Minimal rational interpolation for time-harmonic Maxwell's equations

June 24, 2022

Fabio Matti

To locally approximate

$$\mathbf{u} : \mathbb{C} \ni \omega \mapsto \mathbf{u}(\omega)$$

To locally approximate

$$\mathbf{u} : \mathbb{C} \ni \omega \mapsto \mathbf{u}(\omega)$$

compute

$$\mathbf{u}(\omega_1), \mathbf{u}(\omega_2), \ldots, \mathbf{u}(\omega_S) \qquad \text{(Snapshots)}$$

To locally approximate

$$\mathbf{u} : \mathbb{C} \ni \omega \mapsto \mathbf{u}(\omega)$$

compute

$$\mathbf{u}(\omega_1), \mathbf{u}(\omega_2), \ldots, \mathbf{u}(\omega_S) \qquad \text{(Snapshots)}$$

and build

$$\tilde{\mathbf{u}}(\omega) = \frac{\mathbf{P}(\omega)}{q(\omega)} \qquad \text{(Rational surrogate)}$$

such that

$$\tilde{\mathbf{u}}(\omega) \approx \mathbf{u}(\omega)$$

close to $\omega_1, \omega_2, \ldots, \omega_S$.

- Problem formulation

Outline

- Problem formulation
- Finite element method

- Problem formulation
- Finite element method
- Minimal rational interpolation

- Problem formulation
- Finite element method
- Minimal rational interpolation
- Example applications

- Problem formulation
- Finite element method
- Minimal rational interpolation
- Example applications
- Conclusion and outlook

Time-harmonic vector potential $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \exp(i\omega t)$.

$$\mathbf{B} = \nabla \times \mathbf{u} \qquad \text{(Magnetic field)}$$

$$\mathbf{E} = -i\omega\mathbf{u} \qquad \text{(Electric field)}$$

Time-harmonic vector potential $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \exp(\imath\omega t)$.

$$\mathbf{B} = \nabla \times \mathbf{u} \qquad \text{(Magnetic field)}$$

$$\mathbf{E} = -\imath\omega\mathbf{u} \qquad \text{(Electric field)}$$

Maxwell's equation

$$\nabla \times (\mu^{-1}\mathbf{B}) - \partial_t(\epsilon\mathbf{E}) = \mathbf{j}$$

Time-harmonic vector potential $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \exp(i\omega t)$.

$$\mathbf{B} = \nabla \times \mathbf{u} \qquad \text{(Magnetic field)}$$

$$\mathbf{E} = -i\omega\mathbf{u} \qquad \text{(Electric field)}$$

Maxwell's equation

$$\nabla \times (\mu^{-1}\mathbf{B}) - \partial_t(\epsilon\mathbf{E}) = \mathbf{j}$$

Time-harmonic potential equation

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) - \epsilon\omega^2\mathbf{u} = \mathbf{j} \tag{1}$$

$$\mathsf{H}_{\text{curl}}(\Omega) = \{\mathbf{v} : \Omega \to \mathbb{C}^{\mathsf{3}},\ \text{such that } \mathbf{v} \in \mathsf{L}_{\mathsf{2}}(\mathbb{C})^{\mathsf{3}},\ \nabla \times \mathbf{v} \in \mathsf{L}_{\mathsf{2}}(\mathbb{C})^{\mathsf{3}}\}$$

$$H_{\text{curl}}(\Omega) = \{\mathbf{v} : \Omega \to \mathbb{C}^3,\ \text{such that } \mathbf{v} \in L_2(\mathbb{C})^3,\ \nabla \times \mathbf{v} \in L_2(\mathbb{C})^3\}$$

Weak formulation of the time-harmonic potential equation

Find $\mathbf{u} \in H_{\text{curl}}(\Omega)$, such that

$$\int_\Omega \langle \mu^{-1} \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle - \omega^2 \int_\Omega \epsilon \langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \langle \mathbf{j}, \mathbf{v} \rangle + \int_{\partial\Omega} \langle \mathbf{g}, \mathbf{v} \rangle \quad (2)$$

for all $\mathbf{v} \in H_{\text{curl}}$, where $\mathbf{g} = (\mu^{-1} \nabla \times \mathbf{u}) \times \mathbf{n}$.

Perfectly conducting boundary

$$\mathbf{u} \times \mathbf{n} = 0, \text{ on } \Gamma_{\mathrm{D}}$$

Perfectly conducting boundary

$$\mathbf{u} \times \mathbf{n} = 0, \text{ on } \Gamma_D$$

Inlet, i.e. **B** is given along Γ_N

$$\mathbf{g} = (\mu^{-1}\mathbf{B}) \times \mathbf{n}, \text{ on } \Gamma_N$$

Perfectly conducting boundary

$$\mathbf{u} \times \mathbf{n} = 0, \text{ on } \Gamma_D$$

Inlet, i.e. $\mathbf{B}$ is given along Γ_N

$$\mathbf{g} = (\mu^{-1}\mathbf{B}) \times \mathbf{n}, \text{ on } \Gamma_N$$

Imperfectly conducting boundary

$$\mathbf{g} = i\omega\lambda(\mathbf{n} \times \mathbf{u}) \times \mathbf{n}, \text{ on } \Gamma_I$$

TODO

Rational surrogate

$$\tilde{\mathbf{u}}(\omega) = \sum_{i=1}^{S} \frac{\mathbf{p}_i}{\omega - \omega_i} \Big/ \sum_{i=1}^{S} \frac{q_i}{\omega - \omega_i}$$

in barycentric coordinates with support points $\omega_1, \omega_2, \ldots, \omega_3$.

Rational surrogate

$$\tilde{\mathbf{u}}(\omega) = \sum_{i=1}^{S} \frac{\mathbf{p}_i}{\omega - \omega_i} / \sum_{i=1}^{S} \frac{q_i}{\omega - \omega_i}$$

in barycentric coordinates with support points $\omega_1, \omega_2, \ldots, \omega_3$.
Interpolation property

$$\tilde{\mathbf{u}}(\omega_i) = \mathbf{u}(\omega_i), \ \forall i \in \{1, 2, \ldots, S\}$$

Given snapshots $\mathbf{u}(\omega_1), \mathbf{u}(\omega_2), \ldots, \mathbf{u}(\omega_S)$:

Given snapshots $\mathbf{u}(\omega_1), \mathbf{u}(\omega_2), \ldots, \mathbf{u}(\omega_S)$:

1. Compute the Gramian matrix $\underline{\mathbf{G}}$ with $g_{ij} = \langle \mathbf{u}(\omega_i), \mathbf{u}(\omega_j) \rangle$, $i, j \in \{1, 2, \ldots, S\}$

Given snapshots $\mathbf{u}(\omega_1), \mathbf{u}(\omega_2), \dots, \mathbf{u}(\omega_S)$:

1. Compute the Gramian matrix $\underline{\mathbf{G}}$ with $g_{ij} = \langle \mathbf{u}(\omega_i), \mathbf{u}(\omega_j) \rangle$, $i, j \in \{1, 2, \dots, S\}$
2. Compute the singular value decomposition $\mathbf{G} = \underline{\mathbf{V}}\, \underline{\boldsymbol{\Sigma}}\, \underline{\mathbf{V}}^H$

Given snapshots $\mathbf{u}(\omega_1), \mathbf{u}(\omega_2), \ldots, \mathbf{u}(\omega_S)$:

1. Compute the Gramian matrix $\underline{\mathbf{G}}$ with $g_{ij} = \langle \mathbf{u}(\omega_i), \mathbf{u}(\omega_j) \rangle$, $i, j \in \{1, 2, \ldots, S\}$
2. Compute the singular value decomposition $\mathbf{G} = \underline{\mathbf{V}}\,\underline{\boldsymbol{\Sigma}}\,\underline{\mathbf{V}}^H$
3. Define $\mathbf{q} = (q_1, q_2, \ldots, q_S)^T = \underline{\mathbf{V}}[:, S]$

Given snapshots $\mathbf{u}(\omega_1), \mathbf{u}(\omega_2), \ldots, \mathbf{u}(\omega_S)$:

1. Compute the Gramian matrix $\underline{\mathbf{G}}$ with $g_{ij} = \langle \mathbf{u}(\omega_i), \mathbf{u}(\omega_j) \rangle$, $i, j \in \{1, 2, \ldots, S\}$
2. Compute the singular value decomposition $\mathbf{G} = \underline{\mathbf{V}}\, \underline{\mathbf{\Sigma}}\, \underline{\mathbf{V}}^H$
3. Define $\mathbf{q} = (q_1, q_2, \ldots, q_S)^T = \underline{\mathbf{V}}[:, S]$
4. Define $\tilde{\mathbf{u}}(\omega) = \mathbf{P}(\omega)/Q(\omega)$ with $\mathbf{P}(\omega) = \sum_{j=1}^{S} \frac{q_j \mathbf{u}(\omega_j)}{\omega - \omega_j}$ and $Q(\omega) = \sum_{j=1}^{S} \frac{q_j}{\omega - \omega_j}$

1. Define $\Omega_{\text{test}} = \{\omega_1, \omega_2, \ldots, \omega_M\}$ as candidate support points

1. Define $\Omega_{\text{test}} = \{\omega_1, \omega_2, \ldots, \omega_M\}$ as candidate support points
2. Build surrogate $\tilde{\mathbf{u}}_t$ with initial support points $\omega_1, \omega_2, \cdot, \omega_t \in \Omega_{\text{test}}$

1. Define $\Omega_{\text{test}} = \{\omega_1, \omega_2, \ldots, \omega_M\}$ as candidate support points
2. Build surrogate $\tilde{\mathbf{u}}_t$ with initial support points $\omega_1, \omega_2, \cdot, \omega_t \in \Omega_{\text{test}}$
3. Iteratively add a new support point $\omega_{t+1} = argmin_{\omega \in \Omega_{\text{test}}} |Q_t(\omega)|$ to build $\tilde{\mathbf{u}}_{t+1}$ based on $\mathbf{u}(\omega_1), \mathbf{u}(\omega_2), \ldots, \mathbf{u}(\omega_{t+1})$

1. Define $\Omega_{\text{test}} = \{\omega_1, \omega_2, \ldots, \omega_M\}$ as candidate support points
2. Build surrogate $\tilde{\mathbf{u}}_t$ with initial support points $\omega_1, \omega_2, \cdot, \omega_t \in \Omega_{\text{test}}$
3. Iteratively add a new support point $\omega_{t+1} = \mathrm{argmin}_{\omega \in \Omega_{\text{test}}} |Q_t(\omega)|$ to build $\tilde{\mathbf{u}}_{t+1}$ based on $\mathbf{u}(\omega_1), \mathbf{u}(\omega_2), \ldots, \mathbf{u}(\omega_{t+1})$
4. Stop when relative error $\|\mathbf{u}_{t+1} - \tilde{\mathbf{u}}_{t+1}\| / \|\mathbf{u}_{t+1}\|$ is small enough

Householder, sequential

Norms

We want to find ω, such that

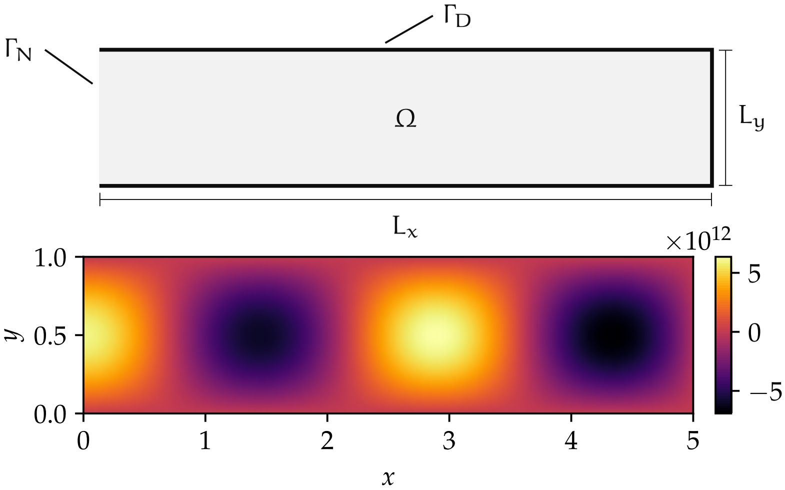
$$0 = Q(\omega) = \sum_{i=1}^{S} \frac{q_i}{\omega - \omega_i}$$

Equivalently eigenvalue problem

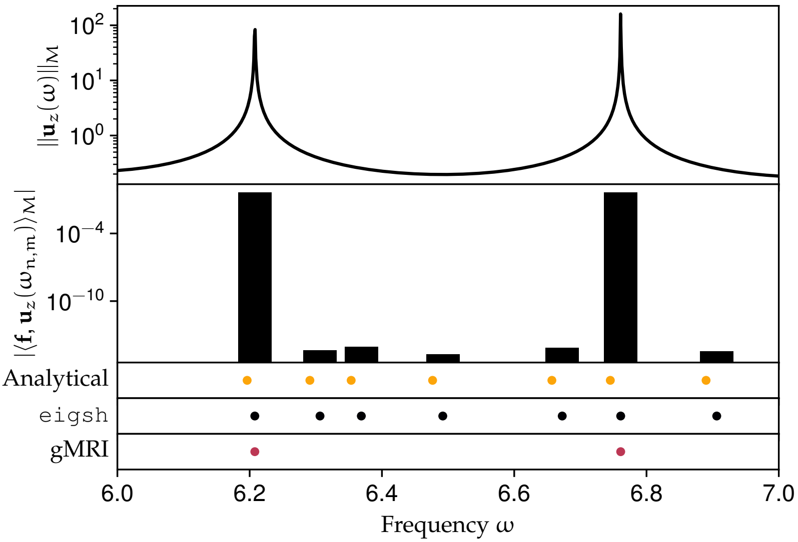
$$\underline{\mathbf{A}}\mathbf{w} = \omega \underline{\mathbf{B}}\mathbf{w}$$

with

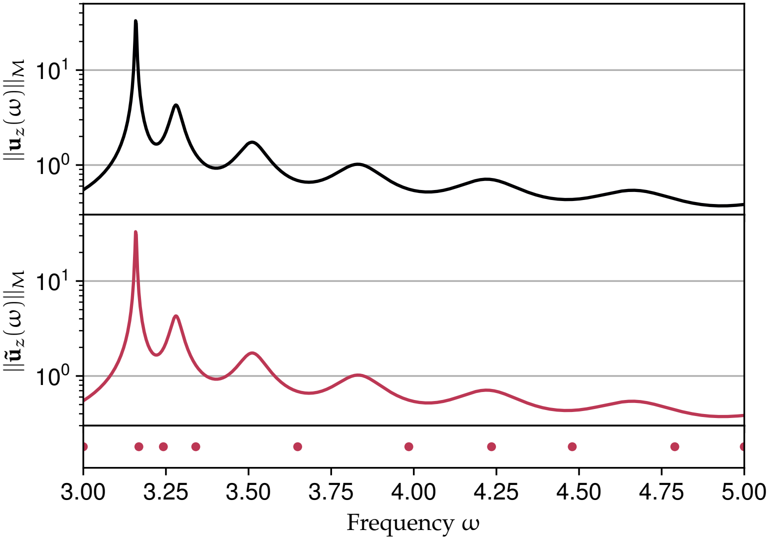
$$\underline{\mathbf{A}} = \begin{pmatrix} 0 & q_1 & q_2 & \dots & q_S \\ 1 & \omega_1 & & & \\ 1 & & \omega_2 & & \\ \vdots & & & \ddots & \\ 1 & & & & \omega_S \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{B}} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \end{pmatrix}$$



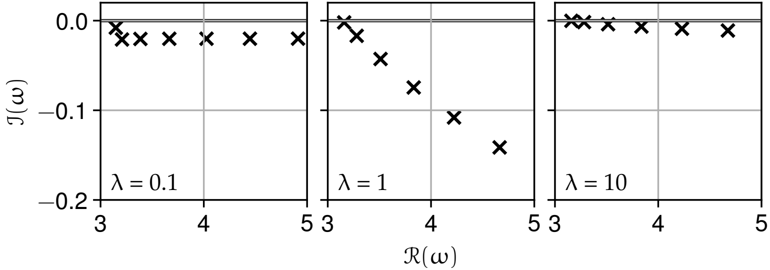
Γ_D
Γ_N
Ω
L_y
L_x
$\times 10^{12}$
1.0
0.5
0.0
y
0
1
2
3
4
5
x
5
0
−5



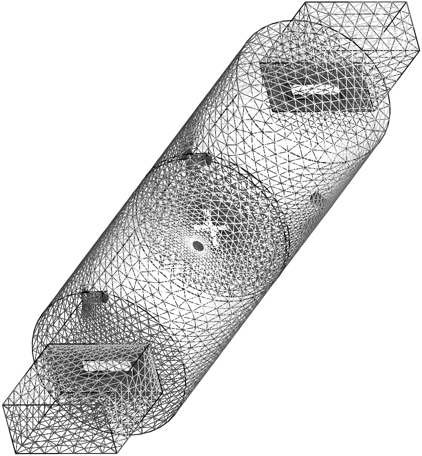
$\|\mathbf{u}_z(\omega)\|_M$
10^2
10^1
10^0
$|\langle \mathbf{f}, \mathbf{u}_z(\omega_{n,m})\rangle_M|$
10^{-4}
10^{-10}
Analytical
eigsh
gMRI
6.0
6.2
6.4
6.6
6.8
7.0
Frequency ω

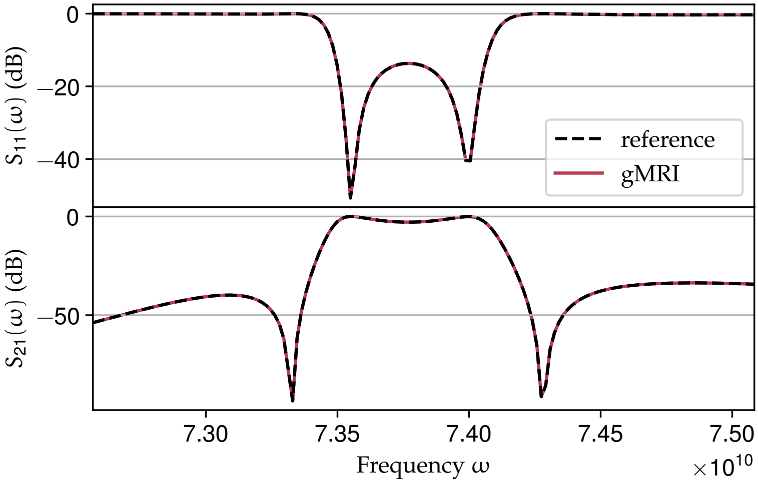

$\|\mathbf{u}_z(\omega)\|_M$
10^1
10^0
$\|\tilde{\mathbf{u}}_z(\omega)\|_M$
10^1
10^0
3.00
3.25
3.50
3.75
4.00
4.25
4.50
4.75
5.00
Frequency ω

Example application 2 | Resonances



Dual-mode circular waveguide filter




0
−20
−40
$S_{11}(\omega)$ (dB)
reference
gMRI
0
−50
$S_{21}(\omega)$ (dB)
7.30
7.35
7.40
7.45
7.50
Frequency ω
$\times 10^{10}$

- xy

Conclusion and outlook

- xy
- xy

- xy
- xy
- DMCWF exact dimensions and reference