EPFL

PROJECT CSE I

Notes

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1 Finite element method

1.1 The general approach

Summarizes Chapter 1 in Quarteroni: Introduction to Finite Elements Method

Usually, the problems may be expressed in a simple equation

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = u_D & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where L denotes a linear differential operator (e.g. $-\Delta$ in the Poisson equation), u the solution to be found, and f is a source term independent of u. Some boundary condition u_D is imposed on the solution u.

However, equation (1.1) usually does not allow all physically significant solutions (particularly non-differentiable ones). Therefore, we convert the problem to a weak form. This is achieved by multiplying (1.1) with a test function $v \in V$, and integrating over the whole domain Ω:

$$\int_\Omega (Lu)v = \int_\Omega fv, \ \forall v \in V \tag{1.2}$$

Usually, integration by parts allows us to "transfer" the derivatives from the Lu term to the test function v, such that the order (with respect to the derivatives taken) is more "balanced" between the two terms. As a trade-off, a boundary term appears and needs to be eliminated to facilitate the finite element solution. This term can often be eliminated by restricting ourselves to test functions from a subspace $V' \subset V$. The weak problem then reads

$$\int_\Omega (L_u u)(L_v v) = \int_\Omega fv, \ \forall v \in V' \tag{1.3}$$

with a linear differential operator L_u acts on the solution u, and a linear differential operator L_v, appearing due to the integration by parts, acts on the test function v. For simplicity, we refer to the left-hand side as the bilinear form

$$a(u, v) = \int_\Omega (L_u u)(L_v v) \tag{1.4}$$

and the right-hand side as the linear form

$$F(v) = \int_\Omega fv \tag{1.5}$$

To solve (1.3), we prefer to look for approximate solutions u_h in a finite dimensional space V_h with $\dim(V_h) = N_h$ (what role does V_h', the space wherein v lies to satisfy the boundary conditions, play? as far as I can tell V_h' is the subspace of V_h, in which all functions vanish at the boundary where u is known). Choosing a basis $\{\varphi_i\}_{i \leqslant N_h}$ then allows us to represent the approximate solution as

$$u_h = \sum_{j \leqslant N_h} u_j \varphi_j \tag{1.6}$$

for some coefficients u_i that need to be determined. We thus see, that (1.3) turns into a linear system, since we only need to test the equality for the basis elements φ_i of V_h:

$$\sum_{j=1}^{N_h} u_j a(\varphi_j, \varphi_i) = F(\varphi_i), \ i \in \{1, \dots, N_h\} \tag{1.7}$$

If we write $A_{ij} = a(\varphi_j, \varphi_i)$ and $\mathbf{F} = (F(\varphi_1), \dots, F(\varphi_{N_h}))^T$, we have reduced the problem to finding $\mathbf{u} = (u_1, \dots, u_{N_h})^T$, such that

$$\mathbb{A}\mathbf{u} = \mathbf{F} \tag{1.8}$$

and have identified an approximate solution to (1.1) as $u \approx u_h = \sum_{j \leqslant N_h} u_j \varphi_j$.

Choosing the space V_h is fundamental to have an accurate method that gives a good approximation u_h of u. Furthermore, the choice of basis $\{\varphi_i\}_{i \leqslant N_h}$ influences how $\mathbb{A}$ ends up looking. Of particular interest are bases for which $a(\varphi_j, \varphi_i)$ vanishes for almost all i and j, thus yielding a sparse matrix $\mathbb{A}$. The choice of basis also controls the conditioning of $\mathbb{A}$ (need to find an example to illustrate this).

1.2 The Poisson equation

Taken from FEniCS manual (too lazy for bibtex...)

We aim to solve an equation of the form

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}) \tag{1.9}$$

on a domain $\mathbf{x} \in \Omega$, with a solution $u(\mathbf{x})$ that satisfies a certain boundary condition $u(\mathbf{x}) = u_d(\mathbf{x})$ for all $x \in \partial\Omega$ that lie on the border of Ω.

To do this, we first convert this equation to its weak form by multiplying both sides with a arbitrary test function $v(\mathbf{x})$, which vanishes on the border (i.e. $v(mathbfx) = 0, \forall \mathbf{x} \in \partial\Omega$), and by then integrating over all of Ω:

$$-\int_\Omega \Delta u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \tag{1.10}$$

We may now rearrange the gradient product rule $\nabla(ab) = (\nabla a)b + a(\nabla b)$ and Gauss' theorem (as long as $v(x)$ is differentiable in a neighborhood of Ω) combined with the fact that $v(\mathbf{x})$ vanishes on $\partial\Omega$ to convert the right-hand side to

$$\begin{aligned} -\int_\Omega \Delta u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} &= -\int_\Omega \nabla(\nabla u(\mathbf{x}) v(\mathbf{x})) d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \\ &= -\int_{\partial\Omega} \nabla u(\mathbf{x}) v(\mathbf{x}) d\boldsymbol{\omega} + \int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \\ &= \int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \end{aligned} \tag{1.11}$$

Consequently, the weak formulation of the problem is to find $u(\mathbf{x})$, such that for arbitrary $v(\mathbf{x})$, we have

$$\int_\Omega \nabla u(\mathbf{x})\nabla v(\mathbf{x})d\mathbf{x} = \int_\Omega f(\mathbf{x})v(\mathbf{x})d\mathbf{x} \tag{1.12}$$

To simplify and generalize the notation, we may use the linear form $L : V \to \mathbb{R}$ as

$$L(v) = \int_\Omega f(\mathbf{x})v(\mathbf{x})d\mathbf{x} \tag{1.13}$$

and also the bilinear form $a : V \times V \to \mathbb{R}$

$$a(u,v) = \int_\Omega \nabla u(\mathbf{x})\nabla v(\mathbf{x})d\mathbf{x} \tag{1.14}$$

1.3 Example: One dimensional Poisson equation

Initial idea taken from Wikipedia article about FEM.

To illustrate the choice of basis functions, we will now consider the simple one dimensional case $\Omega = [a,b]$, such that the weak formulation of the problem turns into

$$\int_a^b u'(x)v'(x)dx = \int_a^b f(x)v(x)dx \tag{1.15}$$

We now subdivide the domain $[a,b]$ into M subintervals, each of length $h = (b-a)/M$, with nodes at $x_k = a + hk$, $k \in \{0,1,\ldots,M\}$. We proceed to choose as the basis functions the class of the piecewise linear Lagrange interpolating polynomials on $[x_k, x_{k+1}]$, $k \in \{0,1,\ldots,M\}$, defined as

$$v_k(x) = \frac{x - x_{k-1}}{x_k - x_{k-1}}\mathbf{1}_{\{x\in[x_{k-1},x_k]\}} + \frac{x_{k+1} - x}{x_{k+1} - x_k}\mathbf{1}_{\{x\in[x_k,x_{k+1}]\}} \tag{1.16}$$

If we now interpolate $f(x)$ and $u(x)$ as piecewise linear Lagrange polynomaials, we get the representation

$$\begin{aligned} f(x) &\approx \sum_{i=1}^{M} f(x_{i-1})\frac{x - x_i}{x_{i-1} - x_i} + f(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}} \\ &= \sum_{i=1}^{M-1} f(x_i)v_i(x) \end{aligned} \tag{1.17}$$

and analogously

$$u(x) = \sum_{i=1}^{M-1} u(x_i)v_i(x) \tag{1.18}$$

We now restricted ourselves to the discrete variational formulation of the problem

$$\sum_{i=1}^{M-1} u(x_i) \int_a^b v_i'(x) v_j'(x) dx = \sum_{i=1}^{M-1} f(x_i) \int_a^b v_i(x) v_j(x) dx \tag{1.19}$$

which needs to be satisfied for all $j \in \{0, 1, \ldots, M\}$.

This equation can be rewritten in terms of two matrices **K** and **L** which we define as

$$K_{ij} = \int_a^b v_i(x) v_j(x) dx \tag{1.20}$$

$$L_{ij} = \int_a^b v_i'(x) v_j'(x) dx \tag{1.21}$$

such that we get

$$\sum_{i=1}^{M-1} u(x_i) L_{ij} = \sum_{i=1}^{M-1} f(x_i) K_{ij} \tag{1.22}$$

Notice, that we only need the entries K_{ij} and L_{ij} with $i \in \{1, 2, \ldots, M-1\}$, since we already know the boundary conditions of $u(x)$ at $x = x_0$ and $x = x_M$.

We realize, that the L_2 inner product of $v_i(x)$ with $v_j(x)$ (and consequently also the one of $v_i'(x)$ with $v_j'(x)$) is zero for all $|i - j| > 1$, hence, we distinguish two different cases.

1. $i = j$: Here, the inner product turns out to be

$$\begin{aligned}
\int_a^b v_i(x) v_i(x) dx &= \int_a^b \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right)^2 \mathbf{1}_{\{x \in [x_{i-1}, x_i]\}} + \left(\frac{x_{i+1} - x}{x_{i+1} - x_i} \right)^2 \mathbf{1}_{\{x \in [x_i, x_{i+1}]\}} dx \\
&= 2 \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right)^2 dx \\
&= \frac{2}{h^2} \int_{x_{i-1} - x_{i-1}}^{x_i - x_{i-1}} u^2 du \\
&= \frac{2}{h^2} \frac{1}{3} h^3 \\
&= \frac{2h}{3}
\end{aligned} \tag{1.23}$$

and for the derivatives it is

$$
\begin{aligned}
\int_a^b v_i'(x)v_i'(x)dx &= \int_a^b \left(\frac{1}{x_i - x_{i-1}}\right)^2 \mathbf{1}_{\{x\in[x_{i-1},x_i]\}} + \left(\frac{-1}{x_{i+1}-x_i}\right)^2 \mathbf{1}_{\{x\in[x_i,x_{i+1}]\}} dx \\
&= 2\int_{x_{i-1}}^{x_i} \left(\frac{1}{x_i - x_{i-1}}\right)^2 dx \\
&= \frac{2}{h^2}\int_0^h 1 du \\
&= \frac{2}{h} \qquad (1.24)
\end{aligned}
$$

2. $|i-j| = 1$: Here, we can limit ourselves to the case where $j = i+1$, since the other case is fully symmetric. We calculate

$$
\begin{aligned}
\int_a^b v_i(x)v_{i+1}(x)dx &= \int_a^b \frac{x_{i+1}-x}{x_{i+1}-x_i}\frac{x-x_i}{x_{i+1}-x_i}\mathbf{1}_{\{x\in[x_i,x_{i+1}]\}}dx \\
&= \int_{x_i}^{x_{i+1}} \frac{x_{i+1}-x}{x_{i+1}-x_i}\frac{x-x_i}{x_{i+1}-x_i}dx \\
&= \frac{1}{h^2}\int_{x_i-x_i}^{x_{i+1}-x_i}(x_{i+1}-x_i-u)udu \\
&= \frac{1}{h^2}\int_0^h (h-u)udu \\
&= \frac{1}{h^2}(\frac{h^3}{2}-\frac{h^3}{3}) \\
&= \frac{h}{6} \qquad (1.25)
\end{aligned}
$$

and for the derivative it is

$$
\begin{aligned}
\int_a^b v_i'(x)v_{i+1}'(x)dx &= \int_a^b \frac{-1}{x_{i+1}-x_i}\frac{1}{x_{i+1}-x_i}\mathbf{1}_{\{x\in[x_i,x_{i+1}]\}}dx \\
&= -\frac{1}{h^2}\int_{x_i}^{x_{i+1}} 1 dx \\
&= -\frac{1}{h} \qquad (1.26)
\end{aligned}
$$

Now, using the previously defined matrices $\mathbf{K}_{ij}$ and $\mathbf{L}_{ij}$, we get the matrix equation

$$\mathbf{L}u = \mathbf{K}f \qquad (1.27)$$

with

$$u = (u_0, u(x_1), \dots, u_M)^T \tag{1.28}$$

$$f = (f(x_0), f(x_1), \dots, f(x_M))^T \tag{1.29}$$

$$\mathbf{L} = \begin{pmatrix} 1 & & & & \\ \frac{2}{h} & -\frac{1}{h} & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & -\frac{1}{h} & \frac{2}{h} & \ddots & \\ & & -\frac{1}{h} & \ddots & -\frac{1}{h} \\ & & & \ddots & \frac{2}{h} \\ & & & & 1 \end{pmatrix} \tag{1.30}$$

$$\mathbf{K} = \begin{pmatrix} \frac{u_0}{f(x_0)} & & & & \\ \frac{2h}{3} & \frac{h}{6} & & & \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & & \\ & \frac{h}{6} & \frac{2h}{3} & \ddots & \\ & & \frac{h}{6} & \ddots & \frac{h}{6} \\ & & & \ddots & \frac{2h}{3} \\ & & & & \frac{u_M}{f(x_M)} \end{pmatrix} \tag{1.31}$$

$$\tag{1.32}$$

Here, we have adjusted the first rows in **L** and **K**, such that the boundary conditions are necessarily satisfied. To obtain the finite element solution, we simply solve this linear system.

2 Maxwell's equations

Let $\mathbf{E} = (E_1, E_2, E_3)^T$ denote the electric field, $\mathbf{B} = (B_1, B_2, B_3)^T$ the magnetic field strength, and $\mathbf{j} = (j_1, j_2, j_3)^T$ the electric current density. We suppose Maxwell's equations hold:

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{2.3}$$

$$\nabla \times (\mu^{-1} \mathbf{B}) = \partial_t (\epsilon \mathbf{E}) + \mathbf{j} \tag{2.4}$$

2.1 Vector potential formulation

We can therefore use (2.2) to write $\mathbf{B} = \nabla \times \mathbf{A}$ for some vector potential $\mathbf{A}$. Furthermore, we can identify from (2.3) that $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$ can be written for some scalar potential ϕ.

The physical quantities **E** and **B** remain unchanged if we transform $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\psi$ and $\phi \to \phi' = \phi - \partial_t \psi$ (gauge transformations), as can be explicitly verified by plugging these transformed potentials into the definitions of **E** and **B**. Choosing as the gauge field as

$$\psi = \int_0^t \phi \, dt' \tag{2.5}$$

we see that $\phi' = \phi - \partial_t \int_0^t \phi \, dt' = \phi - \phi = 0$. Hence, if we now express the electric field **E** in terms of these transformed potentials, we realize that $\mathbf{E} = -\nabla\phi' - \partial_t \mathbf{A}' = -\partial_t \mathbf{A}'$ because we have transformed ϕ exactly in such a way, which makes ϕ' vanish (not symbolically, but due to its actual value being zero). As for the magnetic field **B**, we have $\mathbf{B} = \nabla \times \mathbf{A}'$.

Plugging these identities into (2.4), and for simplicity replacing the symbol $\mathbf{A}'$ with **A** again, we get

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{A}) = -\epsilon \partial_t^2 \mathbf{A} + \mathbf{j} \tag{2.6}$$

We may want to introduce a harmonic time dependence of **A** and **j** with frequencies ω, such that $\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}) \exp(i\omega t)$ and $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x}) \exp(i\omega t)$. Plugging this into (2.6) yields us

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{A}) - \epsilon \omega^2 \mathbf{A} = \mathbf{j} \tag{2.7}$$

We reduce this equation to its weak formulation, by multiplying it with a vector-valued function $\mathbf{v} \in H_{\mathrm{curl}}(\Omega)$, where we denoted

$$H_{\mathrm{curl}}(\Omega) = \{u : \Omega \to \mathbb{C}, \text{ such that } u \in L^2(\mathbb{C})^3, \nabla \times u \in L^2(\mathbb{C})^3\} \tag{2.8}$$

and by integrating over all of Ω:

$$\int_\Omega (\nabla \times (\mu^{-1} \nabla \times \mathbf{A})) \cdot \mathbf{v} - \epsilon \omega^2 \int_\Omega \mathbf{A} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} \tag{2.9}$$

To further simplify this expression, we will derive an identity for the scalar product of a vector-valued function **v** with the curl of a vector-valued function **u**. For this, we use the completely antisymmetric tensor ε_{ijk} (frequently referred to as the Levi-Civita tensor), to rewrite the k-th component of the curl as

$$(\nabla \times \mathbf{u})_k = \sum_i \sum_j \varepsilon_{ijk} \partial_i u_j \tag{2.10}$$

where ∂_i denotes the partial derivative with respect to the i-th coordinate direction. Rewriting the scalar product as a sum and identifying $\mathbf{u} = \mu^{-1} \nabla \times \mathbf{A}$, we apply the product rule to get

$$
\begin{aligned}
(\nabla \times \mathbf{u}) \cdot \mathbf{v} &= \sum_k (\nabla \times \mathbf{u})_k v_k \\
&= \sum_k (\sum_i \sum_j \varepsilon_{ijk} \partial_i u_j) v_k \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{ijk} u_j v_k) - \sum_k \sum_i \sum_j u_j (\varepsilon_{ijk} \partial_i v_k) \\
&= \sum_k \sum_i \sum_j \partial_i (\varepsilon_{jki} u_j v_k) - \sum_k \sum_i \sum_j u_j ((-\varepsilon_{ikj}) \partial_i v_k) \\
&= \sum_i \partial_i (\mathbf{u} \times \mathbf{v})_i + \sum_j u_j (\nabla \times \mathbf{v})_j \\
&= \nabla \cdot (\mathbf{u} \times \mathbf{v}) + \mathbf{u} \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{2.11}
$$

Consequently, we may rewrite the double curl term as

$$
\begin{aligned}
\int_\Omega (\nabla \times (\mu^{-1} \nabla \times \mathbf{A})) \cdot \mathbf{v} &= \int_\Omega \nabla \cdot ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{v}) + \int_\Omega (\mu^{-1} \nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{v}) \cdot \mathbf{n} + \int_\Omega (\mu^{-1} \nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{2.12}
$$

Again denoting $\mathbf{u} = \mu^{-1} \nabla \times \mathbf{A}$, we can rearrange

$$
\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n} &= \sum_k (\sum_i \sum_j \varepsilon_{ijk} u_i v_j) n_k \\
&= \sum_j (\sum_i \sum_k (-\varepsilon_{ikj}) u_i n_k) v_j \\
&= -(\mathbf{u} \times \mathbf{n}) \cdot \mathbf{v}
\end{aligned} \tag{2.13}
$$

and therefore have

$$
\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{v}) \cdot \mathbf{n} = -\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{n}) \cdot \mathbf{v} \tag{2.14}
$$

We can finally identify the weak form of the problem as

$$
\int_\Omega (\mu^{-1} \nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) - \epsilon\omega^2 \int_\Omega \mathbf{A} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} + \underbrace{\int_{\partial\Omega} ((\mu^{-1} \nabla \times \mathbf{A}) \times \mathbf{n}) \cdot \mathbf{v}}_{\text{boundary integral term}} \tag{2.15}
$$

where $\mathbf{n}$ denotes the normal vector to $\partial\Omega$. Notice how we placed the boundary integral term on the right hand side, since along the border $\partial\Omega$ we usually either know $\nabla \times \mathbf{A} = \mathbf{B} = \mathbf{B}_D$ (or even only $\mathbf{B} \times \mathbf{n}$) as the Neumann boundary condition or $-i\omega\mathbf{A} = \mathbf{E}_D$ as the Dirichlet boundary condition.

2.2 Electric field formulation

Multiplying both sides of (2.3) with μ^{-1} and taking the curl allows us to substitute (2.4) in, such that we get

$$\begin{aligned}\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) &= -\partial_t(\nabla \times (\mu^{-1}\mathbf{B})) \\ &= -\partial_t(\partial_t(\epsilon\mathbf{E}) + \mathbf{j}) \\ &= -\partial_t^2(\epsilon\mathbf{E}) + \partial_t\mathbf{j} \end{aligned} \tag{2.16}$$

Introducing a harmonic time dependence of $\mathbf{E}$ and $\mathbf{j}$ with frequencies ω, such that $\mathbf{E}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x})\exp(i\omega t)$ and $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x})\exp(i\omega t)$, we obtain

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) - \omega^2\epsilon\mathbf{E} = i\omega\mathbf{j} \tag{2.17}$$

The i on the right-hand side is giving me (and most likely FEniCS too) a major headache, hence why I will try my luck with finding a formulation involving $\mathbf{B}$ in the next section.

2.3 Magnetic field formulation

Multiply both sides of (2.4) with ϵ^{-1} and take the curl to obtain, with the help of (2.3), the expression

$$\begin{aligned}\nabla \times (\epsilon^{-1}\nabla \times (\mu^{-1}\mathbf{B})) &= \nabla \times (\partial_t\mathbf{E} + \epsilon^{-1}\mathbf{j}) \\ &= \partial_t(\nabla \times \mathbf{E}) + \nabla \times (\epsilon^{-1}\mathbf{j}) \\ &= -\partial_t^2\mathbf{B} + \nabla \times (\epsilon^{-1}\mathbf{j}) \end{aligned} \tag{2.18}$$

Introducing a harmonic time dependence of $\mathbf{B}$ and $\mathbf{j}$ with frequencies ω, such that $\mathbf{B}(\mathbf{x}, t) = \mathbf{B}(\mathbf{x})\exp(i\omega t)$ and $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x})\exp(i\omega t)$, we obtain

$$\nabla \times (\epsilon^{-1}\nabla \times (\mu^{-1}\mathbf{B})) - \omega^2\mathbf{B} = \nabla \times (\epsilon^{-1}\mathbf{j}) \tag{2.19}$$

Converting this equation to its weak form by multiplying both sides with a vector valued function $\mathbf{v} \in H_{\text{curl}}(\Omega)$ and integrating over all of Ω yields

$$\int_\Omega (\nabla \times (\epsilon^{-1}\nabla \times (\mu^{-1}\mathbf{B}))) \cdot \mathbf{v} - \omega^2 \int_\Omega \mathbf{B} \cdot \mathbf{v} = \int_\Omega (\nabla \times (\epsilon^{-1}\mathbf{j})) \cdot \mathbf{v} \tag{2.20}$$

Using the above derived identity

$$(\nabla \times \mathbf{u}) \cdot \mathbf{v} = \nabla \cdot (\mathbf{u} \times \mathbf{v}) + \mathbf{u} \cdot (\nabla \times \mathbf{v}) \tag{2.21}$$

we may write using Gauss' theorem

$$
\begin{aligned}
\int_\Omega (\nabla \times (\epsilon^{-1}\nabla \times (\mu^{-1}\mathbf{B}))) \cdot \mathbf{v} &= \int_\Omega \nabla \cdot ((\epsilon^{-1}\nabla \times (\mu^{-1}\mathbf{B})) \times \mathbf{v}) \\
&+ \int_\Omega (\epsilon^{-1}\nabla \times (\mu^{-1}\mathbf{B})) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\epsilon^{-1}\nabla \times (\mu^{-1}\mathbf{B})) \times \mathbf{v}) \cdot \mathbf{n} \\
&+ \int_\Omega (\epsilon^{-1}\nabla \times (\mu^{-1}\mathbf{B})) \cdot (\nabla \times \mathbf{v})
\end{aligned}
\tag{2.22}
$$

and

$$
\begin{aligned}
\int_\Omega (\nabla \times (\epsilon^{-1}\mathbf{j})) \cdot \mathbf{v} &= \int_\Omega \nabla \cdot ((\epsilon^{-1}\mathbf{j}) \times \mathbf{v}) + \int_\Omega (\epsilon^{-1}\mathbf{j}) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\epsilon^{-1}\mathbf{j}) \times \mathbf{v}) \cdot \mathbf{n} + \int_\Omega (\epsilon^{-1}\mathbf{j}) \cdot (\nabla \times \mathbf{v})
\end{aligned}
\tag{2.23}
$$

Since $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{n})$ (as was shown above), the boundary integrals vanish if we have

$$
\mathbf{v} \times \mathbf{n} = 0, \text{ on } \partial\Omega \tag{2.24}
$$

In that case, we end up with the following (simplified) weak form

$$
\int_\Omega (\epsilon^{-1}\nabla \times (\mu^{-1}\mathbf{B})) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_\Omega \mathbf{B} \cdot \mathbf{v} = \int_\Omega (\epsilon^{-1}\mathbf{j}) \cdot (\nabla \times \mathbf{v}) \tag{2.25}
$$

3 Waveguides

In Section 2 we have derived the weak form of the time-harmonic Maxwell equation for the vector potential $\mathbf{A}$. Let us now apply it to a small collection of illustrative examples.

3.1 Two-dimensional perfectly conducting waveguide

Consider a two-dimensional rectangular box of length l and width w enclosing a vacuum (depicted in Figure 3.1).

Defining $\mathbf{g} = (\mu^{-1}\nabla \times \mathbf{A}) \times \mathbf{n} = \mu^{-1}\mathbf{B} \times \mathbf{n}$ on the boundary $\partial\Omega$ and setting $\mathbf{j} = 0$, we can convert (2.15) into

$$
\int_\Omega (\mu^{-1}\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) - \epsilon\omega^2 \int_\Omega \mathbf{A} \cdot \mathbf{v} = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v} \tag{3.1}
$$

If we only consider $\mathbf{A} = (0, 0, A_z)^\mathsf{T}$ and $\mathbf{v} = (0, 0, v_z)^\mathsf{T}$, the two curls can be rewritten as the dot product of two gradients, since

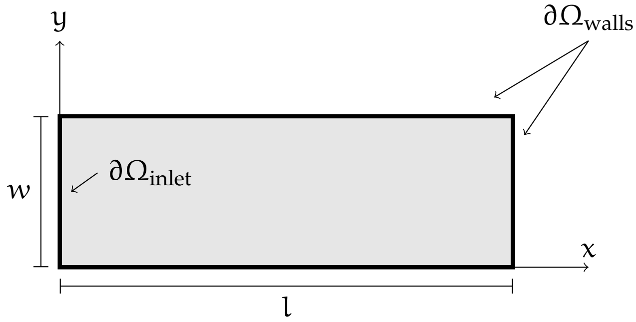

FIGURE 3.1 – Waveguide

$$\nabla \times \mathbf{A} = (\partial_y A_z, -\partial_x A_z, 0) \tag{3.2}$$

and similarly for $\nabla \times \mathbf{v}$, such that

$$(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) = (\nabla A_z) \cdot (\nabla v_z) \tag{3.3}$$

We thus end up with the following simplified weak form of the problem

$$\int_\Omega (\mu^{-1} \nabla A_z) \cdot (\nabla v_z) - \epsilon\omega^2 \int_\Omega A_z v_z = \int_{\partial\Omega} g_z v_z \tag{3.4}$$

3.1.1 Constant field (old and flawed version)

Suppose we observe an electromagnetic wave at frequency ω and with the magnetic field of strength B_0 aligned with the y-direction incident on the waveguide at $x = 0$. Symbolically, we thus have $\mathbf{B}(t) = (0, B_0, 0)^T \exp(i\omega t)$ at $x = 0$ for $y \in [0, w]$.

From this, we can calculate the function g_z on the "inlet" boundary to be (knowing that we have already taken account of the harmonic time-dependence of $\mathbf{B}$ in deriving (3.4))

$$\mathbf{g}_{\text{inlet}} = \mu^{-1} \mathbf{B} \times \mathbf{n} = \mu^{-1} B_0 (\mathbf{e}_y \times \mathbf{e}_x) = -\mu^{-1} B_0 \mathbf{e}_z \tag{3.5}$$

Referring to Section 3.2, we know that for perfectly conducting walls the magnetic field $\mathbf{B}$ in our case (with walls on top and bottom) must satisfy

$$n \cdot \mathbf{B} = 0 \implies B_y = 0 \tag{3.6}$$

This, in turn, means that different orientation of $\mathbf{n}$ for bottom wall!

$$\mathbf{g}_{\text{walls}} = \mu^{-1} \mathbf{B} \times \mathbf{n} = \mu^{-1} B_x (\mathbf{e}_x \times e_y) = \mu^{-1} B_x e_z \tag{3.7}$$

Therefore, in general, we have (assuming all other walls to be perfect conductors)

$$g_z = \begin{cases} -\mu^{-1} B_0, & \text{at the inlet} \\ 0 & \text{at the walls} \end{cases} \tag{3.8}$$

Ok, this is not entirely right, because this implicitly assumes that $B_x = 0$ at the walls (which would be correct for purely transversal waves in the waveguide, but not in general). Will somehow have to find a way to correct this later.

Alternatively, the problem can be equivalently solved using Dirichlet instead of Neumann boundary conditions (mental note: the function $\mathbf{g}$ encodes a Neumann boundary condition, since it contains $\nabla \times \mathbf{A}$, hence partial derivatives of $\mathbf{A}$). Here, we no longer have a right-hand side term in (3.4), because for Dirichlet boundary conditions we require the test functions $\mathbf{v}$ to vanish wherever we already know the exact value of $\mathbf{A}$ (i.e. on the boundary $\partial\Omega$).

Because we only consider time-harmonic electric fields $\mathbf{A} = \mathbf{A}(\mathbf{x}) \exp(i\omega t)$, and the relation $\mathbf{E} = -\partial_t \mathbf{A} = -i\omega\mathbf{A}$ holds, we should be able to impose boundary conditions on $\mathbf{A}$ based on an unput field $\mathbf{E}$. Again making use of the two-dimensionality of the problem, we allow an input field $\mathbf{E}(t) = (0, 0, E_0)^T \sin(\omega t)$ (because I currently do not feel too comfortable dealing with complex numbers in FEniCS), and have to therefore impose a boundary condition $A_z = -E_0/\omega$ at $x = 0$ for $y \in [0, w]$.

3.1.2 Constant field (new version)

We consider two types of boundaries. An inlet $\partial\Omega_{\text{inlet}}$ at $x = 0$ (left boundary) and perfectly conducting walls $\partial\Omega_{\text{walls}}$ at all other boundaries.

Suppose we observe an electromagnetic wave at frequency ω and with the magnetic field of strength B_0 aligned with the y-direction incident on the waveguide at $x = 0$. Symbolically, we thus have $\mathbf{B}(t) = (0, B_0, 0)^T \exp(i\omega t)$ at $\partial\Omega_{\text{inlet}}$.

From this, we can calculate the function g_z on the inlet boundary $\partial\Omega_{\text{inlet}}$ to be (knowing that we have already taken account of the harmonic time-dependence of $\mathbf{B}$ in deriving (3.4))

$$g_z = (\mu^{-1}\mathbf{B} \times \mathbf{n})_z = \mu^{-1} B_0 (\mathbf{e}_y \times (-\mathbf{e}_x))_z = \mu^{-1} B_0, \text{ on } \partial\Omega_{\text{inlet}} \tag{3.9}$$

For the other boundaries $\partial\Omega_{\text{walls}}$, we preferably use the boundary conditions on $\mathbf{A}$ derived in Section 3.2. Particularly the one following from looking at $\mathbf{E}$, i.e.

$$\mathbf{n} \times \mathbf{A} = 0 \tag{3.10}$$

is useful, since it tells us that $\mathbf{A} = (0, 0, A_z)^T = 0 \implies A_z = 0$ on $\partial\Omega_{\text{walls}}$ (because $\mathbf{n}$ points either in the x- or y-direction, so the curl of $\mathbf{n}$ with $\mathbf{A}$ will always explicitly contain A_z in one of its components that are required to vanish).

Table 3.1 – Boundary conditions for a waveguide with a field $\mathbf{B}(t) = (0, B_y, 0)^T \exp(i\omega t)$ incident on an inlet $\partial\Omega_{\text{inlet}}$ and perfectly conducting walls $\partial\Omega_{\text{walls}}$ otherwise.

Type	$\partial\Omega_{\text{inlet}}$	$\partial\Omega_{\text{walls}}$
Dirichlet	$A_z = -\frac{1}{i\omega} E_z$	$A_z = 0$
Neumann	$g_z = \mu^{-1} B_y$	$g_z = \pm\mu^{-1} B_x$ (-$\mu^{-1} B_y$ for outlet)

Only the combination of Neumann boundary conditions for $\partial\Omega_{\text{inlet}}$ and Dirichlet boundary conditions for $\partial\Omega_{\text{walls}}$ seem to be practical for now, since we do not have to deal with complex numbers, and do not have to know the B_x component at the walls (since setting it to zero is only really viable for **B**-fields that exclusively oscillate in the y-direction).

(Remark: The Neumann boundary condition for $\partial\Omega$ may be compactly summarized as $g_z = \mu^{-1}(-B_y, B_x, 0)^T \cdot \mathbf{n} = (\mathbf{e}_z \times \mu^{-1}\mathbf{B}) \cdot \mathbf{n}$.)

3.2 Boundary conditions

Monk2003, Page 8

At an interface $\partial\Omega$, separating the waveguide Ω from its environment, the electric field **E** and the magnetic field **B** satisfy the following boundary conditions:

$$\mathbf{n} \times (\mathbf{E} - \mathbf{E}_{\text{ext}}) = 0, \text{ on } \partial\Omega \tag{3.11}$$

$$\mathbf{n} \cdot (\mathbf{B} - \mathbf{B}_{\text{ext}}) = 0, \text{ on } \partial\Omega \tag{3.12}$$

https://farside.ph.utexas.edu/teaching/jk1/Electromagnetism/node112.html Inside perfect conductors, the electric fields vanish. Therefore, also the curl of the electric field vanishes, and for time-harmonic problems it follows that also the magnetic field is zero.

Consequently, supposing the waveguide's walls are perfectly conducting, we end up with the simplified boundary conditions

$$\mathbf{n} \times \mathbf{E} = 0, \text{ on } \partial\Omega \tag{3.13}$$

$$\mathbf{n} \cdot \mathbf{B} = 0, \text{ on } \partial\Omega \tag{3.14}$$

Expressing the fields **E** and **B** in terms of the vector potential **A** (using the time-harmonic relations $\mathbf{E} = -i\omega\mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$), we have

$$\mathbf{n} \times \mathbf{A} = 0, \text{ on } \partial\Omega \tag{3.15}$$

$$\mathbf{n} \cdot (\nabla \times \mathbf{A}) = 0, \text{ on } \partial\Omega \tag{3.16}$$

3.3 Eigenvalue problem

We want to find the resonant modes of a waveguide. For the unforced case, we have only sinusoidal oscillations in the x- and y-direction. Therefore, the axial frequencies must be integer multiples of π/w or π/l respectively, yielding the superimposed frequency modes

$$\omega_{nm} = \sqrt{\left(\frac{n}{w}\right)^2 + \left(\frac{m}{l}\right)^2} \tag{3.17}$$

Also, we could numerically obtain these eigenfrequencies using the generalized eigenvalue problem

$$\int_\Omega (\mu^{-1}\nabla \times \boldsymbol{\varphi}) \cdot (\nabla \times \mathbf{v}) = \lambda \int_\Omega \boldsymbol{\varphi} \cdot \mathbf{v} \tag{3.18}$$

for all $\mathbf{v} \in V$, which then yields $\omega = \sqrt{\lambda}$. With the finite element discretization of the above integrals, we obtain a generalized eigenvalue problem of the form

$$\mathbb{L}\boldsymbol{\varphi} = \lambda \mathbb{M}\boldsymbol{\varphi} \tag{3.19}$$

There is a small pitfall when using scipy's sparse hermitian eigenvalue solver: When trying to find the smallest magnitude eigenvalue (which is zero, because similarly to what was seen in Section 1.3, the stiffness matrix $\mathbb{L}$ has row sums identically equal to 0, such that a constant solution would always yield zero eigenvalues. This is also obvious from the explicit problem, since constant functions let the left-hand side vanish, and for non-zero $\boldsymbol{\varphi}$, the eigenvalue must essentially be zero.) the method does not seem to converge (why is that so?). However, when switching to the shift-invert mode, this problem no longer presists.

For the forced problem, the inlet acts as an oscillation amplitude. The x-direction modes must therefore lie between two integer multiples of π/l, for a resonant mode. The general rule is

$$\omega_{nm} = \sqrt{\left(\frac{n}{w}\right)^2 + \left(\frac{m+1/2}{l}\right)^2} \tag{3.20}$$

3.4 Three-dimensional perfectly conducting waveguide

We again start from

$$\int_\Omega (\mu^{-1}\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_\Omega \epsilon \mathbf{A} \cdot \mathbf{v} = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v} \tag{3.21}$$

where we defined $\mathbf{g} = (\mu^{-1}\nabla \times \mathbf{A}) \times \mathbf{n} = \mu^{-1}\mathbf{B} \times \mathbf{n}$ on the boundary $\partial\Omega$.

Yet again, for perfectly (electrically) conducting walls we have (see boundary conditions from Section 3.2)

$$\mathbf{n} \times \mathbf{A} = 0, \text{ on } \partial\Omega \tag{3.22}$$

We can split $\mathbf{A}$ into its normal and tangential component $\mathbf{A} = A_n\mathbf{n} + A_t\mathbf{t}$ with $\mathbf{n} \perp \mathbf{t}$. (**??**) then reads $A_t = 0$, meaning all tangential components of the vector potential $\mathbf{A}$ with respect to the boundary $\partial\Omega$ must vanish. (Problem: Only two degrees of freedom are restricted in this way, when implementing the Dirichlet boundary condition. Need to find an expression or condition on A_n as well…)

As for the inlet, if $\mathbf{n}|_{\text{inlet}} = \mathbf{e}_z$, then we have

$$\mathbf{g} = \mu^{-1}\mathbf{B} \times \mathbf{e}_z = \mu^{-1}(B_y, 0, -B_x)^T \tag{3.23}$$

4 WEAK DERIVATIVE

Taken from Quarteroni: Introduction to Finite Elements Method

Let $\Omega \subset \mathbb{R}^d$ open. The support of $f : \Omega \to \mathbb{R}$ is defined as

$$\operatorname{supp}(f) = \overline{\{x \in \Omega \mid f(x) \neq 0\}} \tag{4.1}$$

f has compact support, if there exists a compact subset $K \subset \Omega$, such that supp(f) $\subset K$, and define

$$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) \mid f \text{ has compact support}\} \tag{4.2}$$

(If I remember correctly, extending this notion to $f \in C^1(\Omega)$ should yield an almost identical treatment, unless we also include higher order (weak) partial derivatives). Let $T : \mathcal{D} \to \mathbb{R}$, $\varphi \mapsto \langle T, \varphi \rangle = T(\varphi)$ be a linear map. We say that T is continuous, if

$$\lim_{n\to\infty} \langle T, \varphi_n \rangle = \langle T, \varphi \rangle \tag{4.3}$$

with $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ converging to φ. Such (linear and continuous) maps are called distribution on $\mathcal{D}(\Omega)$, and they form the space of distributions $\mathcal{D}'(\Omega)$.

The (weak) partial coordinate-derivatives of T (namely $\partial_i T,\ i \in \{1, \ldots, d\}$) are characterized by distributions that satisfy

$$\langle \partial_i T, \varphi \rangle = -\langle T, \partial_i \varphi \rangle \tag{4.4}$$

for all $\varphi \in \mathcal{D}(\Omega)$.

Interesting for us is mainly the following case: Given a function $f \in L^2(\Omega)$, we define a distribution $T_f \in \mathcal{D}'(\Omega)$ to be

$$\langle T_f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx \tag{4.5}$$

for all $\varphi \in \mathcal{D}(\Omega)$.

This allows us to define a weak derivative to functions that are (in the classical sense) not differentiable (i.e. not in $C^1(\Omega)$). Consider for example the absolute value function $|\cdot| \in L^2(K)$ where $K \subset \mathbb{R}$ is compact. Since

$$\begin{aligned}
\int_K (\partial_x |x|)\varphi(x)dx &= -\int_K |x|\varphi'(x)dx \\
&= -\int_{K\cap\mathbb{R}_+} x\varphi'(x)dx - \int_{K\cap\mathbb{R}_-} (-x)\varphi'(x)dx \\
&= \int_{K\cap\mathbb{R}_+} \varphi(x)dx + \int_{K\cap\mathbb{R}_-} (-1)\varphi(x)dx \\
&= \int_K \operatorname{sign}(x)\varphi(x)dx
\end{aligned} \tag{4.6}$$

we may conclude that the weak derivative of the absolute value function is therefore the signum function. Notice, how the derivative of the absolute value function is only not well-defined at $x = 0$, i.e. on a set of zero measure. This nuisance is circumvented when talking about the weak derivative, since the measure zero sets have zero integral.

5 Random

5.1 Modified potential

Consider, for instance, solving the following equation:

$$\nabla \times (\nabla \times \mathbf{A}) - \mathbf{A} = \mathbf{j} \tag{5.1}$$

Even if we only care about the quantity $\mathbf{B} = \nabla \times \mathbf{A}$, and therefore would not even "realize" if $\mathbf{A}$ was modified to $\mathbf{A}' = \mathbf{A} + \mathbf{c}$ for a constant vector $\mathbf{c}$, this modification significantly changes the solution to the equation, because

$$\nabla \times (\nabla \times \mathbf{A}') - \mathbf{A}' = \mathbf{j} \tag{5.2}$$

turns, when plugging in the above-mentioned expression for $\mathbf{A}'$, into

$$\nabla \times (\nabla \times \mathbf{A}) - \mathbf{A} = \mathbf{j} + \mathbf{c} \tag{5.3}$$

yielding a possibly entirely different solution $\mathbf{A}$ than initially wanted.

Therefore, we cannot adjust $\mathbf{A}$ in such a way that it vanishes on the boundary of a radial symmetric domain, because we would implicitly change the physical outcome of the solution.

5.2 Kernel of curl operator

Supposedly, the kernel of the curl operator $\nabla\times$ is precisely ∇F in simply connected regions, meaning $\nabla \times f = 0$ iff $f = \nabla F$ for some twice differentiable F. Vague parallel to complex analysis: On simply connected domains any holomorphic function f may be written as $f = F'$ for a holomorphic antiderivative F.

5.3 Complex systems

We want to be able to solve $\mathbb{A}\mathbf{x} = \mathbf{b}$ for a complex-valued system. Since

$$wz = (\Re(w) + i\Im(w))(\Re(z) + i\Im(z)) \quad = \Re(w)\Re(z) - \Im(w)\Im(z) + i(\Re(w)\Im(z) + \mathrm{Im}(w)\Re(z)) \tag{5.4}$$

and both $\Re$ and $\Im$ are linear, we can split the system into its real and imaginary components:

$$\Re(\mathbb{A}\mathbf{x}) = \Re(\mathbb{A})\Re(\mathbf{x}) - \Im(\mathbb{A})\Im(\mathbf{x}) = \Re(\mathbf{b}) \tag{5.5}$$

$$\Im(\mathbb{A}\mathbf{x}) = \Re(\mathbb{A})\Im(\mathbf{x}) + \Im(\mathbb{A})\Re(\mathbf{x}) = \Im(\mathbf{b}) \tag{5.6}$$

This is easily rewritten as a purely real system twice the size of the old one

$$\begin{pmatrix} \Re(\mathbb{A}) & -\Im(\mathbb{A}) \\ \Im(\mathbb{A}) & \Re(\mathbb{A}) \end{pmatrix} \begin{pmatrix} \Re(\mathbf{x}) \\ \Im(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \Re(\mathbf{b}) \\ \Im(\mathbf{b}) \end{pmatrix} \tag{5.7}$$

This now also allows us to enforce complex boundary conditions on $\mathbf{x}$, which we can do by appropriately conditioning on the real or imaginary part of it.

6 Ideas

What might be really interesting is to instead look at the problem in space-time using the Maxwell tensor

$$\mathbb{F} = \begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{bmatrix} \tag{6.1}$$

In the covariant formulation of the Maxwell theory, the inhomogeneous Maxwell equations reduce to a single equation

$$\partial_a F^{ab} = -J^b \tag{6.2}$$

with the four current density $\mathbf{J} = (\mu c \rho, \mu \mathbf{j})$. The weak formulation of the problem could then be stated as (using Einstein's sum convention, i.e. summing over repeated indices)

$$\int_{\Omega \times \mathbb{R}} F^{ab} \partial_a v_b = \int_{\Omega \times \mathbb{R}} J^b v_b \tag{6.3}$$

where boundary conditions are yet to be determined. If we somehow would manage to find a suitable function space for the four-dimensional $\mathbf{v}$, it might be possible to find both $\mathbf{E}$ and $\mathbf{B}$ from a finite element method.