Minimal rational interpolation for time-harmonic Maxwell's equations

June 24, 2022

Fabio Matti

- Problem formulation
- Finite element method
- Minimal rational interpolation
- Example applications
- Conclusion and outlook

Time-harmonic vector potential $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \exp(i\omega t)$.

$$\mathbf{B} = \nabla \times \mathbf{u} \qquad \text{(Magnetic field)}$$

$$\mathbf{E} = -i\omega\mathbf{u} \qquad \text{(Electric field)}$$

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Time-harmonic potential equation

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) - \epsilon\omega^2\mathbf{u} = \mathbf{j}$$

Want to approximate $\mathbf{u} : \mathbb{C} \ni \omega \mapsto \mathbf{u}(\omega) \in H_{\text{curl}}(\Omega)$ with

$$H_{\text{curl}}(\Omega) = \{\mathbf{v} : \Omega \to \mathbb{C}^3, \text{ such that } \mathbf{v} \in L_2(\Omega)^3, \nabla \times \mathbf{v} \in L_2(\Omega)^3\}$$

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Weak formulation of the time-harmonic potential equation

Find $\mathbf{u} \in H_{\text{curl}}(\Omega)$, such that

$$\int_\Omega \langle \mu^{-1} \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle - \omega^2 \int_\Omega \epsilon \langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \langle \mathbf{j}, \mathbf{v} \rangle + \int_{\partial\Omega} \langle \mathbf{g}, \mathbf{v} \rangle$$

for all $\mathbf{v} \in H_{\text{curl}}$, where $\mathbf{g} = (\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}$.

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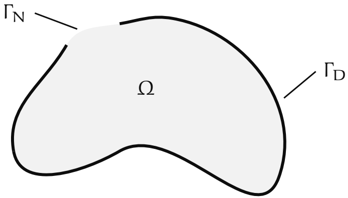
FEniCS [2] with Nédélec elements of the first kind

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \epsilon \omega^2 \mathbf{u} = \mathbf{j}$$

Perfectly conducting boundary

$$\mathbf{g} = \mathbf{0} \text{ and } \mathbf{E} \times \mathbf{n} = \mathbf{0}, \text{ on } \Gamma_D$$



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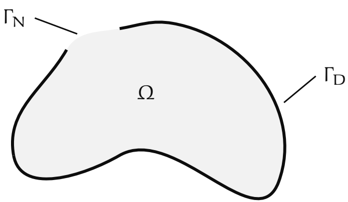

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Imperfectly conducting boundary [3]

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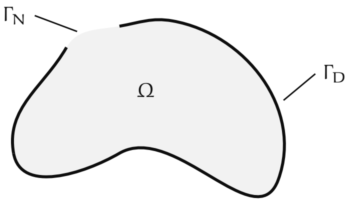

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Rational surrogate

$$\tilde{\mathbf{u}}(\omega) = \frac{\mathbf{P}(\omega)}{Q(\omega)} = \sum_{j=1}^{S} \frac{\mathbf{p}_j}{\omega - \omega_j} \Big/ \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$$

in barycentric coordinates with support points $\omega_1, \omega_2, \ldots, \omega_S$.

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Interpolation property

$$\tilde{\mathbf{u}}(\omega_j) = \mathbf{u}(\omega_j),\ \forall j \in \{1, 2, \ldots, S\}$$

if $\mathbf{p}_j = q_j \mathbf{u}(\omega_j), \forall j$.

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3. Define $\mathbf{q} = (q_1, q_2, \ldots, q_S)^T = \underline{\mathbf{V}}[:, S]$
4. Define the minimal rational surrogate $\tilde{\mathbf{u}}(\omega) = \mathbf{P}(\omega)/Q(\omega)$ with

$$\mathbf{P}(\omega) = \sum_{j=1}^{S} \frac{q_j \mathbf{u}(\omega_j)}{\omega - \omega_j} \quad \text{and} \quad Q(\omega) = \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$$

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$$\omega^{(t+1)} = \operatorname{argmin}_{\omega \in \Omega_{\text{test}}} |Q^{(t)}(\omega)|$$

   from Ω_{test} to build the minimal rational surrogate $\tilde{\mathbf{u}}_{t+1}$ based on $\mathbf{u}(\omega^{(0)}), \mathbf{u}(\omega^{(1)}), \ldots, \mathbf{u}(\omega^{(t+1)})$ and increment t

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3. Stop when relative error

$$\|\mathbf{u}(\omega_{t+1}) - \tilde{\mathbf{u}}_t(\omega_{t+1})\|_M / \|\mathbf{u}(\omega_{t+1})\|_M$$

   is small enough

With the QR-decomposition of the snapshot matrix $\underline{\mathbf{U}} = [\mathbf{u}(\omega_1), \ldots, \mathbf{u}(\omega_S)]^\mathsf{T}$.

$$\underline{\mathbf{U}} = \underline{\mathbf{Q}}\,\underline{\mathbf{R}}$$

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- Improved conditioning of SVD with $\underline{\mathbf{R}}$
- $\underline{\mathbf{R}}$ can be built sequentially (modified Householder triangularization for gMRI [5])

Alternative representations of the surrogate ($\mathbf{r}_j = \underline{\mathbf{R}}[:, j]$)

$$\mathring{\mathbf{u}}(\omega) = \sum_{j=1}^{S} \frac{q_j \mathbf{e}_j}{\omega - \omega_j} \Big/ \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$$

$$\hat{\mathbf{u}}(\omega) = \sum_{j=1}^{S} \frac{q_j \mathbf{r}_j}{\omega - \omega_j} \Big/ \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$$

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Proposed way of approximating relative error in gMRI

$$\frac{\|\mathbf{u}(\omega_{t+1}) - \tilde{\mathbf{u}}_t(\omega_{t+1})\|_M}{\|\mathbf{u}(\omega_{t+1})\|_M} \approx \frac{\|\mathbf{r}_{t+1} - \hat{\mathbf{u}}_t(\omega_{t+1})\|}{\|\hat{\mathbf{u}}_t(\omega_{t+1})\|}$$

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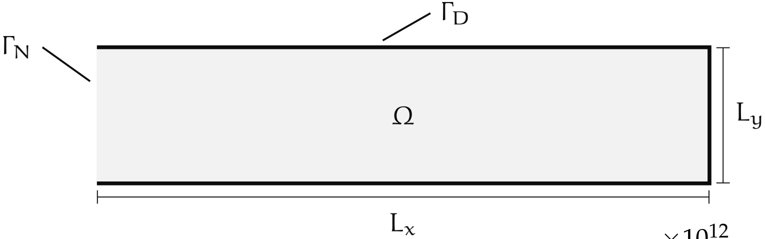
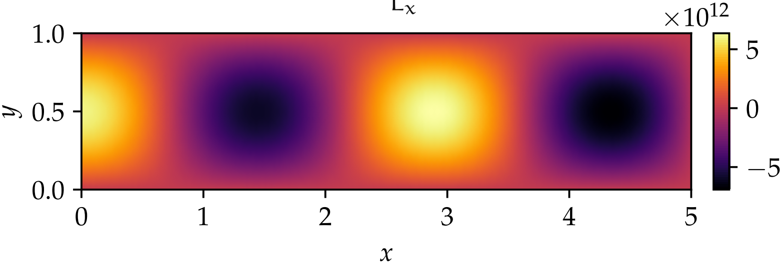
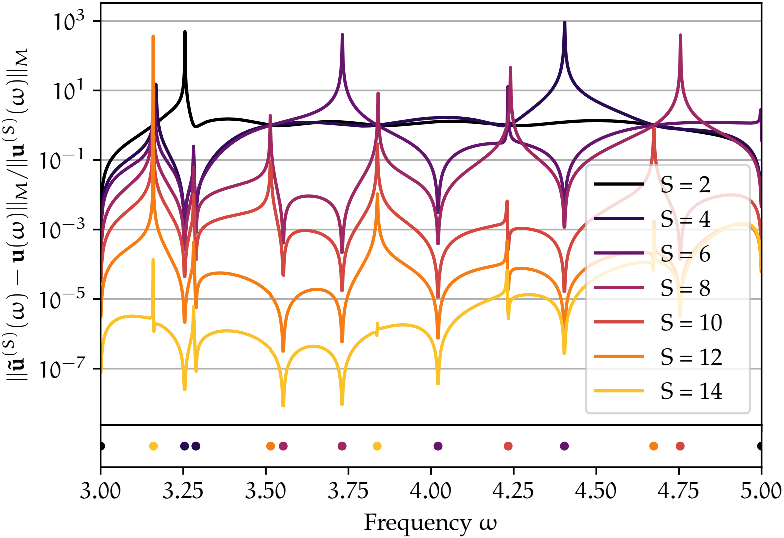
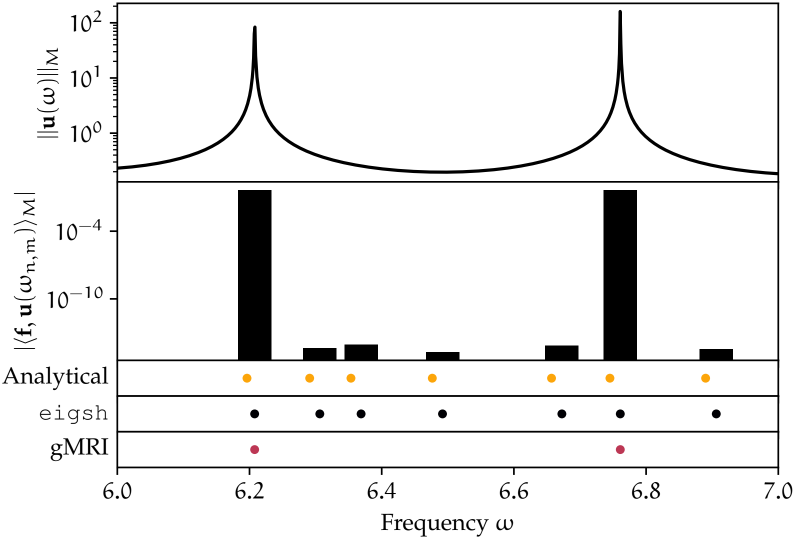


Γ_D
Γ_N
Ω
L_y
L_x

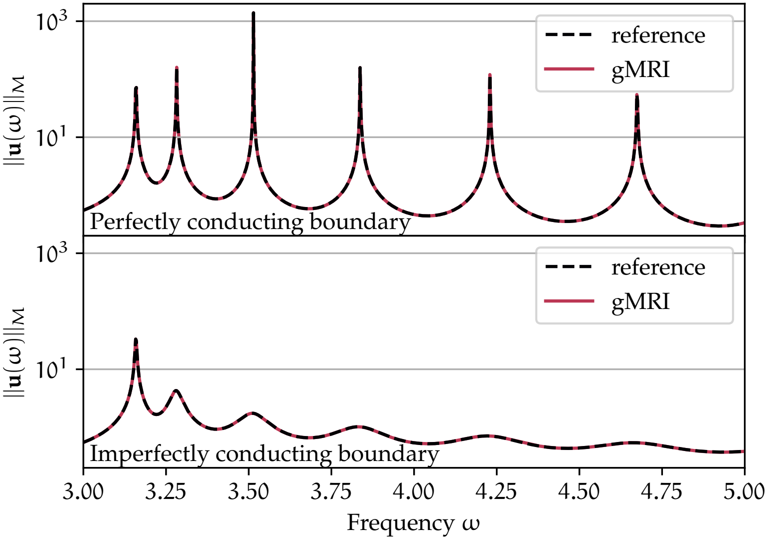


$\times 10^{12}$
1.0
0.5
0.0
y
0
1
2
3
4
5
x
5
0
−5


10^3
10^1
10^{-1}
10^{-3}
10^{-5}
10^{-7}
$\|\tilde{\mathbf{u}}^{(S)}(\omega) - \mathbf{u}(\omega)\|_M / \|\mathbf{u}^{(S)}(\omega)\|_M$
S = 2
S = 4
S = 6
S = 8
S = 10
S = 12
S = 14
3.00
3.25
3.50
3.75
4.00
4.25
4.50
4.75
5.00
Frequency ω

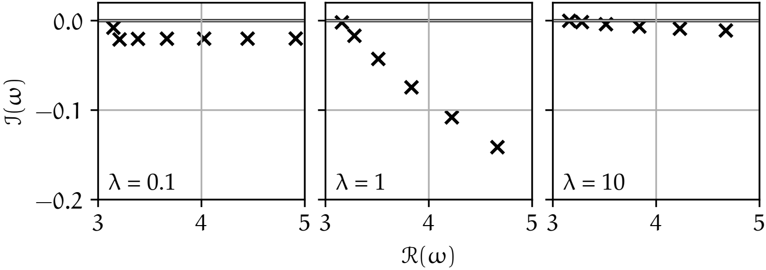


10^2
10^1
10^0
$\|\mathbf{u}(\omega)\|_M$
10^{-4}
10^{-10}
$|\langle \mathbf{f}, \mathbf{u}(\omega_{n,m}) \rangle_M|$
Analytical
eigsh
gMRI
6.0
6.2
6.4
6.6
6.8
7.0
Frequency ω



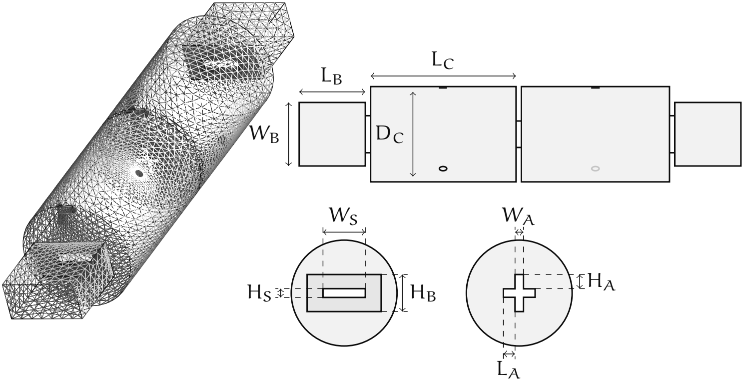
10^3
10^1
$\|\mathbf{u}(\omega)\|_M$
reference
gMRI
Perfectly conducting boundary
10^3
10^1
$\|\mathbf{u}(\omega)\|_M$
reference
gMRI
Imperfectly conducting boundary
3.00
3.25
3.50
3.75
4.00
4.25
4.50
4.75
5.00
Frequency ω

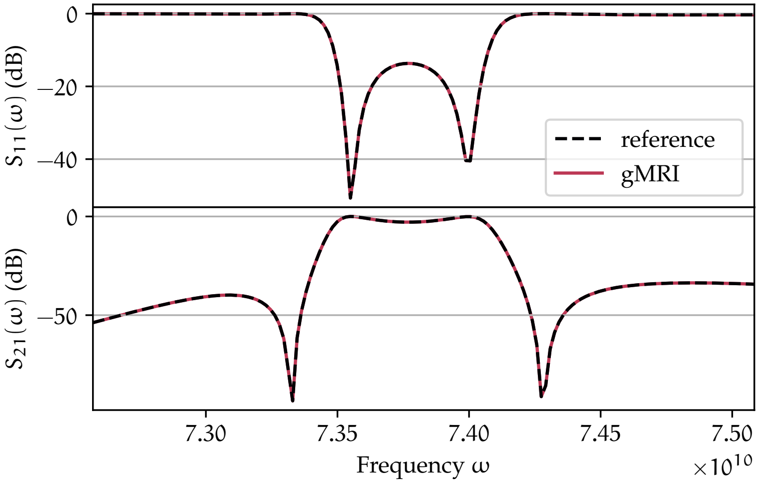
Resonances are shifted into the complex plane



$$\mathbf{g} = i\omega\lambda(\mathbf{n} \times \mathbf{u}) \times \mathbf{n}$$

Dual-mode circular waveguide filter (DMCWF)




0
−20
−40
$S_{11}(\omega)$ (dB)
reference
gMRI
0
−50
$S_{21}(\omega)$ (dB)
7.30
7.35
7.40
7.45
7.50
Frequency ω
$\times 10^{10}$

- Simplicity of the algorithms

Conclusion and outlook

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- Robust and fast for finding resonant frequencies

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- Problem with highly symmetric meshes

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- Simplicity of the algorithms
- Robust and fast for finding resonant frequencies
- Problem with highly symmetric meshes
- Exact dimensions and reference needed for DMCWF

[1] F. Bonzzoni, D. Pradovera, and M. Ruggeri. Rational-based model order reduction of helmholtz frequency response problems with adaptive finite element snapshots. 2021. doi: 10.48550/arXiv.2112.04302.

[2] H. P. Langtangen and A. Logg. *Solving PDEs in Python: The FEniCS Tutorial I*. Springer, 2016. ISBN 978-3-319-52461-0. doi: 10.1007/978-3-319-52462-7.

[3] P. Monk. *Finite Element Methods for Maxwell's Equations*. Oxford Science Publications, 2003. ISBN 0-19-850888-3.

[4] D. Pradovera and F. Nobile. Frequency-domain non-intrusive greedy model order reduction based on minimal rational approximation. pages 159–167, 2021. doi: 10.1007/978-3-030-84238-3_16.

[5] L. N. Trefethen. Householder triangularization of a quasimatrix. *IMA Journal of Numerical Analysis*, 30(4): 887–897, 2010. doi: 10.1093/imanum/drp018.

FEniCS [2] is used to obtain FEM solutions of the form

$$\mathbf{u}_h(\omega) = \sum_{i=1}^{N_h} u_i(\omega) \boldsymbol{\phi}_h^{(i)} \tag{1}$$

for a basis $\{\boldsymbol{\phi}_h^{(i)}\}_{i=1}^{N_h}$ of the finite dimensional subspace $H_{\text{curl},h}(\Omega) \subset H_{\text{curl}}(\Omega)$ (Nédélec finite elements of the first kind).

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$$\mathbf{u} = (u_1, u_2, \ldots, u_{N_h})^T$$

with the $L_2(\Omega)$ inner product in $H_{\text{curl},h}(\Omega)$ represented by

$$\langle \mathbf{u}, \mathbf{v} \rangle_M = \mathbf{u}^H \underline{\mathbf{M}} \mathbf{v}$$

and the norm

$$\|\mathbf{u}\|_M = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle_M}$$

Find ω, such that

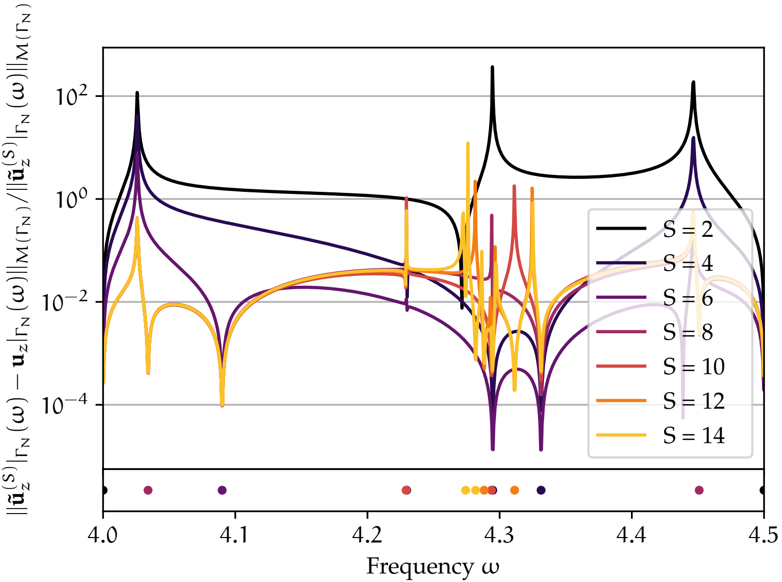
$$0 = Q(\omega) = \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$$

Equivalent eigenvalue problem

$$\underline{\mathbf{A}}\mathbf{w} = \omega\underline{\mathbf{B}}\mathbf{w}$$

with

$$\underline{\mathbf{A}} = \begin{pmatrix} 0 & q_1 & q_2 & \cdots & q_S \\ 1 & \omega_1 & & & \\ 1 & & \omega_2 & & \\ \vdots & & & \ddots & \\ 1 & & & & \omega_S \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{B}} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$


$\|\tilde{\mathbf{u}}_z^{(S)}|_{\Gamma_N}(\omega) - \mathbf{u}_z|_{\Gamma_N}(\omega)\|_{M(\Gamma_N)} / \|\tilde{\mathbf{u}}_z^{(S)}|_{\Gamma_N}(\omega)\|_{M(\Gamma_N)}$
10^2
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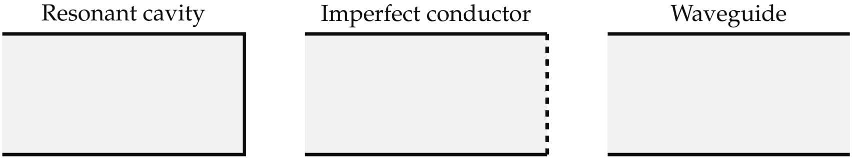
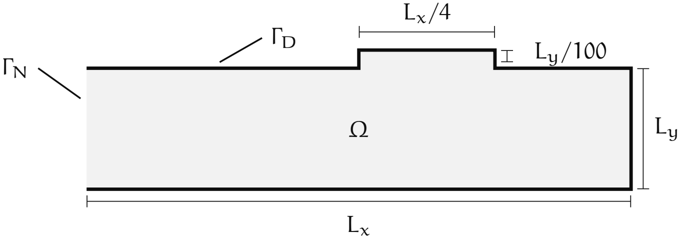
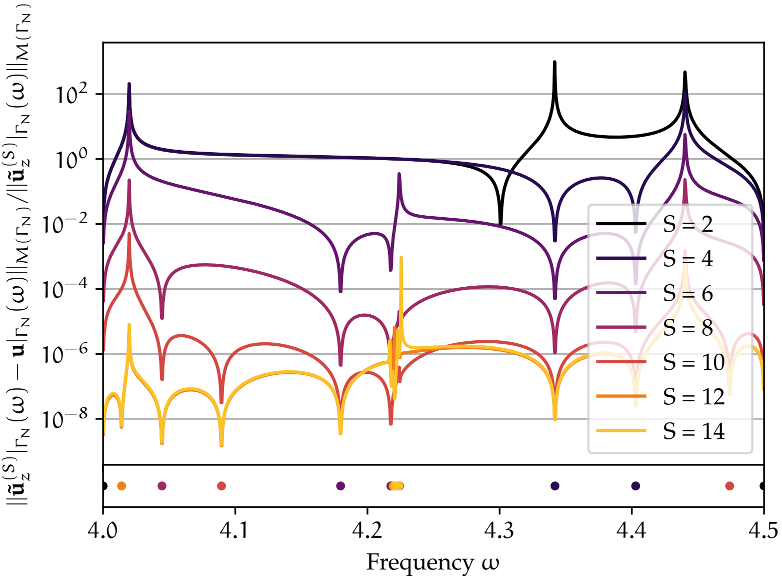
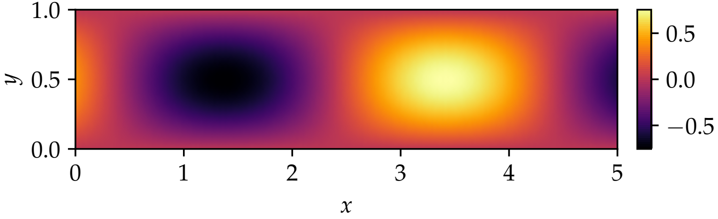
4.5
Frequency ω


Resonant cavity
Imperfect conductor
Waveguide

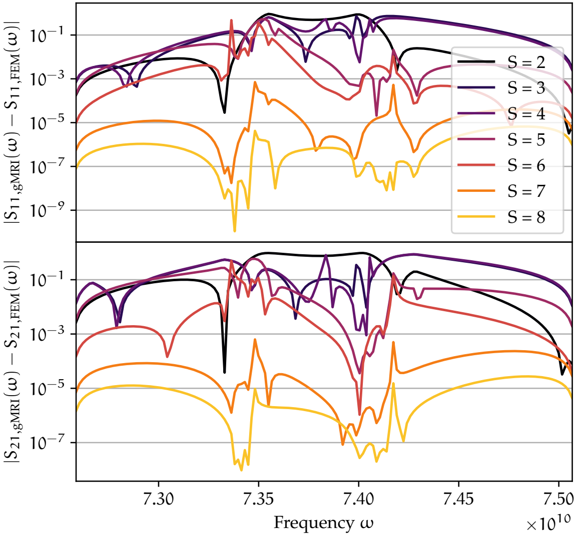

$L_x/4$
Γ_D
Γ_N
$L_y/100$
Ω
L_y
L_x


$\|\tilde{\mathbf{u}}_z^{(S)}|_{\Gamma_N}(\omega) - \mathbf{u}|_{\Gamma_N}(\omega)\|_{M(\Gamma_N)} / \|\tilde{\mathbf{u}}_z^{(S)}|_{\Gamma_N}(\omega)\|_{M(\Gamma_N)}$
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
S = 2
S = 4
S = 6
S = 8
S = 10
S = 12
S = 14
4.0
4.1
4.2
4.3
4.4
4.5
Frequency ω

	`eigsh`		gMRI	
DOF	$\bar{\Delta}$	t	$\bar{\Delta}$	t
713	1.950×10^{-2}	25.9 ± 1.1 ms	1.950×10^{-2}	61.9 ± 3.6 ms
7412	1.826×10^{-3}	199.0 ± 9.9 ms	1.827×10^{-3}	410.0 ± 16.8 ms
74722	1.817×10^{-4}	3.5 ± 0.1 s	1.820×10^{-4}	5.2 ± 0.2 s
745513	1.811×10^{-5}	75.0 ± 1.6 s	1.846×10^{-5}	104.0 ± 1.1 s


1.0
0.5
0.0
y
0
1
2
3
4
5
x
0.5
0.0
−0.5

	`eigs`	gMRI
DOF	t	t
713	57.8 ± 2.35 ms	62.8 ± 0.8 ms
7412	861.0 ± 42.4 ms	498.0 ± 11.7 ms
74722	21.8 ± 1.1 s	5.9 ± 0.3 s


$|S_{11,\mathrm{gMRI}}(\omega) - S_{11,\mathrm{FEM}}(\omega)|$
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
S = 2
S = 3
S = 4
S = 5
S = 6
S = 7
S = 8
$|S_{21,\mathrm{gMRI}}(\omega) - S_{21,\mathrm{FEM}}(\omega)|$
10^{-1}
10^{-3}
10^{-5}
10^{-7}
7.30
7.35
7.40
7.45
7.50
Frequency ω
$\times 10^{10}$