

EPFL

PROJECT CSE I

Notes

Fabio Matti

supervised by
Prof. Fabio Nobile
Dr. Davide Pradovera

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1 FINITE ELEMENT METHOD

1.1 THE GENERAL APPROACH

Summarizes Chapter 1 in Quarteroni: Introduction to Finite Elements Method

Usually, the problems may be expressed in a simple equation

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = u_D & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where L denotes a linear differential operator (e.g. $-\Delta$ in the Poisson equation), u the solution to be found, and f is a source term independent of u . Some boundary condition u_D is imposed on the solution u .

However, equation (1.1) usually does not allow all physically significant solutions (particularly non-differentiable ones). Therefore, we convert the problem to a weak form. This is achieved by multiplying (1.1) with a test function $v \in V$, and integrating over the whole domain Ω :

$$\int_{\Omega} (Lu)v = \int_{\Omega} fv, \quad \forall v \in V \quad (1.2)$$

Usually, integration by parts allows us to “transfer” the derivatives from the Lu term to the test function v , such that the order (with respect to the derivatives taken) is more “balanced” between the two terms. As a trade-off, a boundary term appears and needs to be eliminated to facilitate the finite element solution. This term can often be eliminated by restricting ourselves to test functions from a subspace $V' \subset V$. The weak problem then reads

$$\int_{\Omega} (L_u u)(L_v v) = \int_{\Omega} fv, \quad \forall v \in V' \quad (1.3)$$

with a linear differential operator L_u acts on the solution u , and a linear differential operator L_v , appearing due to the integration by parts, acts on the test function v . For simplicity, we refer to the left-hand side as the bilinear form

$$a(u, v) = \int_{\Omega} (L_u u)(L_v v) \quad (1.4)$$

and the right-hand side as the linear form

$$F(v) = \int_{\Omega} fv \quad (1.5)$$

To solve (1.3), we prefer to look for approximate solutions u_h in a finite dimensional space V_h with $\dim(V_h) = N_h$ (what role does V'_h , the space wherein v lies to satisfy the boundary conditions, play? as far as I can tell V'_h is the subspace of V_h , in which all functions vanish at the boundary where u is known). Choosing a basis $\{\varphi_i\}_{i \leq N_h}$ then allows us to represent the approximate solution as

$$u_h = \sum_{j \leq N_h} u_j \varphi_j \quad (1.6)$$

for some coefficients u_i that need to be determined. We thus see, that (1.3) turns into a linear system, since we only need to test the equality for the basis elements φ_i of V_h :

$$\sum_{j=1}^{N_h} u_j a(\varphi_j, \varphi_i) = F(\varphi_i), \quad i \in \{1, \dots, N_h\} \quad (1.7)$$

If we write $A_{ij} = a(\varphi_j, \varphi_i)$ and $\mathbf{F} = (F(\varphi_1), \dots, F(\varphi_{N_h}))^T$, we have reduced the problem to finding $\mathbf{u} = (u_1, \dots, u_{N_h})^T$, such that

$$\mathbb{A} \mathbf{u} = \mathbf{F} \quad (1.8)$$

and have identified an approximate solution to (1.1) as $u \approx u_h = \sum_{j \leq N_h} u_j \varphi_j$.

Choosing the space V_h is fundamental to have an accurate method that gives a good approximation u_h of u . Furthermore, the choice of basis $\{\varphi_i\}_{i \leq N_h}$ influences how \mathbb{A} ends up looking. Of particular interest are bases for which $a(\varphi_j, \varphi_i)$ vanishes for almost all i and j , thus yielding a sparse matrix \mathbb{A} . The choice of basis also controls the conditioning of \mathbb{A} (need to find an example to illustrate this).

1.2 THE POISSON EQUATION

Taken from FEniCS manual (too lazy for bibtex...)

We aim to solve an equation of the form

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}) \quad (1.9)$$

on a domain $\mathbf{x} \in \Omega$, with a solution $u(\mathbf{x})$ that satisfies a certain boundary condition $u(\mathbf{x}) = u_d(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$ that lie on the border of Ω .

To do this, we first convert this equation to its weak form by multiplying both sides with an arbitrary test function $v(\mathbf{x})$, which vanishes on the border (i.e. $v(\mathbf{x}) = 0, \forall \mathbf{x} \in \partial\Omega$), and by then integrating over all of Ω :

$$-\int_{\Omega} \Delta u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \quad (1.10)$$

We may now rearrange the gradient product rule $\nabla(ab) = (\nabla a)b + a(\nabla b)$ and Gauss' theorem (as long as $v(\mathbf{x})$ is differentiable in a neighborhood of Ω) combined with the fact that $v(\mathbf{x})$ vanishes on $\partial\Omega$ to convert the right-hand side to

$$\begin{aligned} -\int_{\Omega} \Delta u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} &= -\int_{\Omega} \nabla(\nabla u(\mathbf{x}) v(\mathbf{x})) d\mathbf{x} + \int_{\Omega} \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \\ &= -\int_{\partial\Omega} \nabla u(\mathbf{x}) v(\mathbf{x}) d\boldsymbol{\omega} + \int_{\Omega} \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \\ &= \int_{\Omega} \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \end{aligned} \quad (1.11)$$

Consequently, the weak formulation of the problem is to find $u(\mathbf{x})$, such that for arbitrary $v(\mathbf{x})$, we have

$$\int_{\Omega} \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \quad (1.12)$$

To simplify and generalize the notation, we may use the linear form $L : V \rightarrow \mathbb{R}$ as

$$L(v) = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \quad (1.13)$$

and also the bilinear form $a : V \times V \rightarrow \mathbb{R}$

$$a(u, v) = \int_{\Omega} \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \quad (1.14)$$

1.3 EXAMPLE: ONE DIMENSIONAL POISSON EQUATION

Initial idea taken from Wikipedia article about FEM.

To illustrate the choice of basis functions, we will now consider the simple one dimensional case $\Omega = [a, b]$, such that the weak formulation of the problem turns into

$$\int_a^b u'(x) v'(x) dx = \int_a^b f(x) v(x) dx \quad (1.15)$$

We now subdivide the domain $[a, b]$ into M subintervals, each of length $h = (b - a)/M$, with nodes at $x_k = a + hk, k \in \{0, 1, \dots, M\}$. We proceed to choose as the basis functions the class of the piecewise linear Lagrange interpolating polynomials on $[x_k, x_{k+1}], k \in \{0, 1, \dots, M\}$, defined as

$$v_k(x) = \frac{x - x_{k-1}}{x_k - x_{k-1}} \mathbf{1}_{\{x \in [x_{k-1}, x_k]\}} + \frac{x_{k+1} - x}{x_{k+1} - x_k} \mathbf{1}_{\{x \in [x_k, x_{k+1}]\}} \quad (1.16)$$

If we now interpolate $f(x)$ and $u(x)$ as piecewise linear Lagrange polynomials, we get the representation

$$\begin{aligned} f(x) &\approx \sum_{i=1}^M f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + f(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}} \\ &= \sum_{i=1}^{M-1} f(x_i) v_i(x) \end{aligned} \quad (1.17)$$

and analogously

$$u(x) = \sum_{i=1}^{M-1} u(x_i) v_i(x) \quad (1.18)$$

We now restricted ourselves to the discrete variational formulation of the problem

$$\sum_{i=1}^{M-1} u(x_i) \int_a^b v_i'(x) v_j'(x) dx = \sum_{i=1}^{M-1} f(x_i) \int_a^b v_i(x) v_j(x) dx \quad (1.19)$$

which needs to be satisfied for all $j \in \{0, 1, \dots, M\}$.

This equation can be rewritten in terms of two matrices \mathbf{K} and \mathbf{L} which we define as

$$K_{ij} = \int_a^b v_i(x) v_j(x) dx \quad (1.20)$$

$$L_{ij} = \int_a^b v_i'(x) v_j'(x) dx \quad (1.21)$$

such that we get

$$\sum_{i=1}^{M-1} u(x_i) L_{ij} = \sum_{i=1}^{M-1} f(x_i) K_{ij} \quad (1.22)$$

Notice, that we only need the entries K_{ij} and L_{ij} with $i \in \{1, 2, \dots, M-1\}$, since we already know the boundary conditions of $u(x)$ at $x = x_0$ and $x = x_M$.

We realize, that the L_2 inner product of $v_i(x)$ with $v_j(x)$ (and consequently also the one of $v_i'(x)$ with $v_j'(x)$) is zero for all $|i - j| > 1$, hence, we distinguish two different cases.

1. $i = j$: Here, the inner product turns out to be

$$\begin{aligned} \int_a^b v_i(x) v_i(x) dx &= \int_a^b \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right)^2 \mathbf{1}_{\{x \in [x_{i-1}, x_i]\}} + \left(\frac{x_{i+1} - x}{x_{i+1} - x_i} \right)^2 \mathbf{1}_{\{x \in [x_i, x_{i+1}]\}} dx \\ &= 2 \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right)^2 dx \\ &= \frac{2}{h^2} \int_{x_{i-1} - x_{i-1}}^{x_i - x_{i-1}} u^2 du \\ &= \frac{2}{h^2} \frac{1}{3} h^3 \\ &= \frac{2h}{3} \end{aligned} \quad (1.23)$$

and for the derivatives it is

$$\begin{aligned}
\int_a^b v_i'(x) v_i'(x) dx &= \int_a^b \left(\frac{1}{x_i - x_{i-1}} \right)^2 \mathbf{1}_{\{x \in [x_{i-1}, x_i]\}} + \left(\frac{-1}{x_{i+1} - x_i} \right)^2 \mathbf{1}_{\{x \in [x_i, x_{i+1}]\}} dx \\
&= 2 \int_{x_{i-1}}^{x_i} \left(\frac{1}{x_i - x_{i-1}} \right)^2 dx \\
&= \frac{2}{h^2} \int_0^h 1 du \\
&= \frac{2}{h}
\end{aligned} \tag{1.24}$$

2. $|i - j| = 1$: Here, we can limit ourselves to the case where $j = i + 1$, since the other case is fully symmetric. We calculate

$$\begin{aligned}
\int_a^b v_i(x) v_{i+1}(x) dx &= \int_a^b \frac{x_{i+1} - x}{x_{i+1} - x_i} \frac{x - x_i}{x_{i+1} - x_i} \mathbf{1}_{\{x \in [x_i, x_{i+1}]\}} dx \\
&= \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{x_{i+1} - x_i} \frac{x - x_i}{x_{i+1} - x_i} dx \\
&= \frac{1}{h^2} \int_{x_i - x_i}^{x_{i+1} - x_i} (x_{i+1} - x_i - u) u du \\
&= \frac{1}{h^2} \int_0^h (h - u) u du \\
&= \frac{1}{h^2} \left(\frac{h^3}{2} - \frac{h^3}{3} \right) \\
&= \frac{h}{6}
\end{aligned} \tag{1.25}$$

and for the derivative it is

$$\begin{aligned}
\int_a^b v_i'(x) v_{i+1}'(x) dx &= \int_a^b \frac{-1}{x_{i+1} - x_i} \frac{1}{x_{i+1} - x_i} \mathbf{1}_{\{x \in [x_i, x_{i+1}]\}} dx \\
&= -\frac{1}{h^2} \int_{x_i}^{x_{i+1}} 1 dx \\
&= -\frac{1}{h}
\end{aligned} \tag{1.26}$$

Now, using the previously defined matrices \mathbf{K}_{ij} and \mathbf{L}_{ij} , we get the matrix equation

$$\mathbf{L} \mathbf{u} = \mathbf{K} \mathbf{f} \tag{1.27}$$

with

$$\mathbf{u} = (u_0, u(x_1), \dots, u_M)^T \quad (1.28)$$

$$\mathbf{f} = (f(x_0), f(x_1), \dots, f(x_M))^T \quad (1.29)$$

$$\mathbf{L} = \begin{pmatrix} 1 & & & & \\ \frac{2}{h} & -\frac{1}{h} & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & -\frac{1}{h} & \frac{2}{h} & \ddots & \\ & & -\frac{1}{h} & \ddots & -\frac{1}{h} \\ & & & \ddots & \frac{2}{h} \\ & & & & 1 \end{pmatrix} \quad (1.30)$$

$$\mathbf{K} = \begin{pmatrix} \frac{u_0}{f(x_0)} & & & & \\ \frac{2h}{3} & \frac{h}{6} & & & \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & & \\ & \frac{h}{6} & \frac{2h}{3} & \ddots & \\ & & \frac{h}{6} & \ddots & \frac{h}{6} \\ & & & \ddots & \frac{2h}{3} \\ & & & & \frac{u_M}{f(x_M)} \end{pmatrix} \quad (1.31)$$

$$(1.32)$$

Here, we have adjusted the first rows in \mathbf{L} and \mathbf{K} , such that the boundary conditions are necessarily satisfied. To obtain the finite element solution, we simply solve this linear system.

2 MAXWELL'S EQUATIONS

Let $\mathbf{E} = (E_1, E_2, E_3)^T$ denote the electric field, $\mathbf{B} = (B_1, B_2, B_3)^T$ the magnetic field strength, and $\mathbf{j} = (j_1, j_2, j_3)^T$ the electric current density. We suppose Maxwell's equations hold:

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \quad (2.1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.2)$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \quad (2.3)$$

$$\nabla \times (\mu^{-1} \mathbf{B}) = \partial_t (\epsilon \mathbf{E}) + \mathbf{j} \quad (2.4)$$

2.1 VECTOR POTENTIAL FORMULATION (FLAWED)

We can therefore use (2.2) to write $\mathbf{B} = \nabla \times \mathbf{A}$ for some vector potential \mathbf{A} . Furthermore, we can identify from (2.3) that $\mathbf{E} = -\nabla \phi - \partial_t \mathbf{A}$ can be written for some scalar potential ϕ . Plugging these identities into (2.4), we get

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{A}) = -\epsilon(\partial_t \nabla \phi + \partial_t^2 \mathbf{A}) + \mathbf{j} \quad (2.5)$$

Here is where the whole derivation falls apart. We cannot possibly justify letting $\nabla \phi$ vanish, except if we are in a Coulomb gauge and if μ is a constant. In particular, there exists no gauge transformation $\mathbf{A}' = \mathbf{A} + \nabla f$ and $\phi' = \phi - \partial_t f$ for which $\mathbf{E} = -\nabla \phi - \partial_t \mathbf{A} = -\partial_t \mathbf{A}'$, since (obviously) \mathbf{E} is supposed to remain invariant under such transformations... Furthermore the ϵ , whose name, let alone its value, I cannot remember was unintentionally dropped from here on.

We may want to introduce a harmonic time dependence of \mathbf{A} and \mathbf{j} with frequencies ω , such that $\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}) \exp(i\omega t)$ and $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x}) \exp(i\omega t)$. Plugging this into (2.5) yields us

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{A}) - \omega^2 \mathbf{A} = \mathbf{j} \quad (2.6)$$

We reduce this equation to its weak formulation, by multiplying it with a vector-valued function $\mathbf{v} \in H_{\text{curl}}(\Omega)$, where we denoted

$$H_{\text{curl}}(\Omega) = \{\mathbf{u} : \Omega \rightarrow \mathbb{C}, \text{ such that } \mathbf{u} \in L^2(\mathbb{C})^3, \nabla \times \mathbf{u} \in L^2(\mathbb{C})^3\} \quad (2.7)$$

and by integrating over all of Ω :

$$\int_{\Omega} (\nabla \times (\mu^{-1} \nabla \times \mathbf{A})) \cdot \mathbf{v} - \omega^2 \int_{\Omega} \mathbf{A} \cdot \mathbf{v} = \int_{\Omega} \mathbf{j} \cdot \mathbf{v} \quad (2.8)$$

To further simplify this expression, we will derive an identity for the scalar product of a vector-valued function \mathbf{v} with the curl of a vector-valued function \mathbf{u} . For this, we use the completely antisymmetric tensor ε_{ijk} (frequently referred to as the Levi-Civita tensor), to rewrite the k -th component of the curl as

$$(\nabla \times \mathbf{u})_k = \sum_i \sum_j \varepsilon_{ijk} \partial_i u_j \quad (2.9)$$

where ∂_i denotes the partial derivative with respect to the i -th coordinate direction. Rewriting the scalar product as a sum and identifying $\mathbf{u} = \mu^{-1} \nabla \times \mathbf{A}$, we apply the product rule to get

$$\begin{aligned} (\nabla \times \mathbf{u}) \cdot \mathbf{v} &= \sum_k (\nabla \times \mathbf{u})_k v_k \\ &= \sum_k \left(\sum_i \sum_j \varepsilon_{ijk} \partial_i u_j \right) v_k \\ &= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{ijk} u_j v_k) - \sum_k \sum_i \sum_j u_j (\varepsilon_{ijk} \partial_i v_k) \\ &= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{jki} u_j v_k) - \sum_k \sum_i \sum_j u_j ((-\varepsilon_{ikj}) \partial_i v_k) \\ &= \sum_i \partial_i (\mathbf{u} \times \mathbf{v})_i + \sum_j u_j (\nabla \times \mathbf{v})_j \\ &= \nabla \cdot (\mathbf{u} \times \mathbf{v}) + \mathbf{u} \cdot (\nabla \times \mathbf{v}) \end{aligned} \quad (2.10)$$

Consequently, we may rewrite the double curl term in the weak formulation as

$$\begin{aligned} \int_{\Omega} (\nabla \times (\mu^{-1} \nabla \times \mathbf{A})) \cdot \mathbf{v} &= \int_{\Omega} \nabla \cdot ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{v}) + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) \\ &= \int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{v}) \cdot \mathbf{n} + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) \end{aligned} \quad (2.11)$$

We will now have a look at what conditions \mathbf{v} needs to satisfy, such that the boundary term (first integral) vanishes, and we would end up with

$$\int_{\Omega} (\mu^{-1} \nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_{\Omega} \mathbf{A} \cdot \mathbf{v} = \int_{\Omega} \mathbf{j} \cdot \mathbf{v} \quad (2.12)$$

Let \mathbf{n} denote the normal vector to $\partial\Omega$ at a point $\mathbf{x} \in \partial\Omega$. For the boundary term to vanish, we require

$$((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{v}) \cdot \mathbf{n} = 0 \quad (2.13)$$

for all $\mathbf{x} \in \partial\Omega$. Denoting $\mathbf{u} = \mu^{-1} \nabla \times \mathbf{A}$, we rearrange

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n} &= \sum_k \left(\sum_i \sum_j \varepsilon_{ijk} u_i v_j \right) n_k \\ &= \sum_i u_i \left(\sum_j \sum_k \varepsilon_{jki} v_j n_k \right) \\ &= \mathbf{u} \cdot (\mathbf{v} \times \mathbf{n}) \end{aligned} \quad (2.14)$$

For non-trivial \mathbf{u} and \mathbf{v} , this expression is zero if and only if $\mathbf{v} \perp \mathbf{n}$, meaning \mathbf{v} is orthogonal to $\partial\Omega$ for all $\mathbf{x} \in \partial\Omega$.

2.2 ELECTRIC FIELD FORMULATION

Multiplying both sides of (2.3) with μ^{-1} and taking the curl allows us to substitute (2.4) in, such that we get

$$\begin{aligned} \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) &= -\partial_t (\nabla \times (\mu^{-1} \mathbf{B})) \\ &= -\partial_t (\partial_t (\epsilon \mathbf{E}) + \mathbf{j}) \\ &= -\partial_t^2 (\epsilon \mathbf{E}) + \partial_t \mathbf{j} \end{aligned} \quad (2.15)$$

Introducing a harmonic time dependence of \mathbf{E} and \mathbf{j} with frequencies ω , such that $\mathbf{E}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x}) \exp(i\omega t)$ and $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x}) \exp(i\omega t)$, we obtain

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \omega^2 \epsilon \mathbf{E} = i\omega \mathbf{j} \quad (2.16)$$

The i on the right-hand side is giving me (and most likely FEniCS too) a major headache, hence why I will try my luck with finding a formulation involving \mathbf{B} in the next section.

2.3 MAGNETIC FIELD FORMULATION

Multiply both sides of (2.4) with ϵ^{-1} and take the curl to obtain, with the help of (2.3), the expression

$$\begin{aligned}\nabla \times (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) &= \nabla \times (\partial_t \mathbf{E} + \epsilon^{-1} \mathbf{j}) \\ &= \partial_t (\nabla \times \mathbf{E}) + \nabla \times (\epsilon^{-1} \mathbf{j}) \\ &= -\partial_t^2 \mathbf{B} + \nabla \times (\epsilon^{-1} \mathbf{j})\end{aligned}\quad (2.17)$$

Introducing a harmonic time dependence of \mathbf{B} and \mathbf{j} with frequencies ω , such that $\mathbf{B}(\mathbf{x}, t) = \mathbf{B}(\mathbf{x}) \exp(i\omega t)$ and $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x}) \exp(i\omega t)$, we obtain

$$\nabla \times (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) - \omega^2 \mathbf{B} = \nabla \times (\epsilon^{-1} \mathbf{j}) \quad (2.18)$$

Converting this equation to its weak form by multiplying both sides with a vector valued function $\mathbf{v} \in H_{\text{curl}}(\Omega)$ and integrating over all of Ω yields

$$\int_{\Omega} (\nabla \times (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B}))) \cdot \mathbf{v} - \omega^2 \int_{\Omega} \mathbf{B} \cdot \mathbf{v} = \int_{\Omega} (\nabla \times (\epsilon^{-1} \mathbf{j})) \cdot \mathbf{v} \quad (2.19)$$

Using the above derived identity

$$(\nabla \times \mathbf{u}) \cdot \mathbf{v} = \nabla \cdot (\mathbf{u} \times \mathbf{v}) + \mathbf{u} \cdot (\nabla \times \mathbf{v}) \quad (2.20)$$

we may write using Gauss' theorem

$$\begin{aligned}\int_{\Omega} (\nabla \times (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B}))) \cdot \mathbf{v} &= \int_{\Omega} \nabla \cdot ((\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) \times \mathbf{v}) \\ &\quad + \int_{\Omega} (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) \cdot (\nabla \times \mathbf{v}) \\ &= \int_{\partial\Omega} ((\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) \times \mathbf{v}) \cdot \mathbf{n} \\ &\quad + \int_{\Omega} (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) \cdot (\nabla \times \mathbf{v})\end{aligned}\quad (2.21)$$

and

$$\begin{aligned}\int_{\Omega} (\nabla \times (\epsilon^{-1} \mathbf{j})) \cdot \mathbf{v} &= \int_{\Omega} \nabla \cdot ((\epsilon^{-1} \mathbf{j}) \times \mathbf{v}) + \int_{\Omega} (\epsilon^{-1} \mathbf{j}) \cdot (\nabla \times \mathbf{v}) \\ &= \int_{\partial\Omega} ((\epsilon^{-1} \mathbf{j}) \times \mathbf{v}) \cdot \mathbf{n} + \int_{\Omega} (\epsilon^{-1} \mathbf{j}) \cdot (\nabla \times \mathbf{v})\end{aligned}\quad (2.22)$$

Since $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{n})$ (as was shown above), the boundary integrals vanish if we have

$$\mathbf{v} \times \mathbf{n} = 0, \text{ on } \partial\Omega \quad (2.23)$$

In that case, we end up with the following (simplified) weak form

$$\int_{\Omega} (\epsilon^{-1} \nabla \times (\mu^{-1} \mathbf{B})) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_{\Omega} \mathbf{B} \cdot \mathbf{v} = \int_{\Omega} (\epsilon^{-1} \mathbf{j}) \cdot (\nabla \times \mathbf{v}) \quad (2.24)$$

3 WEAK DERIVATIVE

Taken from Quarteroni: Introduction to Finite Elements Method

Let $\Omega \subset \mathbb{R}^d$ open. The support of $f : \Omega \rightarrow \mathbb{R}$ is defined as

$$\text{supp}(f) = \overline{\{x \in \Omega \mid f(x) \neq 0\}} \quad (3.1)$$

f has compact support, if there exists a compact subset $K \subset \Omega$, such that $\text{supp}(f) \subset K$, and define

$$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) \mid f \text{ has compact support}\} \quad (3.2)$$

(If I remember correctly, extending this notion to $f \in C^1(\Omega)$ should yield an almost identical treatment, unless we also include higher order (weak) partial derivatives). Let $T : \mathcal{D} \rightarrow \mathbb{R}$, $\varphi \mapsto \langle T, \varphi \rangle = T(\varphi)$ be a linear map. We say that T is continuous, if

$$\lim_{n \rightarrow \infty} \langle T, \varphi_n \rangle = \langle T, \varphi \rangle \quad (3.3)$$

with $\{\varphi_k\}_{k \in \mathbb{N}} \subset \mathcal{D}(\Omega)$ converging to φ . Such (linear and continuous) maps are called distribution on $\mathcal{D}(\Omega)$, and they form the space of distributions $\mathcal{D}'(\Omega)$.

The (weak) partial coordinate-derivatives of T (namely $\partial_i T$, $i \in \{1, \dots, d\}$) are characterized by distributions that satisfy

$$\langle \partial_i T, \varphi \rangle = -\langle T, \partial_i \varphi \rangle \quad (3.4)$$

for all $\varphi \in \mathcal{D}(\Omega)$.

Interesting for us is mainly the following case: Given a function $f \in L^2(\Omega)$, we define a distribution $T_f \in \mathcal{D}'(\Omega)$ to be

$$\langle T_f, \varphi \rangle = \int_{\Omega} f(x) \varphi(x) dx \quad (3.5)$$

for all $\varphi \in \mathcal{D}(\Omega)$.

This allows us to define a weak derivative to functions that are (in the classical sense) not differentiable (i.e. not in $C^1(\Omega)$). Consider for example the absolute value function $|\cdot| \in L^2(K)$ where $K \subset \mathbb{R}$ is compact. Since

$$\begin{aligned}
\int_K (\partial_x |x|) \varphi(x) dx &= - \int_K |x| \varphi'(x) dx \\
&= - \int_{K \cap \mathbb{R}_+} x \varphi'(x) dx - \int_{K \cap \mathbb{R}_-} (-x) \varphi'(x) dx \\
&= \int_{K \cap \mathbb{R}_+} \varphi(x) dx + \int_{K \cap \mathbb{R}_-} (-1) \varphi(x) dx \\
&= \int_K \text{sign}(x) \varphi(x) dx
\end{aligned} \tag{3.6}$$

we may conclude that the weak derivative of the absolute value function is therefore the signum function. Notice, how the derivative of the absolute value function is only not well-defined at $x = 0$, i.e. on a set of zero measure. This nuisance is circumvented when talking about the weak derivative, since the measure zero sets have zero integral.

4 IDEAS

What might be really interesting is to instead look at the problem in space-time using the Maxwell tensor

$$\mathbb{F} = \begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{bmatrix} \tag{4.1}$$

In the covariant formulation of the Maxwell theory, the inhomogeneous Maxwell equations reduce to a single equation

$$\partial_a F^{ab} = -J^b \tag{4.2}$$

with the four current density $\mathbf{J} = (\mu c \rho, \mu \mathbf{j})$. The weak formulation of the problem could then be stated as (using Einstein's sum convention, i.e. summing over repeated indices)

$$\int_{\Omega \times \mathbb{R}} F^{ab} \partial_a v_b = \int_{\Omega \times \mathbb{R}} J^b v_b \tag{4.3}$$

where boundary conditions are yet to be determined. If we somehow would manage to find a suitable function space for the four-dimensional \mathbf{v} , it might be possible to find both \mathbf{E} and \mathbf{B} from a finite element method.