MAT 458-Design of Experiments

(VI) The 2^k Factorial Design

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Outline

1. Introduction

Factorial designs are widely used in experiments involving **several factors** where it is necessary to study the joint effect of the factors on a response.

In last Chapter, we have considered general methods for the analysis of factorial designs.

Here we focus on the special and important case of the general factorial design: k factors, all at two levels. The two levels are usually called low and high (they could be either quantitative or qualitative).

A complete replicate of such a design requires $2 \times 2 \times \cdots \times 2 = 2^k$ observations and is called a 2^k **factorial design.**

The 2^k factorial design is very widely used in industrial experimentation. It is particularly useful in the early stages of experimental work when many factors are likely to be investigated. It provides the smallest number of runs with which k factors can be studied in a complete factorial design.

Here we assume that

(i) the factors are fixed,

(ii) the designs are completely randomized, and

(iii) the usual normality assumptions are satisfied.

2. The 2^2 Factorial Design

The simplest design in the 2^k series is one with only two factors, say A and B, each run at two levels. This design is called a 2^2 **factorial design**. The levels of the factors may be arbitrarily called low and high.

An example:

Consider an investigation into the effect of the concentration of the reactant and the amount of the catalyst on the conversion (yield) in a chemical process. The objective of the experiment was to determine if adjustments to either of these two factors would increase the yield. Let the reactant concentration be factor A and let the two levels of interest be 15 and 25 percent. The catalyst is factor B, with the high level denoting the use of 2 pounds of the catalyst and the low level denoting the use of only 1 pound.

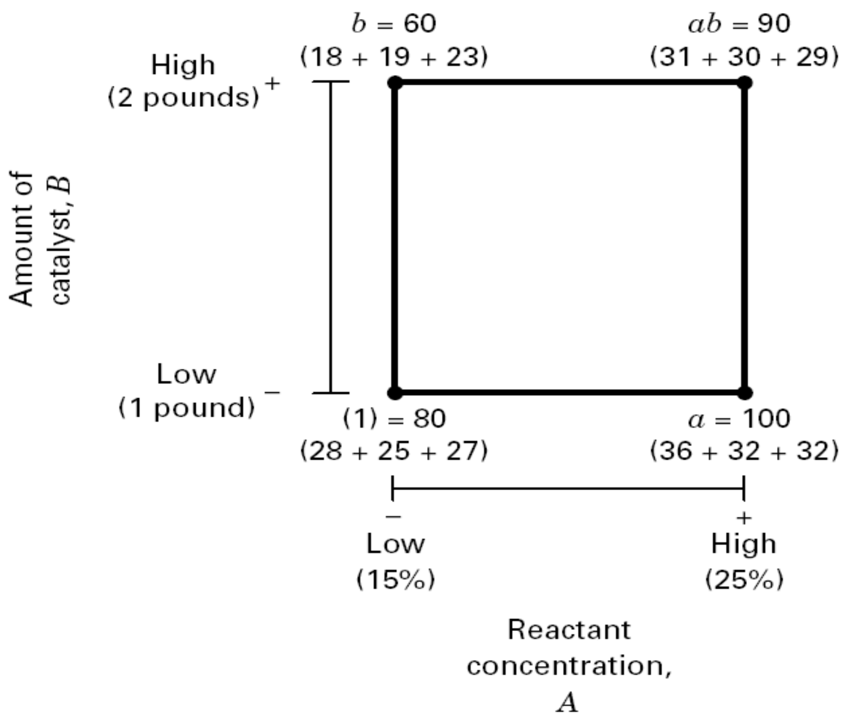
The experiment is replicated three times, so there are 12 runs. The order in which the runs are made is random, so this is a completely randomized experiment. The data obtained are reported in the following table.

Factor			Replicate			
A	B	Tr. Combination	I	II	III	Total
-	-	A low,B low	28	25	27	80
+	-	A high,B low	36	32	32	100
-	+	A low,B high	18	19	23	60
+	+	A high,B high	31	30	29	90

- and + denote the low and high levels of a factor, respectively.

The four treatment combinations in this design are shown graphically in the following Figure 6.1.

The Simplest Case: The 2^2



■ **FIGURE 6.1** Treatment combinations in the 2^2 design

By convention, we denote the effect of a factor by a capital Latin letter. Thus "A" refers to the effect of factor A, "B" refers to the effect of factor B, and "AB" refers to the AB interaction.

The four treatment combinations in the design are also represented by lowercase letters, as shown in Figure 6.1. We see from the figure that the high level of any factor in the treatment combination is denoted by the corresponding lowercase letter and that the low level of a factor in the treatment combination is denoted by the absence of the corresponding letter.

Thus, a represents the treatment combination of A at the high level and B at the low level, b represents A at the low level and B at the high level, and ab represents both factors at the high level. By convention, (1) is used to denote both factors at the low level.

In a two-level factorial design, we may define the average effect of a factor as the change in response produced by a change in the level of that factor averaged over the levels of the other factor. Also, the symbols $(1), a, b$, and ab now represent the total of the response observation at all n replicates taken at the treatment combination, as illustrated in Figure 6.1.

Now the effect of A at the low level of B is $[a-(1)]/n$, and the effect of A at the high level of B is $(ab-b)/n$. Averaging these two quantities yields the main effect of A.

The effect of B is found by the average of the effect of B at the low level of A (i.e.,$[b-(1)]/n$) and at the high level of A (i.e., $(ab-a)/n$).

We define the interaction effect AB as the **average difference** between the effect of A at the high level of B and the effect of A at the low level of B. Alternatively, we may define AB as the average difference between the effect of B at the high level of A and the effect of B at the low level of A.

The formulas for the effects of A, B, and AB may be derived by another method.

$$A = \bar{Y}_{A+} - \bar{Y}_{A-}$$

This is exactly the same result as above.

Similarly, the effect of B can be defined as

$$B = \bar{Y}_{B+} - \bar{Y}_{B-}$$

and, the interaction effect AB is the average of the right-to-left diagonal treatment combinations in the square [ab and (1)] minus the average of the left-to-right diagonal treatment combinations (a and b).

$$
\begin{aligned}
A &= \bar{y}_{A^+} - \bar{y}_{A^-} \\
&= \frac{ab+a}{2n} - \frac{b+(1)}{2n} \\
&= \tfrac{1}{2n}[ab+a-b-(1)] \\
B &= \bar{y}_{B^+} - \bar{y}_{B^-} \\
&= \frac{ab+b}{2n} - \frac{a+(1)}{2n} \\
&= \tfrac{1}{2n}[ab+b-a-(1)] \\
AB &= \frac{ab+(1)}{2n} - \frac{a+b}{2n} \\
&= \tfrac{1}{2n}[ab+(1)-a-b]
\end{aligned}
$$

For the example, the effect estimates are:

A = 8.33, B = -5.00, AB = 1.67

The effect of A (reactant concentration) is positive; this suggests that increasing A from the low level (15%) to the high level (25%) will increase the yield.

The effect of B (catalyst) is negative; this suggests that increasing the amount of catalyst added to the process will decrease the yield.

The interaction effect appears to be small relative to the two main effects.

■ TABLE 6.1

Analysis of Variance for the Experiment in Figure 6.1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_q	P-Value
A	208.33	1	208.33	53.15	0.0001
B	75.00	1	75.00	19.13	0.0024
AB	8.33	1	8.33	2.13	0.1826
Error	31.34	8	3.92		
Total	323.00	11			

In experiments involving 2^k designs, it is always important to examine the magnitude and direction of the factor effects to determine which variables are likely to be important.

Effect magnitude and direction should always be considered along with the ANOVA, because the ANOVA alone does not convey this information.

Let's consider determining the sums of squares for A, B, and AB.

Note that a contrast is used in estimating A, namely

$$Contrast_A = ab + a - b - (1)$$

We usually call this contrast the **total effect of A**.

Similarly, we see that contrasts are also used to estimate B and AB.

$$Contrast_B = ab + b - a - (1)$$

$$Contrast_{AB} = ab + (1) - a - b$$

Furthermore, these three contrasts are orthogonal.

Notice that sum of squares for any contrast is equal to the contrast squared divided by the number of observations in each total in the contrast times the sum of the squares of the contrast coefficients.

Consequently, we have the sums of squares for A,B, and AB

$$\begin{aligned} SS_A &= \frac{(ab + a - b - (1))^2}{4n} \\ SS_B &= \frac{(ab + b - a - (1))^2}{4n} \end{aligned}$$

and

$$SS_{AB} = \frac{(ab + (1) - a - b)^2}{4n}$$

Notice how simple the above equations (for SS) are. Here we can compute sums of squares by only squaring one number.

For the experiment in the above example, we may find the sums of squares easily as follows.

$$\begin{aligned} SS_A &= \frac{50^2}{4 * 3} = 208.33 \\ SS_B &= \frac{(-30)^2}{4 * 3} = 75.00 \\ SS_{AB} &= \frac{10^2}{4 * 3} = 8.33 \end{aligned}$$

The total sum of squares is computed, as usual, i.e.

$$SS_T = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{3} Y_{ijk}^2 - \frac{Y_{\dots}^2}{4n},$$

It is easy to see that SS_T has $4n - 1$ degrees of freedom.

The error sum of squares, with $4(n-1)$ degrees of freedom, is usually computed by subtraction as

$$SS_E = SS_T - SS_A - SS_B - SS_{AB}$$

For the experiment in the above example, we obtain

$$SS_T = 9398.00 - 9075.00 = 323.00$$

and

$$SS_E = 323.00 - 208.33 - 75.00 - 8.33 = 31.34$$

The complete ANOVA is summarized in Table 6.1. On the basis of the P-values, we conclude that the main effects are statistically significant and that there is no interaction between these factors. This confirms our initial interpretation of the data based on the magnitudes of the factor effects.

■ **TABLE 6.1**

Analysis of Variance for the Experiment in Figure 6.1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_q	*P*-Value
A	208.33	1	208.33	53.15	0.0001
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Total	323.00	11			

It is often convenient to write down the treatment combinations in the order $(1), a, b, ab$. This is referred to as standard order.

Using this standard order, we see that the contrast coefficients used in estimating the effects are

Effects	(1)	a	b	ab
A	-1	1	-1	1
B	-1	-1	1	1
AB	1	-1	-1	1

The Regression Model:

In a 2^k factorial design, it is easy to express the results of the experiment in terms of a **regression model**, although we could also use either an effects or a means model.

Here we consider the regression model approach because it is much more natural and intuitive.

For the above chemical process experiment in Figure 6.1, the regression model is

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

where x_1 is a coded variable that represents the reactant concentration, x_2 is a coded variable that represents the amount of catalyst, the β's are regression coefficients, and ε is random error.

$$x_1 = \{ \begin{array}{ll} 1, & \text{if the concentration is at the high level (Conc=25\%)} \\ -1, & \text{if the concentration is at the low level (Conc=15\%)} \end{array}$$

$$x_2 = \{ \begin{array}{ll} 1, & \text{if the catalyst is at the high level (Catalyst=2 pounds))} \\ -1, & \text{if the catalyst is at the low level (Catalyst=1 pound)} \end{array}$$

The fitted regression model is

$$\widehat{Y} = 27.5 + \frac{8.33}{2} x_1 + \frac{-5.00}{2} x_2$$

where the intercept is the grand average of all 12 observations, and the estimated coefficients $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are **one-half** the corresponding factor effect estimates.

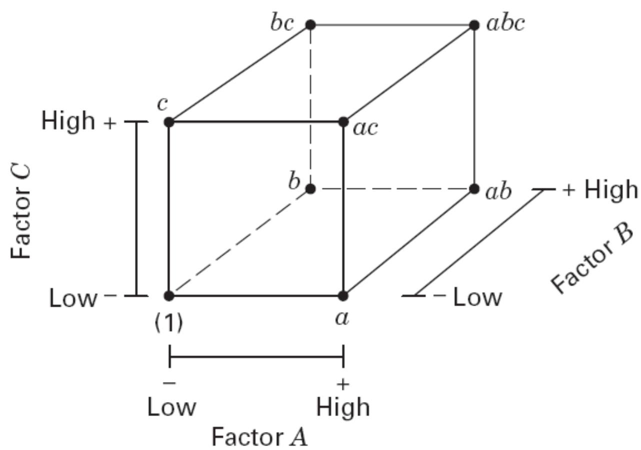
Note that regression coefficient measures the effect of a one-unit change in x on the mean of Y, and the effect estimate is based on a two-unit change (from -1 to 1).

3. The 2^3 Factorial Design

Suppose that three factors, A, B, and C, each at two levels, are of interest. The design is called a 2^3 factorial design, and the eight treatment combinations can now be displayed geometrically as a cube, as shown in Figure 6.4(a).

The 2^3 Factorial Design

■ **FIGURE 6.4**
The 2^3 factorial design



(a) Geometric view

Run	Factor A	Factor B	Factor C
1	–	–	–
2	+	–	–
3	–	+	–
4	+	+	–
5	–	–	+
6	+	–	+
7	–	+	+
8	+	+	+

(b) Design matrix

Using the “+” and “-” to represent the low and high levels of the factors, we may list the eight runs in the 2^3 design as in Figure 6.4(b)

This is sometimes called the design matrix. Extending the label notation discussed in previous Section, we write the treatment combinations in standard order as $(1), a, b, ab, c, ac, bc,$ and abc.

Remember that these symbols also represent the total of all n observations taken at that particular treatment combination.

Table of – and + Signs for the 2^3 Factorial Design

■ **TABLE 6.3**

Algebraic Signs for Calculating Effects in the 2^3 Design

Treatment Combination	Factorial Effect							
	I	A	B	AB	C	AC	BC	ABC
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

4. The General 2^k Factorial Design