

# Minimal rational interpolation for time-harmonic Maxwell's equations

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June 24, 2022  
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and build the rational surrogate

$$\tilde{\mathbf{u}}(\omega) = \frac{\mathbf{P}(\omega)}{Q(\omega)}$$

such that  $\tilde{\mathbf{u}}(\omega) \approx \mathbf{u}(\omega)$  close to  $\omega_1, \omega_2, \dots, \omega_S$ .

- ▶ Problem formulation
- ▶ Finite element method
- ▶ Minimal rational interpolation
- ▶ Example applications
- ▶ Conclusion and outlook

Time-harmonic vector potential  $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \exp(i\omega t)$ .

$$\mathbf{B} = \nabla \times \mathbf{u} \quad (\text{Magnetic field})$$

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Time-harmonic potential equation

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \epsilon \omega^2 \mathbf{u} = \mathbf{j}$$



$$H_{\text{curl}}(\Omega) = \{\mathbf{v} : \Omega \rightarrow \mathbb{C}^3, \text{ such that } \mathbf{v} \in L_2(\Omega)^3, \nabla \times \mathbf{v} \in L_2(\Omega)^3\}$$

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### Weak formulation of the time-harmonic potential equation

Find  $\mathbf{u} \in H_{\text{curl}}(\Omega)$ , such that

$$\int_{\Omega} \langle \mu^{-1} \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle - \omega^2 \int_{\Omega} \epsilon \langle \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \langle \mathbf{j}, \mathbf{v} \rangle + \int_{\partial\Omega} \langle \mathbf{g}, \mathbf{v} \rangle$$

for all  $\mathbf{v} \in H_{\text{curl}}$ , where  $\mathbf{g} = (\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}$ .

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Perfectly conducting boundary

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Imperfectly conducting boundary [4]

$$\mathbf{g} = i\omega\lambda(\mathbf{n} \times \mathbf{u}) \times \mathbf{n}, \text{ on } \Gamma_I$$

$$\int_{\Omega} \langle \mu^{-1} \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle - \omega^2 \int_{\Omega} \epsilon \langle \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \langle \mathbf{j}, \mathbf{v} \rangle + \int_{\partial\Omega} \langle \mathbf{g}, \mathbf{v} \rangle$$

FEniCS [3] is used to obtain FEM solutions of the form

$$\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$$

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where  $N$  is the number of degrees of freedom. Inner product in this representation is

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle_M = \bar{\mathbf{u}}^H \underline{\mathbf{M}} \bar{\mathbf{v}} \approx \int_{\Omega} \langle \mathbf{u}, \mathbf{v} \rangle$$

and the norm

$$\|\bar{\mathbf{u}}\|_M = \sqrt{\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle_M} \approx \|\mathbf{u}\|_{L_2(\Omega)}$$

Rational surrogate

$$\tilde{\mathbf{u}}(\omega) = \frac{\mathbf{P}(\omega)}{Q(\omega)} = \sum_{j=1}^S \frac{\mathbf{p}_j}{\omega - \omega_j} / \sum_{j=1}^S \frac{q_j}{\omega - \omega_j}$$

in barycentric coordinates with support points  $\omega_1, \omega_2, \dots, \omega_S$ .



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Interpolation property

$$\tilde{\mathbf{u}}(\omega_i) = \mathbf{u}(\omega_i), \forall i \in \{1, 2, \dots, S\}$$

if and only if  $\mathbf{p}_i = q_i \mathbf{u}(\omega_i), \forall i$ .

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3. Define  $\mathbf{q} = (q_1, q_2, \dots, q_S)^T = \underline{\mathbf{V}}[:, S]$
4. Define the minimal rational surrogate  $\tilde{\mathbf{u}}(\omega) = \mathbf{P}(\omega)/Q(\omega)$  with

$$\mathbf{P}(\omega) = \sum_{j=1}^S \frac{q_j \mathbf{u}(\omega_j)}{\omega - \omega_j} \quad \text{and} \quad Q(\omega) = \sum_{j=1}^S \frac{q_j}{\omega - \omega_j}$$

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2. Starting with  $t = 2$ , iteratively take a new support point

$$\omega^{(t)} = \operatorname{argmin}_{\omega \in \Omega_{\text{test}}} |Q^{(t)}(\omega)|$$

from  $\Omega_{\text{test}}$  to build the minimal rational surrogate  $\tilde{\mathbf{u}}_{t+1}$  based on  $\mathbf{u}(\omega^{(0)}), \mathbf{u}(\omega^{(1)}), \dots, \mathbf{u}(\omega^{(t+1)})$  and increment  $t$

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3. Stop when relative error

$$\|\mathbf{u}(\omega_{t+1}) - \tilde{\mathbf{u}}_t(\omega_{t+1})\|_M / \|\mathbf{u}(\omega_{t+1})\|_M$$

is small enough

With the QR-decomposition of the snapshot matrix  $\underline{\mathbf{U}} = [\mathbf{u}(\omega_1), \dots, \mathbf{u}(\omega_S)]$ .

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- ▶  $\underline{\mathbf{G}}$  and  $\underline{\mathbf{R}}$  have the same right-singular vector (exactly what is needed for MRI)
- ▶  $\underline{\mathbf{R}}$  can be built sequentially (modified Householder triangularization for gMRI)

Efficient way of storing the surrogate ( $\mathbf{e}_j$  canonical basis vector)

$$\mathbf{\hat{u}}(\omega) = \sum_{j=1}^S \frac{q_j \mathbf{e}_j}{\omega - \omega_j} / \sum_{j=1}^S \frac{q_j}{\omega - \omega_j}$$

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The original surrogate can be recovered with

$$\tilde{\mathbf{u}}(\omega) = \underline{\mathbf{U}} \mathring{\mathbf{u}}(\omega)$$

Neat helper quantity ( $\mathbf{r}_j = \mathbf{R}[:, S]$  from QR-decomposition)

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Proposed way of approximating relative error in gMRI

$$\frac{\|\mathbf{u}(\omega_{t+1}) - \tilde{\mathbf{u}}_t(\omega_{t+1})\|_{\mathbf{M}}}{\|\mathbf{u}(\omega_{t+1})\|_{\mathbf{M}}} \approx \frac{\|\mathbf{r}_{t+1} - \hat{\mathbf{u}}_t(\omega_{t+1})\|}{\|\hat{\mathbf{u}}_t(\omega_{t+1})\|}$$

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$$\tilde{\mathbf{u}}(\omega) = \underline{\mathbf{Q}} \hat{\mathbf{u}}(\omega)$$

We want to find  $\omega$ , such that

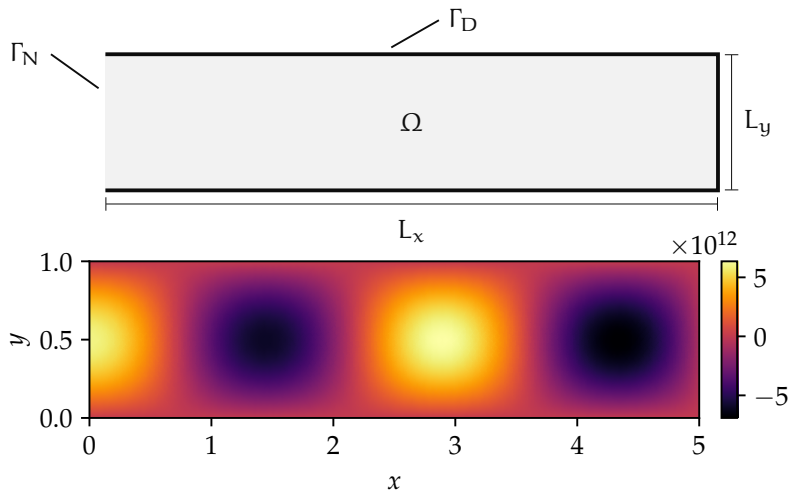
$$0 = Q(\omega) = \sum_{i=1}^S \frac{q_i}{\omega - \omega_i}$$

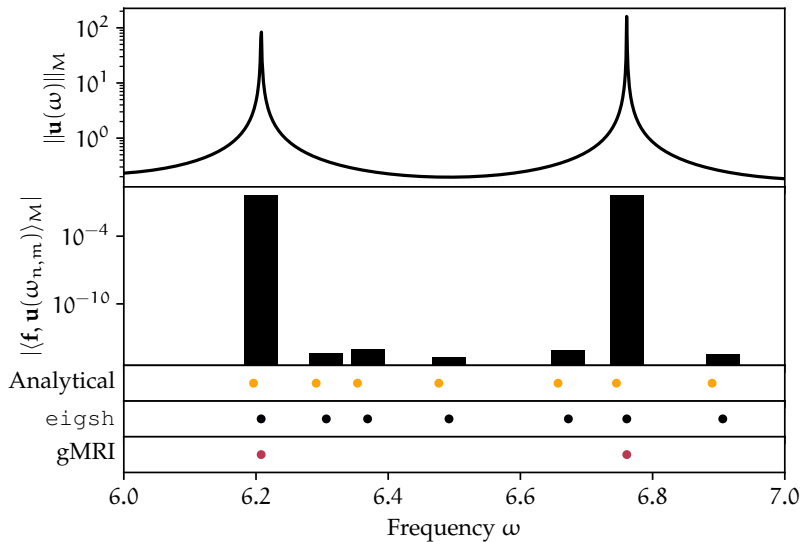
Equivalent eigenvalue problem [2]

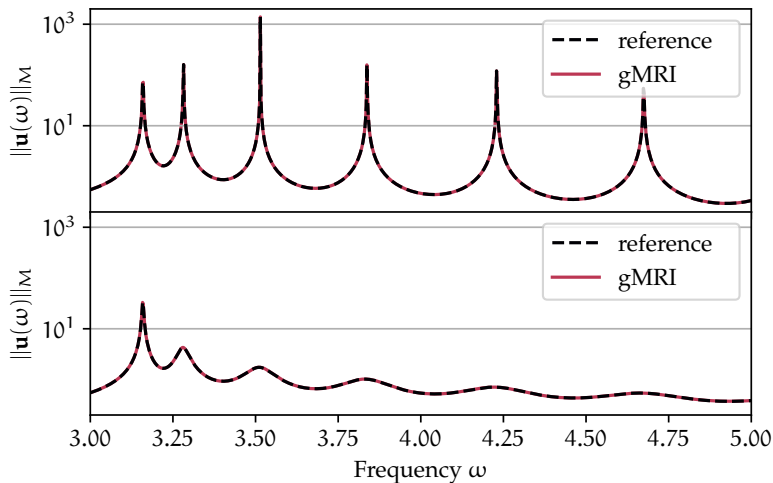
$$\underline{\mathbf{A}}\mathbf{w} = \omega \underline{\mathbf{B}}\mathbf{w}$$

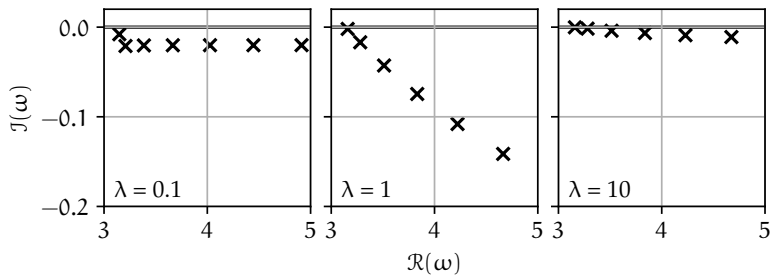
with

$$\underline{\mathbf{A}} = \begin{pmatrix} 0 & q_1 & q_2 & \dots & q_S \\ 1 & \omega_1 & & & \\ 1 & & \omega_2 & & \\ \vdots & & & \ddots & \\ 1 & & & & \omega_S \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{B}} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

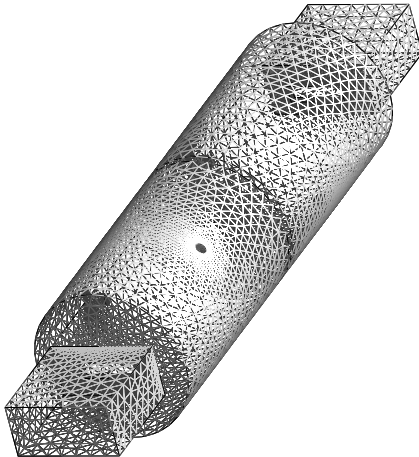




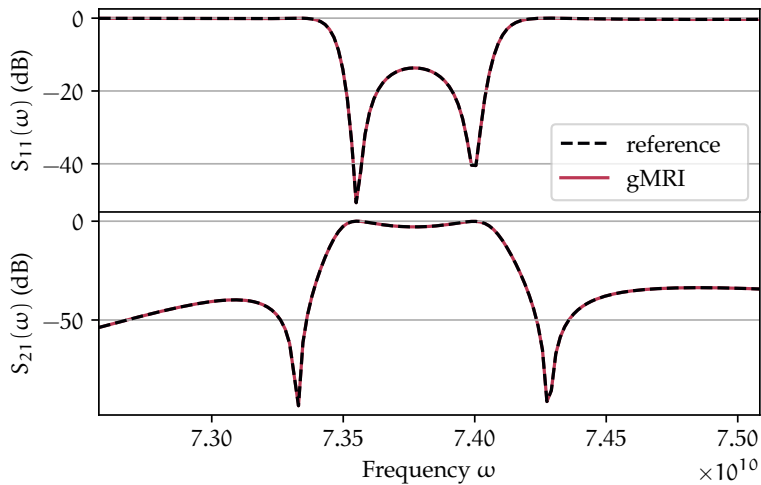




## Dual-mode circular waveguide filter







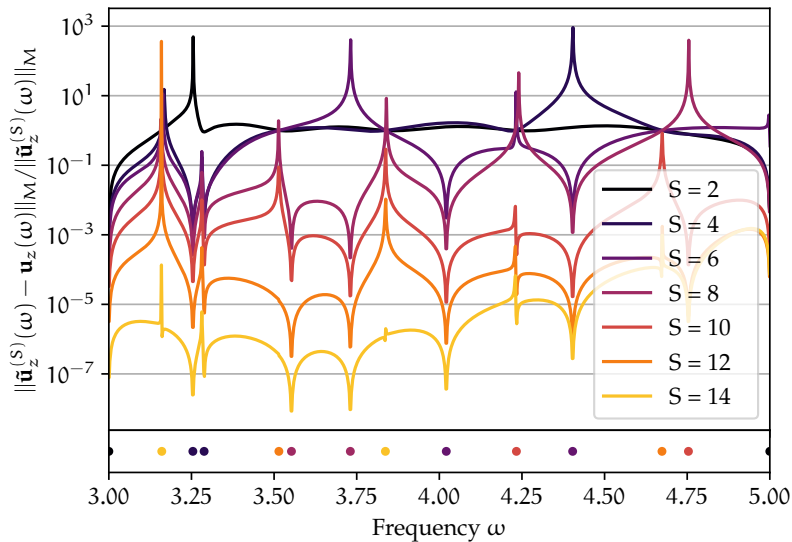
- ▶ Speed and efficiency

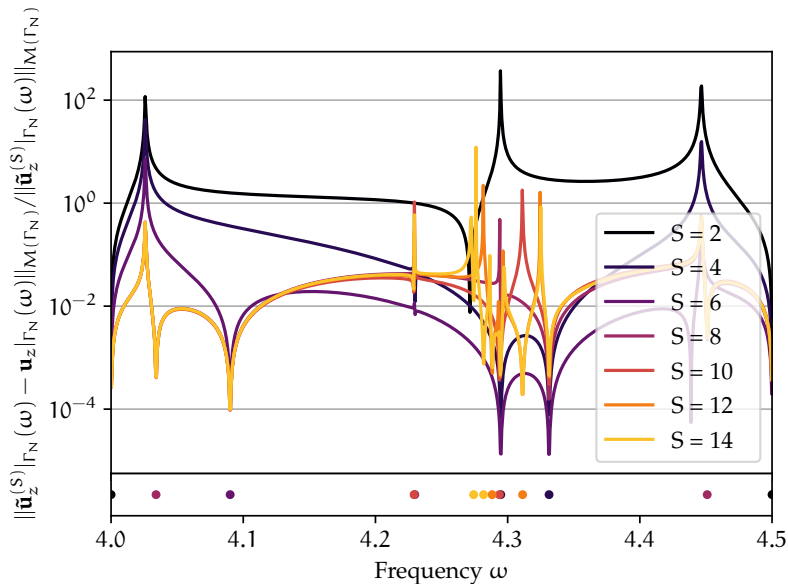
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- ▶ DMCWF exact dimensions and reference needed

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# Rectangular cavity | Breaking symmetry with cubby24

