EPFL

PROJECT CSE I

# Notes

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# 1 FINITE ELEMENT METHOD

## 1.1 THE POISSON EQUATION

*Taken from FEniCS manual (too lazy for bibtex...)*

We aim to solve an equation of the form

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}) \tag{1.1}$$

on a domain $\mathbf{x} \in \Omega$, with a solution $u(\mathbf{x})$ that satisfies a certain boundary condition $u(\mathbf{x}) = u_d(\mathbf{x})$ for all $x \in \partial\Omega$ that lie on the border of $\Omega$.

To do this, we first convert this equation to its weak form by multiplying both sides with a arbitrary test function $v(\mathbf{x})$, which vanishes on the border (i.e. $v(mathbfx) = 0, \forall \mathbf{x} \in \partial\Omega$), and by then integrating over all of $\Omega$:

$$-\int_\Omega \Delta u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \tag{1.2}$$

We may now rearrange the gradient product rule $\nabla(ab) = (\nabla a)b + a(\nabla b)$ and Gauss' theorem (as long as $v(x)$ is differentiable in a neighborhood of $\Omega$) combined with the fact that $v(\mathbf{x})$ vanishes on $\partial\Omega$ to convert the right-hand side to

$$\begin{aligned} -\int_\Omega \Delta u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} &= -\int_\Omega \nabla(\nabla u(\mathbf{x}) v(\mathbf{x})) d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \\ &= -\int_{\partial\Omega} \nabla u(\mathbf{x}) v(\mathbf{x}) d\boldsymbol{\omega} + \int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \\ &= \int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \end{aligned} \tag{1.3}$$

Consequently, the weak formulation of the problem is to find $u(\mathbf{x})$, such that for arbitrary $v(\mathbf{x})$, we have

$$\int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \tag{1.4}$$

To simplify and generalize the notation, we may use the linear form $L : V \to \mathbb{R}$ as

$$L(v) = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \tag{1.5}$$

and also the bilinear form $a : V \times V \to \mathbb{R}$

$$a(u, v) = \int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \tag{1.6}$$

## 1.2 Example: One dimensional poisson equation

*Initial idea taken from Wikipedia article about FEM.*

To illustrate the choice of basis functions, we will now consider the simple one dimensional case $\Omega = [a, b]$, such that the weak formulation of the problem turns into

$$\int_a^b u'(x)v'(x)dx = \int_a^b f(x)v(x)dx \tag{1.7}$$

We now subdivide the domain $[a, b]$ into $M$ subintervals, each of length $h = (b - a)/M$, with nodes at $x_k = a + hk, k \in \{0, 1, \ldots, M\}$. We proceed to choose as the basis functions the class of the piecewise linear Lagrange interpolating polynomials on $[x_k, x_{k+1}], k \in \{0, 1, \ldots, M\}$, defined as

$$v_k(x) = \frac{x - x_{k-1}}{x_k - x_{k-1}} \mathbf{1}_{\{x \in [x_{k-1}, x_k]\}} + \frac{x_{k+1} - x}{x_{k+1} - x_k} \mathbf{1}_{\{x \in [x_k, x_{k+1}]\}} \tag{1.8}$$

If we now interpolate $f(x)$ and $u(x)$ as piecewise linear Lagrange polynomaials, we get the representation

$$\begin{aligned} f(x) &\approx \sum_{i=1}^{M} f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + f(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}} \\ &= \sum_{i=1}^{M-1} f(x_i) v_i(x) \end{aligned} \tag{1.9}$$

and analogously

$$u(x) = \sum_{i=1}^{M-1} u(x_i) v_i(x) \tag{1.10}$$

We now restricted ourselves to the discrete variational formulation of the problem

$$\sum_{i=1}^{M-1} u(x_i) \int_a^b v_i'(x) v_j'(x) dx = \sum_{i=1}^{M-1} f(x_i) \int_a^b v_i(x) v_j(x) dx \tag{1.11}$$

which needs to be satisfied for all $j \in \{0, 1, \ldots, M\}$.

This equation can be rewritten in terms of two matrices **K** and **L** which we define as

$$K_{ij} = \int_a^b v_i(x) v_j(x) dx \tag{1.12}$$

$$L_{ij} = \int_a^b v_i'(x) v_j'(x) dx \tag{1.13}$$

such that we get

$$\sum_{i=1}^{M-1} u(x_i)L_{ij} = \sum_{i=1}^{M-1} f(x_i)K_{ij} \tag{1.14}$$

Notice, that we only need the entries $K_{ij}$ and $L_{ij}$ with $i \in \{1, 2, \dots, M-1\}$, since we already know the boundary conditions of $u(x)$ at $x = x_0$ and $x = x_M$.

We realize, that the $L_2$ inner product of $\nu_i(x)$ with $\nu_j(x)$ (and consequently also the one of $\nu_i'(x)$ with $\nu_j'(x)$) is zero for all $|i-j| > 1$, hence, we distinguish two different cases.

1. $i = j$: Here, the inner product turns out to be

$$\begin{aligned}
\int_a^b \nu_i(x)\nu_i(x)dx &= \int_a^b \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2 \mathbf{1}_{\{x\in[x_{i-1},x_i]\}} + \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)^2 \mathbf{1}_{\{x\in[x_i,x_{i+1}]\}} dx \\
&= 2\int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2 dx \\
&= \frac{2}{h^2}\int_{x_{i-1}-x_{i-1}}^{x_i - x_{i-1}} u^2 du \\
&= \frac{2}{h^2}\frac{1}{3}h^3 \\
&= \frac{2h}{3}
\end{aligned} \tag{1.15}$$

and for the derivatives it is

$$\begin{aligned}
\int_a^b \nu_i'(x)\nu_i'(x)dx &= \int_a^b \left(\frac{1}{x_i - x_{i-1}}\right)^2 \mathbf{1}_{\{x\in[x_{i-1},x_i]\}} + \left(\frac{-1}{x_{i+1} - x_i}\right)^2 \mathbf{1}_{\{x\in[x_i,x_{i+1}]\}} dx \\
&= 2\int_{x_{i-1}}^{x_i} \left(\frac{1}{x_i - x_{i-1}}\right)^2 dx \\
&= \frac{2}{h^2}\int_0^h 1du \\
&= \frac{2}{h}
\end{aligned} \tag{1.16}$$

2. $|i-j| = 1$: Here, we can limit ourselves to the case where $j = i + 1$, since the

other case is fully symmetric. We calculate

$$
\begin{aligned}
\int_a^b v_i(x)v_{i+1}(x)dx &= \int_a^b \frac{x_{i+1}-x}{x_{i+1}-x_i}\frac{x-x_i}{x_{i+1}-x_i}\mathbf{1}_{\{x\in[x_i,x_{i+1}]\}}dx \\
&= \int_{x_i}^{x_{i+1}} \frac{x_{i+1}-x}{x_{i+1}-x_i}\frac{x-x_i}{x_{i+1}-x_i}dx \\
&= \frac{1}{h^2}\int_{x_i-x_i}^{x_{i+1}-x_i}(x_{i+1}-x_i-u)udu \\
&= \frac{1}{h^2}\int_0^h (h-u)udu \\
&= \frac{1}{h^2}(\frac{h^3}{2}-\frac{h^3}{3}) \\
&= \frac{h}{6}
\end{aligned} \tag{1.17}
$$

and for the derivative it is

$$
\begin{aligned}
\int_a^b v_i'(x)v_{i+1}'(x)dx &= \int_a^b \frac{-1}{x_{i+1}-x_i}\frac{1}{x_{i+1}-x_i}\mathbf{1}_{\{x\in[x_i,x_{i+1}]\}}dx \\
&= -\frac{1}{h^2}\int_{x_i}^{x_{i+1}} 1dx \\
&= -\frac{1}{h}
\end{aligned} \tag{1.18}
$$

Now, using the previously defined matrices $\mathbf{K}_{ij}$ and $\mathbf{L}_{ij}$, we get the matrix equation

$$\mathbf{L}u = \mathbf{K}f \tag{1.19}$$

with

$$u = (u_0, u(x_1), \dots, u_M)^T \tag{1.20}$$

$$f = (f(x_0), f(x_1), \dots, f(x_M))^T \tag{1.21}$$

$$\mathbf{L} = \begin{pmatrix} 1 & & & & \\ \frac{2}{h} & -\frac{1}{h} & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & -\frac{1}{h} & \frac{2}{h} & \ddots & \\ & & -\frac{1}{h} & \ddots & -\frac{1}{h} \\ & & & \ddots & \frac{2}{h} \\ & & & & 1 \end{pmatrix} \tag{1.22}$$

$$\mathbf{K} = \begin{pmatrix} \frac{u_0}{f(x_0)} & & & & \\ \frac{2h}{3} & \frac{h}{6} & & & \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & & \\ & \frac{h}{6} & \frac{2h}{3} & \ddots & \\ & & \frac{h}{6} & \ddots & \frac{h}{6} \\ & & & \ddots & \frac{2h}{3} \\ & & & & \frac{u_M}{f(x_M)} \end{pmatrix} \tag{1.23}$$

$$\tag{1.24}$$

Here, we have adjusted the first rows in $\mathbf{L}$ and $\mathbf{K}$, such that the boundary conditions are necessarily satisfied. To obtain the finite element solution, we simply solve this linear system.

## 2 Maxwell's equations

Let $\mathbf{E} = (E_1, E_2, E_3)^T$ denote the electric field, $\mathbf{B} = (B_1, B_2, B_3)^T$ the magnetic field strength, and $\mathbf{j} = (j_1, j_2, j_3)^T$ the electric current density. We suppose Maxwell's equations hold:

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{2.3}$$

$$\nabla \times (\mu^{-1} \mathbf{B}) = \partial_t (\epsilon \mathbf{E}) + \mathbf{j} \tag{2.4}$$

We can therefore write $\mathbf{B} = \nabla \times \mathbf{A}$ for some vector potential $\mathbf{A}$, and $\mathbf{E} = -\nabla \phi - \partial_t \mathbf{A}$ for some scalar potential $\phi$. Plugging these identities into (2.4), we get

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{A}) = \partial_t \nabla \phi - \partial_t^2 \mathbf{A} + \mathbf{j} \tag{2.5}$$

We may choose $\nabla\phi = 0$ as a gauge, and introduce a harmonic time dependence of $\mathbf{A}$ and $\mathbf{j}$ with frequencies $\omega$, such that $\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}) \exp(i\omega t)$ and $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x}) \exp(i\omega t)$. Plugging this into (2.5) yields us

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{A}) - \omega^2 \mathbf{A} = \mathbf{j} \tag{2.6}$$

We reduce this equation to its weak formulation, by multiplying it with a vector-valued function $\mathbf{v} \in H_{\text{curl}}(\Omega)$, where we denoted

$$H_{\text{curl}}(\Omega) = \{u : \Omega \to \mathbb{C}, \text{ such that } u \in L^2(\mathbb{C}), \nabla \times u \in L^2(\mathbb{C})\} \tag{2.7}$$

and by integrating over all of $\Omega$:

$$\int_\Omega (\nabla \times (\mu^{-1}\nabla \times \mathbf{A})) \cdot \mathbf{v} - \omega^2 \int_\Omega \mathbf{A} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} \tag{2.8}$$

To further simplify this expression, we will derive an identity for the scalar product of a vector-valued function $\mathbf{v}$ with the curl of a vector-valued function $\mathbf{u}$. For this, we use the completely antisymmetric tensor $\varepsilon_{ijk}$ (frequently referred to as the Levi-Civita tensor), to rewrite the k-th component of the curl as

$$(\nabla \times \mathbf{u})_k = \sum_i \sum_j \varepsilon_{ijk} \partial_i u_j \tag{2.9}$$

where $\partial_i$ denotes the partial derivative with respect to the i-th coordinate direction. Rewriting the scalar product as a sum, we apply the product rule to get

$$\begin{aligned}
(\nabla \times \mathbf{u}) \cdot \mathbf{v} &= \sum_k (\nabla \times \mathbf{u})_k v_k \\
&= \sum_k (\sum_i \sum_j \varepsilon_{ijk} \partial_i u_j) v_k \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{ijk} u_j v_k) - \sum_k \sum_i \sum_j u_j (\varepsilon_{ijk} \partial_i v_k) \\
&= \sum_i \partial_i (\mathbf{u} \times \mathbf{v})_i - \sum_j u_j (\nabla \times \mathbf{v})_j \\
&= \nabla \cdot (\mathbf{u} \times \mathbf{v}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{2.10}$$

Consequently, we may rewrite the double curl term in the weak formulation as

$$\begin{aligned}
\int_\Omega (\nabla \times (\mu^{-1}\nabla \times \mathbf{A})) \cdot \mathbf{v} &= \int_\Omega \nabla \cdot ((\mu^{-1}\nabla \times \mathbf{A}) \times \mathbf{v}) - \int_\Omega (\mu^{-1}\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\mu^{-1}\nabla \times \mathbf{A}) \times \mathbf{v}) \cdot \mathbf{n} - \int_\Omega (\mu^{-1}\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{2.11}$$

We will now have a look at what conditions $\mathbf{v}$ needs to satisfy, such that the boundary term (first integral) vanishes, and we would end up with

$$-\int_\Omega (\mu^{-1}\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_\Omega \mathbf{A} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} \tag{2.12}$$

Let $\mathbf{n}$ and $\mathbf{t}$ denote the normal and tangential vectors to $\partial\Omega$ at a point $\mathbf{x} \in \partial\Omega$. For the boundary term to vanish, we require

$$((\mu^{-1}\nabla \times \mathbf{A}) \times \mathbf{v}) \cdot \mathbf{n} = 0 \tag{2.13}$$

In other words, we need

$$((\mu^{-1}\nabla \times \mathbf{A}) \times \mathbf{v}) \perp \mathbf{n} \tag{2.14}$$

which is satisfied for all $\mathbf{v} \parallel \mathbf{n}$ on $\partial\Omega$.
[Todo: Formally derive this condition using Levi-Civita tensor]

# 3 Weak derivative

*Taken from Quarteroni: Introduction to Finite Elements Method*

Let $\Omega \subset \mathbb{R}^d$ open. The support of $f : \Omega \to \mathbb{R}$ is defined as

$$\text{supp}(f) = \overline{\{x \in \Omega \mid f(x) \neq 0\}} \tag{3.1}$$

f has compact support, if there exists a compact subset $K \subset \Omega$, such that supp(f) $\subset K$, and define

$$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) \mid f \text{ has compact support}\} \tag{3.2}$$

(If I remember correctly, extending this notion to $f \in C^1(\Omega)$ should yield an almost identical treatment, unless we also include higher order (weak) partial derivatives). Let $T : \mathcal{D} \to \mathbb{R}$, $\varphi \mapsto \langle T, \varphi \rangle = T(\varphi)$ be a linear map. We say that T is continuous, if

$$\lim_{n\to\infty} \langle T, \varphi_n \rangle = \langle T, \varphi \rangle \tag{3.3}$$

with $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ converging to $\varphi$. Such (linear and continuous) maps are called distribution on $\mathcal{D}(\Omega)$, and they form the space of distributions $\mathcal{D}'(\Omega)$.

The (weak) partial coordinate-derivatives of T (namely $\partial_i T,\ i \in \{1, \dots, d\}$) are characterized by distributions that satisfy

$$\langle \partial_i T, \varphi \rangle = -\langle T, \partial_i \varphi \rangle \tag{3.4}$$

for all $\varphi \in \mathcal{D}(\Omega)$.

Interesting for us is mainly the following case: Given a function $f \in L^2(\Omega)$, we define a distribution $T_f \in \mathcal{D}'(\Omega)$ to be

$$\langle T_f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx \tag{3.5}$$

for all $\varphi \in \mathcal{D}(\Omega)$.

This allows us to define a weak derivative to functions that are (in the classical sense) not differentiable (i.e. not in $C^1(\Omega)$). Consider for example the absolute value function $|\cdot| \in L_2(K)$ where $K \subset \mathbb{R}$ is compact. Since

$$\begin{aligned}
\int_K (\partial_x |x|)\varphi(x)\,dx &= -\int_K |x|\varphi'(x)\,dx \\
&= -\int_{K\cap\mathbb{R}_+} x\varphi'(x)\,dx - \int_{K\cap\mathbb{R}_-} (-x)\varphi'(x)\,dx \\
&= \int_{K\cap\mathbb{R}_+} \varphi(x)\,dx + \int_{K\cap\mathbb{R}_-} (-1)\varphi(x)\,dx \\
&= \int_K \operatorname{sign}(x)\varphi(x)\,dx
\end{aligned} \tag{3.6}$$

we may conclude that the weak derivative of the absolute value function is therefore the signum function. Notice, how the derivative of the absolute value function is only not well-defined at $x = 0$, i.e. on a set of zero measure. This nuisance is circumvented when talking about the weak derivative, since the measure zero sets have zero integral.

# 4 IDEAS

What might be really interesting is to instead look at the problem in space-time using the Maxwell tensor

$$\mathbb{F} = \begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{bmatrix} \tag{4.1}$$

In the covariant formulation of the Maxwell theory, the inhomogeneous Maxwell equations reduce to a single equation

$$\partial_a F^{ab} = -J^b \tag{4.2}$$

with the four current density $\mathbf{J} = (\mu c\rho, \mu \mathbf{j})$. The weak formulation of the problem could then be stated as (using Einstein's sum convention, i.e. summing over repeated indices)

$$\int_{\Omega\times\mathbb{R}} F^{ab}\partial_a v_b = \int_{\Omega\times\mathbb{R}} J^b v_b \tag{4.3}$$

where boundary conditions are yet to be determined. If we somehow would manage to find a suitable function space for the four-dimensional $\mathbf{v}$, it might be possible to find both $\mathbf{E}$ and $\mathbf{B}$ from a finite element method.