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MAT 458 - HOMEWORK #3

① using the Randomized Complete block design (RCBD) the statistical model (effect model) for this is

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \begin{cases} i = 1, 2, 3, \ldots, a \\ j = 1, 2, 3, \ldots, b \end{cases}$$ where

y_{ij} is the ith observation belonging to the jth block, μ is the overall mean, τ_i is the ith treatment effect, β_j is the jth Block effect and the ε_{ij} is the random error present in the experiment.

Here, the treatment is the distance (ft) and blocks the subject.

At 5% level of significants ($\alpha = 0.05$), we test the hypothesis $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$

$H_1: \mu_i \neq \mu_j$ for at least one pair

Distance (ft)	Subject					$y_{i.}$
	1	2	3	4	5	
4	10	6	6	6	6	34
6	7	6	6	1	6	26
8	5	3	3	2	5	18
10	6	4	4	2	3	19
$y_{.j}$	28	19	19	11	20	$y_{..} = 97$

$N = ab = 4 \times 5 = 20$; $a = 4$, $b = 5$

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{N} = [10^2+6^2+6^2+6^2+6^2+7^2+6^2+6^2+1^2+6^2+5^2+3^2+3^2+2^2+5^2+6^2+4^2+4^2+2^2+3^2] - \left[\frac{97^2}{20}\right]$$

$$= 555 - 470.45 = \underline{\underline{84.55}}$$

$$SS_{Tr} = \frac{1}{b}\sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N} = \frac{1}{5}[34^2+26^2+18^2+19^2] - \left(\frac{97^2}{20}\right) = \underline{\underline{32.95}}$$

$$SS_{BL} = \frac{1}{a}\sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N} = \frac{1}{4}[28^2+19^2+19^2+11^2+20^2] - \left(\frac{97^2}{20}\right) = \underline{\underline{36.3}}$$

$SS_E = SS_T - SS_{Tr} - SS_{BL} = [84.55 - 32.95 - 36.3] = \underline{15.3}$

The test statistics is $F_0 = \dfrac{MS_{Tr}}{MS_E}$ which is distributed as $F_0 \sim f_{(a-1),\,(a-1)(b-1)}$ if the null hypothesis H_0 is true.

But $MS_{Tr} = \dfrac{SS_{Tr}}{a-1} = \dfrac{32.95}{4-1} = \underline{10.98333}$

$MS_E = \dfrac{SS_E}{(a-1)(b-1)} = \dfrac{15.3}{(4-1)(5-1)} = \underline{\underline{1.275}}$

$\therefore F_0 = \dfrac{10.98333}{1.275} = \underline{\underline{8.6144}}$ and $f_{\alpha,\,a-1,\,(a-1)(b-1)} = f_{0.05,3,12} = \underline{\underline{3.49}}$

$MS_{BL} = \dfrac{SS_{BL}}{b-1} = \dfrac{36.3}{5-1} = \underline{9.075}$

The completed ANOVA table is given by:

Source	SS	df	MS	F_0	P-value
Distance	32.95	3	10.9833	8.6144	0.0025
Subject	36.3	4	9.075		
Error	15.3	12	1.275		
TOTAL	84.55	19			

DECISION: Reject H_0 if $F_0 \geq f_{\alpha,(a-1),(a-1)(b-1)} = f_{0.05,3,12}$ which we conclude that since $F_0 = 8.6144 > 3.49 = f_{\alpha(a-1),(a-1)(b-1)}$ we ~~conclude that~~ hence that, we reject H_0 and conclude that there is a significant difference between the means of the four distances showing ~~difference in~~ the effect of the distance of the object from the eye on the focus time.

Note that the degrees of freedoms (df) are calculated by:

Treatment $df = (a-1) = (4-1) = 3$

Block $df = (b-1) = (5-1) = 4$

Error $df = (a-1)(b-1) = 3 \times 4 = 12$

Total $df = N-1 = (20-1) = 19$

The p-value $= P(F_{3,12} > 8.6144) \approx \underline{0.0025} < 0.05$

This confirms our decision to reject H_0.

(2) Using the Latin square design (LSD) the statistical model (Effect model) is given by:

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \end{cases}$$ where

y_{ijk} is the observation in the ith row and kth column for the jth treatment, μ is the overall mean, α_i is the ith row effect, τ_j is the jth treatment effect and β_k is the kth column effect and ε_{ijk} is the random error present in the experiment.

At 5% level of significants ($\alpha = 0.05$), the hypothesis of interest is:

$H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$

H_1: At least one of τ_i not zero

order of Assembly	operators 1	2	3	4	$y_{i..}$
1	C = 10	D = 14	A = 7	B = 8	39
2	B = 7	C = 18	D = 11	A = 8	44
3	A = 5	B = 10	C = 11	D = 9	35
4	D = 10	A = 10	B = 12	C = 14	46
$y_{..k}$	32	52	41	39	$y_{...} = 164$

Here number of factors $p = 4$ and so $N = p \times p = p^2 = 4^2 = 16$

The row is the order of assembly, column is operator, treatment is ~~method~~.

$$SS_T = \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} y_{ijk}^2 - \frac{y_{...}^2}{N} = [10^2 + 14^2 + 7^2 + 8^2 + 7^2 + 18^2 + 11^2 + 8^2 + 5^2 + 10^2 + 11^2 + 9^2 + 10^2 + 10^2 + 12^2 + 14^2] - \left(\frac{164^2}{16}\right) = \underline{\underline{153.0}}$$

$$SS_{Row} = \frac{1}{p}\sum_{i=1}^{p} y_{i..}^2 - \frac{y_{...}^2}{N} = \frac{1}{4}[39^2 + 44^2 + 35^2 + 46^2] - \left(\frac{164^2}{16}\right) = \underline{\underline{18.50}}$$

$$SS_{Col} = \frac{1}{p}\sum_{k=1}^{p} y_{..k}^2 - \frac{y_{...}^2}{N} = \frac{1}{4}[32^2 + 52^2 + 41^2 + 39^2] - \left(\frac{164^2}{16}\right) = \underline{\underline{51.50}}$$

$$SS_{Tr} = \frac{1}{p}\sum_{j=1}^{p} y_{.j.}^2 - \frac{y_{...}^2}{N} = \frac{1}{4}[30^2 + 37^2 + 53^2 + 44^2] - \left(\frac{164^2}{16}\right) = \underline{\underline{72.5}}$$

note: $A = (5+10+7+8) = 30 = y_{.1.}$, $C = (10+18+11+14) = 53 = y_{.3.}$ as totals.
$B = (7+10+12+8) = 37 = y_{.2.}$, $D = (10+14+11+9) = 44 = y_{.4.}$

$SS_E = SS_T - SS_{Row} - SS_{Col} - SS_{Tr} = 153 - 18.5 - 51.5 - 72.5 = \underline{\underline{10.5}}$

The test statistics is $F_0 = \frac{MS_{Tr}}{MS_E}$ which is distributed as $F_0 \sim F_{p-1,(p-2)(p-1)}$ if the null hypothesis H_0 is true.

But $MS_{Tr} = \frac{SS_{Tr}}{p-1} = \frac{72.5}{4-1} = \underline{\underline{24.167}}$

$MS_E = \frac{SS_E}{(p-2)(p-1)} = \frac{10.5}{(4-2)(4-1)} = \frac{10.5}{6} = \underline{\underline{1.75}}$

$\therefore F_0 = \frac{24.167}{1.75} = \underline{\underline{13.8097}}$ and $F_{\alpha, p-1,(p-2)(p-1)} = F_{0.05, 3, 6} = \underline{\underline{4.76}}$

$MS_{Row} = \frac{SS_{Row}}{p-1} = \frac{18.5}{4-1} = \underline{\underline{6.1667}}$; $MS_{Col} = \frac{SS_{Col}}{p-1} = \frac{51.5}{4-1} = \boxed{17.1667}$

P-value $= P(F_{3,6} > 13.8097) \approx 0.0042 < 0.05$.

<u>df's</u>: Treatment df $= p-1 = (4-1) = 3$; Error df $= (p-2)(p-1) = (4-2)(4-1) = 6$

Rows df : $p-1 = (4-1) = 3$

Column df : $p-1 = (4-1) = 3$ Total df $= p^2-1 = (4^2-1) = 15$

<u>DECISION</u>: we reject H_0 if $f_0 \geq f_{\alpha, p-1,(p-2)(p-1)}$.

we observe that $f_0 = 13.8097 > f_{0.05, 3, 6} = 4.76$ hence we reject H_0 and conclude that the average due to order of assembly differ significantly as well as the operators; but the average due to column do not differ significantly since $f_{Coln} = 3.5238 < F_0 = \cancel{4.76}$.

note $F_{Row} = \frac{MS_{Row}}{MS_E} = \frac{6.1667}{1.75} = \underline{\underline{3.5238}}$

our decision to reject H_0 above is confirmed by the small p-value of $0.0042 < 0.05$.

(3) We determine if there possibly exist BIB design given $a = 5$ treatment (one factor), block of size 3 $(k = 3)$, r = number of replicates, b is the number of blocks, λ is number of blocks where each pair of treatment appear.

i) For $r = 2$, if there possibly exist BIBD, λ and b must be integers. Now, $\lambda = \frac{r(k-1)}{a-1} = \frac{2(3-1)}{5-1} = \frac{4}{4} = 1$ which is an integer.
Also $bk = ar = N$. $\Rightarrow b(3) = 5(2) = 10$
$\Rightarrow b = \left(\cancel{\frac{10}{3}}\right)$ which is not an integer hence $\underline{\underline{No}}$ there is no exist of BIB design. b must be an integer.

ii) For $r = 3$ if there possibly exist BIBD, λ and b must be integers. Now, $\lambda = \frac{r(k-1)}{a-1} = \frac{3(3-1)}{5-1} = \frac{6}{4} = \left(\frac{3}{2}\right)$ which is not an integer.
Now $bk = ar = N$ $\Rightarrow b(3) = 3(5) \Rightarrow b = 5$
Since λ is not an integer, $\underline{\underline{No}}$ there is no exist BIBD.

iii) For $r = 4$ if there possibly exist BIBD λ and b must be integers. Now $\lambda = \frac{r(k-1)}{a-1} = \frac{4(3-1)}{5-1} = \frac{8}{4} = 2$
which is not an integer, Now $bk = ar = N \Rightarrow b(3) = 5(4)$
$\Rightarrow b = \frac{20}{3}$ not an integer. In fact there is $\underline{\underline{NO}}$ exist of BIB design.

iv) For $\lambda = 2$, $\lambda = \frac{r(k-1)}{a-1} \Rightarrow 2 = \frac{r(3-1)}{5-1} \Rightarrow r = \frac{2(4)}{2}$
$\therefore r = 4$ an integer.
Next $bk = ar = N$; Then $\frac{b(3)}{3} = \frac{5(4)}{3} = \left(\frac{20}{3}\right)$ not an integer so $\underline{\underline{NO}}$ there is no existence of BIB design since b is not an integer.

v) For $\lambda = 3$; $\lambda = \frac{r(k-1)}{a-1} \Rightarrow 3 = \frac{r(3-1)}{5-1} \Rightarrow r = \frac{4(3)}{2}$
$\Rightarrow r = 6$ an integer.
Next $bk = ar = N$ and so $\frac{b(3)}{3} = \frac{6(5)}{3} = 10$ an integer. Since λ, r, b, k are all integer $\underline{\underline{\text{Yes}}}$ there possibly exist BIB design with $\lambda = 3$, $r = 6$, $b = 10$ and $k = 3$.

vi) For $\lambda = 4$; $\lambda = \frac{r(k-1)}{a-1} \Rightarrow 4 = \frac{r(3-1)}{5-1} \Rightarrow$
$\Rightarrow \frac{4 \cdot (4)}{2} = \frac{\cancel{2}r}{\cancel{2}} \Rightarrow 16/2 = 8 = r$ which is an integer
Next, $bk = ar = N$. Then $\frac{b(3)}{3} = \frac{5(8)}{3} = 40/3 = b$
which is not an integer so there is no exist BIB design. b must be an integer.

4 i) The table presents $a=5$, $b=5$, $k=4$

we have $N = ar = bk$. So $\frac{\cancel{5}r}{\cancel{5}} = \frac{\cancel{5}(4)}{\cancel{5}} = 4 \therefore r = 4$

$\Rightarrow r = \left(\frac{bk}{a}\right)$

$\lambda = \frac{r(k-1)}{a-1} \Rightarrow \lambda = \frac{4(4-1)}{5-1} = \frac{4(3)}{4} = 3$

Here, a is the total number of treatments.

b is the total number of blocks,

k is the total number of treatments each block contains.

r is number of times each treatment occurs in the design,

λ is the number of times each pair of treatment appear in ~~the~~ same block.

ii) The model for the BIB design is $y_{ij} = \mu + T_i + B_j + \varepsilon_{ij}$ This is the effect model,

$\begin{cases} i = 1, 2, \cdots 5 \\ j = 1, 2, \cdots 5 \end{cases}$ where y_{ij} is the ith observation for the jth block, μ is the overall mean,

$\varepsilon_{ij} \sim N(0, \sigma^2)$ and is i.i.d

T_i is the ith treatment effect and B_j is the j^{th} Block effect ε_{ij} is the random error present in the experiment and that $\sum_{i=1}^{a} T_i = 0$, $\sum_{j=1}^{b} B_j = 0$.

At 5% level of significance ($\alpha = 0.05$), the hypothesis of interest is $H_0: T_1 = T_2 = T_3 = T_4 = T_5 = 0$

H_1: At least one of $T_i \neq 0$.

Now we find the ANOVA entries.

Additive	Car					$y_{i.}$
	1	2	3	4	5	
1	—	17	14	13	12	56
2	14	14	—	13	10	51
3	12	—	13	12	9	46
4	13	11	11	12	—	47
5	11	12	10	—	8	41
$y_{.j}$	50	54	48	50	39	$y_{..} = 241$

from the table above, we calculate the quantities:

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{N} = [17^2 + 14^2 + 13^2 + 12^2 + 13^2 + 12^2 + 14^2 + 14^2 + 13^2 + 10^2 + 12^2 + 13^2 + 12^2 + 9^2 + 13^2 + 11^2 + 11^2 + 12^2 + 11^2 + 12^2 + 10^2 + 8^2] - \frac{241^2}{20} = 2981 - 2904.05$$

$$= \underline{\underline{76.95}}$$

$$SS_{BL} = \frac{1}{K}\sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N} = \frac{1}{4}[50^2 + 54^2 + 48^2 + 50^2 + 39^2] - 2904.0$$

$$= \underline{\underline{31.2}}$$

$$SS_{Tr(adj)} = \frac{K\sum_{i=1}^{a} Q_i^2}{\lambda a} \quad \text{where } Q_i = y_{i.} - \frac{1}{K}\sum_{j=1}^{b} n_{ij} y_{.j}$$

$$Q_1 = y_{1.} - \frac{1}{4}[54 + 48 + 50 + 39] \Rightarrow (56 - 47.75) = \frac{33}{4}$$

$$Q_2 = y_{2.} - \frac{1}{4}[50 + 54 + 50 + 39] = (51 - 48.25) = \frac{11}{4}$$

$$Q_3 = y_{3.} - \frac{1}{4}[50 + \cancel{48} + 50 + 39] = (46 - 46.75) = -\frac{3}{4}$$

$$Q_4 = y_{4.} - \frac{1}{4}[50 + 54 + 48 + 50) = (47 - 50.5) = -\frac{7}{2}$$

$$Q_5 = y_{5.} - \frac{1}{4}[50 + 54 + 48 + 39) = (41 - 47.75) = -6.75 = -\frac{27}{4}$$

we see that $\sum_{i=1}^{a} Q_i = 0 = \frac{33}{4} + \frac{11}{4} - \frac{3}{4} - \frac{7}{2} - \frac{27}{4} = 0$

Then $SS_{Tr(adjusted)} = \dfrac{4\left[\left(\frac{33}{4}\right)^2+\left(\frac{1}{4}\right)^2+\left(\frac{-3}{4}\right)^2+\left(\frac{-7}{2}\right)^2+\left(\frac{-27}{4}\right)^2\right]}{3(5)}$

$= \underline{35.7333}$

$SS_E = \left[SS_T - SS_{Tr(adjusted)} - SS_{BL}\right] = (76.95 - 35.733 - 31.2)$

$= \underline{10.017}$

$MS_{Tr(adj)} = \dfrac{SS_{Tr(adj)}}{(a-1)}$, $MS_{BL} = \dfrac{SS_{BL}}{(b-1)}$, $MS_E = \dfrac{SS_E}{(N-a-b+1)}$

where $(a-1)$, $(b-1)$, $(N-a-b+1)$ are df for treatment adj., block, and error respectively. $N-1$ is the df for the total. So $(a-1)=4$, $(b-1)=4$, $(N-a-b+1)=11$.

Then $MS_{Tr(adj)} = \dfrac{35.7333}{(5-1)} = \underline{8.9333}$, $MS_E = \dfrac{10.017}{(20-5-5+1)} = \underline{0.9106}$

$MS_{BL} = \dfrac{31.2}{(5-1)} = \underline{7.80}$

The ~~critical region~~ test statistic is $F_0 = \dfrac{MS_{Tr(adjusted)}}{MS_E} \sim$ under H_0 $F_{\alpha, (a-1), (N-a-b+1)}$.

$\Rightarrow F_0 = \dfrac{8.9333}{0.9106} = \underline{9.80999}$

The critical region is $F_{0.05, 4, 11} = \underline{3.36}$

Decision rule: Reject H_0 if $F_0 \geq F_{0.05, 4, 11}$

From the above, $F_0 = 9.8099 > 3.36 = F_{0.05, 4, 11}$ so we reject H_0 and conclude that there is a significant effect of additive on milage.