

1 Single Systems

1.1 Classical States and Probability Vectors

For a system of interest X , let Σ refer to the set of classical states of X . We can assume that Σ is finite and non-empty as it does not make sense for a physical system to have no states at all. For example, if X is a bit, then $\Sigma = \{0, 1\}$. Suppose X is a bit with known probabilities represented as:

$$Pr(X = 0) = \frac{3}{4} \quad \text{and} \quad Pr(X = 1) = \frac{1}{4}$$

A more compact way to represent this probability is using a column vector $\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$ where the following two properties must be satisfied:

1. The entries of the vector must be non-negative
2. The sum of the entries must equal to one for probability

1.2 Measuring Probabilistic States

By measuring a system, we mean that we look at the system and unambiguously recognize whatever classical state it is in. We can never "see" a system in a probabilistic state; a measurement will yield exactly one of the allowed classical states. Measurements change our knowledge of the system, correspondingly the probabilistic state we associate with that system. If X is recognized to be in the classical state $a \in \Sigma$ then the new probability vector representing our knowledge of X becomes a vector with 1 in the entry corresponding to a and 0 for all entries. A vector having 1 in entry corresponding to a and 0 for all other entries is denoted as $|a\rangle$, known as "ket a ." The standard basis vectors for a bit is given by:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For the previous system, the two-dimensional column vector can be expressed as such:

$$\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix} = \frac{3}{4}|0\rangle + \frac{1}{4}|1\rangle$$

Any column vector is a linear combination over the classical states. The probabilistic state of a coin flip in a set: $\Sigma = \{\text{heads}, \text{tails}\}$ can be expressed as:

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{2}|\text{heads}\rangle + \frac{1}{2}|\text{tails}\rangle \quad \textbf{where} \quad |\text{heads}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\text{tails}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A note on the measurement of probabilistic states: they may only describe knowledge, not something that's real. The state of the coin after being flipped before we look at it is either heads or tails, we simply cannot tell until we look at it. Looking at it doesn't change the state but only our knowledge of it. Upon seeing that the classical state is tails, we naturally update our knowledge by assigning the vector $|\text{tails}\rangle$ to the coin, but to someone else who didn't see the coin when it was uncovered, the probabilistic state remains unchanged.

1.3 Deterministic Operations

Deterministic operations are when each classical state $a \in \Sigma$ is transformed into $f(a)$ for some function f of the form $f : \Sigma \rightarrow \Sigma$. If $\Sigma = \{0, 1\}$, four functions (f_1, f_2, f_3, f_4) can be represented by following table of values:

a	$f_1(a)$	a	$f_2(a)$	a	$f_3(a)$	a	$f_4(a)$
0	0	0	0	0	1	0	1
1	0	1	1	1	0	1	1

The first two functions are constant while the middle two are *balanced*, where the two possible output values occur the same number of times ranged over the possible inputs. f_2 is known as the identity function that returns the same value: $f_2(a) = a$ and f_3 is the NOT function that returns the opposite value: $f_3(0) = 1$.

Deterministic operations on probabilistic states can be represented by matrix-vector multiplication such that Matrix \mathbf{M} represents a given function $f : \Sigma \rightarrow \Sigma$ that satisfies

$$M|a\rangle = |f(a)\rangle$$

for every $a \in \Sigma$. Such a matrix \mathbf{M} always exists and is unique. The matrices corresponding to the previous 4 functions can be expressed as:

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note: This operation is not straight-up matrix multiplication equals output, **remember** it is Matrix \mathbf{M} multiplied by "ket a" that equals "ket" of $f(a)$.

Example for M_1 :

$$M_1|0\rangle = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle = |f_1(a)\rangle$$

We can represent these matrices with $\langle a|$ as the row vector with 1 in entry a and zero for all other entries, known as "bra a." For set $\Sigma = \{0, 1\}$:

$$\langle 0| = (1 \ 0) \quad \text{and} \quad \langle 1| = (0 \ 1)$$

For an arbitrary classical state set Σ , row vectors and column vectors make up a matrix. Performing matrix multiplication $|b\rangle\langle a|$ results in a square matrix with a 1 in entry (b, a) for row b and column a with 0 for all other entries. For example:

$$|0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Using this notation, for any function $f : \Sigma \rightarrow \Sigma$, Matrix \mathbf{M} can be expressed correspondingly to function f as such:

$$M = \sum_{a \in \Sigma} |f(a)\rangle\langle a|$$

Performing $\langle a||b\rangle$ for a set Σ results in a 1x1 matrix, a scalar. This operation can be written as $\langle a|b\rangle$. which satisfied the following condition:

$$\langle a|b\rangle = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

From this observation, with the properties of matrix multiplication being associative and linear, it can be shown that:

1.4 Summary

At this Dirac notation explains the representation of "bra" $\langle a|$ and "ket" $|b\rangle$ combines to "bracket" $\langle a|b\rangle$.

2 Probabilistic Operations and Stochastic Matrices

2.1 Probabilistic Operations

In addition to deterministic operations, there are probabilistic operations. Consider an operation on a bit where if the classical state of the bit is 0 it remains 0 and if the classical state of the bit is 1 then it is flipped to 0 with probability $\frac{1}{2}$. This operation can be performed by multiplying matrix \mathbf{M} on the standard basis:

$$M|0\rangle = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \quad \text{and} \quad M|1\rangle = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

For an arbitrary classical state set Σ , the set of all probabilistic operations can be described in mathematical terms represented by stochastic (random) matrices that satisfy:

1. The entries of the matrix must be non-negative.
2. The sum of each column vector of the matrix must equal to 1 for probability.

Stochastic matrices are matrices whose columns all form probability vectors.

Column Picture: Each column of the stochastic matrix can be viewed as a probability vector of the classical state input that corresponds to that column. (The classical input state that is mapped to the corresponding probability vector of the stochastic matrix.) For the previous example:

$$|0\rangle \text{ maps to } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle \text{ maps to } \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

"Row Picture" or Combination of Deterministic Operations: The probabilistic operations can also be seen as random choices of deterministic operations. For the previous example this can be seen as applying the constant 0 functions on bit 0 $|0\rangle$ and bit 1 $|1\rangle$ which can be expressed as such:

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

2.2 Compositions of Probabilistic Operations

High Level Overview: Matrix multiplication is associative **NOT** commutative.

3 Quantum Information

When dealing with quantum information, a different choice for the type of vector that represents a state, now being a quantum state of the system, needs to be considered.

3.1 Quantum State Vectors

Similar to probabilistic states, a quantum state of a system is also represented by a column vector. Quantum state vectors are characterized by the following two properties:

1. The entries of a quantum state vector are complex numbers.
2. The sum of the absolute values squared of the quantum state vector entries is equal to 1 for probability.

The condition for absolute values squared in a quantum state vector allows for the possibility of negative real number entries.

The **Euclidean Norm** (square root of the sum of the squares of the elements) of a column vector:

$$v = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ is defined as: } \|\mathbf{v}\| = \sqrt{\sum_{k=1}^n |\alpha_k|^2}$$

Thus, quantum state vectors are unit vectors with respect to the Euclidean Norm.

3.2 Examples of Qubit States

A qubit refers to a quantum system whose classical state set is $\{0, 1\}$. A qubit is essentially a bit that can be in the quantum state. Here are a few examples of quantum states of a qubit:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle, \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle, \quad \begin{pmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{pmatrix} = \frac{1+2i}{3} |0\rangle - \frac{2}{3} |1\rangle.$$

Recall that the absolute value/magnitude (modulus) of a complex number is defined as:

$$|a + bi| = \sqrt{a^2 + b^2} \rightarrow |a + bi|^2 = \left(\sqrt{a^2 + b^2}\right)^2 = a^2 + b^2$$

Thus for the last example:

$$\left|\frac{1+2i}{3}\right|^2 + \left|-\frac{2}{3}\right|^2 = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \left(\frac{1}{9}\right) + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right) = 1$$

Note: they are linear combinations of the states $|0\rangle$ and $|1\rangle$. They are also known as superpositions of the 0 and 1 states.

The first examples of a qubit state vector is called the plus and minus state:

$$|+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$$

If we have some vector $|\psi\rangle$ whose indices correspond to some classical state set Σ , where $a \in \Sigma$ is an element of this classical set, then the matrix product $\langle a|\psi\rangle$ is equal to the entry of vector $|\psi\rangle$ whose index corresponds to a . For example, if $\Sigma = \{0, 1\}$ and:

$$|\psi\rangle = \frac{1+2i}{3} |0\rangle - \frac{2}{3} |1\rangle = \begin{pmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{pmatrix}, \quad \text{then} \quad \langle 0|\psi\rangle = \frac{1+2i}{3} \quad \text{and} \quad \langle 1|\psi\rangle = -\frac{2}{3}.$$

3.3 Generalized form for a Quantum State of a System

From Dirac Notation, for an arbitrary classical state set Σ , we consider the quantum state vector:

$$\frac{1}{\sqrt{|\Sigma|}} = \sum_{a \in \Sigma} |a\rangle$$

that is a uniform superposition of the classical states in Σ , where in this case $|\Sigma|$ refers to the number of elements in Σ .

3.4 Measuring Quantum States

Let us consider what happens when a quantum state of a system is measured, primarily with a simple type of measurement known as standard basis measurement. When the quantum state of a system is measured, the observer will see the classical state vector not the quantum state. In a sense, measurements act as the interface between quantum and classical information, where classical information is extracted from quantum states. When a quantum state is measured, each classical state of the system will have probabilities absolute value squared of each corresponding entry in the quantum state vector. This is known as *Born Rule* in quantum mechanics. As an example, measuring the plus state:

$$|+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$$

results in two possible outcomes for 0 and 1 with probabilities:

$$\Pr(\text{outcome is 0}) = |\langle 0|+\rangle|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

$$\Pr(\text{outcome is 1}) = |\langle 1|+\rangle|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

The exact same is true when measuring the minus state:

$$|-\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$$

$$\Pr(\text{outcome is 0}) = |\langle 0|-\rangle|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

$$\Pr(\text{outcome is 1}) = |\langle 1|-\rangle|^2 = \left| -\frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

If the standard basis measurements of the plus and minus states are the exact same, then why make a distinction?

3.5 Unitary Operations

The set of allowable operations performable on quantum states is different than it is for classical information. Similar to probabilistic settings, quantum states operations are also linear mappings, but the operations are represented by unitary matrices instead of stochastic matrices.

A square matrix U with complex number entries is *unitary* if it satisfies:

$$UU^\dagger = I \quad \text{and} \quad U^\dagger U = I$$

where U^\dagger is the conjugate transpose of U and I is the identity matrix. The conjugate transpose (Hermitian transpose) of a matrix is a combined operation consisting of two steps:

1. **Transpose:** Flipping the matrix over its diagonal, changing the row index to the column index for each element.
2. **Complex Conjugation:** Changes the sign of the imaginary parts of each complex number in the matrix such that for a complex number $a + bi$, its complex conjugate is $a - bi$.

Example: For matrix A , its conjugate transpose A^\dagger is as follows:

$$A = \begin{pmatrix} 1+i & 2 \\ 3 & 4-i \end{pmatrix} \rightarrow A^\dagger = \begin{pmatrix} 1-i & 3 \\ 2 & 4+i \end{pmatrix}.$$

The equivalence of the two above equalities are only true for square matrices. U being a unitary matrix holds the condition where multiplication by U does not change the Euclidean norm of any vector. An $n \times n$ matrix U is unitary if and only if:

$$\|U|\psi\rangle\| = \||\psi\rangle\| \quad \text{for every } n\text{-dim column vector with complex number entries.}$$

3.6 Important examples of Unitary Operations on Qubits

1. **Pauli Operations:** four Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For simpler notation, $X = \sigma_x$, $Y = \sigma_y$, and $Z = \sigma_z$ — but be aware that the letters X , Y , and Z are also commonly used for other purposes. The X operation is also called a *bit flip* or a *NOT operation* because it induces this action on bits:

$$X|0\rangle = |1\rangle \quad \text{and} \quad X|1\rangle = |0\rangle.$$

The Z operation is also called a *phase flip* because it has this action:

$$Z|0\rangle = |0\rangle \quad \text{and} \quad Z|1\rangle = -|1\rangle.$$

2. **Hadamard Operation:** geometrically a 90 degree rotation around the Y-axis, then a 180 degree rotation around the X-axis.

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$