

Minimal rational interpolation for time-harmonic Maxwell's equations

June 24, 2022
Fabio Matti

- ▶ Problem formulation
- ▶ Finite element method
- ▶ Minimal rational interpolation
- ▶ Example applications
- ▶ Conclusion and outlook

Time-harmonic vector potential $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \exp(i\omega t)$.

$$\mathbf{B} = \nabla \times \mathbf{u} \quad (\text{Magnetic field})$$

$$\mathbf{E} = -i\omega \mathbf{u} \quad (\text{Electric field})$$

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Time-harmonic potential equation

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \epsilon \omega^2 \mathbf{u} = \mathbf{j}$$

Want to approximate $\mathbf{u} : \mathbb{C} \ni \omega \mapsto \mathbf{u}(\omega) \in H_{\text{curl}}(\Omega)$ with

$$H_{\text{curl}}(\Omega) = \{\mathbf{v} : \Omega \rightarrow \mathbb{C}^3, \text{ such that } \mathbf{v} \in L_2(\Omega)^3, \nabla \times \mathbf{v} \in L_2(\Omega)^3\}$$

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Weak formulation of the time-harmonic potential equation

Find $\mathbf{u} \in H_{\text{curl}}(\Omega)$, such that

$$\int_{\Omega} \langle \mu^{-1} \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle - \omega^2 \int_{\Omega} \epsilon \langle \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \langle \mathbf{j}, \mathbf{v} \rangle + \int_{\partial\Omega} \langle \mathbf{g}, \mathbf{v} \rangle$$

for all $\mathbf{v} \in H_{\text{curl}}$, where $\mathbf{g} = (\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}$.

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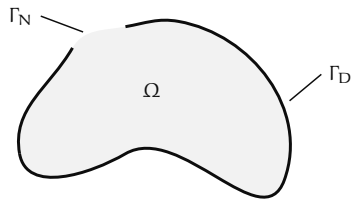
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FEniCS [2] with Nédélec elements of the first kind

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \epsilon \omega^2 \mathbf{u} = \mathbf{j}$$

Perfectly conducting boundary

$$\mathbf{g} = \mathbf{0} \text{ and } \mathbf{E} \times \mathbf{n} = \mathbf{0}, \text{ on } \Gamma_D$$



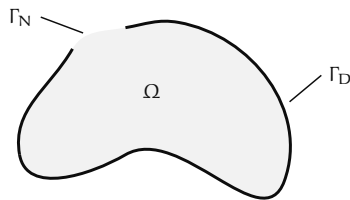
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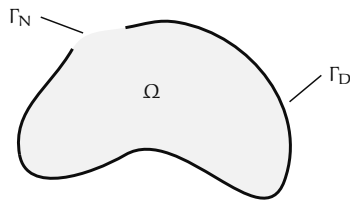
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Imperfectly conducting boundary [3]

$$\mathbf{g} = i\omega\lambda(\mathbf{n} \times \mathbf{u}) \times \mathbf{n}, \text{ on } \Gamma_I$$



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Rational surrogate

$$\tilde{\mathbf{u}}(\omega) = \frac{\mathbf{P}(\omega)}{Q(\omega)} = \sum_{j=1}^S \frac{\mathbf{p}_j}{\omega - \omega_j} / \sum_{j=1}^S \frac{q_j}{\omega - \omega_j}$$

in barycentric coordinates with support points $\omega_1, \omega_2, \dots, \omega_S$.

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Interpolation property

$$\tilde{\mathbf{u}}(\omega_j) = \mathbf{u}(\omega_j), \quad \forall j \in \{1, 2, \dots, S\}$$

if $\mathbf{p}_j = q_j \mathbf{u}(\omega_j), \forall j$.

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4. Define the minimal rational surrogate $\tilde{\mathbf{u}}(\omega) = \mathbf{P}(\omega)/Q(\omega)$ with

$$\mathbf{P}(\omega) = \sum_{j=1}^S \frac{q_j \mathbf{u}(\omega_j)}{\omega - \omega_j} \quad \text{and} \quad Q(\omega) = \sum_{j=1}^S \frac{q_j}{\omega - \omega_j}$$

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$$\omega^{(t+1)} = \operatorname{argmin}_{\omega \in \Omega_{\text{test}}} |Q^{(t)}(\omega)|$$

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3. Stop when relative error

$$\|\mathbf{u}(\omega^{(t+1)}) - \tilde{\mathbf{u}}^{(t)}(\omega^{(t+1)})\|_{\mathcal{M}} / \|\mathbf{u}(\omega^{(t+1)})\|_{\mathcal{M}}$$

is small enough

With the QR-decomposition of the snapshot matrix

$$\underline{\mathbf{U}} = [\mathbf{u}(\omega_1), \dots, \mathbf{u}(\omega_S)]^T.$$

$$\underline{\mathbf{U}} = \underline{\mathbf{Q}} \underline{\mathbf{R}}$$

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- ▶ Improved conditioning of SVD with $\underline{\mathbf{R}}$
- ▶ $\underline{\mathbf{R}}$ can be built sequentially (modified Householder triangularization for gMRI [5])

Alternative representations of the surrogate ($\mathbf{r}_j = \mathbf{R}[:, j]$)

$$\mathring{\mathbf{u}}(\omega) = \sum_{j=1}^S \frac{q_j \mathbf{e}_j}{\omega - \omega_j} / \sum_{j=1}^S \frac{q_j}{\omega - \omega_j}$$

$$\hat{\mathbf{u}}(\omega) = \sum_{j=1}^S \frac{q_j \mathbf{r}_j}{\omega - \omega_j} / \sum_{j=1}^S \frac{q_j}{\omega - \omega_j}$$

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The original surrogate can easily be recovered with

$$\tilde{\mathbf{u}}(\omega) = \underline{\mathbf{U}} \mathring{\mathbf{u}}(\omega) \quad \text{or} \quad \tilde{\mathbf{u}}(\omega) = \underline{\mathbf{Q}} \hat{\mathbf{u}}(\omega)$$

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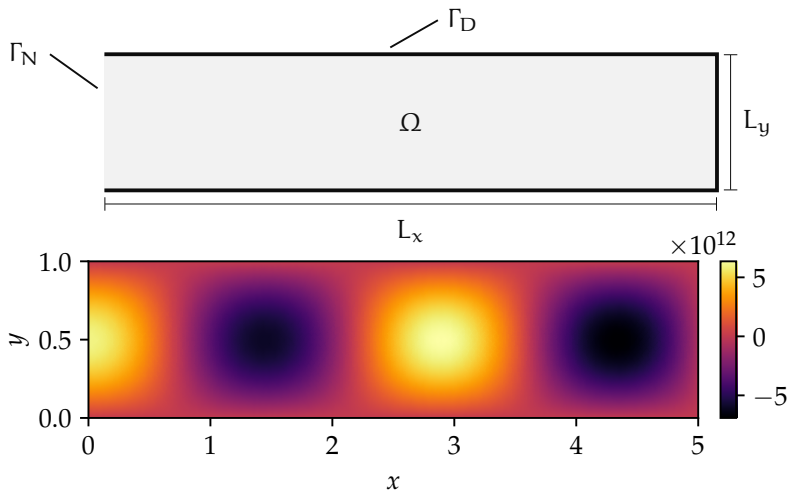
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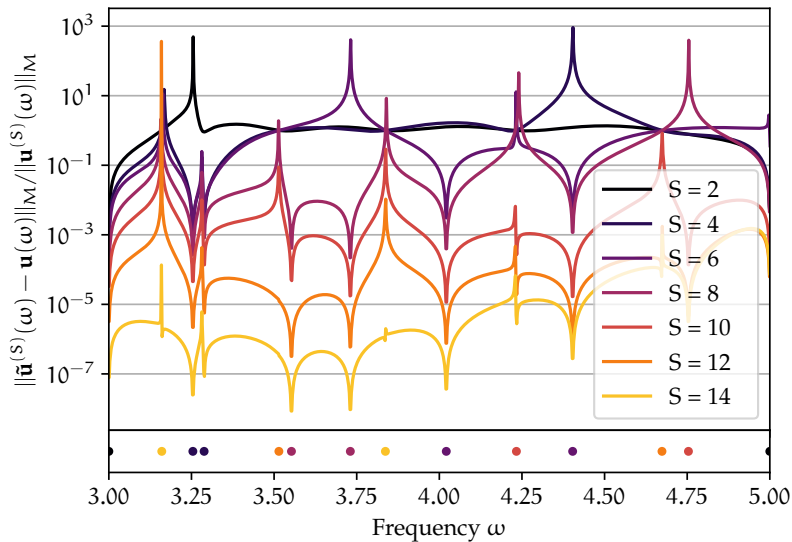
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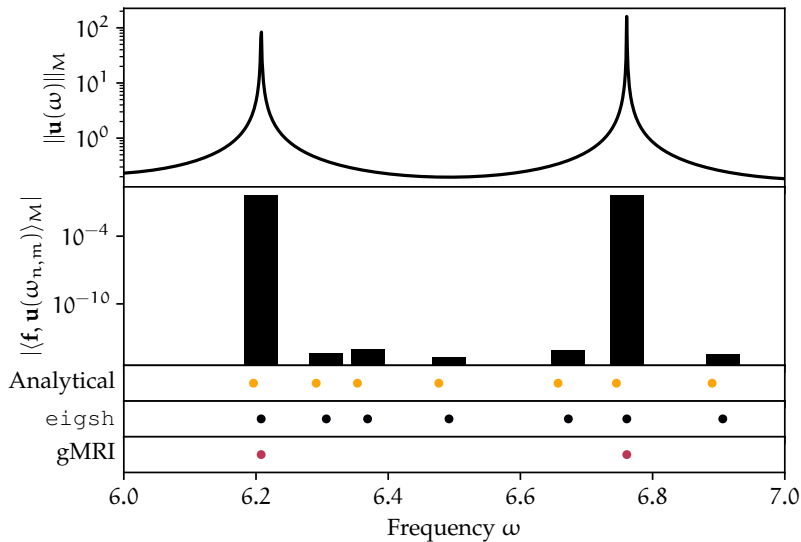
Proposed way of computing the stopping criterion in gMRI

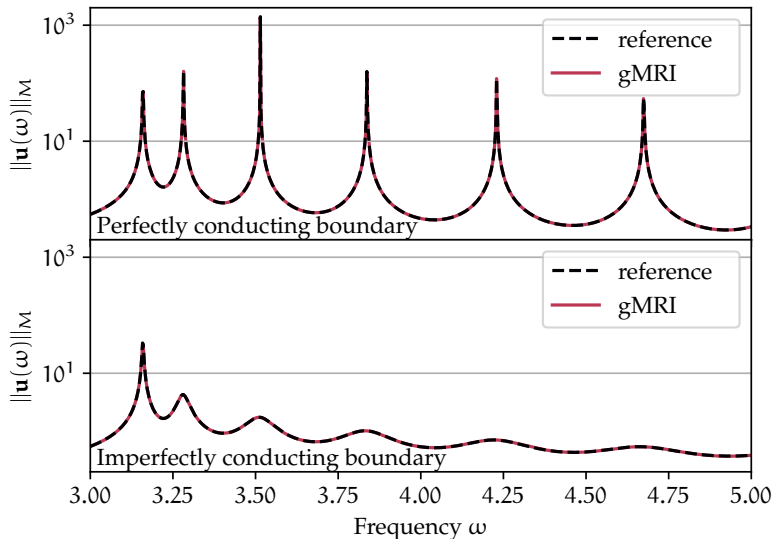
$$\frac{\|\mathbf{u}(\omega^{(t+1)}) - \tilde{\mathbf{u}}^{(t)}(\omega^{(t+1)})\|_{\mathcal{M}}}{\|\mathbf{u}(\omega^{(t+1)})\|_{\mathcal{M}}} = \frac{\|\mathbf{r}_{t+1} - \hat{\mathbf{u}}^{(t)}(\omega^{(t+1)})\|}{\|\mathbf{r}_{t+1}\|}$$

$$\tilde{\mathbf{u}}(\omega) = \sum_{j=1}^S \frac{q_j \mathbf{u}(\omega_j)}{\omega - \omega_j} / \sum_{j=1}^S \frac{q_j}{\omega - \omega_j}$$

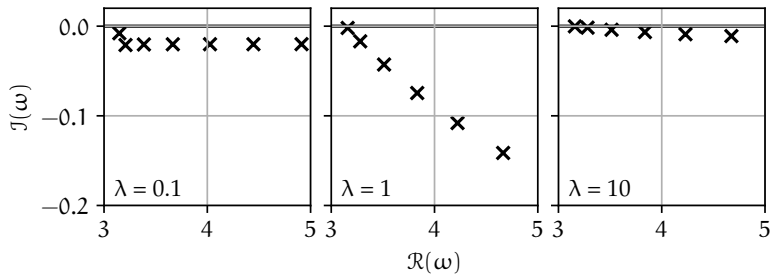






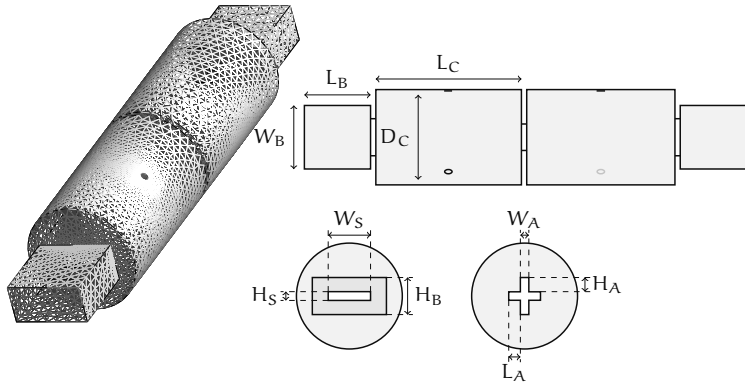


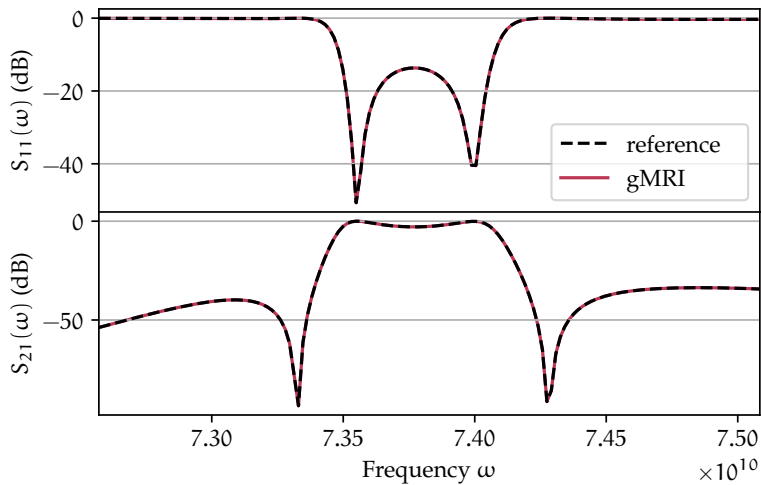
Resonances are shifted into the complex plane



$$\mathbf{g} = i\omega\lambda(\mathbf{n} \times \mathbf{u}) \times \mathbf{n}$$

Dual-mode circular waveguide filter (DMCWF)





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- ▶ Exact dimensions and reference needed for DMCWF

- [1] F. Bonzzoni, D. Pradovera, and M. Ruggeri. Rational-based model order reduction of helmholtz frequency response problems with adaptive finite element snapshots. 2021. doi: 10.48550/arXiv.2112.04302.
- [2] H. P. Langtangen and A. Logg. *Solving PDEs in Python: The FEniCS Tutorial I*. Springer, 2016. ISBN 978-3-319-52461-0. doi: 10.1007/978-3-319-52462-7.
- [3] P. Monk. *Finite Element Methods for Maxwell's Equations*. Oxford Science Publications, 2003. ISBN 0-19-850888-3.
- [4] D. Pradovera and F. Nobile. Frequency-domain non-intrusive greedy model order reduction based on minimal rational approximation. pages 159–167, 2021. doi: 10.1007/978-3-030-84238-3_16.
- [5] L. N. Trefethen. Householder triangularization of a quasimatrix. *IMA Journal of Numerical Analysis*, 30(4): 887–897, 2010. doi: 10.1093/imanum/drp018.

FEniCS [2] is used to obtain FEM solutions of the form

$$\mathbf{u}_h(\omega) = \sum_{i=1}^{N_h} u_i(\omega) \boldsymbol{\phi}_h^{(i)} \quad (1)$$

for a basis $\{\boldsymbol{\phi}_h^{(i)}\}_{i=1}^{N_h}$ of the finite dimensional subspace $H_{\text{curl},h}(\Omega) \subset H_{\text{curl}}(\Omega)$ (Nédélec finite elements of the first kind).

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From now on

$$\mathbf{u} = (u_1, u_2, \dots, u_{N_h})^T$$

with the $L_2(\Omega)$ inner product in $H_{\text{curl},h}(\Omega)$ represented by

$$\langle \mathbf{u}, \mathbf{v} \rangle_M = \mathbf{u}^H \underline{\mathbf{M}} \mathbf{v}$$

and the norm

$$\|\mathbf{u}\|_M = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle_M}$$

Find ω , such that

$$0 = Q(\omega) = \sum_{j=1}^s \frac{q_j}{\omega - \omega_j}$$

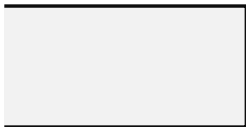
Equivalent eigenvalue problem

$$\underline{\mathbf{A}}\mathbf{w} = \omega \underline{\mathbf{B}}\mathbf{w}$$

with

$$\underline{\mathbf{A}} = \begin{pmatrix} 0 & q_1 & q_2 & \dots & q_s \\ 1 & \omega_1 & & & \\ 1 & & \omega_2 & & \\ \vdots & & & \ddots & \\ 1 & & & & \omega_s \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{B}} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Resonant cavity

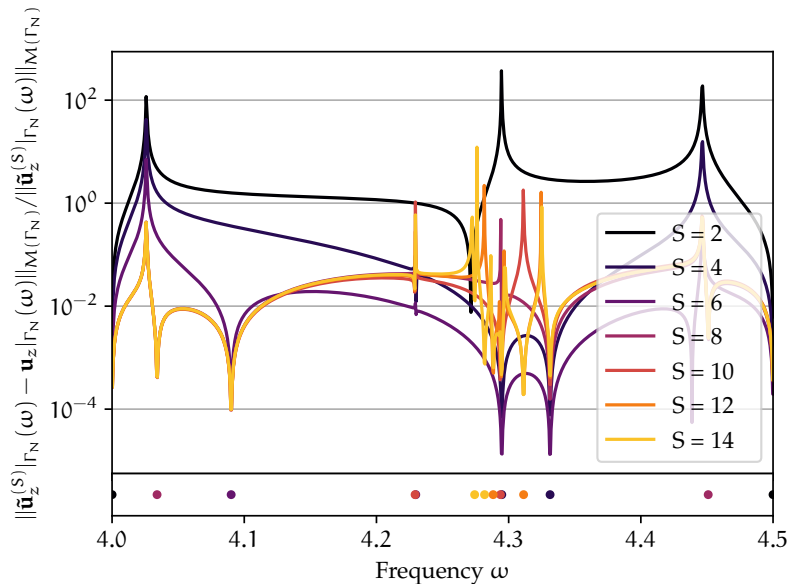


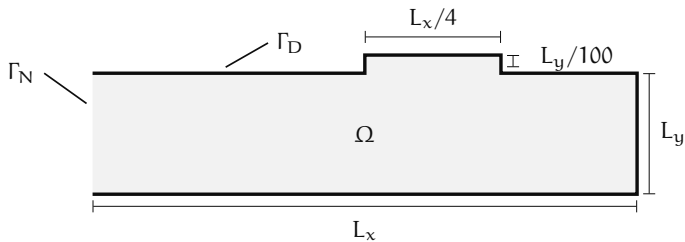
Imperfect conductor

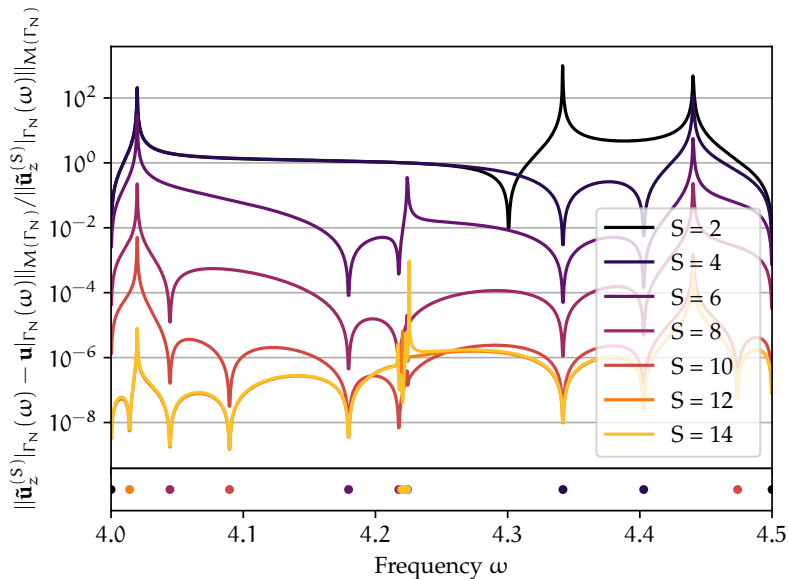


Waveguide

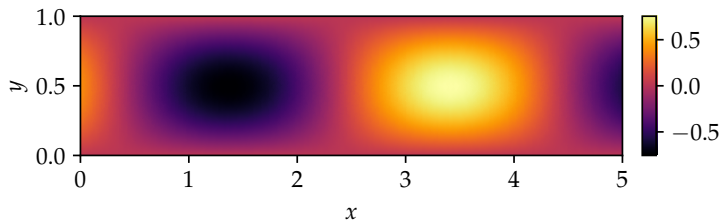








	eigsh		gMRI	
DOF	$\bar{\Delta}$	t	$\bar{\Delta}$	t
713	1.950×10^{-2}	25.9 ± 1.1 ms	1.950×10^{-2}	61.9 ± 3.6 ms
7412	1.826×10^{-3}	199.0 ± 9.9 ms	1.827×10^{-3}	410.0 ± 16.8 ms
74722	1.817×10^{-4}	3.5 ± 0.1 s	1.820×10^{-4}	5.2 ± 0.2 s
745513	1.811×10^{-5}	75.0 ± 1.6 s	1.846×10^{-5}	104.0 ± 1.1 s



	eigs	gMRI
DOF	t	t
713	57.8 ± 2.35 ms	62.8 ± 0.8 ms
7412	861.0 ± 42.4 ms	498.0 ± 11.7 ms
74722	21.8 ± 1.1 s	5.9 ± 0.3 s

