OLG model with multiple sources of risk*

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1 Household problem

Households live for a span of $j = 1, 2, ..., J$ periods, with each period corresponding to five years. Households differ by age j. They begin their life cycle at the age of 20, denoted as $j = 1$, a simplification to ignore the timing of labor market entry and educational choice. Households supply labor to the market until they reach a retirement age, $\bar{J}$.

Each household is subject to age-specific survival rates π_j, representing the conditional probabilities of living up to a given age, j. Consumers who survive until the age of $J = 16$ face certain death.

Households have preferences over stochastic streams of consumption $c_j > 0$, labor supply $\ell_j \geq 0$. Individuals discount the future with the time discount factor δ and incur a utility loss from labor supply. They maximize a discounted expected utility:

$$U_0 = E_0 \left[\sum_{j=1}^{J} \left(\prod_{i=1}^{j} \delta_i \pi_i \right) u(c_j, \ell_j) \right]$$

where expectations are formed with respect to survival risk and idiosyncratic productivity risk, discount factor risk and capital return risk.

Households also face risks related to discount factors, labor income, and returns on capital. The productivity, interest rate and discount factor shocks realize at the beginning of the period.

1.1 Stochastic discount rate

Households discount rate consist of the deterministic $\bar{\delta}$ and stochastic component ϵ_δ. The deterministic component is homogeneous across all household. The stochastic component ϵ_δ that follows AR(1) process with persistence parameter ϱ_δ and $\varepsilon_\delta \sim \mathbf{N}(0, \sigma_\delta^2)$:

$$\ln(\epsilon_\delta^+) = \varrho_\delta \ln(\epsilon_\delta) + \varepsilon_\delta^+. \tag{1}$$

Therefore households discount rate is given by

$$\delta_i = \bar{\delta} + \epsilon_\delta.$$

*This documentation is an outcome of the project "WEALTH: Wealth inequality, demographic change and family". The support of National Center for Science (grant #2016/22/E/HS4/00129) is gratefully acknowledged. For details check https://grape.org.pl/project/wealth-inequality.

1.2 Labor productivity

Households are ex ante heterogeneous, and differ in their productivity type t. There is also age and type specific deterministic productivity profile $e_{j,t}$. Furthermore, individual productivity is differentiated by the stochastic component, ϵ_ℓ that follows AR(1) process with persistence parameter ϱ_ℓ and $\varepsilon_\ell \sim \mathbf{N}(0, \sigma_\ell^2)$:

$$\ln(\epsilon_\ell^+) = \varrho \ln(\epsilon_\ell) + \varepsilon_\ell^+. \tag{2}$$

Individual productivity evolves over the lifetimes according to the following formula, $\omega = e_{j,t}\epsilon_\ell$. Individuals also differ in their initial wage shock realisation.

1.3 Interest rate risk

Households differ with respect to the interest rate. The average interest rate in the economy is given by $\bar{r}$. The stochastic component of capital returns follow the normal distribution, $\varepsilon_r \sim \mathbf{N}(0, \sigma_r^2)$. Individual interest rate r is characterize by:

$$r = \bar{r} + \varepsilon_r.$$

1.4 Budget constraint

Households aged below the retirement age earn gross labor income $(1-\tau)\omega w \ell$, where w is is the marginal productivity of aggregate labor, ℓ denotes labor supply, and τ is social security contribution. In addition to salary, income also consists of capital gain ra. Let us denote $R = 1 + r$. Moreover, since survival rates π_j are lower than one, there are unintended bequests, which are evenly distributed within cohort, Γ. During retirement stage instead of labor income households receive social security benefit b.

The households face the following instantaneous budget constraint:
during working period:

$$a^+ + c = \omega w \ell + Ra + \Gamma, \tag{3}$$

during retirement

$$a^+ + c = b + Ra + \Gamma, \tag{4}$$

1.5 Dynamic optimization

An individual state of each household at $s = \{j, a, \epsilon_\delta, \epsilon_\ell, \varepsilon_r\} \in \Omega$ can be summarized by age j, the level of private assets a, discount rate shock ϵ_δ, individual productivity shock ϵ_ℓ, and interest rate shock ε_r.

A newborn household enters the economy with no assets ($a_{1,t} = 0$) and at the state $s_{j,t}$ the household maximizes the expected value of the lifetime utility. We define the optimization problem of the household in a recursive form as:

$$V(s) = \max_{(c,\ell,a^+)} u(c,\ell) + \delta \pi_j \mathbf{E}\big(V(s^+) \mid s\big), \tag{5}$$

subject to the budget constraint given by equations (3) and (4). The total time endowment is normalized to one.

1.6 First order conditions

Household problem for retiree:

$$\begin{aligned} & V(s) = \max_{(c,a^+)} u(c,0) + \delta\pi_j \mathbf{E}\big(V(s^+) \mid s\big), \\ s.b. \quad & a^+ + c = Ra + \Gamma \end{aligned}$$

It directly leads us to the first-order conditions

$$u_c\left(c\left(s\right),0\right) = \delta\pi_j E\left[V_a(s^+) \mid s)\right].$$

Based on envelop theorem we have:

$$V_a(s^+) = Ru_c\left(c\left(s^+\right),0\right).$$

Therefore we get:

$$u_c\left(c\left(s\right),0\right) = \delta\pi_j R \cdot E\left[u_c\left(c\left(s^+\right),0\right) \mid s)\right].$$

Household problem for working households:

$$\begin{aligned} & V(s) = \max_{(c,\ell,a^+)} u(c,\ell) + \delta\pi_j \mathbf{E}\big(V(s^+) \mid s\big), \\ s.b. \quad & a^+ + c = \omega w\ell + Ra + \Gamma. \end{aligned}$$

We obtain the set of equations:

$$\begin{aligned} & u_c\left(c\left(s\right),l\left(s\right)\right) = \delta\pi_j R \cdot E\left[u_c\left(c\left(s^+\right),l\left(s^+\right)\right) \mid s)\right], \\ & u_l\left(c\left(s\right),l\left(s\right)\right) = \omega w \cdot u_c\left(c\left(s\right),l\left(s\right)\right). \end{aligned}$$

2 Numerical implementation

Building risk into the decision-making process makes it impossible to derive analytical solutions for the consumer problem. However, it is still possible to identify the optimal allocation using the value function approach with numerical methods. The development of this branch gained momentum when the endogenous grid points method (EGM) was developed ed by Christopher Carroll. His modification of the algorithm significantly reduces the numerical complexity of the consumer's problem.

2.1 Discretization of the state-space

The state-space Ω consists of five dimensions. The first one, age j is already discrete. For the remaining four: private assets A, discount rate E_δ, individual productivity shock E_ℓ, and interest rate shock we need to apply discretization. The discretized state-space is denoted by $\hat{\Omega}$.

Assets The discretization of the private assets defines the exogenous grid. Since consumers face non-borrowing constraint, the first point on assets grid $a^0 \in \hat{A}$ is equal to zero. The last point on the grid a^{n_a} is chosen, such that it never bounds policy function for assets. Since for low levels of wealth, policy functions are highly nonlinear, higher density of assets grid close to zero speeds the performance of EGM. To build an assets grid, we use the algorithm based on [1], where:

$$\forall i \in \{0, ..., n_a\} : a^i = a^0 + (a^{n_a} - a^0)\frac{a^i_{growth} - 1}{a^{n_a - 1}_{growth} - 1},$$

the number a_{growth} is set to 1.04 base on computational performance.

Discount rate The discretization of the idiosyncratic discount rate shock follows the Tauchen method for AR(1) process, see [6]. The method relies on a fact that conditional on today's shock realization ϵ_δ, tomorrow's shock ϵ^+_δ is normally distributed. We use five grid points for discretization. Period corespondents to five years; thus, the persistence of the process used in the model is much lower compare to one-year process estimated form the data, especially $\rho_\eta < 0.9$. Following [3] for processes with higher persistence method described in [5] is more recommended. As an effect of discretization, we obtain grid for shock realization $\{\epsilon^1_\delta, ..., \epsilon^5_\delta\}$ and corresponding Markow matrix Π_δ where $\Pi_\delta(id, id^+)$ is the probability of receiving productivity shock $\epsilon^{id^+}_\delta$ in the next period condition of having shock ϵ^{id}_δ in the current period.

Labor productivity The discretization of the idiosyncratic labor productivity shock is based on the same approach as shock for discount rate. We use five grid points to discretize state space $\{\epsilon^1_\ell, ..., \epsilon^5_\ell\}$ and corresponding Markow matrix Π_ℓ where $\Pi_\ell(ip, ip^+)$ is the probability of receiving productivity shock $\epsilon^{ip^+}_\ell$ in the next period condition of having shock ϵ^{ip}_ℓ in the current period. To better capture wealth inequality in a parsimonious form, we allow for so-called superstars shock captured on two additional grid points, see for example [2].

Interest rate The discretization of idiosyncratic insert rate is based on method presented in [4]. We use five grid points state space $\{\varepsilon^1_\ell, ..., \varepsilon^5_\ell\}$ and corresponding Markow matrix Π_r where $\Pi_r(ir, ir^+)$ is the probability of receiving productivity shock $\varepsilon^{ir^+}_r$ in the next period condition of having shock ε^{ir}_r in the current period.

2.2 Endogenous grid point methods[1]

The EGM offers a substantial numerical advantages for solving dynamic stochastic consumption/savings problems compare to traditional approaches. Searching for optimal decision in each point of the state space (traditional approaches) requires numerical root-finding. In the EGM we look for the state variable (level of assets) where arbitrary chosen decision (consumption $\rightarrow$ savings) would be optimal. That can be done without numerical optimization implying higher speed and accuracy.

[1] For the introduction materials to the EGM check https://github.com/dseconf/DSE2021

EGM algorithm For the terminal period we have: $c_J^* = A_J$. For each period $j = J-1, J-2, ..., 1$:

1. Policy function for $j+1$: $c_{j+1} = c_{j+1}^*(A_{j+1})$ is given
2. Inverted Euler equation $c_{j+1} \to c_j$:
   $c_j = \left(\frac{\partial u}{\partial c}\right)^{-1} \left(R\delta\pi_j E\left[u\left(c_{j+1}\left(s^+\right), l_{j+1}\left(s^+\right)\right) \mid s\right]\right)$
   Based on intra-temporal FOC we can determine optimal labor supply
   $\frac{\partial u}{\partial l}\left(c_j, l_j\right) = \omega w \frac{\partial u}{\partial c}\left(c_j, l_j\right)$
   At this stage the policy functions is given for the endogenous assets grid $c_j^*\left(\widetilde{A}_j\right)$, $l_j^*\left(\widetilde{A}_j\right)$, $a_j^*\left(\widetilde{A}_j\right) = A_{j+1}$
3. Inter-temporal budget constraint:
   $r\widetilde{A}_j = A_{j+1} + c_j - \omega w l_j$ for working households, and $r\widetilde{A}_j = A_{j+1} + c_j - b$, for retiree.
   Here we obtain the values for endogenous grid.
4. Linear interpolation:
   $a_j^*\left(\widetilde{A}_j\right) \to a_j^*\left(A_j\right)$
5. Identifying optimal consumption and labor for given policy function for assets accumulation

Since households's do not have bequest motive the policy function for consumption in the last period ($j = J$) is trivial and given by:

```
! households's problem in the last period of lifetime
do ia = 0, n_a, 1 ! assests
    do ip=1, n_sp, 1 ! labor shock realisation
        do ir =1, n_sr,1 ! intrest rate realisation
            do id = 1, n_sd,1 ! discount rate realisation
                c_ss(bigj, ia, ip, ir, id) = max(((1d0+n_sr_value(ir)+r_ss_vfi)*sv(
                    ia)/gam_ss_vfi + b_ss_j_vfi(bigJ) + bequest_ss_j_vfi(bigJ)), 1d
                    -10)
                        [...]

            enddo
        enddo
    enddo
enddo
```

Notice that the policy function are defined on the exogenous grid. In the next step we need to define the right hand side of the Euler equation for the solution at age $j = J-1$. Notice that we need to take into account the transition between space state s at age j and stage s^+at age $j+1$. Therefore for each pair of points $s = \left\{j, a^{ia_{endo}}, \epsilon_d^{id}, \epsilon_\ell^{ip}, \varepsilon_r^{ir}\right\}$ and $s = \left\{j+1, a^{ia}, \epsilon_d^{id^+}, \epsilon_\ell^{ip^+}, \varepsilon_r^{ir^+}\right\}$ we have the probability of the transition is equal to $\Pi_d(id, id^+)\Pi_\ell(ip, ip^+)\Pi_r(ir, ir^+)$. Therefore, for each point s the right hand side, which represents the expected value next period marginal utility form consumption, is defined as a sum over all potential next period shock realizations. In the last step the discount factor is accounted for.

```
! Calculation of Euler equations' Righ Hand Side
do ia=0, n_a, 1 ! assets on egzogenous grid
    do ip = 1, n_sp, 1 ! labor productivity shock at age j
        do ir = 1, n_sr, 1 ! interest rate shock at age j
            do id = 1, n_sd, 1  ! discount rate shock at age j
                EV_prim = 0d0
                [...]
                do ir_r= 1, n_sr, 1 ! interest rate shock at age j+1
                    do id_d= 1, n_sd, 1 ! discount rate shock at age j+1
                        do ip_p = 1, n_sp,1 ! labor productivity shock at age j+1
                            c_help = c_ss(j, ia, ip_p, ir_r, id_d)
                            l_help = l_ss(j, ia, ip_p, ir_r, id_d)
                            if(j<jbar_ss_vf)then ! working, non zero labor supply
                                EV_prim =  EV_prim + (1d0+r_ss_vfi+n_sr_value(ir_r))
                                    /gam_ss_vfi*pi_ip(ip, ip_p)*pi_ir(ir, ir_r)*
                                    pi_id(id,id_d)*((1-l_help)/c_help)**((1d0-theta)
                                    *(1d0-phi))*c_help**(-theta)
                            else ! retiree
                                EV_prim = EV_prim + (1d0+r_ss_vfi+n_sr_value(ir_r))/
                                    gam_ss_vfi*pi_ip(ip, ip_p)*pi_ir(ir, ir_r)*pi_id
                                    (id,id_d)*c_help**(phi -theta*phi -1)
                            endif
                                                [...]
                        enddo
                    enddo
                enddo
                    RHS_ss(j,ia,ip,ir,id) = (delta+n_sd_value(id))*pi_ss_vfi_cond(j)
                        *EV_prim
                    [...]
            enddo
        enddo
    enddo
enddo
```

For given RHS we can calculate the policy function for consumption at age j for endogenous grid point. Furthermore based on intra-temporal condition labor is defined. At this point we can calculate the exact value of the endogenous assets $a^{ia_{endo}}$ that would lead to given consumption, labor supply and future assets a^{ia}.

2.3 Distribution of households

For given the initial distribution $\hat{\mathbb{P}}_1$ at age $j = 1$ and and transition matrix $Pi(\eta_j|\eta_{j-1})$ and the policy functions $\{a_j^+(s)\}_{j=1}^J$, we can compute the distribution in any successive age j .

```
! Initial distribution
call linear_int(0d0, ial, iar, dist, sv, n_a, a_grow)
ial = min(ial, n_a)
iar = min(iar, n_a)
dist = min(dist, 1d0)
prob_ss(1, ial, :, :, :) = dist
prob_ss(1, iar, :,  :, :) = 1d0 - dist
do ia = 0, n_a, 1
    do ip = 1 , n_sp, 1
        do ir=1, n_sr, 1
            do id=1,n_sd,1
                prob_ss(1, ia, ip, ir, id) = prob_ss(1, ia, ip, ir, id) * pi_ip_init
                    (ip) * pi_id_init(id) * pi_ir_init(ir)
```

```
            enddo
        enddo
    enddo
enddo
```

It can be interpreted as a fraction of cohort of age j residing at each state of the state space $\hat{\Omega}$. Once we compute the distributions, we combine that with the policy functions and obtain aggregate quantities of consumption, labor, and savings. To this end, we use the Gaussian quadrature method.

```
! successively compute distribution over ages
do j = 2, bigJ       ! iterate over yesterdays gridpoints
    do ia = 0, n_a, 1
        do ip = 1 , n_sp, 1
            do ir=1, n_sr, 1
                do id=1,n_sd,1
                    ! interpolate yesterday's savings decision
                    call linear_int(svplus_ss(j-1, ia, ip, ir, id), ial, iar, dist,
                        sv, n_a, a_grow)
                    ! restrict values to grid just in case
                    dist = min(abs(dist), 1d0)
                    ! redistribute households
                    do ip_p = 1, n_sp,1
                        do ir_r=1, n_sr, 1
                            do id_d =1, n_sd, 1
                                prob_ss(j, ial, ip_p, ir_r, id_d) = prob_ss(j, ial,
                                    ip_p, ir_r, id_d)+pi_ip(ip, ip_p) *pi_ir(ir,
                                    ir_r) *pi_id(id, id_d) *dist *prob_ss(j-1, ia,
                                    ip, ir, id)
                                prob_ss(j, iar, ip_p, ir_r, id_d) = prob_ss(j, iar,
                                    ip_p, ir_r, id_d)+ pi_ip(ip, ip_p) *pi_ir(ir,
                                    ir_r) *pi_id(id, id_d) *(1d0-dist) *prob_ss(j-1,
                                     ia, ip, ir, id)
                            enddo
                        enddo
                    enddo
                enddo
            enddo
        enddo
    enddo
enddo
```

3 Model parameters

The parameters of our macroeconomic model can be set in the following files: `set_globals.f90` - macroeconomy and household preferences, and `shock_parameters.f90`.

References

[1] Hans Fehr and Fabian Kindermann. *Introduction to computational economics using fortran*. Oxford University Press, 2018.

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