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SEMESTER PROJECT

Minimal Rational Interpolation for Time-Harmonic Maxwell's Equations

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1 Abstract

2 Introduction

3 Finite element discretization of the time-harmonic Maxwell's equations

3.1 Vector potential formulation of the time-harmonic Maxwell's equations

Let **E** denote an electric field, **B** a magnetic field strength, ρ an electric charge density, and **j** an electric current density. Maxwell's equations are stated in **?** as

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \tag{3.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.2}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{3.3}$$

$$\nabla \times (\mu^{-1}\mathbf{B}) = \partial_t(\epsilon \mathbf{E}) + \mathbf{j} \tag{3.4}$$

with ε being the permittivity and μ the permeability.

Equation (3.2) allows for an expression of the magnetic field $\mathbf{B} = \nabla \times \mathbf{u}$ in terms of a vector valued function **u**, the vector potential (in literature commonly denoted with **A**). Similarly, (3.3) suggests rewriting the electric field $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{u}$ using a scalar function ϕ, referred to as the scalar potential.

The physical quantities **E** and **B** remain unchanged if we transform $\mathbf{u} \to \mathbf{u}' = \mathbf{u} + \nabla\psi$ or $\phi \to \phi' = \phi - \partial_t \psi$ for arbitrary functions ψ. A convenient choice of ψ is suggested in **?** to be

$$\psi = \int_0^t \phi \, dt' \tag{3.5}$$

which transforms $\phi \to \phi' = 0$ and $\mathbf{u} \to \mathbf{u}' = \mathbf{u} + \nabla \int_0^t \phi \, dt'$. Thus, the expressions for the electrical and magnetic field become

$$\mathbf{E} = -\partial_t \mathbf{u} \tag{3.6}$$

$$\mathbf{B} = \nabla \times \mathbf{u} \tag{3.7}$$

where I renamed the variable $\mathbf{u}'$ to **u** for simplicity.

Plugging the identities (3.6) and (3.7) into (3.4) yields

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) = \varepsilon \partial_t^2 \mathbf{u} + \mathbf{j} \tag{3.8}$$

For the rest of this report, I restrict myself to vector potentials **u** that exhibit a harmonic dependence on time t, i.e. may be factorized into a term solely depending on the position **x** and a complex exponential: $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \exp(i\omega t)$. Substituting this expression into (3.8) results in the

$$\boxed{\text{Time-harmonic potential equation: } \nabla \times (\mu^{-1}\nabla \times \mathbf{u}) - \varepsilon\omega^2 \mathbf{u} = \mathbf{j}} \tag{3.9}$$

3.2 Weak formulation for the time-harmonic potential equation

Equation (3.9) may be multiplied by a vector-valued function $\mathbf{v} \in H_{\text{curl}}(\Omega)$, where

$$H_{\text{curl}}(\Omega) = \{\mathbf{u} : \Omega \to \mathbb{C}, \text{ such that } \mathbf{u} \in L^2(\mathbb{C})^3, \nabla \times \mathbf{u} \in L^2(\mathbb{C})^3\} \tag{3.10}$$

and then integrated over all of Ω to obtain

$$\int_\Omega (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} - \omega^2 \int_\Omega \epsilon \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} \tag{3.11}$$

This may further be simplified (3.11) to (see Section 1 for details)

$$\boxed{\int_\Omega (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) - \epsilon\omega^2 \int_\Omega \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} + \int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}) \cdot \mathbf{v}} \tag{3.12}$$

where $\mathbf{n}$ denotes the surface normal to the boundary $\partial\Omega$.

3.3 Two-dimensional resonant cavity

A resonant cavity is a region Ω enclosed by a boundary $\partial\Omega$. The boundary is subdivided into one (or more) inlets Γ_N and a perfect electrically conducting wall $\Gamma_D = \partial\Omega \setminus \Gamma_N$.

FIGURE 3.1 – Example of a two-dimensional resonant cavity: The rectangular cavity.

Suppose the current density $\mathbf{j} \equiv 0$ and orient the coordinate system in such a way that $\mathbf{u} = u_z \mathbf{e}_z$ and $\mathbf{v} = v_z \mathbf{e}_z$. Consequently,

$$(\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) = (\mu^{-1} \nabla u_z) \cdot (\nabla v_z) \tag{3.13}$$

Define $g_z = ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n})_z$ along the boundary Γ_N, to convert (3.12) into the weak formulation for a two-dimensional resonant cavity

$$\int_\Omega (\mu^{-1} \nabla u_z) \cdot (\nabla v_z) - \omega^2 \int_\Omega \epsilon u_z v_z = \int_{\partial\Omega} g_z v_z \tag{3.14}$$

3.4 Waveguide

3.5 Imperfect conductor

4 Minimal rational interpolation for the time-harmonic Maxwell's equations

5 Examples

5.1 Two-dimensional rectangular cavity

5.2 Dual mode circular waveguide filter

FIGURE 5.1 – Dual-mode circular waveguide filter.

5.3 Imperfectly conducting boundaries

6 Conclusion and outlook

7 Appendix

7.1 Detailed derivation for the weak formulation of the time-harmonic potential equation

The goal is to rewrite the curl-integral on the left-hand side of (3.11):

$$\int_\Omega (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} \tag{7.1}$$

In order to simplify the curls and apply the Gauss theorem, I first show prove the vector calculus identity

$$(\nabla \times \mathbf{a}) \cdot \mathbf{b} = \nabla \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\nabla \times \mathbf{b}) \tag{7.2}$$

where $\mathbf{a}$, $\mathbf{b}$ are vector-value functions. The completely antisymmetric tensor ε_{ijk}, frequently referred to as the Levi-Civita tensor, may be employed to rewrite the components of the curl of a vector-function $\mathbf{a}$ as the sum

$$(\nabla \times \mathbf{a})_k = \sum_i \sum_j \varepsilon_{ijk} \partial_i u_j \tag{7.3}$$

where ∂_i denotes the partial derivative with respect to the i-th coordinate direction. This yields

$$\begin{aligned}
(\nabla \times \mathbf{a}) \cdot \mathbf{b} &= \sum_k (\nabla \times \mathbf{a})_k b_k \\
&= \sum_k (\sum_i \sum_j \varepsilon_{ijk} \partial_i a_j) b_k \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{ijk} a_j b_k) - \sum_k \sum_i \sum_j a_j (\varepsilon_{ijk} \partial_i b_k) \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{jki} a_j b_k) - \sum_k \sum_i \sum_j a_j ((-\varepsilon_{ikj}) \partial_i b_k) \\
&= \sum_i \partial_i (\mathbf{a} \times \mathbf{b})_i + \sum_j u_j (\nabla \times \mathbf{b})_j \\
&= \nabla \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\nabla \times \mathbf{b})
\end{aligned} \tag{7.4}$$

by expressing the scalar product as a component-sum, using the product rule and applying the symmetry and anti-symmetry properties of the Levi-Civita tensor. Now the identity (7.2) to (7.1) together with Gauss' theorem gives

$$\begin{aligned}
\int_\Omega (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} &= \int_\Omega \nabla \cdot ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) + \int_\Omega (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} + \int_\Omega (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{7.5}$$

For later convenience, the boundary integral can further be simplified using

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} \tag{7.6}$$

This identity follows immediately from a small manipulation with the Levi-Civita tensor:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \sum_k (\sum_i \sum_j \varepsilon_{ijk} a_i b_j) c_k \\ &= \sum_j (\sum_i \sum_k (-\varepsilon_{ikj}) a_i c_k) b_j \\ &= -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} \end{aligned} \tag{7.7}$$

The boundary integral becomes

$$\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} = -\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}) \cdot \mathbf{v} \tag{7.8}$$

This concludes the short derivation, because now (7.1) may be rewritten as

$$-\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \tag{7.9}$$