# 1 Multiple Systems

The general idea is to view multiple systems together as a formation of a single, compound system. This concept leads to a description of how quantum states, measurements, and operations work for multiple systems.

## 1.1 Classical Information

### 1.1.1 Classical States via the Cartesian Product

For simplicity, lets first begin by introducing two systems then generalizing to more than two systems. Let X be a system whose classical state set is $\Sigma$ and Y be a second system with classical state set $\Gamma$.

Reasonably assume that $\Sigma$ and $\Gamma$ are two finite and nonempty sets that do not necessarily equal one another. Now imagine the two systems X and Y, placed side-by-side to one another, with X on the left and Y on the right.

We can view the two systems as a single system denoted by (X, Y) or simply XY. The set of classical states of (X, Y) is the *Cartesian Product* of $\Sigma$ and $\Gamma$, defined as:

$$\Sigma \times \Gamma = \{(a, b) \ : \ a \in \Sigma \textbf{ and } b \in \Gamma\}.$$

Essentially, the Cartesian Product is the mathematical notion that captures the idea of viewing an element of on set and an element of another set together to form a single element of a single (combined) set. To say (X, Y) is in the classical state $(a, b) \in \Sigma \times \Gamma$ means that X is in the classical state $a \in \Sigma$ and Y is in the classical state $b \in \Gamma$. The classical state of the joint system (X, Y) is (a, b).

For more than two systems we can generalize classical states of the joint systems. Suppose $X_1, \ldots, X_n$ are systems with classical state sets $\Sigma_1, \ldots, \Sigma_n$, the cartesian product of the compound system is:

$$\Sigma_1 \times \cdots \times \Sigma_n = \{(\sigma_1, \ldots, \sigma_n) : \sigma_1 \in \Sigma_1, \ldots, \sigma_n \in \Sigma_n\}.$$

For example if $\Sigma_1 = \Sigma_2 = \Sigma_3 = \{0, 1\}$ (a system of 3 bits), then the classical state set of $(X_1, X_2, X_3)$ is:

$$\Sigma_1 \times \Sigma_2 \times \Sigma_3 = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

### 1.1.2 Representing States as Strings

It is convenient to write a classical state. $(a_1, \ldots, a_n)$ as a string $a_1, \cdots, a_n$. Consider a 10 bit system $(X_1, \ldots, X_n)$ with $2^{10} = 1024$ classical states which are the elements of the set:

$$\Sigma_1 \times \cdots \times \Sigma_{10} = \{0, 1\}^{10}$$

The cartesian product is ordered lexicographically (significance of index decreases left to right), where the previous system is represented in strings as such:

0000000000
0000000001
0000000010

$$\begin{gathered}0000000100\\ \vdots\\ 1111111111\end{gathered}$$