}} f_{ib_t}^{b_t} = C_e^{b_{k-1}} C_{e_I(t_{k-1})}^{e_I(t)} C_{b_{k-1}}^{e} C_{b_I(t)}^{b_I(t_{k-1})} f_{ib_t}^{b_t} \tag{144}$$

$$\dot{\alpha}_{k-1}^{t} = \beta_{k-1}^{t} \tag{145}$$

$$\begin{aligned}&\int_{t_{k-1}}^{t_k} C_{b_I(t)}^{b_I(t_{k-1})} f_{ib_t}^{b_t} dt \\ &\approx \int_{t_{i-1}}^{t_i} f_{ib_t}^{b_t}\, dt + \frac{1}{2} \int_{t_{i-1}}^{t_i} \omega_{ib_t}^{b_t}\, dt \times \int_{t_{i-1}}^{t_i} f_{ib_t}^{b_t}\, dt \\ &+ \frac{1}{12}\left(\int_{t_{i-2}}^{t_{i-1}} \omega_{ib_t}^{b_t}\, dt \times \int_{t_{i-1}}^{t_i} f_{ib_t}^{b_t}\, dt + \int_{t_{i-2}}^{t_{i-1}} f_{ib_t}^{b_t}\, dt \times \int_{t_{i-1}}^{t_i} \omega_{ib_t}^{b_t}\, dt\right) \\ &\approx f_{ib_i}^{b_i} \Delta t_{i-1,i} + \frac{1}{2} f_{ib_i}^{b_i} \Delta t_{i-1,i} \times \omega_{ib_i}^{b_i} \Delta t_{i-1,i} \\ &+ \frac{1}{12}\left(\omega_{ib_{i-1}}^{b_{i-1}} \Delta t_{i-2,i-1} \times f_{ib_i}^{b_i} \Delta t_{i-1,i} + f_{ib_{i-1}}^{b_{i-1}} \Delta t_{i-2,i-1} \times \omega_{ib_i}^{b_i} \Delta t_{i-1,i}\right)\end{aligned} \tag{146}$$

$$\gamma_{I,k-1}^{i} = \gamma_{I,k-1}^{i-1} C_{b_I(t_i)}^{b_I(t_{i-1})} = \gamma_{I,k-1}^{i-1} \exp\left(\left(\int_{t_{i-1}}^{t_i} \dot{\rho}_{I,i}^{i-1}\, dt\right)^{\times}\right) \tag{147}$$

$$
\begin{aligned}
\dot{\rho}_{I,i}^{i-1} &= \omega_{ib}^{b} + \frac{1}{2}\rho_{I,i}^{i-1} \times \omega_{ib}^{b} + \frac{1}{\|\rho_{I,i}^{i-1}\| \cdot \rho_{I,i}^{i-1}}\left(1 - \frac{\|\rho_{I,i}^{i-1}\|\sin(\|\rho_{I,i}^{i-1}\|)}{2\left(1-\cos(\|\rho_{I,i}^{i-1}\|)\right)}\right)\rho_{I,i}^{i-1\times}\rho_{I,i}^{i-1\times}\omega_{ib}^{b} \\
&\approx \omega_{ib}^{b} + \frac{1}{2}\rho_{I,i}^{i-1} \times \omega_{ib}^{b} + \frac{1}{12}\rho_{I,i}^{i-1\times}\rho_{I,i}^{i-1\times}\omega_{ib}^{b} \approx \omega_{ib}^{b} + \frac{1}{2}\rho_{I,i}^{i-1} \times \omega_{ib}^{b} \\
&\approx \omega_{ib}^{b} + \frac{1}{2}\int_{t_{i-1}}^{t_i} \omega_{ib}^{b}\, dt \times \omega_{ib}^{b}
\end{aligned}
\tag{148}
$$

$$
\begin{aligned}
\dot{s}_{b_k}^{b_{k-1}} &= C_e^{b_{k-1}} C_{e_I(t_{k-1})}^{e_I(t)} C_{b_{k-1}}^{e} C_{b_I(t)}^{b_I(t_{k-1})} C_v^{b} v_{ev_t}^{v_t}(1+s_o) - C_e^{b_{k-1}} C_{e_I(t_{k-1})}^{e_I(t)} C_{b_{k-1}}^{e} C_{b_I(t)}^{b_I(t_{k-1})} \omega_{eb_t}^{b_t\times} l_{bv}^{b} \\
&= C_e^{b_{k-1}} C_{e_I(t_{k-1})}^{e_I(t)} C_{b_{k-1}}^{e} C_{b_I(t)}^{b_I(t_{k-1})} C_v^{b} v_{ev_t}^{v_t}(1+s_o) \\
&- C_e^{b_{k-1}} C_{e_I(t_{k-1})}^{e_I(t)} C_{b_{k-1}}^{e} C_{b_I(t)}^{b_I(t_{k-1})} \left(\omega_{ib_t}^{b_t} - C_e^{b_t}\omega_{ie}^{e}\right)^{\times} l_{bv}^{b}
\end{aligned}
\tag{149}
$$