EPFL

PROJECT CSE I

Notes

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1 Finite element method

1.1 The general approach

Summarizes Chapter 1 in Quarteroni: Introduction to Finite Elements Method

Usually, the problems may be expressed in a simple equation

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = u_D & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where L denotes a linear differential operator (e.g. $-\Delta$ in the Poisson equation), u the solution to be found, and f is a source term independent of u. Some boundary condition u_D is imposed on the solution u.

However, equation (1.1) usually does not allow all physically significant solutions (particularly non-differentiable ones). Therefore, we convert the problem to a weak form. This is achieved by multiplying (1.1) with a test function $v \in V$, and integrating over the whole domain Ω:

$$\int_\Omega (Lu)v = \int_\Omega fv, \ \forall v \in V \tag{1.2}$$

Usually, integration by parts allows us to “transfer” the derivatives from the Lu term to the test function v, such that the order (with respect to the derivatives taken) is more “balanced” between the two terms. As a trade-off, a boundary term appears and needs to be eliminated to facilitate the finite element solution. This term can often be eliminated by restricting ourselves to test functions from a subspace $V' \subset V$. The weak problem then reads

$$\int_\Omega (L_u u)(L_v v) = \int_\Omega fv, \ \forall v \in V' \tag{1.3}$$

with a linear differential operator L_u acts on the solution u, and a linear differential operator L_v, appearing due to the integration by parts, acts on the test function v. For simplicity, we refer to the left-hand side as the bilinear form

$$a(u, v) = \int_\Omega (L_u u)(L_v v) \tag{1.4}$$

and the right-hand side as the linear form

$$F(v) = \int_\Omega fv \tag{1.5}$$

To solve (1.3), we prefer to look for approximate solutions u_h in a finite dimensional space V_h with $\dim(V_h) = N_h$ (what role does V'_h, the space wherein v lies to satisfy the boundary conditions, play? as far as I can tell V'_h is the subspace of V_h, in which all functions vanish at the boundary where u is known). Choosing a basis $\{\varphi_i\}_{i \leqslant N_h}$ then allows us to represent the approximate solution as

$$u_h = \sum_{j \leqslant N_h} u_j \varphi_j \tag{1.6}$$

for some coefficients u_i that need to be determined. We thus see, that (1.3) turns into a linear system, since we only need to test the equality for the basis elements φ_i of V_h:

$$\sum_{j=1}^{N_h} u_j a(\varphi_j, \varphi_i) = F(\varphi_i),\ i \in \{1, \dots, N_h\} \tag{1.7}$$

If we write $A_{ij} = a(\varphi_j, \varphi_i)$ and $\mathbf{F} = (F(\varphi_1), \dots, F(\varphi_{N_h}))^T$, we have reduced the problem to finding $\mathbf{u} = (u_1, \dots, u_{N_h})^T$, such that

$$\mathbb{A}\mathbf{u} = \mathbf{F} \tag{1.8}$$

and have identified an approximate solution to (1.1) as $u \approx u_h = \sum_{j \leqslant N_h} u_j \varphi_j$.

Choosing the space V_h is fundamental to have an accurate method that gives a good approximation u_h of u. Furthermore, the choice of basis $\{\varphi_i\}_{i \leqslant N_h}$ influences how $\mathbb{A}$ ends up looking. Of particular interest are bases for which $a(\varphi_j, \varphi_i)$ vanishes for almost all i and j, thus yielding a sparse matrix $\mathbb{A}$. The choice of basis also controls the conditioning of $\mathbb{A}$ (need to find an example to illustrate this).

1.2 The Poisson equation

Taken from FEniCS manual (too lazy for bibtex...)

We aim to solve an equation of the form

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}) \tag{1.9}$$

on a domain $\mathbf{x} \in \Omega$, with a solution $u(\mathbf{x})$ that satisfies a certain boundary condition $u(\mathbf{x}) = u_d(\mathbf{x})$ for all $x \in \partial\Omega$ that lie on the border of Ω.

To do this, we first convert this equation to its weak form by multiplying both sides with a arbitrary test function $v(\mathbf{x})$, which vanishes on the border (i.e. $v(mathbfx) = 0, \forall \mathbf{x} \in \partial\Omega$), and by then integrating over all of Ω:

$$-\int_\Omega \Delta u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \tag{1.10}$$

We may now rearrange the gradient product rule $\nabla(ab) = (\nabla a)b + a(\nabla b)$ and Gauss' theorem (as long as $v(x)$ is differentiable in a neighborhood of Ω) combined with the fact that $v(\mathbf{x})$ vanishes on $\partial\Omega$ to convert the right-hand side to

$$\begin{aligned}
-\int_\Omega \Delta u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} &= -\int_\Omega \nabla(\nabla u(\mathbf{x}) v(\mathbf{x})) d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \\
&= -\int_{\partial\Omega} \nabla u(\mathbf{x}) v(\mathbf{x}) d\boldsymbol{\omega} + \int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \\
&= \int_\Omega \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x}
\end{aligned} \tag{1.11}$$

Consequently, the weak formulation of the problem is to find $u(\mathbf{x})$, such that for arbitrary $v(\mathbf{x})$, we have

$$\int_\Omega \nabla u(\mathbf{x})\nabla v(\mathbf{x})d\mathbf{x} = \int_\Omega f(\mathbf{x})v(\mathbf{x})d\mathbf{x} \tag{1.12}$$

To simplify and generalize the notation, we may use the linear form $L : V \to \mathbb{R}$ as

$$L(v) = \int_\Omega f(\mathbf{x})v(\mathbf{x})d\mathbf{x} \tag{1.13}$$

and also the bilinear form $a : V \times V \to \mathbb{R}$

$$a(u, v) = \int_\Omega \nabla u(\mathbf{x})\nabla v(\mathbf{x})d\mathbf{x} \tag{1.14}$$

1.3 Example: One dimensional poisson equation

Initial idea taken from Wikipedia article about FEM.

To illustrate the choice of basis functions, we will now consider the simple one dimensional case $\Omega = [a, b]$, such that the weak formulation of the problem turns into

$$\int_a^b u'(x)v'(x)dx = \int_a^b f(x)v(x)dx \tag{1.15}$$

We now subdivide the domain $[a, b]$ into M subintervals, each of length $h = (b-a)/M$, with nodes at $x_k = a + hk$, $k \in \{0, 1, \ldots, M\}$. We proceed to choose as the basis functions the class of the piecewise linear Lagrange interpolating polynomials on $[x_k, x_{k+1}]$, $k \in \{0, 1, \ldots, M\}$, defined as

$$v_k(x) = \frac{x - x_{k-1}}{x_k - x_{k-1}}\mathbf{1}_{\{x\in[x_{k-1},x_k]\}} + \frac{x_{k+1} - x}{x_{k+1} - x_k}\mathbf{1}_{\{x\in[x_k,x_{k+1}]\}} \tag{1.16}$$

If we now interpolate $f(x)$ and $u(x)$ as piecewise linear Lagrange polynomaials, we get the representation

$$\begin{aligned} f(x) &\approx \sum_{i=1}^{M} f(x_{i-1})\frac{x - x_i}{x_{i-1} - x_i} + f(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}} \\ &= \sum_{i=1}^{M-1} f(x_i)v_i(x) \end{aligned} \tag{1.17}$$

and analogously

$$u(x) = \sum_{i=1}^{M-1} u(x_i)v_i(x) \tag{1.18}$$

We now restricted ourselves to the discrete variational formulation of the problem

$$\sum_{i=1}^{M-1} u(x_i) \int_a^b v_i'(x) v_j'(x) dx = \sum_{i=1}^{M-1} f(x_i) \int_a^b v_i(x) v_j(x) dx \tag{1.19}$$

which needs to be satisfied for all $j \in \{0, 1, \dots, M\}$.

This equation can be rewritten in terms of two matrices **K** and **L** which we define as

$$K_{ij} = \int_a^b v_i(x) v_j(x) dx \tag{1.20}$$

$$L_{ij} = \int_a^b v_i'(x) v_j'(x) dx \tag{1.21}$$

such that we get

$$\sum_{i=1}^{M-1} u(x_i) L_{ij} = \sum_{i=1}^{M-1} f(x_i) K_{ij} \tag{1.22}$$

Notice, that we only need the entries K_{ij} and L_{ij} with $i \in \{1, 2, \dots, M-1\}$, since we already know the boundary conditions of $u(x)$ at $x = x_0$ and $x = x_M$.

We realize, that the L_2 inner product of $v_i(x)$ with $v_j(x)$ (and consequently also the one of $v_i'(x)$ with $v_j'(x)$) is zero for all $|i - j| > 1$, hence, we distinguish two different cases.

1. $i = j$: Here, the inner product turns out to be

$$\begin{aligned}
\int_a^b v_i(x) v_i(x) dx &= \int_a^b \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right)^2 \mathbf{1}_{\{x \in [x_{i-1}, x_i]\}} + \left(\frac{x_{i+1} - x}{x_{i+1} - x_i} \right)^2 \mathbf{1}_{\{x \in [x_i, x_{i+1}]\}} dx \\
&= 2 \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right)^2 dx \\
&= \frac{2}{h^2} \int_{x_{i-1} - x_{i-1}}^{x_i - x_{i-1}} u^2 du \\
&= \frac{2}{h^2} \frac{1}{3} h^3 \\
&= \frac{2h}{3}
\end{aligned} \tag{1.23}$$

and for the derivatives it is

$$
\begin{aligned}
\int_a^b v_i'(x)v_i'(x)dx &= \int_a^b \left(\frac{1}{x_i - x_{i-1}}\right)^2 \mathbf{1}_{\{x\in[x_{i-1},x_i]\}} + \left(\frac{-1}{x_{i+1} - x_i}\right)^2 \mathbf{1}_{\{x\in[x_i,x_{i+1}]\}}dx \\
&= 2\int_{x_{i-1}}^{x_i} \left(\frac{1}{x_i - x_{i-1}}\right)^2 dx \\
&= \frac{2}{h^2}\int_0^h 1du \\
&= \frac{2}{h}
\end{aligned}
\tag{1.24}
$$

2. $|i-j| = 1$: Here, we can limit ourselves to the case where $j = i+1$, since the other case is fully symmetric. We calculate

$$
\begin{aligned}
\int_a^b v_i(x)v_{i+1}(x)dx &= \int_a^b \frac{x_{i+1}-x}{x_{i+1}-x_i}\frac{x-x_i}{x_{i+1}-x_i}\mathbf{1}_{\{x\in[x_i,x_{i+1}]\}}dx \\
&= \int_{x_i}^{x_{i+1}} \frac{x_{i+1}-x}{x_{i+1}-x_i}\frac{x-x_i}{x_{i+1}-x_i}dx \\
&= \frac{1}{h^2}\int_{x_i-x_i}^{x_{i+1}-x_i} (x_{i+1}-x_i-u)udu \\
&= \frac{1}{h^2}\int_0^h (h-u)udu \\
&= \frac{1}{h^2}(\frac{h^3}{2} - \frac{h^3}{3}) \\
&= \frac{h}{6}
\end{aligned}
\tag{1.25}
$$

and for the derivative it is

$$
\begin{aligned}
\int_a^b v_i'(x)v_{i+1}'(x)dx &= \int_a^b \frac{-1}{x_{i+1}-x_i}\frac{1}{x_{i+1}-x_i}\mathbf{1}_{\{x\in[x_i,x_{i+1}]\}}dx \\
&= -\frac{1}{h^2}\int_{x_i}^{x_{i+1}} 1dx \\
&= -\frac{1}{h}
\end{aligned}
\tag{1.26}
$$

Now, using the previously defined matrices $\mathbf{K}_{ij}$ and $\mathbf{L}_{ij}$, we get the matrix equation

$$
\mathbf{L}u = \mathbf{K}f \tag{1.27}
$$

with

$$u = (u_0, u(x_1), \dots, u_M)^T \tag{1.28}$$

$$f = (f(x_0), f(x_1), \dots, f(x_M))^T \tag{1.29}$$

$$\mathbf{L} = \begin{pmatrix} 1 & & & & \\ \frac{2}{h} & -\frac{1}{h} & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & -\frac{1}{h} & \frac{2}{h} & \ddots & \\ & & -\frac{1}{h} & \ddots & -\frac{1}{h} \\ & & & \ddots & \frac{2}{h} \\ & & & & 1 \end{pmatrix} \tag{1.30}$$

$$\mathbf{K} = \begin{pmatrix} \frac{u_0}{f(x_0)} & & & & \\ \frac{2h}{3} & \frac{h}{6} & & & \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & & \\ & \frac{h}{6} & \frac{2h}{3} & \ddots & \\ & & \frac{h}{6} & \ddots & \frac{h}{6} \\ & & & \ddots & \frac{2h}{3} \\ & & & & \frac{u_M}{f(x_M)} \end{pmatrix} \tag{1.31}$$

$$\tag{1.32}$$

Here, we have adjusted the first rows in $\mathbf{L}$ and $\mathbf{K}$, such that the boundary conditions are necessarily satisfied. To obtain the finite element solution, we simply solve this linear system.

2 Maxwell's equations

Let $\mathbf{E} = (E_1, E_2, E_3)^T$ denote the electric field, $\mathbf{B} = (B_1, B_2, B_3)^T$ the magnetic field strength, and $\mathbf{j} = (j_1, j_2, j_3)^T$ the electric current density. We suppose Maxwell's equations hold:

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{2.3}$$

$$\nabla \times (\mu^{-1} \mathbf{B}) = \partial_t (\epsilon \mathbf{E}) + \mathbf{j} \tag{2.4}$$

We can therefore write $\mathbf{B} = \nabla \times \mathbf{A}$ for some vector potential $\mathbf{A}$, and $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$ for some scalar potential ϕ. Plugging these identities into (2.4), we get

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{A}) = \partial_t \nabla \phi - \partial_t^2 \mathbf{A} + \mathbf{j} \tag{2.5}$$

We may choose $\nabla\phi = 0$ (why?) as a gauge, and introduce a harmonic time dependence of $\mathbf{A}$ and $\mathbf{j}$ with frequencies ω, such that $\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x})\exp(i\omega t)$ and $\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x})\exp(i\omega t)$. Plugging this into (2.5) yields us

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{A}) - \omega^2\mathbf{A} = \mathbf{j} \tag{2.6}$$

We reduce this equation to its weak formulation, by multiplying it with a vector-valued function $\mathbf{v} \in H_{\text{curl}}(\Omega)$, where we denoted

$$H_{\text{curl}}(\Omega) = \{u : \Omega \to \mathbb{C}, \text{ such that } u \in L^2(\mathbb{C})^3, \nabla \times u \in L^2(\mathbb{C})^3\} \tag{2.7}$$

and by integrating over all of Ω:

$$\int_\Omega (\nabla \times (\mu^{-1}\nabla \times \mathbf{A})) \cdot \mathbf{v} - \omega^2 \int_\Omega \mathbf{A} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} \tag{2.8}$$

To further simplify this expression, we will derive an identity for the scalar product of a vector-valued function $\mathbf{v}$ with the curl of a vector-valued function $\mathbf{u}$. For this, we use the completely antisymmetric tensor ε_{ijk} (frequently referred to as the Levi-Civita tensor), to rewrite the k-th component of the curl as

$$(\nabla \times \mathbf{u})_k = \sum_i \sum_j \varepsilon_{ijk}\partial_i u_j \tag{2.9}$$

where ∂_i denotes the partial derivative with respect to the i-th coordinate direction. Rewriting the scalar product as a sum and identifying $\mathbf{u} = \mu^{-1}\nabla \times \mathbf{A}$, we apply the product rule to get

$$\begin{aligned}
(\nabla \times \mathbf{u}) \cdot \mathbf{v} &= \sum_k (\nabla \times \mathbf{u})_k v_k \\
&= \sum_k (\sum_i \sum_j \varepsilon_{ijk}\partial_i u_j) v_k \\
&= \sum_k \sum_i \sum_j \partial_i(\varepsilon_{ijk} u_j v_k) - \sum_k \sum_i \sum_j u_j(\varepsilon_{ijk}\partial_i v_k) \\
&= \sum_k \sum_i \sum_j \partial_i(\varepsilon_{jki} u_j v_k) - \sum_k \sum_i \sum_j u_j((-\varepsilon_{ikj})\partial_i v_k) \\
&= \sum_i \partial_i(\mathbf{u} \times \mathbf{v})_i + \sum_j u_j(\nabla \times \mathbf{v})_j \\
&= \nabla \cdot (\mathbf{u} \times \mathbf{v}) + \mathbf{u} \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{2.10}$$

Consequently, we may rewrite the double curl term in the weak formulation as

$$\begin{aligned}
\int_\Omega (\nabla \times (\mu^{-1}\nabla \times \mathbf{A})) \cdot \mathbf{v} &= \int_\Omega \nabla \cdot ((\mu^{-1}\nabla \times \mathbf{A}) \times \mathbf{v}) + \int_\Omega (\mu^{-1}\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) \\
&= \int_{\partial\Omega} ((\mu^{-1}\nabla \times \mathbf{A}) \times \mathbf{v}) \cdot \mathbf{n} + \int_\Omega (\mu^{-1}\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{2.11}$$

We will now have a look at what conditions $\mathbf{v}$ needs to satisfy, such that the boundary term (first integral) vanishes, and we would end up with

$$\int_\Omega (\mu^{-1}\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) - \omega^2 \int_\Omega \mathbf{A} \cdot \mathbf{v} = \int_\Omega \mathbf{j} \cdot \mathbf{v} \tag{2.12}$$

Let $\mathbf{n}$ denote the normal vector to $\partial\Omega$ at a point $\mathbf{x} \in \partial\Omega$. For the boundary term to vanish, we require

$$((\mu^{-1}\nabla \times \mathbf{A}) \times \mathbf{v}) \cdot \mathbf{n} = 0 \tag{2.13}$$

for all $\mathbf{x} \in \partial\Omega$. Denoting $\mathbf{u} = \mu^{-1}\nabla \times \mathbf{A}$, we rearrange

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n} &= \sum_k (\sum_i \sum_j \varepsilon_{ijk} u_i v_j) n_k \\ &= \sum_i u_i (\sum_j \sum_k \varepsilon_{jki} v_j n_k) \\ &= \mathbf{u} \cdot (\mathbf{v} \times \mathbf{n}) \end{aligned} \tag{2.14}$$

For non-trivial $\mathbf{u}$ and $\mathbf{v}$, this expression is zero if and only if $\mathbf{v} \perp \mathbf{n}$, meaning $\mathbf{v}$ is orthogonal to $\partial\Omega$ for all $\mathbf{x} \in \partial\Omega$.

3 WEAK DERIVATIVE

Taken from Quarteroni: Introduction to Finite Elements Method

Let $\Omega \subset \mathbb{R}^d$ open. The support of $f : \Omega \to \mathbb{R}$ is defined as

$$\text{supp}(f) = \overline{\{x \in \Omega \mid f(x) \neq 0\}} \tag{3.1}$$

f has compact support, if there exists a compact subset $K \subset \Omega$, such that supp(f) $\subset K$, and define

$$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) \mid f \text{ has compact support}\} \tag{3.2}$$

(If I remember correctly, extending this notion to $f \in C^1(\Omega)$ should yield an almost identical treatment, unless we also include higher order (weak) partial derivatives). Let $T : \mathcal{D} \to \mathbb{R}$, $\varphi \mapsto \langle T, \varphi \rangle = T(\varphi)$ be a linear map. We say that T is continuous, if

$$\lim_{n\to\infty} \langle T, \varphi_n \rangle = \langle T, \varphi \rangle \tag{3.3}$$

with $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ converging to φ. Such (linear and continuous) maps are called distribution on $\mathcal{D}(\Omega)$, and they form the space of distributions $\mathcal{D}'(\Omega)$.

The (weak) partial coordinate-derivatives of T (namely $\partial_i T,\ i \in \{1, \ldots, d\}$) are characterized by distributions that satisfy

$$\langle \partial_i T, \varphi \rangle = -\langle T, \partial_i \varphi \rangle \tag{3.4}$$

for all $\varphi \in \mathcal{D}(\Omega)$.

Interesting for us is mainly the following case: Given a function $f \in L^2(\Omega)$, we define a distribution $T_f \in \mathcal{D}'(\Omega)$ to be

$$\langle T_f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx \tag{3.5}$$

for all $\varphi \in \mathcal{D}(\Omega)$.

This allows us to define a weak derivative to functions that are (in the classical sense) not differentiable (i.e. not in $C^1(\Omega)$). Consider for example the absolute value function $|\cdot| \in L^2(K)$ where $K \subset \mathbb{R}$ is compact. Since

$$\begin{aligned}\int_K (\partial_x |x|)\varphi(x)dx &= -\int_K |x|\varphi'(x)dx \\ &= -\int_{K\cap\mathbb{R}_+} x\varphi'(x)dx - \int_{K\cap\mathbb{R}_-} (-x)\varphi'(x)dx \\ &= \int_{K\cap\mathbb{R}_+} \varphi(x)dx + \int_{K\cap\mathbb{R}_-} (-1)\varphi(x)dx \\ &= \int_K \operatorname{sign}(x)\varphi(x)dx \end{aligned} \tag{3.6}$$

we may conclude that the weak derivative of the absolute value function is therefore the signum function. Notice, how the derivative of the absolute value function is only not well-defined at $x = 0$, i.e. on a set of zero measure. This nuisance is circumvented when talking about the weak derivative, since the measure zero sets have zero integral.

4 Ideas

What might be really interesting is to instead look at the problem in space-time using the Maxwell tensor

$$\mathbb{F} = \begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{bmatrix} \tag{4.1}$$

In the covariant formulation of the Maxwell theory, the inhomogeneous Maxwell equations reduce to a single equation

$$\partial_a F^{ab} = -J^b \tag{4.2}$$

with the four current density $\mathbf{J} = (\mu c \rho, \mu \mathbf{j})$. The weak formulation of the problem could then be stated as (using Einstein's sum convention, i.e. summing over repeated indices)

$$\int_{\Omega\times\mathbb{R}} F^{ab}\partial_a v_b = \int_{\Omega\times\mathbb{R}} J^b v_b \tag{4.3}$$

where boundary conditions are yet to be determined. If we somehow would manage to find a suitable function space for the four-dimensional **v**, it might be possible to find both **E** and **B** from a finite element method.