EPFL

Semester Project

Minimal Rational Interpolation for Time-Harmonic Maxwell's Equations

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Abstract

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2.1 Vector potential formulation of the time-harmonic Maxwell's equations

Let **E** denote an electric field, **B** a magnetic field strength, ρ an electric charge density, and **j** an electric current density. Maxwell's equations are stated in [Monk, 2003] as

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{2.3}$$

$$\nabla \times (\mu^{-1}\mathbf{B}) = \partial_t(\epsilon \mathbf{E}) + \mathbf{j} \tag{2.4}$$

with ε being the permittivity and μ the permeability.

Equation (2.2) allows for an expression of the magnetic field $\mathbf{B} = \nabla \times \mathbf{u}$ in terms of a vector valued function $\mathbf{u}$, the vector potential (in literature commonly denoted with **A**). Similarly, (2.3) suggests rewriting the electric field $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{u}$ using a scalar function ϕ, referred to as the scalar potential.

The physical quantities **E** and **B** remain unchanged if we transform $\mathbf{u} \to \mathbf{u}' = \mathbf{u} + \nabla\psi$ or $\phi \to \phi' = \phi - \partial_t\psi$ for arbitrary functions ψ. A convenient choice of ψ is suggested in [Kagerer, 2018] to be

$$\psi = \int_0^t \phi \, dt' \tag{2.5}$$

which transforms $\phi \to \phi' = 0$ and $\mathbf{u} \to \mathbf{u}' = \mathbf{u} + \nabla \int_0^t \phi \, dt'$. Thus, the expressions for the electrical and magnetic field become

$$\mathbf{E} = -\partial_t \mathbf{u} \tag{2.6}$$

$$\mathbf{B} = \nabla \times \mathbf{u} \tag{2.7}$$

where I renamed the variable $\mathbf{u}'$ to $\mathbf{u}$ for simplicity.

Plugging the identities (2.6) and (2.7) into (2.4) yields

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) = \epsilon \partial_t^2 \mathbf{u} + \mathbf{j} \tag{2.8}$$

For the rest of this report, I restrict myself to vector potentials $\mathbf{u}$ that exhibit a harmonic dependence on time t, i.e. may be factorized into a term solely depending on the position $\mathbf{x}$ and a complex exponential

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \exp(i\omega t) \tag{2.9}$$

Substituting this expression into (2.8) results in the

Time-harmonic potential equation

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) - \epsilon\omega^2 \mathbf{u} = \mathbf{j} \tag{2.10}$$

2.2 Weak formulation for the time-harmonic potential equation

Equation (2.10) may be multiplied by a vector-valued function $\mathbf{v} \in H_{\text{curl}}(\Omega)$, where

$$H_{\text{curl}}(\Omega) = \{\mathbf{u} : \Omega \to \mathbb{C}, \text{ such that } \mathbf{u} \in L^2(\mathbb{C})^3, \nabla \times \mathbf{u} \in L^2(\mathbb{C})^3\} \tag{2.11}$$

and then integrated over all of Ω to obtain

$$\int_\Omega (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} - \omega^2 \int_\Omega \epsilon \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} \tag{2.12}$$

This may further be simplified (2.12) to (see Section for details)

Weak formulation of the time-harmonic potential equation

$$\int_\Omega (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_\Omega \epsilon \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} + \int_{\partial\Omega} \underbrace{((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n})}_{=\mathbf{g}} \cdot \mathbf{v} \tag{2.13}$$

where $\mathbf{n}$ denotes the surface normal to the boundary $\partial\Omega$.

Boundary conditions on the electric field $\mathbf{E}$ may be enforced in a Dirichlet-type fashion through the relation (2.6) and the assumption (2.9)

$$\mathbf{u}|_{\partial\Omega} = -\frac{1}{i\omega} \mathbf{E}|_{\partial\Omega} \tag{2.14}$$

Those on the magnetic field $\mathbf{B}$ through a Neumann-type condition following from (2.7) and again (2.9)

$$\mathbf{g}|_{\partial\Omega} = (\mu^{-1} \mathbf{B}|_{\partial\Omega}) \times \mathbf{n} \tag{2.15}$$

2.3 Two-dimensional resonant cavity

A resonant cavity is a region Ω enclosed by a boundary $\partial\Omega$. The boundary is subdivided into one (or more) inlets Γ_N and a perfect electrically conducting wall $\Gamma_D = \partial\Omega \setminus \Gamma_N$.

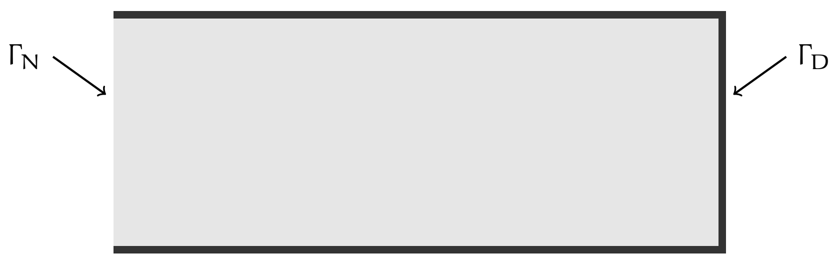

Figure 2.1 – Example of a two-dimensional resonant cavity: The rectangular cavity.

Suppose the current density $\mathbf{j} \equiv 0$ and orient the coordinate system in such a way that $\mathbf{u} = u_z \mathbf{e}_z$ and $\mathbf{v} = v_z \mathbf{e}_z$. Consequently,

$$(\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) = (\mu^{-1} \nabla u_z) \cdot (\nabla v_z) \tag{2.16}$$

Use $g_z = (\mathbf{g})_z$ along the boundary Γ_N, to convert (2.13) into the weak formulation for a two-dimensional resonant cavity

$$\int_\Omega (\mu^{-1}\nabla u_z)\cdot(\nabla v_z) - \omega^2 \int_\Omega \epsilon u_z v_z = \int_{\partial\Omega} g_z v_z \tag{2.17}$$

Let $\mathbf{E}$ and $\mathbf{B}$ refer to the electric and magnetic fields inside the cavity. For now, I distinguish two types of boundaries.

For the perfectly conducting boundary, treated in [Monk, 2003], it holds that

$$\mathbf{n} \times \mathbf{E} = 0, \quad \text{on } \Gamma_D \tag{2.18}$$

For the boundaries in a two-dimensional resonant cavity (see Figure 2.1), this only holds true if $E_z = 0$, which translates to the Dirichlet boundary condition $\mathbf{u}|_{\Gamma_D} = 0$ in light of (2.14).

For the inlet, it is easiest to enforce the boundary condition through the magnetic field $\mathbf{B}$ (considering $\mathbf{n} = -\mathbf{e}_x$ as depicted in Figure 2.1):

$$g_z = ((\mu^{-1}\mathbf{B}) \times (-\mathbf{e}_x))_z = \mu^{-1} B_x \tag{2.19}$$

2.4 Waveguide

2.5 Imperfect conductor

3 Finite element approximation with FEniCS

$a_\omega(u, v) = L(v).$

4 Greedy minimal rational interpolation for the time-harmonic Maxwell's equations

[F. Bonzzoni]

Algorithm 1 Minimal rational interpolation

Require: $\omega_1, \ldots, \omega_S$
Require: $U = [u(\omega_1)| \ldots |u(\omega_S)]$ ▷ Snapshot matrix
Compute G with $g_{ij} = \langle u(\omega_i), u(\omega_j) \rangle_M$, $i, j \in \{1, \ldots, S\}$ ▷ Gramian matrix
Compute the singular value decomposition $G = V\Sigma V^H$
Define $q = V[:, S]$
Define $\tilde{u}(\omega) = P(\omega)/Q(\omega)$ with $P(\omega) = \sum_{j=1}^{S} \frac{q_j u(\omega_j)}{\omega - \omega_j}$ and $Q(\omega) = \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$

[D. Pradovera, 2021]

Algorithm 2 Greedy minimal rational interpolation

Require: $\tau > 0$ ▷ Relative L_2-error tolerance
Require: $\Omega_{\text{test}} = \{\omega_i\}_{i=1}^{M}$ ▷ Set of candidate support points
Require: $a_\omega(u, v) = L(v)$ ▷ Finite element formulation of the problem
Choose $\omega_1, \ldots, \omega_t \in \Omega_{\text{test}}$ ▷ Usually the smallest and largest element
Remove $\omega_1, \ldots, \omega_t$ from Ω_{test}
Solve $a_{\omega_i}(u_i, v) = L(v)$ for $i \in \{1, \ldots, t\}$
Build surrogate $\tilde{u}_t = P_t(\omega)/Q_t(\omega)$ using the solutions $u_1, \ldots, u_t$
while $\Omega_{\text{test}} \neq \emptyset$ **do**
    $\omega_{t+1} \leftarrow \text{argmin}_{\omega \in \Omega_{\text{test}}} |Q_t(\omega)|$
    Solve $a_{\omega_{t+1}}(u_{t+1}, v) = L(v)$
    Build surrogate $\tilde{u}_{t+1} = P_{t+1}(\omega)/Q_{t+1}(\omega)$ using the solutions $u_1, \ldots, u_{t+1}$
    if $\|u_{t+1}(\omega_{t+1}) - \tilde{u}_{t+1}(\omega_{t+1})\|_M / \|u_{t+1}(\omega_{t+1})\|_M < \tau$ **then return**
    end if
    $t \leftarrow t + 1$
end while

[Klein, 2012]
Defining

$$\nu_i = (\omega - \omega_i)^{-1} \tag{4.1}$$

and requiring

$$0 = Q(\omega) = \sum_{i=1}^{S} q_i \nu_i(\omega) \tag{4.2}$$

can be equivalently expressed as a generalized eigenvalue problem

$$\underline{\mathbf{A}}\mathbf{u} = \omega \underline{\mathbf{B}}\mathbf{u} \tag{4.3}$$

with

$$\underline{\mathbf{A}} = \begin{pmatrix} 0 & q_1 & q_2 & \cdots & q_S \\ 1 & \omega_1 & & & \\ 1 & & \omega_2 & & \\ \vdots & & & \ddots & \\ 1 & & & & \omega_S \end{pmatrix} \text{ and } \underline{\mathbf{B}} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \end{pmatrix} \tag{4.4}$$

[Trefethen, 2010]

Algorithm 3 Additive Householder triangularization

Require: $U[1\ldots s, 1\ldots N]$ ▷ Next snapshot matrix
Require: $R[1\ldots S, 1\ldots S]$ ▷ Previous triangular matrix
Require: $E[1\ldots S, 1\ldots N]$ ▷ Previous orthonormal matrix
Require: $V[1\ldots S, 1\ldots N]$ ▷ Previous Householder matrix
  Extend size of R to $(S+s)\times(S+s)$
  Extend E with S orthonormal columns to $(S+s)\times N$
  Extend size of V to $(S+s)\times N$
  for $j = S+1 : S+s$ **do**
    $u = U[j]$
    for $k = 1 : j-1$ **do**
      $u \leftarrow u - 2\langle V[k,:], u\rangle_M V[k,:]$
      $R[k,j] \leftarrow \langle E[k,:], u\rangle_M$
      $u \leftarrow u - R[k,j]E[k,:]$
    end for
    $R[j,j] \leftarrow \|u\|_M$
    $\alpha \leftarrow \langle E[j,:], u\rangle_M$
    if $|\alpha| \neq 0$ **then**
      $E[j,:] \leftarrow E[j,:](-\alpha/|\alpha|)$
    end if
    $V[j,:] \leftarrow R[j,j]E[j,:] - u$
    $V[j,:] \leftarrow V[j,:] - \langle E[S+1:j], V[j,:]\rangle_M E[S+1:j,:]$
    $\sigma \leftarrow \|V[j,:]\|_M$
    if $\sigma \neq 0$ **then**
      $V[j,:] \leftarrow V[j,:]/\sigma$
    else
      $V[j,:] \leftarrow E[j,:]$
    end if
  end for

5 Examples

5.1 Two-dimensional rectangular cavity

5.2 Dual mode circular waveguide filter

Figure 5.1 – Dual-mode circular waveguide filter.

5.3 Imperfectly conducting boundaries

6 Conclusion and outlook

7 Appendix

7.1 Detailed derivation for the weak formulation of the time-harmonic potential equation

The goal is to rewrite the curl-integral on the left-hand side of (2.12):

$$\int_{\Omega} (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} \tag{7.1}$$

In order to simplify the curls and apply the Gauss theorem, I first show the following vector calculus identity:

Curl product rule

$$(\nabla \times \mathbf{a}) \cdot \mathbf{b} = \nabla \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\nabla \times \mathbf{b}) \tag{7.2}$$

where $\mathbf{a}$, $\mathbf{b}$ are vector-value functions. The completely antisymmetric tensor ε_{ijk}, frequently referred to as the Levi-Civita tensor, may be employed to rewrite the components of the curl of a vector-function $\mathbf{a}$ as the sum

$$(\nabla \times \mathbf{a})_k = \sum_i \sum_j \varepsilon_{ijk} \partial_i u_j \tag{7.3}$$

where ∂_i denotes the partial derivative with respect to the i-th coordinate direction. This yields

$$\begin{aligned}
(\nabla \times \mathbf{a}) \cdot \mathbf{b} &= \sum_k (\nabla \times \mathbf{a})_k b_k \\
&= \sum_k (\sum_i \sum_j \varepsilon_{ijk} \partial_i a_j) b_k \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{ijk} a_j b_k) - \sum_k \sum_i \sum_j a_j (\varepsilon_{ijk} \partial_i b_k) \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{jki} a_j b_k) - \sum_k \sum_i \sum_j a_j ((-\varepsilon_{ikj}) \partial_i b_k) \\
&= \sum_i \partial_i (\mathbf{a} \times \mathbf{b})_i + \sum_j u_j (\nabla \times \mathbf{b})_j \\
&= \nabla \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\nabla \times \mathbf{b})
\end{aligned} \tag{7.4}$$

by expressing the scalar product as a component-sum, using the product rule and applying the symmetry and anti-symmetry properties of the Levi-Civita tensor. Now the identity (7.2) to (7.1) together with Gauss' theorem gives

$$\begin{aligned}
\int_{\Omega} (\nabla \times (\mu^{-1} \nabla \times \mathbf{u})) \cdot \mathbf{v} &= \int_{\Omega} \nabla \cdot ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{7.5}$$

For later convenience, the boundary integral can further be simplified using the

Commutative behavior of the scalar triple product

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} \tag{7.6}$$

This identity follows immediately from a small manipulation with the Levi-Civita tensor:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \sum_k (\sum_i \sum_j \varepsilon_{ijk} a_i b_j) c_k \\ &= \sum_j (\sum_i \sum_k (-\varepsilon_{ikj}) a_i c_k) b_j \\ &= -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}\end{aligned} \tag{7.7}$$

The boundary integral becomes

$$\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} = -\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}) \cdot \mathbf{v} \tag{7.8}$$

This concludes the short derivation, because now (7.1) may be rewritten as

$$-\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{n} + \int_{\Omega} (\mu^{-1} \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \tag{7.9}$$

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