EPFL

SEMESTER PROJECT

Minimal Rational Interpolation for Time-Harmonic Maxwell's Equations

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Abstract

CONTENTS

1 Introduction

2 Finite element discretization of the time-harmonic Maxwell's equations

2.1 Vector potential formulation of the time-harmonic Maxwell's equations

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Let $\mathbf{E}$ denote an electric field, $\mathbf{B}$ a magnetic field strength, ρ an electric charge density, and $\mathbf{j}$ an electric current density. Maxwell's equations are stated in [?] as

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{2.3}$$

$$\nabla \times (\mu^{-1} \mathbf{B}) = \partial_t (\epsilon \mathbf{E}) + \mathbf{j} \tag{2.4}$$

with ε being the permittivity and μ the permeability.

Equation (2.2) allows for an expression of the magnetic field $\mathbf{B} = \nabla \times \mathbf{u}$ in terms of a vector valued function $\mathbf{u}$, the vector potential (in literature commonly denoted with $\mathbf{A}$). Similarly, (2.3) suggests rewriting the electric field $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{u}$ using a scalar function ϕ, referred to as the scalar potential.

The physical quantities $\mathbf{E}$ and $\mathbf{B}$ remain unchanged if we transform $\mathbf{u} \to \mathbf{u}' = \mathbf{u} + \nabla\psi$ or $\phi \to \phi' = \phi - \partial_t \psi$ for arbitrary functions ψ. A convenient choice of ψ is suggested in [?] to be

$$\psi = \int_0^t \phi \, dt' \tag{2.5}$$

which transforms $\phi \to \phi' = 0$ and $\mathbf{u} \to \mathbf{u}' = \mathbf{u} + \nabla \int_0^t \phi \, dt'$. Thus, the expressions for the electrical and magnetic field become

$$\mathbf{E} = -\partial_t \mathbf{u} \tag{2.6}$$

$$\mathbf{B} = \nabla \times \mathbf{u} \tag{2.7}$$

where I renamed the variable $\mathbf{u}'$ to $\mathbf{u}$ for simplicity.

Plugging the identities (2.6) and (2.7) into (2.4) yields

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) = \epsilon \partial_t^2 \mathbf{u} + \mathbf{j} \tag{2.8}$$

For the rest of this report, I restrict myself to vector potentials $\mathbf{u}$ that exhibit a harmonic dependence on time t, i.e. may be factorized into a term solely depending on the position $\mathbf{x}$ and a complex exponential

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \exp(i\omega t) \tag{2.9}$$

Substituting this expression into (2.8) results in the

Time-harmonic potential equation

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \epsilon \omega^2 \mathbf{u} = \mathbf{j} \tag{2.10}$$

2.2 Weak formulation for the time-harmonic potential equation

Equation (2.10) may be multiplied by a vector-valued function $\mathbf{v} \in H_{\mathrm{curl}}(\Omega)$, where

$$H_{\mathrm{curl}}(\Omega) = \{\mathbf{u} : \Omega \to \mathbb{C}, \text{ such that } \mathbf{u} \in L^2(\mathbb{C})^3, \nabla \times \mathbf{u} \in L^2(\mathbb{C})^3\} \tag{2.11}$$

and then integrated over all of Ω to obtain

$$\int_\Omega (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} - \omega^2 \int_\Omega \epsilon \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} \tag{2.12}$$

This may further be simplified (2.12) to (see Section for details)

Weak formulation of the time-harmonic potential equation

$$\int_\Omega (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_\Omega \epsilon \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} + \int_{\partial\Omega} \underbrace{((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n})}_{=\mathbf{g}} \cdot \mathbf{v} \tag{2.13}$$

where $\mathbf{n}$ denotes the surface normal to the boundary $\partial\Omega$.

Boundary conditions on the electric field $\mathbf{E}$ may be enforced in a Dirichlet-type fashion through the relation (2.6) and the assumption (2.9)

$$\mathbf{u}|_{\partial\Omega} = -\frac{1}{i\omega} \mathbf{E}|_{\partial\Omega} \tag{2.14}$$

Those on the magnetic field $\mathbf{B}$ through a Neumann-type condition following from (2.7) and again (2.9)

$$\mathbf{g}|_{\partial\Omega} = (\mu^{-1} \mathbf{B}|_{\partial\Omega}) \times \mathbf{n} \tag{2.15}$$

2.3 Examples

We apply this weak formulation to three different but very intimately related problems:

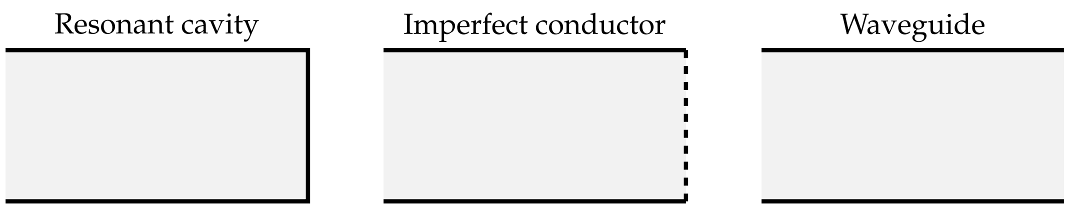


Figure 2.1 – Schematically the most trivial case for each example.

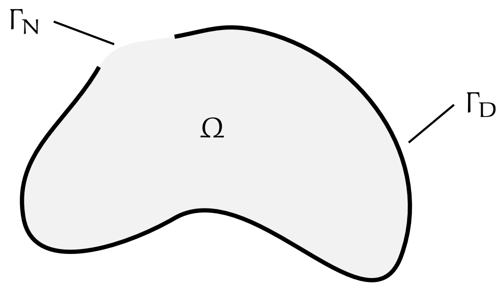


FIGURE 2.2 – 2d resonant cavity.

2.3.1 Two-dimensional resonant cavity

A resonant cavity is a region Ω enclosed by a boundary $\partial\Omega$. The boundary is subdivided into one (or more) inlets Γ_N and a perfect electrically conducting wall $\Gamma_D = \partial\Omega \setminus \Gamma_N$.

Suppose the current density $\mathbf{j} \equiv 0$ and orient the coordinate system in such a way that $\mathbf{u} = u_z\mathbf{e}_z$ and $\mathbf{v} = v_z\mathbf{e}_z$. Consequently,

$$(\mu^{-1}\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) = (\mu^{-1}\nabla u_z) \cdot (\nabla v_z) \tag{2.16}$$

Use $g_z = (\mathbf{g})_z$ along the boundary Γ_N, to convert (2.13) into the weak formulation for a two-dimensional resonant cavity

$$\int_\Omega (\mu^{-1}\nabla u_z) \cdot (\nabla v_z) - \omega^2 \int_\Omega \epsilon u_z v_z = \int_{\partial\Omega} g_z v_z \tag{2.17}$$

Let $\mathbf{E}$ and $\mathbf{B}$ refer to the electric and magnetic fields inside the cavity. For now, I distinguish two types of boundaries.

For the perfectly conducting boundary, treated in [**?**], it holds that

$$\mathbf{n} \times \mathbf{E} = 0, \quad \text{on } \Gamma_D \tag{2.18}$$

For the boundaries in a two-dimensional resonant cavity (see Figure 2.2), this only holds true if $E_z = 0$, which translates to the Dirichlet boundary condition $\mathbf{u}|_{\Gamma_D} = 0$ in light of (2.14).

For the inlet, it is easiest to enforce the boundary condition through the magnetic field $\mathbf{B}$ (considering $\mathbf{n} = -\mathbf{e}_x$ as depicted in Figure 2.2):

$$g_z = ((\mu^{-1}\mathbf{B}) \times (-\mathbf{e}_x))_z = \mu^{-1}B_x \tag{2.19}$$

2.3.2 Imperfect conductor

At an imperfect boundary **?**, (2.6), with (2.9)

$$\mathbf{g} = (\mu^{-1}\nabla \times \mathbf{u}) \times \mathbf{n} = i\omega\lambda(\mathbf{n} \times \mathbf{u}) \times \mathbf{n} \tag{2.20}$$

which, supposing that $\mathbf{u} = u_z \mathbf{e}_z$ and only treating a two-dimensional domain, simplifies (using the fact that $\mathbf{n} \perp \mathbf{u}$ and $\|\mathbf{n}\| = 1$, so $(\mathbf{n} \times \mathbf{u}) \times \mathbf{n} = \mathbf{u}$)

$$g_z = i\omega\lambda u_z \tag{2.21}$$

Therefore, reuse (2.17) as it is, but split boundary integral term for Neumann and impedance boundary.

2.3.3 Waveguide

We go back to (2.13). Again, $\mathbf{j} \equiv 0$, but now we stay in 3d. Supposing that the inlet is located at a constant x-value, such that the surface normal to this inlet is $-\mathbf{e}_x$. For an incoming magnetic field $\mathbf{B}|_{\Gamma_N} = B_0 \mathbf{e}_y$ at the inlet, we see from (2.15) that $\mathbf{g}|_{\Gamma_i} = -\mu^{-1} B_0 \mathbf{e}_z$. At the outlet, we set $\mathbf{g}|_{\Gamma_o} = \mathbf{0}$

3 Finite element approximation with FEniCS

3.1 Theory (come up with better title)

Along the lines of **?**.

See immediately that the weak formulation (2.13) assumes the shape

$$\text{Find } \mathbf{u} \in V, \text{ such that } a_\omega(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}), \ \forall \mathbf{v} \in V_0 \tag{3.1}$$

with the bilinear form

$$a_\omega(\mathbf{u}, \mathbf{v}) = \int_\Omega (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_\Omega \epsilon \mathbf{u} \cdot \mathbf{v} \tag{3.2}$$

and the linear form

$$L(\mathbf{u}) = \int_\Omega \mathbf{j} \cdot \mathbf{v} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v} \tag{3.3}$$

3.2 Demonstration (come up with better title)

In the style of **?**. Problem (2.13) with Ω being a cubic cavity with an inlet on one side and all other sides with $\mu = \epsilon = 0$, $\mathbf{j} = 0$ for simplicity.

Required packages

```python
import numpy as np
import fenics as fen
import matplotlib.pyplot as plt
```

Mesh

```python
nx, ny, nz = 10, 10, 10
mesh = fen.UnitCubeMesh(nx, ny, nz)
```

Function space (Nédélec elements of the first kind **?**)

```python
V = fen.FunctionSpace(mesh, 'N1curl', 1)
```

Inlet (at x = 0)

```python
class Inlet(fen.SubDomain):
    def inside(self, x, on_boundary):
        return on_boundary and fen.near(x[0], 0)
```

PEC boundary

```python
class PECWalls(fen.SubDomain):
    def inside(self, x, on_boundary):
        return on_boundary and not Inlet().inside(x, on_boundary)
```

Boundary ids

```python
boundary_id = fen.MeshFunction('size_t', mesh, mesh.topology().dim()-1)
boundary_id.set_all(0)
Inlet().mark(boundary_id, 1)
PECWalls().mark(boundary_id, 2)
```

Dirichlet boundary (0 for PEC, because Nédélec)

```python
u_D = fen.Expression(('0.0', '0.0', '0.0'), degree=2)
bc = fen.DirichletBC(V, u_D, boundary_id, 2)
```

Neumann boundary (1 for ...)

```
g_N = fen.Expression(('0.0', '0.0', '1.0'), degree=2)
ds = fen.Measure('ds', subdomain_data=boundary_id)
```

Trial and test functions

```
u = fen.TrialFunction(V)
v = fen.TestFunction(V)
```

Neumann boundary integral term

```
N = fen.assemble(fen.dot(g_N, v) * ds(2))
```

Stiffness matrix

```
K = fen.assemble(fen.dot(fen.curl(u), fen.curl(v)) * fen.dx)
bc.apply(K)
```

Mass matrix

```
M = fen.assemble(fen.dot(u, v) * fen.dx)
bc.zero(M)
```

L2 norms

```
def L2_norm(u):
    u_vec = u.vector().get_local()
    return pow(((M * u_vec) * u_vec).sum(), 0.5)
```

Solution at frequencies

```
omegas = np.linspace(6.2, 6.8, 200)
norms = []
u = fen.Function(V)
for omega in omegas:
    fen.solve(K - omega**2 * M, u.vector(), N)
    norms.append(L2_norm(u))
```

Plotting the L2-norms

```
plt.plot(omegas, norms)
plt.yscale('log')
```

4 Minimal rational interpolation for the time-harmonic Maxwell's equations

Let $u : \mathbb{R} \to \mathbb{C}$. Given $u_j = u(\omega_j)$ for $j \in \{1, \dots, S\}$. Want to find a surrogate

$$\tilde{u}(\omega) \approx u(\omega) \tag{4.1}$$

4.1 Motivation

Equations of the type (2.10), in the most simple case (dropping all constants)

$$\nabla \times (\nabla \times \mathbf{u}) - \omega^2 \mathbf{u} = \mathbf{j} \tag{4.2}$$

Writing the double-curl operator in terms of a matrix $\underline{\mathbf{A}}$

$$\mathbf{u} = (\underline{\mathbf{A}} - \omega^2)^{-1} \mathbf{j} \tag{4.3}$$

Eigenvalue decomposition $\underline{\mathbf{A}} = \underline{\mathbf{V}}\,\underline{\mathbf{\Lambda}}\,\underline{\mathbf{V}}^H$ leads us to a form proposed in **?**

$$\mathbf{u} = \underline{\mathbf{V}}(\underline{\mathbf{\Lambda}} - \omega^2 \underline{\mathbf{1}})^{-1} \underline{\mathbf{V}}^H \mathbf{j} = \sum_i \frac{\mathbf{v}_i \mathbf{v}_i^H \mathbf{j}}{\lambda_i - \omega^2} \tag{4.4}$$

which follows from the fact that $\underline{\mathbf{\Lambda}}$ is diagonal, hence also $(\underline{\mathbf{\Lambda}} - \omega^2 \underline{\mathbf{1}})^{-1}$. Here, we denoted the diagonal elements of $\underline{\mathbf{\Lambda}}$ with λ_i (the eigenvalues of $\underline{\mathbf{A}}$) and the columns of $\underline{\mathbf{V}}$ with $\mathbf{v}_i$ (the eigenvectors of $\underline{\mathbf{A}}$).

Hence, it would make sense to search for an approximation of the solution $\mathbf{u}$ that is able to "model" the singularities at $\omega^2 = \lambda_i$, e.g. rational polynomials

$$\tilde{u}(\omega) = \frac{P(\omega)}{Q(\omega)} \tag{4.5}$$

[?]

Algorithm 1 Minimal rational interpolation

Require: $\omega_1, \dots, \omega_S$
Require: $U = [u(\omega_1)| \dots |u(\omega_S)]$ ▷ Snapshot matrix
Compute G with $g_{ij} = \langle u(\omega_i), u(\omega_j) \rangle_M,\ i, j \in \{1, \dots, S\}$ ▷ Gramian matrix
Compute the singular value decomposition $G = V \Sigma V^H$
Define $q = V[:, S]$
Define $\tilde{u}(\omega) = P(\omega)/Q(\omega)$ with $P(\omega) = \sum_{j=1}^{S} \frac{q_j u(\omega_j)}{\omega - \omega_j}$ and $Q(\omega) = \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$

[?]
The rational surrogate $\tilde{\mathbf{u}}$ can be rewritten as

$$\begin{aligned}\tilde{\mathbf{u}}(\omega) &= \sum_{j=1}^{S} \frac{q_j \mathbf{u}(\omega_j)}{\omega - \omega_j} \Big/ \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j} \\ &= \sum_{j=1}^{S} \prod_{\substack{i=0 \\ i \neq j}}^{S} (\omega - \omega_i) q_j \mathbf{u}(\omega_j) \Big/ \sum_{j=1}^{S} \prod_{\substack{i=0 \\ i \neq j}}^{S} (\omega - \omega_i) q_j \end{aligned} \tag{4.6}$$

Algorithm 2 Greedy minimal rational interpolation

Require: $\tau > 0$ ▷ Relative L_2-error tolerance
Require: $\Omega_{\text{test}} = \{\omega_i\}_{i=1}^{M}$ ▷ Set of candidate support points
Require: $a_\omega(u, v) = L(v)$ ▷ Finite element formulation of the problem
  Choose $\omega_1, \dots, \omega_t \in \Omega_{\text{test}}$ ▷ Usually the smallest and largest element
  Remove $\omega_1, \dots, \omega_t$ from Ω_{test}
  Solve $a_{\omega_i}(u_i, v) = L(v)$ for $i \in \{1, \dots, t\}$
  Build surrogate $\tilde{u}_t = P_t(\omega)/Q_t(\omega)$ using the solutions $u_1, \dots, u_t$
  while $\Omega_{\text{test}} \neq \emptyset$ **do**
    $\omega_{t+1} \leftarrow \text{argmin}_{\omega \in \Omega_{\text{test}}} |Q_t(\omega)|$
    Solve $a_{\omega_{t+1}}(u_{t+1}, v) = L(v)$
    Build surrogate $\tilde{u}_{t+1} = P_{t+1}(\omega)/Q_{t+1}(\omega)$ using the solutions $u_1, \dots, u_{t+1}$
    if $\|u_{t+1}(\omega_{t+1}) - \tilde{u}_{t+1}(\omega_{t+1})\|_M / \|u_{t+1}(\omega_{t+1})\|_M < \tau$ **then return**
    end if
    $t \leftarrow t + 1$
  end while

Therefore, if the rational surrogate $\tilde{\mathbf{u}}$ is evaluated at one of the interpolation nodes ω_i, $\mathbf{u}(\omega_i)$ is recovered.

If the rational surrogate $\tilde{\mathbf{u}}$ is evaluated in a zero $\bar{\omega}$ of the denominator $Q(\bar{\omega}) = 0$, we observe a pole, unless $P(\bar{\omega})$ happens to vanish too.

[?]

Defining

$$\nu_i = (\omega - \omega_i)^{-1} \tag{4.7}$$

and requiring

$$0 = Q(\omega) = \sum_{i=1}^{S} q_i \nu_i(\omega) \tag{4.8}$$

can be equivalently expressed as a generalized eigenvalue problem

$$\underline{\mathbf{A}}\mathbf{u} = \omega \underline{\mathbf{B}}\mathbf{u} \tag{4.9}$$

with

$$\underline{\mathbf{A}} = \begin{pmatrix} 0 & q_1 & q_2 & \cdots & q_S \\ 1 & \omega_1 & & & \\ 1 & & \omega_2 & & \\ \vdots & & & \ddots & \\ 1 & & & & \omega_S \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{B}} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \end{pmatrix} \tag{4.10}$$

[?]

Can check stability with singular values $\sigma_1, \dots, \sigma_S$ in Σ which we obtain in Algorithm 1. Need smallest singular values to different from each other. This conditioning can be estimated with the relative spectral range **?**

$$\frac{\sigma_{S-1} - \sigma_S}{\sigma_1 - \sigma_S} \tag{4.11}$$

Algorithm 3 Additive Householder triangularization

Require: $U[1 \ldots s, 1 \ldots N]$ ▷ Next snapshot matrix
Require: $R[1 \ldots S, 1 \ldots S]$ ▷ Previous triangular matrix
Require: $E[1 \ldots S, 1 \ldots N]$ ▷ Previous orthonormal matrix
Require: $V[1 \ldots S, 1 \ldots N]$ ▷ Previous Householder matrix

```
Extend size of R to (S + s) × (S + s)
Extend E with S orthonormal columns to (S + s) × N
Extend size of V to (S + s) × N
for j = S + 1 : S + s do
    u = U[j]
    for k = 1 : j − 1 do
        u ← u − 2⟨V[k, :], u⟩_M V[k, :]
        R[k, j] ← ⟨E[k, :], u⟩_M
        u ← u − R[k, j]E[k, :]
    end for
    R[j, j] ← ||u||_M
    α ← ⟨E[j, :], u⟩_M
    if |α| ≠ 0 then
        E[j, :] ← E[j, :](−α/|α|)
    end if
    V[j, :] ← R[j, j]E[j, :] − u
    V[j, :] ← V[j, :] − ⟨E[S + 1 : j], V[j, :]⟩_M E[S + 1 : j, :]
    σ ← ||V[j, :]||_M
    if σ ≠ 0 then
        V[j, :] ← V[j, :]/σ
    else
        V[j, :] ← E[j, :]
    end if
end for
```

Denote $\underline{\mathbf{U}} = [\mathbf{u}(\omega_1), \dots, \mathbf{u}(\omega_S)]$ snapshot matrix. Let

$$\mathbf{u} = \sum_{j=1}^{S} \frac{q_j \mathbf{e}_j}{\omega - \omega_j} / \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j} \tag{4.12}$$

with the canonical basis vectors $\{\mathbf{e}_j\}_j$. If we perform a QR decomposition on this matrix, we obtain

$$\underline{\mathbf{U}} = \underline{\mathbf{W}}\, \underline{\mathbf{R}} \tag{4.13}$$

5 EXAMPLES

5.1 TWO-DIMENSIONAL RECTANGULAR CAVITY

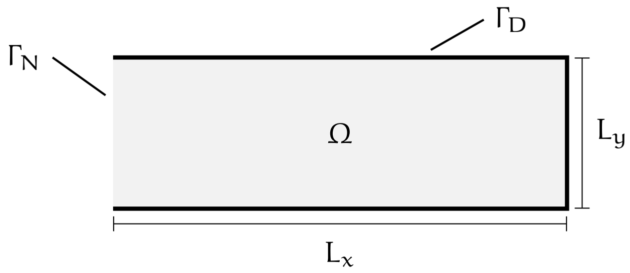
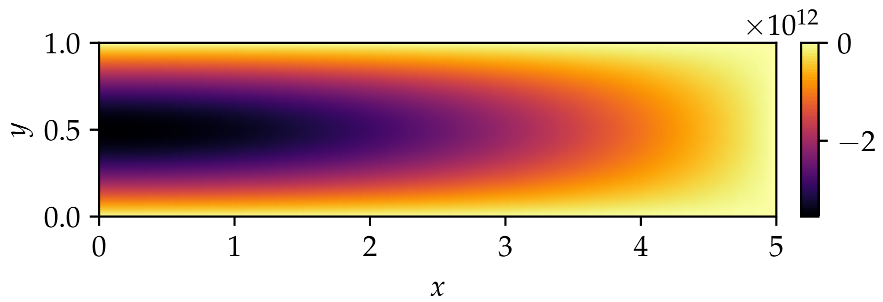
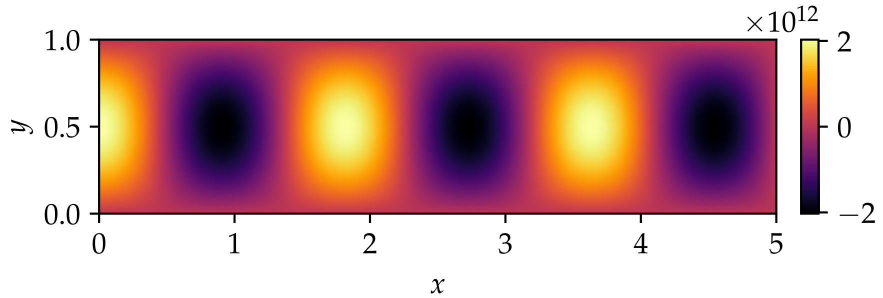


FIGURE 5.1 – TODO.



(A) First resonant frequency $\omega = 3.159$.



(B) Fifth resonant frequency $\omega = 4.675$.

FIGURE 5.2 – Caption.

For $\|\mathbf{u}\|^2_{L_2(\Omega)} = \int_\Omega \|\mathbf{u}\|^2$
Analytical eigenfrequencies

$$\omega_{n,m} = \pi\sqrt{\left(\frac{2n+1}{2L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2},\ n \in \{0,1,\ldots\},\ m \in \{1,2,\ldots\} \tag{5.1}$$

Numerical eigenfrequencies, solve generalized (symmetric) eigenvalue problem

$$\underline{\mathbf{K}}\mathbf{u} = \omega^2\underline{\mathbf{M}}\mathbf{u} \tag{5.2}$$

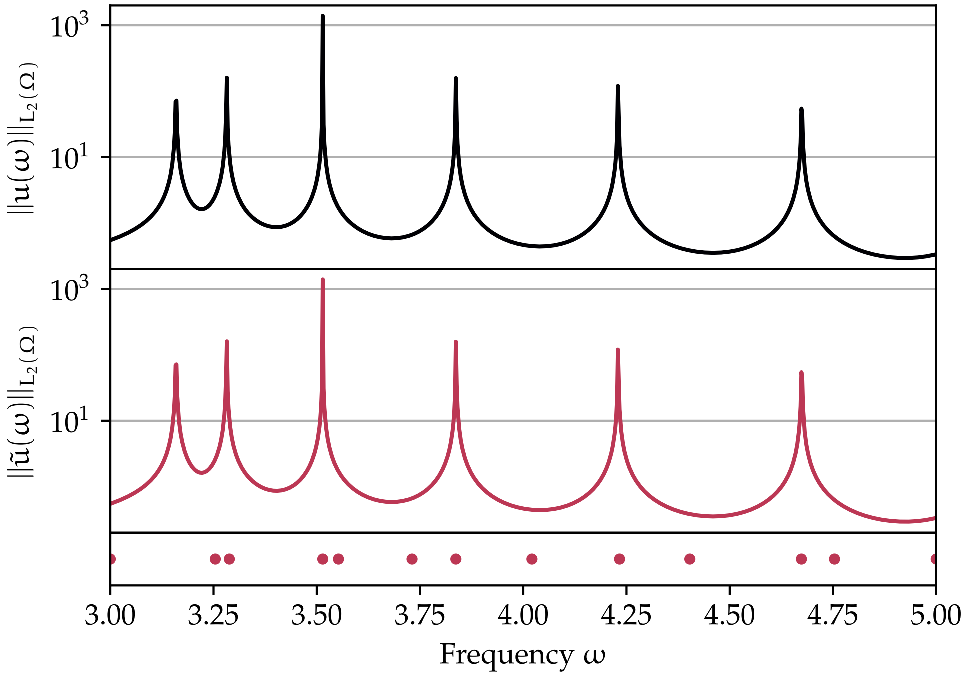


FIGURE 5.3 – Caption.

TABLE 5.1 – Comparison eigsh and gMRI.

	eigsh		gMRI	
DOF	Δ	t	Δ	t
713	1.950×10^{-2}	25.9 ± 1.1 ms	1.950×10^{-2}	61.9 ± 3.6 ms
7412	1.826×10^{-3}	199.0 ± 9.9 ms	1.827×10^{-3}	410 ± 16.8 ms
74722	1.817×10^{-4}	3.5 ± 0.1 s	1.820×10^{-4}	5.2 ± 0.2 s
745513	1.811×10^{-5}	75.0 ± 1.6 s	1.846×10^{-5}	104.0 ± 1.1 s

Take $\{\mathbf{u}_j, \omega_j^2\}_j$ be the resonant modes, i.e. solutions to the eigenvalue problem (5.2), such that

$$\underline{\mathbf{K}}\mathbf{u}_j = \omega_j^2 \underline{\mathbf{M}}\mathbf{u}_j \tag{5.3}$$

Adding a source term $\mathbf{f}$

$$\underline{\mathbf{K}}\mathbf{u} - \omega^2 \underline{\mathbf{M}}\mathbf{u} = \mathbf{f} \tag{5.4}$$

If $\mathbf{u}$ is expressed in terms of the basis $\{\mathbf{u}_j\}_j$, i.e. $\mathbf{u} = \sum_j \alpha_j \mathbf{u}_j$

$$\sum_j \alpha_j (\underline{\mathbf{K}}\mathbf{u}_j - \omega^2 \underline{\mathbf{M}}\mathbf{u}_j) = \mathbf{f} \tag{5.5}$$

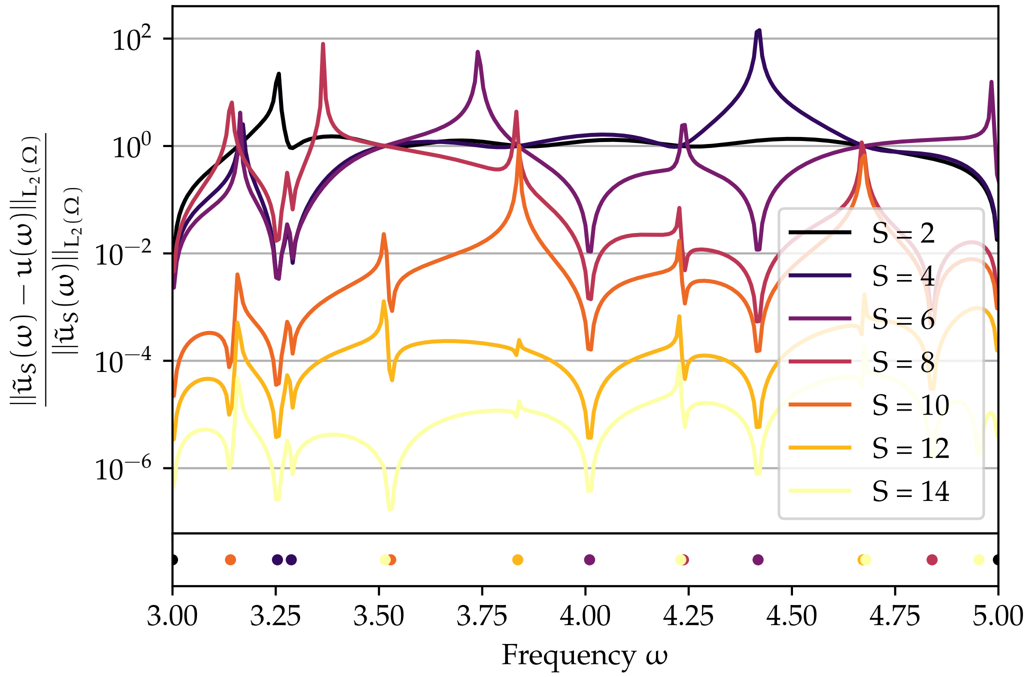


FIGURE 5.4 – Caption.

Using (5.3)

$$\sum_j \alpha_j(\omega_j^2 - \omega^2)\underline{\mathbf{M}}\mathbf{u}_j = \mathbf{f} \tag{5.6}$$

from which we can take the scalar product with $\mathbf{u}_j^H$ to obtain

$$\alpha_j = \frac{\mathbf{u}_j^H \mathbf{f}}{\omega_j^2 - \omega^2} \tag{5.7}$$

If $\mathbf{u}_j^H \mathbf{f} = 0$, then the resonant mode at ω_j is suppressed (fine with MRI, but eigsh will detect a resonant mode).

For $\|\mathbf{u}\|^2_{L_2(\Gamma)} = \int_\Gamma \|\mathbf{u}\|^2$

5.2 IMPERFECTLY CONDUCTING BOUNDARIES

Numerical eigenfrequencies, solve

$$(\underline{\mathbf{K}} - i\omega\underline{\mathbf{I}} - \omega^2\underline{\mathbf{M}})\mathbf{u} = 0 \tag{5.8}$$

Define $\mathbf{v} = \omega\mathbf{u}$, so that we may write this as the generalized eigenvalue problem

$$\begin{bmatrix} & \underline{\mathbf{1}} \\ \underline{\mathbf{K}} & -i\underline{\mathbf{I}} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \omega \begin{bmatrix} \underline{\mathbf{1}} & \\ & \underline{\mathbf{M}} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \tag{5.9}$$

which is, however, no longer Hermitian (LHS chosen as "diagonal" as possible).

5.3 DUAL MODE CIRCULAR WAVEGUIDE FILTER

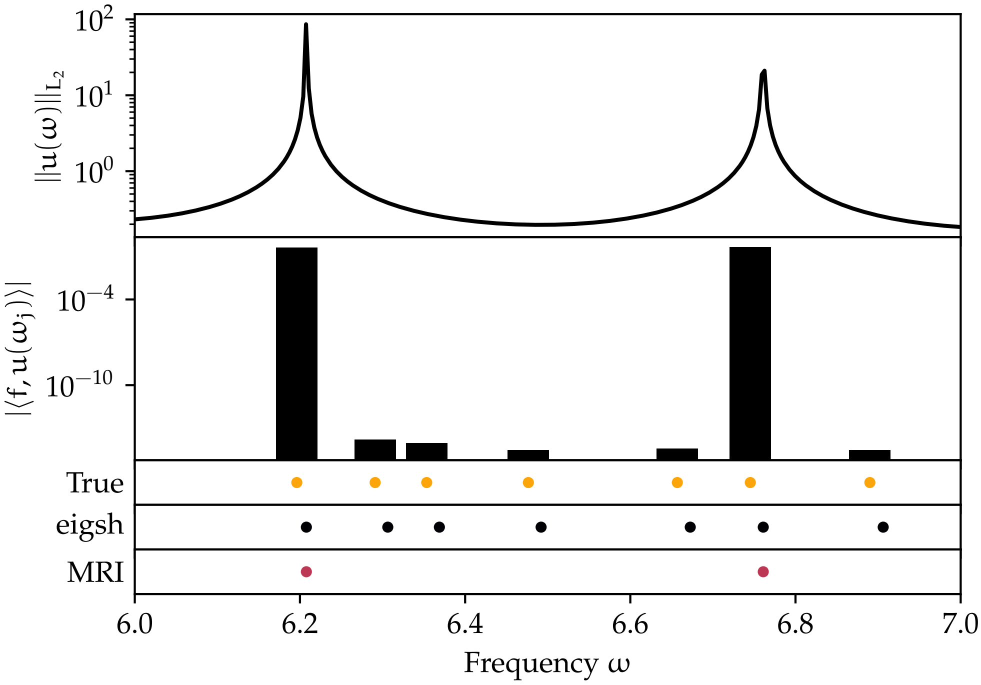


FIGURE 5.5 – Caption.

6 CONCLUSION AND OUTLOOK

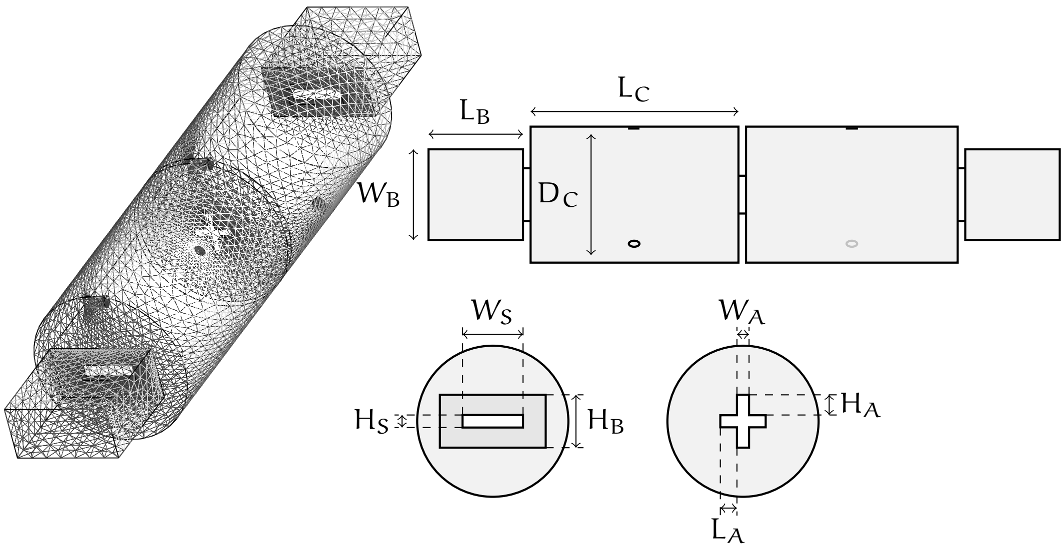


FIGURE 5.6 – Dual-mode circular waveguide filter. **?** $W_C = 43.87$ mm, $D_C = 28.0$ mm, $L_B = 43.87$ mm, $W_B = 19.05$ mm, $H_B = 9.525$ mm, $L_B = 20.0$ mm, $W_S = 10.05$ mm, $H_S = 3.0$ mm, $W_A = 2.0$ mm, $H_A = 3.375$ mm, $L_A = 2.825$ mm, thickness of all irises 2.0 mm, screws half way up the cavity horizontal tuning screws with depth 3.82 mm and coupling screws at angles $\pm 45°$ with depth 3.57 mm.

7 Appendix

7.1 Detailed derivation for the weak formulation of the time-harmonic potential equation

The goal is to rewrite the curl-integral on the left-hand side of (2.12):

$$\int_\Omega (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} \tag{7.1}$$

In order to simplify the curls and apply the Gauss theorem, I first show the following vector calculus identity:

Curl product rule

$$(\nabla \times \mathbf{a}) \cdot \mathbf{b} = \nabla \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\nabla \times \mathbf{b}) \tag{7.2}$$

where $\mathbf{a}$, $\mathbf{b}$ are vector-value functions. The completely antisymmetric tensor ε_{ijk}, frequently referred to as the Levi-Civita tensor, may be employed to rewrite the components of the curl of a vector-function $\mathbf{a}$ as the sum

$$(\nabla \times \mathbf{a})_k = \sum_i \sum_j \varepsilon_{ijk} \partial_i u_j \tag{7.3}$$

where ∂_i denotes the partial derivative with respect to the i-th coordinate direction. This yields

$$\begin{aligned}
(\nabla \times \mathbf{a}) \cdot \mathbf{b} &= \sum_k (\nabla \times \mathbf{a})_k b_k \\
&= \sum_k (\sum_i \sum_j \varepsilon_{ijk} \partial_i a_j) b_k \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{ijk} a_j b_k) - \sum_k \sum_i \sum_j a_j (\varepsilon_{ijk} \partial_i b_k) \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{jki} a_j b_k) - \sum_k \sum_i \sum_j a_j ((-\varepsilon_{ikj}) \partial_i b_k) \\
&= \sum_i \partial_i (\mathbf{a} \times \mathbf{b})_i + \sum_j u_j (\nabla \times \mathbf{b})_j \\
&= \nabla \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\nabla \times \mathbf{b})
\end{aligned} \tag{7.4}$$

by expressing the scalar product as a component-sum, using the product rule and applying the symmetry and anti-symmetry properties of the Levi-Civita tensor. Now the identity (7.2) to (7.1) together with Gauss' theorem gives

$$\begin{aligned}
\int_\Omega (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} &= \int_\Omega \nabla \cdot ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) + \int_\Omega (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} + \int_\Omega (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{7.5}$$

For later convenience, the boundary integral can further be simplified using the

Commutative behavior of the scalar triple product

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} \tag{7.6}$$

This identity follows immediately from a small manipulation with the Levi-Civita tensor:

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \sum_k (\sum_i \sum_j \varepsilon_{ijk} a_i b_j) c_k \\
&= \sum_j (\sum_i \sum_k (-\varepsilon_{ikj}) a_i c_k) b_j \\
&= -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}
\end{aligned} \tag{7.7}$$

The boundary integral becomes

$$\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} = -\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}) \cdot \mathbf{v} \tag{7.8}$$

This concludes the short derivation, because now (7.1) may be rewritten as

$$-\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} + \int_\Omega (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \tag{7.9}$$