

EPFL

PROJECT CSE I

# Notes

*Fabio Matti*

supervised by  
Prof. Fabio Nobile  
Dr. Davide Pradovera

February 26, 2022

# 1 FINITE ELEMENT METHOD

## 1.1 THE GENERAL APPROACH

*Summarizes Chapter 1 in Quarteroni: Introduction to Finite Elements Method*

Usually, the problems may be expressed in a simple equation

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = u_D & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where  $L$  denotes a linear differential operator (e.g.  $-\Delta$  in the Poisson equation),  $u$  the solution to be found, and  $f$  is a source term independent of  $u$ . Some boundary condition  $u_D$  is imposed on the solution  $u$ .

However, equation (1.1) usually does not allow all physically significant solutions (particularly non-differentiable ones). Therefore, we convert the problem to a weak form. This is achieved by multiplying (1.1) with a test function  $v \in V$ , and integrating over the whole domain  $\Omega$ :

$$\int_{\Omega} (Lu)v = \int_{\Omega} fv, \quad \forall v \in V \quad (1.2)$$

Usually, integration by parts allows us to “transfer” the derivatives from the  $Lu$  term to the test function  $v$ , such that the order (with respect to the derivatives taken) is more “balanced” between the two terms. As a trade-off, a boundary term appears and needs to be eliminated to facilitate the finite element solution. This term can often be eliminated by restricting ourselves to test functions from a subspace  $V' \subset V$ . The weak problem then reads

$$\int_{\Omega} (L_u u)(L_v v) = \int_{\Omega} fv, \quad \forall v \in V' \quad (1.3)$$

with a linear differential operator  $L_u$  acts on the solution  $u$ , and a linear differential operator  $L_v$ , appearing due to the integration by parts, acts on the test function  $v$ . For simplicity, we refer to the left-hand side as the bilinear form

$$a(u, v) = \int_{\Omega} (L_u u)(L_v v) \quad (1.4)$$

and the right-hand side as the linear form

$$F(v) = \int_{\Omega} fv \quad (1.5)$$

To solve (1.3), we prefer to look for approximate solutions  $u_h$  in a finite dimensional space  $V_h$  with  $\dim(V_h) = N_h$  (what role does  $V'_h$ , the space wherein  $v$  lies to satisfy the boundary conditions, play? as far as I can tell  $V'_h$  is the subspace of  $V_h$ , in which all functions vanish at the boundary where  $u$  is known). Choosing a basis  $\{\varphi_i\}_{i \leq N_h}$  then allows us to represent the approximate solution as

$$u_h = \sum_{j \leq N_h} u_j \varphi_j \quad (1.6)$$

for some coefficients  $u_i$  that need to be determined. We thus see, that (1.3) turns into a linear system, since we only need to test the equality for the basis elements  $\varphi_i$  of  $V_h$ :

$$\sum_{j=1}^{N_h} u_j a(\varphi_j, \varphi_i) = F(\varphi_i), \quad i \in \{1, \dots, N_h\} \quad (1.7)$$

If we write  $A_{ij} = a(\varphi_j, \varphi_i)$  and  $\mathbf{F} = (F(\varphi_1), \dots, F(\varphi_{N_h}))^T$ , we have reduced the problem to finding  $\mathbf{u} = (u_1, \dots, u_{N_h})^T$ , such that

$$\mathbb{A}\mathbf{u} = \mathbf{F} \quad (1.8)$$

and have identified an approximate solution to (1.1) as  $u \approx u_h = \sum_{j \leq N_h} u_j \varphi_j$ .

Choosing the space  $V_h$  is fundamental to have an accurate method that gives a good approximation  $u_h$  of  $u$ . Furthermore, the choice of basis  $\{\varphi_i\}_{i \leq N_h}$  influences how  $\mathbb{A}$  ends up looking. Of particular interest are bases for which  $a(\varphi_j, \varphi_i)$  vanishes for almost all  $i$  and  $j$ , thus yielding a sparse matrix  $\mathbb{A}$ . The choice of basis also controls the conditioning of  $\mathbb{A}$  (need to find an example to illustrate this).

## 1.2 THE POISSON EQUATION

*Taken from FEniCS manual (too lazy for bibtex...)*

We aim to solve an equation of the form

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}) \quad (1.9)$$

on a domain  $\mathbf{x} \in \Omega$ , with a solution  $u(\mathbf{x})$  that satisfies a certain boundary condition  $u(\mathbf{x}) = u_d(\mathbf{x})$  for all  $\mathbf{x} \in \partial\Omega$  that lie on the border of  $\Omega$ .

To do this, we first convert this equation to its weak form by multiplying both sides with an arbitrary test function  $v(\mathbf{x})$ , which vanishes on the border (i.e.  $v(\mathbf{x}) = 0, \forall \mathbf{x} \in \partial\Omega$ ), and by then integrating over all of  $\Omega$ :

$$-\int_{\Omega} \Delta u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \quad (1.10)$$

We may now rearrange the gradient product rule  $\nabla(ab) = (\nabla a)b + a(\nabla b)$  and Gauss' theorem (as long as  $v(\mathbf{x})$  is differentiable in a neighborhood of  $\Omega$ ) combined with the fact that  $v(\mathbf{x})$  vanishes on  $\partial\Omega$  to convert the right-hand side to

$$\begin{aligned} -\int_{\Omega} \Delta u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} &= -\int_{\Omega} \nabla(\nabla u(\mathbf{x}) v(\mathbf{x})) d\mathbf{x} + \int_{\Omega} \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \\ &= -\int_{\partial\Omega} \nabla u(\mathbf{x}) v(\mathbf{x}) d\boldsymbol{\omega} + \int_{\Omega} \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \\ &= \int_{\Omega} \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \end{aligned} \quad (1.11)$$

Consequently, the weak formulation of the problem is to find  $u(\mathbf{x})$ , such that for arbitrary  $v(\mathbf{x})$ , we have

$$\int_{\Omega} \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \quad (1.12)$$

To simplify and generalize the notation, we may use the linear form  $L : V \rightarrow \mathbb{R}$  as

$$L(v) = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \quad (1.13)$$

and also the bilinear form  $a : V \times V \rightarrow \mathbb{R}$

$$a(u, v) = \int_{\Omega} \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \quad (1.14)$$

### 1.3 EXAMPLE: ONE DIMENSIONAL POISSON EQUATION

*Initial idea taken from Wikipedia article about FEM.*

To illustrate the choice of basis functions, we will now consider the simple one dimensional case  $\Omega = [a, b]$ , such that the weak formulation of the problem turns into

$$\int_a^b u'(x) v'(x) dx = \int_a^b f(x) v(x) dx \quad (1.15)$$

We now subdivide the domain  $[a, b]$  into  $M$  subintervals, each of length  $h = (b - a)/M$ , with nodes at  $x_k = a + hk, k \in \{0, 1, \dots, M\}$ . We proceed to choose as the basis functions the class of the piecewise linear Lagrange interpolating polynomials on  $[x_k, x_{k+1}], k \in \{0, 1, \dots, M\}$ , defined as

$$v_k(x) = \frac{x - x_{k-1}}{x_k - x_{k-1}} \mathbf{1}_{\{x \in [x_{k-1}, x_k]\}} + \frac{x_{k+1} - x}{x_{k+1} - x_k} \mathbf{1}_{\{x \in [x_k, x_{k+1}]\}} \quad (1.16)$$

If we now interpolate  $f(x)$  and  $u(x)$  as piecewise linear Lagrange polynomials, we get the representation

$$\begin{aligned} f(x) &\approx \sum_{i=1}^M f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + f(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}} \\ &= \sum_{i=1}^{M-1} f(x_i) v_i(x) \end{aligned} \quad (1.17)$$

and analogously

$$u(x) = \sum_{i=1}^{M-1} u(x_i) v_i(x) \quad (1.18)$$

We now restricted ourselves to the discrete variational formulation of the problem

$$\sum_{i=1}^{M-1} u(x_i) \int_a^b v_i'(x) v_j'(x) dx = \sum_{i=1}^{M-1} f(x_i) \int_a^b v_i(x) v_j(x) dx \quad (1.19)$$

which needs to be satisfied for all  $j \in \{0, 1, \dots, M\}$ .

This equation can be rewritten in terms of two matrices  $\mathbf{K}$  and  $\mathbf{L}$  which we define as

$$K_{ij} = \int_a^b v_i(x) v_j(x) dx \quad (1.20)$$

$$L_{ij} = \int_a^b v_i'(x) v_j'(x) dx \quad (1.21)$$

such that we get

$$\sum_{i=1}^{M-1} u(x_i) L_{ij} = \sum_{i=1}^{M-1} f(x_i) K_{ij} \quad (1.22)$$

Notice, that we only need the entries  $K_{ij}$  and  $L_{ij}$  with  $i \in \{1, 2, \dots, M-1\}$ , since we already know the boundary conditions of  $u(x)$  at  $x = x_0$  and  $x = x_M$ .

We realize, that the  $L_2$  inner product of  $v_i(x)$  with  $v_j(x)$  (and consequently also the one of  $v_i'(x)$  with  $v_j'(x)$ ) is zero for all  $|i - j| > 1$ , hence, we distinguish two different cases.

1.  $i = j$ : Here, the inner product turns out to be

$$\begin{aligned} \int_a^b v_i(x) v_i(x) dx &= \int_a^b \left( \frac{x - x_{i-1}}{x_i - x_{i-1}} \right)^2 \mathbf{1}_{\{x \in [x_{i-1}, x_i]\}} + \left( \frac{x_{i+1} - x}{x_{i+1} - x_i} \right)^2 \mathbf{1}_{\{x \in [x_i, x_{i+1}]\}} dx \\ &= 2 \int_{x_{i-1}}^{x_i} \left( \frac{x - x_{i-1}}{x_i - x_{i-1}} \right)^2 dx \\ &= \frac{2}{h^2} \int_{x_{i-1} - x_{i-1}}^{x_i - x_{i-1}} u^2 du \\ &= \frac{2}{h^2} \frac{1}{3} h^3 \\ &= \frac{2h}{3} \end{aligned} \quad (1.23)$$

and for the derivatives it is

$$\begin{aligned}
\int_a^b v_i'(x) v_i'(x) dx &= \int_a^b \left( \frac{1}{x_i - x_{i-1}} \right)^2 \mathbf{1}_{\{x \in [x_{i-1}, x_i]\}} + \left( \frac{-1}{x_{i+1} - x_i} \right)^2 \mathbf{1}_{\{x \in [x_i, x_{i+1}]\}} dx \\
&= 2 \int_{x_{i-1}}^{x_i} \left( \frac{1}{x_i - x_{i-1}} \right)^2 dx \\
&= \frac{2}{h^2} \int_0^h 1 du \\
&= \frac{2}{h}
\end{aligned} \tag{1.24}$$

2.  $|i - j| = 1$ : Here, we can limit ourselves to the case where  $j = i + 1$ , since the other case is fully symmetric. We calculate

$$\begin{aligned}
\int_a^b v_i(x) v_{i+1}(x) dx &= \int_a^b \frac{x_{i+1} - x}{x_{i+1} - x_i} \frac{x - x_i}{x_{i+1} - x_i} \mathbf{1}_{\{x \in [x_i, x_{i+1}]\}} dx \\
&= \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{x_{i+1} - x_i} \frac{x - x_i}{x_{i+1} - x_i} dx \\
&= \frac{1}{h^2} \int_{x_i - x_i}^{x_{i+1} - x_i} (x_{i+1} - x_i - u) u du \\
&= \frac{1}{h^2} \int_0^h (h - u) u du \\
&= \frac{1}{h^2} \left( \frac{h^3}{2} - \frac{h^3}{3} \right) \\
&= \frac{h}{6}
\end{aligned} \tag{1.25}$$

and for the derivative it is

$$\begin{aligned}
\int_a^b v_i'(x) v_{i+1}'(x) dx &= \int_a^b \frac{-1}{x_{i+1} - x_i} \frac{1}{x_{i+1} - x_i} \mathbf{1}_{\{x \in [x_i, x_{i+1}]\}} dx \\
&= -\frac{1}{h^2} \int_{x_i}^{x_{i+1}} 1 dx \\
&= -\frac{1}{h}
\end{aligned} \tag{1.26}$$

Now, using the previously defined matrices  $\mathbf{K}_{ij}$  and  $\mathbf{L}_{ij}$ , we get the matrix equation

$$\mathbf{L} \mathbf{u} = \mathbf{K} \mathbf{f} \tag{1.27}$$

with

$$\mathbf{u} = (u_0, u(x_1), \dots, u_M)^T \quad (1.28)$$

$$\mathbf{f} = (f(x_0), f(x_1), \dots, f(x_M))^T \quad (1.29)$$

$$\mathbf{L} = \begin{pmatrix} 1 & & & & \\ \frac{2}{h} & -\frac{1}{h} & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & -\frac{1}{h} & \frac{2}{h} & \ddots & \\ & & -\frac{1}{h} & \ddots & -\frac{1}{h} \\ & & & \ddots & \frac{2}{h} \\ & & & & 1 \end{pmatrix} \quad (1.30)$$

$$\mathbf{K} = \begin{pmatrix} \frac{u_0}{f(x_0)} & & & & \\ \frac{2h}{3} & \frac{h}{6} & & & \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & & \\ & \frac{h}{6} & \frac{2h}{3} & \ddots & \\ & & \frac{h}{6} & \ddots & \frac{h}{6} \\ & & & \ddots & \frac{2h}{3} \\ & & & & \frac{u_M}{f(x_M)} \end{pmatrix} \quad (1.31)$$

$$(1.32)$$

Here, we have adjusted the first rows in  $\mathbf{L}$  and  $\mathbf{K}$ , such that the boundary conditions are necessarily satisfied. To obtain the finite element solution, we simply solve this linear system.

## 2 MAXWELL'S EQUATIONS

Let  $\mathbf{E} = (E_1, E_2, E_3)^T$  denote the electric field,  $\mathbf{B} = (B_1, B_2, B_3)^T$  the magnetic field strength, and  $\mathbf{j} = (j_1, j_2, j_3)^T$  the electric current density. We suppose Maxwell's equations hold:

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \quad (2.1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.2)$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \quad (2.3)$$

$$\nabla \times (\mu^{-1} \mathbf{B}) = \partial_t (\epsilon \mathbf{E}) + \mathbf{j} \quad (2.4)$$

### 2.1 VECTOR POTENTIAL FORMULATION (FLAWED)

We can therefore use (2.2) to write  $\mathbf{B} = \nabla \times \mathbf{A}$  for some vector potential  $\mathbf{A}$ . Furthermore, we can identify from (2.3) that  $\mathbf{E} = -\nabla \phi - \partial_t \mathbf{A}$  can be written for some scalar potential  $\phi$ .

The physical quantities  $\mathbf{E}$  and  $\mathbf{B}$  remain unchanged if we transform  $\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\psi$  and  $\phi \rightarrow \phi' = \phi - \partial_t\psi$  (gauge transformations), as can be explicitly verified by plugging these transformed potentials into the definitions of  $\mathbf{E}$  and  $\mathbf{B}$ . Choosing as the gauge field as

$$\psi = \int_0^t \phi dt' \quad (2.5)$$

we see that  $\phi' = \phi - \partial_t \int_0^t \phi dt' = \phi - \phi = 0$ . Hence, if we now express the electric field  $\mathbf{E}$  in terms of these transformed potentials, we realize that  $\mathbf{E} = -\nabla\phi' - \partial_t\mathbf{A}' = -\partial_t\mathbf{A}'$  because we have transformed  $\phi$  exactly in such a way, which makes  $\phi'$  vanish (not symbolically, but due to its actual value being zero). As for the magnetic field  $\mathbf{B}$ , we have  $\mathbf{B} = \nabla \times \mathbf{A}'$ .

Plugging these identities into (2.4), and for simplicity replacing the symbol  $\mathbf{A}'$  with  $\mathbf{A}$  again, we get

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{A}) = -\epsilon \partial_t^2 \mathbf{A} + \mathbf{j} \quad (2.6)$$

We may want to introduce a harmonic time dependence of  $\mathbf{A}$  and  $\mathbf{j}$  with frequencies  $\omega$ , such that  $\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}) \exp(i\omega t)$  and  $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x}) \exp(i\omega t)$ . Plugging this into (2.6) yields us

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{A}) - \epsilon \omega^2 \mathbf{A} = \mathbf{j} \quad (2.7)$$

We reduce this equation to its weak formulation, by multiplying it with a vector-valued function  $\mathbf{v} \in H_{\text{curl}}(\Omega)$ , where we denoted

$$H_{\text{curl}}(\Omega) = \{\mathbf{u} : \Omega \rightarrow \mathbb{C}, \text{ such that } \mathbf{u} \in L^2(\mathbb{C})^3, \nabla \times \mathbf{u} \in L^2(\mathbb{C})^3\} \quad (2.8)$$

and by integrating over all of  $\Omega$ :

$$\int_{\Omega} (\nabla \times (\mu^{-1} \nabla \times \mathbf{A})) \cdot \mathbf{v} - \epsilon \omega^2 \int_{\Omega} \mathbf{A} \cdot \mathbf{v} = \int_{\Omega} \mathbf{j} \cdot \mathbf{v} \quad (2.9)$$

To further simplify this expression, we will derive an identity for the scalar product of a vector-valued function  $\mathbf{v}$  with the curl of a vector-valued function  $\mathbf{u}$ . For this, we use the completely antisymmetric tensor  $\varepsilon_{ijk}$  (frequently referred to as the Levi-Civita tensor), to rewrite the  $k$ -th component of the curl as

$$(\nabla \times \mathbf{u})_k = \sum_i \sum_j \varepsilon_{ijk} \partial_i u_j \quad (2.10)$$

where  $\partial_i$  denotes the partial derivative with respect to the  $i$ -th coordinate direction. Rewriting the scalar product as a sum and identifying  $\mathbf{u} = \mu^{-1} \nabla \times \mathbf{A}$ , we apply the product rule to get



$$\begin{aligned}
(\nabla \times \mathbf{u}) \cdot \mathbf{v} &= \sum_k (\nabla \times \mathbf{u})_k v_k \\
&= \sum_k \left( \sum_i \sum_j \varepsilon_{ijk} \partial_i u_j \right) v_k \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{ijk} u_j v_k) - \sum_k \sum_i \sum_j u_j (\varepsilon_{ijk} \partial_i v_k) \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{jki} u_j v_k) - \sum_k \sum_i \sum_j u_j ((-\varepsilon_{ikj}) \partial_i v_k) \\
&= \sum_i \partial_i (\mathbf{u} \times \mathbf{v})_i + \sum_j u_j (\nabla \times \mathbf{v})_j \\
&= \nabla \cdot (\mathbf{u} \times \mathbf{v}) + \mathbf{u} \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{2.11}$$

Consequently, we may rewrite the double curl term as

$$\begin{aligned}
\int_{\Omega} (\nabla \times (\mu^{-1} \nabla \times \mathbf{A})) \cdot \mathbf{v} &= \int_{\Omega} \nabla \cdot ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{v}) + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{v}) \cdot \mathbf{n} + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{2.12}$$

Again denoting  $\mathbf{u} = \mu^{-1} \nabla \times \mathbf{A}$ , we can rearrange

$$\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n} &= \sum_k \left( \sum_i \sum_j \varepsilon_{ijk} u_i v_j \right) n_k \\
&= \sum_j \left( \sum_i \sum_k (-\varepsilon_{ikj}) u_i n_k \right) v_j \\
&= -(\mathbf{u} \times \mathbf{n}) \cdot \mathbf{v}
\end{aligned} \tag{2.13}$$

and therefore have

$$\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{v}) \cdot \mathbf{n} = - \int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{n}) \cdot \mathbf{v} \tag{2.14}$$

We can finally identify the weak form of the problem as

$$\int_{\Omega} (\mu^{-1} \nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) - \epsilon \omega^2 \int_{\Omega} \mathbf{A} \cdot \mathbf{v} = \int_{\Omega} \mathbf{j} \cdot \mathbf{v} + \underbrace{\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{n}) \cdot \mathbf{v}}_{\text{boundary integral term}} \tag{2.15}$$

where  $\mathbf{n}$  denotes the normal vector to  $\partial\Omega$ . Notice how we placed the boundary integral term on the right hand side, since along the border  $\partial\Omega$  we usually either know  $\nabla \times \mathbf{A} = \mathbf{B} = \mathbf{B}_D$  (or even only  $\mathbf{B} \times \mathbf{n}$ ) as the Neumann boundary condition or  $-i\omega \mathbf{A} = \mathbf{E}_D$  as the Dirichlet boundary condition.

## 2.2 ELECTRIC FIELD FORMULATION

Multiplying both sides of (2.3) with  $\mu^{-1}$  and taking the curl allows us to substitute (2.4) in, such that we get

$$\begin{aligned}\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) &= -\partial_t (\nabla \times (\mu^{-1} \mathbf{B})) \\ &= -\partial_t (\partial_t (\epsilon \mathbf{E}) + \mathbf{j}) \\ &= -\partial_t^2 (\epsilon \mathbf{E}) + \partial_t \mathbf{j}\end{aligned}\tag{2.16}$$

Introducing a harmonic time dependence of  $\mathbf{E}$  and  $\mathbf{j}$  with frequencies  $\omega$ , such that  $\mathbf{E}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x}) \exp(i\omega t)$  and  $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x}) \exp(i\omega t)$ , we obtain

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \omega^2 \epsilon \mathbf{E} = i\omega \mathbf{j}\tag{2.17}$$

The  $i$  on the right-hand side is giving me (and most likely FEniCS too) a major headache, hence why I will try my luck with finding a formulation involving  $\mathbf{B}$  in the next section.

## 2.3 MAGNETIC FIELD FORMULATION

Multiply both sides of (2.4) with  $\epsilon^{-1}$  and take the curl to obtain, with the help of (2.3), the expression

$$\begin{aligned}\nabla \times (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) &= \nabla \times (\partial_t \mathbf{E} + \epsilon^{-1} \mathbf{j}) \\ &= \partial_t (\nabla \times \mathbf{E}) + \nabla \times (\epsilon^{-1} \mathbf{j}) \\ &= -\partial_t^2 \mathbf{B} + \nabla \times (\epsilon^{-1} \mathbf{j})\end{aligned}\tag{2.18}$$

Introducing a harmonic time dependence of  $\mathbf{B}$  and  $\mathbf{j}$  with frequencies  $\omega$ , such that  $\mathbf{B}(\mathbf{x}, t) = \mathbf{B}(\mathbf{x}) \exp(i\omega t)$  and  $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x}) \exp(i\omega t)$ , we obtain

$$\nabla \times (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) - \omega^2 \mathbf{B} = \nabla \times (\epsilon^{-1} \mathbf{j})\tag{2.19}$$

Converting this equation to its weak form by multiplying both sides with a vector valued function  $\mathbf{v} \in H_{\text{curl}}(\Omega)$  and integrating over all of  $\Omega$  yields

$$\int_{\Omega} (\nabla \times (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B}))) \cdot \mathbf{v} - \omega^2 \int_{\Omega} \mathbf{B} \cdot \mathbf{v} = \int_{\Omega} (\nabla \times (\epsilon^{-1} \mathbf{j})) \cdot \mathbf{v}\tag{2.20}$$

Using the above derived identity

$$(\nabla \times \mathbf{u}) \cdot \mathbf{v} = \nabla \cdot (\mathbf{u} \times \mathbf{v}) + \mathbf{u} \cdot (\nabla \times \mathbf{v})\tag{2.21}$$

we may write using Gauss' theorem

$$\begin{aligned}
\int_{\Omega} (\nabla \times (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B}))) \cdot \mathbf{v} &= \int_{\Omega} \nabla \cdot ((\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) \times \mathbf{v}) \\
&+ \int_{\Omega} (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) \times \mathbf{v}) \cdot \mathbf{n} \\
&+ \int_{\Omega} (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
\int_{\Omega} (\nabla \times (\epsilon^{-1} \mathbf{j})) \cdot \mathbf{v} &= \int_{\Omega} \nabla \cdot ((\epsilon^{-1} \mathbf{j}) \times \mathbf{v}) + \int_{\Omega} (\epsilon^{-1} \mathbf{j}) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\epsilon^{-1} \mathbf{j}) \times \mathbf{v}) \cdot \mathbf{n} + \int_{\Omega} (\epsilon^{-1} \mathbf{j}) \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{2.23}$$

Since  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{n})$  (as was shown above), the boundary integrals vanish if we have

$$\mathbf{v} \times \mathbf{n} = 0, \text{ on } \partial\Omega \tag{2.24}$$

In that case, we end up with the following (simplified) weak form

$$\int_{\Omega} (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_{\Omega} \mathbf{B} \cdot \mathbf{v} = \int_{\Omega} (\epsilon^{-1} \mathbf{j}) \cdot (\nabla \times \mathbf{v}) \tag{2.25}$$

### 3 WAVEGUIDES

In Section 2 we have derived the weak form of the time-harmonic Maxwell equation for the vector potential  $\mathbf{A}$ . Let us now apply it to a small collection of illustrative examples.

#### 3.1 TWO-DIMENSIONAL OPEN-ENDED WAVEGUIDE

Consider a two-dimensional rectangular box of length  $l$  and width  $w$  enclosing a vacuum (depicted in Figure 3.1).

Defining  $\mathbf{g} = (\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{n} = \mu^{-1} \mathbf{B} \times \mathbf{n}$  on the boundary  $\partial\Omega$  and setting  $\mathbf{j} = 0$ , we can convert (2.15) into

$$\int_{\Omega} (\mu^{-1} \nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) - \epsilon \omega^2 \int_{\Omega} \mathbf{A} \cdot \mathbf{v} = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v} \tag{3.1}$$

If we only consider  $\mathbf{A} = (0, 0, A_z)^T$  and  $\mathbf{v} = (0, 0, v_z)^T$ , the two curls can be rewritten as the dot product of two gradients, since

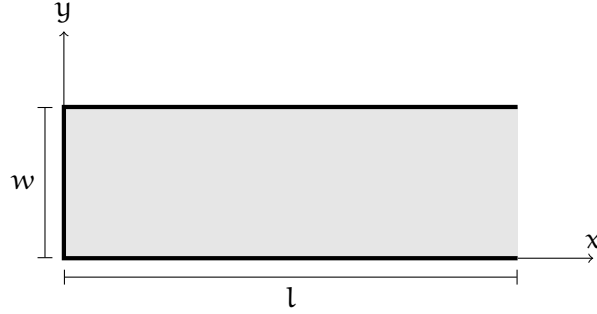


FIGURE 3.1 – Waveguide

$$\nabla \times \mathbf{A} = (\partial_y A_z, -\partial_x A_z, 0) \quad (3.2)$$

and similarly for  $\nabla \times \mathbf{v}$ , such that

$$(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) = (\nabla A_z) \cdot (\nabla v_z) \quad (3.3)$$

We thus end up with the following simplified weak form of the problem

$$\int_{\Omega} (\mu^{-1} \nabla A_z) \cdot (\nabla v_z) - \epsilon \omega^2 \int_{\Omega} A_z v_z = \int_{\partial\Omega} g_z v_z \quad (3.4)$$

### 3.1.1 CONSTANT FIELD

Suppose we observe an electromagnetic wave at frequency  $\omega$  and with the magnetic field of strength  $B_0$  aligned with the  $y$ -direction incident on the waveguide at  $x = 0$ . Symbolically, we thus have  $\mathbf{B}(t) = (0, B_0, 0)^T \exp(i\omega t)$  at  $x = 0$  for  $y \in [0, w]$ .

From this, we can calculate the function  $g_z$  on the “inlet” boundary to be (knowing that we have already taken account of the harmonic time-dependence of  $\mathbf{B}$  in deriving (3.4))

$$\mathbf{g} = \mu^{-1} \mathbf{B} \times \mathbf{n} = \mu^{-1} B_0 (\mathbf{e}_y \times \mathbf{e}_x) = -\mu^{-1} B_0 \mathbf{e}_z \quad (3.5)$$

Therefore, in general, we have (assuming all other walls to be perfect conductors)

$$g_z = \begin{cases} -\mu^{-1} B_0, & \text{at the inlet} \\ 0 & \text{at the walls} \end{cases} \quad (3.6)$$

## 4 WEAK DERIVATIVE

*Taken from Quarteroni: Introduction to Finite Elements Method*

Let  $\Omega \subset \mathbb{R}^d$  open. The support of  $f : \Omega \rightarrow \mathbb{R}$  is defined as

$$\text{supp}(f) = \overline{\{x \in \Omega \mid f(x) \neq 0\}} \quad (4.1)$$

$f$  has compact support, if there exists a compact subset  $K \subset \Omega$ , such that  $\text{supp}(f) \subset K$ , and define

$$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) \mid f \text{ has compact support}\} \quad (4.2)$$

(If I remember correctly, extending this notion to  $f \in C^1(\Omega)$  should yield an almost identical treatment, unless we also include higher order (weak) partial derivatives). Let  $T : \mathcal{D} \rightarrow \mathbb{R}$ ,  $\varphi \mapsto \langle T, \varphi \rangle = T(\varphi)$  be a linear map. We say that  $T$  is continuous, if

$$\lim_{n \rightarrow \infty} \langle T, \varphi_n \rangle = \langle T, \varphi \rangle \quad (4.3)$$

with  $\{\varphi_k\}_{k \in \mathbb{N}} \subset \mathcal{D}(\Omega)$  converging to  $\varphi$ . Such (linear and continuous) maps are called distribution on  $\mathcal{D}(\Omega)$ , and they form the space of distributions  $\mathcal{D}'(\Omega)$ .

The (weak) partial coordinate-derivatives of  $T$  (namely  $\partial_i T$ ,  $i \in \{1, \dots, d\}$ ) are characterized by distributions that satisfy

$$\langle \partial_i T, \varphi \rangle = -\langle T, \partial_i \varphi \rangle \quad (4.4)$$

for all  $\varphi \in \mathcal{D}(\Omega)$ .

Interesting for us is mainly the following case: Given a function  $f \in L^2(\Omega)$ , we define a distribution  $T_f \in \mathcal{D}'(\Omega)$  to be

$$\langle T_f, \varphi \rangle = \int_{\Omega} f(x) \varphi(x) dx \quad (4.5)$$

for all  $\varphi \in \mathcal{D}(\Omega)$ .

This allows us to define a weak derivative to functions that are (in the classical sense) not differentiable (i.e. not in  $C^1(\Omega)$ ). Consider for example the absolute value function  $|\cdot| \in L^2(K)$  where  $K \subset \mathbb{R}$  is compact. Since

$$\begin{aligned} \int_K (\partial_x |x|) \varphi(x) dx &= - \int_K |x| \varphi'(x) dx \\ &= - \int_{K \cap \mathbb{R}_+} x \varphi'(x) dx - \int_{K \cap \mathbb{R}_-} (-x) \varphi'(x) dx \\ &= \int_{K \cap \mathbb{R}_+} \varphi(x) dx + \int_{K \cap \mathbb{R}_-} (-1) \varphi(x) dx \\ &= \int_K \text{sign}(x) \varphi(x) dx \end{aligned} \quad (4.6)$$

we may conclude that the weak derivative of the absolute value function is therefore the signum function. Notice, how the derivative of the absolute value function is only not well-defined at  $x = 0$ , i.e. on a set of zero measure. This nuisance is circumvented when talking about the weak derivative, since the measure zero sets have zero integral.

## 5 RANDOM

Consider, for instance, solving the following equation:

$$\nabla \times (\nabla \times \mathbf{A}) - \mathbf{A} = \mathbf{j} \quad (5.1)$$

Even if we only care about the quantity  $\mathbf{B} = \nabla \times \mathbf{A}$ , and therefore would not even “realize” if  $\mathbf{A}$  was modified to  $\mathbf{A}' = \mathbf{A} + \mathbf{c}$  for a constant vector  $\mathbf{c}$ , this modification significantly changes the solution to the equation, because

$$\nabla \times (\nabla \times \mathbf{A}') - \mathbf{A}' = \mathbf{j} \quad (5.2)$$

turns, when plugging in the above-mentioned expression for  $\mathbf{A}'$ , into

$$\nabla \times (\nabla \times \mathbf{A}) - \mathbf{A} = \mathbf{j} + \mathbf{c} \quad (5.3)$$

yielding a possibly entirely different solution  $\mathbf{A}$  than initially wanted.

Therefore, we cannot adjust  $\mathbf{A}$  in such a way that it vanishes on the boundary of a radial symmetric domain, because we would implicitly change the physical outcome of the solution.

## 6 IDEAS

What might be really interesting is to instead look at the problem in space-time using the Maxwell tensor

$$\mathbb{F} = \begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{bmatrix} \quad (6.1)$$

In the covariant formulation of the Maxwell theory, the inhomogeneous Maxwell equations reduce to a single equation

$$\partial_a F^{ab} = -J^b \quad (6.2)$$

with the four current density  $\mathbf{J} = (\mu c \rho, \mu \mathbf{j})$ . The weak formulation of the problem could then be stated as (using Einstein’s sum convention, i.e. summing over repeated indices)

$$\int_{\Omega \times \mathbb{R}} F^{ab} \partial_a v_b = \int_{\Omega \times \mathbb{R}} J^b v_b \quad (6.3)$$

where boundary conditions are yet to be determined. If we somehow would manage to find a suitable function space for the four-dimensional  $\mathbf{v}$ , it might be possible to find both  $\mathbf{E}$  and  $\mathbf{B}$  from a finite element method.