Introduction to the Kuramoto model

University of Bologna, Applied Physics
Complex Systems course

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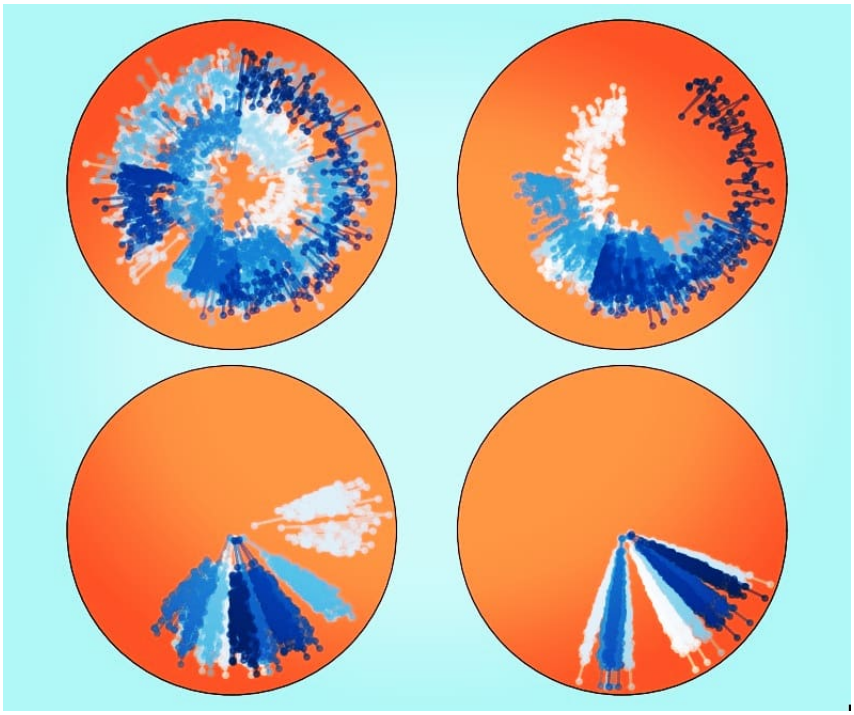

October 2021

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1 Introduction

The Kuramoto model describes a large population of coupled limit-cycle oscillators, each with its natural frequency. If the coupling strength exceeds a certain threshold, the system exhibits a phase transition: some of the oscillators spontaneously synchronize, while others remain incoherent. Due to this emergent property, this model and its numerous variations are the main paradigm for synchronization phenomena. The model has been introduced back in 1975 but still today many of its properties are not fully understood mathematically.
In this section we will briefly list some of the open problems related to this model, along with the cases in which some solutions have been found. In the second section the mean field case without noise is presented, its thermodynamic limit and the original analysis made by Kuramoto (in a more recent formalization). The third section very briefly introduces the problem of the mean field model with White Noise, of the which analytical solutions have been found only under very specific conditions. Finally, in the fourth section some numerical simulations are presented.

Let's consider the following, extremely general, system of differential equations:

$$\dot{\phi}_i = \omega_i + \sum_j^N \Gamma_{ij}(\phi_j - \phi_i), \qquad i = 1, ..., N \tag{1}$$

where ϕ_i represent the phase of the i^{th} oscillator of the population and ω_i its natural frequency. The oscillators are coupled by means of their phase difference, through the interaction functions Γ. In the most general case they have an arbitrary number of Fourier harmonics. The Kuramoto model and its variations employ only sinusoidal couplings:

$$\dot{\phi}_i = \omega_i + \sum_j^N K_{ij} \sin(\phi_j - \phi_i), \qquad i = 1, ..., N \tag{2}$$

This choice makes the model more tractable still retaining the complex properties of interest. Nevertheless, if the connection topology defined by the matrix K is unspecified the problem is yet too general to be solved. In its original analysis Kuramoto considered the mean field case, the most simple one, i.e.

$$\dot{\phi}_i = \omega_i + \frac{k}{N} \sum_j^N \sin(\phi_j - \phi_i), \qquad i = 1, ..., N \tag{3}$$

This system is completely defined by the natural frequencies ω_i, by the initial phases of the oscillators $\phi_i(0)$ and by the parameter k. For sufficiently large k, phase locking

occurs and a group of phases evolve coherently.
In the large N approximation the treatment is made in terms of distributions and the previous system is determined by the initial phases distribution $\rho_0(\phi)$, by the distribution of the natural frequencies g(ω) and by the coupling parameter k. Kuramoto tackled the problem through a self-consistency approach, concerning only purely stationary states. Their stability properties could not be established and in general this is still an open problem. Even if improvements have been done for the incoherent solution, the stability of the partially and complete synchronization is a tough problem. The noisy case (called also diffusive, with reference to the Fokker Planck equation) have been studied under particular conditions of g(ω).
If g(ω) unimodal the incoherent state has been proved to be linearly unstable, but explicit calculations can be done only for Lorentzian (Cauchy) or discrete distributions of natural frequencies. Otherwise, only the simple bimodal distribution have been studied. In that setting, it's possible to get the complete bifurcation diagram of the degree of coherence as a function of k. In this framework, synchronization arises as a bifurcation from the incoherent state.
The finite size effects of the Kuramoto model are currently poorly understood. Nevertheless, the numerical integration of the model can exploit a variety of methods that will be briefly listed in section four.

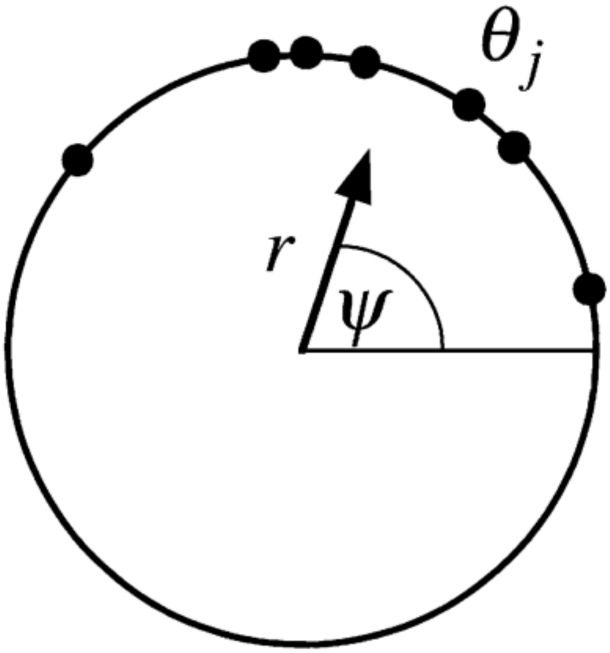


Figure 1: The geometrical interpretation of the Kuramoto model, as a dynamics on S^1 described by a population of vectors $e^{i\phi_j}$, j=1,...,N. Image taken from [2].

2 Mean field model without noise

The mean field Kuramoto model without noise is the following system of differential equations:

$$\dot{\phi}_i = \omega_i + \frac{k}{N}\sum_{j}^{N}\sin(\phi_j - \phi_i), \qquad i = 1, ..., N \tag{4}$$

where the $\phi_i \in [0, 2\pi]$ should be interpreted as the phases of a population of oscillators. Since the amplitudes of the oscillations are not important in the model, let's assume them to be all 1. Then we can give a geometrical interpretation to the system on the complex plane, associating to each oscillator the vector $e^{i\phi_i}$ as in Figure 1. If there's no coupling, each particle undergo a uniform motion on the manifold S^1 with its speed ω_i. Synchronization arises in the presence of coupling, when a group or all the particles have the same effective speed $\dot{\phi}$ and so they change coherently their phases from that point on.

Hamiltonian derivation The Kuramoto model can be also derived from a hamilto-

nian system [1]. Starting from the following Hamiltonian

$$\mathcal{H}(q_1, ..., q_N, p_1, ..., p_N) = \sum_{i=1}^{N} \omega_i \frac{q_i^2 + p_i^2}{2} + \frac{k}{4N} \sum_{i,j=1}^{N} (q_i p_j - q_j p_i)(q_i^2 + p_i^2 - q_j^2 - p_j^2)$$

and applying the canonical transformation to action-angle variables with $I_i = \frac{q_i^2+p_i^2}{2}$ and $\tan(\phi_i) = \frac{q_i}{p_i}$ we get the transformed Hamiltonian

$$\mathcal{H}'(I_1, ..., I_N, \phi_1, ..., \phi_N) = \sum_{i=1}^{N} \omega_i I_i - \frac{k}{N} \sum_{i,j=1}^{N} \sqrt{I_i I_j}(I_i - I_j) \sin(\phi_j - \phi_i)$$

Hamilton's equation of motion become:

$$\begin{aligned} \dot{I}_i &= -\tfrac{2k}{N} \textstyle\sum_{j=1}^{N} \sqrt{I_j I_i}(I_j - I_i) \cos(\phi_j - \phi_i) \\ \dot{\phi}_i &= \omega_i + \tfrac{k}{N} \textstyle\sum_{j=1}^{N} \left[2\sqrt{I_i I_j} \sin(\phi_j - \phi_i) + \sqrt{I_j / I_i}(I_j - I_i) \sin(\phi_j - \phi_i) \right] \end{aligned}$$

The dynamical evolution is not only dependent on the phase differences but also on the actions differences. If all the actions are equal, the first term is identically zero and the second term is the Kuramoto dynamics previously defined. So the Kuramoto system arises as the dynamics onto the invariant submanifold defined by $I_j = 1/2 \quad \forall j$.

2.1 Case N=2

For only two oscillators, the system has the following form:

$$\dot{\phi_1} = \omega_1 + \frac{k}{2}\sin(\phi_2 - \phi_1) \tag{5}$$
$$\dot{\phi_2} = \omega_2 + \frac{k}{2}\sin(\phi_1 - \phi_2) \tag{6}$$

So we have immediately that

$$\dot{\phi_1} + \dot{\phi_2} = \omega_1 + \omega_2 = const \tag{7}$$

that we can assume to be zero since we can always employ a reference frame rotating with angular speed equal to $\omega_1 + \omega_2$. Now, since $\frac{d}{dt}(\phi_1 + \phi_2) = 0$ we know that the mean phase $\psi := (\phi_1 + \phi_2)/2$ is constant. So $\phi_2 = 2\psi - \phi_1 = const - \phi_1$ and the dimensionality of the system is reduced to one equation only. Choosing ϕ_1 we have

$$\phi_1 = \omega_1 + \frac{k}{2}\sin(2(\psi - \phi_1)) = \omega_1 + k\cos(\psi - \phi_1)\sin(\psi - \phi_1) \tag{8}$$

We now define $r := \cos(\frac{\phi_1 - \phi_2}{2}) = \cos(\psi - \phi_1)$ as the *coherence* of the system, since $r = 1$ when $\phi_1 = \phi_2$ and $r = 0$ if $\phi_1 - \phi_2 = \pi$ (i.e. if they have the maximum distance on S^1). Note that $\phi_1 - \phi_2 \in [-\pi, \pi]$ so that $r \in [0, 1]$.
In order to give a geometrical interpretation to the coherence, let's consider the center of mass of the particles in the complex plane (Figure 2), whose phase it's the mean phase ψ and whose amplitude we call r:

$$re^{i\psi} = \tfrac{1}{2}(e^{i\phi_1} + e^{i\phi_2}) \tag{9}$$
$$r = \tfrac{1}{2}(e^{-i\frac{\phi_1 - \phi_2}{2}} + e^{i\frac{\phi_1 - \phi_2}{2}}) = \cos(\tfrac{\phi_1 - \phi_2}{2}) \tag{10}$$

We now can study the system in the (r,ψ) coordinates, recalling that $\dot{\psi} = 0$ and

$$\dot{r} = -\sin(\tfrac{\Delta\phi}{2})\tfrac{1}{2}(\dot{\Delta\phi}) \tag{11}$$
$$(\dot{\Delta\phi}) = \dot{\phi_1} - \dot{\phi_2} = \Delta\omega - k\sin(\Delta\phi) \tag{12}$$

from which it's clear that $\dot{r} = 0$ is equivalent to $\dot{\phi_1} = \dot{\phi_2}$ i.e. a fixed point for the coherence is the state in which the phase locking occurs.
With respect to the control parameter k we have that the following condition must hold

$$\Delta\omega = k\sin(\Delta\phi) \tag{13}$$

that's possible if $k > |\Delta\omega| = |\omega_1 - \omega_2|$ and in such a case, the equilibrium state will have the phases coherently evolving with difference

$$\Delta\phi = \arcsin(\frac{\Delta\omega}{k}) \tag{14}$$

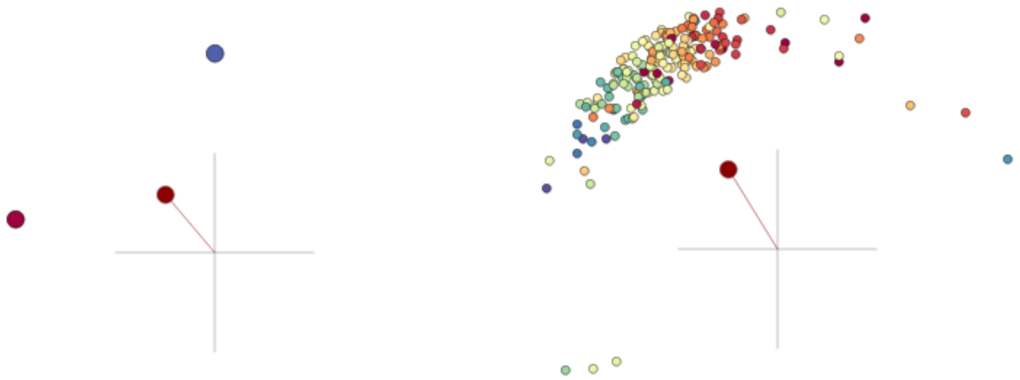


Figure 2: Geometrical interpretation of a Kuramoto system with two particles (left) and N particles (right). In both the center of mass of the system $re^{i\psi}$ is shown in red. For large N, r and ψ are the order parameters of the system. Image taken from [4].

So the critical value for the coupling is $k_c = |\omega_1 - \omega_2|$. Beyond such a value the system reaches the stable state with $\dot{r} = \dot{\psi} = 0$ and fixed point $r_\star = \cos(\arcsin(\frac{\Delta\omega}{k})) = const.$ We would like to compute also the dynamical stability of $r_\star$.
In order to do this, let's write the dynamics for r:

$$\dot{r} = -\sin(\tfrac{\Delta\phi}{2})\tfrac{1}{2}[\Delta\omega - k\sin(\Delta\phi)] = \tag{15}$$

$$-\sin(\tfrac{\Delta\phi}{2})\tfrac{1}{2}[\Delta\omega - 2k\cos(\tfrac{\Delta\phi}{2})\sin(\tfrac{\Delta\phi}{2})] = \tag{16}$$

$$-\tfrac{1}{2}\Delta\omega\sqrt{1-r^2} + kr(1-r^2) \tag{17}$$

Let's notice that $\dot{r}$ is not defined in r=0. With a mechanical analogy we can assume that $\dot{r} = -\frac{\partial}{\partial r}V(r)$ for some potential V. We computed $r_\star$ such as $V'(r) = 0$, in order to find out if it's a maximum or a minimum, we now compute

$$\frac{\partial^2}{\partial r^2}V(r) = -\frac{\partial}{\partial r}\dot{r} = \frac{\Delta\omega r}{2\sqrt{1-r^2}} - k + 3kr^2 \tag{18}$$

We shall study the sign of the previous expression for $r = r_\star$ i.e. for

$$\Delta\omega = 2kr\sqrt{1-r^2} \tag{19}$$

as follows from equation 13 recalling that $r = \cos((\Delta\phi)/2)$. So we get

$$V''(r)\big|_{r_\star} = k[2r^2 - 1] \tag{20}$$

whose sign is given by $2r^2 - 1$ since we assume $k > 0$. The critical value for the sign is $r_{\star\star} = 1/\sqrt{2}$ from which, through equation 19, we get $k_{\star\star} = \Delta\omega$ that's equal to k_c for which the extremal point of V' starts to exist. In words, whenever the phase locking is possible ($k > \Delta\omega$) it's for sure a stable fixed point.
This concludes the analysis of N = 2. It has been useful to introduce some key concepts employed even in the cases with a larger population. The stability analysis, so straightforward with only two oscillators, becomes a really complex task for large N. Already for N=3, similarly to the three body problem, chaos could arise. In the limit for $N \to \infty$ the only hope is to leverage symmetries of the distribution of natural frequencies $g(\omega)$.

2.2 Case N=∞

In this section the original analysis made by Kuramoto is repeated, in a more recent formalization based on densities. Let's recall that the governing equations are

$$\dot{\phi}_i = \omega_i + \frac{k}{N}\sum_{j}^{N}\sin(\phi_j - \phi_i), \qquad i = 1, ..., N \tag{21}$$

The frequencies are distributed according to some probability density g(ω).

Frequency distribution Some assumptions are made for g(ω): it's unimodal (i.e. it has only one local maximum) and symmetric about its mean frequency $\bar{\omega}$. We can employ a rotating reference frame with angular velocity equal to $\bar{\omega}$ in such a way that the governing equations are still the same with each ω_i reduced by $\bar{\omega}$. In practice, we assume that

$$g(\omega) = g(-\omega) \tag{22}$$

Order parameters We define the complex order parameters r and ψ as

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j} \tag{23}$$

whose geometrical meaning is shown in Figure 2. Multiplying both sides for $e^{-i\phi_i}$ gives

$$re^{i(\psi-\phi_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\phi_j-\phi_i)} \tag{24}$$

and the imaginary part of this equation is

$$r\sin(\psi - \phi_i) = \frac{1}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i) \tag{25}$$

We can easily substitute into 21 in order to get

$$\dot{\phi}_i = \omega_i + kr\sin(\psi - \phi_i), \qquad i = 1, ..., N \tag{26}$$

that are the governing equations in terms of the order parameters. Let's notice that now the particles are no more explicitly coupled one to the others, but each with the common mean phase. They still interact, through ψ. Moreover, the effective coupling to ψ is modulated by $r \in [0, 1]$: if the coherence increases also the coupling of the particles to the common mean phase increases causing more particles to join the group. With this mechanism spontaneous synchronization arises.

In term of densities The limit for $N \to \infty$ can be framed as a continuum limit [2]: for each value of ω a continuum of oscillators is distributed along the circle S^1. So for each value of ω a conditional probability $\rho(\phi|\omega, t)$ is defined, normalized on S^1:

$$\int_{-\pi}^{+\pi} \rho(\phi|\omega, t) d\phi = 1 \tag{27}$$

In terms of the joint probability, denoted $\rho(\phi, \omega, t)$, it is

$$\rho(\phi|\omega, t) = \frac{\rho(\phi, \omega, t)}{g(\omega)} \tag{28}$$

Finally, the marginal probability for ϕ, meaning the distribution of oscillators along S^1 with any frequency ω, will be denoted $\rho(\phi)$ with

$$\rho(\phi) = \int_{-\infty}^{+\infty} \rho(\phi, \omega, t) d\omega \tag{29}$$

Clearly it is

$$1 = \int_{-\pi}^{+\pi} \rho(\phi) d\phi = \int_{-\pi}^{+\pi} \int_{-\infty}^{+\infty} \rho(\phi, \omega, t) d\omega d\phi = \int_{-\pi}^{+\pi} \int_{-\infty}^{+\infty} \rho(\phi|\omega, t) g(\omega) d\omega d\phi \tag{30}$$

The evolution of $\rho(\phi|\omega, t)$ is given by the continuity equation

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial \phi}[v\rho] \tag{31}$$

where the velocity v is to be interpreted as the instantaneous speed of each oscillator along S^1 as

$$v = v(\phi, \omega, t) = \omega + kr \sin(\psi - \phi) \tag{32}$$

In this framework, we shall restate also the definition of the order parameters. Equation 23 becomes

$$re^{i\psi} = \int_{-\pi}^{+\pi} e^{i\phi} \rho(\phi) d\phi = \int_{-\pi}^{+\pi} \int_{-\infty}^{+\infty} e^{i\phi} \rho(\phi|\omega, t) g(\omega) d\omega d\phi \tag{33}$$

Within this framework, we now proceed to follow the classic Kuramoto analysis of the system.

Kuramoto's analysis Kuramoto directly looked for steady solutions, where r(t) is constant and $\psi(t)$ rotates at a constant frequency, that is 0 in the co-rotating frame. We

can further assume that $\psi \equiv 0$ thanks to the freedom in the choice of the origin of the co-rotating system. So we have that

$$v(\phi, \omega, t) = \dot{\phi} = \omega + kr\sin(\psi - \phi) = \omega - kr\sin(\phi) \tag{34}$$

Since r is assumed constant, all the oscillators are effectively independent. With respect to the long term behaviour, Kuramoto understood that the oscillators belong to one of two classes, depending on their natural frequency:

- If $|\omega_i| < kr$ the corresponding oscillator will approach a stable fixed point defined by $\omega_i = kr\sin(\phi_i)$. These oscillators will be called "phase-locked" or simply "locked".
- If $|\omega_i| > kr$ the oscillator will be called "drifting" since its dynamics has not a fixed point in S^1. It will keep rotating non-uniformly around S^1.

The locked oscillators corresponds to the central portion of g(ω), where ω is closer to the average natural frequency.
Note that this treatment lies on the assumption that r and ψ are constant. How can this hold in presence of the drifting oscillators? The solution is that the steady state condition holds if the drifting oscillators are uniformly distributed onto S^1. Kuramoto guessed this condition intuitively, while we can be rigorous thanks to the continuous framework. In fact, the steady state condition is equivalent to $\frac{\partial}{\partial t}\rho = 0$ that implies $v\rho = C(\omega)$.
So we have

$$\rho(\phi|\omega, t) = \begin{cases} \delta(\phi - \psi - \arcsin(\frac{\omega}{kr})) & \text{if } |\omega| < kr \\ \frac{C}{\omega - kr\sin(\psi - \phi)} & \text{if } |\omega| > kr \end{cases} \tag{35}$$

where the normalization constant C is fixed by the normalization condition of ρ as $C = \frac{1}{2\pi}\sqrt{\omega^2 - (kr)^2}$.
From the definition of the order parameters, recalling that $\psi = 0$ and using angular brackets to denote population averages, we have

$$re^{i\psi} = r = \langle e^{i\phi}\rangle = \langle e^{i\phi}\rangle_{Lock} + \langle e^{i\phi}\rangle_{Drift} \tag{36}$$

For the phase-locked oscillators holds $\sin(\phi) = \frac{\omega}{kr}$. Since in the large population limit g(ω) = g(-ω) there will be as many oscillators at ϕ as at $-\phi$. So $\langle\sin(\phi)\rangle_{Lock} = 0$ and

$$\langle e^{i\phi}\rangle_{Lock} = \langle\cos(\phi)\rangle_{Lock} = \int_{-kr}^{kr} \cos(\arcsin(\frac{\omega}{kr}))g(\omega)d\omega \tag{37}$$

Changing variables from ω to ϕ yields

$$\langle e^{i\phi}\rangle_{Lock} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\phi)g(kr\sin(\phi))d\phi \tag{38}$$

As for the drifting oscillators, we have explicitly

$$\langle e^{i\phi} \rangle_{Drift} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{|\omega|>kr} e^{i\phi} \rho(\phi|\omega, t) g(\omega) d\omega d\phi \tag{39}$$

Recall the symmetries g(ω) = g(-ω) and $\rho(\phi + \pi| - \omega) = \rho(\phi|\omega)$, implied by the fact that

$$\rho(\phi|\omega)_{Drift} = \frac{C}{\omega - kr\sin(\psi - \phi)} \tag{40}$$

Due to these symmetries, the integral 39 vanishes.
Finally we can state the self-consistency equation for the order parameter r:

$$r = kr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\phi) g(kr\sin(\phi)) d\phi \tag{41}$$

Solutions Equation 41 has a zero solution for every k. It corresponds to r=0 and $\rho(\phi|\omega) = \frac{1}{2\pi}$ for all values of ω, that is the completely incoherent state.
The second branch of solutions is the partial (or total) synchronization, whose corresponding distribution has been found previously (equation 35). Note however that using equation 35 it's not possible to predict the stationary value for r or the critical value k_c for the coupling. In order to do this, we shall study the equation

$$1 = k \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\phi) g(kr\sin(\phi)) d\phi \tag{42}$$

Note that this branch of solutions bifurcates continuously from r=0 at the critical point $k = k_c$. So we can find k_c letting $r \to 0^+$. It's easy to find that

$$k_c = \frac{2}{\pi g(0)} \tag{43}$$

The exact value of r as a function of k is difficult to find in general. Kuramoto integrated equation 42 exactly for the special case of the Cauchy (Lorentzian) density

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)} \tag{44}$$

obtaining

$$r = \sqrt{1 - \frac{k_c}{k}} \tag{45}$$

For general distributions of the natural frequencies, it's possible to expand in series of r the formula 42 near k_c.
As a final note, we recall that Kuramoto in its original work developed this analysis without the benefit of numerical simulations.

3 Mean field model with White Noise

The Kuramoto model has been widely studied in the presence of additive White Noise. It can be interpreted as rapid stochastic fluctuations in the natural frequencies. The governing equations now are

$$\dot{\phi}_i = \omega_i + \xi_i + \frac{k}{N}\sum_{j}^{N}\sin(\phi_j - \phi_i), \qquad i = 1, ..., N \tag{46}$$

where the stochastic variables $\xi_i(t)$ are independent white noise processes, i.e.

$$\langle \xi_i(t) \rangle = 0 \tag{47}$$

$$\langle \xi_j(s)\xi_i(t) \rangle = 2D\delta_{ij}\delta(s-t) \tag{48}$$

where the angular brackets denote an average over the noise realizations and $D \geq 0$ is the noise strength. Equations 46 are a system of Langevin equations and for $N \to \infty$ the density $\rho(\phi, \omega, t)$ should satisfy the Fokker Planck equation

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial \phi^2} - \frac{\partial}{\partial \phi}[v\rho] \tag{49}$$

that is the continuity equation 31 with an additional diffusive term. Also for this equation the simplest solution is the incoherent state $\rho = \frac{1}{2\pi}$. Sakaguchi [5] adapted Kuramoto's analysis to this case showing with similar arguments that

$$k_c = 2\left[\int_{-\infty}^{\infty}\frac{D}{D^2+\omega^2}g(\omega)d\omega\right]^{-1} \tag{50}$$

which is consistent to equation 43 if $D \to 0^+$. Even for the noisy case the main concern of the scientific community is the determination of the stability properties of the solutions. The stability problem of the incoherent state of the Sakaguchi's model has been solved in [6].

4 Numerical integration of the Kuramoto model

Several numerical schemes exist for the integration of stochastic differential equations. Three of them have been chosen and implemented for the case of the (noisy) Kuramoto model.
A general autonomous stochastic differential equation has the following form

$$d\mathbf{X}(t) = \mathbf{f}(\mathbf{X}(t))dt + \mathbf{g}(\mathbf{X}(t))d\mathbf{W}(t) \tag{51}$$

with initial conditions $\mathbf{X}_0 = \mathbf{X}(0)$.
In our setting we have the following correspondences:

$$\mathbf{X}(t) = \vec{\phi}(t) \equiv \phi(t) \tag{52}$$
$$\mathbf{g}(\mathbf{X}(t)) = const := T \tag{53}$$
$$\mathbf{f}(\mathbf{X}(t)) = \omega + \tfrac{k}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i) \tag{54}$$

Where for simplicity of notation ω and ϕ should be treated as vectors if they have no subscript. Using the previous definitions equation 51 becomes equation 46.

Euler method The most simple numerical scheme to be employed is the Euler method:

$$\phi^{n+1} = \phi^n + f(\phi^n)\Delta t + T\Delta W^n \tag{55}$$

where we denote with a superscript the n^{th} time step and ϕ, W and f are vectors. The brownian increments $\Delta W^n = W^n - W^{n-1}$ are random variables distributed with a Gaussian distribution with zero mean and variance Δt. So the evolution of each component W_i of W is an independent random walk. This means that each solution of equation 51 relies on a particular realization of a N-dimensional random walk $(W^0, W^1, ..., W^M)$, if the integration used M time steps. N is the number of particles.

Heun method The Heun method, an improved version of the Euler method, is the following scheme:

$$\langle\phi^{n+1}\rangle = \phi^n + f(\phi^n)\Delta t + T\Delta W^n \tag{56}$$
$$\phi^{n+1} = \phi^n + \tfrac{1}{2}[f(\phi^n) + f(\langle\phi^{n+1}\rangle)]\Delta t + \tfrac{T}{2}\Delta W^n \tag{57}$$

Taylor 3/2 Finally, the stochastic Taylor formula of order 3/2 has been used, whose scheme is

$$\phi^{n+1} = \phi^n + f(\phi^n)\Delta t + T\Delta W^n \tag{58}$$
$$+\tfrac{(\Delta t)^2}{2}\left[f(\phi^n)\tfrac{df}{d\phi}(\phi^n) + \tfrac{T^2}{2}\tfrac{d^2f}{d\phi^2}(\phi^n)\right] + \tfrac{df}{d\phi}(\phi^n)T\Delta Z^n \tag{59}$$

where ΔZ^n are random variables, distributed according to a Gaussian distribution with zero mean and variance $(\Delta t)^3/3$. They are correlated to the ΔW^n variables with $E(\Delta W^n \Delta Z^n) = (\Delta t)^2/2$. The variables ΔW and ΔZ are generated at the same time with a multivariate gaussian distribution with correlation matrix

$$\Sigma = \begin{bmatrix} \Delta t & \frac{(\Delta t)^2}{2} \\ \frac{(\Delta t)^2}{2} & \frac{(\Delta t)^3}{3} \end{bmatrix} \tag{60}$$

For completeness, we recall that a different approach is to integrate directly the Fokker Plack equation

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial \phi^2} - \frac{\partial}{\partial \phi}[v\rho] \tag{61}$$

with the spectral methods available for the integration of PDEs. Clearly in this way only the limiting system resulting from $N \to \infty$ can be integrated. It's impossible to take into account with this method the so-called finite size effects.

4.1 Numerical simulations

Some numerical simulations have been realized involving the noisy Kuramoto model. Finite size effects, the impact of the noise amplitude on the critical coupling and some polar plots of the order parameter have been realized.

Numerical schemes comparison In Figure 3 the three schemes showed in the previous paragraphs are compared, for three population sizes. In the top panel of each figure the time evolution of the order parameter is plot, while in the bottom panel a zoom of the stable regime is made. In the stable regime the order parameter should be constant but fluctuations are present, due to the finite population sizes used. Because of computational limits, no more than 1000 oscillators could be employed. With such few numbers it's difficult to see the trend in fluctuations and accuracy. Nevertheless, it's clear that the different numerical schemes agree on the limiting behaviour of the system, even if they simulate differently the transition state. Increasing the number of oscillators employed, the stable state takes longer to be reached and the fluctuations in the order parameter decrease.
The natural frequency distribution used is the simplest, in which all the frequencies are equal: $g(\omega) = \delta(\omega)$. The temperature T is 0.1 and the coupling is 2. 1000 iterations have been done with $\Delta t = 0.01$.

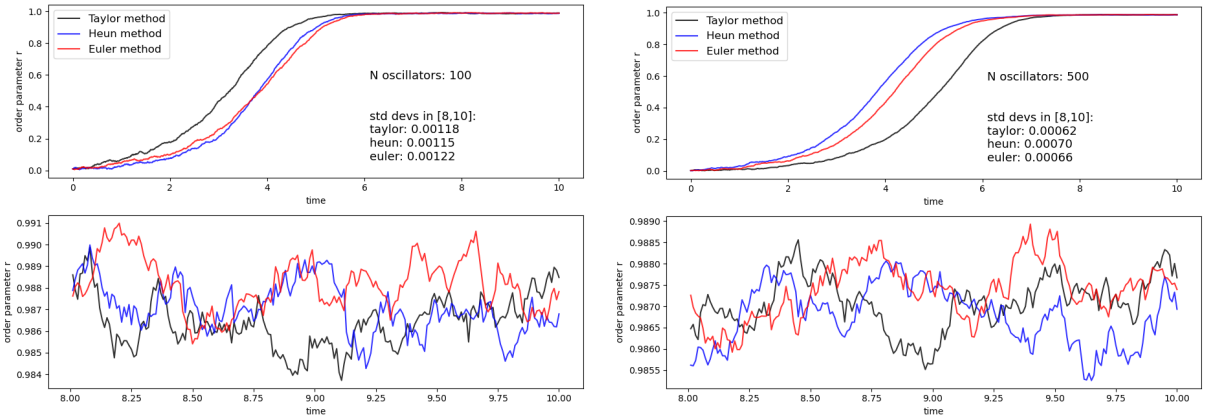
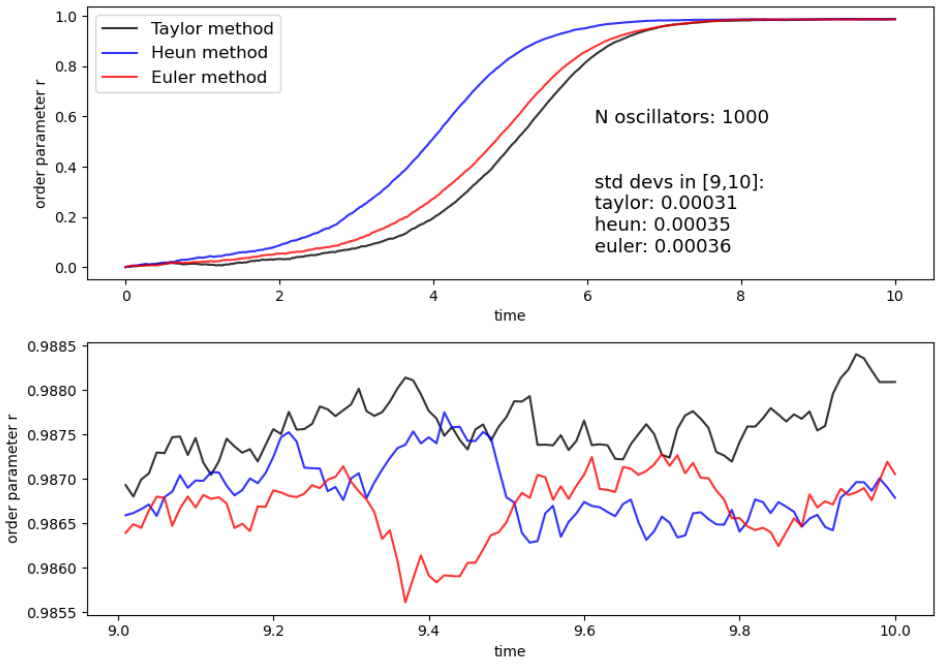


Figure 3: Three simulations have been realized with population sizes equal to 100 (top left), 500 (top right) and 1000 (bottom). The order parameter is plotted as a function of time. Note that the fluctuations decreases as N increases. The three schemes doesn't agree on the transition state but they does on the stable state of the system. In the bottom panel of each figure a zoom is plot of the stable state of the order parameter.



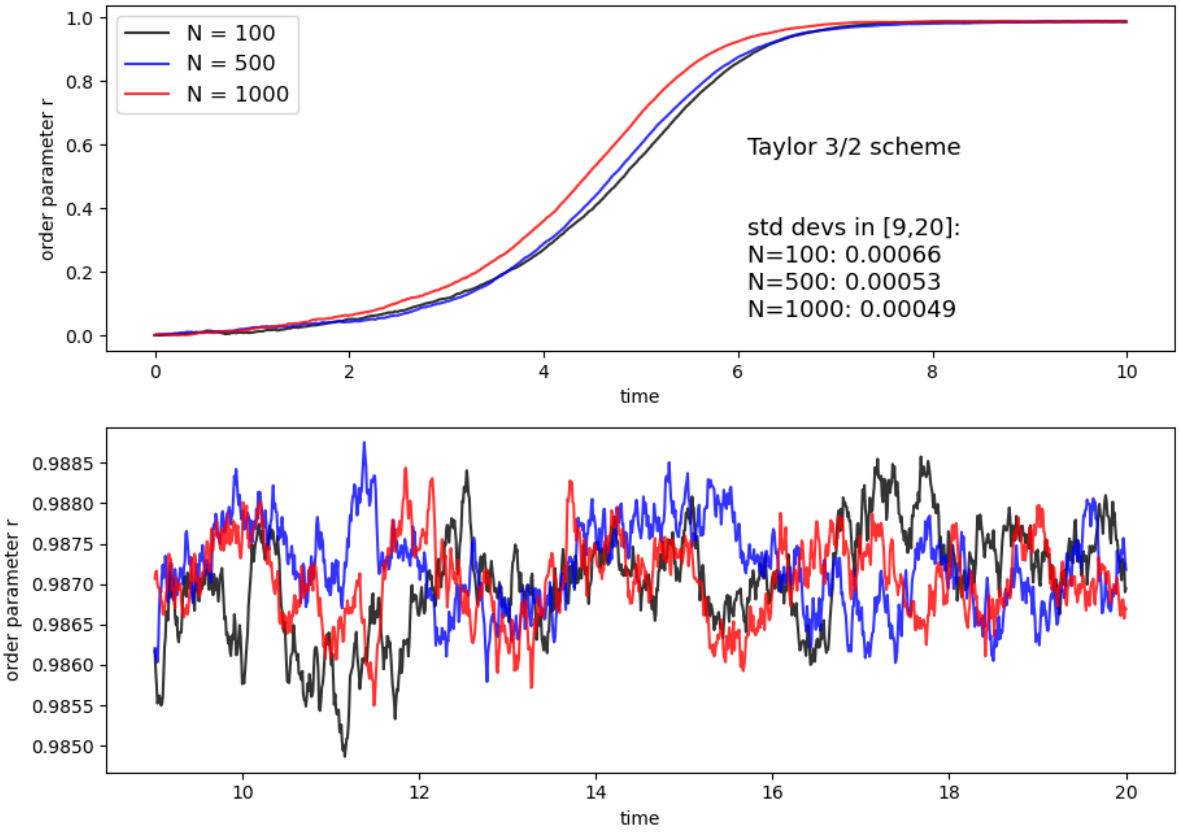


Figure 4: Caption

Population size comparison In Figure 4 the Taylor method of order 3/2 has been employed with the same parameters of Figure 3. The total number of iterations is 2000 in order to have more data for the computation of the standard deviation of the fluctuations of the stable order parameter. The fluctuations for population sizes of 100, 500 and 1000 decrease very slightly. Other studies as [3] show, using up to 10^5 oscillators, that $\Delta r = |r_N - r_\infty| \sim N^{-1/2}$ as N grows. As in Figure 3 the top panel shows the time evolution of the order parameter up to the stable state, while the bottom panel shows the fluctuations around the mean value of the order parameter after the stabilization. Ideally it should be constant, the fluctuations are due to the finite size effects.

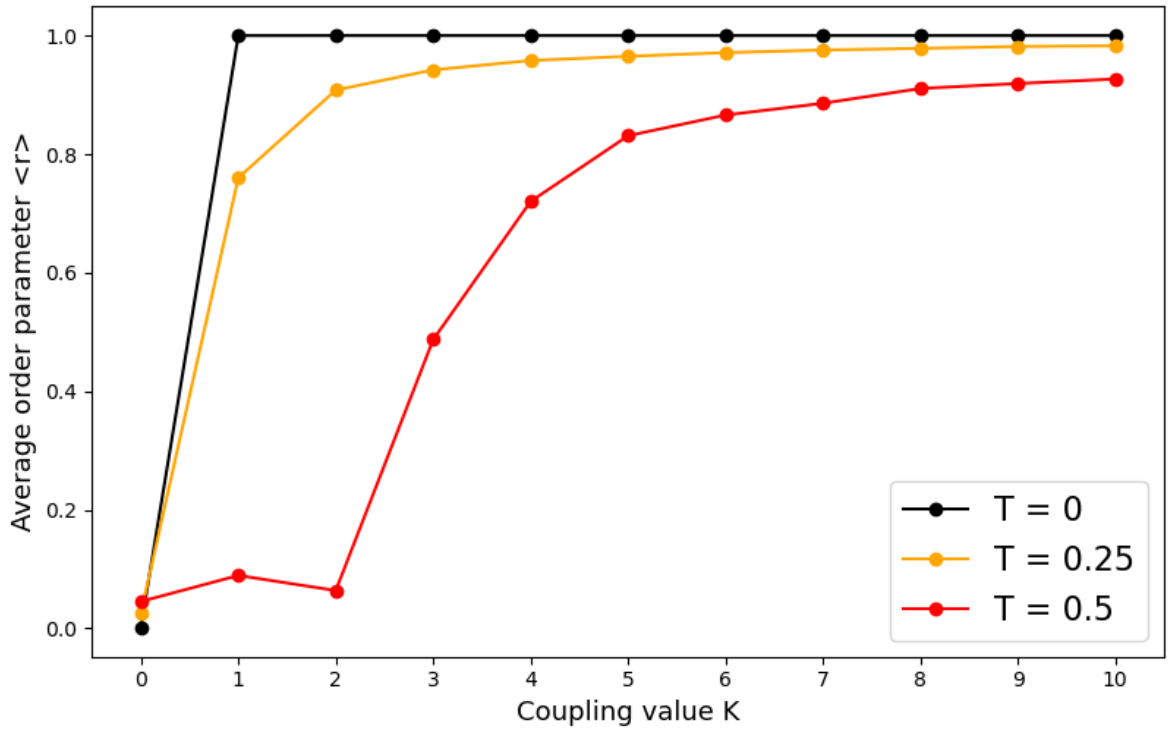


Figure 5: Plot of the average value of the order parameter (after stabilization on its fixed point) as a function of the coupling, for three values of temperature. Increasing the temperature, the synchronization becomes harder to obtain and the critical value of the coupling increases.

Behaviour of the critical coupling with respect to the temperature The behaviour of the system as the temperature increases is studied. For T equal to 0, 0.25 and 0.5 the numerical simulation has been run for 2000 iterations, with 500 oscillators and $\Delta t = 0.01$. The Taylor scheme has been used. In Figure 5 the average value of r in the last 100 iterations (when the system stabilized after the transition state) has been plotted as a function of the coupling value k. It's clear that the critical value for k increases as the temperature grows, in a non-linear way. Moreover, the value of the order parameter at which the system finally stabilizes decreases. For k=1, the three temperatures lead to the three possible states of the system. Total synchronization arises in the noise-free system ($r_\star$=1). With T=0.1 the system reaches a partial synchronization, with $r_\star$=0.76. If the temperature is increased up to 0.5, the synchronization cannot arise anymore and the incoherent state persists.

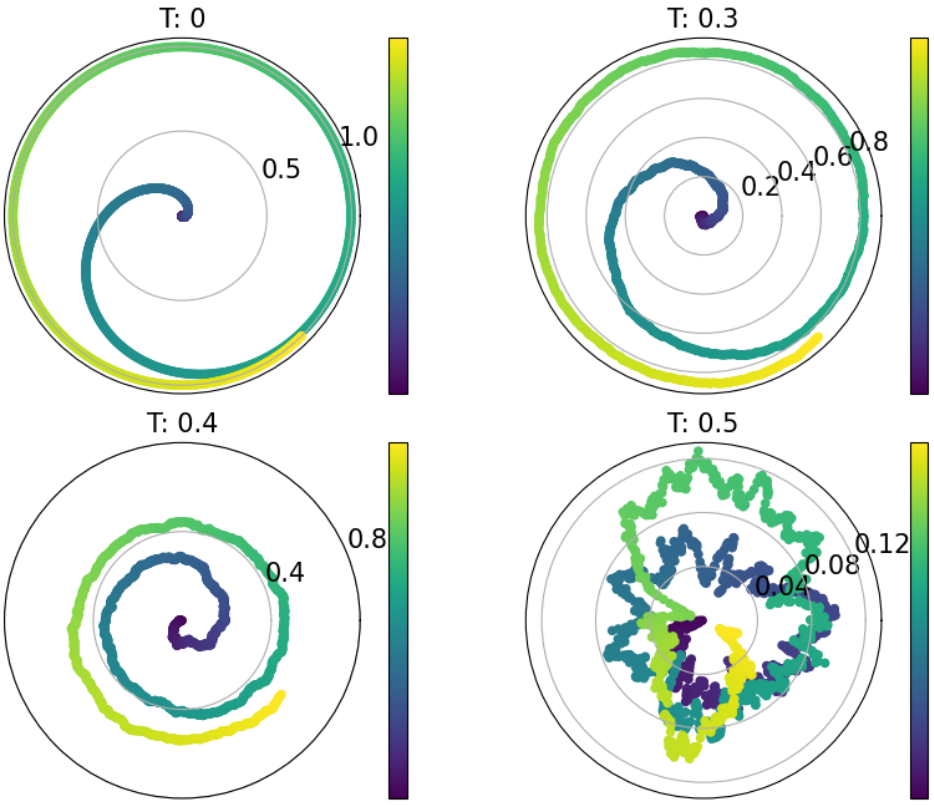


Figure 6: Polar plots of the time evolution of the complex order parameter for increasing values of the temperature: 0 (top left), 0.3 (top right), 0.4 (bottom left), 0.5 (bottom right). Time goes from blue to yellow.

Polar plots of the order parameters Finally the time evolution of the complex order parameter $re^{i\psi}$ has been plotted in the complex plane, for different values of the temperature: 0, 0.3, 0.4 and 0.5. In Figure 6 the resulting graphs are shown. ψ always keeps moving uniformly, because in the simulation it's not used the co-moving reference frame and all the frequencies are equal to 1. The other parameters are set as follows: k=2, N=1000 and $\Delta t = 0.01$. The time goes from blue to yellow. In the noise-free case (top left) the evolution is smooth and the amplitude of the order parameter stabilizes on 1. For T=0.3 (top right panel) yet a stable point for r is reached, but at a lower level of 0.75. For T=0.4 it seems that k=2 is close to the critical coupling, since the order parameter slightly increases as time passes, until it stabilizes at $r_\star = 0.59$. Finally, for T=0.5 the amplitude of the order parameter is highly noisy and in any case it cannot increase beyond 0.12: the state is totally incoherent and the rise of synchronization is prevented by the noise.

5 Conclusions

In this report the mean field Kuramoto model has been briefly reviewed. The most important analytical results have been presented, along with some numerical simulations, in order to give the idea of the different possible approaches to this kind of problems. The main feature of the model is the emergence of synchronization, what is called an *emergent property* in the field of complex systems. Being a paradigmatic example of a non-linear dynamical system, also the issues related to the stability of the solutions retain a big importance. Moreover, it's fundamental the introduction of stochastic fluctuations in the model, that lead in the large population limit to the Fokker Planck equation.
The analysis of the mean field Kuramoto model passes on through the identification of a *control parameter* (the coupling k in this case) and the study of its effect onto an *order parameter*. The order parameter somehow quantifies the emergent property and it has a central role in the analysis.
Anyway, moving away from the most simple cases is immediately a tough problem. The connection topology of the oscillators could be not a mean field, the distribution of the natural frequencies could be arbitrary, the number of oscillators could be large but not infinite, just to list few of the possible deviations from the base model. Even sticking to the nowadays "classical" case, many issues about the stability of the partially coherent solution are to be solved. Some researchers have indicated similarities of these problems to the phenomenon of the Landau damping.
Despite it has been introduced 46 years ago, the Kuramoto model still proves to be an useful and surprising mathematical challenge, that keeps stimulating new advancements in the field.

References

[1] Whitthaut, Timme. "Kuramoto dynamics in Hamiltonian systems." Physical Review E 90(3)

[2] Strogatz. "From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators." Physica D 143 (2000) 1–20

[3] Acebrón, Bonilla, Vicente, Ritort, Spigler. "The Kuramoto model: a simple paradigm for synchronization phenomena." Rev. Mod. Phys. 77, 137

[4] `https://www.complexity-explorables.org/explorables/ride-my-kuramotocycle/`

[5] H. Sakaguchi, Progr. Theoret. Phys. 79 (1988) 39.

[6] S.H. Strogatz, R.E. Mirollo, J. Statist. Phys. 63 (1991) 613.

Code

All the code used for the numerical simulations and the plotting is available at:
`https://github.com/FMagnani/MeanFieldKuramotoModel`