Wordle is NP-Complete

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1 Dominating Set Cover

Let $k \geq 1$, and $G = (V, E)$ an undirected graph. Does there exists $D \subseteq V$ of size $\leq k$ such that for every $v \notin D$ there exists a $w \in D$ such that $(v, w) \in E$.

2 Wordle

Let $\mathcal{A} = \{\alpha_i\}_{i=1}^{l}$ be an alphabet, $\mathcal{A}^m = \{\alpha_{i_1} \cdots \alpha_{i_m} | 1 \leq j \leq m \ \& \ 1 \leq i_j \leq l\}$, $S \subseteq A^m$, $t \in S$, and $f_t : S \to \{0, 1, 2\}^m$ is a scoring function, defined as $f_t(w) = \left\langle f_t^1(w), \cdots, f_t^m(w) \right\rangle$ where

$$f_t^i(w) = \begin{cases} 0 & w_i = t_i \\ 1 & w_i \neq t_i \ \& \ \exists j (w_i = t_j \ \& \ \forall k < i(w_k = w_i \to \exists r < j(w_k = t_r))) \\ 2 & otherwise \end{cases}$$

Does there exists a subset S' of $S \setminus \{t\}$ such that $|S'| \leq n < |S| - 1$ and t is the only word not excluded by the set of constraints $\{f_t(w) | w \in S'\}$?

Definition 1 *For $w, w' \in S$, we say that $f_t(w)$ excludes a word w' if one of the following holds:*

- $f_t^i(w) = 0$ *and* $w_i \neq w'_i$.
- $f_t^i(w) = 1$ *and* $w_i \notin w'$.
- $f_t^i(w) = 2$, $\forall j (w_i = w_j \to f_t^j(w) = 2)$ *and* $w_i \in w'$.
- $f_t^i(w) = 2$, $\exists j (w_i = w_j \ \& \ f_t^j(w) < 2 \ \& \ |\{j | w_i = w_j\}| \leq |\{j | w_i = w'_j\}|)$.

3 Reduction from dominating set to Wordle

First we need to construct a graph $K = (V, E')$ such that $E' = B \cup R$ where $B = E$ and $R = \{(v, w) | v, w \in V \ and \ (v, w) \notin E\}$. Let $NR(v) = \{(v, w) | w \in V, \ and \ (v, w) \in R\}$ and $NB(v) = \{(v, w) | w \in V, \ and \ (v, w) \in B\}$.

Note that K is complete. Our alphabet $\mathcal{A}$ contains $|V|^2|B|$ letters which we denote as follows

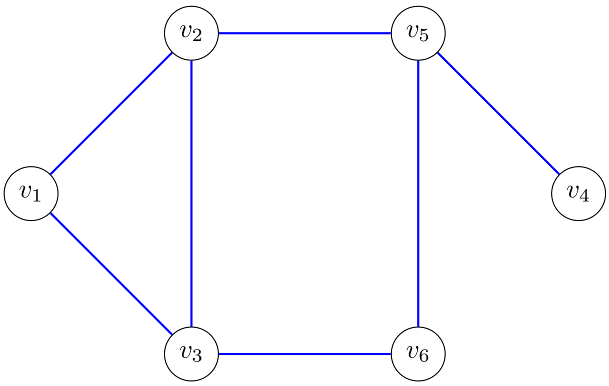
$$\{\alpha_i\}_{i=1}^{|B|} \cup \{\beta_i\}_{i=1}^{(|V|^2-1)|B|}$$

In total we construct $|V|+1$ words denoted by $\{t\} \cup \{w^i\}_{i=1}^{|V|}$. Let $B = \{b_i\}_{i=1}^{|B|}$, $R = \{r_i\}_{i=1}^{|R|}$, and $V = \{v_i\}_{i=1}^{|V|}$, then

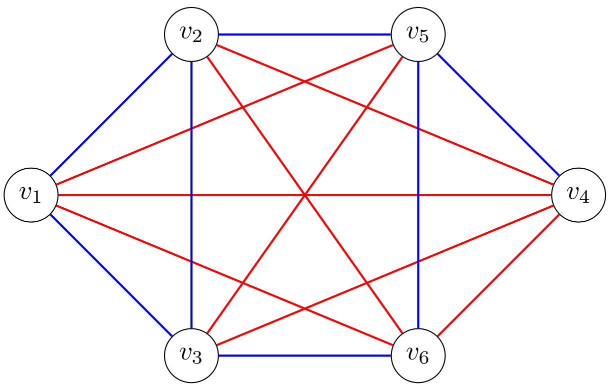
- for $1 \leq i \leq |B|$, if $b_i = (v_j, v_k)$, then $w_i^j = w_i^k = \alpha_i$.
- for $1 \leq i \leq |V|$ and $|B| + 1 + \sum_{k=1}^{i-1} NR(V_k) \leq l \leq j \leq |B| + 1 + \sum_{k=1}^{i} NR(V_k)$, $w_j^i \in \{\alpha_i\}_{i=1}^{|B|}$ such that $w_j^i \neq w_l^i$ iff $j \neq l$, and for all $1 \leq q \leq |B|$, $w_j^i \neq w_q^i$.
- $1 \leq i \leq |B|$, $t_{i+|B|+1+\sum_{k=1}^{|V|} NR(V_k)} = \alpha_i$
- All positions not mentioned contain a unique symbol from $\{\beta_i\}_{i=1}^{(|V|^2-1)|B|}$. Notice, each of these words is $|V||B|$ long.

4 example

The graph below has a minimal dominating sets of $\{v_3, v_5\}$ and $\{v_2, v_5\}$.



We transform the graph preserving the edges using colors.



From the transformed graph we need to construct words for each vertex. The idea is that words connected by a blue edge exclude each other and words connected by a red edge don't.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
v_1	$\bullet$	$\bullet$	α_3	$\bullet$	$\bullet$	$\bullet$	α_7	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_2	α_1	α_2	α_3	$\bullet$	$\bullet$	$\bullet$	$\bullet$	α_4	α_5	α_6	α_7	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_3	$\bullet$	α_2	$\bullet$	$\bullet$	$\bullet$	α_6	α_7	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_4	$\bullet$	$\bullet$	$\bullet$	α_4	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	α_1
v_5	α_1	$\bullet$	$\bullet$	α_4	α_5	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	α_2	α_3	α_6	α_7	$\bullet$
v_6	$\bullet$	$\bullet$	$\bullet$	$\bullet$	α_5	α_6	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
t	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$

	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
v_1	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	α_1	α_2
v_2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_3	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	α_1	α_3	α_4	α_5	$\bullet$	$\bullet$
v_4	α_2	α_3	α_5	α_6	α_7	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_5	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_6	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	α_1	α_2	α_3	α_4	α_7	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
t	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$

	33	34	35	36	37	38	39	40	41	42
v_1	α_4	α_5	α_6	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_3	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_4	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_5	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
v_6	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
t	$\bullet$	$\bullet$	$\bullet$	α_1	α_2	α_3	α_4	α_5	α_6	α_7

Bullets are chosen uniquely from $\{\beta_i\}_{i=1}^{245}$. Notice that $\{w_2, w_5\}$, and $\{w_2, w_5\}$ cover all words but the target t. All other sets of words do not cover the whole set or contain more words.