Minimal rational interpolation for time-harmonic Maxwell's equations

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and build the rational surrogate

$$\tilde{\mathbf{u}}(\omega) = \frac{\mathbf{P}(\omega)}{Q(\omega)}$$

such that $\tilde{\mathbf{u}}(\omega) \approx \mathbf{u}(\omega)$ close to $\omega_1, \omega_2, \dots, \omega_S$.

- Problem formulation
- Finite element method
- Minimal rational interpolation
- Example applications
- Conclusion and outlook

Time-harmonic vector potential $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \operatorname{exp}(i\omega t)$.

$$\mathbf{B} = \nabla \times \mathbf{u} \qquad \text{(Magnetic field)}$$

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Time-harmonic potential equation

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) - \epsilon\omega^2\mathbf{u} = \mathbf{j}$$

$$H_{\mathrm{curl}}(\Omega) = \{\mathbf{v} : \Omega \to \mathbb{C}^3,\ \text{such that } \mathbf{v} \in L_2(\Omega)^3,\ \nabla \times \mathbf{v} \in L_2(\Omega)^3\}$$

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Weak formulation of the time-harmonic potential equation

Find $\mathbf{u} \in H_{\text{curl}}(\Omega)$, such that

$$\int_\Omega \langle \mu^{-1} \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle - \omega^2 \int_\Omega \epsilon \langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \langle \mathbf{j}, \mathbf{v} \rangle + \int_{\partial\Omega} \langle \mathbf{g}, \mathbf{v} \rangle$$

for all $\mathbf{v} \in H_{\text{curl}}$, where $\mathbf{g} = (\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}$.

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \epsilon\omega^2 \mathbf{u} = \mathbf{j}$$

Perfectly conducting boundary

$$\mathbf{g} = \mathbf{0} \;\text{ and }\; \mathbf{E} \times \mathbf{n} = \mathbf{0}, \;\text{ on } \Gamma_D$$

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Imperfectly conducting boundary

$$\mathbf{g} = i\omega\lambda(\mathbf{n} \times \mathbf{u}) \times \mathbf{n}, \text{ on } \Gamma_I$$

$$\int_\Omega \langle \mu^{-1}\nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle - \omega^2 \int_\Omega \epsilon \langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \langle \mathbf{j}, \mathbf{v} \rangle + \int_{\partial\Omega} \langle \mathbf{g}, \mathbf{v} \rangle$$

FEniCS is used to obtain FEM solutions of the form

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where N is the number of degrees of freedom. Inner product in this representation is

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle_M = \bar{\mathbf{u}}^H \underline{\mathbf{M}} \bar{\mathbf{v}} \approx \int_\Omega \langle \mathbf{u}, \mathbf{v} \rangle$$

and the norm

$$\|\bar{\mathbf{u}}\|_M = \sqrt{\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle_M} \approx \|\mathbf{u}\|_{L_2(\Omega)}$$

Rational surrogate

$$\tilde{\mathbf{u}}(\omega) = \frac{\mathbf{P}(\omega)}{Q(\omega)} = \sum_{j=1}^{S} \frac{\mathbf{p}_j}{\omega - \omega_j} \Big/ \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$$

in barycentric coordinates with support points $\omega_1, \omega_2, \ldots, \omega_S$.

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Interpolation property

$$\tilde{\mathbf{u}}(\omega_i) = \mathbf{u}(\omega_i), \ \forall i \in \{1, 2, \ldots, S\}$$

if and only if $\mathbf{p}_i = q_i \mathbf{u}(\omega_i), \forall i$.

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3. Define $\mathbf{q} = (q_1, q_2, \ldots, q_S)^T = \underline{\mathbf{V}}[:, S]$
4. Define the minimal rational surrogate $\tilde{\mathbf{u}}(\omega) = \mathbf{P}(\omega)/Q(\omega)$ with

$$\mathbf{P}(\omega) = \sum_{j=1}^{S} \frac{q_j \mathbf{u}(\omega_j)}{\omega - \omega_j} \quad \text{and} \quad Q(\omega) = \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$$

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$$\omega^{(t)} = \operatorname{argmin}_{\omega \in \Omega_{\text{test}}} |Q^{(t)}(\omega)|$$

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3. Stop when relative error
$$\|\mathbf{u}(\omega_{t+1}) - \tilde{\mathbf{u}}_t(\omega_{t+1})\|_M / \|\mathbf{u}(\omega_{t+1})\|_M$$
is small enough

With the QR-decomposition of the snapshot matrix $\underline{\mathbf{U}} = [\mathbf{u}(\omega_1), \ldots, \mathbf{u}(\omega_S)]$.

$$\underline{\mathbf{U}} = \underline{\mathbf{Q}}\,\underline{\mathbf{R}}$$

the Gramian matrix can be expressed as

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- $\underline{\mathbf{G}}$ and $\underline{\mathbf{R}}$ have the same right-singular vector (exactly what is needed for MRI)
- $\underline{\mathbf{R}}$ can be built sequentially (modified Householder triangularization for gMRI)

Efficient way of storing the surrogate ($\mathbf{e}_j$ canonical basis vector)

$$\mathring{\mathbf{u}}(\omega) = \sum_{j=1}^{S} \frac{q_j \mathbf{e}_j}{\omega - \omega_j} \Big/ \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$$

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The original surrogate can be recovered with

$$\tilde{\mathbf{u}}(\omega) = \underline{\mathbf{U}} \mathring{\mathbf{u}}(\omega)$$

Neat helper quantity ($\mathbf{r}_j = \underline{\mathbf{R}}[:, S]$ from QR-decomposition)

$$\hat{\mathbf{u}}(\omega) = \sum_{j=1}^{S} \frac{q_j \mathbf{r}_j}{\omega - \omega_j} \Big/ \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$$

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Proposed way of approximating relative error in gMRI

$$\frac{\|\mathbf{u}(\omega_{t+1}) - \tilde{\mathbf{u}}_t(\omega_{t+1})\|_M}{\|\mathbf{u}(\omega_{t+1})\|_M} \approx \frac{\|\mathbf{r}_{t+1} - \hat{\mathbf{u}}_t(\omega_{t+1})\|}{\|\hat{\mathbf{u}}_t(\omega_{t+1})\|}$$

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The original surrogate can be recovered with

$$\tilde{\mathbf{u}}(\omega) = \underline{\mathbf{Q}}\hat{\mathbf{u}}(\omega)$$

We want to find ω, such that

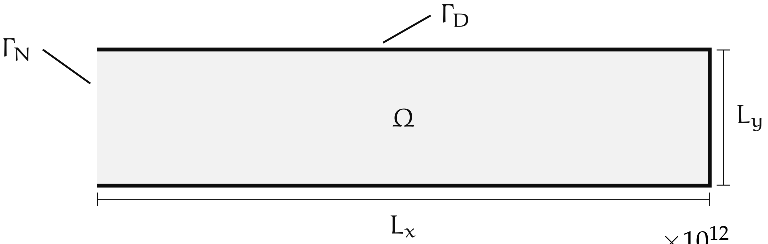
$$0 = Q(\omega) = \sum_{i=1}^{S} \frac{q_i}{\omega - \omega_i}$$

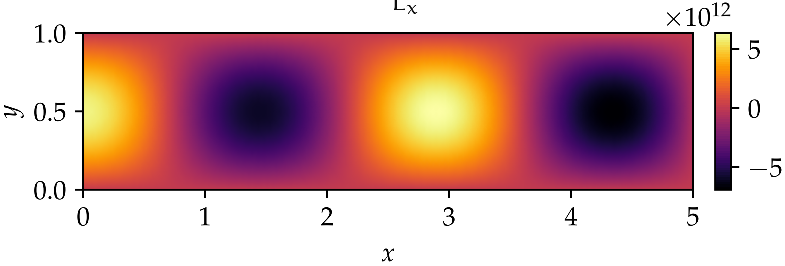
Equivalent eigenvalue problem

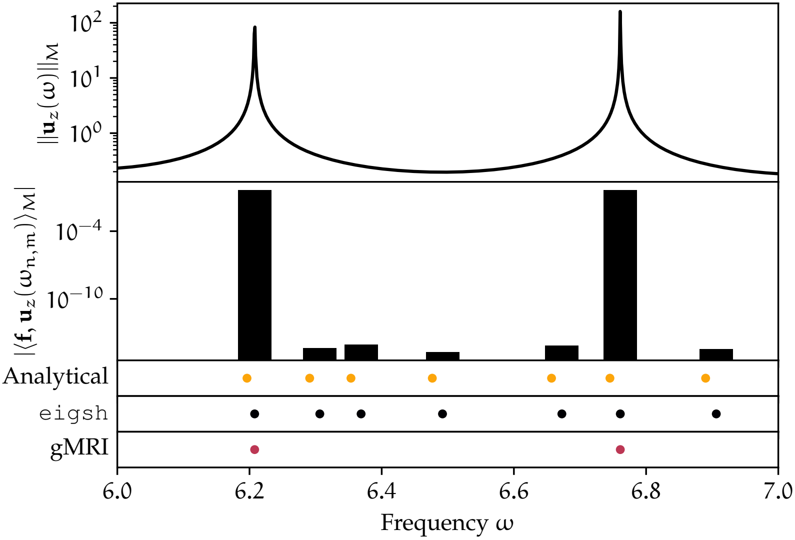
$$\underline{\mathbf{A}}\mathbf{w} = \omega\underline{\mathbf{B}}\mathbf{w}$$

with

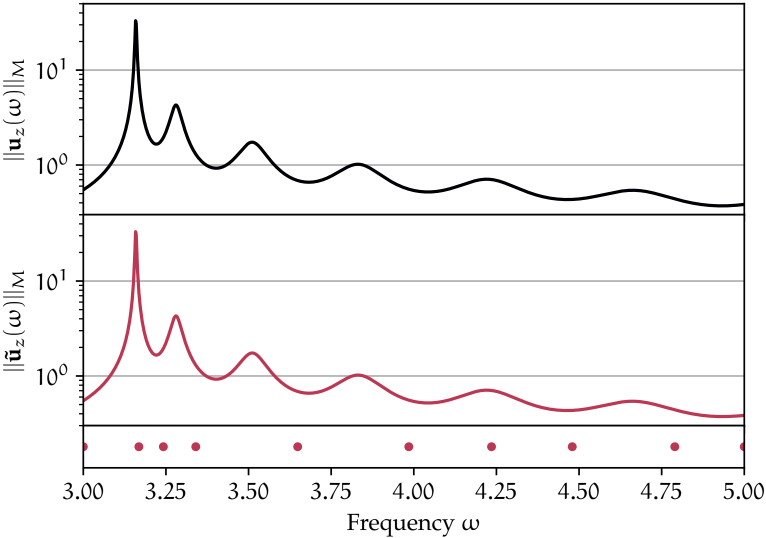
$$\underline{\mathbf{A}} = \begin{pmatrix} 0 & q_1 & q_2 & \cdots & q_S \\ 1 & \omega_1 & & & \\ 1 & & \omega_2 & & \\ \vdots & & & \ddots & \\ 1 & & & & \omega_S \end{pmatrix} \text{ and } \underline{\mathbf{B}} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

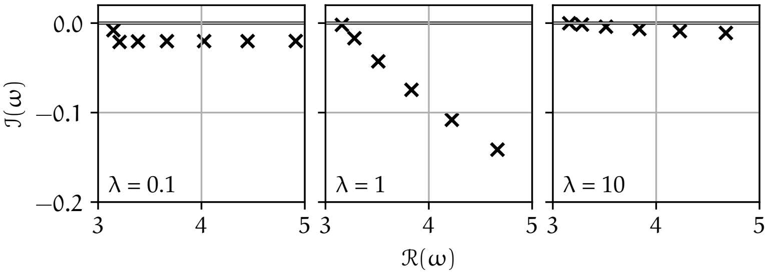

Γ_D
Γ_N
Ω
L_y
L_x


$\times 10^{12}$
1.0
0.5
0.0
y
0
1
2
3
4
5
x
5
0
−5

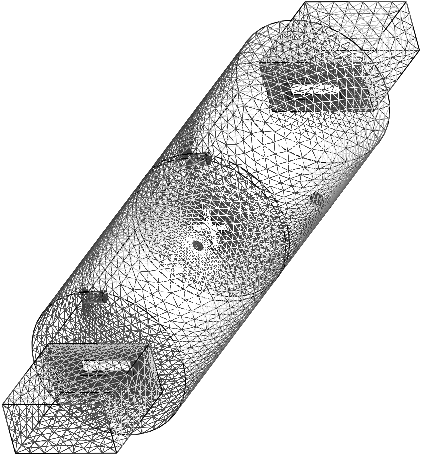


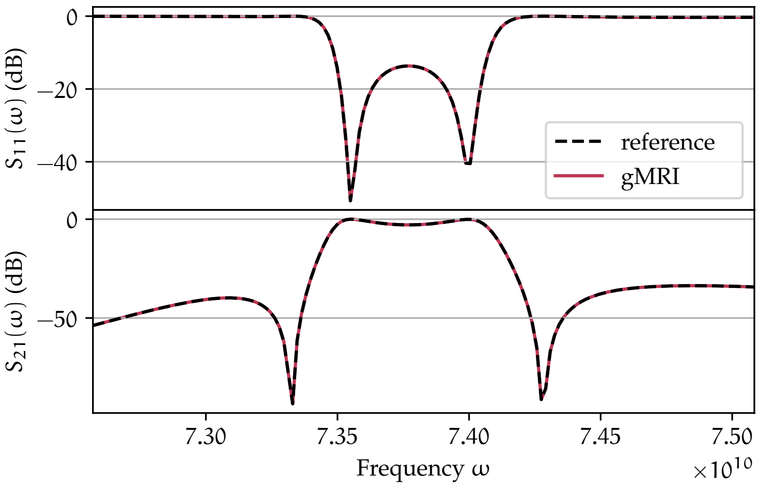
10^2
10^1
10^0
$\|\mathbf{u}_z(\omega)\|_M$
10^{-4}
10^{-10}
$|\langle \mathbf{f}, \mathbf{u}_z(\omega_{n,m})\rangle_M|$
Analytical
eigsh
gMRI
6.0
6.2
6.4
6.6
6.8
7.0
Frequency ω


10^1
10^0
$\|\mathbf{u}_z(\omega)\|_M$
$\|\tilde{\mathbf{u}}_z(\omega)\|_M$
3.00
3.25
3.50
3.75
4.00
4.25
4.50
4.75
5.00
Frequency ω


0.0
−0.1
−0.2
$\mathcal{I}(\omega)$
λ = 0.1
λ = 1
λ = 10
3
4
5
$\mathcal{R}(\omega)$

Dual-mode circular waveguide filter




0
−20
−40
$S_{11}(\omega)$ (dB)
reference
gMRI
0
−50
$S_{21}(\omega)$ (dB)
7.30
7.35
7.40
7.45
7.50
Frequency ω
$\times 10^{10}$

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