

EPFL

SEMESTER PROJECT

Minimal Rational Interpolation for Time-Harmonic Maxwell's Equations

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May 20, 2022

ABSTRACT

CONTENTS

1	Introduction	3
2	Finite element discretization of the time-harmonic Maxwell's equations	4
2.1	Vector potential formulation of the time-harmonic Maxwell's equations	4
2.2	Weak formulation for the time-harmonic potential equation	5
2.3	Two-dimensional resonant cavity	5
2.4	Waveguide	6
2.5	Imperfect conductor	6
3	Finite element approximation with FEniCS	7
4	Minimal rational interpolation for the time-harmonic Maxwell's equations	9
4.1	Motivation	9
5	Examples	13
5.1	Two-dimensional rectangular cavity	13
5.2	Dual mode circular waveguide filter	14
5.3	Imperfectly conducting boundaries	14
6	Conclusion and outlook	15
7	Appendix	16
7.1	Detailed derivation for the weak formulation of the time-harmonic potential equation	16

1 INTRODUCTION

2 FINITE ELEMENT DISCRETIZATION OF THE TIME-HARMONIC MAXWELL'S EQUATIONS

2.1 VECTOR POTENTIAL FORMULATION OF THE TIME-HARMONIC MAXWELL'S EQUATIONS

Let \mathbf{E} denote an electric field, \mathbf{B} a magnetic field strength, ρ an electric charge density, and \mathbf{j} an electric current density. Maxwell's equations are stated in [?] as

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \quad (2.1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.2)$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \quad (2.3)$$

$$\nabla \times (\mu^{-1} \mathbf{B}) = \partial_t (\epsilon \mathbf{E}) + \mathbf{j} \quad (2.4)$$

with ϵ being the permittivity and μ the permeability.

Equation (2.2) allows for an expression of the magnetic field $\mathbf{B} = \nabla \times \mathbf{u}$ in terms of a vector valued function \mathbf{u} , the vector potential (in literature commonly denoted with \mathbf{A}). Similarly, (2.3) suggests rewriting the electric field $\mathbf{E} = -\nabla \phi - \partial_t \mathbf{u}$ using a scalar function ϕ , referred to as the scalar potential.

The physical quantities \mathbf{E} and \mathbf{B} remain unchanged if we transform $\mathbf{u} \rightarrow \mathbf{u}' = \mathbf{u} + \nabla \psi$ or $\phi \rightarrow \phi' = \phi - \partial_t \psi$ for arbitrary functions ψ . A convenient choice of ψ is suggested in [?] to be

$$\psi = \int_0^t \phi dt' \quad (2.5)$$

which transforms $\phi \rightarrow \phi' = 0$ and $\mathbf{u} \rightarrow \mathbf{u}' = \mathbf{u} + \nabla \int_0^t \phi dt'$. Thus, the expressions for the electrical and magnetic field become

$$\mathbf{E} = -\partial_t \mathbf{u} \quad (2.6)$$

$$\mathbf{B} = \nabla \times \mathbf{u} \quad (2.7)$$

where I renamed the variable \mathbf{u}' to \mathbf{u} for simplicity.

Plugging the identities (2.6) and (2.7) into (2.4) yields

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) = \epsilon \partial_t^2 \mathbf{u} + \mathbf{j} \quad (2.8)$$

For the rest of this report, I restrict myself to vector potentials \mathbf{u} that exhibit a harmonic dependence on time t , i.e. may be factorized into a term solely depending on the position \mathbf{x} and a complex exponential

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \exp(i\omega t) \quad (2.9)$$

Substituting this expression into (2.8) results in the

Time-harmonic potential equation

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \epsilon \omega^2 \mathbf{u} = \mathbf{j} \quad (2.10)$$

2.2 WEAK FORMULATION FOR THE TIME-HARMONIC POTENTIAL EQUATION

Equation (2.10) may be multiplied by a vector-valued function $\mathbf{v} \in \mathcal{H}_{\text{curl}}(\Omega)$, where

$$\mathcal{H}_{\text{curl}}(\Omega) = \{\mathbf{u} : \Omega \rightarrow \mathbb{C}, \text{ such that } \mathbf{u} \in L^2(\mathbb{C})^3, \nabla \times \mathbf{u} \in L^2(\mathbb{C})^3\} \quad (2.11)$$

and then integrated over all of Ω to obtain

$$\int_{\Omega} (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} - \omega^2 \int_{\Omega} \epsilon \mathbf{u} \cdot \mathbf{v} = \int_{\Omega} \mathbf{j} \cdot \mathbf{v} \quad (2.12)$$

This may further be simplified (2.12) to (see Section for details)

Weak formulation of the time-harmonic potential equation

$$\int_{\Omega} (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_{\Omega} \epsilon \mathbf{u} \cdot \mathbf{v} = \int_{\Omega} \mathbf{j} \cdot \mathbf{v} + \int_{\partial\Omega} \underbrace{((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}) \cdot \mathbf{v}}_{=\mathbf{g}} \quad (2.13)$$

where \mathbf{n} denotes the surface normal to the boundary $\partial\Omega$.

Boundary conditions on the electric field \mathbf{E} may be enforced in a Dirichlet-type fashion through the relation (2.6) and the assumption (2.9)

$$\mathbf{u}|_{\partial\Omega} = -\frac{1}{i\omega} \mathbf{E}|_{\partial\Omega} \quad (2.14)$$

Those on the magnetic field \mathbf{B} through a Neumann-type condition following from (2.7) and again (2.9)

$$\mathbf{g}|_{\partial\Omega} = (\mu^{-1} \mathbf{B}|_{\partial\Omega}) \times \mathbf{n} \quad (2.15)$$

2.3 EXAMPLES

We apply this weak formulation to three different but very intimately related problems:

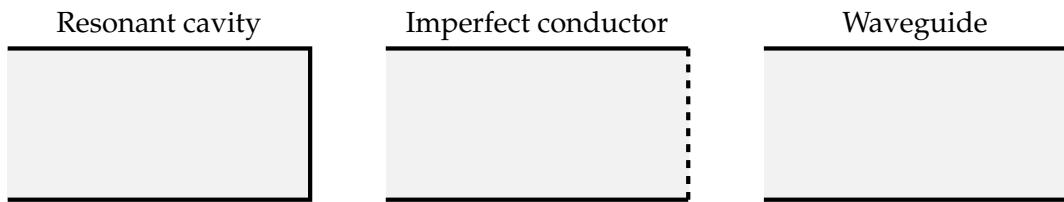


FIGURE 2.1 – Schematically the most trivial case for each example.

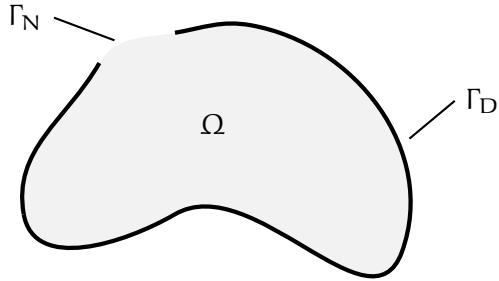


FIGURE 2.2 – 2d resonant cavity.

2.3.1 TWO-DIMENSIONAL RESONANT CAVITY

A resonant cavity is a region Ω enclosed by a boundary $\partial\Omega$. The boundary is subdivided into one (or more) inlets Γ_N and a perfect electrically conducting wall $\Gamma_D = \partial\Omega \setminus \Gamma_N$.

Suppose the current density $\mathbf{j} \equiv 0$ and orient the coordinate system in such a way that $\mathbf{u} = u_z \mathbf{e}_z$ and $\mathbf{v} = v_z \mathbf{e}_z$. Consequently,

$$(\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) = (\mu^{-1} \nabla u_z) \cdot (\nabla v_z) \quad (2.16)$$

Use $g_z = (\mathbf{g})_z$ along the boundary Γ_N , to convert (2.13) into the weak formulation for a two-dimensional resonant cavity

$$\int_{\Omega} (\mu^{-1} \nabla u_z) \cdot (\nabla v_z) - \omega^2 \int_{\Omega} \epsilon u_z v_z = \int_{\partial\Omega} g_z v_z \quad (2.17)$$

Let \mathbf{E} and \mathbf{B} refer to the electric and magnetic fields inside the cavity. For now, I distinguish two types of boundaries.

For the perfectly conducting boundary, treated in [?], it holds that

$$\mathbf{n} \times \mathbf{E} = 0, \text{ on } \Gamma_D \quad (2.18)$$

For the boundaries in a two-dimensional resonant cavity (see Figure 2.1), this only holds true if $E_z = 0$, which translates to the Dirichlet boundary condition $\mathbf{u}|_{\Gamma_D} = 0$ in light of (2.14).

For the inlet, it is easiest to enforce the boundary condition through the magnetic field \mathbf{B} (considering $\mathbf{n} = -\mathbf{e}_x$ as depicted in Figure 2.1):

$$g_z = ((\mu^{-1} \mathbf{B}) \times (-\mathbf{e}_x))_z = \mu^{-1} B_x \quad (2.19)$$

2.3.2 IMPERFECT CONDUCTOR

At an imperfect boundary ?, (2.6), with (2.9)

$$\mathbf{g} = (\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n} = i\omega\lambda(\mathbf{n} \times \mathbf{u}) \times \mathbf{n} \quad (2.20)$$

which, supposing that $\mathbf{u} = u_z \mathbf{e}_z$ and only treating a two-dimensional domain, simplifies (using the fact that $\mathbf{n} \perp \mathbf{u}$ and $\|\mathbf{n}\| = 1$, so $(\mathbf{n} \times \mathbf{u}) \times \mathbf{n} = \mathbf{u}$)

$$g_z = i\omega\lambda u_z \quad (2.21)$$

Therefore, reuse (2.17) as it is, but split boundary integral term for Neumann and impedance boundary.

2.3.3 WAVEGUIDE

A waveguide We go back to (2.13).

3 FINITE ELEMENT APPROXIMATION WITH FENICS

$$a_\omega(u, v) = L(v).$$

In the style of ?. Problem (2.13) with Ω being a cubic cavity with an inlet on one side and all other sides with $\mu = \epsilon = 0, \mathbf{j} = 0$ for simplicity.

Required packages

```
0 | import numpy as np
1 | import fenics as fen
2 | import matplotlib.pyplot as plt
```

Mesh

```
5 | nx, ny, nz = 10, 10, 10
6 | mesh = fen.UnitCubeMesh(nx, ny, nz)
```

Function space (Nédélec elements of the first kind ?)

```
9 | V = fen.FunctionSpace(mesh, 'N1curl', 1)
```

Inlet (at $x = 0$)

```
12 | class Inlet(fen.SubDomain):
13 |     def inside(self, x, on_boundary):
14 |         return on_boundary and fen.near(x[0], 0)
```

PEC boundary

```
17 | class PECWalls(fen.SubDomain):
18 |     def inside(self, x, on_boundary):
19 |         return on_boundary and not Inlet().inside(x, on_boundary)
```

Boundary ids

```
22 | boundary_id = fen.MeshFunction('size_t', mesh, mesh.topology().dim()-1)
23 | boundary_id.set_all(0)
24 | Inlet().mark(boundary_id, 1)
25 | PECWalls().mark(boundary_id, 2)
```

Dirichlet boundary (0 for PEC, because Nédélec)

```
28 | u_D = fen.Expression((0.0, 0.0, 0.0), degree=2)
29 | bc = fen.DirichletBC(V, u_D, boundary_id, 2)
```

Neumann boundary (1 for ...)

```
32 | g_N = fen.Expression((0.0, 0.0, 1.0), degree=2)
33 | ds = fen.Measure('ds', subdomain_data=boundary_id)
```

Trial and test functions

```
36 | u = fen.TrialFunction(V)
37 | v = fen.TestFunction(V)
```

Neumann boundary integral term

```
40 | N = fen.assemble(fen.dot(g_N, v) * ds(2))
```

Stiffness matrix

```
43 | K = fen.assemble(fen.dot(fencurl(u), fencurl(v)) * fen.dx)
44 | bc.apply(K)
```

Mass matrix

```
47 | M = fen.assemble(fen.dot(u, v) * fen.dx)
48 | bc.zero(M)
```

L2 norms

```
51 | def L2_norm(u):
52 |     u_vec = u.vector().get_local()
53 |     return pow(((M * u_vec) * u_vec).sum(), 0.5)
```

Solution at frequencies

```
56 | omegas = np.linspace(6.2, 6.8, 200)
57 | norms = []
58 | u = fen.Function(V)
59 | for omega in omegas:
60 |     fen.solve(K - omega**2 * M, u.vector(), N)
61 |     norms.append(L2_norm(u))
```

Plotting the L2-norms

```
64 | plt.plot(omegas, norms)
65 | plt.yscale('log')
```

4 MINIMAL RATIONAL INTERPOLATION FOR THE TIME-HARMONIC MAXWELL'S EQUATIONS

Let $\mathbf{u} : \mathbb{R} \rightarrow \mathbb{C}$. Given $\mathbf{u}_j = \mathbf{u}(\omega_j)$ for $j \in \{1, \dots, S\}$. Want to find a surrogate

$$\tilde{\mathbf{u}}(\omega) \approx \mathbf{u}(\omega) \quad (4.1)$$

4.1 MOTIVATION

Equations of the type (2.10), in the most simple case (dropping all constants)

$$\nabla \times (\nabla \times \mathbf{u}) - \omega^2 \mathbf{u} = \mathbf{j} \quad (4.2)$$

Writing the double-curl operator in terms of a matrix $\underline{\mathbf{A}}$

$$\mathbf{u} = (\underline{\mathbf{A}} - \omega^2 \mathbf{1})^{-1} \mathbf{j} \quad (4.3)$$

Eigenvalue decomposition $\underline{\mathbf{A}} = \underline{\mathbf{V}} \underline{\Lambda} \underline{\mathbf{V}}^H$ leads us to a form proposed in ?

$$\mathbf{u} = \underline{\mathbf{V}} (\underline{\Lambda} - \omega^2 \mathbf{1})^{-1} \underline{\mathbf{V}}^H \mathbf{j} = \sum_i \frac{\mathbf{v}_i \mathbf{v}_i^H \mathbf{j}}{\lambda_i - \omega^2} \quad (4.4)$$

which follows from the fact that $\underline{\Lambda}$ is diagonal, hence also $(\underline{\Lambda} - \omega^2 \mathbf{1})^{-1}$. Here, we denoted the diagonal elements of $\underline{\Lambda}$ with λ_i (the eigenvalues of $\underline{\mathbf{A}}$) and the columns of $\underline{\mathbf{V}}$ with \mathbf{v}_i (the eigenvectors of $\underline{\mathbf{A}}$).

Hence, it would make sense to search for an approximation of the solution \mathbf{u} that is able to "model" the singularities at $\omega^2 = \lambda_i$, e.g. rational polynomials

$$\tilde{\mathbf{u}}(\omega) = \frac{P(\omega)}{Q(\omega)} \quad (4.5)$$

[?]

Algorithm 1 Minimal rational interpolation

Require: $\omega_1, \dots, \omega_S$

Require: $\mathbf{U} = [\mathbf{u}(\omega_1) | \dots | \mathbf{u}(\omega_S)]$ ▷ Snapshot matrix

Compute \mathbf{G} with $g_{ij} = \langle \mathbf{u}(\omega_i), \mathbf{u}(\omega_j) \rangle_M$, $i, j \in \{1, \dots, S\}$

▷ Gramian matrix

Compute the singular value decomposition $\mathbf{G} = \mathbf{V} \Sigma \mathbf{V}^H$

Define $\mathbf{q} = \mathbf{V}[:, S]$

Define $\tilde{\mathbf{u}}(\omega) = P(\omega)/Q(\omega)$ with $P(\omega) = \sum_{j=1}^S \frac{q_j \mathbf{u}(\omega_j)}{\omega - \omega_j}$ and $Q(\omega) = \sum_{j=1}^S \frac{q_j}{\omega - \omega_j}$

[?]

The rational surrogate $\tilde{\mathbf{u}}$ can be rewritten as

$$\begin{aligned} \tilde{\mathbf{u}}(\omega) &= \sum_{j=1}^S \frac{q_j \mathbf{u}(\omega_j)}{\omega - \omega_j} / \sum_{j=1}^S \frac{q_j}{\omega - \omega_j} \\ &= \sum_{j=1}^S \prod_{\substack{i=0 \\ i \neq j}}^S (\omega - \omega_i) q_j \mathbf{u}(\omega_j) / \sum_{j=1}^S \prod_{\substack{i=0 \\ i \neq j}}^S (\omega - \omega_i) q_j \end{aligned} \quad (4.6)$$

Algorithm 2 Greedy minimal rational interpolation

Require: $\tau > 0$ ▷ Relative L_2 -error tolerance
Require: $\Omega_{\text{test}} = \{\omega_i\}_{i=1}^M$ ▷ Set of candidate support points
Require: $a_\omega(u, v) = L(v)$ ▷ Finite element formulation of the problem
 Choose $\omega_1, \dots, \omega_t \in \Omega_{\text{test}}$ ▷ Usually the smallest and largest element
 Remove $\omega_1, \dots, \omega_t$ from Ω_{test}
 Solve $a_{\omega_i}(u_i, v) = L(v)$ for $i \in \{1, \dots, t\}$
 Build surrogate $\tilde{u}_t = P_t(\omega)/Q_t(\omega)$ using the solutions u_1, \dots, u_t
while $\Omega_{\text{test}} \neq \emptyset$ **do**
 $\omega_{t+1} \leftarrow \operatorname{argmin}_{\omega \in \Omega_{\text{test}}} |Q_t(\omega)|$
 Solve $a_{\omega_{t+1}}(u_{t+1}, v) = L(v)$
 Build surrogate $\tilde{u}_{t+1} = P_{t+1}(\omega)/Q_{t+1}(\omega)$ using the solutions u_1, \dots, u_{t+1}
 if $\|u_{t+1}(\omega_{t+1}) - \tilde{u}_{t+1}(\omega_{t+1})\|_M / \|u_{t+1}(\omega_{t+1})\|_M < \tau$ **then return**
 end if
 $t \leftarrow t + 1$
end while

Therefore, if the rational surrogate \tilde{u} is evaluated at one of the interpolation nodes ω_i , $u(\omega_i)$ is recovered.

If the rational surrogate \tilde{u} is evaluated in a zero $\bar{\omega}$ of the denominator $Q(\bar{\omega}) = 0$, we observe a pole, unless $P(\bar{\omega})$ happens to vanish too.

[?]

Defining

$$v_i = (\omega - \omega_i)^{-1} \quad (4.7)$$

and requiring

$$0 = Q(\omega) = \sum_{i=1}^s q_i v_i(\omega) \quad (4.8)$$

can be equivalently expressed as a generalized eigenvalue problem

$$\underline{A}\underline{u} = \omega \underline{B}\underline{u} \quad (4.9)$$

with

$$\underline{A} = \begin{pmatrix} 0 & q_1 & q_2 & \dots & q_s \\ 1 & \omega_1 & & & \\ 1 & & \omega_2 & & \\ \vdots & & & \ddots & \\ 1 & & & & \omega_s \end{pmatrix} \text{ and } \underline{B} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad (4.10)$$

[?]

Can check stability with singular values $\sigma_1, \dots, \sigma_s$ in Σ which we obtain in Algorithm 1. Need smallest singular values to different from each other. This conditioning can be estimated with the relative spectral range ?

$$\frac{\sigma_{s-1} - \sigma_s}{\sigma_1 - \sigma_s} \quad (4.11)$$

Algorithm 3 Additive Householder triangularization

Require: $U[1 \dots s, 1 \dots N]$ ▷ Next snapshot matrix
Require: $R[1 \dots S, 1 \dots S]$ ▷ Previous triangular matrix
Require: $E[1 \dots S, 1 \dots N]$ ▷ Previous orthonormal matrix
Require: $V[1 \dots S, 1 \dots N]$ ▷ Previous Householder matrix

Extend size of R to $(S + s) \times (S + s)$
Extend E with S orthonormal columns to $(S + s) \times N$
Extend size of V to $(S + s) \times N$

for $j = S + 1 : S + s$ **do**

$u = U[j]$

for $k = 1 : j - 1$ **do**

$u \leftarrow u - 2\langle V[k, :], u \rangle_M V[k, :]$

$R[k, j] \leftarrow \langle E[k, :], u \rangle_M$

$u \leftarrow u - R[k, j]E[k, :]$

end for

$R[j, j] \leftarrow \|u\|_M$

$\alpha \leftarrow \langle E[j, :], u \rangle_M$

if $|\alpha| \neq 0$ **then**

$E[j, :] \leftarrow E[j, :](-\alpha/|\alpha|)$

end if

$V[j, :] \leftarrow R[j, j]E[j, :] - u$

$V[j, :] \leftarrow V[j, :] - \langle E[S + 1 : j], V[j, :] \rangle_M E[S + 1 : j, :]$

$\sigma \leftarrow \|V[j, :]\|_M$

if $\sigma \neq 0$ **then**

$V[j, :] \leftarrow V[j, :]/\sigma$

else

$V[j, :] \leftarrow E[j, :]$

end if

end for

Denote $\underline{\mathbf{U}} = [\mathbf{u}(\omega_1), \dots, \mathbf{u}(\omega_S)]$ snapshot matrix. Let

$$\mathbf{u} = \sum_{j=1}^S \frac{q_j \mathbf{e}_j}{\omega - \omega_j} / \sum_{j=1}^S \frac{q_j}{\omega - \omega_j} \quad (4.12)$$

with the canonical basis vectors $\{\mathbf{e}_j\}_j$. If we perform a QR decomposition on this matrix, we obtain

$$\underline{\mathbf{U}} = \underline{\mathbf{W}} \underline{\mathbf{R}} \quad (4.13)$$

5 EXAMPLES

5.1 TWO-DIMENSIONAL RECTANGULAR CAVITY

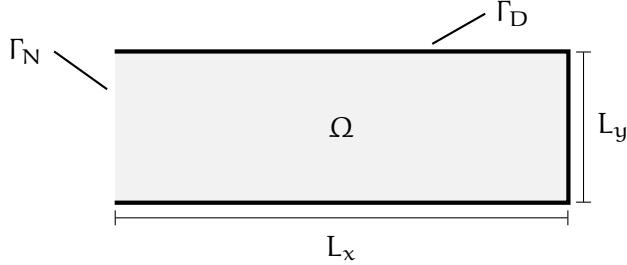


FIGURE 5.1 – TODO.

For $\|\mathbf{u}\|_{L_2(\Omega)}^2 = \int_{\Omega} \|\mathbf{u}\|^2$

Analytical eigenfrequencies

$$\omega_{n,m} = \pi \sqrt{\left(\frac{2n+1}{2L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2}, \quad n \in \{0, 1, \dots\}, \quad m \in \{1, 2, \dots\} \quad (5.1)$$

Numerical eigenfrequencies, solve generalized (symmetric) eigenvalue problem

$$\underline{\mathbf{K}}\mathbf{u} = \omega^2 \underline{\mathbf{M}}\mathbf{u} \quad (5.2)$$

Take $\{\mathbf{u}_j, \omega_j^2\}_j$ be the resonant modes, i.e. solutions to the eigenvalue problem (5.2), such that

$$\underline{\mathbf{K}}\mathbf{u}_j = \omega_j^2 \underline{\mathbf{M}}\mathbf{u}_j \quad (5.3)$$

Adding a source term \mathbf{f}

$$\underline{\mathbf{K}}\mathbf{u} - \omega^2 \underline{\mathbf{M}}\mathbf{u} = \mathbf{f} \quad (5.4)$$

If \mathbf{u} is expressed in terms of the basis $\{\mathbf{u}_j\}_j$, i.e. $\mathbf{u} = \sum_j \alpha_j \mathbf{u}_j$

$$\sum_j \alpha_j (\underline{\mathbf{K}}\mathbf{u}_j - \omega^2 \underline{\mathbf{M}}\mathbf{u}_j) = \mathbf{f} \quad (5.5)$$

Using (5.3)

$$\sum_j \alpha_j (\omega_j^2 - \omega^2) \underline{\mathbf{M}}\mathbf{u}_j = \mathbf{f} \quad (5.6)$$

from which we can take the scalar product with \mathbf{u}_j^H to obtain

$$\alpha_j = \frac{\mathbf{u}_j^H \mathbf{f}}{\omega_j^2 - \omega^2} \quad (5.7)$$

If $\mathbf{u}_j^H \mathbf{f} = 0$, then the resonant mode at ω_j is suppressed (fine with MRI, but eigsh will detect a resonant mode).

For $\|\mathbf{u}\|_{L_2(\Gamma)}^2 = \int_{\Gamma} \|\mathbf{u}\|^2$

5.2 IMPERFECTLY CONDUCTING BOUNDARIES

Numerical eigenfrequencies, solve

$$(\underline{\mathbf{K}} - i\omega \underline{\mathbf{I}} - \omega^2 \underline{\mathbf{M}}) \mathbf{u} = 0 \quad (5.8)$$

Define $\mathbf{v} = \omega \mathbf{u}$, so that we may write this as the generalized eigenvalue problem

$$\begin{bmatrix} \underline{\mathbf{1}} & \\ \underline{\mathbf{K}} & -i\underline{\mathbf{I}} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} = \omega \begin{bmatrix} \underline{\mathbf{1}} & \\ \underline{\mathbf{M}} & \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} \quad (5.9)$$

which is, however, no longer Hermitian (LHS chosen as “diagonal” as possible).

5.3 DUAL MODE CIRCULAR WAVEGUIDE FILTER

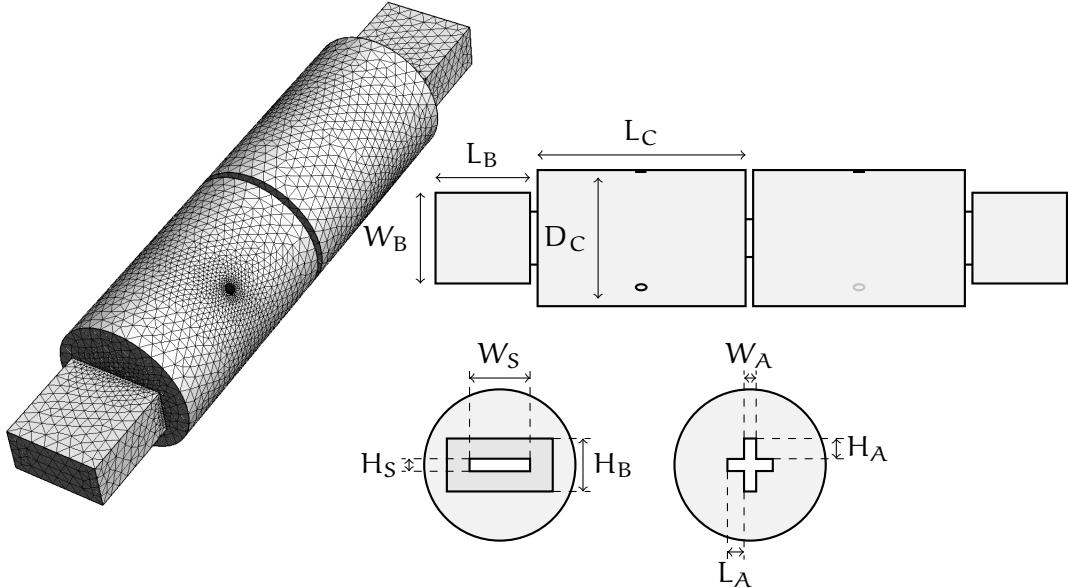


FIGURE 5.2 – Dual-mode circular waveguide filter. ? $W_C = 43.87$ mm, $D_C = 28.0$ mm, $L_B = 43.87$ mm, $W_B = 19.05$ mm, $H_B = 9.525$ mm, $L_B = 20.0$ mm, $W_S = 10.05$ mm, $H_S = 3.0$ mm, $W_A = 2.0$ mm, $H_A = 3.375$ mm, $L_A = 2.825$ mm, thickness of all irises 2.0 mm, screws half way up the cavity horizontal tuning screws with depth 3.82 mm and coupling screws at angles $\pm 45^\circ$ with depth 3.57 mm.

6 CONCLUSION AND OUTLOOK

7 APPENDIX

7.1 DETAILED DERIVATION FOR THE WEAK FORMULATION OF THE TIME-HARMONIC POTENTIAL EQUATION

The goal is to rewrite the curl-integral on the left-hand side of (2.12):

$$\int_{\Omega} (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} \quad (7.1)$$

In order to simplify the curls and apply the Gauss theorem, I first show the following vector calculus identity:

Curl product rule

$$(\nabla \times \mathbf{a}) \cdot \mathbf{b} = \nabla \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\nabla \times \mathbf{b}) \quad (7.2)$$

where \mathbf{a}, \mathbf{b} are vector-value functions. The completely antisymmetric tensor ε_{ijk} , frequently referred to as the Levi-Civita tensor, may be employed to rewrite the components of the curl of a vector-function \mathbf{a} as the sum

$$(\nabla \times \mathbf{a})_k = \sum_i \sum_j \varepsilon_{ijk} \partial_i a_j \quad (7.3)$$

where ∂_i denotes the partial derivative with respect to the i -th coordinate direction. This yields

$$\begin{aligned} (\nabla \times \mathbf{a}) \cdot \mathbf{b} &= \sum_k (\nabla \times \mathbf{a})_k b_k \\ &= \sum_k \left(\sum_i \sum_j \varepsilon_{ijk} \partial_i a_j \right) b_k \\ &= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{ijk} a_j b_k) - \sum_k \sum_i \sum_j a_j (\varepsilon_{ijk} \partial_i b_k) \\ &= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{jki} a_j b_k) - \sum_k \sum_i \sum_j a_j ((-\varepsilon_{ikj}) \partial_i b_k) \\ &= \sum_i \partial_i (\mathbf{a} \times \mathbf{b})_i + \sum_j a_j (\nabla \times \mathbf{b})_j \\ &= \nabla \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\nabla \times \mathbf{b}) \end{aligned} \quad (7.4)$$

by expressing the scalar product as a component-sum, using the product rule and applying the symmetry and anti-symmetry properties of the Levi-Civita tensor. Now the identity (7.2) to (7.1) together with Gauss' theorem gives

$$\begin{aligned} \int_{\Omega} (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} &= \int_{\Omega} \nabla \cdot ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \\ &= \int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \end{aligned} \quad (7.5)$$

For later convenience, the boundary integral can further be simplified using the

Commutative behavior of the scalar triple product

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} \quad (7.6)$$

This identity follows immediately from a small manipulation with the Levi-Civita tensor:

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \sum_k (\sum_i \sum_j \epsilon_{ijk} a_i b_j) c_k \\ &= \sum_j (\sum_i \sum_k (-\epsilon_{ikj}) a_i c_k) b_j \\ &= -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} \end{aligned} \quad (7.7)$$

The boundary integral becomes

$$\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} = - \int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}) \cdot \mathbf{v} \quad (7.8)$$

This concludes the short derivation, because now (7.1) may be rewritten as

$$-\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \quad (7.9)$$