MAT 458-Design of Experiments

(IV) Randomized Blocks, Latin Squares, and Related Designs

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Outline

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1. The Randomized Complete Block Design (RCBD)

- Blocking and nuisance factors

Blocking is a technique for dealing with nuisance factors. In any experiment, variability arising from a nuisance factor can affect the results.

Question: What is a nuisance factor ?

A nuisance factor is a factor that probably has some effect on the response, but it's of no interest to the experimenter. However, the variability it transmits to the response needs to be minimized.

Question: How to do it ?

(i) If the nuisance variable is known and controllable, we use blocking.

(ii) If the nuisance factor is known and uncontrollable, sometimes we can use the analysis of covariance (see Chapter 15) to remove the effect of the nuisance factor from the analysis.

(iii) If the nuisance factor is unknown and uncontrollable (a "lurking variable), we hope that randomization balances out its impact across the experiment.

The RCBD is one of the most widely used experimental designs.

Situations for which the RCBD is appropriate are numerous. Units of test equipment or machinery are often different in their operating characteristics and would be a typical blocking factor. Batches of raw material, people,

and time are also common nuisance sources of variability in an experiment that can be systematically controlled through blocking.

- The Hardness Testing Example:

  (See pg. 136)

Consider a hardness testing machine that presses a rod with a pointed tip into a metal specimen with a known force. By measuring the depth of the depression caused by the tip, the hardness of the specimen is determined.

We wish to determine whether 4 different tips produce different (mean) hardness.

Assignment of the tips to an **experimental unit**; that is, a test coupon.

The test coupons are a source of **nuisance variability**.

The experimenter may want to test the tips across coupons of various hardness levels.

There is need for blocking.

To conduct this experiment as a RCBD, assign all 4 tips to each coupon.

Each coupon is called a **block**; that is, it's a more homogenous experimental unit on which to test the tips.

Variability **between blocks** can be large, variability **within a block** should be relatively small.

In general, a **block** is a specific level of the nuisance factor.

A complete replicate of the basic experiment is conducted in each block.

A block represents a **restriction on randomization**.

All **runs** within a block are **randomized**.

Suppose that we use $b = 4$ blocks:

■ **TABLE 4.1**

Randomized Complete Block Design for the Hardness Testing Experiment

Test Coupon (Block)			
1	**2**	**3**	**4**
Tip 3	Tip 3	Tip 2	Tip 1
Tip 1	Tip 4	Tip 1	Tip 4
Tip 4	Tip 2	Tip 3	Tip 2
Tip 2	Tip 1	Tip 4	Tip 3

Notice the two-way structure of the experiment.

Once again, we are interested in testing the equality of treatment means, but now we have to remove the variability associated with the nuisance factor (the blocks).

- Statistical Analysis of the RCBD Model

Here we consider to extend the ANOVA to the RCBD.

Suppose that there are a treatments (factor levels) and b blocks.

A statistical model (effects model) for the RCBD is

$$Y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}\,, \quad \begin{cases} i = 1, 2, \cdots, a \\ j = 1, 2, \cdots, b \end{cases}$$

where μ is the overall mean; τ_i is the i^{th} treatment effect; β_j is the j^{th} block effect; ε_{ij} is experiment error and i.i.d. $N(0, \sigma^2)$.

We usually think of the treatment and block effects as deviations from the overall mean. Thus for the fixed design, we require that

$$\sum_{i=1}^{a} \tau_i = 0 \quad \text{and} \quad \sum_{j=1}^{b} \beta_j = 0.$$

The relevant (fixed effects) hypotheses are

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_a$$

$$H_1 : \text{at least for one parir} \quad \mu_i \neq \mu_j$$

Because the i^{th} treatment mean $\mu_i = \frac{1}{b}\sum_{j=1}^{b} \mu_{ij} = \frac{1}{b}\sum_{j=1}^{b}(\mu + \tau_i + \beta_j) = \mu + \tau_i$, an equivalent way to write the above hypotheses is in terms of the treatment effects, i.e.,

$$H_0 : \tau_1 = \tau_2 = \cdots = \tau_a = 0$$

$$H_1 : \tau_i \neq 0 \quad \text{at least for one } i$$

The analysis of variance can be easily extended to the RCBD.

Let $N = ab$ be the total number of observations.

Note that $\bar{Y}_{i.}$ is the average of the observations taken under treatment i, $\bar{Y}_{.j}$ is the average of the observations in block j, and $\bar{Y}_{..}$ is the grand average of all observations.

We may express the total corrected sum of

squares as

$$
\begin{aligned}
SS_T &= \sum_{i=1}^{a}\sum_{j=1}^{b}(Y_{ij}-\bar{Y}_{..})^2 \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\Big[(\bar{Y}_{i.}-\bar{Y}_{..})+(\bar{Y}_{.j}-\bar{Y}_{..}) \\
&\qquad\qquad +(Y_{ij}-\bar{Y}_{i.}-\bar{Y}_{.j}+\bar{Y}_{..})\Big]^2 \\
&= b\sum_{i=1}^{a}(\bar{Y}_{i.}-\bar{Y}_{..})^2+a\sum_{j=1}^{b}(\bar{Y}_{.j}-\bar{Y}_{..})^2 \\
&\quad +\sum_{i=1}^{a}\sum_{j=1}^{b}(Y_{ij}-\bar{Y}_{i.}-\bar{Y}_{.j}+\bar{Y}_{..})^2 \\
&\quad +2\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{Y}_{i.}-\bar{Y}_{..})(\bar{Y}_{.j}-\bar{Y}_{..}) \\
&\quad +2\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{Y}_{i.}-\bar{Y}_{..})(Y_{ij}-\bar{Y}_{i.}-\bar{Y}_{.j}+\bar{Y}_{..}) \\
&\quad +2\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{Y}_{.j}-\bar{Y}_{..})(Y_{ij}-\bar{Y}_{i.}-\bar{Y}_{.j}+\bar{Y}_{..}) \\
&= b\sum_{i=1}^{a}(\bar{Y}_{i.}-\bar{Y}_{..})^2+a\sum_{j=1}^{b}(\bar{Y}_{.j}-\bar{Y}_{..})^2 \\
&\quad +\sum_{i=1}^{a}\sum_{j=1}^{b}(Y_{ij}-\bar{Y}_{i.}-\bar{Y}_{.j}+\bar{Y}_{..})^2
\end{aligned}
$$

$$= SS_{Tr} + SS_{Bl} + SS_E$$

where the treatment sum of square

$$SS_{Tr} = b \sum_{i=1}^{a} (\bar{Y}_{i.} - \bar{Y}_{..})^2,$$

the block sum of square

$$SS_{Bl} = a \sum_{j=1}^{b} (\bar{Y}_{.j} - \bar{Y}_{..})^2,$$

and the error sum of squares

$$SS_E = \sum_{i=1}^{a} \sum_{j=1}^{b} (Y_{ij} - \bar{Y}_{i.} - \bar{Y}_{.j} + \bar{Y}_{..})^2.$$

Thus we have

$$SS_T = SS_{Tr} + SS_{Bl} + SS_E$$

and the corresponding degrees of freedom, mean

squares are as follows.

$$\begin{aligned} df_{Total} &= df_{Treatment} + df_{Block} + df_{Error} \\ N-1 = ab-1 &= (a-1) + (b-1) + (a-1)(b-1) \\ MS_{Tr} &= \frac{SS_{Tr}}{a-1}, \quad MS_{Bl} = \frac{SS_{Bl}}{b-1} \end{aligned}$$

and

$$MS_E = \frac{SS_E}{(a-1)(b-1)}$$

Remark: Under the H_0, SS_{Tr}/σ^2, SS_{Bl}/σ^2 and SS_E/σ^2, are independently distributed chi-square random variables

The expected value of the mean squares, if treatments and blocks are fixed, can be shown to be

$$E(MS_{Tr}) = \sigma^2 + \frac{b\sum_{i=1}^{a}\tau_i^2}{a-1}$$

$$E(MS_{Bl}) = \sigma^2 + \frac{a\sum_{j=1}^{b}\beta_j^2}{b-1}$$

$$E(MS_E) = \sigma^2$$

Therefore, to test the equality of treatment means, we would use the test statistic

$$F = \frac{MS_{Tr}}{MS_E}.$$

which is distributed as $F \sim F_{a-1,(a-1)(b-1)}$ if the null hypothesis H_0 is true.

We would reject H_0 at level α if the test statistic satisfying

$$f \geq F_{\alpha,(a-1),(a-1)(b-1)}.$$

Question: What is the p-value of the test ?

■ **TABLE 4.2**

Analysis of Variance for a Randomized Complete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a - 1$	$\frac{SS_{\text{Treatments}}}{a - 1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Blocks	SS_{Blocks}	$b - 1$	$\frac{SS_{\text{Blocks}}}{b - 1}$	
Error	SS_E	$(a - 1)(b - 1)$	$\frac{SS_E}{(a - 1)(b - 1)}$	
Total	SS_T	$N - 1$		

EXAMPLE 4.1

A medical device manufacturer produces vascular grafts (artificial veins). These grafts are produced by extruding billets of polytetrafluoroethylene (PTFE) resin combined with a lubricant into tubes. Frequently, some of the tubes in a production run contain small, hard protrusions on the external surface. These defects are known as "flicks." The defect is cause for rejection of the unit.

The product developer responsible for the vascular grafts suspects that the extrusion pressure affects the occurrence of flicks and therefore intends to conduct an experiment to investigate this hypothesis. However, the resin is manufactured by an external supplier and is delivered to the medical device manufacturer in batches. The engineer also suspects that there may be significant batch-to-batch variation, because while the material should be consistent with respect to parameters such as molecular weight, mean particle size, retention, and peak height ratio, it probably isn't due to manufacturing variation at the resin supplier and natural variation in the material. Therefore, the product developer decides to investigate the effect of four different levels of extrusion pressure on flicks using a randomized complete block design considering batches of resin as blocks. The RCBD is shown in Table 4.3. Note that there are four levels of extrusion pressure (treatments) and six batches of resin (blocks). Remember that the order in which the extrusion pressures are tested within each block is random. The response variable is yield, or the percentage of tubes in the production run that did not contain any flicks.

To conduct this experiment as a RCBD, assign all 4 pressures to each of the 6 batches of resin.

Each batch of resin is called a block; that is, it's a more homogenous experimental unit on which to test the extrusion pressures.

■ **TABLE 4.3**

Randomized Complete Block Design for the Vascular Graft Experiment

Extrusion Pressure (PSI)	Batch of Resin (Block)						Treatment Total
	1	2	3	4	5	6	
8500	90.3	89.2	98.2	93.9	87.4	97.9	556.9
8700	92.5	89.5	90.6	94.7	87.0	95.8	550.1
8900	85.5	90.8	89.6	86.2	88.0	93.4	533.5
9100	82.5	89.5	85.6	87.4	78.9	90.7	514.6
Block Totals	350.8	359.0	364.0	362.2	341.3	377.8	$y_{..} = 2155.1$

To perform the analysis of variance, we need the following sum of squares:

$$SS_T = \sum_{i=1}^{4}\sum_{j=1}^{6} y_{ij}^2 - \frac{y_{..}^2}{N}$$
$$= 193{,}999.31 - \frac{(2155.1)^2}{24} = 480.31$$

$$SS_{\text{Treatments}} = \frac{1}{b}\sum_{i=1}^{4} y_{i.}^2 - \frac{y_{..}^2}{N}$$
$$= \frac{1}{6}[(556.9)^2 + (550.1)^2 + (533.5)^2 + (514.6)^2] - \frac{(2155.1)^2}{24} = 178.17$$

$$SS_{\text{Blocks}} = \frac{1}{a}\sum_{j=1}^{6} y_{.j}^2 - \frac{y_{..}^2}{N}$$
$$= \frac{1}{4}[(350.8)^2 + (359.0)^2 + \cdots + (377.8)^2] - \frac{(2155.1)^2}{24} = 192.25$$

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}}$$
$$= 480.31 - 178.17 - 192.25 = 109.89$$

The ANOVA is shown in Table 4.4. Using $\alpha = 0.05$, the critical value of F is $F_{0.05,3,15} = 3.29$. Because $8.11 > 3.29$, we conclude that extrusion pressure affects the mean yield. The P-value for the test is also quite small. Also, the resin batches (blocks) seem to differ significantly, because the mean square for blocks is large relative to error.

Now let's consider **Multiple Comparisons**.

If the treatments in an RCBD are fixed, and the analysis indicates a significant difference in treatment means, the experimenter is usually interested in multiple comparisons to discover which treatment means differ.

Any of the multiple comparison procedures discussed in last Chapter may be used for this purpose. We can simply replace the number of replicates in the single-factor completely randomized design n by the number of blocks b. Also, remember to use the number of error degrees of freedom for the randomized block $(a-1)(b-1)$ instead of those for the completely randomized design $a(n-1)$.

For the Vascular Graft Example, we notice the highest pressure (9100 psi) results in a mean yield that is much lower than other means.

Overall, we would conclude that lower extrusion pressures (8500 psi and 8700 psi) lead to fewer defects.

- Adequacy Checking

As in the completely randomized design, residual analysis is the major tool used in checking the adequacy of the assumed model for the RCBD.

The residuals are calculated from

$$e_{ij} = Y_{ij} - \hat{Y}_{ij}$$

where the fitted values are $\hat{Y}_{ij} = \hat{Y}_{i.} + \hat{Y}_{.j} - \hat{Y}_{..}$ Thus we have

$$e_{ij} = Y_{ij} - \hat{Y}_{i.} - \hat{Y}_{.j} + \hat{Y}_{..}$$

Generally, we should be alert for potential problems with the normality assumption, unequal

error variance by treatment or block, and block-treatment interaction.

Based on the residual analysis for the RCBD in Example 4.1, we don't find obvious problems.

- Some Other Aspects of the RCBD

i) The RCBD utilizes an additive model no interaction between treatments and blocks Treatments.

ii) Treatments and/or blocks are random effects

iii) Sample sizing in the RCBD

In the RCBD, it is important to choos the sample size, or the number of blocks to run, is an important decision when using an RCBD.

Increasing the number of blocks increases the number of replicates and the number of error degrees of freedom, which makes design more sensitive. Any of the techniques discussed for selecting the number of replicates to run in a completely randomized single-factor experiment may be applied directly to the RCBD.

The OC curve approach can be used to determine the number of blocks to run. For the case of a fixed factor, the operating characteristic curves may be used with

$$\Phi^2 = \frac{b\sum_{i=1}^{a}\tau_i^2}{a\sigma^2}$$

where there are $a - 1$ numerator degrees of freedom and $(a-1)(b-1)$ denominator degrees of freedom.

The minimum value of Φ^2 is

$$\Phi^2 = \frac{bD^2}{2a\sigma^2}$$

if D is the maximum difference we wish to detect.

2. The Latin Square Design

Here we consider designs that are used to simultaneously control (or eliminate) two sources of nuisance variability.

A significant assumption is that the three factors (treatment, nuisance factors) do not interact. If this assumption is violated, the following Latin square design will not produce valid results.

By using RCBD, we can reduce the residual error in an experiment by removing variability due to a known and controllable nuisance variable. There are other types of designs that utilize the blocking principle.

For example, suppose that an experimenter is studying the effects of five different formulations of a rocket propellant used in aircrew escape systems on the observed burning rate.

Each formulation is mixed from a batch of raw material that is only large enough for five formulations to be tested. Furthermore, the formulations are prepared by several operators, and there may be substantial differences in the skills and experience of the operators. Thus, it would seem that there are two nuisance factors to be averaged out in the design: batches of raw material and operators.

The appropriate design for this problem consists of testing each formulation exactly once in each batch of raw material and for each formulation to be prepared exactly once by each of five operators. The resulting design, shown in the following table , is called a **Latin square design**.

Notice that the design is a square arrangement and that the five formulations (or treatments)

are denoted by the Latin letters A, B, C, D, and E; hence the name Latin square.

The above Rocket Propellant Problem is designed as a 5×5 Latin Square Design.

■ **TABLE 4.9**
Latin Square Design for the Rocket Propellant Problem

Batches of Raw Material	Operators				
	1	**2**	**3**	**4**	**5**
1	$A = 24$	$B = 20$	$C = 19$	$D = 24$	$E = 24$
2	$B = 17$	$C = 24$	$D = 30$	$E = 27$	$A = 36$
3	$C = 18$	$D = 38$	$E = 26$	$A = 27$	$B = 21$
4	$D = 26$	$E = 31$	$A = 26$	$B = 23$	$C = 22$
5	$E = 22$	$A = 30$	$B = 20$	$C = 29$	$D = 31$

The Latin square design is used to eliminate two nuisance sources of variability; that is, it systematically allows blocking in two directions. Thus, the rows and columns actually represent two restrictions on **randomization**.

In general, a **Latin square for p factors**, or a $p \times p$ Latin square, is a square containing p rows and p columns. Each of the resulting p^2 cells contains one of the p letters that corresponds to the treatments, and each letter occurs **once** and **only once** in each row and column.

A Latin square in which the first row and column consists of the letters written in alphabetical order is called a **standard Latin square**, which is the design shown in the above Example.

A standard Latin square can always be obtained by writing the first row in alphabetical

order and then writing each successive row as the row of letters just above shifted one place to the left.

See below for some examples of 3×3, 4×4, 5×5 and 6×6 Latin squares.

Example of Latin squares	3×3	4×4	5×5	6×6
	ABC	ABDC	ADBEC	ADCEBF
	BCA	BCAD	DACBE	BAECFD
	CAB	CDBA	CBEDA	CEDFAB
		DACB	BEACD	DCFBEA
			ECDAB	FBADCE
				EFBADC
# of standard squares	1	4	56	9408
Total # of Latin squares	12	576	161280	818851200

As with any experimental design, the observations in the Latin square should be taken in random order. The proper randomization procedure is to select the particular square employed at random.

As we see in the above table, there are a large number of Latin squares of a particular size,

so it is impossible to enumerate all the squares and select one randomly.

The usual procedure is to select an arbitrary Latin square from a table of such designs, or start with a standard square, and then arrange the order of the rows, columns, and letters at random.

The statistical (effect) model is

$$Y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \varepsilon_{ijk}\,, \quad \begin{cases} i = 1, 2, \cdots, p \\ j = 1, 2, \cdots, p \\ k = 1, 2, \cdots, p \end{cases}$$

where Y_{ijk} is the observation in the i^{th} row and k^{th} column for the j^{th} treatment, μ is the overall mean, α_i is the i^{th} row effect, τ_j is the j^{th} treatment effect, β_k is the k^{th} column effect, and ε_{ijk} is the random error.

Note that this is an effect model. The model is completely additive; that is, there is no interaction between rows, columns, and treatments.

Because there is only one observation in each cell, only two of the three subscripts i, j, and k are needed to denote a particular observation.

For example, referring to the rocket propellant problem, if $i = 3$ and $k = 2$, we automatically find $j = 4$ (formulation D). This is a consequence of each treatment appearing exactly once in each row and column.

The analysis of variance consists of partitioning the total sum of squares of the $N = p^2$ observations into components for rows, columns, treatments, and error, for example,

$$SS_T = SS_{Row} + SS_{Col} + SS_{Tr} + SS_E$$

with respective degrees of freedom

$$p^2 - 1 = (p-1) + (p-1) + (p-1) + (p-2)(p-1).$$

Under the usual assumption that ε_{ijk} is $NID(0, \sigma^2)$, each sum of squares on the right-hand side of

the above equation is, upon division by σ^2, an independently distributed chi-square random variable.

The appropriate statistic for testing for no differences in treatment means is

$$F = \frac{MS_{Tr}}{MS_E}$$

which is distributed as $F_{p-1,(p-2)(p-1)}$ under the null hypothesis.

The statistical analysis (ANOVA) is much like the analysis for the RCBD.

The computational procedure for the ANOVA in terms of treatment, row, and column totals is shown in the following ANOVA table.

See the ANOVA table, and the analysis for the rocket propellant example.

■ **TABLE 4.10**

Analysis of Variance for the Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}} = \frac{1}{p}\sum_{j=1}^{p} y_{\cdot j\cdot}^2 - \frac{y_{\cdots}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i\cdot\cdot}^2 - \frac{y_{\cdots}^2}{N}$	$p-1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{k=1}^{p} y_{\cdot\cdot k}^2 - \frac{y_{\cdots}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Error	SS_E (by subtraction)	$(p-2)(p-1)$	$\frac{SS_E}{(p-2)(p-1)}$	
Total	$SS_T = \sum_i\sum_j\sum_k y_{ijk}^2 - \frac{y_{\cdots}^2}{N}$	p^2-1		

■ **TABLE 4.12**
Analysis of Variance for the Rocket Propellant Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Formulations	330.00	4	82.50	7.73	0.0025
Batches of raw material	68.00	4	17.00		
Operators	150.00	4	37.50		
Error	128.00	12	10.67		
Total	676.00	24			

By the ANOVA, we conclude that there is a significant difference in the mean burning rate generated by the different rocket propellant formulations.

As in any design problem, the experimenter should investigate the adequacy of the model by inspecting and plotting the residuals. For a Latin square, the residuals are given by

$$\begin{aligned} e_{ijk} &= y_{ijk} - \hat{y}_{ijk} \\ &= y_{ijk} - \overline{y}_{i..} - \overline{y}_{.j.} - \overline{y}_{..k} + 2\overline{y}_{...} \end{aligned}$$

Replication of Latin Squares

A disadvantage of small Latin squares is that they provide a relatively small number of error degrees of freedom.

For example, a 3×3 Latin square has only two error degrees of freedom, a 4×4 Latin square has only six error degrees of freedom, and so forth.

When small Latin squares are used, it is frequently desirable to replicate them to **increase the error degrees of freedom**.

A Latin square may be replicated in several ways. To illustrate, suppose that the 5×5 Latin square (for the Rocket Propellant Problem) is replicated n times. This could have been done as follows:

Case 1: Use the same batches and operators in each replicate.

Case 2: Use the same batches but different operators in each replicate (or, equivalently, use the same operators but different batches).

Case 3: Use different batches and different operators.

The ANOVA depends on the method of replication.

For case 1, where the same levels of the row and column blocking factors are used in each replicate. Let Y_{ijkl} be the observation in row i, treatment j, column k, and replicate l. There are $N = np^2$ total observations.

The following is the Analysis of Variance for a Replicated Latin Square, Case 1.

CASE 1:

Source	df	SS	Mean Square	f
Treatment	p-1	$\frac{1}{np}\sum_{j=1}^{p} Y_{.j..}^2 - \frac{Y_{....}^2}{N}$	MS_{Tr}	$\frac{MS_{Tr}}{MS_E}$
Row	p-1	$\frac{1}{np}\sum_{i=1}^{p} Y_{i...}^2 - \frac{Y_{....}^2}{N}$	MS_{Row}	
Column	p-1	$\frac{1}{np}\sum_{k=1}^{p} Y_{..k.}^2 - \frac{Y_{....}^2}{N}$	MS_{Col}	
Replicates	n-1	$\frac{1}{p^2}\sum_{l=1}^{n} Y_{...l}^2 - \frac{Y_{....}^2}{N}$	MS_{Rep}	
Error	(p-1)[n(p+1)-3]	Subtraction	MS_E	
Total	np^2-1	$\sum\sum\sum\sum Y_{ijkl}^2 - \frac{Y_{....}^2}{N}$		

For case 2, we assume that new batches of raw material but the same operators are used in each replicate. Thus, there are now five new rows (in general, p new rows) within each replicate. The ANOVA is summarized in the following table. Note that the source of variation for the rows really measures the variation between rows within the n replicates.

CASE 2:

Source	df	SS	MS	f
Treatment	p-1	$\frac{1}{np}\sum_{j=1}^{p} Y_{.j..}^2 - \frac{Y_{....}^2}{N}$	MS_{Tr}	$\frac{MS_{Tr}}{MS_E}$
Row	n(p-1)	$\frac{1}{p}\sum_{i=1}^{p}\sum_{l=1}^{n} Y_{i..l}^2 - \frac{\sum_{l=1}^{n} Y_{...l}^2}{p^2}$	MS_{Row}	
Column	p-1	$\frac{1}{np}\sum_{k=1}^{p} Y_{..k.}^2 - \frac{Y_{....}^2}{N}$	MS_{Col}	
Replicates	n-1	$\frac{1}{p^2}\sum_{l=1}^{n} Y_{...l}^2 - \frac{Y_{....}^2}{N}$	MS_{Rep}	
Error	(p-1)(np- 2)	Subtraction	MS_E	
Total	np^2-1	$\sum\sum\sum\sum Y_{ijkl}^2 - \frac{Y_{....}^2}{N}$		

Finally, consider case 3, where new batches of raw material and new operators are used in each replicate. Now the variation that results from both the rows and columns measures the variation resulting from these factors within the replicates. The ANOVA is summarized as follows.

CASE 3:

Source	df	SS	Mean Square
Treatment	p-1	$\frac{1}{np}\sum_{j=1}^{p} Y_{.j..}^2 - \frac{Y_{....}^2}{N}$	MS_{Tr}
Row	n(p-1)	$\frac{1}{p}\sum_{i=1}^{p}\sum_{l=1}^{n} Y_{i..l}^2 - \frac{\sum_{l=1}^{n} Y_{...l}^2}{p^2}$	MS_{Row}
Column	n(p-1)	$\frac{1}{p}\sum_{k=1}^{p}\sum_{l=1}^{n} Y_{..kl}^2 - \frac{\sum_{l=1}^{n} Y_{...l}^2}{p^2}$	MS_{Col}
Replicates	n-1	$\frac{1}{p^2}\sum_{l=1}^{n} Y_{...l}^2 - \frac{Y_{....}^2}{N}$	MS_{Rep}
Error	(p-1)[n(p-1)-1]	Subtraction	MS_E
Total	$np^2 - 1$	$\sum\sum\sum\sum Y_{ijkl}^2 - \frac{Y_{....}^2}{N}$	

3. The Balanced Incomplete Block Design (BIBD)

In some experiments using randomized block designs, we may not be able to run all the treatment combinations in each block.

For example, in the vascular graft experiment (Example 4.1), suppose that each batch of material is only large enough to accommodate testing 3 extrusion pressures. Therefore, each pressure can't be tested in each batch.

For this type of problem it is possible to use randomized block designs in which every treatment is not present in every block. These designs are known as **randomized incomplete block designs**.

When all treatment comparisons are equally important, the treatment combinations used

in each block should be selected in a balanced manner, so that any pair of treatments occur together the same number of times as any other pair.

Thus, a **balanced incomplete block design** (BIBD) is an incomplete block design in which **any two treatments** appear together an equal number of times.

Suppose that there are a treatments and that each block can hold exactly $k (k < a)$ treatments. A balanced incomplete block design may be constructed by taking $\binom{a}{k}$ blocks and assigning a different combination of treatments to each block.

However, Frequently, balance can be obtained with fewer than $\binom{a}{k}$ blocks. Tables of BIBDs are given in Fisher and Yates (1953), Davies (1956), and Cochran and Cox (1957).

Example: Suppose that a chemical engineer thinks that the time of reaction for a chemical process is a function of the type of catalyst employed. Four catalysts are currently being investigated. The experimental procedure consists of selecting a batch of raw material, loading the pilot plant, applying each catalyst in a separate run of the pilot plant, and observing the reaction time. Because variations in the batches of raw material may affect the performance of the catalysts, the engineer decides to use batches of raw material as blocks. However, each batch is only large enough to permit three catalysts to be run. Therefore, a randomized incomplete block design must be used.

The balanced incomplete block design for the above experiment, along with the observations recorded, is shown in the following table. The order in which the catalysts are run in each block is randomized.

Treatment (Catalyst)	Block 1	2	3	4	$y_{i.}$
1	73	74	-	71	218
2	-	75	67	72	214
3	73	75	68	-	216
4	75	-	72	75	222
$y_{.j}$	221	224	207	218	870=$y_{..}$

Statistical Analysis of the BIBD

For the statistical analysis of the BIBD, we assume that there are a treatments and b blocks.

In addition, we assume that each block contains k treatments, that each treatment occurs r times in the design (or is replicated r times). **Question:** what is the number of total observations ?

$$N = ar = bk$$

Let λ denotes the number of times each pair of treatments appears in the same block. **Question:** what is λ?

It is obvious that the parameter λ must be an integer. We can show that it satisfies that

$$(a-1)\lambda = r(k-1).$$

To derive the above relationship for λ, consider any treatment, say treatment 1. Because treatment 1 appears in r blocks and there are $k-1$ other treatments in each of those blocks, there are r(k-1) other observations in blocks containing treatment 1. These $r(k-1)$ observations also have to represent the remaining $a-1$ treatments λ times.

Therefore, we have

$$\lambda = \frac{r(k-1)}{a-1}$$

If $a = b$, the design is said to be symmetric.

The **statistical model** for the BIBD is

$$Y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij},$$

whereμ is the overall mean, τ_i is the effect of the i^{th} treatment, β_j is the effect of the j^{th} block, and ε_{ij} is the $NID(0, \sigma^2)$ random error component.

The total variability in the data is expressed by the total corrected sum of squares:

$$SS_T = \sum_i \sum_j (Y_{ij} - \bar{Y}_{..})^2 = \sum_i \sum_j Y_{ij}^2 - \frac{Y_{..}^2}{N}$$

The total variability can be partitioned as follows

$$SS_T = SS_{Tr(adjusted)} + SS_{Bl} + SS_E$$

where the sum of squares for treatments is adjusted to separate the treatment and the block effects.

This adjustment is necessary because each treatment is represented in a different set of r blocks. Thus, differences between unadjusted treatment totals $Y_{1.}, Y_{2.}, ..., Y_{a.}$ are also affected by differences between blocks.

The block sum of squares is

$$SS_{Bl} = \frac{1}{k}\sum_{j} Y_{.j}^2 - \frac{Y_{..}^2}{N}$$

with $b - 1$ degrees of freedom.

The adjusted treatment sum of squares is

$$SS_{Tr(ad)} = \frac{k\sum_{i=1}^{a} Q_i^2}{\lambda a}$$

where Q_i is the adjusted total for the i^{th} treatment, which is computed as

$$Q_i = Y_{i.} - \frac{1}{k}\sum_{j=1}^{b} n_{ij} Y_{.j}$$

with

$$n_{ij} = \begin{cases} 1 & \text{if treatment i appears in block j} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that

$$\sum_{i=1}^{a} Q_i = 0,$$

i.e., the adjusted treatment totals will always sum to zero.

$SS_{Tr(adjusted)}$ has $a - 1$ degrees of freedom.

The error sum of squares is computed by subtraction as

$$SS_E = SS_T - SS_{Tr(adjusted)} - SS_{Bl}$$

and has $N - a - b + 1$ degrees of freedom.

The appropriate statistic for testing the equality of the treatment effects is

$$F = \frac{MS_{Tr(adjusted)}}{MS_E}.$$

The ANOVA table is as follows.

Source of variation	df	SS	Mean Square	f
Treatments(adjusted)	a-1	$SS_{Tr(adj)}$	$MS_{Tr(adj)}$	$\frac{MS_{Tr(adj)}}{MS_E}$
Blocks	b-1	SS_{Bl}	MS_{Bl}	
Error	N-a-b+1	SS_E	MS_E	
Total	N-1	SS_T		

Example: Consider the data in the table for the catalyst experiment. This is a BIBD with $a = 4, b = 4, k = 3, r = 3, \lambda = 2$,and $N = 12$. The analysis of this data is as follows.

$$SS_T = \sum_i \sum_j Y_{ij}^2 - \frac{Y_{..}^2}{12} = 81.00$$

$$SS_{Bl} = \frac{1}{3} \sum_j Y_{.j}^2 - \frac{Y_{..}^2}{12} = 55.00$$

To compute the treatment sum of squares adjusted for blocks, we first determine the adjusted treatment totals by $Q_i = Y_{i.} - \frac{1}{k} \sum_{j=1}^{b} n_{ij} Y_{.j}$

We obtain that

$$Q_1 = -3, Q_2 = -7/3, Q_3 = -4/3, Q_4 = 20/3.$$

The adjusted sum of squares for treatments is

$$SS_{Treatments(adjusted)} = 22.75$$

The error sum of squares is obtained by subtraction as

$$SS_E = SS_T - SS_{Tr(adjusted)} - SS_{Bl}$$

$$= 81.00 - 22.75 - 55.00 = 3.25$$

The ANOVA table is as follows.

Source of variation	df	SS	Mean Square	f	P_{value}
Treatments(adjusted)	3	22.75	7.58	11.66	0.0107
Blocks	3	55	18.33		
Error	5	3.25	0.65		
Total	11	81			

Because the P-value is small, we conclude that the catalyst employed has a significant effect on the time of reaction.