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Symplectic Topology and Floer Homology

Volume 2: Floer Homology and its Applications

Yong-Geun Oh

Symplectic Topology and Floer Homology

Volume 2

Published in two volumes, this is the first book to provide a thorough and systematic explanation of symplectic topology, and the analytical details and techniques used in applying the machinery arising from Floer theory as a whole.

Volume 1 covers the basic materials of Hamiltonian dynamics and symplectic geometry and the analytic foundations of Gromov's pseudoholomorphic curve theory. One novel aspect of this treatment is the uniform treatment of both closed and open cases and a complete proof of the boundary regularity theorem of weak solutions of pseudoholomorphic curves with totally real boundary conditions. Volume 2 provides a comprehensive introduction to both Hamiltonian Floer theory and Lagrangian Floer theory, including many examples of their applications to various problems in symplectic topology.

Symplectic Topology and Floer Homology is a comprehensive resource suitable for experts and newcomers alike.

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Contents of Volume 1

Preface

This is a two-volume series of monographs. This series provides a self-contained exposition of basic Floer homology in both open and closed string contexts, and systematic applications to problems in Hamiltonian dynamics and symplectic topology. The basic objects of study in these two volumes are the geometry of Lagrangian submanifolds and the dynamics of Hamiltonian diffeomorphisms and their interplay in symplectic topology.

The classical Darboux theorem in symplectic geometry reveals the *flexibility* of the group of symplectic transformations. On the other hand, Gromov and Eliashberg's celebrated theorem (El87) reveals the subtle *rigidity* of symplectic transformations: *the subgroup* $\mathrm{Symp}(M, \omega)$ *consisting of symplectomorphisms is closed in* $\mathrm{Diff}(M)$ *with respect to the* C^0 *topology.* This demonstrates that the study of symplectic topology is subtle and interesting. Eliashberg's theorem relies on a version of the non-squeezing theorem, such as the one proved by Gromov (Gr85) using the machinery of pseudoholomorphic curves. Besides Eliashberg's original combinatorial proof of this non-squeezing result, there is another proof given by Ekeland and Hofer (EkH89) using the classical variational approach of Hamiltonian systems. The interplay between these two facets of symplectic geometry, namely the analysis of pseudoholomorphic curves and Hamiltonian dynamics, has been the main driving force in the development of symplectic topology since Floer's pioneering work on his semi-infinite dimensional homology theory, which we now call *Floer homology* theory.

Hamilton's equation $\dot{x} = X_H(t, x)$ arises in Hamiltonian mechanics and the study of its dynamics has been a fundamental theme of investigation in physics since the time of Lagrange, Hamilton, Jacobi, Poincaré and others. Many mathematical tools have been developed in the course of understanding its dynamics and finding explicit solutions of the equation. One crucial tool for the study of the questions is the *least action principle*: a solution of Hamilton's equation corresponds to a critical point of some action functional. In this variational principle, there are two most important boundary conditions considered for the equation $\dot{x} = X_H(t, x)$ on a general symplectic manifold: one is the *periodic* boundary condition $\gamma(0) = \gamma(1)$, and the other is the *Lagrangian* boundary condition $\gamma(0) \in L_0$, $\gamma(1) \in L_1$ for a given pair (L_0, L_1) of two Lagrangian submanifolds. A submanifold $i : L \hookrightarrow (M, \omega)$ is called Lagrangian if $i^*\omega = 0$ and $\dim L = \frac{1}{2} \dim M$. This replaces the *two-point* boundary condition in classical mechanics.

A diffeomorphism ϕ of a symplectic manifold (M, ω) is called a Hamiltonian diffeomorphism if ϕ is the time-one map of $\dot{x} = X_H(t, x)$ for some (time-dependent) Hamiltonian H. The set of such diffeomorphisms is denoted by $\mathrm{Ham}(M, \omega)$. It forms a subgroup of $\mathrm{Symp}(M, \omega)$. However, in the author's opinion, it is purely a historical accident that Hamiltonian diffeomorphisms are studied because the definition of $\mathrm{Ham}(M, \omega)$ is not a priori natural. For example, it is not a structure group of any geometric structure associated with the smooth manifold M (or at least not of any structure known as yet), unlike the case of $\mathrm{Symp}(M, \omega)$, which is the automorphism group of the symplectic structure ω. Mathematicians' interest in $\mathrm{Ham}(M, \omega)$ is largely motivated by the celebrated conjecture of Arnol'd (Ar65), and Floer homology was invented by Floer (Fl88b, Fl89b) in his attempt to prove this conjecture.

Since the advent of Floer homology in the late 1980s, it has played a fundamental role in the development of symplectic topology. (There is also the parallel notion in lower-dimensional topology which is not touched upon in these two volumes. We recommend for interested readers Floer's original article (Fl88c) and the masterpiece of Kronheimer and Mrowka (KM07) in this respect.) Owing to the many technicalities involved in its rigorous definition, especially in the case of Floer homology of Lagrangian intersections (or in the context of 'open string'), the subject has been quite inaccessible to beginning graduate students and researchers coming from other areas of mathematics. This is partly because there is no existing literature that systematically explains the problems of symplectic topology, the analytical details and the techniques involved in applying the machinery embedded in the Floer theory as a whole. In the meantime, Fukaya's categorification of Floer homology, i.e., his introduction of an A_∞ category into symplectic geometry (now called the Fukaya category), and Kontsevich's homological mirror symmetry proposal, followed by the development of open string theory of D branes in physics, have greatly enhanced the Floer theory and attracted much attention from other mathematicians and physicists as well as the traditional symplectic geometers and topologists. In addition, there has also been considerable research into applications of symplectic ideas to various problems in (area-preserving) dynamical systems in two dimensions.

Our hope in writing these two volumes is to remedy the current difficulties to some extent. To achieve this goal, we focus more on the foundational materials of Floer theory and its applications to various problems arising in symplectic topology, with which the author is more familiar, and attempt to provide complete analytic details assuming the reader's knowledge of basic elliptic theory of (first-order) partial differential equations, second-year graduate differential geometry and first-year algebraic topology. In addition, we also try to motivate various constructions appearing in Floer theory from the historical context of the classical Lagrange–Hamilton variational principle and Hamiltonian mechanics. The choice of topics included in the book is somewhat biased, partly due to the limitations of the author's knowledge and confidence level, and also due to his attempt to avoid too much overlap with the existing literature on symplectic topology. We would like to particularly cite the following three monographs among others and compare these two volumes with them:

(1) *J-Holomorphic Curves and Symplectic Topology*, McDuff, D., Salamon, D., 2004.
(2) *Fukaya Categories and Picard–Lefschetz Theory*, Seidel, P., 2008.
(3) *Lagrangian Intersection Floer Theory: Anomaly and Obstruction*, volumes I & II, Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K, 2009.

(There is another more recent monograph by Audin and Damian (AD14), which was originally written in French and then translated into English.)

First of all, Parts 2 and 3 of these two volumes could be regarded as the prerequisite for graduate students or post-docs to read the book (3) (FOOO09) in that the off-shell setting of Lagrangian Floer theory in Volume 2 presumes the presence of non-trivial instantons, or non-constant holomorphic discs or spheres. However, we largely limit ourselves to the monotone case and avoid the full-fledged obstruction–deformation theory of Floer homology which would inevitably involve the theory of A_∞-structures and the abstract perturbation theory of virtual moduli technique such as the Kuranishi structure, which is beyond the scope of these two volumes. Luckily, the books (2) (Se08) and (3) (FOOO09) cover this important aspect of the theory, so we strongly encourage readers to consult them. We also largely avoid any extensive discussion on the Floer theory of exact Lagrangian submanifolds, except for the cotangent bundle case, because Seidel's book (Se08) presents an extensive study of the Floer theory and the Fukaya category in the context of exact symplectic geometry to which we cannot add anything. There is much overlap of the materials in Part 2 on the basic pseudoholomorphic curve theory with Chapters 1–6 of the book by McDuff and Salamon (1) (MSa04). However, our exposition of the materials is quite different from that of (MSa04). For example, from the beginning, we deal with pseudoholomorphic curves of arbitrary genus and with a boundary and unify the treatment of both closed and open cases, e.g., in the regularity theory of weak solutions and in the removal singularity theorem. Also we discuss the transversality issue after that of compactness, which seems to be more appropriate for accommodating the techniques of Kuranishi structure and abstract perturbation theory when the readers want to go beyond the semi-positive case. There are also two other points that we are particularly keen about in our exposition of pseudoholomorphic curve theory. One is to make the relevant geometric analysis resemble the style of the more standard geometric analysis in Riemannian geometry, emphasizing the tensor calculations via the canonical connection associated with the almost-Kähler property whenever possible. In this way we derive the relevant $W^{k,p}$-coercive estimates, especially an optimal $W^{2,2}$-estimate with Neumann boundary condition, by pure tensor calculations and an application of the Weitzenböck formula. The other is to make the deformation theory of pseudoholomorphic curves resemble that of holomorphic curves on (integrable) Kähler manifolds. We hope that this style of exposition will widen the readership beyond the traditional symplectic geometers to graduate students and researchers from other areas of mathematics and enable them to more easily access important developments in symplectic topology and related areas.

Now comes a brief outline of the contents of each part of the two volumes. The first volume consists of Parts 1 and 2.

Part 1 gives an introduction to symplectic geometry starting from the classical variational principle of Lagrange and Hamilton in classical mechanics and introduces the main concepts in symplectic geometry, i.e., Lagrangian submanifolds, Hamiltonian diffeomorphisms and symplectic fibrations. It also introduces Hofer's geometry of Hamiltonian diffeomorphisms. Then the part ends with the proof of the Gromov–Eliashberg C^0-rigidity theorem (El87) and the introduction to continuous Hamiltonian dynamics and the concept of Hamiltonian homeomorphisms introduced by Müller and the present author (OhM07).

Part 2 provides a mostly self-contained exposition of the analysis of pseudoholomorphic curves and their moduli spaces. We attempt to provide the optimal form of a-priori elliptic estimates for the nonlinear Cauchy–Riemann operator $\overline{\partial}_J$ *in the off-shell setting*. For this purpose, we emphasize our usage of the canonical connection of the almost-Kähler manifold (M, ω, J). Another novelty of our treatment of the analysis is a complete proof of the boundary regularity theorem of weak solutions (in the sense of Ye (Ye94)) of J-holomorphic curves with totally real boundary conditions. As far as we know, this regularity proof has not been given before in the existing literature. We also give a complete proof of compactness of the stable map moduli space following the approach taken by Fukaya and Ono (FOn99). The part ends with an explanation of how compactness–cobordism analysis of the moduli space of (perturbed) pseudoholomorphic curves combined with a bit of symplectic topological data give rise to the proofs of two basic theorems in symplectic topology; Gromov's non-squeezing theorem and the nondegeneracy of Hofer's norm on $\mathrm{Ham}(M, \omega)$ (for tame symplectic manifolds).

The second volume consists of Parts 3 and 4. Part 3 gives an introduction to Lagrangian Floer homology restricted to the special cases of monotone Lagrangian submanifolds. It starts with an overview of Lagrangian intersection Floer homology on cotangent bundles and introduces all the main objects of study that enter into the recent Lagrangian intersection Floer theory without delving too much into the technical details. Then it explains the compactification of Floer moduli spaces, the details of which are often murky in the literature. The part ends with the construction of a spectral sequence, a study of Maslov class obstruction to displaceable Lagrangian submanifolds and Polterovich's theorem on the Hofer diameter of $\mathrm{Ham}(S^2)$.

Part 4 introduces Hamiltonian Floer homology and explains the complete construction of spectral invariants and various applications. The applications include construction of the spectral norm, Usher's proofs of the minimality conjecture in the Hofer geometry and the optimal energy–capacity inequality. In particular, this part contains a complete self-contained exposition of the Entov–Polterovich construction of spectral quasimorphisms and the associated symplectic quasi-states. The part ends with further discussion of topological Hamiltonian flows and their relation to the geometry of area-preserving homeomorphisms in two dimensions.

The prerequisites for the reading of these two volumes vary part by part. A standard first-year graduate differentiable manifold course together with a little bit of knowledge on the theory of fiber bundles should be enough for Part 1. However, Part 2, the most technical part of the book, which deals with the general theory of pseudoholomorphic curves, the moduli spaces thereof and their stable map compactifications, assumes readers' knowledge of the basic language of Riemannian geometry (e.g., that of Volumes 1 and 2 of Spivak (Spi79)), basic functional analysis (e.g., Sobolev embedding and Reillich compactness and others), elliptic (first-order) partial differential equations and first-year algebraic topology. The materials in Parts 3 and 4, which deal with the main topics of Floer homology both in the open and in the closed string context, rely on the materials of Parts 1 and 2 and should be readable on their own. Those who are already familiar with basic symplectic geometry and analysis of pseudoholomorphic curves should be able to read Parts 3 and 4 immediately. This book can be used as a graduate textbook for the introduction to Gromov and Floer's analytic approach to modern symplectic topology. Readers who would like to learn more about various deeper aspects of symplectic topology and mirror symmetry are strongly encouraged to read the books (1)–(3) mentioned above in addition, depending on their interest.

The author would like to end this preface by recalling his personal experience and perspective, which might not be shared by others, but which he hopes may help readers to see how the author came up with the current shape of these two volumes. The concept of symplectic topology emerged from Eliashberg and Gromov's celebrated symplectic rigidity theorem. Eliashberg's original proof was based on the existence of some C^0-type invariant of the symplectic diffeomorphism which measures the size of domains in the symplectic vector space. The existence of such an invariant was first established by Gromov as a corollary of his fundamental *non-squeezing theorem* that was proven by using the analytical method of pseudoholomorphic curves. With the advent of the method of pseudoholomorphic curves developed by Gromov and Floer's subsequent invention of elliptic Morse theory that resulted in Floer homology, the landscape of symplectic geometry changed drastically. Many previously intractable problems in symplectic geometry were solved by the techniques of pseudoholomorphic curves, and the concept of symplectic topology gradually began to take shape.

There are two main factors determining how the author shaped the structure of the present book. The first concerns how the analytical materials are treated in Volume 1. The difficulty, or the excitement, associated with the method of pseudoholomorphic curves at the time of its appearance was that it involves a mathematical discipline of a nature very different from the type of mathematics employed by the mainstream symplectic geometers at that time. As a result the author feels that it created some discontinuity between the symplectic geometry before and after Gromov's paper appeared, and the analysis presented was given quite differently from how such matters are normally treated by geometric analysts of Riemannian geometry. For example, the usage of tensor calculations is not emphasized as much as in Riemannian geometry. Besides, in the author's personal experience, there were two stumbling blocks hindering getting into the Gromov–Witten–Floer theory as a graduate student and as a beginning researcher working in the area of symplectic topology. The first was the need to get rid of some phobia towards the abstract algebraic geometric materials like the Deligne–Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves, and the other was the need to absorb the large amount of analytical materials that enter into the study of moduli spaces of pseudoholomorphic maps from the original sources in the literature of the relevant mathematics whose details are often too sketchy. It turns out that many of these details are in some sense standard in the point of view of geometric analysts and can be treated in a more effective way using the standard tensorial methods of Riemannian geometry.

The second concerns how the Floer theory is presented in the book. In the author's personal experience, it seems to be most effective to learn the Floer theory both in the closed and in the open string context simultaneously. Very often problems on the Hamiltonian dynamics are solved via the corresponding problems on the geometry of Lagrangian intersections. For this reason, the author presents the Floer theory of the closed and the open string context at the same time. While the technical analytic details of pseudoholomorphic curves are essentially the same for both closed and open string contexts, the relevant geometries of the moduli space of pseudoholomorphic curves are different for the closed case and the open case of Riemann surfaces. This difference makes the Floer theory of Lagrangian intersection very different from that of Hamiltonian fixed points.

Acknowledgments

This book owes a great debt to many people whose invaluable help cannot be over-emphasized. The project of writing these two volumes started when I offered a three-semester-long course on symplectic geometry and pseudo-holomorphic curves at the University of Wisconsin–Madison in the years 2002–2004 and another quarter course on Lagrangian Floer homology while I was on sabbatical leave at Stanford University in the year 2004–2005.

I learned most of the materials on basic symplectic geometry from Alan Weinstein as his graduate student in Berkeley. Especially, the majority of the materials in Part 1 of Volume 1 are based on the lecture notes I had taken for his year-long course on symplectic geometry in 1987–1988. One could easily see the widespread influence of his mathematics throughout Part 1. I would like to take this chance to sincerely thank him for his invaluable support and encouragement throughout my graduate study.

I also thank Yasha Eliashberg for making my sabbatical leave in Stanford University possible and giving me the opportunity of offering the Floer homology course. Many thanks also go to Peter Spaeth and Cheol-Hyun Cho, whose lecture notes on the author's course in 2002–2004 became the foundation of Volume 1. I also thank Bing Wang for his excellent job of typing the first draft of Volume 1, without which help the appearance of these two volumes would be in great doubt. Thanks also go to Dongning Wang and Erkao Bao for their proofreading of earlier versions of Parts 2 and 4, respectively, and Rui Wang and Yoosik Kim for their proofreading of earlier versions of Parts 1 and 2.

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I thank Michael Entov and Leonid Polterovich for their astounding construction of symplectic quasi-states and quasimorphisms, which I believe starts to unveil the true symplectic nature of the analytical construction of Floer homology and spectral invariants. I also thank Claude Viterbo, Michael Usher, Lev Buhovsky and Soban Seyfaddini for their important mathematical works, which are directly relevant to the author's more recent research on spectral invariants and topological Hamiltonian dynamics and which the author very much appreciates.

I also thank various institutions in which I spent some time during the writing of the book. Special thanks go to the University of Wisconsin–Madison, where the majority of the writing of the present volumes was carried out while I was a member of the faculty there. I also thank the Korea Institute for Advanced Study (KIAS), the National Institute of Mathematical Sciences (NIMS) and Seoul National University, which provided financial support and an excellent research environment during my stay there.

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List of conventions

We follow the conventions of (Oh05c, Oh09a, Oh10) for the definition of Hamiltonian vector fields and action functionals and others appearing in the Hamiltonian Floer theory and in the construction of spectral invariants and Entov–Polterovich Calabi quasimorphisms. They are different from, e.g., those used in (EnP03, EnP06, EnP09) in one way or another.

(1) The canonical symplectic form ω_0 on the cotangent bundle T^*N is given by

$$\omega_0 = -d\Theta = \sum_{i=1}^{n} dq^i \wedge dp_i,$$

where Θ is the Liouville one-form given by $\Theta = \sum_{i=1}^n p_i\, dq^i$ in the canonical coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ associated with a coordinate system $(q^1, \ldots, q^n)$ on N.

(2) The Hamiltonian vector field X_H on a symplectic manifold (M, ω) is defined by $dH = \omega(X_H, \cdot)$.

(3) The action functional $\mathcal{A}_H : \widetilde{\mathcal{L}}_0(M) \to \mathbb{R}$ is defined by

$$\mathcal{A}_H([\gamma, w]) = -\int w^*\omega - \int_0^1 H(t, \gamma(t))dt.$$

(4) In particular, $\mathcal{A}_H$ is reduced to the classical Hamilton action functional on the path space of T^*N,

$$\int_\gamma p\, dq - \int_0^1 H(t, \gamma(t))dt,$$

which *coincides with* the standard definition in the literature of classical mechanics.

(5) An almost-complex structure is called J-positive if $\omega(X, JX) \geq 0$ and J-compatible if the bilinear form $\omega(\cdot, J\cdot)$ defines a Riemannian metric.

(6) Note that $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ carries three canonical bilinear forms: the symplectic form ω_0, the *Euclidean inner product* g and the Hermitian inner product $\langle \cdot, \cdot \rangle$. We take the Hermitian inner product to be complex linear in the first argument and anti-complex linear in the second argument. Our convention for the relation among these three is

$$\langle \cdot, \cdot \rangle = g(\cdot, \cdot) - i\omega_0(\cdot, \cdot).$$

Comparison with the Entov–Polterovich convention (EnP09)

In (EnP03, EnP06, EnP09), Entov and Polterovich used different sign conventions from the ones in (Oh05c) and the present book. If we compare our convention with the one from (EnP09), the following is the list of differences.

(1) The canonical symplectic form on T^*N in their convention is

$$\widetilde{\omega}_0 := d\Theta = \sum_{i=1}^n dp_i \wedge dq^i.$$

(2) The definition of the Hamiltonian vector field in (EnP09) is

$$dH = \omega(\cdot, X_H).$$

Therefore, by replacing H by $-H$, one has the same set of closed loops as the periodic solutions of the corresponding Hamiltonian vector fields on a given symplectic manifold (M, ω) in both conventions.

(3) Combination of (1) and (2) makes the Hamiltonian vector field associated with a function $H = H(t, q, p)$ on the cotangent bundle give rise to the *same* vector field. In particular, the classical Hamiltonian vector field on the phase space $\mathbb{R}^{2n}$ with canonical coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ is given by the expression

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right).$$

(4) For the definition of the action functional (EnP03) and (EnP09) take

$$-\int w^*\omega + \int_0^1 H(t, \gamma(t))dt. \tag{0.0.1}$$

We denote the definition (0.0.1) by $\widetilde{\mathcal{A}}_H([\gamma, w])$ for the purpose of comparison of the two below.

(5) In particular, $\widetilde{\mathcal{A}}_H$ is reduced to

$$-\int_\gamma p\, dq + \int_0^1 H(t, \gamma(t))dt$$

on the path space of T^*N, which *is the negative* of the standard definition of Hamilton's action functional in the literature of classical mechanics (e.g., (Ar89) and (Go80)).

(6) Since these two conventions use the same associated almost-Kähler metric $\omega(\cdot, J\cdot)$, the associated perturbed Cauchy–Riemann equations have exactly the same form; in particular, they have the same sign in front of the Hamiltonian vector field.

(7) In addition, Entov and Polterovich (EnP03, EnP06) use the notation $c(a, H)$ for the spectral numbers, where a is the quantum *homology* class, while we use a to denote a quantum cohomology class. The comparison is the following:

$$\rho(H; a) = c(a^\flat; \widetilde{H}) = c(a^\flat; \overline{H}), \tag{0.0.2}$$

where $a^\flat$ is the homology class dual to the cohomology class a and $\overline{H}$ is the inverse Hamiltonian of H given by

$$\overline{H}(t, x) = -H(t, \phi_H^t(x)). \tag{0.0.3}$$

(8) The relationship among the three bilinear forms on $\mathbb{R}^{2n}$ is given by

$$\langle \cdot, \cdot \rangle = g(\cdot, \cdot) + i\widetilde{\omega}_0(\cdot, \cdot),$$

for which the the inner product is complex linear in the second argument and anti-complex linear in the first argument.

With these understood, one can translate every statement in (EnP03, EnP06) into ones in terms of our notations.

PART 3

Lagrangian intersection Floer homology

12

Floer homology on cotangent bundles

In the 1960s, Arnol'd first predicted (Ar65) the existence of *Lagrangian intersection theory* (on the cotangent bundle) as the intersection-theoretic version of the Morse theory and posed *Arnol'd's conjecture*: the geometric intersection number of the zero section of T^*N for a compact manifold N is bounded from below by the one given by the number of critical points provided by the Morse theory on N. This original version of the conjecture is still open due to the lack of understanding of the latter Morse-theoretic invariants. However, its cohomological version was proven by Hofer (H85) using the *direct approach* of the classical variational theory of the action functional. This was inspired by Conley and Zehnder's earlier proof (CZ83) of Arnol'd's conjecture on the number of fixed points of Hamiltonian diffeomorphisms. Around the same time Chaperon (Ch84) and Laudenbach and Sikorav (LS85) used the *broken geodesic* approximation of the action functional and the method of *generating functions* in their proof of the same result. This replaced Hofer's complicated technical analytic details by simple more or less standard Morse theory.

The proof published by Chaperon and by Laudenbach and Sikorav is reminiscent of Conley and Zehnder's proof (CZ83) in that both proofs reduce the infinite-dimensional problem to a finite-dimensional one. (Laudenbach and Sikorav's method of generating functions was further developed by Sikorav (Sik87) and then culminated in Viterbo's theory of *generating functions quadratic at infinity* (Vi92).)

In the meantime, Floer introduced in (Fl88b) a general infinite-dimensional homology theory, now called the Floer homology, which is based on the study of the moduli space of an elliptic equation of the Cauchy–Riemann type that occurs as the L^2-gradient flow of the action integral associated with the variational problem. In particular Hofer's theorem mentioned above is a special case of Floer's (Fl88a) (at least up to the orientation problem, which was solved later in (Oh97b)), if we set $L_0 = \phi(o_N)$, $L_1 = o_N$ in the cotangent bundle. (Floer's construction is applicable not only to the action functional in symplectic geometry but also to the various first-order elliptic systems that appear in low-dimensional topology, e.g., the anti-self-dual Yang–Mills equation and the Seiberg–Witten monopole equation, and has been a fundamental ingredient in recent developments in low-dimensional topology as well as in symplectic topology.)

In (Oh97b), the present author exploits the natural filtration present in the Floer complex associated with the classical action functional and provides a Floer-theoretic construction of Viterbo's invariants. This construction is partially motivated by Weinstein's observation that the classical action functional

$$\mathcal{A}_H(\gamma) = \int_\gamma p\,dq - \int_0^1 H(t,\gamma(t))dt$$

is a canonical 'generating function' of the Lagrangian submanifold $\phi_H^1(L)$.

In this chapter we give a brief summary of the Floer theory on cotangent bundles in which we can illustrate essentially all the aspects of known applications of Floer homology to symplectic topology, which does not require a study of the bubbling phenomenon because bubbling does not occur. We postpone the study of this crucial aspect of the bubbling phenomenon and its technical underpinning in general to later chapters.

12.1 The action functional as a generating function

We start with the first variation formula (3.4.15) restricted to the case of cotangent bundles with $\alpha = -\theta$, $\theta = \sum_{i=1}^n p_i\,dq^i$ in which the Liouville one-form

$$d\mathcal{A}_H(\gamma)\xi = \int_0^1 \omega_0(\dot\gamma(t) - X_H(t,\gamma(t)), \xi(t))dt + \langle\theta(\gamma(1)), \xi(1)\rangle - \langle\theta(\gamma(0)), \xi(0)\rangle \tag{12.1.1}$$

for $\gamma : [0,1] \to T^*N$ and $\xi \in \Gamma(\gamma^*T(T^*N))$ is a vector field along γ.

A moment's reflection on this formula gives rise to several important consequences.

First, we consider the set of paths $\gamma : [0,1] \to T^*N$ issued at a point in the zero section. We denote

$$\Omega(0; o_N) = \{\gamma \mid \gamma(0) \in o_N\}.$$

There is a natural map $\pi \mathrm{ev}_1 : \Omega(0; o_N) \to N$ defined by

$$\pi \circ \mathrm{ev}_1(\gamma) = \pi(\gamma(1)). \tag{12.1.2}$$

This defines a fibration of $\Omega(0; o_N)$ over N whose fiber at $q \in N$ is given by

$$\Omega(o_N, T_q^*N) = \{\gamma \mid \gamma(0) \in o_N,\ \gamma(1) \in T_q^*N\}.$$

The fiber derivative of $\mathcal{A}_H$ for $\pi \mathrm{ev}_1$ at q is nothing but the first variation of $\mathcal{A}_H : \Omega(o_N, T_q^*N) \to \mathbb{R}$. This shows that we have

$$d^{\mathrm{fiber}}\mathcal{A}_H(\gamma)(\xi) = \int_0^1 \omega_0(\dot\gamma(t) - X_H(t, \gamma(t)), \xi(t))dt$$

for all $\xi \in T_\gamma\Omega(o_N, T_q^*N)$. Therefore the fiber critical set thereof, denoted by $\Sigma_{\mathcal{A}_H} \subset \Omega(0; o_N)$, is given by the set of solutions of Hamilton's equation

$$\dot x = X_H(t, x),\ x(0) \in o_N,\ x(1) \in T_q^*N.$$

We note that on $\Sigma_{\mathcal{A}_H}$ we have

$$d\mathcal{A}_H(\gamma)(\xi) = \langle \gamma(1), d\pi(\xi(1))\rangle$$

from (12.1.1).

Exercise 12.1.1 Complete a formal heuristic argument to derive from this formula that the push-forward of the one-form $d\mathcal{A}_H(\gamma)$ is nothing but $\gamma(1) \in T_q^*N$.

This completes the heuristic proof of the following proposition, which was observed by Weinstein.

Proposition 12.1.2 (Weinstein) *The pair $(\mathcal{A}_H, \Omega(0; o_N))$ is a generating function of $\phi_H^1(o_N)$ in the above sense.*

In fact, there is the natural finite-dimensional reduction of this generating function which we call the *basic generating function* of $L_H = \phi_H^1(o_N)$ and denote by $h_H : L_H \to \mathbb{R}$. For given $x \in L_H$, we denote

$$z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x)),$$

which is a Hamiltonian trajectory such that

$$z_x^H(0) \in o_N, \quad z_x^H(1) = x \tag{12.1.3}$$

by definition. We recall the following basic lemmata used in (Oh97b), whose proofs we leave as an exercise.

Lemma 12.1.3 *The function $h_H : L_H \to \mathbb{R}$ defined by*

$$h_H(x) = \mathcal{A}_H(z_x^H), \quad z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x))$$

satisfies

$$i_H^*\theta = dh_H,$$

i.e., h_H is a generating function of L_H. We call h_H the basic generating function *of L_H.*

Exercise 12.1.4 Prove this lemma.

Another consequence of the formula (12.1.1) is the characterization of the natural boundary conditions for the variational theory of $\mathcal{A}_H$. One obvious natural boundary condition is the periodic boundary condition $\gamma(0) = \gamma(1)$. A more interesting class of natural boundary conditions is the following.

Proposition 12.1.5 *Consider the conormal bundle $i_S : \nu^*S \hookrightarrow T^*N$ for a submanifold $S \subset N$. Then $i_S^*\theta = 0$.*

Proof Let $\xi \in T_\alpha(T^*N)$ and $\pi(\alpha) = x$. We have

$$i_S^*\theta(\xi) = \alpha(x)(d\pi(\xi(\alpha))).$$

But this pairing vanishes because $d\pi(\xi(\alpha)) \in T_xS$ and $\alpha(x) \in \nu_x^*S$ by definition. This finishes the proof. □

By this proposition, if we restrict the action functional $\mathcal{A}_H$ to the subset

$$\Omega_{S_0S_1} = \Omega(\nu^*S_0, \nu^*S_1) = \{\gamma : [0,1] \to T^*N \mid \gamma(0) \in \nu^*S_0,\ \gamma(1) \in \nu^*S_1\}$$

the first variation of $d\mathcal{A}_H$ is reduced to

$$d\mathcal{A}_H(\gamma)(\xi) = \int_0^1 \omega_0(\dot\gamma(t) - X_H(t,\gamma(t)), \xi(t))dt. \tag{12.1.4}$$

We point out that the path space $\Omega(o_N, T_q^*N)$ defined above is a special case of a conormal boundary condition corresponding to $S_0 = N$ and $S_1 = \{q\}$.

An immediate corollary of Proposition 12.1.5 is the following characterization of the critical-point equation: $d\mathcal{A}_H|_{\Omega(S_0,S_1)}(\gamma) = 0$ if and only if γ satisfies

$$\dot\gamma = X_H(t,\gamma(t)), \quad \gamma(0) \in \nu^*S_0,\ \gamma(1) \in \nu^*S_1. \tag{12.1.5}$$

Definition 12.1.6 (Action spectrum) We define $\mathrm{Spec}(H; S_0, S_1)$ to be the set of critical values and call it the action spectrum of $\mathcal{A}_H$ on $\Omega_{S_0S_1}$, i.e.,

$$\mathrm{Spec}(H; S_0, S_1) = \{\mathcal{A}_H(z) \mid \dot{z} = X_H(t, z(t)), \quad z(0) \in \nu^*S_0,\ z(1) \in \nu^*S_1\}.$$

Proposition 12.1.7 *The subset* $\mathrm{Spec}(H; S_0, S_1) \subset \mathbb{R}$ *is compact and has measure zero.*

Proof We have a one-to-one correspondence between the solutions of (12.1.5) and the intersection

$$\phi_{H^1}(\nu^*S_0) \cap \nu^*S_1,$$

as shown in Section 3.1. Clearly this set is compact. On the other hand, the function from ν^*S_1 to $\mathbb{R}$

$$h : x \mapsto z_x^H \mapsto \mathcal{A}_H(z_x^H)$$

is smooth and hence its image of $\phi_{H^1}(\nu^*S_0) \cap \nu^*S_1$ is compact.

Exercise 12.1.8 The set of critical points of h coincides with the intersection set $\phi_{H^1}(\nu^*S_0) \cap \nu^*S_1$ and the set of critical values of h with $\mathrm{Spec}(H; S_0, S_1)$.

Sard's theorem applied to the smooth function $h : \nu^*S_1 \to \mathbb{R}$ then finishes the proof. □

We also have the transversality result that $\phi_{H^1}(\nu^*S_0) \pitchfork \nu^*S_1$ if and only if the linearization operator of (12.1.5)

$$\nabla_{\gamma'} - DX_H(\gamma) : \Gamma_{S_0S_1}(\gamma^*(T(T^*N))) \to \Gamma(\gamma^*T(T^*N)) \tag{12.1.6}$$

is surjective, where $\Gamma_{S_0S_1}(\gamma^*(T(T^*N)))$ is the subset of $\Gamma(\gamma^*T(T^*N))$ defined by

$$\Gamma_{S_0S_1}(\gamma^*(T(T^*N))) = \{\xi \in \Gamma(\gamma^*T(T^*N)) \mid \\ \xi(0) \in T_{\gamma(0)}(\nu^*S_0),\ \xi(1) \in T_{\gamma(1)}(\nu^*S_1)\}.$$

The subset $\Gamma_{S_0S_1}(\gamma^*(T(T^*N)))$ is also the tangent space $T_\gamma\Omega_{S_0S_1}$ of $\Omega_{S_0S_1}$ at γ. This correspondence is the key to the relationship between the *dynamics* of Hamilton's equations and the *geometry* of Lagrangian intersections.

12.2 L^2-gradient flow of the action functional

We first note that, if a Riemannian metric g is given to N, the associated Levi-Civita connection induces a natural almost-complex structure on T^*N, which we denote by J_g and call the *canonical almost-complex structure* (in terms of the metric g on N).

Definition 12.2.1 Let g be a Riemannian metric on N. The canonical almost complex structure J_g on T^*N is defined as follows. Consider the splitting

$$T_{(q,p)}(T^*N) = T^h_{(q,p)}(T^*N) \oplus T^v_{(q,p)}(T^*N)$$

with respect to the Levi-Civita connection of g. For every $(q,p) \in T^*N$, J_g maps the horizontal unit tangent vectors to vertical unit vectors.

We will fix the Riemannian metric g on N once and for all. We leave the proof of the following proposition as an exercise.

Proposition 12.2.2 *We have the following.*

(1) *J_g is compatible with the canonical symplectic structure ω_0 of T^*N.*
(2) *On the zero section $o_N \subset T^*N \cong T_{(q,0)}o_N$, J_g assigns to each $v \in T_qN \subset T_{(q,0)}(T^*N)$ the cotangent vector $J_g(v) = g(v,\cdot) \in T^*_qN \subset T_{(q,0)}(T^*N)$. Here we use the canonical splitting*

$$T_{(q,0)}(T^*N) \cong T_qN \oplus T^*_qN.$$

(3) *The metric $g_{J_g} := \omega_0(\cdot, J_g\cdot)$ on T^*N defines a Riemannian metric that has bounded curvature and injectivity radius bounded away from 0.*
(4) *J_g is invariant under the anti-symplectic reflection $\mathfrak{r} : T^*N \to T^*N$ mapping $(q,p) \mapsto (q,-p)$.*

We consider the class of compatible almost-complex structures J on T^*N such that

$$J \equiv J_g \text{ outside a compact set in } T^*N,$$

and denote the class by

$$\mathcal{J}^c := \{J \mid J \text{ is compatible to } \omega \text{ and } J \equiv J_g \\ \text{outside a compact subset in } T^*N\}.$$

We define and denote the *support* of J by

$$\operatorname{supp} J := \text{ the closure of } \{x \in T^*N \mid J(x) \neq J_g(x)\}.$$

Define

$$\mathcal{P}(\mathcal{J}^c) := C^\infty([0,1], \mathcal{J}^c) = \{J : [0,1] \to \mathcal{J}^c \mid J = \{J_t\}_{0\le t\le 1}\}.$$

For each given $J = \{J_t\}_{0\le t\le 1}$, we consider the associated family of compatible metrics g_{J_t}. This family induces an L^2-metric on the space of paths on T^*N defined by

$$\langle\langle \xi_1, \xi_2 \rangle\rangle_J = \int_0^1 g_{J_t}(\xi_1(t), \xi_2(t))dt = \int_0^1 \omega(\xi_1(t), J_t\, \xi_2(t))dt. \tag{12.2.7}$$

From now on, we will always denote by J a $[0,1]$-*family* of compatible almost-complex structures unless stated otherwise.

Next we consider Hamiltonians $H = H(t,x)$ such that H_t is asymptotically constant, i.e., ones whose Hamiltonian vector field X_H is compactly supported. We define

$$\operatorname{supp}_{asc} H = \operatorname{supp} X_H := \bigcup_{t\in[0,1]} \operatorname{supp} X_{H_t}.$$

For each given compact set $K \subset T^*N$ and $R \in \mathbb{R}_+$, we define

$$\mathcal{H}_R = \mathcal{P}C^\infty_R(T^*N, \mathbb{R}) = \{H \in C^\infty([0,1]\times T^*N, \mathbb{R}) \mid \operatorname{supp}_{asc} H \subset D^R(T^*N)\}, \tag{12.2.8}$$

which provides a natural filtration of the space $\mathcal{H}$. Then we have

$$\mathcal{H} := C^\infty([0,1]\times T^*N, \mathbb{R}) = \bigcup_R \mathcal{H}_R$$

and equip the union $\cup_R \mathcal{H}_R$ with the direct limit topology of $\{\mathcal{H}_R\}_{R>0}$.

If we denote by $\operatorname{grad}_J \mathcal{A}_H$ the associated L^2-gradient vector field, the formula for $d\mathcal{A}_H(\gamma)$ and (12.1.4) imply that $\operatorname{grad}_J \mathcal{A}_H$ has the form

$$\operatorname{grad}_J \mathcal{A}_H(\gamma)(t) = J_t(\dot\gamma(t) - X_H(t, \gamma(t))), \tag{12.2.9}$$

which we simply write $J(\dot\gamma - X_H(\gamma))$. Therefore the *negative* gradient flow equation of a path $u : \mathbb{R} \to \Omega_{S_0S_1}$ has the form

$$\begin{cases} \partial u/\partial\tau + J\Big(\partial u/\partial t - X_H(u)\Big) = 0, \\ u(\tau,0) \in \nu^* S_0,\ u(\tau,1) \in \nu^* S_1, \end{cases} \tag{12.2.10}$$

if we regard u as a map $u : \mathbb{R}\times[0,1] \to M$. We call this equation *Floer's perturbed Cauchy–Riemann equation* or simply the perturbed Cauchy–Riemann equation associated with the quadruple $(H, J; S_0, S_1)$.

The general Floer theory largely relies on the study of the moduli spaces of solutions $u : \mathbb{R}\times[0,1] \to T^*N$ with *finite energy* and of *bounded image* of the kind (12.2.10) of perturbed Cauchy–Riemann equations. The relevant energy function is given by the following definition.

Definition 12.2.3 For a given smooth map $u : \mathbb{R} \times [0,1] \to M$, we define the energy, denoted by $E_{(H,J)}(u)$, of u by

$$E_{(H,J)}(u) = \frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_{J_t}^2 + \left| \frac{\partial u}{\partial t} - X_H(u) \right|_{J_t}^2 \right) dt\, d\tau.$$

We denote by

$$\widetilde{\mathcal{M}}(H, J; S_0, S_1)$$

the set of bounded finite-energy solutions of (12.2.10) for general H not necessarily nondegenerate. The following lemma is an easy consequence of the condition of finite energy and bounded image.

Lemma 12.2.4 *Suppose that $J \in \mathcal{P}(\mathcal{J}^c)$ and H is any smooth Hamiltonian. If u satisfies*

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0$$

and has bounded image and finite energy, there exists a sequence $\tau_k \to \infty$ (respectively $\tau_k \to -\infty$) such that the path $z_k := u(\tau_k) = u(\tau_k, \cdot)$ converges in C^∞ to a solution $z : [0,1] \to T^*N$ of the Hamilton equation $\dot{x} = X_H(x)$.

Proof The proof of this lemma is similar to that of Lemma 11.2.2 and hence will be brief. Since u satisfies (12.2.10), we have $E_{(H,J)}(u) = \int |\partial u/\partial t - X_H(u)|_{J_t}^2\, dt\, d\tau < \infty$. Therefore there exists a sequence $\tau_k \to \infty$ such that

$$\int_0^1 \left| \frac{\partial u}{\partial t}(\tau_k, \cdot) - X_H(u(\tau)) \right|_{J_t}^2 \to 0. \tag{12.2.11}$$

Denote $z_k := u(\tau_k, \cdot)$. *Since u is assumed to have a bounded image*, $|X_H(t, z_k(t))|$ is uniformly bounded over k. Then this boundedness and (12.2.11) imply that $\|z_k\|_{W^{1,2}} \leq C$, with C independent of k. By Sobolev embedding $W^{1,2} \hookrightarrow C^\epsilon$ with $0 < \epsilon < \frac{1}{2}$, z_k is pre-compact in C^ϵ and so has a sequence converging to z_∞ in C^ϵ. From the equation $\dot{z}_k(t) = X_H(t, z_k(t))$, it follows that $\dot{z}_k(t)$ converges to $X_H(t, z_\infty(t))$ in C^ϵ. Now, by the same boot-strap argument as that of Lemma 11.2.2, we derive that z_k converges to z_∞ in C^∞. This finishes the proof. □

For $\widetilde{\mathcal{M}}(H, J; S_0, S_1)$ to be well-behaved when the parameter $(H, J; S_0, S_1)$ varies, we need to establish an a-priori energy bound and a C^0 bound for the map u satisfying (12.2.10). For this purpose, we need to impose a certain tameness of (H, J) at infinity. (We have mentioned the tameness condition on J before.)

Definition 12.2.5 A function $h : T^*N \to \mathbb{R}$ is called *conic at infinity* if it satisfies

$$R_\lambda^* \, dh = \lambda \, dh, \quad \lambda > 0$$

(i.e., if it is homogeneous of order 1) outside some compact subset of T^*N, where $R_\lambda : T^*N \to T^*N$ is the fiber multiplication map $(q, p) \mapsto (q, \lambda p)$. We say that a time-dependent Hamiltonian H is of conic form at the end if there exists a function h of conic form at the end

$$H_t \equiv h$$

for all $t \in [0, 1]$ outside a compact set.

We note that

$$\nu^* S_0 \cap \nu^* S_1 = \bigcup_{q \in S_0 \cap S_1} \nu_q^* S_0 \cap \nu_q^* S_1,$$

which is a conic subset of T^*N, but not necessarily an isolated subset. In particular, it is always non-compact except when S_0 and S_1 have complementary dimensions. From now on, we consider only the case

$$S_0 = o_N,$$

which is compact.

Lemma 12.2.6 *Let $S \subset N$ be a compact submanifold. There is a dense subset of Hamiltonians H conic at infinity such that $\phi_H^1(o_N) \cap \nu^* S$ is transversal and compact.*

12.3 C^0 bounds of Floer trajectories

If u satisfies (12.2.10), the energy of u can be rewritten as

$$E_{(H,J)}(u) = \int\int \left| \frac{\partial u}{\partial t} - X_H(u) \right|_{J_t}^2 dt \, d\tau = \mathcal{A}_H(u(-\infty)) - \mathcal{A}_H(u(+\infty)), \tag{12.3.12}$$

as in the general case of the gradient flow equation.

We now derive several consequences of this identity. Denote by $\widetilde{\mathcal{M}}(H, J; z_-, z_+)$ the subset of $\widetilde{\mathcal{M}}(H, J)$ consisting of u satisfying the *fixed* asymptotic condition

$$u(-\infty) = z_-, \quad u(\infty) = z_+.$$

(1) If $\widetilde{\mathcal{M}}(H, J; z_-, z_+) \neq \emptyset$, then we have $\mathcal{A}_H(z_-) - \mathcal{A}_H(z_+) \geq 0$ and the equality holds only when $z_- = z_+$ and u is stationary.
(2) The energy $E_{(H,J)}(u)$ depends only on the action values of the asymptotic Hamilton's orbit $u(\pm\infty)$: we have the *a-priori energy bound*

$$E_{(H,J)}(u) = \mathcal{A}_H(z_-) - \mathcal{A}_H(z_+) < \infty \tag{12.3.13}$$

for all $u \in \widetilde{\mathcal{M}}(H, J; z_-, z_+)$.
(3) There exists a natural free $\mathbb{R}$-action on $\widetilde{\mathcal{M}}(H, J; z_-, z_+)$ by τ-translations when $z_- \neq z_+$. We denote

$$\mathcal{M}(H, J; z_-, z_+) = \widetilde{\mathcal{M}}(H, J; z_-, z_+)/\mathbb{R}.$$

We have developed the analysis needed in the study of the moduli space in the context of genuine J-holomorphic equations in Part 2 of Volume 1. For example, the bubbling-off analysis applies equally to these perturbed J-holomorphic equations. However, for an exact Lagrangian pair such as ν^*S_0, ν^*S^1 in T^*N, bubbling-off of non-constant spheres or discs cannot occur. Now all the necessary Fredholm and compactness properties developed for the case of *compact* Lagrangian submanifolds will apply to the current non-compact target space, *provided* that the intersection set is compact and certain C^0-estimates for the solution u are present. This C^0-estimation is an essential step for the Floer theory on *non-compact* Lagrangian submanifolds in non-compact symplectic manifolds. We will also need to consider the parameterized versions of (12.2.10) in order to establish some invariance property under the deformation of Hamiltonians or of submanifolds S.

We now divide our discussion into two cases: one for the construction of the Floer boundary map and the other for the construction of the chain map, especially under the change of boundary ν^*S_s for the isotopy of submanifolds $\{S_s\}_{0\leq s\leq 1}$.

In the first case we study Equation (12.2.10), which involves only the fixed boundary. In this case, one can apply the strong maximum principle since we consider the Hamiltonian H such that X_H is compactly supported, an explanation of which is now in order.

We first recall the notion of J-convexity in the sense of Gromov (Gr85).

Definition 12.3.1 Let (Z, ω) be a symplectic manifold with boundary $Q = \partial Z$. We say that (Z, ω) has a contact-type boundary if there exists a contact from λ on Z such that

(1) $\omega|_Q = d\lambda$ and $\xi = \ker \lambda$, and
(2) the orientation of Q defined by $\lambda \wedge (d\lambda)^{n-1}$ coincides with the boundary orientation of $Q = \partial W$.

Definition 12.3.2 Let (Z, ω, J) be an almost-Kähler structure with contact-type boundary. We say ∂Z is J-convex if there exists a contact form λ on ∂Z such that the two-form $d\lambda(\cdot, J\cdot)$ is positive-definite on ξ.

With these definitions prepared, we have the following.

Proposition 12.3.3 *Let (Z, ω, J) be an almost-Kähler structure with its boundary ∂Z J-convex. Then no J-holomorphic curve u in Z can touch ∂Z from inside at an interior point of the domain of u. More precisely, there is no point $z_0 \in \operatorname{Int}\operatorname{Dom} u$ with $u(z_0) \in \partial Z$ such that it carries an open neighborhood $U \subset \operatorname{Dom} u$ satisfying $u(U \setminus \{z_0\}) \subset \operatorname{Int} Z$.*

Proof The J-convexity of the boundary implies that the distance function $\psi(x) := \operatorname{dist}(x, \partial Z)^2$ becomes a pluri-subharmonic function, i.e., it satisfies the criterion that $d(d\psi \circ J)$ is positive definite.

Suppose to the contrary that there exists such an interior point z_0 and let U be an associated open neighborhood. Then the function $\psi \circ u$ becomes a subharmonic function. This violates the maximum principle and thus finishes the proof. □

The following gives the main energy and C^0 bounds. Here we write $u(\tau, t) = (q(\tau, t), p(\tau, t))$ in T^*N and $|\cdot|_g$ is the norm on $T^*_{q(\tau,t)}N$ induced from the metric g on N.

Theorem 12.3.4 *Assume that $S_0, S_1 \subset N$ intersect transversely. Then there exists a constant $C = C(H, J, S_0, S_1) > 0$ such that every bounded finite-energy solution u of* (12.2.10) *satisfies*

$$\sup_{(\tau,t)\in\Theta} |p(\tau, t)|_g \leq R_0, \tag{12.3.14}$$

where $R_0 > 0$ is given by

$$R_0 := \inf\{R \in \mathbb{R}_+ \mid T^*N \backslash D_R \subset T^*N \backslash (\operatorname{Supp} H \cup \operatorname{Supp} J)\}.$$

Proof By the finite-energy assumption, we have

$$\int_{-\infty}^{\infty}\int_0^1 \left|\frac{\partial u}{\partial \tau}\right|^2_{J_t} + \left|\frac{\partial u}{\partial t} - X_K(u)\right|^2_{J_t} dt\, d\tau < \infty.$$

Then by Lemma 12.2.4 and the exponential decay estimates for u (see Section 14.1 for the proof in a slightly different context, or Appendix B.1 for the general scheme of the exponential estimates), we obtain

$$\lim_{\tau \to -\infty} u(\tau) = z_-, \quad \lim_{\tau \to -\infty} u(\tau) = z_+ \text{ uniformly,}$$

where z_- and z_+, respectively, satisfy the equations,

$$\dot{z}_\pm = X_H(t, z), \; z_\pm(0) \in \nu^* S_0, \; z_\pm(1) \in \nu^* S_1. \tag{12.3.15}$$

Then it must be the case that either the $\sup_{(\tau,t)} |p(\tau, t)|_g$, where $u(\tau, t) = (q(\tau, t), p(\tau, t))$ is realized at $\tau = \pm\infty$, or the supremum is realized at some point $(\tau_0, t_0) \in \mathbb{R} \times [0, 1]$.

Since one can easily derive the C^0-estimate of the Hamiltonian paths z^α from the assumption that X_H has compact support and hence $\phi_H^1(\nu^* S_0) \cap \nu^* S_1$ is compact, obtaining the C^0-estimate of the paths z^α finishes the proof for the first case. Therefore we consider only the second case. It will suffice to prove the following.

Proposition 12.3.5 *Suppose* $(\phi_H^1(\nu^* S_0) \cap \nu^* S_1) \cup \operatorname{supp} H \cup \operatorname{supp} J \subset D_R$ *for some* $R > 0$, *where* $D_R \subset T^*N$ *is the disc bundle*

$$D_R := \{p \in T^*N \mid |p|_g \le R\}.$$

Then we have $u(\tau_0, t_0) \in D_R$ *for all solutions* u *of* (12.2.10).

Proof Suppose to the contrary that $u(\tau_0, t_0) \notin D_R$, say $u(\tau_0, t_0) \in \partial D_{R'}$, $R' > R$ and that there exists an open neighborhood B_ϵ of (τ_0, t_0) in $\mathbb{R} \times [0, 1]$ for some $\epsilon > 0$ such that

$$u(B_\epsilon) \subset T^*N \backslash D_R \subset T^*N \backslash (\operatorname{Supp} H \cup \operatorname{Supp} J)$$

and hence u satisfies on B_ϵ

$$\frac{\partial u}{\partial \tau} + J_g \frac{\partial u}{\partial t} = 0.$$

We consider two cases separately: the cases where $(\tau_0, t_0) \in \operatorname{Int} \mathbb{R} \times [0, 1]$ and $(\tau_0, t_0) \in \partial\mathbb{R} \times [0, 1]$.

Proposition 12.3.3 immediately rules out the possibility $(\tau_0, t_0) \in \operatorname{Int} \mathbb{R} \times [0, 1]$.

Now consider the case where $(\tau_0, t_0) \in \partial(\mathbb{R} \times [0, 1])$, i.e., $t_0 = 0, 1$. Since the cases $t_0 = 0$ will be the same, we will consider the case $t_0 = 1$ only. Consider the boundary curve

$$\tau \to u(\tau, 1) = (q(\tau, 1), p(\tau, 1)),$$

which becomes tangent to $\partial D_{R'} \cap \nu^* S$ at $(\tau_0, 1)$. Since $\nu^* S \cap \partial D_{R'}$ is Legendrian in $\partial D_{R'}$, the curve is tangent to the contact distribution of ∂D_r at $(\tau_0, 1)$, which is given by

$$\{\xi \in T(\partial D_{R'}) \mid \xi \perp J_g \, \partial/\partial r, \partial/\partial r \text{ is the radial field on } T^*N\}.$$

Since u is J_g-holomorphic and ξ is J_g-invariant, u is also tangent to the contact distribution at $(\tau_0, 1)$, and in particular we have

$$\frac{\partial}{\partial t}|p(\tau, t)|^2\Big|_{(\tau_0,1)} = 0.$$

However, this contradicts the strong maximum principle applied to the subharmonic function (with respect to J_g)

$$(\tau, t) \mapsto |p(\tau, t)|^2 \quad \text{on } B_\epsilon(\tau_0, 1),$$

since we can assume $\operatorname{Image} u\big|_{B_\epsilon(\tau_0,1)} \not\subset \partial D_{R'}$ by choosing ϵ slightly bigger if necessary. This finishes the assertion of the proposition and thus the proof of Theorem 12.3.4. □

Exercise 12.3.6 Prove that the function $(\tau, t) \mapsto |p(\tau, t)|^2$ is a subharmonic function.

This finishes the proof of the theorem. □

12.4 Floer-regular parameters

In this section, we explain the meaning of the 'generic' parameters for which the various moduli spaces we consider become smooth manifolds. This in turn gives rise to the definitions of various Floer operators.

We consider pairs (H, S) that satisfy

$$\phi_H^1(o_N) \pitchfork \nu^* S.$$

For any such pair, there are only finitely many solutions of

$$\dot{z} = X_H(z), \quad z(0) \in o_N,\ z(1) \in \nu^* S,$$

i.e., critical points of $\mathcal{A}_H$ on Ω_S. We denote by $\operatorname{Emb}(S_0 : N)$ the set of embeddings of S_0 into N for a given abstract compact manifold S_0 and introduce its subset

$$\begin{aligned}\operatorname{Emb}^H(S_0 : N)& \\ &= \{S \mid \nu^* S \pitchfork \phi_H^1(o_N),\ S \subset N \text{ is the image of an embedding of } S_0\}.\end{aligned} \tag{12.4.16}$$

For such $S \in \operatorname{Emb}^H(S_0 : N)$, we consider also the isotopy class of a given embedding S_0, which we denote by

$$\operatorname{Iso}^H(S_0 : N) \subset \operatorname{Emb}^H(S_0 : N).$$

By the standard transversality theorem, it follows that $\mathrm{Emb}^H(S_0 : N)$ is dense in $\mathrm{Emb}(S_0 : N)$ in the C^∞ topology. Next, we consider the Fredholm-regular property of the space of solutions of (12.2.10).

We now prove the following theorem similarly to Theorem 10.4.1. The kind of transversality result under the change of boundary conditions was proved in (Oh96a), to whose proof we refer the reader.

Theorem 12.4.1 *Let $H \in \mathcal{H}(S)$, where*

$$\mathcal{H}(S) := \{H \in \mathcal{H} \mid \phi_H(o_M) \pitchfork \nu^*S\} \tag{12.4.17}$$

and $S \in \mathrm{Emb}^H(S_0 : M)$. Then there exists a dense subset $\mathcal{J}_{H,S} \subset \mathcal{J}^c$ such that all the solutions of (12.2.10) *are regular.*

We will also need the parameterized versions of this theorem.

Exercise 12.4.2 Formulate and prove the parameterized version of Theorem 12.4.1.

We call a triple (H, J, S) *Floer-regular* if it satisfies $J \in \mathcal{J}_{H,S}$ and $H \in \mathcal{H}(S)$.

12.5 Floer homology of submanifold $S \subset N$

Let $H \in \mathcal{H}_0$, $S \in \mathrm{Emb}^H(S_0; N)$ and $J \in \mathcal{J}_{H,S}$. The gradient trajectories of $\mathcal{A}_H|_{\Omega_{o_N \nu^* S}}$ with respect to the L^2-metric $\langle\langle\ ,\ \rangle\rangle_J$ are solutions of (12.2.10). We denote by $\widetilde{\mathcal{M}}(H, J, S; N)$ the set of finite-energy solutions that satisfy

$$\inf_{\tau\in\mathbb{R}} \mathcal{A}_H(u(\tau)) > -\infty, \qquad \sup_{\tau\in\mathbb{R}} \mathcal{A}_H(u(\tau)) < \infty.$$

12.5.1 Boundary map and Floer complex

We denote

$$\begin{aligned}\mathrm{Chord}(H; o_N, \nu^*S) = \{z : [0,1] \to T^*N \mid \dot z = X_H(t,z),\\ z(0) \in o_N,\ z(1) \in \nu^*S\}.\end{aligned} \tag{12.5.18}$$

Recall that $\mathrm{Chord}(H; o_N, \nu^*S)$ has a one-to-one correspondence with the set $o_N \cap \nu^*S$ via the map

$$p \in o_N \cap \nu^*S \mapsto z_p^H; \qquad z_p^H(t) = \phi_H^t(\phi_H^1)^{-1}(p).$$

Because of the choice of $S \in \mathrm{Emb}^H(S_0; N)$ and $H \in \mathcal{H}(S)$ from (12.4.16) and (12.4.17), respectively, there are only finitely many elements in $\mathrm{Chord}(H; o_N, \nu^*S)$. We now consider the Floer equation

$$\begin{cases} \partial u/\partial\tau + J(\partial u/\partial t - X_H(u)) = 0, \\ u(\tau, 0) \in o_N, \quad u(\tau, 1) \in \nu^* S_1. \end{cases} \tag{12.5.19}$$

The following two theorems will be assumed; for their proofs we refer the reader to (Oh97b). We will discuss corresponding theorems in a more general context later.

Theorem 12.5.1 (Canonical grading) *For each solution z,*

$$\dot{z} = X_H(t, z), \quad z(0) \in o_N,\ z(1) \in \nu^* S_1,$$

there exists a canonically assigned Maslov index that has the values in $\frac{1}{2}\mathbb{Z}$. We denote this map by

$$\mu_S : \mathrm{Chord}(H; o_N, \nu^*S) \to \frac{1}{2}\mathbb{Z}.$$

Furthermore, μ_S satisfies the following properties.

(1) *For each solution u of* (12.2.10) *with $u(-\infty) = z^\alpha$, $u(+\infty) = z^\beta$, we have the Fredholm index of u given by*

$$\mathrm{Index}\, u = \mu_S(z^\alpha) - \mu_S(z^\beta).$$

(2) *Consider the time-independent Hamiltonian $F = f \circ \pi$, $f \in C^\infty(N)$, where f is a Morse function on N. Let $p \in \mathrm{Graph}\, df \cap \nu^*S$ and hence $x = \pi(p) \in \mathrm{Crit}(f|_S)$. Denote $z_x(t) = (x, t\, df(x))$, which is the Hamiltonian path of F with $z_x(0) \in o_N$, $z_x(1) \in \nu^*S$. Then we have*

$$\mu_S(z_x) = \frac{1}{2}\dim S - \mu_f^S(x),$$

where μ_f^S is the Morse index of $f|_S$ at x on S.

Theorem 12.5.2 (Coherent orientation)

(1) *Let (H, S, J) be generic in the isotopy class $[H, S, J]$. For each $z^\alpha, z^\beta \in \mathrm{Chord}(H; o_N, \nu^*S)$, there exists an orientation of $\mathcal{M}_J(z^\alpha, z^\beta)$, i.e., the determinant bundle*

$$\mathrm{Det} \to \mathcal{M}_J(z^\alpha, z^\beta)$$

whose fiber at $u \in \mathcal{M}_J(z^\alpha, z^\beta)$ is the one-dimensional real vector space

$$\det(D\bar{\partial}_{J,H}(u)) := \Lambda^{\max}(\mathrm{Ker}\, D\bar{\partial}_{J,H}(u)) \otimes \Lambda^{\max}(\mathrm{Coker}\, D\bar{\partial}_{J,H}(u))^*$$

is trivial. The same is true for the parameterized Floer moduli spaces $\mathcal{M}(\overline{H}, \overline{S}, \overline{J})$ *for the generic paths* $(\overline{H}, \overline{S}, \overline{J})$.

(2) *Furthermore, there exists a coherent orientation (in the sense of (FH93) or Section 15.6) on the set of all* $\mathcal{M}_J(H, S)$ *and* $\mathcal{M}(\overline{H}, \overline{S}, \overline{J})$ *over* (H, S, J) *and the paths* $(\overline{H}, \overline{S}, \overline{J})$ *in each isotopy class. We denote the set of such coherent orientations by* $\mathrm{or}([H, S, J] : N) = \mathrm{or}([S] : N)$.

We form a $\frac{1}{2}\mathbb{Z}$-graded free abelian group (i.e., $\mathbb{Z}$-module)

$$CF_*(H, S : N) = \mathbb{Z}\{\mathrm{Chord}(H; o_N, \nu^* S)\}.$$

In fact, we can also consider a free G-module for any abelian group G.

We fix a system of coherent orientations $\sigma \in \mathrm{or}([S] : N)$. Then for each $z^\alpha, z^\beta \in \mathrm{Chord}(H; o_N, \nu^* S)$ with $\mu(z^\alpha) - \mu(z^\beta) = 1$ each element $u \in \mathcal{M}(z^\alpha, z^\beta) = \mathcal{M}(H, J; z^\alpha, z^\beta)$ defines its $[u_\tau]$. We compare this $[u_\tau]$ of the flow with the orientation $\sigma(u)$ induced from the coherent orientation defined in Theorem 12.5.2, and we define the sign $\tau(u) \in \{1, -1\}$ by the equation

$$\sigma(u) = \tau(u)[u_\tau].$$

We define an integer

$$n^\sigma_{(H,J)}(z^\alpha, z^\beta) := \sum_{u \in \mathcal{M}(z^\alpha, z^\beta)} \tau(u) \tag{12.5.20}$$

for such a pair (z^α, z^β), and a homomorphism $\partial_{(H,J)} : CF_*(H, S : N) \to CF_*(H, S : N)$ by

$$\partial^\sigma_{(H,J)}(z^\alpha) = \sum_\beta n^\sigma_{(H,J)}(z^\alpha, z^\beta) z^\beta.$$

By definition, $\partial^\sigma_{(H,J)}$ has degree -1 with respect to the grading given above. By the compactness and cobordism argument, which we explain later in the more general context, we prove that $\partial^\sigma_{(H,J)}$ satisfies

$$\partial^\sigma_{(H,J)} \circ \partial^\sigma_{(H,J)} = 0$$

and thus we obtain a graded complex $(CF_*(H, S : N), \partial^\sigma_{(H,J)})$.

Definition 12.5.3 For each Floer regular parameter (H, S, J), we define

$$HF^\sigma_*(H, S, J; N) = \mathrm{Ker}\, \partial^\sigma_{(H,J)} / \mathrm{Im}\, \partial^\sigma_{(H,J)}$$

and call it the *Floer homology* of (H, S, J) on N (with respect to the coherent orientation σ).

12.5.2 The chain map and its isomorphism property

The following theorem is the main theorem we establish in this subsection.

Theorem 12.5.4 *For two regular parameters* $(H^\alpha, S^\alpha, J^\alpha)$ *and* $(H^\beta, S^\beta, J^\beta)$ *isotopic to each other, there is a natural chain map*

$$h^\sigma_{\alpha\beta} : CF^\sigma_*(H^\alpha, S^\alpha, J^\alpha; N) \to CF^\sigma_*(H^\beta, S^\beta, J^\beta; N)$$

that preserves the grading.

We fix a monotone function $\rho : \mathbb{R} \to [0,1]$ such that

$$\rho(\tau) = \begin{cases} 0, & \text{if } \tau \le 0, \\ 1, & \text{if } \tau \ge 1. \end{cases}$$

Now consider the path $(\overline{H}, \overline{S}, \overline{J}) = \{(H^s, S^s, J^s)\}_{s \in [0,1]}$. We take its elongation

$$(\overline{H}, \overline{S}, \overline{J}) = \{(H^{\rho(\tau)}, S^{\rho(\tau)}, J^{\rho(\tau)})\}_{-\infty \le \tau \le \infty}$$

and consider the non-autonomous version of (12.2.10)

$$\begin{cases} \partial u/\partial\tau + J^\rho\Big(\partial u/\partial t - X_{H^\rho}(u)\Big) = 0, \\ u(\tau, 0) \in o_N,\ u(\tau, 1) \in \nu^* S^{\rho(\tau)}. \end{cases} \tag{12.5.21}$$

For each given $z^\alpha \in \mathrm{Chord}(o_N, \nu^* S^\alpha)$ and $z^\beta \in \mathrm{Chord}(o_N, \nu^* S^\beta)$, we define

$$\mathcal{M}(z^\alpha, z^\beta) = \{u : \mathbb{R} \times [0,1] \to T^*N \mid u \text{ solves (12.5.21) and } \lim_{\tau \to -\infty} u(\tau) = z^\alpha,\ \lim_{\tau \to +\infty} u(\tau) = z^\beta\}.$$

Using the orientations provided by Theorem 12.5.2, we define an integer similarly to what we did in (12.5.20),

$$n_{\alpha\beta}(z^\alpha, z^\beta) := \#(\mathcal{M}(z^\alpha, z^\beta)) \quad \text{for } \mu(z^\alpha) - \mu(z^\beta) = 0,$$

and the chain map $h_{\alpha\beta} : CF_*(H^\alpha, S^\alpha) \to CF_*(H^\beta, S^\beta)$ by

$$h_{\alpha\beta}(z^\alpha) = \sum_{\alpha,\beta} n_{\alpha\beta}(z^\alpha, z^\beta) z^\beta. \tag{12.5.22}$$

We would like to emphasize that, from the definition, *only those pairs* (z^α, z^β) *for which Equation* (12.5.21) *has a solution with the given asymptotic condition*

$$\lim_{\tau \to -\infty} u(\tau) = z^\alpha, \quad \lim_{\tau \to +\infty} u(\tau) = z^\beta$$

give non-trivial contributions to (12.5.22). This is what enables one to estimate the change of the filtration under the various homomorphisms between the Floer homology for different parameters.

Here is one crucial point about the Cauchy–Riemann equation with a *moving boundary* condition with *non-compact* Lagrangian submanifolds in the above construction of the chain map using (12.5.21): to apply compactness arguments, one has to establish an a-priori C^0 bound for the solutions of (12.5.21). Proving this C^0 bound is quite non-trivial (see (Oh09b) for the proof in a more general context of conic Lagrangian submanifolds in general Weinstein manifolds). Here, instead of directly using the moduli space of solutions of (12.5.21) in the construction of a chain map

$$h^\sigma_{\alpha\beta} : CF^\sigma_*(H^\alpha, S^\alpha, J^\alpha; N) \to CF^\sigma_*(H^\beta, S^\beta, J^\beta; N),$$

we will go around this subtlety by the trick used by Nadler in (Na09), (NaZ09). The main idea of this construction is to express the isotopy

$$s \mapsto \nu^* S^s =: L^s$$

as a composition of two different isotopies of $\nu^* S^\alpha = L^\alpha$. Since we assume X_H are compactly supported, there exists $R_1 > 0$ such that

$$\mathrm{Image}\, z \subset D_{R_1}$$

for any $z \in \mathrm{Chord}(H^s; o_N, \nu^* S^s)$ for all $s \in [0,1]$. This implies that $\mathrm{Image}\, u(\pm\infty) \subset D_{R_1}$ for all finite-energy solutions u for all $s \in [0,1]$.

First, we choose a sufficiently large constant R_2 with $R_2 > R_1$ so that

$$\mathrm{supp}\, H^s \subset D_{R_2}.$$

Then we take another sufficiently large constant $R_3 > R_2$ and consider an isotopy of $\psi^\rho_{(1)}(\tau)(L^\alpha)$ such that

$$\mathrm{supp}\, \psi^1_{(1)} \subset D_{R_3}, \quad \psi^s_{(1)}(L^\alpha) \cap D_{R_2} = \nu^* S^s \cap D_{R_2}.$$

Because this isotopy fixes L^α outside the disc bundle D_{R_3}, the C^0 bound in Theorem 12.3.4 still applies and hence we can construct a chain map

$$h_{\psi_{(1)}} : CF^\sigma_*(H^\alpha, L^\alpha) \to CF^\sigma_*(H^\beta, \psi^1(L^\alpha))$$

by considering the associated Floer moduli space.

Next, we fix the above isotopy on D_{R_2} and move L^α by another isotopy so that $\psi_{(2)}$ satisfies

$$\mathrm{supp}\, \psi_{(2)} \subset T^*N \setminus D_{R_2}, \quad \psi^1_{(2)}\psi^1_{(1)}(L^\alpha) = \nu^* S^\beta$$

and

$$\psi^s_{(2)}(L^\alpha) \cap (T^*N \setminus D_{R_3}) = \nu^* S^s \cap (T^*N \setminus D_{R_3}). \tag{12.5.23}$$

We note that this isotopy does not change the set of Hamiltonian trajectories

$$\begin{aligned}&\{z : [0,1] \to T^*N \mid \dot z = X_{H^\alpha}(t,z),\ z(0) \in 0_S,\ z(1) \in L^\alpha\}\\ &\quad \bigcup \{z : [0,1] \to T^*N \mid \dot z = X_{H^\alpha}(t,z),\ z(0) \in 0_S,\ z(1) \in \psi^1_{(1)}(L^\alpha)\}\end{aligned}$$

during the isotopy $\psi_{(2)}$. We now construct another chain map,

$$h_{\psi_{(2)}} : CF^\sigma_*(H^\alpha, \psi^1_{(1)}(L^\alpha)) \to CF^\sigma_*(H^\beta, \psi^1_{(2)}\psi^1_{(1)}(L^\alpha)),$$

in a way different from that of $h_{\psi(1)}$. Because the generators of the complexes of both the domain and the target of this map are fixed, we can consider the following *parameterized* moduli space

$$\mathcal{M}^{\text{para}}(z^-, z^+) := \bigcup_{s \in [0,1]} \{s\} \times \mathcal{M}^s(z^-, z^+), \tag{12.5.24}$$

where $\mathcal{M}^s(z^-, z^+)$ is the moduli space of solutions of

$$\begin{cases}\partial u/\partial \tau + J^{\rho(\tau)}\,(\partial u/\partial t - X_{H^\rho}(u)) = 0,\\ u(\tau, 0) \in o_N,\ u(\tau, 1) \in \psi^s_{(2)}\psi^1_{(1)}(L^\alpha)\end{cases}$$

for $s \in [0,1]$. *Because we do not move the boundary conditions* when we study $\mathcal{M}^{\text{para}}(z^-, z^+)$ but study $\mathcal{M}^s(z^-, z^+)$ for each fixed s, we have a uniform C^0 bound for the elements in $\mathcal{M}^{\text{para}}(z^-, z^+)$. When the index condition $\mu(z^-) = \mu(z^+)$ holds for the pair (z^-, z^+) as in the construction of the chain map, $\mathcal{M}^{\text{para}}(z^-, z^+)$ becomes a compact one-dimensional manifold for a generic choice of $\overline{J}$. By the cobordism argument, we obtain

$$\#(\mathcal{M}^0(z^-, z^+)) = \#(\mathcal{M}^1(z^-, z^+)).$$

Motivated by this, we define $h_{\psi_{(2)}}$ so that its matrix associated with the bases of $CF^\sigma_*(H^\alpha, \psi^1_{(1)}(L^\alpha))$ and $CF^\sigma_*(H^\beta, \psi^1_{(2)}\psi^1_{(1)}(L^\alpha))$ becomes the identity matrix. This makes sense because the two vector spaces have generators consisting of the *same* set of Hamiltonian trajectories.

Exercise 12.5.5 Prove that the map $h_{\psi_{(2)}}$ is a chain map. (**Hint**. Consider the pairs (z^-, z^+) with $\mu(z^-) = \mu(z^+) + 1$, and examine the compactification of $\mathcal{M}^{\text{para}}(z^-, z^+)$ and analyze its boundary components. This time $\mathcal{M}^{\text{para}}(z^-, z^+)$ has dimension 2.)

By composing the two chain maps $h_{\psi_{(2)}}, h_{\psi_{(1)}}$, we obtain the chain map required in Theorem 12.5.4. □

12.5.3 Chain homotopy map

We now want to show that the chain map constructed in Theorem 12.5.4 induces an isomorphism in homology.

For this purpose, we consider concatenation of the isotopy $s \mapsto S^s$ and its inversion $s \mapsto S^{1-s}$. We then elongate the concatenation by substituting $s = \rho_\kappa = \rho(\tau - \kappa)$ for $\kappa \geq 1$. We denote the concatenation function by χ_κ,

$$\chi_\kappa(\tau) = \begin{cases} \rho_\kappa(\tau), & \tau \geq 0, \\ 1 - \rho_\kappa(\tau), & \tau \leq 0, \end{cases}$$

when $\kappa \geq 1$. We further deform $\chi_{\kappa=1}$ to the zero function as $\kappa \to 0$ so that $\chi_{\kappa'} \leq \chi_\kappa$ if $0 \leq \kappa' \leq \kappa \leq 1$.

We apply the same construction as used for the chain map $h_{\psi(1)}$ by replacing ρ by χ_κ for each $\kappa \in [0, \infty)$. This will give rise to a chain map

$$h_{\chi_\kappa} : CF_*^\sigma(H^\alpha, S^\alpha, J^\alpha; N) \to CF_*^\sigma(H^\alpha, S^\alpha, J^\alpha; N)$$

for $\kappa = 0$ and for any generic parameter κ, and the associated parameterized moduli space can be proven to be a chain homotopy map by the same proof as that of Exercise 12.5.5. We note that $\kappa = 0$ corresponds to the identity map

$$\mathrm{id} : CF_*^\sigma(H^\alpha, S^\alpha, J^\alpha; N) \to CF_*^\sigma(H^\alpha, S^\alpha, J^\alpha; N).$$

This is because, due to the $\mathbb{R}$-translation, only the stationary solution is allowed when the index difference is 0 under the genericity hypothesis that there is no non-stationary solution for the pair with index difference 0.

On the other hand, as $\kappa \to \infty$, the gluing theorem (see Section 15.5) implies that

$$h_{\chi_\kappa} = h_{\psi(1);\beta\alpha} \circ h_{\psi(1);\alpha\beta}$$

if κ is sufficiently large. This proves

$$(h_{\psi(1);\beta\alpha})_* \circ (h_{\psi(1);\alpha\beta})_* = \mathrm{id}$$

in homology. Since we can change the role of α, β, this implies that $(h_{\psi(1);\alpha\beta})_*$ is an isomorphism, which finishes the proof.

The isomorphism property of $h_{\psi(2);\alpha\beta}$ is even clearer by definition and so we have proved that the composition $h_{\alpha\beta} = h_{\psi(2);\alpha\beta} h_{\psi(1);\alpha\beta}$ induces an isomorphism in homology.

We now just state the following basic theorem, for whose proof we refer the reader to the original articles by Floer (Fl89a) and (Oh96b).

Theorem 12.5.6 *Consider the pair (f, g) of a Morse function f and a Riemannian metric g on N. There exists some $\delta = \delta(f, g) > 0$ such that, whenever $\|f\|_{C^2} \leq \delta$,*

(1) *there exists a one-to-one correspondence between the Morse moduli space* $\mathcal{M}^{\mathrm{Morse}}(f, g)$ *and the Floer moduli space* $\mathcal{M}(\phi_{f\circ\pi}, J_g)$*, and*
(2) *the pair* (f, g) *is Morse–Smale if and only if* $\mathcal{M}(\phi_{f\circ\pi}, J_g)$ *is Fredholm-regular.*

An immediate corollary of this theorem is the following isomorphism.

Corollary 12.5.7 *There exists a natural isomorphism*

$$H^*(L; \mathbb{Z}) \cong H^*_{\mathrm{Morse}}(f; L) \to HF_*(-f \circ \pi, J_g).$$

Here we remind the reader that our convention of the action functional

$$\mathcal{A}_H(\gamma) = \int \gamma^*\theta - \int_0^1 H(t, \gamma(t))dt$$

and the Hamiltonian vector field makes the *positive* Morse gradient trajectory χ of f which satisfies $\dot{\chi}(\tau) - \mathrm{grad}_g f(\chi(\tau)) = 0$ correspond to the t-independent *negative* Floer gradient trajectory of $H = -f$ which satisfies

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0.$$

(Recall the formula (18.5.40).) This forces the appearance of cohomology for the Morse side and that of homology for the Floer side.

12.6 Lagrangian spectral invariants

We fix the canonical coherent orientation $\sigma \in \mathrm{or}([S] : N)$ so that $HF^\sigma_*(H, S; N)$ is isomorphic to the singular cohomology $H^*(S, \mathbb{Z})$. We denote this isomorphism by

$$\Phi_{(H,S)} : H^*(S, \mathbb{Z}) \to HF^\sigma_*(H, S; N).$$

With this, we will also suppress σ from the notation $HF^\sigma_*(H, S, J; N)$. We omit the construction of this isomorphism but refer readers to (MiO97) and (Mi00) for the complete details.

In this section, assuming the existence of this canonical isomorphism, we apply the mini-max theory of an action functional by detecting the linking property of the mini-maxing sets using the *Floer homology* machinery. This replaces the direct approach of the classical mini-max theory of the action functional on the Sobolev space $H^{1/2}(S^1, \mathbb{R}^{2n})$ developed in (BnR79) and (Bn82), which uses the global gradient flow of the action functional to push down the mini-maxing sets and detects the linking property by *classical topology*.

We first note that (12.2.10) is the negative gradient flow of $\mathcal{A}_H|_{\Omega(S)}$ with respect to the L^2 metric $\langle\langle\cdot,\cdot\rangle\rangle_J$ on $\Omega(S)$ and hence preserves the downward filtration given by the values of the action functional $\mathcal{A}_H$. This fact is analytically encoded in the identity

$$\frac{d}{dt}\mathcal{A}_H(u(\tau)) = -\int \left|\frac{\partial u}{\partial t} - X_H(u)\right|_J^2 dt \le 0. \tag{12.6.25}$$

Let $S \subset N$ be a given compact submanifold and let $(H,J) \in \mathcal{N}_{\mathrm{reg}}(S)$ defined as before. For $a \in \mathbb{R}$, we define $CF_*^a(H,S;N)$ to be the $\mathbb{Z}$-free module generated by $z \in \mathrm{Crit}\,\mathcal{A}_H|_{\Omega(S)}$ with $\mathcal{A}_H(z) \le \lambda_1$ and

$$CF_*^{[\lambda_1,\lambda_2)} = CF_*^{\lambda_2}/CF_*^{\lambda_1}.$$

Then the boundary map we defined in (12.2.10),

$$\partial_{(H,J)} : CF_*(H,S:N) \to CF_*(H,S:N),$$

induces the (relative) boundary map

$$\partial_{(H,J)} = \partial_{(H,J)}^{[\lambda_1,\lambda_2)} : CF_*^{[\lambda_1,\lambda_2)}(H,S:N) \to CF_*^{[\lambda_1,\lambda_2)}(H,S:N)$$

for any $b > a$, which will obviously satisfy

$$\partial_{(H,J)}^{[\lambda_1,\lambda_2)} \circ \partial_{(H,J)}^{[\lambda_1,\lambda_2)} = 0.$$

Hence, we can define the relative homology groups by

$$HF_*^{[\lambda_1,\lambda_2)}(H,S,J;N) := \mathrm{Ker}\,\partial_{(H,J)}^{[\lambda_1,\lambda_2)}/\mathrm{Im}\,\partial_{(H,J)}^{[\lambda_1,\lambda_2)}.$$

From the definition, there is a natural homomorphism

$$j_* : HF_*^{[\lambda_1,\lambda_2)} \to HF_*^{[\mu_1,\mu_2)}$$

when $\lambda_1 \le \mu_1$ and $\lambda_2 \le \mu_2$. In particular, there exists a natural homomorphism

$$j_*^\lambda : HF_*^{(-\infty,\lambda)} \to HF_* = HF_*^{(-\infty,\infty)}. \tag{12.6.26}$$

Definition 12.6.1 Let $S \subset N$ be a compact submanifold, and let $(H,J) \in \mathcal{N}_{\mathrm{reg}}(S)$. For each given $0 \neq a \in H^*(S)$, we define the real number $\rho(H,J;S,a)$ by

$$\rho(H,J;S,a) := \inf_\lambda\{\lambda \in \mathbb{R} \mid \Phi_{(H,S,J)}(a) \in \mathrm{im}\, j_*^\lambda \subset HF_*(H,S,J:N)\}.$$

A priori it is not obvious whether $\rho(H,J;S,a)$ is finite, i.e., $\rho(H,J;S,a) \neq -\infty$.

In this regard, the following lemma is an important one that establishes the 'linking property' of the 'fundamental cycle' of $\Omega(S;N)$. This is the

Floer-theoretic version of the linking property and the Palais–Smale condition combined. (See (BnR79) for the definitions of these conditions in classical variational theory.)

Lemma 12.6.2 *For $H \in \mathcal{H}(S)$ with* $\operatorname{supp} X_H \subset D^R(T^*N)$*, we have*

$$\rho(H; S, a) \geq -\left(R|dH|_{C^0} + |H|_{C^0;D^R(T^*N)}\right) > -\infty.$$

Proof For any $z \in \mathrm{Chord}(H; o_N, \nu^*S)$,

$$\begin{aligned}\mathcal{A}_H(z) &= \int z^*\theta - \int_0^1 H(t, z(t))dt \\ &= \int_0^1 \langle T\pi(\dot{z}), z\rangle dt - \int_0^1 H(t, z(t))dt.\end{aligned}$$

But we have $|\langle T\pi(\dot{z}), z\rangle| \leq |T\pi(X_H(t, z(t)))||p(z)|$, where $z(t) = (q(t), p(t))$. Therefore

$$|\mathcal{A}_H(z)| \leq \int_0^1 \max_{x \in D^R(T^*N)} (|X_{H_t}(x)|R + |H_t|(x))dt$$

for all $z \in \mathrm{Chord}(H; o_N, \nu^*S)$. If we denote

$$\lambda_H = \inf \mathrm{Spec}(H; S) := \min\{\mathcal{A}_H(z) \mid z \in \mathrm{Chord}(H; o_N, \nu^*S)\},$$

clearly

$$\lambda_H \geq -\left(R|dH|_{C^0} + |H|_{C^0;D^R(T^*N)}\right) > -\infty.$$

On the other hand, by definition $HF_*^{(-\infty,\lambda)} = 0$ if $\lambda < \lambda_H$. On combining these two statements, we have proved

$$\rho(H; S, a) \geq -\left(R|dH|_{C^0} + |H|_{C^0;D^R(T^*N)}\right) > -\infty.$$

This finishes the proof. □

The number $\rho(H, J; S, a)$ can be realized also as a mini-max value in a more intuitive geometric way as follows. Recall that each Floer chain α is a finite linear combination

$$\alpha = \sum_{z \in \mathrm{Crit}\, \mathcal{A}_H|_{\Omega(S)}} a_z[z],\ a_z \in \mathbb{Z}.$$

We define the support of α by

$$\operatorname{supp} \alpha = \{z \in \mathrm{Crit}\, \mathcal{A}_H|_{\Omega(S)} \mid a_z \neq 0\}.$$

A Floer cycle α is a Floer chain with $\partial_{(H,S,J)}(\alpha) = 0$.

Now, for each given Floer cycle α, we define its level by

$$\lambda_{(H,S:N)}(\alpha) = \max_{z \in \operatorname{supp} \alpha} \mathcal{A}_H(z).$$

Then, by definition, we have

$$\rho(H, J; S, a) = \inf_{\alpha; [\alpha] = \Phi_{(H,S,J)}(a)} \lambda_{(H,S;N)}(\alpha). \tag{12.6.27}$$

Lemma 12.6.3 *Let (H, S, J) be Floer-regular. Then $\rho(H, J; S, a)$ is a critical value $\mathcal{A}_H|_{\Omega(S)}$.*

Proof Since $\nu^* S \pitchfork \phi^1_H(o_N)$, there are only finitely many solutions of

$$\dot{z} = X_H(t, z), \quad z(0) \in o_N,\ z(1) \in \nu^* S, \tag{12.6.28}$$

and so there are finitely many critical values of $\mathcal{A}_H|_{\Omega(S)}$. Denote by $\lambda^-_S > -\infty$ the minimum critical value thereof. Obviously any non-zero Floer chain α has its level greater than or equal to λ^-_S. On the other hand, (12.6.27) can be rewritten as

$$\rho(H, J; S, a) = \inf_{\beta \in CF_*(H)} \{\lambda_{(H,S;N)}(\alpha - \partial_{(H,S,J)}(\beta))\}.$$

Since $[\alpha] \neq 0$, $\alpha - \partial_{(H,S,J)}\beta \neq 0$ and so

$$\lambda_{(H,S;N)}(\alpha - \partial_{(H,S,J)}) \geq \lambda^-_S$$

for all $\beta \in CF_*(H)$. This proves $\rho(H, J; S, a) \geq \lambda^-_S$. Since there are only finitely many critical values, $\rho(H, J; S, a)$ is achieved by a solution of (12.6.28) and hence the proof has been obtained. □

Next, we study the J-dependence of $\rho(H, J; S, a)$ for fixed S and $H \in \mathcal{H}(S)$ when J varies among $\mathcal{J}_{(S,H)}$.

Lemma 12.6.4 *Let J^α, $J^\beta \in \mathcal{J}_{(S,H)}$. Then we have*

$$\rho(H, J^\alpha; S, a) = \rho(H, J^\beta; S, a).$$

Proof Using the fact that $\mathcal{J}^c$ is contractible and in particular connected, we can choose a path $\overline{J} = \{J^s\}_{0 \leq s \leq 1}$ in $\mathcal{J}^c$ connecting J^α and J^β so that the solution set of

$$\begin{cases} \partial u/\partial \tau + J^{\rho(\tau)}(\partial u/\partial t - X_H(u)) = 0, \\ u(\tau, 0) \in o_N,\ u(\tau, 1) \in \nu^* S \end{cases} \tag{12.6.29}$$

satisfies the (H, S)-regular property required before. Recall that the canonical homomorphism

$$h_{\alpha\beta;\overline{J}} : CF_*(H, J^\alpha; S) \to CF_*(H, J^\beta; S)$$

is defined by

$$h^{\rho}_{\alpha\beta}(z^{\alpha}) = \sum n(\overline{J}; z^{\alpha}, z^{\beta}) z^{\beta},$$

where $n(\overline{J}; z^{\alpha}, z^{\beta}) = \#(\mathcal{M}_{\rho}(z^{\alpha}, z^{\beta}))$ induces an isomorphism $HF_*(H, J^{\alpha}, S; N) \to HF_*(H, J^{\beta}, S; N)$.

To see how $\rho(H, J; S, a)$ vary under the change of J, we need to estimate the difference

$$\mathcal{A}_H(z^{\beta}) - \mathcal{A}_H(z^{\alpha})$$

whenever $n(\overline{J}; z^{\alpha}, z^{\beta}) \neq 0$ and thus in particular when there exists a solution u of (12.6.29) with

$$\lim_{\tau\to-\infty} u(\tau) = z^{\alpha}, \qquad \lim_{\tau\to\infty} u(\tau) = z^{\beta}.$$

For such u, we write

$$\mathcal{A}_H(z^{\beta}) - \mathcal{A}_H(z^{\alpha}) = \int_{-\infty}^{\infty} \frac{d}{d\tau} \mathcal{A}_H(u(\tau)) d\tau.$$

However, we have

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{A}_H(u(\tau)) &= d\mathcal{A}_H(u(\tau)) \cdot \frac{\partial u}{\partial \tau} \\
&= \int_0^1 \left(\omega(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial \tau}) - dH_t(u) \frac{\partial u}{\partial \tau} \right) dt \\
&= \int_0^1 \left\langle J^{\rho(\tau)} \left(\frac{\partial u}{\partial t} - X_H(u) \right), \frac{\partial u}{\partial \tau} \right\rangle_{J_t^{\rho(\tau)}} dt \\
&= - \int_0^1 \left| \frac{\partial u}{\partial t} - X_H(u) \right|^2_{J_t^{\rho(\tau)}} \leq 0,
\end{aligned}$$

where we use the equation $\partial u/\partial \tau = -J^{\rho(\tau)}(\partial u/\partial t - X_H(u))$ for the third equality. Hence, we have proved that, whenever there exists a solution u as above,

$$\mathcal{A}_H(z^{\beta}) \leq \mathcal{A}_H(z^{\alpha}).$$

This shows that the map $h_{\alpha\beta,J} : CF_*(H, S, J^{\alpha}) \to CF_*(H, S, J^{\beta})$ restricts to a map

$$h^{\rho}_{\alpha\beta;\overline{J}} : CF_*^{(-\infty,\lambda)}(H, S, J^{\alpha}) \to CF_*^{(-\infty,\lambda)}(H, S, J^{\beta})$$

for any $\lambda \in \mathbb{R}$ and so induces a homomorphism

$$h^{\rho}_{\alpha\beta;\overline{J}} : HF_*^{(-\infty,\lambda)}(H, S, J_{\alpha} : N) \to HF_*^{(-\infty,\lambda)}(H, S, J^{\beta} : N).$$

Now consider the commutative diagram

$$\begin{tikzcd} HF_*^{(-\infty,\lambda)}(H,S,J^\alpha:N) \arrow[r] \arrow[d] & HF_*(H,S,J^\alpha:N) \arrow[d] \\ HF_*^{(-\infty,\lambda)}(H,S,J^\beta:N) \arrow[r] & HF_*(H,S,J^\beta:N) \end{tikzcd}$$

where all downward arrows are induced by the canonical homomorphisms $h_{\alpha\beta;\overline{J}}$ and the horizontal ones by the canonical inclusion-induced map j_*^λ.

Since $h_{\alpha\beta;\overline{J}}$ on the right-hand side is an isomorphism, if $[a]^\alpha := \Phi_{(H,S,J^\alpha)}(a) \in \mathrm{Im}(j_*^\lambda)_\alpha$, then $[a]^\beta \in \mathrm{Im}(j_*^\lambda)_\beta$. Therefore, we have proved

$$\rho(H, J_\alpha; S, a) \geq \rho(H, J_\beta; S, a).$$

By exchanging the roles of α and β, we also obtain

$$\rho(H, J_\beta; S, a) \geq \rho(H, J_\alpha; S, a),$$

which finishes the proof of $\rho(H, J_\alpha; S, a) = \rho(H, J_\beta; S, a)$. □

Definition 12.6.5 Let $S \subset N$ be a compact submanifold and $H \in \mathcal{H}(S)$. We define

$$\rho(H; S, a) := \rho(H, J; S, a)$$

for a $J \in \mathcal{J}_{(S,H)}$.

Now we study the dependence of $\rho(H; S, a)$ on (H, S). We first study the dependence of $\rho(H; S, a)$ on H. We fix $S \subset N$ and consider the Hs in $\mathcal{H}(S)$.

The following theorem summarizes the basic properties of $\rho(H; S, a)$.

Theorem 12.6.6 *Let $S \subset N$ be a compact submanifold and H, $H^\alpha, H^\beta \in \mathcal{H}(S)$. Then the following properties hold.*

(1) *We have*

$$\int_0^1 -\max_x(H^\beta - H^\alpha)dt \leq \rho(H^\beta; S, a) - \rho(H^\alpha; S, a)$$
$$\leq \int_0^1 -\min_x(H^\beta - H^\alpha)dt. \tag{12.6.30}$$

In particular, we have

$$\int_0^1 -\max_x H\, dt \leq \rho(H; S, a) \leq \int_0^1 -\min_x H\, dt. \tag{12.6.31}$$

(2) *We also have from (1)*

$$|\rho(H^\beta; S, a) - \rho(H^\alpha; S, a)| \leq \operatorname{osc}(H_\beta - H_\alpha),$$

which in particular implies that, for fixed S, one can extend the assignment $H \mapsto \rho(H; S, a)$ to all $\mathcal{H}$ as a continuous function in the C^0 topology of $\mathcal{H}$. We will continue to denote the extension by $\rho(H; S, a)$.

Proof The proof of (2) immediately follows from (12.6.30) and so we need prove (1) only. Consider the linear homotopy

$$H^s := (1-s)H^\alpha + sH^\beta.$$

Although this homotopy might not be Floer-regular, we will pretend it is so for the moment and explain the necessary justification at the end. Consider the equation

$$\begin{cases} \partial u/\partial\tau + J(\partial u/\partial t - X_{H^{\rho(\tau)}}(u)) = 0, \\ u(\tau, 0) \in o_N,\ u(\tau, 1) \in \nu^* S, \\ \lim_{\tau\to-\infty} u(\tau) = z^\alpha,\ \lim_{\tau\to\infty} u(\tau) = z^\beta. \end{cases}$$

As before, we compute

$$\mathcal{A}_{H^\beta}(z^\beta) - \mathcal{A}_{H^\alpha}(z^\alpha) = \int_{-\infty}^{\infty} \frac{d}{d\tau}\Big(\mathcal{A}_{H^{\rho(\tau)}}(u(\tau))\Big)\, d\tau$$

for the pair (z^α, z^β) with $n_{\alpha\beta}(z^\alpha, z^\beta) \neq 0$. But we have

$$\frac{d}{d\tau}(\mathcal{A}_{H^{\rho(\tau)}}(u(\tau))) = d\mathcal{A}_{H^{\rho(\tau)}}(u(\tau))\left(\frac{du}{d\tau}\right) - \int_0^1 \left(\frac{\partial H^{\rho(\tau)}}{\partial\tau}\right)(u, t)dt$$

and

$$d\mathcal{A}_{H^{\rho(\tau)}}(u(\tau))\left(\frac{du}{d\tau}\right) = -\int_0^1 \left|\frac{\partial u}{\partial t} - X_{H^{\rho(\tau)}}(u)\right|_J^2 dt \leq 0.$$

Furthermore,

$$\begin{aligned} \int_0^1 \left(\frac{\partial H^{\rho(\tau)}}{\partial\tau}\right)(u, t)dt &= -\int_0^1 \rho'(\tau)(H^\beta - H^\alpha)(u, t)dt \\ &\leq -\rho'(\tau)\int_0^1 \min_x (H^\beta - H^\alpha)dt. \end{aligned} \tag{12.6.32}$$

Here we have used the inequality $\rho'(\tau) \geq 0$. Therefore we have obtained

$$\mathcal{A}_{H^\beta}(z^\beta) - \mathcal{A}_{H^\alpha}(z^\alpha) \leq \int_0^1 -\min_x (H^\beta - H^\alpha)dt$$

by the same calculation as in the proof of Lemma 14.4.5.

By arguments similar to those used in the proof of Lemma 12.6.4 before, this estimate implies

$$\rho(H^\beta; S, a) \le \rho(H^\alpha; S, a) + \int_0^1 -\min_x (H^\beta - H^\alpha) dt,$$

i.e.,

$$\rho(H^\beta; S, a) - \rho(H^\alpha; S, a) \le \int_0^1 -\min_x (H^\beta - H^\alpha) dt \tag{12.6.33}$$

by considering the homomorphism

$$h_{\alpha\beta} : HF_*^{(-\infty,\lambda)}(H^\alpha, S, J : N) \to HF_*^{(-\infty,\lambda+\epsilon^{\alpha\beta})}(H^\beta, S, J : N).$$

where $\epsilon^{\alpha\beta} = -\int_0^1 \min_x (H^\beta - H^\alpha) dt$. By exchanging the roles of α and β, we also have

$$\rho(H^\alpha; S, a) \le \rho(H^\beta; S, a) + \int_0^1 -\min_x (H^\alpha - H^\beta) dt,$$

i.e.,

$$\begin{aligned} \rho(H^\beta; S, a) - \rho(H^\alpha; S, a) &\ge \int_0^1 \min_x (H^\alpha - H^\beta) dt \\ &= \int_0^1 -\max_x (H^\beta - H^\alpha) dt. \end{aligned} \tag{12.6.34}$$

On combining (22.5.66) and (22.5.63), we will have finished the proof if we can justify the use of the linear homotopy, which might not be regular. To justify this, we proceed as follows. For each given $\epsilon > 0$, we approximate the above linear homotopy by C^1-close regular homotopies H so that for all $t \in [0, 1]$

$$\max_{x,s} \left| \frac{\partial H}{\partial s}(x, t, s) - (H^\beta - H^\alpha)(x, t) \right| \le \epsilon.$$

For such a homotopy, (12.6.32) is replaced by

$$\begin{aligned} -\int_0 \left(\frac{\partial H^{\rho(\tau)}}{\partial \tau} \right)(u, t) dt &= -\int_0^1 \rho'(\tau) \frac{\partial H}{\partial s}(u, t, \rho(\tau)) dt \\ &\le \rho'(\tau) \int_0^1 -\min_{x,s} \frac{\partial H}{\partial s}(s, t, s) dt \\ &\le \rho'(\tau) \left(\int_0^1 -\min_{x,s} (H^\beta - H^\alpha) + \epsilon \right) dt, \end{aligned}$$

from which we derive

$$\mathcal{A}_{H^\beta}(z^\beta) - \mathcal{A}_{H^\alpha}(z^\alpha) \le \int_0^1 -\min_x (H^\beta - H^\alpha) + \epsilon.$$

By letting $\epsilon \to 0$, we are done for (12.6.30). To prove (12.6.31), we set $H^\beta = H$ and $H^\alpha \to 0$ *in C^1 topology* and apply Lemma 12.6.2 and (12.6.30). This finishes the proof. □

One immediate consequence of (12.6.30) is that the map $H \mapsto \rho(H; S, a)$ can be continuously extended to arbitrary Hamiltonian $H \in \mathcal{H}$, not just in $\mathcal{H}(S)$: take any C^2 approximation $H_i \in \mathcal{H}(S)$ of H and then take the limit

$$\rho(H; S, a) := \lim_{i\to\infty} \rho(H_i; S, a).$$

The following is an improvement of Lemma 12.6.3

Proposition 12.6.7 *For any $H \in \mathcal{H}$, the value $\rho(H; S, a)$ lies in* $\mathrm{Spec}(H; o_N, \nu^* S)$.

Proof By the definition of $\mathcal{H}$, $H \in \mathcal{H}_R$ for some sufficiently large $R > 0$. Let H_i be a C^2-approximation of H such that $H_i \in \mathcal{H}_{R+1}$.

It follows, from Lemma 12.6.2 and C^2-continuity, that the map

$$H \mapsto \left(R|dH|_{C^0} + |H|_{C^0;D^R(T^*N)}\right)$$

is continuous and satisfies the inequality

$$\rho(H; S, a) \geq -\left((R+1)|dH|_{C^0} + |H|_{C^0;D^{R+1}(T^*N)}\right).$$

(For this part, we need only C^1-approximation, but C^2-approximation will be needed in the later part of the proof where we need convergence of Hamiltonian trajectories of X_H.) In particular, $\rho(H; S, a)$ is a finite value. It remains to prove that it lies in $\mathrm{Spec}(H; o_N, \nu^* S)$. By definition, we have that $\rho(H; S, a) = \lim_{i\to\infty} \rho(H_i; S, a)$. By Lemma 12.6.3, $\rho(H_i; S, a) = \mathcal{A}_{H_i}(z_i)$ for some path $z_i : [0, 1] \to T^*N$ satisfying

$$\dot z_i = X_{H_i}(t, z), \quad z_i(0) \in o_N,\ z_i(1) \in \nu^* S.$$

Since $H_i \to H$ in C^2, $X_{H_i} \to X_H$ in C^1. Furthermore, since we have $\mathrm{supp}\, X_{H_i} \subset D^{R+1}(T^*N)$, it follows that $|\dot z_i(t)| \leq C$ for some $C > 0$ independently of i and $t \in [0, 1]$, and so z_i are equi-continuous. Finally, given the boundary condition $z_i(0) \in o_N$ with o_N being compact, there exists a subsequence, again denoted by z_i, converging to a smooth path z that satisfies $\dot z = X_H(t, z)$, $z(0) \in o_N$, $z(1) \in \nu^* S$. This implies that

$$\rho(H; S, a) = \lim_{i\to\infty} \rho(H_i; S, a) = \lim_{i\to\infty} \mathcal{A}_{H_i}(z_i) = \mathcal{A}_H(z),$$

which proves that $\rho(H; S, a) \in \mathrm{Spec}(H; S)$. □

Now we study the dependence of $\rho(H; S, a)$ under a change of S.

Proposition 12.6.8 *Let $S_0 \subset N$ and let $\mathrm{Iso}(S_0; N)$ be the isotopy class of S_0 in N. Then the assignment*

$$S^\alpha \mapsto \rho(H; S^\alpha, a)$$

on $S^\alpha \in \mathrm{Iso}^H(S_0 : N)$ is continuous on S^α in the C^1 topology of $\mathrm{Iso}(S_0 : N)$. Hence we can extend the definition of $\rho(S, H)$ to all $S \in \mathrm{Iso}(S_0; N)$ by continuity in C^1 topology of $\mathrm{Iso}(S_0; N)$.

Proof The idea of the proof of this proposition is similar to that of Lemma 12.6.4. Let S^α and $S^\beta \in \mathrm{Iso}^H(S_0 : N)$ and let S^s be a generic isotopy between them. On partitioning the isotopy so that the C^0 and energy bound proved in Section 12.5.2 apply, it suffices to consider the case where S^α and S^β are sufficiently C^1-close that the map $\phi : S^\alpha \to S^\beta$ defined by the nearest point becomes a diffeomorphism.

We now consider the chain map $h_{\alpha\beta}$ constructed in Theorem 12.5.4.

For the first chain map $h_{(\psi(1))}$, we have the C^0 bound $|p(\tau, 1)|_g \le C_2$ for some constant $C_2 = C_2(H, J, \overline{S}) > 0$ and derive from (12.1.1)

$$\begin{aligned}
\mathcal{A}_H(z^\beta) - \mathcal{A}_H(z^\alpha) &= -\int_{\mathbb{R}}\int_0^1 \left|\frac{\partial u}{\partial t} - X_H(u)\right|_{J_t}^2 dt - \int_{\mathbb{R}} \left\langle T\pi\frac{\partial u}{\partial \tau}(\tau, 1), u(\tau, 1)\right\rangle d\tau \\
&\le -\int_{\mathbb{R}} \left\langle \frac{\partial}{\partial \tau}(\pi \circ u)(\tau, 1), p(\tau, 1)\right\rangle \\
&= -\int_R \rho'(\tau)\left\langle p(\tau, 1), \frac{\partial q}{\partial \tau}(\tau, 1)\right\rangle d\tau \\
&= -\int_R \rho'(\tau)\langle p(\tau, 1), X_{\rho(\tau)}(q(\tau, 1))\rangle d\tau.
\end{aligned}$$

By applying the C^0 bound

$$|p(\tau, 1)|_g \le C_2$$

and recalling that $\int_{\mathbb{R}} \rho'(\tau) d\tau = 1$, we obtain

$$|\mathcal{A}_H(z^\beta) - \mathcal{A}_H(z^\alpha)| \le C_2 |X|_{C^0}.$$

Obviously, one can choose the path $\{S^s\}_{0 \le s \le 1}$ so that

$$\max_{s \in [0,1]} \left|\frac{\partial S^s}{\partial s}\right| \sim d_{C^1}(S^\alpha, S^\beta)$$

and hence we get

$$|\mathcal{A}_H(z^\beta) - \mathcal{A}_H(z^\alpha)| \le C d_{C^1}(S^\alpha, S^\beta) =: \epsilon^{\alpha\beta}$$

as long as S^α and S^β are sufficiently C^1-close. As before, we have the natural homomorphism $h_{(\psi(1))} : HF_*(H, S^\alpha, J : N) \to HF_*(H, S^\beta, J : N)$ inducing the commutative diagram

$$\begin{tikzcd} HF_*^{(-\infty,\lambda)}(H, S^\alpha, J : N) \arrow[r] \arrow[d] & HF_*(H, S^\alpha, J : N) \arrow[d] \\ HF_*^{(-\infty,\lambda+\epsilon^{\alpha\beta})}(H, S^\beta, J : N) \arrow[r] & HF_*(H, S^\beta, J : N) \end{tikzcd}$$

Again, since $h_{\alpha\beta}$ on the right-hand side is an isomorphism, we conclude

$$\rho(H; S_\beta, a) \le \rho(H; S_\alpha, a) + Cd_{C^1}(S^\alpha, S^\beta).$$

By changing the role of α and β, we prove the other side of the inequality and so

$$|\rho(H; S^\beta, a) - \rho(H; S^\alpha, a)| \le Cd_{C^1}(S^\alpha, S^\beta), \tag{12.6.35}$$

which in particular proves the continuity of $S \mapsto \rho(H; S, u)$ in C^1 topology of S.

On the other hand, the chain map $h_{\psi(2)}$ fixes the filtration by construction and so finishes the proof of continuity. □

We would like to note that in the proof of Proposition 12.6.8, we used the a-priori C^0-estimate in an essential way.

Remark 12.6.9 By combining Proposition 12.6.8 and Theorem 12.6.6, we can now extend the definition of $\rho(H; S, u)$ to the set

$$\mathcal{H}_{C^0} \times \mathrm{Iso}_{C^1}(S_0; N),$$

where $\mathcal{H}_{C^0}$ is the set of asymptotically constant C^0-functions on $T^*N \times [0, 1]$ and $\mathrm{Iso}_{C^1}(S_0; N)$ is the set of C^1-embeddings that are isotopic to S_0. In fact, we can even extend the definition to

$$\mathcal{H}_{C^0} \times \mathrm{Iso}_{Lip}(S_0; N).$$

It would be interesting to study the geometric meaning of $\rho(H; S, u)$ for the cases where $H \in C^0$ but not in C^1.

12.7 Deformation of Floer equations

So far we have looked at the Hamiltonian-perturbed Cauchy–Riemann equation (12.5.19), which we call the *dynamical version* as in (Oh97b).

On the other hand, one can also consider the *genuine* Cauchy–Riemann equation

$$\begin{cases} \partial v/\partial\tau + J^H \partial v/\partial t = 0, \\ v(\tau, 0) \in \phi_H^1(o_N),\ v(\tau, 1) \in \nu^* S \end{cases} \tag{12.7.36}$$

for the path $u : \mathbb{R} \to \Omega(L, \nu^* S)$, where $L = \phi_H^1(o_N)$ and

$$\Omega(L, \nu^* S) = \{\gamma : [0, 1] \to T^* N \mid \gamma(0) \in L,\ \gamma(1) \in \nu^* S\}$$

and $J_t^H = (\phi_H^t(\phi_H^1)^{-1})_* J_t$. We call this version the *geometric version*.

We now describe the geometric version of the Floer homology in more detail. The upshot is that there is a filtration preserving isomorphisms between the dynamical version and the geometric version of the Lagrangian Floer theories.

We denote by $\widetilde{\mathcal{M}}(L_H, \nu^* S; J^H)$ the set of finite-energy solutions and write as $\mathcal{M}(L_H, \nu^* S; J^H)$ its quotient by $\mathbb{R}$-translations. This gives rise to the geometric version of the Floer homology $HF_*(\phi_H^1(o_N), \nu^* S; \widetilde{J})$ whose generators are the intersection points of $\phi_H^1(o_N) \cap \nu^* S$. An advantage of this version is that it depends only on the Lagrangian submanifold $L = \phi_H^1(o_N)$, and hence only loosely on H.

The following lemma is straightforward to check but crucial.

Lemma 12.7.1 *Let $L = \phi_H^1(o_S)$.*

(1) *The map $\Phi_H : o_N \cap L \to \mathrm{Chord}(H; o_N, \nu^* S)$ defined by*

$$x \mapsto z_x^H(t) = \phi_H^t(\phi_H^1)^{-1}(x)$$

gives rise to the one-to-one correspondence between the set $L \cap \nu^* S$ as constant paths and the set of solutions of Hamilton's equation of H.

(2) *The map $a \mapsto \Phi_H(a)$ also defines a one-to-one correspondence from the set of solutions of* (12.5.19) *to that of*

$$\begin{cases} \partial v/\partial\tau + J^H\, \partial v/\partial t = 0, \\ v(\tau, 0) \in \phi_H^1(o_N),\ v(\tau, 1) \in \nu^* S, \end{cases} \tag{12.7.37}$$

where $J^H = \{J_t^H\}, J_t^H := (\phi_H^t(\phi_H^1)^{-1})^* J_t$. Furthermore, (12.7.37) *is regular if and only if* (12.5.19) *is regular.*

Once we have transformed (12.5.19) into (12.7.37), we can further deform J^H to the constant family J_0 and consider

$$\begin{cases} \partial v/\partial\tau + J_0\,\partial v/\partial t = 0, \\ v(\tau,0) \in \phi_H^1(o_N),\ v(\tau,1) \in \nu^* S. \end{cases} \tag{12.7.38}$$

This deformation preserves the filtration of the associated Floer complexes, which can be shown as in the proof of Lemma 12.6.4.

The following proposition provides the action functional associated with Equations (12.7.37) and (12.7.38) that determines a natural filtration of the associated Floer homology $HF(L, o_N)$.

Proposition 12.7.2 *Let L and h_L be as in Lemma 12.1.3. Let $\Omega(L, \nu^*S; T^*N)$ be the space of paths $\gamma : [0,1] \to \mathbb{R}$ satisfying $\gamma(0) \in L, o_N$, $\gamma(1) \in o_N$. Consider the effective action functional*

$$\mathcal{A}^{\mathrm{eff}}(\gamma) = \int \gamma^*\theta + h_H(\gamma(0)).$$

Then $d\mathcal{A}^{\mathrm{eff}}(\gamma)(\xi) = \int_0^1 \omega(\xi(t), \dot\gamma(t))dt$. In particular,

$$\mathcal{A}^{\mathrm{eff}}(c_x) = h_H(x) = \mathcal{A}_H^{cl}(z_x^H) \tag{12.7.39}$$

for the constant path $c_x \equiv x \in L \cap o_N$, i.e., for any critical path c_x of $\mathcal{A}^{\mathrm{eff}}$.

We would like to highlight the presence of the 'boundary contribution' $h_H(\gamma(0))$ in the definition of the effective action functional above. This addition is needed in order to make the Cauchy–Riemann equation (12.7.36) or (12.7.38) into a *gradient trajectory equation* of the relevant action functional.

In the same way as we defined $\rho(H, J; S, u)$, we can define $\widetilde{\rho}(L, \widetilde{J}; u, S)$ first for regular $\widetilde{J}$ and then prove its independence on $\widetilde{J}$ for the effective action functional $\mathcal{A}^{\mathrm{eff}}$. It then follows from construction that we have

$$\rho(H, S, J) = \widetilde{\rho}(L, S, J^H).$$

An immediate corollary of the above discussion is the following.

Theorem 12.7.3 *Consider two Hamiltonians H and H' such that $\phi_H^1(o_N) = \phi_{H'}^1(o_N)$ and $h_H = h_{H'}$ for the associated generating functions of $\phi_H^1(o_N)$ and $\phi_{H'}^1(o_N)$, respectively. Then*

$$\rho(H, S, J) = \rho(H', S, J).$$

Note that the condition $\phi_H^1(o_N) = \phi_{H'}^1(o_N)$ depends only on the Hamiltonian vector field or the differential of the Hamiltonians, while the condition

$h_H = h_{H'}$ depends on the actual values of the Hamiltonian itself. There is one particular natural circumstance in which both requirements automatically hold.

Proposition 12.7.4 *Assume N is connected. Let $B \subset N$ be a subset with nonempty interior and consider Hamiltonians H such that* $\operatorname{supp} H \subset T^*N \setminus T$*, where T is a tubular neighborhood of o_B in T^*N. If $\phi_H^1(o_N) = \phi_{H'}^1(o_N)$, then $h_H = h_{H'}$. In particular, for such a Hamiltonian H, $\rho(H, S, J)$ depends only on the time-one image $\phi_H^1(o_N)$.*

Proof It follows from the equation $dh_H = i_H^*\theta$ that the identity $\phi_H^1(o_N) = \phi_{H'}^1(o_N)$ implies that $dh_H = dh_{H'}$ and hence $h_H - h_{H'}$ is constant. Since $H,\ H' \equiv 0$ on $T \supset B$, it follows that $z_q^H \equiv q \equiv z_q^{H'}$ for $q \in B$. Therefore $h_H(q) = h_{H'}(q) = 0$. This finishes the proof. □

Example 12.7.5 Consider a symplectic manifold (M, ω) and let $B \subset M$ be a subset of nonempty interior. Let $H = H(t, x)$ be a time-dependent Hamiltonian on M such that $\operatorname{supp} H \subset M \setminus B$. Suppose the flow $t \mapsto \phi_H^t$ is C^0-small so that there exists a Darboux–Weinstein neighborhood V_Δ of the diagonal $\Delta_M \subset M \times M$ such that

$$\operatorname{Graph} \phi_H^t \subset V_\Delta.$$

If we define $\mathbb{H}(t, \mathbf{x}) = H(t, x)$ for $\mathbf{x} = (x, y) \in V_\Delta$ and extended to $T^*\Delta$ by a cut-off function outside of V_Δ, then $\mathbb{H}$ satisfies all the hypotheses required in Proposition 12.7.4. In this way, we can define a local version of the Hamiltonian spectral invariant which was denoted by $\rho_{\mathcal{U}}^{\mathrm{ham}}(H; a)$ in (Oh11b), which depends only on the graph of the time-one map ϕ_H^1. Here $\mathcal{U} \subset \mathcal{L}_0(M)$ is the subset consisting of short paths $\gamma : [0, 1] \to M$ such that $(\gamma(t), \gamma(0)) \subset V_\Delta$ for all $t \in [0, 1]$.

12.8 The wave front and the basic phase function

The case $S = \{pt\}$ is particularly interesting in that it is closely related to the structure of the *wave front* of the Lagrangian submanifold $L = \phi_H(o_N)$.

When $S = \{pt\}$, we denote by $\rho(H; \{q\})$ the invariant associated with the unique (modulo sign) generator of $H_0(\{q\}) \cong \mathbb{Z}$ for each given H. Then the assignment

$$q \mapsto \rho(H, \{q\}) \quad \text{on} \quad N$$

defines a continuous function on N, which is a consequence of Proposition 12.6.8.

Definition 12.8.1 We denote this function by $f_H : N \to \mathbb{R}$, i.e.,

$$f_H(q) := \rho(H, \{q\})$$

and call it the *basic phase function* of H or of $L = \phi_H^1(o_N)$.

The usage of 'phase function' is motivated by the appearance of an analogous function in the micro-local analysis of Fourier integral operators (Hor71), (GS77) as the phase function. We can extend the definition of f_H by continuity to an arbitrary smooth Hamiltonian H that is not necessarily nondegenerate, and then to the set of topological Hamiltonians in the sense of Section 6.2. Then the assignment

$$H \mapsto f_H : \quad \mathcal{H} \to C^0(N)$$

defines a continuous map with respect to the Hamiltonian topology of $\mathcal{H}$ and the C^0 topology of f_H, respectively. Furthermore, it has the property

$$\operatorname{osc} f_H = \max f_H - \min f_H \leq \|H\|,$$

where $\|H\|$ is the Hofer norm. By taking the infimum $\inf_{H \mapsto L} \|H\|$ here, where $H \mapsto L$ means $\phi_H^1(o_N) = L$, we have derived the inequality

$$\operatorname{osc} f_H \leq d(o_N, L) := \inf_{H \mapsto L} \|H\|. \tag{12.8.40}$$

Definition 12.8.2 Let $L \subset T^*N$ be any exact Lagrangian submanifold and $h : L \to \mathbb{R}$ be a function with $i^*\theta = dh$. The *wave front* of L (associated with h) is the image of the map

$$L \to \mathbb{R} \times N; \quad x \mapsto (h(x), \pi(x)),$$

where $\pi : T^*N \to N$ is the projection.

Recall that h is determined up to addition of a constant on each connected component of L. In general h does not define a (single-valued) function on N. Away from the *caustics* of L, i.e., the subset of N consisting of the critical values of the projection $\pi|_L : L \to N$, L can be locally represented by the graph $(q, df(q))$, where $h = f \circ \pi$. However, in general f cannot be globally defined as a differentiable function. We would like to emphasize that $\#(\pi^{-1}(q) \cap L)$ may jump as q varies. Eliashberg (El87) calls a subset of the wave front of L a *semi-simple part* if it is a graph of a continuous function $f : N \to \mathbb{R}$. The question of whether the wave front of an exact Lagrangian submanifold L contains a semi-simple part at all is not trivial.

It turns out that the construction of spectral invariants provides a canonical choice of such a semi-simple part, whose explanation is now in order.

Theorem 12.8.3 *Let G_{f_H} be the graph of f_H. Then $G_{f_H} \subset N \times \mathbb{R}$ is a semi-simple part of the wave front of L. Furthermore, f_H is smooth away from a measure-zero subset for Floer-regular H and Lipschitz continuous for general H.*

Proof The first statement follows from the definition of $f_H = \rho(H, \{q\})$ and the fact that $f_H(q)$ is a critical value of $\mathcal{A}_H|_{\Omega(\{q\})}$ and so

$$f_H(q) = \mathcal{A}_H(z_p^H) \quad \text{for some} \quad p \in \phi_H(o_N) \cap T_q^*N.$$

The Lipschitz continuity for Floer-regular H is an immediate consequence of Proposition 12.6.8 applied to $S = \{pt\}$. For general smooth H, we need only take a sequence $H_i \to H$ of Floer-regular smooth Hamiltonians and note from the proof of Proposition 12.6.8 that the constant C in (12.6.35) depends only on $\operatorname{supp} H$.

Finally, the smoothness of f_H away from a measure-zero subset is obvious from the construction for Floer-regular H. In this case, the locus of non-differentiable points of f_H is a compact stratified submanifold of N. For general H, we first note that $f_{H_i} \to f_H$ is of Lipschitz topology if $H_i \to H$ in smooth topology (in fact C^2-convergence is enough for this). Then the statement follows from Radermacher's classical theorem (see (Fe69, Theorem 3.1.6), for example), which states that a Lipschitz function is differentiable almost everywhere. □

Theorem 12.8.3 gives rise to an easy proof of the nondegeneracy of the Hofer's distance defined in the set of Hamiltonian deformations of o_N.

Theorem 12.8.4 *Let $d(L_1, L_2)$ be the Hofer distance. Then $d(L_1, L_2) = 0$ if and only if $L_1 = L_2$.*

Proof We first consider the case when $L_1 = o_N$. In this case, we have by definition

$$d(o_N, L_2) = \inf_{H \mapsto L_2} \|H\|.$$

Now suppose that $d(o_N, L_2) = 0$. Then (12.8.40) implies

$$\operatorname{osc} f_H = 0,$$

i.e., the basic phase function f_H is a constant function on N and so f_H is smooth everywhere and

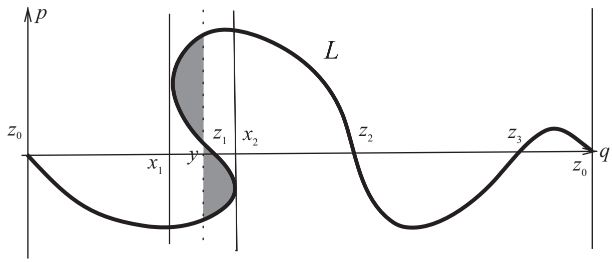

Figure 12.1 A multi-section Lagrangian L.

$$df_H(q) = 0$$

for all $q \in N$. Therefore, we have proved

$$o_N \subset L_2 = \phi_H(o_N). \tag{12.8.41}$$

Using the compactness and connectedness of N, it is easy to prove that (12.8.41) does indeed imply

$$o_N = L_2,$$

which finishes the proof for the case when $L_1 = o_N$. The other direction of the proof is obvious.

For the general pairs

$$L_1 = \phi(o_N) \quad \text{and} \quad L_2 = \psi(o_N),$$

we first note that

$$d(L_1, L_2) = d(\eta(L_1), \eta(L_2))$$

for any $\eta \in \mathcal{D}_\omega(T^*N)$. Therefore one can reduce the general case to the special case $L_1 = o_N$. □

To illustrate the meaning of the f_H, we give an example for the case when $N = S^1$.

Example 12.8.5 Let us consider the Lagrangian submanifold $L \subset T^*S^1$ as pictured in Figure 12.1. Here we denote by zs the intersections of L with the zero section, by xs the caustics and by y the point at which the two shaded regions in the picture have the same area. The corresponding wave front can be easily drawn as in Figure 12.2.

Note that the zs correspond to critical points of the action functional, the xs to the cusp points of the wave front and y to the point where two different

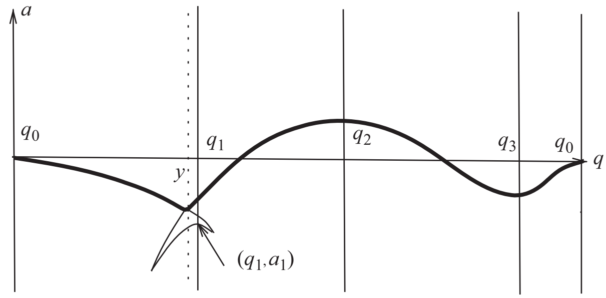

Figure 12.2 The wave front of L and the graph of f_H.

branches of the wave front cross. Using the continuity of the basic phase function f_H where $H \mapsto L$, one can easily see that the graph of f_H is the one in bold in Figure 12.2. We would like to note that the value $\min_{q \in N} F_H(q)$ is *not* a critical value of $\mathcal{A}_H$ but a value associated to the wave front cross.

13

The off-shell framework of a Floer complex with bubbles

The well-known Riemann–Roch formula for the $\overline{\partial}$-operator over the sections of a complex vector bundle provides the formula for the virtual dimension of the moduli space of holomorphic maps from a closed Riemann surface. The virtual dimension formula involves the first Chern number of the vector bundle.

In this chapter, we review the construction of the relative version of the first Chern number, which is ubiquitously called the *Maslov index* in general. We organize our exposition by combining those from (Gr85), (Oh95b) and (KL01).

13.1 Lagrangian subspaces versus totally real subspaces

In relation to the study of the boundary version of the Riemann–Roch formula, we consider the (maximally) *totally real subspace* of $\mathbb{C}^n$. A real subspace $V \subset \mathbb{C}^n$ is called totally real if $V \cap iV = \{0\}$ and $\dim_{\mathbb{R}} V = n$. We denote the set of totally real subspaces by $\mathcal{R}(n)$.

Any totally real subspace V can be written as

$$V = A \cdot \mathbb{R}^n$$

for some $A \in GL(n, \mathbb{C})$ and $A_1 \cdot \mathbb{R}^n = A_2 \cdot \mathbb{R}^n$ if and only if

$$A_2^{-1} A_1 \in GL(n, \mathbb{R}). \tag{13.1.1}$$

Therefore the set $\mathcal{R}(n)$ of totally real subspaces is a homogeneous space

$$\mathcal{R}(n) = GL(n, \mathbb{C})/GL(n, \mathbb{R}).$$

The following lemma is a useful fact for the study of the index problem. See Proposition 4.4 in (Oh95b) for its proof.

Lemma 13.1.1 *Consider the subset*

$$\widetilde{\mathcal{R}}(n) = \{D \in GL(n, \mathbb{C}) \mid D\overline{D} = I_n\},$$

where I_n is the identity matrix. Then the map

$$B : \mathcal{R}(n) \cong GL(n, \mathbb{C})/GL(n, \mathbb{R}) \to \widetilde{\mathcal{R}}(n); \quad A \cdot \mathbb{R}^n \mapsto A\overline{A}^{-1}$$

is a diffeomorphism.

Proof We consider the map

$$A \mapsto A\overline{A}^{-1} \tag{13.1.2}$$

defined on $GL(n, \mathbb{C})$. Note that any matrix D of the form $A\overline{A}^{-1}$ satisfies $D\overline{D} = Id_n$ and so the image of this map lies in $\widetilde{\mathcal{R}}(n)$.

The equation (13.1.1) is equivalent to

$$A_1\overline{A_1}^{-1} = A_2\overline{A_2}^{-1}.$$

This shows that the map B is well defined. We leave the proof of the one-to-one correspondence of the map (13.1.2) as an exercise. □

Exercise 13.1.2 Prove the map (13.1.2) is a one-to-one correspondence.

Next we consider the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$ with

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

in terms of the canonical coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. We denote by $\Lambda(n) := \mathcal{L}(\mathbb{R}^{2n}, \omega_0)$ the set of Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$. When we equip $\mathbb{R}^{2n}$ with the standard complex multiplication and identify it with $\mathbb{C}^n$ by the map

$$(x_i, y_i) \mapsto z_i = x_i + \sqrt{-1}y_i$$

then any Lagrangian subspace $V \subset \mathbb{C}^n$ can be written as

$$V = A \cdot \mathbb{R}^n$$

for a complex matrix $A \in U(n)$. This shows that $\Lambda(n)$ is a homogeneous space

$$\Lambda(n) \cong U(n)/O(n).$$

This in particular shows that the natural inclusion $\Lambda(n) \hookrightarrow \mathcal{R}(n)$ is a homotopy equivalence.

It is shown in (Ar67) that $H^1(\Lambda(n), \mathbb{Z}) \cong \mathbb{Z}$ and $\Lambda(n)$ carries the well-known characteristic class $\mu \in H^1(\Lambda(n), \mathbb{Z})$, the *Maslov class* (Ar67). This assigns to each given loop $\gamma : S^1 \to \Lambda(n)$ an integer given by

$$\mu(\gamma) = \frac{1}{2\pi} \det{}^2(A).$$

We refer to Section 3.2 for the coordinate-free description of the class. Furthermore, two loops γ_1, γ_2 are homotopic if and only if $\mu(\gamma_1) = \mu(\gamma_2)$. Furthermore, the value $\mu(\gamma)$ does not depend on the identification $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ as long as the bilinear form $\omega_0(\cdot, J_0\cdot)$, where J_0 is the almost complex structure associated with the complex multiplication on $\mathbb{C}^n$, remains *positive*.

Note that when A is a unitary matrix we have $\overline{A}^{-1} = A^{\mathrm{T}}$, where A^{T} is the transpose of A. Therefore we have the following corollary of Lemma 13.1.1.

Corollary 13.1.3 *Denote by $\widetilde{\Lambda}(n) \subset \widetilde{\mathcal{R}}(n)$ the image of B restricted to $\Lambda(n) \subset \mathcal{R}(n)$, i.e.,*

$$\widetilde{\Lambda}(n) := \{D \in U(n) \mid D = D^{\mathrm{T}}\}, \tag{13.1.3}$$

the set of symmetric unitary matrices. Then B restricts to a diffeomorphism on $\Lambda(n)$.

The following generalizes the Maslov index to the loops of totally real subspaces, which restricts to the Maslov index of loops in $\Lambda(n)$.

Definition 13.1.4 (Generalized Maslov index) Let $\gamma : S^1 \to \mathcal{R}(n)$ be a loop. The *generalized Maslov index* $\mu(\gamma)$ is defined to be the winding number of

$$\det \circ B \circ \gamma : S^1 \to \mathbb{C} \setminus \{0\}.$$

13.2 The bundle pair and its Maslov index

Let Σ be an oriented compact surface with boundary $\partial\Sigma$. We denote by g the genus of Σ and by h the number of connected components of $\partial\Sigma$.

13.2.1 The case of complex vector bundles

Consider a complex vector bundle $E \to \Sigma$. Note that, if $\partial\Sigma \neq \emptyset$, then any complex vector bundle $E \to \Sigma$ is topologically trivial.

Definition 13.2.1 A *complex bundle pair* (E, λ) is a complex vector bundle $E \to \Sigma$ with a real subbundle $\lambda \to \partial\Sigma$ of $E|_{\partial\Sigma}$ equipped with an isomorphism

$$E|_{\partial\Sigma} \cong \lambda \otimes \mathbb{C}. \tag{13.2.4}$$

Equivalently, a complex bundle pair is a complex vector bundle $E \to \Sigma$ with a totally real subbundle $\lambda \subset E_{\mathbb{R}}|_{\partial\Sigma} \to \partial\Sigma$, where $E_{\mathbb{R}}$ is the realization of the bundle E.

We fix a trivialization $\Phi : E \to \Sigma \times \mathbb{C}^n$ and let $R_1, \ldots, R_h$ be the connected components of $\partial\Sigma$ with boundary orientation induced from Σ. Then, owing to the given isomorphism, $E|_{\partial\Sigma} \cong \lambda \otimes \mathbb{C}$, $\Phi(\lambda|_{R_i})$ gives rise to a loop $\gamma_{\Phi,i} : S^1 \to \mathcal{R}(n)$. Setting $\mu(\Phi, R_i) = \mu(\gamma_{\Phi,i})$, we derive the following proposition.

Proposition 13.2.2 *Let (E, λ) be a complex bundle pair of rank n over an oriented compact surface Σ. Then the sum $\sum_{j=1}^h \mu(\Phi, R_i)$ is independent of the choice of trivialization $\Phi : E \to \Sigma \times \mathbb{C}^n$.*

Proof Let Φ_1, Φ_2 be two trivializations of E. Then

$$\Phi_2 \circ \Phi_1^{-1} : \Sigma \times \mathbb{C}^n \to \Sigma \times \mathbb{C}^n$$

is given by $(x, v) \mapsto (x, g(x)v)$ for a map

$$g : \Sigma \to GL(n, \mathbb{C}).$$

It follows from the definition of $\gamma_{\Phi,i}$ that we have

$$g\overline{g}^{-1} \cdot B \circ \gamma_{\Phi_1,i} = B \circ \gamma_{\Phi_2,i}$$

and hence $\deg(B \circ \gamma_{\Phi_2,i}) = \deg(\det(g\overline{g}^{-1})|_{\partial\Sigma}) + \deg(B \circ \gamma_{\Phi_1,i})$, i.e.,

$$\mu(\Phi_2, R_i) - \mu(\Phi_1, R_i) = \deg(\det(g\overline{g}^{-1})|_{\partial\Sigma}).$$

However, since $\partial\Sigma = \cup_i R_i$ and the map $\det(g\overline{g}^{-1})|_{\partial\Sigma} : \partial\Sigma \to S^1$ obviously extends to Σ, we have $\sum_i^h \deg(\det(g\overline{g}^{-1})|_{\partial\Sigma}) = 0$ by the cobordism invariance of the degree. This finishes the proof. □

Proposition 13.2.2 says that the following definition is well defined.

Definition 13.2.3 The Maslov index of the bundle pair (E, λ) is defined by

$$\mu(E, \lambda) = \sum_{i=1}^h \mu(\Phi, R_i), \tag{13.2.5}$$

where $\Phi : E \to \Sigma \times \mathbb{C}^n$ is a particular (and hence any) trivialization.

13.2.2 The case of symplectic vector bundles

Consider a symplectic vector bundle $E \to \Sigma$, i.e., assume that each fiber E_x carries a symplectic inner product ω_x depending smoothly on $x \in \Sigma$.

Again, if $h > 0$, any symplectic vector bundle on Σ is symplectically trivial. We fix a trivialization

$$\Psi : E \to \Sigma \times (\mathbb{R}^{2n}, \omega_0).$$

Definition 13.2.4 A *symplectic bundle pair* is a pair (E, λ), where $E \to \Sigma$ is a symplectic vector bundle and $\lambda \to \partial\Sigma$ is a Lagrangian subbundle of $E|_{\partial\Sigma}$.

We recall that the complexification of any Lagrangian subspace of $\mathbb{C}^n$ is canonically isomorphic to $\mathbb{C}^n$. Compare this with Definition 13.2.1. As in the complex case, the restriction $\Psi(\lambda|_{R_i})$ gives rise to a loop $\gamma_{\Psi,i} : S^1 \to \Lambda(n)$. We denote $\mu(\Psi, R_i) = \mu(\gamma_{\Psi,i})$. Then we have the following.

Proposition 13.2.5 *Let (E, λ) be a symplectic bundle pair over an oriented compact surface Σ. Then the sum $\sum_{j=1}^{h} \mu(\Psi, R_i)$ is independent of the choice of symplectic trivialization $\Psi : E \to E \times \mathbb{C}^n$.*

Proof Let $\Psi_1, \Psi_2 : E \to (\mathbb{R}^{2n}, \omega_0)$ be symplectic trivializations. The map

$$\Psi_2 \circ \Psi_1^{-1} : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$$

is given by the assignment $(x, v) \to (x, g(x)v)$ for a map $g : \Sigma \to Sp(2n, \mathbb{R})$. We write the natural action of $Sp(2n, \mathbb{R})$ on $\Lambda(n)$ by $g \cdot V$. Then we have

$$g(x)\gamma_{\Psi_1,i}(x) = \gamma_{\Psi_2,i}(x) \quad \text{for } x \in R_i.$$

Since any Lagrangian subspace of $\mathbb{C}^n$ is totally real for the standard complex multiplication and the generalized Maslov index is invariant under the homotopy inside $\mathcal{R}(n)$, this identity gives rise to

$$\mu(\Psi_2, R_i) = \mu(\Psi_1, R_i) + 2\deg(g|_{R_i}),$$

where $\deg(g|_{R_i})$ is the degree of the loop $g|_{R_i} : S^1 \to Sp(2n, \mathbb{R})$ defined in Section 2.2.2. Since $g|_{\partial\Sigma} = \coprod_{i=1}^{h} g|_{R_i}$, it again follows $\sum_{i=1}^{h} \deg(g|_{R_i}) = 0$ and hence the proof. □

This proposition allows one to define the Maslov index $\mu(E, \lambda)$ for the symplectic bundle pair (E, λ).

Definition 13.2.6 The Maslov index of the symplectic bundle pair (E, λ) is defined by

$$\mu(E, \lambda) = \sum_{i=1}^{h} \mu(\Phi, R_i), \tag{13.2.6}$$

where $\Phi : E \to \Sigma \times \mathbb{C}^n$ is a particular (and hence any) trivialization.

It is straightforward to prove the invariance of $\mu(E, \lambda)$ under a homotopy. In fact, a stronger invariance property holds, which we now formulate.

Let (E_0, λ_0), (E_1, λ_1) be symplectic bundle pairs over Σ_1, Σ_2, respectively.

Proposition 13.2.7 *Suppose that the real vector bundle*

$$\lambda_0|_{\partial\Sigma_0} \coprod \lambda_1|_{\partial\Sigma_1}$$

extends to a real vector bundle $\lambda_{12} \to \Sigma_{12}$ *for an oriented cobordism* Σ_{12} *with*

$$\partial\Sigma_{12} = -\partial\Sigma_1 \coprod \partial\Sigma_2$$

and λ_{12} *embeds to a symplectic vector bundle* $E_{12} \to \Sigma_{12}$ *as a Lagrangian subbundle restricted to the bundle pairs*

$$(E_0, \lambda_0)|_{\partial\Sigma_0} \coprod (E_1, \lambda_1)|_{\partial\Sigma_1}$$

on its boundary $\partial\Sigma_{12}$. *Then* $\mu(E_1, \lambda_1) = \mu(E_2, \lambda_2)$.

Proof Choose any trivialization of $E_{12} \to \Sigma_{12}$

$$\Phi_{12} : E_{12} \to \Sigma_{12} \times (\mathbb{R}^{2n}, \omega_0).$$

This obviously defines trivializations of E_1 and E_2. By computing $\mu(E_i, \lambda_i)$ with $i = 1, 2$ and $\mu(E_{12}, \lambda_{12})$ in this trivialization, the proof follows. □

We call the statement of this proposition the cobordism invariance of $\mu(E, \lambda)$. Obviously this proposition implies the homotopy invariance of $\mu(E, \lambda)$.

Now let $L \subset (M, \omega)$ be a Lagrangian submanifold and consider a smooth map $w : (\Sigma, \partial\Sigma) \to (M, L)$. Denote by $\partial w : \partial\Sigma \to L$ the restriction $w|_{\partial\Sigma}$. Then $(w, \partial w)$ induces a canonical bundle pair

$$E = w^*TM, \quad \lambda = (\partial w)^*TL.$$

Definition 13.2.8 Let w be as above. The *Maslov index* of w, denoted by $\mu_L(w)$, is defined to be

$$\mu_L(w) = \mu(w^*TM, (\partial w)^*TL).$$

An immediate corollary of the homotopy invariance of $\mu(E, \lambda)$ is the homotopy invariance of $\mu_L(w)$ for the map w.

13.3 Maslov indices of polygonal maps

In this section, we assign a topological index to the *polygonal maps* following the exposition given in (FOOO10a).

We first explain how we associate a natural bundle pair with each polygonal map attached to a transversal Lagrangian chain using Proposition 3.2.14.

Consider the disc $\Sigma = D^2$ with a set of marked points $\{u_0, u_1, \ldots, u_k\} \subset \partial D^2$. Denote the (boundary) punctured disc by

$$\dot{\Sigma} = D^2 \setminus \{u_i\}_{0 \le i \le k}.$$

We fix a small closed neighborhood $U_i \subset \Sigma$ of u_i for $i = 0, \ldots, k$ so that the points u_i are disjoint, and each carries a conformal isomorphism

$$U_i \setminus \{u_i\} \to (-\infty, 0] \times [0, 1].$$

Let $\mathcal{L} = (L_0, L_1, \ldots, L_k)$ be an ordered $(k+1)$-tuple of Lagrangian submanifolds L_i, $i = 0, \ldots, k$.

Definition 13.3.1 We say that $\mathcal{L}$ is a transversal chain if it is pairwise transversal and $L_i \cap L_j \cap L_k = \emptyset$ for any distinct triple i, j, k. We call $k + 1$ the length of the chain.

Now we consider a chain of intersection points

$$\vec{p} = \{p_{i(i+1)} \in L_i \cap L_{i+1} \mid i = 0, \ldots, k\}$$

and a smooth map $w : \dot{\Sigma} \to M$ that extends to Σ continuously and satisfies the condition

$$\lim_{z \to u_i} w(z) = p_{i(i+1)}, \quad w(\overline{u_i u_{i+1}}) \subset L_i. \tag{13.3.7}$$

We denote by

$$\pi_2(\mathcal{L}; \vec{p})$$

the set of homotopy classes of such continuous maps and by B an element from $\pi_2(\mathcal{L}; \vec{p})$. We next want to assign a topological index of such maps w relative to the data $\mathcal{L}$, $\vec{p}$ and B. Consider a symplectic trivialization of w^*TM

$$w^*TM \cong \dot{\Sigma} \times (\mathbb{R}^{2n}, \omega_0).$$

Under this trivialization, the map

$$\alpha_w : S^1 = \dot{\Sigma} \to \Lambda(n); \qquad t \mapsto T_{w(t)}L_i \quad \text{if } t \in \overline{u_{i-1}u_i}$$

defines a piecewise smooth path with discontinuities at $k+1$ points $u_i \in \partial D^2$ for $i = 0,\ 1, \ldots, k$. By the transversality hypothesis, the two Lagrangian subspaces $T_{w(u_i)}L_i$ and $T_{w(u_{i+1})}L_{i+1}$ in $(\mathbb{R}^{2n}, \omega_0)$ satisfy

$$T_{w(u_i)}L_i \cap T_{w(u_{i+1})}L_{i+1} = \{0\}.$$

From these data, we will associate a smooth symplectic bundle pair (E_w, λ_w) over a suitably constructed bordered Riemann surface, which we will denote by $\widehat{\dot{\Sigma}}$.

First we consider the symplectic vector space $(S, \Omega) = (T_{w(u_i)}M, \omega_{w(u_i)})$ and fix a smooth path $\alpha_{i(i+1)} : [0,1] \to \mathcal{L}(S, \Omega)$ chosen as in Proposition 3.2.14 for each $i = 0, \ldots, k$ so that

$$\alpha_{i(i+1)}(0) = T_{w(u_i)}L_i, \quad \alpha_{i(i+1)}(1) = T_{w(u_{i+1})}L_{i+1}.$$

At each $p_{i(i+1)} \in L_i \cap L_{i+1}$, there exists a Darboux chart $\Phi_i : V_i \to T_{p_{i(i+1)}}M$ on V_i such that V_i is a neighborhood of $p_{i(i+1)}$ in M and

(1) $\Phi(p_{i(i+1)}) = 0$ in $T_{p_{i(i+1)}}M$,
(2) $d\Phi_i(p_{i(i+1)}) : T_{p_{i(i+1)}}M \to T_{p_{i(i+1)}}M$ is the identity,
(3) $\Phi_i(V_i \cap L_i) = \Phi(V_i) \cap T_{p_{i(i+1)}}L_i$, $\Phi_i(V_i \cap L_{i+1}) = \Phi(V_i) \cap T_{p_{i(i+1)}}L_{i+1}$.

Exercise 13.3.2 Prove the existence of such a Darboux chart.

Then, by the convergence (13.3.7), we can decompose the compact bordered Riemann surface $\dot{\Sigma}$ into the union

$$\widehat{\dot{\Sigma}} := \Big(\dot{\Sigma} \cup \bigcup \widehat{U_i}\Big),$$

where $w(U_i) \subset V_i$. We set $\widehat{U_i} \cong [-\infty, 0] \times [0,1]$ to be the natural completion of $(-\infty, 0] \times [0,1] \cong U_i$ for each $i = 0, \ldots, k$. We call this union the *real blow-up* of $\dot{\Sigma}$ along its punctures with respect to the conformal chart at the ends

$$U_i \cong (-\infty, 0] \times [0,1].$$

For the Darboux chart Φ_i given in a neighborhood of $p_{i(i+1)}$, the map $\Phi_i \circ w$ satisfies

$$d\Phi_i(T_{w(\tau,0)}L_i) \equiv T_{p_{i(i+1)}}L_i, \quad d\Phi_i(T_{w(\tau,1)}L_{i+1}) \equiv T_{p_{i(i+1)}}L_{i+1} \tag{13.3.8}$$

and $\Phi_i \circ w(\infty, t) \equiv p_{i(i+1)}$. We denote the continuous extension of w to $\widehat{\dot{\Sigma}}$ by $\widehat{w} : \widehat{\dot{\Sigma}} \to M$.

Lemma 13.3.3 *The pull-back bundle w^*TM over $\dot\Sigma$ can be smoothly extended to $\widehat{w}^*TM$ so that $\widehat{w}^*TM|_{U_i}$ has a canonical trivialization*

$$((\Phi \circ \widehat{w})^*M; (\partial_i(\Phi \circ \widehat{w}))^*TL_i, (\partial_{i+1}(\Phi \circ \widehat{w}))^*TL_{i+1})|_{U_i}$$
$$\cong (U_i; \partial_0 U_i, \partial_1 U_i) \times (T_{p_{i(i+1)}}M; T_{p_{i(i+1)}}L_i, T_{p_{i(i+1)}}L_{i+1}).$$

Proof By (13.3.8), we have the trivialization of w^*TM so that the triple

$$(w^*M; (\partial_i w)^*TL_i, (\partial_{i+1}w)^*TL_{i+1})$$

restricted to U_i is mapped to the trivial bundle

$$(U_i; \partial_0 U_i, \partial_1 U_i) \times (T_{p_{i(i+1)}}M; T_{p_{i(i+1)}}L_i, T_{p_{i(i+1)}}L_{i+1})$$

under the derivative $d\Phi_i$. The lemma immediately follows from this. □

Now we have extended the pull-back bundle $w^*TM \to \dot\Sigma$ smoothly to $\widehat{w}^*TM \to \widehat{\dot\Sigma}$. We denote the corresponding symplectic vector bundle by

$$E_w = \widehat{w}^*TM.$$

Next we describe a Lagrangian subbundle $\lambda \subset E_w|_{\partial\widehat{\Sigma}}$. We insert the path $\alpha_{i(i+1)} : [0,1] \to \Lambda(T_{w(u_i)}M)$ into $\{-\infty\} \times [0,1]$ at the puncture u_i of $\dot\Sigma$ between the Gauss maps $\alpha_w|_{\overline{u_{i-1}u_i}}$ and $\alpha_w|_{\overline{u_i u_{i+1}}}$, whereupon we obtain a Lagrangian subbundle

$$\lambda_w \to \partial\widehat{\Sigma}$$

of $E_w|_{\partial\widehat{\Sigma}}$.

Definition 13.3.4 We define the *polygonal Maslov index*, denoted by $\mu(w; \mathcal{L}, \vec{p})$, as the Maslov index

$$\mu(w; \mathcal{L}, \vec{p}) = \mu(E_w, \lambda_w).$$

When the length of the Lagrangian chain $\mathcal{L}$ is two, i.e., $k = 1$, this index is related to the *Maslov–Viterbo index* (Vi88) for the pair (L_0, L_1) in a natural way, which we will provide later in Definition 13.6.2.

Exercise 13.3.5 Find out the precise relationship between the Maslov index and the Maslov–Viterbo index.

Proposition 13.3.6 *$\mu(w; \mathcal{L}, \vec{p})$ depends only on the homotopy class $[w] \in \pi_2(\mathcal{L}, \vec{p})$ and the choice of the $\alpha_{i(i+1)}$. We denote the corresponding common index by $\mu(\mathcal{L}, \vec{p}; B)$ for $B = [w]$.*

Proof Let w_0 and w_1 be two polygonal maps defined on $\dot{\Sigma} = D^2 \setminus \{u_i\}_{i=0,\cdot,k}$ attached to the datum $(\mathcal{L}, \vec{p})$ homotopic to each other. A smooth homotopy $\{w_s\}_{0\le s\le 1}$ between them induces a family of symplectic vector bundles w_s^*TM. Since $[0, 1]$ is compact, we can find a common neighborhood U_i of u_i such that w_s^*TM is trivialized by Φ so that

$$\begin{aligned} d\Phi : (w_s^*TM|_{U_i}; (\partial w_s)^*TL_i, (\partial w_s)^*TL_{i+1}) \\ \to (U_i; \partial_0 U_i, \partial_1 U_i) \times (T_{w(u_i)}M; T_{w(u_i)}L_i, T_{w(u_i)}L_{i+1}) \end{aligned}$$

for all $s \in [0, 1]$. We fill the segment $\{\infty_i\} \times [0, 1] \subset [0, \infty] \times [0, 1] \cong \widehat{U}_i$ by the same path $\alpha_{i(i+1)}$ and, upon extending the trivialization $d\Phi$ above to the rest of D^2, we obtain a homotopy of bundle pairs (E_{w_s}, λ_{w_s}). By virtue of the homotopy invariance of the Maslov index of bundle pairs, we obtain

$$\mu(E_0, \lambda_0) = \mu(E_1, \lambda_1),$$

which finishes the proof. □

This topological index forms the basis in the computation of the Fredholm index of the linearization of the $\overline{\partial}$-operator appearing in the construction of the Fukaya category. (See (FOOO10a).)

13.4 Novikov covering and Novikov ring

Let (L_0, L_1) be a pair of connected compact Lagrangian submanifolds of (M, ω) that are transversal. We would like to note that we do *not* assume our Lagrangian submanifolds are connected at the moment.

Consider the space of paths

$$\Omega = \Omega(L_0, L_1) = \{\ell : [0, 1] \to P \mid \ell(0) \in L_0, \ell(1) \in L_1\}.$$

We define a one-form (the *action one-form*) α on Ω by

$$\alpha(\ell)(\xi) = \int_0^1 \omega(\dot{\ell}(t), \xi(t))dt \qquad (13.4.9)$$

for $\xi \in T_\ell\Omega$, which we know becomes a '*closed* one-form' by Corollary 3.8.4.

$\Omega(L_0, L_1)$ is not connected but has countably many connected components. We will always work on a particular fixed connected component of $\Omega(L_0, L_1)$ when we study Floer moduli spaces later. We specify the particular component by choosing a *base path*, which we denote by ℓ_0. Denote the corresponding component by

$$\Omega(L_0, L_1; \ell_0) \subset \Omega(L_0, L_1).$$

The base path ℓ_0 automatically picks out a connected component from each of L_0 and L_1 as its initial and final points

$$x_0 = \ell_0(0) \in L_0, \quad x_1 = \ell_0(1) \in L_1.$$

Then $\Omega(L_0, L_1; \ell_0)$ is nothing but the space of paths between the corresponding connected components of L_0 and L_1. Because of this *we will always assume that L_0 and L_1 are connected from now on*, unless stated otherwise.

Next we describe a covering space, which we call *the Novikov covering* of the component $\Omega(L_0, L_1; \ell_0)$ of $\Omega(L_0, L_1)$. We start by describing the universal covering space of $\Omega(L_0, L_1; \ell_0)$. Consider the set of all pairs (ℓ, w) such that w satisfies the boundary condition

$$w(0, \cdot) = \ell_0, \quad w(1, \cdot) = \ell, \tag{13.4.10}$$

$$w(s, 0) \in L_0,\ w(s, 1) \in L_1 \quad \text{for all } s \in [0, 1]. \tag{13.4.11}$$

Considering w as a continuous path in $\Omega(L_0, L_1; \ell_0)$ from ℓ_0 and ℓ, the fiber at ℓ of the universal covering space of $\Omega(L_0, L_1; \ell_0)$ can be represented by the set of path homotopy classes of w relative to the end $s = 0,\ 1$. We denote this homotopy class of w in $\Omega(L_0, L_1; \ell_0)$ satisfying (13.4.10) by $[\ell, w]$.

Now we define a smaller covering space of $\Omega(L_0, L_1; \ell_0)$ by modding out the latter by another equivalence relation that is weaker than the homotopy. This is an analog to the Novikov covering space of the contractible loop space of a symplectic manifold (M, ω), which we will study later in Part 4.

Note that, when we are given two pairs (ℓ, w) and (ℓ, w') with a given common path ℓ from $\Omega(L_0, L_1; \ell_0)$, the concatenation

$$\overline{w} \# w' : [0, 1] \times [0, 1] \to M$$

defines a loop $c : S^1 \to \Omega(L_0, L_1; \ell_0)$. One may regard this loop as an annular map

$$C : S^1 \times [0, 1] \to M$$

satisfying the boundary condition

$$C(s, 0) \in L_0, \quad C(s, 1) \in L_1. \tag{13.4.12}$$

Obviously the symplectic area of C, denoted by

$$I_\omega(C) = \int_C \omega, \tag{13.4.13}$$

depends only on the homotopy class of C satisfying (13.4.12) and so defines a homomorphism, which we denote by

$$I_\omega : \pi_1(\Omega(L_0, L_1; \ell_0)) \to \mathbb{R}.$$

Next we note that, for the map $C : S^1 \times [0,1] \to M$ satisfying (13.4.12), it associates a symplectic bundle pair (E, λ) given by

$$E_C = C^*TM, \qquad \lambda_C = R_0^*TL_0 \coprod R_1^*TL_1,$$

where $R_i : S^1 \to L_i$ is the map given by $R_i(s) = C(s,i)$ for $i = 0,\ 1$. This allows us to define another homomorphism,

$$I_\mu : \pi_1(\Omega_{\ell_0}(L_0, L_1), \ell_0) \to \mathbb{Z}; \quad I_\mu(c) = \mu(E_C, \lambda_C),$$

where $\mu(E_C, \lambda_C)$ is the Maslov index of the bundle pair (E_C, λ_C).

Using I_μ and the symplectic form I_ω, we define an equivalence relation $\sim$ on the set of all pairs (ℓ, w) satisfying (13.4.12). For a given such pair w, w', we denote by $\overline{w}\# w'$ the concatenation of $\overline{w}$ and w' along ℓ, which defines a loop in $\Omega(L_0, L_1; \ell_0)$ based at ℓ_0.

Definition 13.4.1 We say that (ℓ, w) is Γ-equivalent to (ℓ, w') and write $(\ell, w) \sim (\ell, w')$ if the following conditions are satisfied:

$$I_\omega(\overline{w}\# w') = 0 = I_\mu(\overline{w}\# w'). \tag{13.4.14}$$

We denote the set of equivalence classes $[\ell, w]$ by $\widetilde{\Omega}(L_0, L_1; \ell_0)$ and call this the Γ-*covering space* of $\Omega(L_0, L_1; \ell_0)$.

There is a canonical lifting of $\ell_0 \in \Omega(L_0, L_1; \ell_0)$ to $\widetilde{\Omega}(L_0, L_1; \ell_0)$: this is just

$$[\ell_0, \widetilde{\ell}_0] \in \widetilde{\Omega}(L_0, L_1; \ell_0),$$

where $\widetilde{\ell}_0$ is the map $\widetilde{\ell}_0 : [0,1]^2 \to M$ with $\widetilde{\ell}_0(s,t) = \ell_0(t)$. In this way, $\widetilde{\Omega}(L_0, L_1; \ell_0)$ also has a natural base point, which we suppress from the notation.

Now we denote by $G(L_0, L_1; \ell)$ the group of deck transformations on $\pi^{-1}(\ell)$. It is easy to see that there is a canonical isomorphism between the groups for two different ℓs if they are homotopic. Then it follows that the two isomorphisms I_ω and I_μ push down to homomorphisms

$$\nu : G(L_0, L_1; \ell_0) \to \mathbb{R}, \quad d : G(L_0, L_1; \ell_0) \to \mathbb{Z}$$

defined by

$$\nu(g) = \omega[g], \quad d(g) = \mu(g). \tag{13.4.15}$$

Proposition 13.4.2 *The group $G(L_0, L_1; \ell_0)$ is an abelian group.*

Proof By definition of $G(L_0, L_1; \ell_0)$, the map $\nu \times d : G(L_0, L_1; \ell_0) \to \mathbb{R} \times \mathbb{Z}$ is an injective group homomorphism. Therefore we conclude that $G(L_0, L_1; \ell_0)$ is abelian since $\mathbb{R} \times \mathbb{Z}$ is abelian. □

Definition 13.4.3 (Period group) We define the *period group* of the group $G(L_0, L_1; \ell_0)$ or of the component $\widetilde{\Omega}(L_0, L_1; \ell_0)$ by the image

$$\Gamma_\omega(L_0, L_1; \ell_0) = \{\omega[g] \mid g \in G(L_0, L_1; \ell_0)\} \subset \mathbb{R}.$$

Now we define a Novikov ring associated with the abelian group $G(L_0, L_1; \ell_0) \subset \mathbb{R} \times \mathbb{Z}$. We first consider the group ring $R[G(L_0, L_1; \ell_0)]$ for a commutative ring R. We will use the topology induced by ν to define a completion of the ring $R[G(L_0, L_1; \ell_0)]$.

Consider the formal sum

$$\sum_{g \in G(L_0, L_1; \ell_0)} a_g g =: \sigma$$

and set

$$\mathrm{supp}(\sigma) = \{g \in G(L_0, L_1; \ell_0) \mid a_g \neq 0\}.$$

Definition 13.4.4 Define the *Novikov ring* associated with the group $G(L_0, L_1; \ell_0)$ by

$$\Lambda(L_0, L_1; \ell_0) = \{\sigma \mid \#(\mathrm{supp}(\sigma) \cap \nu^{-1}((C, \infty))) < \infty\}$$

with the obvious ring structure on it.

It is easy to see that $\Lambda(L_0, L_1; \ell_0)$ forms a graded R-module

$$\Lambda(L_0, L_1; \ell_0) = \bigoplus_{k \in \mathbb{Z}} \Lambda^k(L_0, L_1; \ell_0),$$

where $\Lambda^k(L_0, L_1; \ell_0)$ is the R-submodule generated by g with $\mu(g) = k$ in σ above.

13.5 Action functional

We define the action for a given pair (ℓ, w) by the formula

$$\mathcal{A}(\ell, w) = -\int w^*\omega,$$

which defines the *action functional*

$$\mathcal{A} : \widetilde{\Omega}(L_0, L_1; \ell_0) \to \mathbb{R}.$$

It follows from the definition of $G(L_0, L_1; \ell_0)$ that the integral depends only on the Γ-equivalence class $[\ell, w]$ and so pushes down to a well-defined functional on the covering space $\widetilde{\Omega}(L_0, L_1; \ell_0)$.

Proposition 13.5.1 *Let $\pi : \widetilde{\Omega}(L_0, L_1; \ell_0) \to \Omega(L_0, L_1; \ell_0)$ be the Γ-covering space and α be the action one-form (13.4.9) on $\Omega(L_0, L_1; \ell_0)$. Then we have*

$$d\mathcal{A} = \pi^* \alpha. \tag{13.5.16}$$

Proof Obviously $d\pi : T_{[\ell,w]}\widetilde{\Omega}(L_0, L_1; \ell_0) \to T_\ell \Omega(L_0, L_1; \ell_0)$ induces an isomorphism. Let $[\ell, w] \in \widetilde{\Omega}(L_0, L_1; \ell_0)$ and $\widetilde{\xi} \in T_{[\ell,w]}\widetilde{\Omega}(L_0, L_1; \ell_0)$. We need to show that

$$d\mathcal{A}([\ell, w])(\widetilde{\xi}) = \alpha(d\pi(\widetilde{\xi})). \tag{13.5.17}$$

Denote $d\pi(\widetilde{\xi}) = \xi$. We choose a path

$$u : (-\epsilon, \epsilon) \to \Omega(L_0, L_1; \ell_0)$$

satisfying

$$u(0) = \ell, \qquad \left.\frac{\partial u}{\partial s}\right|_{s=0} = \xi. \tag{13.5.18}$$

Then we consider the local lifting $\widetilde{u} : (-\epsilon, \epsilon) \to \widetilde{\Omega}(L_0, L_1; \ell_0)$ of u through $\widetilde{u}(0) = [\ell, w]$. By an abuse of notation, we also denote the corresponding map by $u : (-\epsilon, \epsilon) \times [0, 1] \to M$.

Note that (13.5.17) is equivalent to

$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{A}(\widetilde{u}(s)) = \int_0^1 \omega(\dot{\ell}(t), \xi(t))dt. \tag{13.5.19}$$

To compute the left-hand side, we need an explicit representation of $\widetilde{u}(r)$ as a pair $[\ell(r), w(r)]$, where $\ell(r) = u(r, \cdot)$ and $w(r) : [0, 1] \times [0, 1] \to M$ is a map with $w(r)(0, t) = \ell_0(t)$, $w(r)(1, t) = u(r, t)$. We note that the integral $\int_w \omega$ is independent of a parameterization of w and invariant under the homotopy fixing the end points. Motivated by these two invariance properties, we define $\widetilde{u} : (-\epsilon, \epsilon) \to \widetilde{\Omega}(L_0, L_1; \ell_0)$ by the pair $[u(r), w(r)]$, where $w(r)$ is the concatenation of w and $u|_{[0,r]}$. (One can write down an explicit formula for this concatenation but it is not necessary for the following discussion.) Then we have

$$\begin{aligned}
d\mathcal{A}([\ell, w])(\widetilde{\xi}) &= -\left.\frac{d}{dr}\right|_{r=0} \int_{w(r)} \omega \\
&= -\left.\frac{d}{dr}\right|_{r=0} \left\{ \int w^*\omega + \int_0^r \int_0^1 u^*\omega \right\} dt\, ds \\
&= \left.\frac{d}{dr}\right|_{r=0} \int_0^r \int_0^1 \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) dt\, ds \\
&= \int_0^1 \omega\left(\frac{\partial u}{\partial t}(0, t), \frac{\partial u}{\partial s}(0, t)\right) dt = \int_0^1 \omega(\dot{\ell}(t), \xi(t))dt,
\end{aligned}$$

where we use the initial condition (13.5.18) for the last equality. Hence we have proved (13.5.19) and thereby finished the proof of (13.9.1). □

An immediate corollary of this proposition is the following characterization of the critical point set of $\mathcal{A}$.

Corollary 13.5.2 *The set* $\mathrm{Crit}(L_0, L_1; \ell_0)$ *of critical points of* $\mathcal{A}$ *consists of the pairs of the type* $[\widehat{x}, w]$, *where* $\widehat{x}$ *is the constant path with* $x \in L_0 \cap L_1$. *Also it is invariant under the action of* $G(L_0, L_1; \ell_0)$ *and hence forms a* $G(L_0, L_1; \ell_0)$-*principal bundle (or* $G(L_0, L_1; \ell_0)$-*torsor) over* $L_0 \cap L_1$.

For each $x \in L_0 \cap L_1$, we denote by

$$\pi_2(x; \ell_0) = \pi_2(L_0, L_1; x, \ell_0)$$

the set of homotopy classes of the ws appearing in this corollary.

13.6 The Maslov–Morse index

In this subsection, we assign an absolute Morse index to each critical point of $\mathcal{A}$. In general, assigning such an absolute index is a delicate matter because the obvious Morse index of $\mathcal{A}$ at any critical point is infinite. We call this Morse index the *Maslov–Morse index* of the critical point. The definition of this index will somewhat resemble that of $\mathcal{A}$. However, to define it, we also need to fix a section λ^0 of $\ell_0^*\Lambda(M)$ such that

$$\lambda^0(0) = T_{\ell_0(0)}L_0, \quad \lambda^0(1) = T_{\ell_0(1)}L_1.$$

Here we denote $\Lambda(M)$ to be the bundle of Lagrangian Grassmannians of TM,

$$\Lambda(M) = \bigcup_{x \in M} \Lambda(T_xM),$$

where $\Lambda(T_xM) := \mathcal{L}(T_xM, \omega_x)$ is the set of Lagrangian subspaces of the symplectic vector space (T_xM, ω_x). (Here, to avoid notational confusion with the Lagrangian chain $\mathcal{L}$ in the next section, we change the notation for the Lagrangian Grassmannian.)

We associate a (continuous) symplectic bundle pair (E_w, λ_w) over $[0, 1]^2$. Using Proposition 3.2.14 of Volume 1, we associate a symplectic bundle pair (E_w, λ_w) as before, which is defined uniquely up to the homotopy. We first choose $E_w = w^*TM$. To define λ_w, we choose a vector field ξ^x along $\Lambda_1(T_xM, \omega_x; T_xL_0) \subset \Lambda(T_xM, \omega_x)$ satisfying (see Lemma 3.2.12)

$$\xi^x \in C_0(T_xL_0)$$

and a path $\alpha^x : [0,1] \to \Lambda(S,\Omega)$ satisfying

$$\alpha^x(0) = T_x L_0,\ \alpha^x(1) = T_x L_1 \subset T_x M$$

and $(\alpha^x)'(0) = \xi^x(T_x L_0)$ as in Proposition 3.2.14. It also follows therefrom that any two such choices of (ξ^x, α^x) are homotopic to each other.

Then we consider a continuous Lagrangian subbundle $\lambda_w \to \partial[0,1]^2$ of $E|_{\partial[0,1]^2}$ given by the following formula: the fiber at each point of $\partial[0,1]^2$ is given as

$$\lambda_w(s,0) = T_{w(s,0)}L_0, \quad \lambda_w(1,t) = \alpha^x(t),$$
$$\lambda_w(s,1) = T_{w(s,1)}L_1, \quad \lambda_w(0,t) = \lambda^0_{(0,t)}.$$

Again Proposition 3.2.14 implies that λ_w is uniquely defined up to homotopy.

Definition 13.6.1 We define the *Maslov–Morse index* of $[\widehat{x}, w]$ by

$$\mu([\widehat{x}, w]; (\ell_0, \lambda^0)) = \mu(E_w, \lambda_w).$$

We note that this absolute index depends on the choices of the (ℓ_0, λ_0) and $[w] \in \pi_2(x)$. The following relative version of the index, called the Maslov–Viterbo index, does not require any such choices but depends only on the homotopy class $B \in \pi_2(x,y)$. Let $u : [0,1]^2 \to M$ be a smooth map representing the homotopy class B.

We pick a trivialization of u^*TM and consider the bundle pair (E_u, λ_u), where λ_u is the Lagrangian subbundle defined by

$$\lambda_u(s,0) = T_{u(s,0)}L_0, \quad \lambda_u(1,t) = \alpha^y(t),$$
$$\lambda_u(s,1) = T_{u(s,1)}L_1, \quad \lambda_u(0,t) = \alpha^x(t).$$

Definition 13.6.2 (Maslov–Viterbo index) Let $B \in \pi_2(x,y)$ and let u be as above. The Maslov–Viterbo index, denoted by $\mu(L_0, L_1; B)$, is defined as the Maslov index of the bundle pair (E_u, λ_u).

We would like to point out that the definition of this relative index does not require any choices in its definition. This definition of the index $\mu(E_u, \lambda_u)$ is precisely Viterbo's in (Vi88).

The following proposition relates the Maslov–Morse index to the Maslov–Viterbo index, which immediately follows from the definition.

Proposition 13.6.3 *Let $\widehat{x}, \widehat{y} \in \Omega(L_0, L_1; \ell_0)$ and $B \in \pi_2(x,y)$. Then*

$$\mu(L_0, L_1; B) = \mu([\widehat{y}, w\#B]; (\ell_0, \lambda_0)) - \mu([\widehat{x}, w]; (\ell_0, \lambda^0)).$$

In particular, the difference on the right-hand side does not depend on the choices of (ℓ_0, λ^0) *or* w.

Proof We pick a trivialization of u^*TM and consider the bundle pair (E_u, λ_u), where λ_u is the Lagrangian subbundle defined as above. Represent $\mu([\widehat{x}, w]; (\ell_0, \lambda^0))$ as $\mu(E_w, \lambda_w)$ for a bundle pair (E_w, λ_w) constructed as above.

Then we represent $\mu([\widehat{y}, w\#B]; (\ell_0, \lambda_0))$ as $\mu(E_{w\#u}, \lambda_{w\#u})$ for the (homotopically) unique bundle pair $(E_{w\#u}, \lambda_{w\#u})$ constructed as the concatenation of the above two bundle pairs. Then, by construction, they satisfy the relation

$$(E_{w\#u}, \lambda_{w\#u}) = (E_w, \lambda_w)\#(E_u, \lambda_u).$$

Since the Maslov index is additive under concatenation of bundle pairs, we have

$$\mu(E_{w\#u}, \lambda_{w\#u}) = \mu(E_w, \lambda_w) + \mu(E_u, \lambda_u),$$

or equivalently

$$\mu(E_u, \lambda_u) = \mu(E_{w\#u}, \lambda_{w\#u}) - \mu(E_w, \lambda_w)$$

and hence the proof. □

13.7 Anchored Lagrangian submanifolds

Now we would like to relate $\mu(\mathcal{L}, \vec{p}; B)$ to the Maslov–Morse index of a critical point $[\widehat{x}, w]$ of the action functional $\mathcal{A}$. Recall that the Maslov–Morse index depends on the choice of the base path $\ell_0 \in \Omega(L_0, L_1)$ and a Lagrangian subbundle $\lambda_0 \subset \ell_0^* TM$. At this point, one might wonder whether Proposition 13.6.3 generalizes to arbitrary polygonal maps. Because of the dependence of the Maslov–Morse index on the choice of ℓ_0 and λ^0, the above proof should be suitably reformulated. This requires a procedure of choosing the reference pair (ℓ_0, λ^0) in the definition of the Maslov–Morse index. To carry out this comparison in a coherent way, we will take the point of view of *pointed Lagrangian submanifolds* (L, x), not just L itself.

We also fix a base point y of the ambient symplectic manifold (M, ω) once and for all. Then it is easy to see that any homotopy class of path in $\Omega(L, L')$ can be realized by a path that passes through the given point y. We call any such path an anchor of L and the set of anchors $\{\gamma_i\}$ associated with the Lagrangian chain $\mathcal{L}$ an anchor system. We take a Lagrangian chain

$$\mathcal{L} = (L_0, L_1, \ldots, L_k)$$

equipped with a path $\gamma_i : [0,1] \to M$ satisfying

$$\gamma_i(0) \in L, \quad \gamma_i(1) = y$$

for each $i = 0, \ldots, k$.

They provide a systematic choice of a base path

$$\ell_{ij} = \gamma_i \# \overline{\gamma}_j \in \Omega(L_i, L_j),$$

the concatenation of γ_i and the time-reversal $\overline{\gamma}_j$ of γ_j, in the obvious sense that $\overline{\gamma}_j$ is the time-reversal of γ_j.

The upshot of this construction is the following overlapping property:

$$\begin{aligned} \ell_{ij}(t) &= \ell_{i\ell}(t) \quad \text{for } 0 \le t \le \frac{1}{2}, \\ \ell_{ij}(t) &= \ell_{\ell j}(t) \quad \text{for } \frac{1}{2} \le t \le 1. \end{aligned} \tag{13.7.20}$$

Definition 13.7.1 We say that a homotopy class $B \in \pi_2(\mathcal{L}; \vec{p})$ is admissible to the anchor system if it can be represented by a polygon that is a gluing of k bounding bands $w_{i(i+1)} : [0,1] \times [0,1] \to M$ satisfying

$$\begin{aligned} &w_{i(i+1)}(0,t) = \begin{cases} \gamma_i(2t) & 0 \le t \le \frac{1}{2}, \\ \gamma_{i+1}(2-2t) & \frac{1}{2} \le t \le 1, \end{cases} \\ &w_{i(i+1)}(s,0) \in L_i, \quad w_{i(i+1)}(s,1) \in L_{i+1}, \\ &w_{i(i+1)}(1,t) = p_{i(i+1)}. \end{aligned}$$

When this is the case, we denote the homotopy class B as

$$B = [w_{01}; \ell_{01}] \# [w_{12}; \ell_{12}] \# \cdots \# [w_{k0}; \ell_{k0}]$$

and by $\pi_2^{ad}(\mathcal{L}; \vec{p})$.

We call such a tuple $\mathcal{L}$ an *anchored Lagrangian chain*. Denote by $\mathcal{L}$ a chain $(L_0, \ldots, L_k)$ of Lagrangian submanifolds and by $\mathcal{E}$ that of anchored Lagrangian submanifolds.

When the collection $\mathcal{E} = \{(L_i, \gamma_i)\}_{0 \le i \le k}$ is given, we note that not all homotopy classes in $\pi_2(\mathcal{L}; \vec{p})$ are admissible. But we have the following basic lemma which will suffice for the construction of higher products in Floer homology, whose proof is easy and hence has been omitted. This is needed for the construction of the Fukaya category in general (FOOO10a).

Lemma 13.7.2 *Let $w_{i(i+1)}$ be given for $i = 1, \dots, k$ and $B \in \pi_2(\mathcal{L}; \vec{p})$. Then they canonically define a class $[w_{k0}] \in \pi_1(\ell_{k0}, p_{k0})$ by*

$$[w_{k0}] := [\overline{w}_{01}]\# \cdots \#[\overline{w}_{(k-1)k}]\#B$$

so that B becomes admissible to the given anchor system $\{\gamma_i\}_{i=0}^k$.

In the same spirit, we choose the Lagrangian subbundle $\lambda_i \subset \gamma_i^* TM$ for each $i = 0, \dots, k$ and define $\lambda_{ij} = \lambda_i \# \overline{\lambda}_j$ in the obvious way. Then it follows that the λ_{ij} also satisfy the overlapping property

$$\lambda_{ij}|_{[0,\frac{1}{2}]} = \lambda_{i\ell}|_{[0,\frac{1}{2}]},$$
$$\lambda_{ij}|_{[\frac{1}{2},1]} = \lambda_{\ell j}|_{[\frac{1}{2},1]}.$$

The following basic identity immediately follows from the definition.

Proposition 13.7.3 *Suppose $B \in \pi_2^{ad}(\mathcal{L}, \vec{p})$ given as Lemma 13.7.1 and $B = [w_0; \ell_{01}] + \cdots + [w_k; \ell_{k0}]$. Then we have*

$$\mu(\mathcal{L}, \vec{v}; B) = \sum_{i=0}^{k} \mu([p_i, w_i]; \lambda_{i(i+1)}), \tag{13.7.21}$$

$$\omega(B) = -\sum_{i=0}^{k} \mathcal{A}([p_i, w_i]). \tag{13.7.22}$$

Proof It is clear from the definition of the Maslov index of the bundle pair that the polygonal Maslov index is additive under the obvious fiber sum of the bundle pairs over the gluing of the bases. Therefore the proposition follows from the definition of the Lagrangian loop $\widetilde{\alpha}_w$ because the latter implies that

$$\widetilde{\alpha}_w \sim \sum_{i=0}^{k} w_{i(i+1)}\big|_{\partial([0,1]\times[0,1])}$$

in $\pi_2(\mathcal{L}, \vec{p})$. Here the negative sign in (13.7.22) appears by the definition of the action functional $\mathcal{A}([\ell, w]) = -\int w^* w$. □

Because these choices play an important role in coherent organization of the grading and the filtration problem in the Fukaya category, they would seem to deserve a name.

Definition 13.7.4 Let (M, ω) be an arbitrary symplectic manifold and $y \in M$ be a base point. A path $\ell : [0, 1] \to M$ with

$$\ell(0) = y, \qquad \ell(1) \in L$$

is called an anchor of L (relative to y). We call a pair (γ, λ) a *graded anchor*, where λ is a section of $\gamma^*\mathcal{L}(M, \omega) \to [0, 1]$. We call the pair $(L, (\gamma, \lambda))$ an anchored Lagrangian submanifold.

Remark 13.7.5 It appears that the anchored Lagrangian submanifolds play an important role in the study of the family version of the Floer homology that enters into the construction of mirror objects in the homological mirror symmetry of SYZ-fibrations. The presence of the Lagrangian section of the torus fibrations is known to be important for the construction mirror. It turns out that the section also plays the role of base points above in the construction of the Floer complex (Ab14).

13.8 Abstract Floer complex and its homology

In this section, we provide a purely algebraic context behind the construction of all the known general Floer homology. We follow the description provided by M. Usher (Ush08), with some amplification.

Definition 13.8.1 A *graded filtered Floer–Novikov complex* c over a ring R consists of the following data.

(1) A groupoid Γ and a principal Γ-bundle (with discrete topology) $P \to S$, where
  (a) S is a finite set, and
  (b) Γ is a finitely generated abelian groupoid, written multiplicatively, and P is a *G-set* (or a *G-act*).

(2) An 'action functional' $\mathcal{A} : P \to \mathbb{R}$ and a 'period groupoid homomorphism' $\omega : \Gamma \to \mathbb{R}$ satisfying

$$\mathcal{A}(g \cdot p) = \mathcal{A}(p) - \omega(g) \quad (g \in \Gamma,\ p \in P).$$

(3) A 'grading' $\mathrm{gr} : P \to \mathbb{Z}$ and a 'degree homomorphism' $d : \Gamma \to \mathbb{Z}$ satisfying

$$\mathrm{gr}(g \cdot p) = \mathrm{gr}(p) + d(g) \quad (g \in \Gamma,\ p \in P).$$

(4) A map $n : P \times P \to R$ satisfying the following conditions.
  (a) $n(p, p') = 0$ unless both $\mathcal{A}(p) > \mathcal{A}(p')$ and $\mathrm{gr}(p) = \mathrm{gr}(p') + 1$.
  (b) $n(g \cdot p, g \cdot p') = n(p, p')$ for all $p,\ p' \in P,\ g \in \Gamma$.

(c) For each formal sum

$$\beta = \sum_{q \in P} a_q q, \quad a_q \in R$$

we define its *support* by

$$\operatorname{supp}(\beta) = \{q \in P \mid a_q \neq 0\}.$$

Now we define the Floer chain group by

$$C_*(\mathfrak{c}) := \left\{ \sum_{q \in P} a_q q \,\middle|\, a_q \in R,\ \forall C \#(\operatorname{supp}(\beta)) \cap \mathcal{A}^{-1}((C, \infty)) < \infty \right\}.$$

Then, for each $p \in P$, the formal sum

$$\partial p = \sum_{p \in P} n(p, q) q$$

belongs to the Floer chain group $C_*(\mathfrak{c})$.

(d) Where the Novikov ring of Γ is defined by

$$\Lambda^{\uparrow}_{\Gamma,\omega} = \left\{ \sum_{g \in \Gamma} b_g g \,\middle|\, b_g \in R,\ \#(\operatorname{supp}(\beta)) \cap \omega^{-1}((-\infty, C)) < \infty \right\}$$

and where C_* inherits the structure of a $\Lambda_{\Gamma,\omega}$-module in the obvious way from the Γ-action on P, the operator $\partial : P \to C_*$ defined above extends to a $\Lambda_{\Gamma,\omega}$-module homomorphism $\partial : C_* \to C_*$, which moreover satisfies $\partial \circ \partial = 0$.

We call the image of $\mathcal{A}$ the *action spectrum* of the complex $\mathfrak{c}$ and that of ω the *period group* of Γ. We denote them by $\operatorname{Spec}(\mathfrak{c})$ and $\Gamma_\omega \subset \mathbb{R}$, respectively.

Note that the Novikov finite condition implies that each Floer–Novikov chain

$$\beta = \sum_{q \in P} a_q q, \quad a_q \in R$$

has a ‘highest peak’ and so we can define its *level* by

$$\ell(\beta) := \max_{q \in P} \{\mathcal{A}(q) \mid q \in \operatorname{supp}(\beta)\}.$$

We set $\ell(\beta) = -\infty$ if $\beta = 0$. This defines the *level function*

$$\ell : C_*(\mathfrak{c}) \to \mathbb{R}$$

and induces a filtration on $C_*(\mathfrak{c})$ by

$$C_*^\lambda(\mathfrak{c}) := \{\beta \mid \ell(\beta) \le \lambda\}.$$

We note that by definition the map ∂ preserves the filtration, i.e., $\partial(C_*^\lambda(\mathfrak{c})) \subset C_*^\lambda(\mathfrak{c})$, and also induces a topology on $C_*(\mathfrak{c})$, which is induced by the non-Archimedean metric defined by

$$d(\beta_1, \beta_2) = e^{\ell(\beta_1 - \beta_2)}. \tag{13.8.23}$$

Similarly, we define the *valuation* on the Novikov ring $\Lambda_{\Gamma,\omega}$ by

$$\nu(a) = \min_g\{\omega(g) \mid g \in \mathrm{supp}(a)\},$$

where $a = \sum_{g\in\Gamma} b_g g$, which in turn induces a metric on $\Lambda_{\Gamma,\omega}$ by

$$d(a_1, a_2) = e^{-\nu(a_1 - a_2)}.$$

We summarize the above discussion in the following proposition.

Proposition 13.8.2 *The map $\partial : C_*(\mathfrak{c}) \to C_*(\mathfrak{c})$ is continuous with respect to the distance d in (13.8.23). The action of $\Lambda_{\Gamma,\omega}$ on $C_*(\mathfrak{c})$,*

$$\Lambda_{\Gamma,\omega} \times C_*(\mathfrak{c}) \to C_*(\mathfrak{c}),$$

is continuous. Moreover, the map ∂ is equivariant under this action and hence the action induces a $\Lambda_{\Gamma,\omega}$-module structure on $H_*(\mathfrak{c}, \partial)$.

The restriction $\partial : C_*^\lambda(\mathfrak{c}) \to C_*^\lambda(\mathfrak{c})$ and the inclusion-induced chain map $\iota : C_*^\lambda(\mathfrak{c}) \to C_*(\mathfrak{c})$ induces the map

$$\iota_* : H_*^\lambda(\mathfrak{c}) \to H_*(\mathfrak{c}).$$

This enables us to introduce the following definition.

Definition 13.8.3 For $a \in H_*(\mathfrak{c})$, we define the *spectral number* of a by

$$\rho(a) = \inf\{\lambda \in \mathbb{R} \mid a \in \mathrm{Im}(\iota_* : H_*^\lambda(\mathfrak{c}) \to H_*(\mathfrak{c}))\}.$$

Equivalently, we can define $\rho(a)$ by the mini-max value

$$\rho(a) = \inf\{\ell(\beta) \mid \beta \in C_*(\mathfrak{c}),\ [\beta] = a\}, \tag{13.8.24}$$

where $[\beta]$ is the homology class of β. Obviously we have $\rho(0) = -\infty$. But it is not completely obvious whether $\rho(a) > -\infty$ even when $a \neq 0$ and whether the value $\rho(a)$ is realized by a cycle β representing the homology class a when $\rho(a) > -\infty$. The finiteness of $\rho(a)$ corresponds to some 'linking' property of the cycles representing the given homology class a in the classical critical point theory (BnR79), and realizability of the spectral number $\rho(a)$ corresponds to some kind of 'Palais–Smale' condition for the action functional $\mathcal{A}$. In this general abstract setting, Usher proved both properties in (Ush08).

Theorem 13.8.4 (Usher (Ush08)) *For any* $0 \neq a \in H_*(\mathfrak{c})$*,* $\rho(a) > -\infty$ *and there exists a cycle* $\alpha \in C_*(\mathfrak{c})$ *such that* $\rho(a) = \ell(\alpha)$*. In particular,*

$$\rho(a) \in \mathrm{Spec}(\mathfrak{c})$$

for all $a \in H_*(\mathfrak{c})$*.*

This extends a result in (Oh05c, Oh09a) which was proven in the context of Hamiltonian Floer homology using a more geometric approach. We will give the proof of this theorem in this geometric context in Section 21.3 and refer readers to (Ush08) for the proof of the theorem in the general abstract context. One important ingredient entering into the proof is the notion of the boundary depth, which was formally introduced into the Floer theory by Usher in (Ush11). (See (FOOO09), (Oh09a) for some usages of the same concept in the chain-level Floer theory.)

Definition 13.8.5 Let (C, ∂) be a $\mathbb{R}$-filtered chain complex. The boundary depth $\beta(\partial)$ is defined to be

$$\beta(\partial) = \inf\{\beta \in \mathbb{R} \mid \forall \lambda \in \mathbb{R},\ C^\lambda \cap \partial C \subset \partial(C^{\lambda+\beta})\}.$$

We now prove a useful structure theorem of $\Lambda_{\Gamma,\omega}$ that can be derived from the finiteness condition thereof. First we have the exact sequence

$$0 \to \ker\omega \to \Gamma \xrightarrow{\omega} \Gamma_\omega \to 0.$$

Therefore we can rewrite $\sigma = \sum_{g\in\Gamma} a_g g \in \Lambda_{\Gamma,\omega}$ as

$$\sigma = \sum_{\lambda\in\Gamma_\omega} \left(\sum_{g;\omega(g)=\lambda} a_g g \right).$$

We note that each summand $\sum_{g;\omega(g)=\lambda} a_g g$ is a finite sum.

Proposition 13.8.6 *Consider the ring* $K = R[\ker\omega]$ *and define a ring*

$$\Lambda^\uparrow_{\Gamma_\omega,K} = \left\{ \sum_{\lambda\in\Gamma_\omega} p_\lambda T^\lambda \,\middle|\, p_\lambda \in K,\ \forall C \#\{\lambda \in \Gamma_\omega \cap (-\infty, C) \mid p_\lambda \neq 0\} < \infty \right\}.$$

Then the map

$$\sigma = \sum_{g \in \Gamma} a_g g \to \sum_{\lambda \in \Gamma_\omega} p_\lambda(\sigma) T^\lambda$$

is a ring isomorphism.

13.9 Floer chain modules

Now we attempt to associate with each pair (L_0, L_1) of Lagrangian submanifolds the data required in the abstract definition of a graded filtered Floer complex given in Section 13.8.

For a given pair (L_0, L_1) of compact Lagrangian submanifolds in (M, ω) intersecting transversely, we set

$$\begin{aligned} S &= \widehat{L_0 \cap L_1} = \{\widehat{p} \mid x \in L_0 \cap L_1\} \cong L_0 \cap L_1, \\ P &= \mathrm{Crit}\, \mathcal{A}_{(L_0, L_1; \ell_0)}, \\ \Gamma &= G(L_0, L_1; \ell_0). \end{aligned} \tag{13.9.25}$$

The period and the degree homomorphisms on Γ are given by the assignments

$$g \mapsto \omega(g), \quad \mu(g),$$

respectively, where $\omega(g)$ is the symplectic area $\int u^*\omega$ and the Maslov index $\mu(u)$ of a particular (and hence any) annular map $u : S^1 \times [0, 1] \to M$ satisfying

$$u(\theta, 0) \in L_0, \quad u(\theta, 1) \in L_1.$$

Here $\mu(u)$ is the Maslov index of the bundle pair associated with map u as defined before. Both maps are defined on $\pi_1(\Omega(L_0, L_1); \ell_0)$ and push down to

$$G(L_0, L_1; \ell_0) = \frac{\pi_1(\Omega(L_0, L_1); \ell_0)}{\ker I_\omega \cap \ker I_\mu}$$

by definition. We emphasize that the group $G(L_0, L_1; \ell_0)$ does not depend on the choice of ℓ_0 as long as it is chosen from the same connected component. In this regard, the ring depends only on the connected component of ℓ_0.

The action functional $\mathcal{A} : P \to \mathbb{R}$ is nothing but the evaluation of critical values of $\mathcal{A}_{(L_0, L_1; \ell_0)}$

$$\mathcal{A}_{(L_0, L_1; \ell_0)}([\widehat{p}, w]) = -\int w^*\omega$$

for $x \in L_0 \cap L_1$ and $w : [0, 1]^2 \to M$ satisfying the defining boundary condition of the Γ-covering space $\widetilde{\Omega}(L_0, L_1; \ell_0) \to \Omega(L_0, L_1; \ell_0)$.

We recall the action of $G(L_0, L_1; \ell_0)$ on $\widetilde{\Omega}(L_0, L_1; \ell_0)$. For any $g \in G(L_0, L_1; \ell_0)$ and $[\ell, w]$ we define $g \cdot [\ell, w]$ by the equivalence class of the map

$$[\ell, u\#w]$$

where $u : S^1 \times [0,1] \to M$ is a representative of $g \in G(L_0, L_1; \ell_0)$ and $u\#w : [0,1]^2 \to M$ is defined by the formula

$$u\#w(s,t) = \begin{cases} u(2s, t) & \text{for } 0 \le s \le \frac{1}{2}, \\ w(2s-1, t) & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$

This action then restricts to one on $P = \operatorname{Crit} \mathcal{A}_{(L_0, L_1; \ell_0)}$ and induces a structure of a $G(L_0, L_1; \ell_0)$-torsor (i.e., of a principal $G(L_0, L_1; \ell_0)$-bundle) over $L_0 \cap L_1$. We note that

$$\mathcal{A}([\ell, u\#w]) = \mathcal{A}([\ell, w]) - \omega(u),$$

where $\omega(u) = \int u^*\omega$. We provide the grading map $\mathrm{gr} : \operatorname{Crit} \mathcal{A}_{(L_0, L_1; \ell_0)} \to \mathbb{Z}$ by the Maslov–Morse index. Then a simple Maslov calculation also shows that

$$\mathrm{gr}([\widehat{p}, u\#w]) = \mathrm{gr}([\widehat{p}, w]) - \mu(u).$$

Exercise 13.9.1 Prove this grading formula from the definition of the Maslov–Morse index.

These give rise to the formulae $\mathcal{A}(g \cdot p) = \mathcal{A}(p) - \omega(g)$ and $gr(g \cdot p) = gr(p) - d(g)$ required in the abstract definition of a Floer complex.

Now we can define the Novikov ring and the Floer complex associated with the given data above as in the abstract definition. We denote them by $\Lambda(L_0, L_1; \ell_0)$ and $CF_*(L_0, L_1; \ell_0)$, respectively. For the sake of completeness, we include a discussion on the grading. For each given $k \in \mathbb{Z}$, we define

$$\Lambda^k(L_0, L_1; \ell_0) = \{\sigma \in \Lambda(L_0, L_1; \ell_0) \mid \mu(g) = k\}.$$

Then it follows that $\Lambda(L_0, L_1; \ell_0)$ is decomposed into

$$\Lambda(L_0, L_1; \ell_0) = \bigoplus_{k \in \mathbb{Z}} \Lambda^k(L_0, L_1; \ell_0).$$

Similarly, we define $CF_k(L_0, L_1; \ell_0)$ to be the set of elements

$$\beta = \sum_{[\widehat{p}, w] \in \operatorname{Crit} \mathcal{A}} a_{[\widehat{p}, w]} [\widehat{p}, w], \quad a_{[\widehat{p}, w]} \in R,$$

with $\mu([\widehat{p}, w]) = k$ for all $[\widehat{p}, w]$ contributing non-trivially in the above sum. Then we have

$$CF_*(L_0, L_1; \ell_0) = \bigoplus_{k \in \mathbb{Z}} CF_k(L_0, L_1; \ell_0).$$

This is the Floer chain module associated with the pair (L_0, L_1) relative to the anchor (ℓ_0, γ_0).

Next we would like to construct a map $\partial : CF_*(L_0, L_1; \ell_0) \to CF_{*-1}(L_0, L_1; \ell_0)$ that satisfies $\partial^2 = 0$. For this purpose, we need to construct the map $n : P \times P \to \mathbb{Z}$ in the abstract definition so that the associated ∂ satisfies $\partial^2 = 0$. In terms of $n(p, q)$, $\partial^2 = 0$ is equivalent to

$$\sum_{q \in P} n(p, q) n(q, r) = 0 \tag{13.9.26}$$

for all given p, r with $\mu(p) - \mu(r) = 2$.

So far, we have been regarding $CF_*(L_0, L_1; \ell_0)$ as a downward-completed Q-vector space with an infinite number of generators $[\widehat{p}, w]$. Using the $\Gamma(L_0, L_1; \ell_0)$-equivariance of ∂, we can also regard $CF_*(L_0, L_1; \ell_0)$ as a $\Lambda(L_0, L_1; \ell_0)$-module over the free basis $L_0 \cap L_1$. For this purpose, we need to choose a lifting

$$\widehat{L_0 \cap L_1} \to \widetilde{\Omega}_0(L_0, L_1).$$

Proposition 13.9.2 *Let* $\widehat{L_0 \cap L_1} \to \widetilde{\Omega}_0(L_0, L_1)$ *be an embedding and denote by* $[\widehat{p}, w_p]$ *its image for* $p \in L_0 \cap L_1$. *Consider the free module*

$$\Lambda(L_0, L_1; \ell_0)\{L_0 \cap L_1\}.$$

Then the homomorphism induced by the map of the generators

$$p \mapsto [\widehat{p}, w_p], \quad p \in L_0 \cap L_1,$$

induces an isomorphism $\Lambda(L_0, L_1; \ell_0)\{L_0 \cap L_1\} \cong CF_*(L_0, L_1; \ell_0)$ *as a* $\Lambda(L_0, L_1; \ell_0)$*-module.*

To make the above isomorphism preserve the filtration, we need only define a non-Archimedean norm on the free module $\Lambda(L_0, L_1; \ell_0)\{L_0 \cap L_1\}$ by assigning the norm

$$\mathfrak{v}(\sigma p) = -\omega(\sigma) + \mathcal{A}([\widehat{p}, w_p])$$

to the monomial σp with $\sigma \in \Lambda(L_0, L_1; \ell_0)$ and $p \in L_0 \cap L_1$.

13.9.1 The L^2 metrics on $\Omega(L_0, L_1)$

We have shown that

$$d\mathcal{A} = \pi^* \alpha$$

in Proposition 13.5.1 for the action one-form α in (13.4.9) defined on the Γ-covering space $\widetilde{\Omega}(L_0, L_1; \ell_0)$. As in the finite-dimensional Morse theory, we will study the gradient flow of $\mathcal{A}$ in terms of a given 'Riemannian metric' on $\widetilde{\Omega}(L_0, L_1)$.

For each given pair (L_0, L_1) and a pair (J_0, J_1) of compatible almost-complex structures on (M, ω), we consider a path of compatible almost-complex structure $J = \{J(t)\}_{0 \le t \le 1}$ on (X, ω) satisfying

$$J(0) = J_0, \quad J(1) = J_1. \tag{13.9.27}$$

We denote by

$$j_{(J_0,J_1)}$$

the set of such one-parameter families. Then, for each given $J \in j_{(J_0,J_1)}$, we define an L^2 metric on Ω by the formula

$$\langle \xi_1, \xi_2 \rangle_J := \int_0^1 \omega(\xi_1(t), J_t \xi_2(t))dt,$$

using the one-parameter family of Riemannian metrics

$$g_t := \omega(\cdot, J_t \cdot).$$

According to (13.5.16), the push-down of the negative L^2-gradient of the action functional $\mathcal{A}$ to $\Omega(L_0, L_1; \ell_0)$ becomes

$$\ell \mapsto J\dot{\ell}$$

and hence the negative L^2-gradient trajectory $u : \mathbb{R} \times [0, 1] \to M$ of the action functional $\mathcal{A}$ becomes

$$\begin{cases} \partial u/\partial \tau + J_t\, \partial u/\partial t = 0, \\ u(\tau, 0) \in L_0, \quad u(\tau, 1) \in L_1, \end{cases} \tag{13.9.28}$$

for a map $u : \mathbb{R} \times [0, 1] \to M$ if one considers u as the path $\tau \to u(\tau)$ in $\Omega_0(L_0, L_1; \ell_0)$.

We will study the set of *bounded gradient trajectories* $u : \mathbb{R} \to \Omega(L_0, L_1; \ell_0)$ of the action functional $\mathcal{A} : \Omega(L_0, L_1; \ell_0) \to \mathbb{R}$ for each given transversal pair (L_0, L_1). Owing to the non-compactness of the domain $\mathbb{R} \times [0, 1]$, one needs to impose a certain decay condition of the derivatives of u in order to study the compactness property and the deformation theory of solutions u. This will be achieved by imposing a certain *boundedness* of the trajectories. The boundedness of the trajectories is measured in terms of an L^2 integral of the derivatives of the path u. The L^2-energy is in turn defined with respect to Riemannian metrics $g := \omega(\cdot, J\cdot)$ on M, where J is an almost-complex structure compatible with ω. When J is t-independent, as a map $u : \mathbb{R} \times [0, 1] \to M$, the L^2-energy is nothing but the harmonic energy of u with respect to the metric g, and coincides with the symplectic area $\int u^*\omega$. Often we will also need to consider a time-dependent family of almost-complex structures J_t for $t \in [0,1]$.

It is important to distinguish the following three different forms of boundedness of the trajectory.

(1) In the point of view of analysis, controlling the geometric energy of u defined by

$$E_J(u) := \frac{1}{2}\int \left(\left|\frac{\partial u}{\partial \tau}\right|_{J_t}^2 + \left|\frac{\partial u}{\partial t}\right|_{J_t}^2 \right) dt\, d\tau$$

is essential for the deformation theory of solutions of (13.9.28).

(2) For general maps $u : \mathbb{R} \times [0,1] \to M$ satisfying the Lagrangian boundary condition, the symplectic area $\int u^*\omega$ is invariant under the homotopy of u satisfying the boundary condition

$$u(\tau, 0) \in L_0, \quad u(\tau, 1) \in L_1 \tag{13.9.29}$$

and the asymptotic condition

$$\lim_{\tau \to \pm\infty} u(\tau) = x_\pm \tag{13.9.30}$$

in a suitable topology. In physics terminology, we have the *off-shell* bound for the symplectic area $\int u^*\omega$. Such an off-shell bound does not exist for the geometric energy $E_J(u)$.

(3) On the other hand, we have the following identity

$$E_J(u) = \int u^*\omega \tag{13.9.31}$$

for the maps satisfying

$$\frac{\partial u}{\partial \tau} + J_t \frac{\partial u}{\partial t} = 0$$

as long as $J = \{J_t\}_{0 \le t \le 1}$ is a family of compatible almost-complex structures. In physics terminology, the identity (13.9.31) holds *on shell*.

13.9.2 Floer moduli spaces

We start with an off-shell description of the relevant maps $u : \mathbb{R} \times [0,1] \to M$. We remind readers that the main purpose of studying the Floer homology is to develop some tools with which one can study the symplectic topology of a symplectic manifold (M, ω) or its Lagrangian submanifolds. In that regard, the almost-complex structure J should be an auxiliary tool and should not enter into the off-shell set-up of the mapping space where the Cauchy–Riemann equation (13.9.28) is defined. We fix a Riemannian metric once and for all that will be used in the various analytic estimates on the maps studied. All the norms, both the pointwise norm $|\cdot|$ and the Sobolev norms $\|\cdot\|_{k,p}$, will be defined in terms of this fixed metric. We will always assume

$$k - \frac{2}{p} > 0$$

so that imposing the boundary condition (13.9.29) makes sense, and mostly consider $k = 1$ for estimates and the necessary Fredholm theory.

Now let $L_0,\ L_1$ be Lagrangian submanifolds of (M, ω) that do not necessarily intersect each other transversely. Denote $\Theta = \mathbb{R} \times [0, 1]$ and consider the space of maps

$$\mathcal{P}^p_{k:\mathrm{loc}}(L_0, L_1 : M) = \left\{ u \in L^p_{k:\mathrm{loc}}(\Theta, M) \,\middle|\, u(\mathbb{R} \times \{0\}) \subset L_0, u(\mathbb{R} \times \{1\}) \subset L_1 \right\}$$

and

$$\mathcal{J}_\omega = \mathcal{J}_\omega(M) = C^\infty(S_\omega),$$
$$j_\omega = C^\infty([0, 1], \mathcal{J}_\omega) = C^\infty([0, 1] \times S_\omega).$$

For each given $J \in j_\omega$, we study Equation (13.9.28) and define the space of bounded solutions thereof by

$$\widetilde{\mathcal{M}}(L_0, L_1; J) = \left\{ u \in \mathcal{P}^p_{k:\mathrm{loc}} \,\middle|\, u \text{ satisfies (13.9.28) and } \int_\Theta E_J(u) < \infty \right\}. \tag{13.9.32}$$

When L_0 is transverse to L_1, exponential convergence as $\tau \to \pm\infty$, which will be proved in Proposition 14.1.5 later, provides a decomposition of $\widetilde{\mathcal{M}}(L_0, L_1; J)$ into

$$\widetilde{\mathcal{M}}(L_0, L_1; J) = \bigcup_{p,q \in L_0 \cap L_1} \widetilde{\mathcal{M}}(p, q; J),$$

where we define

$$\widetilde{\mathcal{M}}(p, q; J) = \left\{ u \in \widetilde{\mathcal{M}}(L_0, L_1; J) \,\middle|\, \lim_{\tau \to \infty} u(\tau) = \widehat{p}, \lim_{\tau \to -\infty} u(\tau) = \widehat{q} \right\}$$

for each given pair $p, q \in L_0 \cap L_1$. This $\widetilde{\mathcal{M}}(p, q; J)$ plays the role of the space of *connecting orbits*, i.e., the trajectories connecting two critical points of a smooth function in the finite-dimensional Morse theory.

There is a natural $\mathbb{R}$ action of τ translations on $\mathcal{M}(p, q; J)$ and we denote its quotient by

$$\mathcal{M}(p, q; J) = \widetilde{\mathcal{M}}(p, q; J)/\mathbb{R}$$

for each given pair $p,\ q \in L_0 \cap L_1$. In fact, the space $\mathcal{M}(p, q; J)$ has a further decomposition by the homotopy class of the elements. It is important to study this decomposition in order to properly encode the quantum contributions in the Floer homology theory in the non-exact case.

Definition 13.9.3 We denote by $\pi_2(p, q)$ this set of homotopy classes of maps $u : [0, 1]^2 \to M$ with the boundary conditions

$$u(0, t) \equiv p,\ u(1, t) \equiv q, \quad u(s, 0) \in L_0,\ u(s, 1) \in L_1.$$

For each given $B \in \pi_2(p, q)$, we denote by $\mathrm{Map}(p, q; B)$ the set of such us in class B.

Each element $B \in \pi_2(p, q)$ induces a map

$$(\cdot)\#B : \pi_2(p; \ell_0) \to \pi_2(q; \ell_0)$$

given by the obvious gluing map

$$w \mapsto w\#u.$$

More specifically, we define the map $w\#u : [0, 1]^2 \to M$ by the formula

$$w\#u : (s, t) = \begin{cases} w(2s, t) & \text{for } 0 \le s \le \frac{1}{2}, \\ u(2s - 1, t) & \text{for } \frac{1}{2} \le s \le 1 \end{cases} \tag{13.9.33}$$

once and for all. There is also the natural gluing map

$$\pi_2(p, q) \times \pi_2(q, r) \to \pi_2(p, r)$$

and

$$(u_1, u_2) \mapsto u_1\#u_2.$$

We also explicitly represent the map $u_1\#u_2 : [0, 1]^2 \to M$ in the standard way once and for all, similarly to (13.9.33).

By the Lagrangian property of L_0 and L_1, the following lemma is immediately evident from Stokes' formula.

Lemma 13.9.4 *Let $p,\ q \in L_0 \cap L_1$ be given. Then the symplectic area of $u \in \mathrm{Map}(p, q; B)$ is constant. We denote by $\omega(B)$ the common area of the elements from $\mathrm{Map}(p, q; B)$.*

The Maslov–Viterbo index will govern the dimension of the Floer moduli space in the transversal case, whose discussion we postpone to later chapters.

We now denote by

$$\mathcal{M}(L_0, L_1; p, q; B)$$

the set of finite-energy solutions of the Cauchy–Riemann equation (13.9.28) with the asymptotic condition and the homotopy condition

$$u(-\infty) \equiv \widehat{p}, \quad u(\infty) \equiv \widehat{q}; \quad [u] = B.$$

Here we remark that, although u is a priori defined on $\mathbb{R} \times [0, 1]$, it can be compactified into a continuous map $\overline{u} : [0, 1] \times [0, 1] \to M$ with the corresponding boundary condition due to the exponential decay property of solutions u of the Cauchy–Riemann equation when L_0 and L_1 intersect transversely. Consider the map

$$\rho^+(\tau) = \begin{cases} 1 & \text{for } \tau = \infty, \\ 0 & \text{for } \tau = -\infty, \\ \frac{1}{2}\left(1 + \tau(1 + \tau^2)^{-1/2}\right) & \text{for } \tau \in \mathbb{R}. \end{cases}$$

Thus, we have $\overline{u} : [0, 1] \times [0, 1] \to M$ defined by

$$\overline{u}(s, t) = u((\rho^+)^{-1}(s), t).$$

Owing to the exponential decay property of u at $\pm\infty$, this map is continuous on $[0, 1] \times [0, 1]$ and lies in $\mathrm{Map}(L_0, L_1; p, q)$. (See [Schw93] for further explanation.) We call $\overline{u}$ the *compactified map* of u. With some abuse of notation, we also denote by $[u]$ the class $[\overline{u}] \in \pi_2(p, q)$ of the compactified map $\overline{u}$.

13.9.3 The structure of the $\Lambda(L_0, L_1)$-module

So far we have associated a Floer complex $CF(L_0, L_1; \ell_0)$ with each connected component $\Omega(L_0, L_1; \ell_0)$, which has the structure of a $\Lambda(L_0, L_1; \ell_0)$-module depending on the element $[\ell_0] \in \pi_0(\Omega(L_0, L_1))$. In this section, we explain how we extend this to the Floer chain module $CF(L_0, L_1)$ incorporating all components of $\Omega(L_0, L_1)$ over the Novikov ring $\Lambda(L_0, L_1)$.

According to Usher's abstract framework given in Section 13.8, we identify the corresponding quintuple

$$\mathfrak{c} = (\pi : P \to S, \mathcal{A}, \omega, \deg, \partial)$$

in the abstract definition of a graded filtered Floer complex.

Definition 13.9.5 We define the *period group* of $\Omega(L_0, L_1)$ by

$$G_\omega(L_0, L_1) = \{\omega(u) \mid u : S^1 \times [0, 1] \to M,\ u(s, 0) \in L_0,\ u(s, 1) \in L_1\}.$$

Similarly to before, we define the Novikov ring $\Lambda(L_0, L_1)$ associated with $G(L_0, L_1)$.

For a given pair (L_0, L_1) of compact Lagrangian submanifolds in (M, ω) intersecting transversely, we set S, Γ, P as in (13.9.25). The period and degree homomorphisms on Γ are given by the assignments $g \mapsto \mu(g)$ as before and the real grading by

$$\mathcal{A}(g \cdot p) = -\omega(g). \tag{13.9.34}$$

In other words, we put grading 0 on any generator p. Since the Floer boundary map is $\Gamma(L_0, L_1; \ell_0)$-equivariant, these assignments provide a graded filtered Floer complex, which we denote by $CF(L_0, L_1; \Lambda_{(L_0,L_1)})$, whenever we have $\partial^2 = 0$.

We note that this real grading does *not* come from the value of the action functional $\mathcal{A}_{\ell_0}$. Recall that to define a real grading on the Floer complex we need only associate a grading on its generators, not on the full path space, so that the grading becomes $G(L_0, L_1)$-equivariant. With these choices made, we can define a $\Lambda(L_0, L_1)$-module structure on $CF(L_0, L_1)$ by considering the convolution product

$$\Lambda(L_0, L_1) \times CF(L_0, L_1) \to CF(L_0, L_1).$$

14

On-shell analysis of Floer moduli spaces

For the construction of various operators that enter into Lagrangian Floer theory, one needs to study the fine structure of the moduli space of solutions for Floer's trajectory equations. Two most basic operators are Floer's (pre)-boundary map and Floer's (pre)-chain map. Since the domains of the corresponding pseudoholomorphic maps are non-compact, one needs to control the behavior at infinity of finite-energy solutions. This analytic study requires the geometric condition of transversal intersection of Lagrangian submanifolds and the relevant exponential convergence property of the solutions. We start with the proof of the exponential decay property of finite-energy solutions.

14.1 Exponential decay

In this section, we prove an important exponential decay estimate of the derivative $|du|$ as $\tau \to +\infty$, *when* that L_0 intersects L_1 transversely. This is the analog to the similar exponential decay property of *bounded* gradient trajectories of a Morse function on a finite-dimensional manifold.

We start with the following a-priori estimates on the classical linear $\bar{\partial}$-operator. We denote by $W^{k+1,p}(u^*TM)$ the set of $L^{k+1,p}$-vector fields along a map $u \in W^{k+1,p}(\Theta, M)$ satisfying the boundary condition

$$\xi(\tau, 0) \in T_{u(\tau,0)}L_0, \quad \xi(\tau, 1) \in T_{u(\tau,1)}L_1,$$

while $L^{k,p}(u^*TM)$ is the set of $W^{k,p}$-vector fields without the boundary condition imposed. (We refer the reader to Section 8.2 or (RS01) for its derivation.)

Lemma 14.1.1 *Let Ω and Ω' be bounded open subsets of the upper half space $\mathbb{H} \subset \mathbb{C}$ such that $\overline{\Omega} \subset \Omega'$. Then, for any $k \geq 0$, there exists $C = C(k, p; \Omega, \Omega') > 0$ such that*

$$\|\xi\|_{W^{k+1,p}(\Omega)} \leq C(\|\bar{\partial}\xi\|_{W^{k,p}(\Omega')} + \|\xi\|_{L^{k,p}(\Omega')}) \tag{14.1.1}$$

for every smooth map $\xi : \Omega' \to \mathbb{C}^n$ *satisfying*

$$\xi(\tau, 0) \in \mathbb{R}^n + 0 \cdot i \subset \mathbb{C}^n.$$

Denote $\Theta = \mathbb{R} \times [0, 1] \subset \mathbb{C}$ and

$$\begin{aligned}\mathcal{F}^{k+1,p}(\Theta, M) &:= \mathcal{F}^{k+1,p}((\Theta, \partial_0\Theta, \partial_1\Theta); (M, L_0, L_1)) \\ &= \{u \in L^{k+1,p} \mid u(\tau, 0) \in L_0,\ u(\tau, 1) \in L_1\}.\end{aligned}$$

We regard the assignment $u \mapsto \overline{\partial}_J u$,

$$\overline{\partial}_J u := \frac{\partial u}{\partial \tau} + J_t \frac{\partial u}{\partial t},$$

as a map from $\mathcal{F}^{k+1,p}(\Theta, M)$ to $L^{k,p}(u^*TM)$.

By considering $\xi = \partial u/\partial \tau$, using the equation $\partial u/\partial \tau + J\,\partial u/\partial t = 0$ and applying the interpolation theorem as before, starting from Proposition 8.3.4, to the t-dependent J, which corresponds to $k = 1$, $p = 2$ here, we obtain the following proposition.

Proposition 14.1.2 *Let* $J = \{J_t\}_{0 \le t \le 1}$ *be a one-parameter family of compatible almost-complex structures and* $\Omega,\ \Omega' \subset \mathbb{R} \times [0, 1]$ *be open subsets. Suppose* $\|du\|_p \le K < \infty$. *Then, for any* $k \ge 1$, *there exists* $C_j = C_j(J, k, K; \Omega, \Omega')$ *for* $j = 1,\ 2$ *such that*

$$\|du\|_{W^{k,p}(\Omega)} \le C_1(\|\overline{\partial}_J u\|_{L^{k,p}(\Omega')} + \|du\|_{W^{k-1,p}(\Omega')}), \tag{14.1.2}$$

where the constant C_1 *depends continuously on* J.

We recall that we have fixed a reference metric on M and do all the estimates in terms of this metric.

Now we perform the main exponential decay estimates. We start with the following subsequence convergence result.

Lemma 14.1.3 *Suppose that* $u : \mathbb{R} \times [0, 1] \to M$ *is a finite-energy solution of (13.9.28). Then there exist a sequence* $\tau_i \to \pm\infty$ *and* $p_-,\ p_+ \in L_0 \cap L_1$ *such that*

$$\lim_{i \to \infty} \mathrm{dist}(u(\tau_i), p_\pm) = 0,$$

where $u(\tau) : [0, 1] \to M$ *is the path given by* $u(\tau)(t) = u(\tau, t)$.

Proof We will prove the statement as $\tau \to +\infty$. The case as $\tau \to -\infty$ will be the same. The finite-energy condition implies the existence of a sequence $\tau_i \to \infty$ such that $\|u_t(\tau_i)\|^2 \to 0$ as $\tau_i \to 0$. Since $W^{1,2}([0, 1]) \hookrightarrow C^\epsilon([0, 1])$

for $0 < \epsilon < \frac{1}{2}$, $u(\tau_i) : [0,1] \to M$ converges uniformly to a continuous path $z_\infty : [0,1] \to M$ and weakly satisfies $\partial z_\infty/\partial t = 0$. By Fatou's lemma, we have

$$\int_0^1 \left|\frac{\partial z_\infty}{\partial t}\right|^2 dt \leq \lim \int_0^1 \left|\frac{\partial u(\tau_i)}{\partial t}\right|^2 dt = 0.$$

Therefore $z_\infty \equiv$ constant. Denote this constant point by p_+. This finishes the proof. □

Next we prove the following uniform convergence.

Lemma 14.1.4 *We have*

$$\lim_{\tau\to\infty} \left\|\frac{\partial u}{\partial \tau}\right\|_{C^k([\tau,\infty)\times[0,1])} = 0 \tag{14.1.3}$$

for all $k = 0, 1, \ldots$. *The same holds for* $(-\infty, \tau]$.

Proof We first show (14.1.3) for $k = 0$. Suppose to the contrary that there exist sequences $\tau_j \to \infty$, $t_i \in [0,1]$ and a constant $\delta > 0$ such that

$$\left|\frac{\partial u}{\partial \tau}(\tau_j, t_i)\right| \geq \delta. \tag{14.1.4}$$

The finite-energy condition implies

$$\int_{[\tau_j-1,\tau_j+1]\times[0,1]} \left|\frac{\partial u}{\partial \tau}\right|_J^2 \to 0 \tag{14.1.5}$$

as $j \to \infty$.

Now we consider the translated sequence

$$u_j(\tau, t) := u(\tau + \tau_j, t)$$

and apply the local convergence argument using the a-priori estimates. This produces a limit u_∞ to which u_j converges locally uniformly in C^k topology. Choosing a subsequence, we may assume $t_j \to t_\infty$ as $j \to \infty$. Then we have convergence,

$$\frac{\partial u_j}{\partial \tau}(0, t_j) \to \frac{\partial u_\infty}{\partial \tau}(0, t_\infty),$$

and

$$\begin{aligned}\int_{-1}^1 \int_0^1 \left|\frac{\partial u_\infty}{\partial \tau}\right|^2 dt\, d\tau &= \lim_{j\to\infty} \int_{-1}^1 \int_0^1 \left|\frac{\partial u_j}{\partial \tau}\right|^2 dt\, d\tau \\ &= \lim_{j\to\infty} \int_{\tau_j-1}^{\tau_j+1} \int_0^1 \left|\frac{\partial u}{\partial \tau}\right|^2 dt\, d\tau = 0.\end{aligned}$$

The latter implies that $\partial u_\infty/\partial\tau = 0$ on $[-1,1]\times[0,1]$, which in turn implies

$$\lim_{j\to\infty}\frac{\partial u}{\partial\tau}(\tau_j,t_j) = \lim_{j\to\infty}\frac{\partial u_j}{\partial\tau}(0,t_j) = 0.$$

This contradicts (14.1.4), which proves (14.1.7) for $k = 0$. The same kind of argument applies to $k \geq 1$ and will be omitted. □

The following is the main proposition concerning the exponential decay.

Proposition 14.1.5 *Suppose $u : \mathbb{R}\times[0,1] \to M$ is a finite-energy solution of* (13.9.28) *and let p_+ be the intersection points obtained in Lemma 14.1.3. Suppose that $u(\tau,t) = p_+$ uniformly over $t \in [0,1]$ as $\tau \to \infty$. Then there exists a constant $R > 0$ and C_k, $\sigma_+ > 0$ for $k = 1, 2, \ldots$ such that*

$$\mathrm{dist}(u(\tau,t),p_+) \leq C_0 e^{-\sigma_+\tau} \tag{14.1.6}$$

and

$$\|\nabla^k u(\tau)\|_{C^k} \leq C_k e^{-\sigma_+\tau} \tag{14.1.7}$$

for all $\tau \geq R$. A similar conclusion applies at $-\infty$.

This exponential decay relies on the transversality of intersection $L_0 \cap L_1$, which implies the *asymptotic operator $J\,d/dt$*, which is a self-adjoint operator with trivial kernel on

$$\begin{aligned}&W^{1,2}([0,1],\{0,1\};T_pM,T_pL_0,T_pL_1)\\ &\quad = \{\xi \in L^{1,2}([0,1],T_pM) \mid \xi(0)\in T_pL_0,\ \xi(1)\in T_pL_1\}\end{aligned}$$

for $p = p_\pm$. For computational purposes, we use the metric g of the type given in Lemma 8.3.1. By applying this lemma to (L_0,J_0), (L_1,J_1) we obtain two metrics g_0, g_1 of this type. Now we connect g_0 to g_1 by a smooth family $g = \{g_t\}_{0\leq t\leq 1}$ satisfying $g_t(J_t\cdot,J_t\cdot) = g_t(\cdot,\cdot)$. Then we consider the family of Levi-Civita connections $\nabla = \{\nabla^t\}_{0\leq t\leq 1}$ of g. We note that the Levi-Civita connection ∇^t of g_t satisfies $\nabla^t g_t = 0$ and has zero torsion but might not preserve J_t. We denote by $K = \{K^t\}_{0\leq t\leq 1}$ the corresponding curvature tensor of g_t. With this notation set-up, we will omit the superscript t from the notation of ∇^t and K^t, and the subscript t from J_t and g_t, whenever there is no danger of confusion.

By (8.3.24) in Lemma 8.3.1, we have the orthogonality condition

$$T_pL_i \perp J_iT_pL_i, \quad i = 0, 1. \tag{14.1.8}$$

We define the L^2-norm of a vector field ξ along a path $\gamma : [0,1] \to M$ by

$$\|\xi\|_g^2 := \int_0^1 |\xi|_{g_t}^2\,dt$$

and the associated L^2 inner product by $\langle\langle\cdot,\cdot\rangle\rangle_g$.

Proof of Proposition 14.1.5. Again we will focus on $+\infty$. By the uniform convergence $u(\tau) \to p$ as $\tau \to \infty$, the image $u(\tau, t)$ lies in a Darboux neighborhood of p_+ and its derivative converges to 0 as $\tau \to \infty$.

We denote by $\epsilon(\tau)$ a function with $\epsilon(\tau) \to 0$ as $\tau \to \infty$ uniformly that may vary at different places.

Differentiating $\partial u/\partial\tau + J\,\partial u/\partial t = 0$ in τ, we obtain

$$\nabla_\tau \frac{\partial u}{\partial \tau} + J\,\nabla_\tau \frac{\partial u}{\partial t} + \nabla J\left(\frac{\partial u}{\partial \tau}\right)\frac{\partial u}{\partial t} = 0,$$

where we denote $\nabla J(\partial u/\partial\tau) := \nabla_{\partial u/\partial t} J$. If we write $\xi = \partial u/\partial\tau$, this equation becomes

$$\nabla_\tau \xi + J\,\nabla_\tau \xi + \nabla J(\xi) J\xi = 0 \tag{14.1.9}$$

by the equations $\partial u/\partial t = J\,\partial u/\partial\tau = J\xi$ and

$$\nabla_\tau \frac{\partial u}{\partial t} = \nabla_t \frac{\partial u}{\partial \tau}.$$

We consider the function $f(\tau) = \frac{1}{2}\|\xi(\tau)\|_g^2$. We will derive a certain second-order differential inequality of f. For this purpose, we will actually need to compute the full Laplacian Δe of the function

$$e(\tau, t) := \langle \xi(\tau, t), \xi(\tau, t)\rangle_{g_t},$$

where $\Delta = \partial^2/\partial\tau^2 + \partial^2/\partial t^2$. The computation of Δe is somewhat similar to what we did for the derivation of (7.3.27), with three differences.

(1) Here we use the standard flat cylindrical metric for the domain Θ and so $K \equiv 0$.
(2) We also use the (t-dependent) Levi-Civita connection here rather than the canonical metric, mainly to get rid of the boundary contribution that later arises from the application of Green's formula to the integral $\int_\Theta \Delta e$. See Lemma 14.1.6.
(3) Unlike the case of (7.3.27), J depends on the domain parameter t.

Because of these differences, we need to derive Δe again, which is somewhat simpler than the derivation of (7.3.27) thanks to the presence of the global isothermal coordinates (τ, t) of the flat metric.

We have

$$\frac{\partial^2 e}{\partial t^2} = 2\langle \nabla_t^2 \xi, \xi\rangle_g + 2\langle \nabla_t \xi, \nabla_t \xi\rangle_g.$$

On substituting (14.1.9) into $\nabla_t \xi$ and using $\nabla_t J = (\nabla J)(J\xi)$, we obtain

$$\begin{aligned}
\langle \nabla_t^2 \xi, \xi\rangle_g &= \langle \nabla_t (J\,\nabla_\tau \xi + J\,(\nabla J)(\xi)J\xi), \xi\rangle_g \\
&= \langle \nabla_t (J\,\nabla_\tau \xi), \xi\rangle_g + \langle \nabla_t (J\,(\nabla J)(\xi)J\xi), \xi\rangle_g \\
&= \langle (\nabla_t J)\,\nabla_\tau \xi + J\,\nabla_t \nabla_\tau \xi, \xi\rangle_g + \langle \nabla_t (J\,(\nabla J)(\xi)J\xi), \xi\rangle_g \\
&= \langle (\nabla J)(J\xi)\nabla_\tau \xi, \xi\rangle_g + \langle J\nabla_t \nabla_\tau \xi, \xi\rangle_g \\
&\quad + \langle \nabla(J\,\nabla J)(J\xi)(\xi)J\xi, \xi\rangle_g + \langle J\,(\nabla J)(\nabla_t(\xi))J\xi, \xi\rangle_g \\
&\quad + \langle J\,(\nabla J)(\xi)(\nabla J)(J\xi)\xi, \xi\rangle_g + \langle J\,(\nabla J)(\xi)J\,\nabla_t \xi, \xi\rangle_g \\
&= \langle J\,\nabla_t \nabla_\tau \xi, \xi\rangle_g + \epsilon(\tau)|\xi|_g^2, \quad \epsilon(\tau) \to 0 \text{ as } |\tau| \to \infty. \qquad (14.1.10)
\end{aligned}$$

Here the last identity holds since $\|\xi(\tau)\|_\infty$, $\|\nabla \xi(\tau)\|_\infty \to 0$ uniformly and

$$\|J\|_{\infty,M},\ \|\nabla J\|_{\infty,M},\ \|\nabla^2 J\|_{\infty,M} < C$$

and each term, except the first one, of the sum a line above the last involves at least three factors of ξ or its derivatives and at least two factors of ξ itself. Therefore we have

$$\frac{\partial^2 e}{\partial t^2} = 2\langle J\,\nabla_t \nabla_\tau \xi, \xi\rangle_g + \epsilon(\tau)|\xi|_g^2 + 2\langle \nabla_t \xi, \nabla_t \xi\rangle_g. \qquad (14.1.11)$$

Next we compute

$$\frac{\partial^2 e}{\partial \tau^2} = 2\langle \nabla_\tau \xi, \nabla_\tau \xi\rangle_g + 2\langle \nabla_\tau^2 \xi, \xi\rangle_g. \qquad (14.1.12)$$

From (14.1.9), we obtain

$$\begin{aligned}
\nabla_\tau \xi &= -J\,\nabla_\tau \frac{\partial u}{\partial t} - \nabla J(\xi)J\xi \\
&= -J\,\nabla_t \frac{\partial u}{\partial \tau} - \nabla J(\xi)J\xi \\
&= -J\,\nabla_t \xi - \nabla J(\xi)J\xi, \qquad (14.1.13)
\end{aligned}$$

where for the second equality we use the fact that ∇ is the Levi-Civita connection and so has zero torsion. Using this, we compute

$$\begin{aligned}
\nabla_\tau^2 \xi = \nabla_\tau(\nabla_\tau \xi) &= -\nabla_\tau\,(J(\nabla_t \xi + \nabla J(\xi)J\xi)) \\
&= -J\,(\nabla_\tau \nabla_t \xi) - (\nabla_\tau J)\nabla_t \xi + \nabla_\tau(\nabla J(\xi)J\xi).
\end{aligned}$$

Here, by our notational convention, the second term is

$$\nabla_\tau J\,\nabla_t \xi = \nabla J(\xi)\nabla_t \xi.$$

Also we have

$$\nabla_\tau \nabla_t \xi = \nabla_t \nabla_\tau \xi + K\left(\frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial t}\right)\xi = \nabla_t \nabla_\tau \xi + K(\xi, J\xi)\xi$$

and

$$\nabla_\tau(J\,\nabla J(\xi)J\xi) = \nabla J(\xi)J\xi + J\nabla^2 J(\xi)(\xi)J\xi + J\,\nabla J(\xi)\nabla_\tau\xi J\xi$$
$$+ J\,\nabla J(\xi)\nabla J(\xi)\xi + J\,\nabla J(\xi)J\,\nabla_\tau\xi.$$

Again, by virtue of the uniform convergence $\|\xi(\tau)\|_\infty, \|\nabla\xi(\tau)\|_\infty \to 0$ and

$$\|J\|_{\infty,M}, \|\nabla J\|_{\infty,M}, \|\nabla^2 J\|_{\infty,M} < C$$

we obtain

$$\langle\nabla_\tau^2\xi, \xi\rangle_g = -\langle J\,\nabla_t\nabla_\tau\xi, \xi\rangle_g + \epsilon(\tau)|\xi|_g^2.$$

Therefore we have

$$\frac{\partial^2 e}{\partial\tau^2} = 2\langle\nabla_\tau\xi, \nabla_\tau\xi\rangle_g - 2\langle J\,\nabla_t\nabla_\tau\xi, \xi\rangle_g + \epsilon(\tau)|\xi|_g^2. \tag{14.1.14}$$

Upon adding (14.1.11) and (14.1.14), we obtain

$$\Delta e = 2|\nabla_\tau\xi|_g^2 + 2|\nabla_t\xi|_g^2 + \epsilon(\tau)e.$$

On the other hand, it is easy to see, by substituting (14.1.13) into $\nabla_\tau\xi$, that

$$\langle\nabla_\tau\xi, \nabla_\tau\xi\rangle_g = |J\nabla_t\xi|_g^2 + \epsilon(\tau)|\xi|_g^2$$

and $|\nabla_t\xi|_g^2 = |J\,\nabla_t\xi|_g^2$. This implies that

$$\Delta e = 4|J\nabla_t\xi|_g^2 + \epsilon(\tau)e. \tag{14.1.15}$$

Now we recall

$$f(\tau) := \frac{1}{2}\|\xi(\tau)\|_g^2 = \frac{1}{2}\int_0^1 e(\tau,t)dt.$$

By integrating (14.1.15) over $t \in [0,1]$ we obtain

$$f''(\tau) + \frac{1}{2}\int_0^1 \frac{\partial^2 e}{\partial t^2}(\tau,t)dt = 2\|J\,\nabla_t\xi(\tau)\|_g^2 + \epsilon(\tau)f(\tau).$$

Here we derive

$$\int_0^1 \frac{\partial^2 e}{\partial t^2}(\tau,t)\,dt = \frac{\partial e}{\partial t}(\tau,1) - \frac{\partial e}{\partial t}(\tau,0) = 2(\langle\nabla_t\xi,\xi\rangle_{g_1}(\tau,1) - \langle\nabla_t\xi,\xi\rangle_{g_0}(\tau,0)).$$

Lemma 14.1.6 *We have*

$$\langle\nabla_t\xi(\tau,i), \xi(\tau,i)\rangle_{g_i} \equiv 0$$

for both $i = 0, 1$ and hence

$$\int_0^1 \frac{\partial^2 e}{\partial t^2}(\tau,t)dt = 0.$$

Proof Recall $\xi(\tau,i) = (\partial u/\partial\tau)(\tau,i)$ is tangent to L_i, and

$$\frac{\partial u}{\partial t}(\tau,i) = J(i,u(\tau,i))\frac{\partial u}{\partial\tau}(\tau,i)$$

is perpendicular to L_i. Then, by the definition of the second fundamental form B_i of L_i with respect to g_i, we have

$$\langle \nabla_t \xi(\tau, i), \xi(\tau, i)\rangle_g = \left\langle B_i\left(\frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial \tau}\right), \frac{\partial u}{\partial t}\right\rangle_g.$$

But B_i vanishes by virtue of the requirement that L_i be totally geodesic with respect to g_i. This finishes the proof. □

Summarizing the above computations, we obtain

$$f''(\tau) = 2\|J\,\nabla_t \xi(\tau)\|_g^2 + \epsilon(\tau) f(\tau). \tag{14.1.16}$$

Now we consider the first-order linear differential operator

$$J\,\nabla_t : W^{k+1,2}(u(\tau)^* TM) \to L^{k,2}(u(\tau)^* TM).$$

This operator has closed range and converges to

$$J\frac{d}{dt} : W^{1,2}([0,1]\{0,1\}; T_pM, T_pL_0, T_pL_1) \to L^2([0,1], T_pM)$$

as $\tau \to \infty$. More precisely, under a unitary trivialization of the bundle

$$\Phi : u^* TM|_{[R, \times\infty)\times[0,1]} \to [R,\infty) \times [0,1] \times \mathbb{R}^{2n},$$

the push-forward operator

$$\Phi_{\tau,*}(J\,\nabla_t) : W^{1,2}([0,1], \{0,1\}; T_pM, T_pL_0, T_pL_1) \to L^2([0,1], T_pM)$$

can be shown to have the form

$$J\frac{d}{dt} + A(\tau),$$

where $A(\tau)$ is a smooth family of compact operators with $A(\tau) \to 0$ uniformly in the norm topology as $\tau \to \infty$ by the uniform C^1-convergence of $u(\tau) \to p$. In particular, we derive that $\Phi_{\tau,*}(J\,\nabla_t)$ has a closed range and $\Phi_\tau^*(J\nabla_t) \to J\,d/dt$ in the norm topology as $\tau \to \infty$.

Exercise 14.1.7 Prove that $\ker J\,d/dt$ has a one-to-one correspondence with $T_pL_0 \cap T_pL_1$.

Since $T_pL_0 \cap T_pL_1 = \{0\}$, $\ker J\,d/dt = 0$ and $J\xi \in W^{1,2}(u(\tau)^* TM)$

$$\left\|J\frac{dJ\xi}{dt}\right\|_g^2 \geq 2\sigma \|J\xi\|_g^2$$

for some $\sigma > 0$ by the open mapping theorem. Therefore we have obtained

$$\|J\,\nabla_t \xi(\tau)\|_g^2 \geq \frac{4\sigma}{3}\|\xi(\tau)\|_g^2$$

for all sufficiently large τ by the C^1 convergence (Proposition 14.1.5) of u as $|\tau| \to \infty$ and in turn by the convergence of $J\nabla_t \to J\,d/dt$. By substituting this into (14.1.16), we obtain

$$f''(\tau) \geq \left(\frac{4\sigma}{3} - \epsilon(\tau)\right) f(\tau) \geq \sigma f(\tau).$$

From this and the fact that $f(\tau) \to 0$ as $\tau \to \infty$, we prove

$$f(\tau) \leq C e^{-\sigma\tau} \tag{14.1.17}$$

for all sufficiently large τ by the maximum principle. Integrating this over τ from R to ∞, we obtain

$$\|u\|_{W^{1,2}([R,\infty)\times[0,1])} \leq \frac{C}{\sigma} e^{-\sigma R}.$$

On combining (14.1.17) and (14.1.3) (for $k = 0$), we also have

$$\|u\|_{1,p} < \infty.$$

Now we apply Proposition 14.1.2 twice to $\Omega' = [0,3] \times [0,1]$ and $\Omega = [1,2] \times [0,1]$ to obtain the inequality

$$\|u\|_{W^{3,2}([1,2]\times[0,1])} \leq C_2 \|u\|_{W^{1,2}([0,3]\times[0,1])}$$

for any u satisfying $\overline{\partial}_J u = 0$, $u(\tau, 0) \in L_0$, $u(\tau, 1) \in L_1$. Because C_2 does not depend on u as long as $\|du\|_{1,p} \leq K$ with K fixed and also because the translated sequence $u \circ \tau$ has the same $W^{1,p}$-norm as u, we obtain

$$\|u\|_{W^{3,2}([\tau+1,\tau+2]\times[0,1])} \leq C_2 \|u\|_{W^{1,2}([\tau,\tau+3]\times[0,1])}.$$

Therefore, by considering $\tau \geq R$, we obtain

$$\|u\|_{W^{3,2}([\tau+1,\tau+2]\times[0,1])} \leq \frac{CC_2}{\sigma} e^{-\sigma\tau}.$$

By the Sobolev embedding $W^{3,2}([\tau+1, \tau+2]\times[0,1]) \hookrightarrow C^1([\tau+1, \tau+2]\times[0,1])$, we obtain the C^1-estimate

$$\|u\|_{C^1([\tau+1,\tau+2]\times[0,1])} \leq C' e^{-\sigma\tau}$$

for all $\tau \geq R$ with C' independent of τ. This proves

$$\|u\|_{C^1([\tau,\infty)\times[0,1])} \leq C' e^{-\sigma\tau},$$

which corresponds to (14.1.7) for $k = 1$. A similar boot-strap argument proves (14.1.7) for general $k \geq 1$. This finishes the proof. □

Remark 14.1.8 In the above proof, a careful treatment will show that we can choose σ determined by

$$\sqrt{\sigma} = \min\{|\lambda| \mid \lambda \text{ is an eigenvalue of } J\,d/dt\}.$$

In particular, when $J_t \equiv J_0$ is t-independent, this $\sqrt{\sigma}$ will coincide with the *Kähler angle* at $p \in L_0 \cap L_1$ between T_pL_0 and T_pL_1 in the Hermitian inner product space $(T_pM, J_{0,p}, g_p)$.

Exercise 14.1.9 Prove the last statement in this remark.

14.2 Splitting ends of $\mathcal{M}(x, y; B)$

In this section, we will give a precise description of the splitting ends of Floer trajectory moduli spaces $\mathcal{M}(x, y; B)$ for a pair (L_0, L_1) of compact Lagrangian submanifolds. The other type of ends, bubbling ends, will be described in the next section.

Denote by

$$e_i : j_\omega \to \mathcal{J}_\omega, \quad \text{for } i = 0,\ 1$$

the evaluation maps $J \mapsto J(i)$ for $i = 0,\ 1$. We define

$$j_{(J_0,J_1)} = \{J \in \mathcal{P}([0,1], \mathcal{J}_\omega) \mid e_i(J) = J_i\}$$

i.e., the set of paths in $\mathcal{J}_\omega$ with fixed end points J_0, J_1, respectively. We associate the Floer moduli space

$$\begin{aligned}\widetilde{\mathcal{M}}(p, q; J; B) = \{u : \Theta \to M \mid \overline{\partial}_J u = 0,\ u(\tau, 0) \in L_0,\ u(\tau, 1) \in L_1,\\ E_J(u) < \infty, [u] = B\}\end{aligned}$$

and its quotient

$$\mathcal{M}(p, q; J; B) = \widetilde{\mathcal{M}}(p, q; J; B)/\mathbb{R}$$

when either $p \neq q$ or $p = q$ and $0 \neq B \in \pi_2(p, p)$. (The remaining cases are not stable and hence have been omitted.) We suppress J from our notations $\widetilde{\mathcal{M}}(p, q; J; B)$ and $\mathcal{M}(p, q; J; B)$ unless we need to specify it to avoid confusion.

In general, the quotient space $\mathcal{M}(p, q; B)$ will not be compact. We first note that the $\mathbb{R}$-action preserves the homotopy class and so preserves the symplectic area. We will analyze failures of compactness of a given sequence $u_i \in \widetilde{\mathcal{M}}(p, q; B)$ modulo the τ-translations.

We start with the following lemma.

Lemma 14.2.1 *Suppose that L_0, L_1 intersect transversely. Then there exists a constant $\sigma = \sigma_{(L_0, L_1; J)} > 0$ depending only on the pair (L_0, L_1) and J such that*

$$E_J(u) \geq \sigma \tag{14.2.18}$$

for any non-stationary solution u of

$$\frac{\partial u}{\partial \tau} + J_t \frac{\partial u}{\partial t} = 0, \quad u(\tau, 0) \in L_0, \ \ u(\tau, 1) \in L_1.$$

Proof We prove this by contradiction. Suppose there exists a sequence u_i of non-stationary solutions with $E_J(u_i) \to 0$ as $i \to \infty$. This in particular rules out bubbling and so we obtain a uniform derivative bound

$$|du_i|_\infty < C.$$

By the same argument as was used in the proof of Lemma 14.1.4, we extract a subsequence, again denoted by u_i, such that

$$\lim_{i \to \infty} |du_i|_\infty = 0. \tag{14.2.19}$$

We claim that there exists a neighborhood $U \supset L_0 \cap L_1$ such that Image $u_i \subset U$ for all sufficiently large is. Otherwise we can find a subsequence and points (τ_i, t_i) such that

$$u_i(\tau_i, t_i) \in M \setminus U.$$

We may assume $\tau_i = 0$ by translating u_i by τ_i, $t_i \to t_\infty \in [0, 1]$ and $u_i(0, t_i) \to x_\infty \in M \setminus U$. Using (14.2.19), we prove that u_i converges to a Floer trajectory u_∞ with $E_J(u_\infty) = 0$ and hence u_∞ must be a constant map and thus its image must be contained in $L_0 \cap L_1$. On the other hand, the convergence also implies that $u_\infty(0, t_\infty) = x_\infty \in M \setminus U$. These two statements obviously contradict each other and hence the proof has been achieved. □

Next recall that Lemma 14.1.3 implies that, as $\tau \to \pm\infty$, $u(\tau)$ converges to $p_\pm$, respectively, for some $p_\pm \in L_0 \cap L_1$. In particular, we can write u in the form

$$u(\tau, t) = \exp_{p_\pm}(\xi_\pm(\tau, t))$$

for some maps

$$\xi_+ : (R_+, \infty) \times [0, 1] \to T_{p_+} M, \tag{14.2.20}$$

$$\xi_- : (-\infty, R_-) \times [0,1] \to T_{p_-}M. \tag{14.2.21}$$

Here the exponential maps $\exp_{p\pm}$ are with respect to any Riemannian metric that has the property that L_0 and L_1 become *totally geodesic* near $p_\pm$. $R_\pm$ *depend on u and may vary as u varies.*

Then Lemma 14.1.3 and Proposition 14.1.5 imply

Proposition 14.2.2 *Let L_0, L_1 be transverse and $\xi_\pm$ and $R_\pm > 0$ be as above. Then, for each $k \in \mathbb{N}$, there exist $C_k > 0$ and $\delta_\pm = \delta_\pm > 0$ depending only on $p_\pm$, respectively, such that*

$$|\nabla^k \xi_+|(\tau,t) \le C_k e^{-\delta_+|\tau|} \quad \textit{for } R_+ < \tau < \infty, \tag{14.2.22}$$
$$|\nabla^k \xi_-|(\tau,t) \le C_k e^{-\delta_-|\tau|} \quad \textit{for } -\infty < \tau < R_- \tag{14.2.23}$$

for each given $k \ge 0$.

We now define a topology of $\widetilde{\mathcal{M}}(p,q;B)$. The topology on $\mathcal{M}(p,q;B)$ will then be defined to be the quotient topology thereof.

To study this convergence more effectively in practice, the following notion of *local energy* will be useful.

Definition 14.2.3 (Local energy) We define the local energy of u on $[R_1,R_2] \times [0,1]$ by

$$E_{J,[R_1,R_2]}(u) = \int_{R_1}^{R_2}\int_0^1 |du|_J^2\, dt\, d\tau.$$

Owing to Proposition 14.2.2 and by the boot-strap argument, we have a more useful version of the convergence statement.

Proposition 14.2.4 *Let $u_i \in \widetilde{\mathcal{M}}(p,q;B)$ be a sequence that satisfies the following criteria.*

(1) *$u_i \in \widetilde{\mathcal{M}}(p,q;B)$ converges to a map $u \in \widetilde{\mathcal{M}}(L_0,L_1;J)$ in C^1 topology uniformly on compact sets.*
(2) *The local energy converges to $E(u_i) = \omega(B)$, i.e.,*

$$\lim_{R\to\infty} E_{J,[-R,R]}(u_i) = \omega(B).$$

Then the local limit u lies in $\widetilde{\mathcal{M}}(p,q;B)$ and u_i converges to u in the fine C^∞ topology.

Exercise 14.2.5 Prove this proposition.

By the definition of quotient topology, a sequence u_i, or rather its equivalence classes $[u_i]$, will converge in $\mathcal{M}(p,q;B)$ if there exists a sequence

$\tau_j \in \mathbb{R}$ such that the translated sequence $u_j \circ \tau_j$ satisfies the above two properties.

This proposition indicates what will cause the failure of convergence of a given sequence. Motivated by this proposition, we will introduce the notion of *cusp-trajectories* and *broken cusp-trajectories* following Floer (Fl88b), which will be used to compactify $\mathcal{M}(p, q; B)$.

Since we have shown that the failure of convergence in local topology is due to the bubbling which has already been studied, we assume that we already have the uniform bound

$$|du_i(\tau, t)| < C \tag{14.2.24}$$

for all $(\tau, t) \in \mathbb{R} \times [0, 1]$ and $i \in \mathbb{N}$. We denote by $I \subset \mathbb{N}$ a subsequence of $\mathbb{N}$ in general. The following is an immediate consequence of the Ascoli–Arzelà theorem.

Proposition 14.2.6 *Suppose the sequence $u_i \in \widetilde{\mathcal{M}}(p, q; B)$ satisfies (14.2.24) for some $C > 0$. Then, for any given sequence $\{\tau_i\} \subset \mathbb{R}$, there exists a subsequence $I = \{i_j\}_{j \in \mathbb{N}}$ of $\mathbb{N}$ and a map $u \in \widetilde{\mathcal{M}}(L_0, L_1; J)$ such that the translated sequence*

$$u_{i_j} \circ \tau_{i_j} = u_{i_j}(\cdot - \tau_{i_j}, \cdot)$$

converges to u locally uniformly on $\mathbb{R} \times [0, 1]$.

We denote by

$$\tau_I = \{\tau_{i_j}\}_{j \in I}$$

the sequence accompanied by I in this proposition, and the corresponding local limit by $u = u_{(I, \tau_I)}$. We next have the following convergence result of local energy.

Proposition 14.2.7 *Let u_{i_j} be a subsequence chosen as above and $u_{(I, \tau_I)}$ be its local limit. Then we have*

$$\limsup_{R \to \infty} \lim_{j} |E_{J, [-R, R]}(u_{i_j}) - E_J(u_{(I, \tau_I)})| = 0. \tag{14.2.25}$$

Proof By the local convergence, we have

$$E_{J, [-R, R]}(u_{i_j}) \to E_{J, [-R, R]}(u_{(I, \tau_I)})$$

as $j \to \infty$ for all $R > 0$. Obviously the Lebesgue dominated convergence theorem implies

$$\lim_{R \to \infty} E_{J, [-R, R]}(u_{(I, \tau_I)}) = E_J(u_{(I, \tau_I)})$$

by the finiteness of $E_J(u_{(I, \tau_I)}) < 0$. On combining the two, (14.2.25) follows. □

It turns out to be useful to introduce the following definition.

Definition 14.2.8 Let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $\widetilde{\mathcal{M}}(p,q;B)$. We call a pair (I,τ_I) given as in Proposition 14.2.6 a *recovering pair* of $\{u_i\}$. We say it is *non-stationary* if the corresponding local limit $u_{(I,\tau_I)}$ is not stationary. We also call any such limit $u_{(I,\tau_I)} \in \widetilde{\mathcal{M}}(L_0,L_1;J)$ a *local limit* of the sequence u_i (associated with the recovering pair (I,τ_I)).

Now, for a given sequence $\{u_i\}$, we collect all the non-stationary recovering pairs and the local limits of $\{u_i\}$ and denote the set of them by

$$\mathcal{R}cov(\{u_i\}) = \{(I,\tau_I) \mid \text{is a non-stationary recovering pair of } \{u_i\}\},$$
$$\widetilde{\mathcal{L}oc}(\{u_i\}) = \{u \in \widetilde{\mathcal{M}}(L_0,L_1) \mid u = u_{(I,\tau_I)} \text{ for some } (I,\tau_I) \in \mathcal{R}cov\}.$$

Definition 14.2.9 Let v, $v' \in \widetilde{\mathcal{L}oc}(\{u_i\})$ be two non-stationary local limits of $\{u_i\}$. We say $v \sim v'$ if there exists $\tau_0 \in \mathbb{R}$ such that

$$v' = v \circ \tau_0.$$

We denote by $\mathcal{L}oc(\{u_i\})$ the set of equivalence classes

$$\mathcal{L}oc(\{u_i\}) = \widetilde{\mathcal{L}oc}(\{u_i\})/\sim$$

and by $[v]$ the equivalence class associated with $v \in \mathcal{L}oc(\{u_i\})$. We will just call $[v]$ itself a local limit and denote it by v, whenever the meaning is clear.

Next we need the following definition.

Definition 14.2.10 Let $\{u_i\} \subset \widetilde{\mathcal{M}}(p,q;B)$. A *chain* of local limits of $\{u_i\}$ is defined to be a sequence of local limits of

$$([v_1],[v_2],\ldots,[v_n])$$

such that there exists a subsequence I and sequences

$$\tau_j^1 < \cdots < \tau_j^n, \quad j \in I$$

such that $v_\ell = u_{(I,\tau_I^\ell)}$. Denote $\vec{\tau}_I = \{(\tau_j^1,\ldots,\tau_j^n)\}_{j\in I}$.

We denote by $\widetilde{Ch}(\{u_i\})$ the set of chains of local limits of $\{u_i\}$. If a chain $(v_1,\ldots,v_n)$ is associated with a subsequence $(I,\vec{\tau}_I)$ as in Definition 14.2.10, then we denote

$$C_{(I,\vec{\tau}_I)} = (v_1,\ldots,v_n).$$

Definition 14.2.11 Let C, C' be chains of local limits of $\{u_i\}$. A chain C is called a subchain of C' if

$$C = C_{(I,\vec{\tau}_I)} = (v_1, \ldots, v_n),$$

I has a subsequence I' such that

$$C' = C_{(I',\vec{\tau}_{I'})} = (v'_1, \ldots, v'_{n'})$$

and $\{v_1, \ldots, v_n\} \subset \{v'_1, \ldots, v'_{n'}\}$. We say that two chains C and C' are equivalent if C is a subchain of C' and vice versa.

Denote by $Ch(\{u_i\})$ the set of equivalence classes of chains and by $[C]$ the equivalence class of C. Now we define a partial order $[C] \leq [C']$ by C being a subchain of C'.

Lemma 14.2.12 *Let $C = (v_1, \ldots, v_n)$ be a chain of local limits with a finite length. Then we have*

$$\sum_{\ell=1}^{n} \omega([v_\ell]) \leq \omega(B). \tag{14.2.26}$$

In particular, the length of the chain n is bounded by

$$n \leq \left[\frac{\omega(B)}{\sigma}\right] + 1. \tag{14.2.27}$$

Using this lemma, we prove in the following the existence of a maximal chain.

Proposition 14.2.13 *Assume either $p \neq q$ or $p = q$ and $B \neq 0$. Let $u_i \in \mathcal{M}(p, q; B)$ be a sequence satisfying*

$$\|du_i\|_\infty < C$$

for a uniform constant $C > 0$. Then there exists a maximal chain in $Ch(\{u_i\})$ whose length is $\leq [\omega(B)/\sigma] + 1$.

Proof Under the given hypothesis, the uniform C^1-bound on u_i implies that $\mathcal{R}cov(\{u_i\})$ is nonempty. Let (I, τ_I) be a recovering pair and let $u_{(I,\tau_I)}$ be its local limit. If $\omega(u_{(I,\tau_I)}) = \omega(B)$, then Proposition 14.2.4 implies that the single-element chain $\{(I, \tau_I)\}$ is already maximal. Therefore we assume that

$$\omega(u_{(I,\tau_I)}) < \omega(B). \tag{14.2.28}$$

Then it follows from (14.2.25) that, for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$|\omega(u_{(I,\tau_I)}) - E_{[-R_\epsilon,R_\epsilon]}(u_{i_j}(\cdot - \tau_{i_j}, \cdot)| \leq \epsilon \tag{14.2.29}$$

for all $R \geq R_\epsilon$. We fix $\epsilon > 0$ such that

$$\epsilon < \frac{\omega(B) - \omega(u_{(I,\tau_I)})}{3}.$$

Denote by $u_{i_j,\tau_{i_j}}$ the map defined by $u_{i_j,\tau_{i_j}}(\tau, t) = u_{i_j}(\tau - \tau_{i_j}, t)$. Define $\mu_{\pm,j} > R$ to be the unique numbers in $-\infty < -\mu_{-,j} < -R$ and $R < \mu_{+,j} < \infty$ such that

$$\begin{aligned} E_{J,(-\infty,-\mu_{-,j}]}(u_{i_j,\tau_{i_j}}) &= \frac{1}{2} E_{J,(-\infty,-R]}(u_{i_j,\tau_{i_j}}), \\ E_{J,[\mu_{+,j},\infty)}(u_{i_j,\tau_{i_j}}) &= \frac{1}{2} E_{J,[R,\infty)}(u_{i_j,\tau_{i_j}}), \end{aligned} \tag{14.2.30}$$

respectively.

Lemma 14.2.14

$$\max\{\mu_{+,j} - R, \mu_{-,j} - R\} \to \infty.$$

Proof Otherwise there exists some subsequence, which we again denote by j,

$$\mu_{\pm,j} \leq R + C,$$

for some $C > 0$ for all j. Then, on the one hand, (14.2.29) implies

$$\begin{aligned} E_{J,(-\infty,-R-C]}(u_{i_j}) + E_{J,[R+C,\infty)}(u_{i_j}) &= E(u_{i_j}) - E_{[-R-C,R+C]}(u_{i_j}) \\ &\geq \omega(B) - \omega(u_{(I,\tau_I)}) - \epsilon. \end{aligned}$$

On the other hand,

$$\begin{aligned} E_{J,(-\infty,-R-C]}(u_{i_j}) + E_{J,[R+C,\infty)}(u_{i_j}) &\leq E_{J,(-\infty,-\mu_{-,j}-R]}(u_{i_j}) + E_{J,[\mu_{+,j}+R,\infty)}(u_{i_j}) \\ &= \frac{1}{2}\left(E_{J,(-\infty,-R]}(u_{i_j}) + E_{J,[R,\infty)}(u_{i_j})\right) \\ &\leq \frac{1}{2}(\omega(B) - E_{[-R,R]}(u_{i_j})) \\ &\leq \frac{1}{2}(\omega(B) - \omega(u_{(I,\tau_I)}) + \epsilon). \end{aligned}$$

Combining the two, we obtain

$$\omega(B) - \omega(u_{(I,\tau_I)}) - \epsilon \leq \frac{1}{2}(\omega(B) - \omega(u_{(I,\tau_I)}) + \epsilon)$$

and so

$$\omega(B) - \omega(u_{(I,\tau_I)}) \leq 3\epsilon.$$

Since we chose $\epsilon < [\omega(B) - \omega(u_{(I,\tau_I)})]/3$, we get a contradiction. This finishes the proof. □

Furthermore, using (14.2.30), we derive

$$\begin{aligned}
&\frac{1}{2}(E_{J,(-\infty,-R]}(u_{i_j,\tau_j}) + E_{J,[R,\infty)}(u_{i_j,\tau_j})) \\
&\quad = E_{J,(-\infty,-\mu_{-,j}]}(u_{i_j,\tau_{i_j}}) + E_{J,[\mu_{+,j},\infty)}(u_{i_j,\tau_{i_j}}) \\
&\quad = \omega(B) - E_{J,[-\mu_{-,j},\mu_{+,j}]}(u_{i_j,\tau_j}) \\
&\quad \geq \omega(B) - \omega(u_{(I,\tau_I)}) - \epsilon > \frac{2}{3}(\omega(B) - \omega(u_{(I,\tau_I)})),
\end{aligned}$$

and hence one of $E_{J,(-\infty,-R]}(u_{i_j,\tau_{+,j}})$ and $E_{J,[R,\infty)}(u_{i_j,\tau_j})$ is greater than or equal to $\frac{2}{3}(\omega(B) - \omega(u_{(I,\tau_I)})) > 0$, after choosing a subsequence of js, if necessary.

It is easy to see from the local convergence of u_{i_j,τ_j} that this must happen for $* = \pm$ for which $\mu_{*,j} - R \to \infty$. We set μ_j to be one among $\mu_{*,j}$ for which $\lim_{j\to\infty} \mu_{*,j} = \infty$. Without loss of any generality, we may assume $* = -$ and so $\mu_j = \mu_{-,j}$.

Now we consider the translated sequence v_k defined by $w_k = u_{i_k} \circ \mu_k$. By definition and again by (14.2.30), we have

$$E_{J,(-\infty,0]}(w^k) > \frac{1}{3}(\omega(B) - \omega(u_{(I,\tau_I)}))$$

for all k, and so we can choose a sequence $(\nu_k, t_k) \in (-\infty, 0] \times [0, 1]$ such that

$$w_k(\nu_k, t_k) = u_{i_k}(\nu_k - \mu_k, t_k) \in M \setminus U, \tag{14.2.31}$$

where U is a sufficiently small neighborhood of $L_0 \cap L_1$. We consider the map

$$\tilde{w}_k = w_k \circ \nu_k = u_{i_k} \circ (-\mu_k + \nu_k)$$

on $(-\infty, \mu_k - R - \nu_k] \times [0, 1]$. By definition, we have $\tilde{w}_k(0, t_k) \in M \setminus U$. Note that, since $\nu_k \leq 0$, we still have

$$\mu_k - R - \nu_k \to \infty.$$

Since $[0, 1]$ and M are compact, we may assume both that $t_k \to t_\infty$ and that $\tilde{w}(0, t_k) \to x$ for a subsequence $I' \subset \{i_k\}_{k\in\mathbb{N}}$. Then $\tilde{w}_k$ has a local limit v'_∞ that satisfies $v'_\infty(0, t_\infty) = x$ lies in $M \setminus L_0 \cap L_1$. Since we know that $\tilde{w}_\infty(\pm\infty) \in L_0 \cap L_1$, $\tilde{w}_\infty$ cannot be stationary. We denote by $(I', \tau'_{I'})$ the corresponding recovering pair defined by

$$\tau'_\ell = \mu_\ell + \tau_\ell, \quad \ell \in I'.$$

By construction, we have

$$[\tilde{w}_\infty] < [(\tilde{w}_\infty, u_{(I,\tau_I)})].$$

If $\omega([\tilde{w}_\infty]) + \omega([u_{(I,\tau_I)}]) = \omega(B)$, again $(\tilde{w}_\infty, u_{(I,\tau_I)})$ forms a maximal chain. Otherwise we can repeat the above process to produce a larger chain. By Lemma

14.2.12, this process must stop at a finite stage and hence the proof has been obtained. □

Definition 14.2.15 A chain $(v_1, \ldots, v_N)$ of Floer trajectories is said to be gluable if $v_i(\infty) = v_{i+1}(-\infty)$ for all $i = 1, \ldots, N$.

Now we state the following basic property on the maximal chain of recovering sequences and their corresponding local limits.

Proposition 14.2.16 *Let $u_i \in \mathcal{M}(p, q; B)$ be a sequence satisfying*

$$\|du_i\|_\infty < C$$

for a uniform constant $C > 0$. For each maximal chain

$$C_{(I,\vec{\tau}_I)} = (u_{(I,\tau_1)}, \ldots, u_{(I,\tau_\ell)})$$

the corresponding local limits $(u_{(I,\tau_1)}, \ldots, u_{(I,\tau_\ell)})$ are consecutively gluable and satisfy

$$u_{I_1}(-\infty) = x, \ldots, u_{I_N}(\infty) = y.$$

Proof By Proposition 14.2.4, we need only prove that (I_i, τ_{I_i}) is gluable to $(I_{i+1}, \tau_{I_{i+1}})$ for all i. We denote

$$I_i = \{n^i_\ell\}_{\ell\in\mathbb{N}}, \qquad \tau_{I_i} = \{\tau^i_\ell\}_{\ell\in\mathbb{N}},$$
$$I_{i+1} = \{n^{i+1}_\ell\}_{\ell\in\mathbb{N}}, \quad \tau_{I_{i+1}} = \{\tau^{i+1}_\ell\}_{\ell\in\mathbb{N}}.$$

Suppose to the contrary that there exists i_0 such that $(I_{i_0}, \tau_{I_{i_0}})$ is not gluable to $(I_{i_0+1}, \tau_{I_{i_0+1}})$. By definition, there exists $I'' \subset I$ such that

$$\tau^{i+1}_{\ell_k} - \tau^i_{\ell_k} \nearrow \infty \quad \text{as } k \to \infty, \tag{14.2.32}$$

where $I'' = \{\ell_k\}_{k\in\mathbb{N}}$. We now consider the subsequence u_{ℓ_k}. We denote the average

$$\mu'_k = \frac{\tau^{i+1}_{\ell_k} + \tau^i_{\ell_k}}{2} \tag{14.2.33}$$

and consider the translated sequences

$$u''_k = u_{\ell_k} \circ \mu'_k.$$

Then we can extract a subsequence of u''_k that has the local limit u''_∞. Now we denote the corresponding recovering pair by $(I^{(3)}, \tau_{I^{(3)}})$, which satisfies $I^{(3)} \subset I''$. By virtue of the choice of μ'_k made in (14.2.32), (14.2.33), we have

$$(I_{i_0}, \tau_{I_{i_0}}) < (I^{(3)}, \tau_{I^{(3)}}) < (I_{i_0+1}, \tau_{I_{i_0+1}}).$$

This implies that, *unless the local limit $u^{(3)}$ corresponding to $(I^{(3)}, \tau_{I^{(3)}})$ is stationary*, we will have a contradiction to the hypothesis that $C = C_{(I,\vec{\tau}_I)}$ is a maximal chain. This proves that $\{(I_i, \tau_{I_i})\}$ is consecutively gluable. A similar proof applies to prove $v_1(-\infty) = x$ or $v_N(\infty) = y$. This finishes the proof. □

Next we prove that any maximal chain recovers the homotopy class.

Theorem 14.2.17 *Let $u_i \in \mathcal{M}(p, q; B)$ be any given sequence satisfying*

$$\|du_i\|_\infty < C$$

and let

$$(v^1, \ldots, v^N)$$

be a chain of local limits of $\{u_i\}$ corresponding to a maximal chain of recovering sequences. Then we have

$$B = [v^1]\# \cdots \#[v^N]. \tag{14.2.34}$$

Proof In the course of proving Lemma 14.2.12 and Proposition 14.2.13, we have established

$$\omega(B) = \sum_{n=1}^{N} \int (v^n)^* \omega.$$

This implies that there exists a sequence

$$\tau_{1,i} < \tau_{2,i} < \cdots < \tau_{N,i}$$

such that

(1) $\tau_{\ell+1,i} - \tau_{\ell,i} \to \infty$ as $i \to \infty$ and
(2) there exists $R_{\ell,i} > 0$ for $i = 1, \ldots$ such that $R_{\ell,i} \to \infty$ and

$$R_{\ell,i} + R_{\ell+1,i} < \tau_{\ell+1,i} - \tau_{\ell,i},$$

and

$$\int_{[-R_{\ell,i}, R_{\ell,i}] \times [0,1]} (u_i \circ \tau_{\ell,i})^* \omega \to 0 \quad \text{as } i \to \infty. \tag{14.2.35}$$

Once we have (14.2.35), the ϵ-regularity theorem implies that the image of u_i on the region

$$\bigcup_{\ell=1}^{N} [\tau_{\ell,i} + R_{\ell,i}, \tau_{\ell+1,i} - R_{\ell,i}] \times [0, 1]$$

is contained in a small neighborhood of $L_0 \cap L_1$. By a simple topological argument using the isolatedness of $L_0 \cap L_1$, we derive $B = [v^1]\#\cdots\#[v^N]$, which finishes the proof. □

We consider the set of the equivalence classes of gluable chains starting at x and ending at y. We will provide a neighborhood basis at each point in this splitting end of $\mathcal{M}(x, y; B)$. We recall that $\mathcal{M}(x, y; B) = \widetilde{\mathcal{M}}(x, y; B)/\mathbb{R}$ is a quotient space. We represent an element in $\mathcal{M}(x, y; B)$ by the unique representative of the map u in the equivalence class that satisfies the energy normalization

$$E_J(u|_{(-\infty,0]\times[0,1]}) = E_J(u|_{[0,\infty)\times[0,1]}). \tag{14.2.36}$$

We can always uniquely achieve this normalization condition by translating any given u along the τ-direction. We denote the subset of $\widetilde{\mathcal{M}}$ consisting of us satisfying this normalization condition by $\widetilde{\mathcal{M}}_0$, which we call a slice. Then the quotient map restricted to $\widetilde{\mathcal{M}}_0$ defines a diffeomorphism onto $\mathcal{M}$. To provide the above-mentioned neighborhood basis at each point in the splitting end, we generalize the construction of this slice to the gluable chains.

Definition 14.2.18 Consider a gluable chain of maps $v^\ell \in \widetilde{\mathcal{M}}_0(x_{\ell-1}, x_\ell; B_\ell)$

$$(v^1, \ldots, v^N); \quad v^1(-\infty) = x,\ v^N(\infty) = y$$

with $B_1\#\ldots\#B_N = B$. We say that a sequence $[u_i] \in \mathcal{M}(x, y; B)$ converges to $[v^1]\#\ldots\#[v^N]$ if there exists a sequence $(\tau_{1,i}, \ldots, \tau_{N,i})$ with

$$\tau_{1,i} < \cdots < \tau_{N,i}$$

such that $(v^1, \ldots, v^N) = C_{(\mathbb{N},\vec{\tau}_\mathbb{N})}$ for a maximal chain $C_{(\mathbb{N},\vec{\tau}_\mathbb{N})}$ of the sequence u_i. Equivalently, the vector $\vec{\tau}_{N,i} = (\tau_{1,i}, \ldots, \tau_{N,i})$ satisfies the two properties (1) and (2) stated in the proof of Theorem 14.2.17.

We can also construct a neighborhood basis (at infinity) of $\mathcal{M}(x, y; B)$ of $(v^1, \ldots, v^N)$ (from infinity) as follows. Consider a collection

$$\widetilde{V}^\epsilon_\delta((v^1, \ldots, v^N))$$

of open subsets of $\mathcal{F}(x, y; B)$ consisting of us that satisfies

$$d_{W^{1,p}([\tau_\ell - R_\ell, \tau_\ell + R_\ell])}(u_i \circ \tau_{\ell,i}, v^\ell) < \epsilon, \tag{14.2.37}$$

$$E_J(u_i \circ \tau_{\ell,i}|_{\tau_\ell + R_\ell \le \tau \le \tau_{\ell+1} - R_{\ell+1}}) < \delta \tag{14.2.38}$$

for all $\ell = 1, \ldots, N$. We note that $V^\epsilon_\delta((v^1, \ldots, v^N))$ is invariant under the translations of the domain in the τ-direction.

Then the following lemma immediately follows from the definition of the above neighborhood basis and the exponential decay.

Lemma 14.2.19 *A sequence $[u_i]$ converges to $[v^1]\#\dots\#[v^N]$ in the sense of Definition 14.2.18 if and only if there exists some ϵ, $\delta > 0$ and $N_0 \in \mathbb{N}$ such that $u_i \in \widetilde{V}^{\epsilon}_{\delta}((v^1, \dots, v^N))$ for all $i \geq N_0$.*

Exercise 14.2.20 Prove this lemma.

Therefore the collection

$$\widetilde{V}^{\epsilon}_{\delta}((v^1, \dots, v^N)) \cap \widetilde{\mathcal{M}}(x, y; B)/\mathbb{R} =: V^{\epsilon}_{\delta}((v^1, \dots, v^N)) \tag{14.2.39}$$

provides a neighborhood basis of $\mathcal{M}(x, y; B)$ at $[v^1]\#\dots\#[v^N]$ (at infinity).

So far we have assumed the C^1-bound for the sequence $u_i \in \mathcal{M}(p, q; B)$. In general this C^1-bound will not hold for a general pair (L_0, L_1) of Lagrangian submanifolds when there are non-trivial holomorphic discs or spheres around. However we can still apply the Sachs–Uhlenbeck bubbling argument and prove that the same kind of convergence holds after one has taken away bubbles during the convergence.

In the next section, we will provide a precise description of the objects to be added in the compactfication of $\mathcal{M}(p, q; B)$.

14.3 Broken cusp-trajectory moduli spaces

Let $p, q \in L_0 \cap L_1$. As indicated in Section 14.2, a compactification of $\mathcal{M}(p, q; B)$ requires one to add an object consisting of the union of Floer trajectories and bubble components that are either pseudoholomorphic spheres or discs. We will call the ones consisting of Floer trajectories among the irreducible components *principal components*.

We start with the case where there exists a unique principal component. Intuitively, an (unbroken) *Floer cusp-trajectory* from x to y in class B is the union of a Floer trajectory $u_0 \in \mathcal{M}(p, q; B_0)$ and a finite number of moduli spaces of sphere bubbles and of disc bubbles attached to a finite number of points in $\mathbb{R} \times [0, 1]$. A precise definition of these objects is in order.

We first note that modulo the Novikov equivalence relation in Definition 13.4.1, $\pi_2(M)$ acts on $\pi_2(p, q)$ as a monoidal action, and the pair $(\pi_2(M, L_0), \pi_2(M, L_1))$ act on $\pi_2(p, q)$ as a bi-monoidal action in an obvious way. We denote an element of $\pi_2(M)$ by α and that of $\pi_2(M, L_i)$ by β_i for $i = 0, 1$. We denote the image of the above-mentioned action on a class $B_0 \in \pi_2(p, q)$ by

$$B = B_0 + \sum_{\ell} \alpha_{\ell} + \sum_{j} \beta_{0,j} + \sum_{k} \beta_{1,k}.$$

Definition 14.3.1 A *configuration on* $\mathbb{R} \times [0,1]$ is the set of of finite points consisting of

$$(\tau_i, t_i) \in \mathbb{R} \times (0,1) \quad \text{for } i = 1, \dots, m,$$
$$(\tau_j, 0) \in \mathbb{R} \times \{0\} \quad \text{for } j = 1, \dots, n_0,$$
$$(\tau_k, 1) \in \mathbb{R} \times \{1\} \quad \text{for } k = 1, \dots, n_1.$$

We denote by $\mathfrak{C}_{(m;n_0,n_1)}$ the set of such configurations.

Denote such a triple by

$$C = (\{(\tau_i, t_i)\}; \{(\tau_j, 0)\}, \{(\tau_k, 1)\})$$

in general. We note that there is no non-trivial holomorphic automorphism as long as $C \neq \emptyset$. It is convenient to include the *empty* configuration $C = \emptyset$ in our discussion. We call a pair

$$(u, C) \in \widetilde{\mathcal{M}}(p, q; B) \times \mathfrak{C}_{(m;n_0,n_1)}$$

a *marked Floer trajectory*, where

$$u : \mathbb{R} \times [0,1] \to M, \quad C \in \mathfrak{C}_{(m;n_0,n_1)}.$$

There is a natural $\mathbb{R}$-action on the product $\mathcal{M}(p, q; B) \times \mathfrak{C}_{(m;n_0,n_1)}$ defined by

$$(u, C) \mapsto (u \circ \tau_0, C \circ \tau_0),$$

where $\tau_0 \in \mathbb{R}$ and $u \circ \tau_0$ is defined by

$$u \circ \tau_0(\tau, t) = u(\tau - \tau_0, t)$$

and $C \circ \tau_0$ by

$$(\{(\tau_i + \tau_0, t_i)\}; \{(\tau_j + \tau_0, 0)\}, \{(\tau_k + \tau_0, 1)\}).$$

We define $\mathcal{M}_{(m;n_0,n_1)}(p, q; B)$ to be the quotient

$$\mathcal{M}_{(m;n_0,n_1)}(p, q; B) = \widetilde{\mathcal{M}}(p, q; B) \times \mathfrak{C}_{(m;n_0,n_1)}/\mathbb{R},$$

and

$$\mathcal{M}_{\emptyset}(p, q; B) = \widetilde{\mathcal{M}}(p, q; B)/\mathbb{R}$$

when $(m; n_0, n_1) = (0; 0, 0)$. We note that when $C \neq \emptyset$ we have the natural evaluation maps

$$\mathrm{ev} : \mathcal{M}_{(m;n_0,n_1)}(p, q; B) \to M^m \times L_0^{n_0} \times L_1^{n_1}$$

defined by

$$\mathrm{ev}(u, C) = (\{u(\tau_i, t_i)\}, \{u(\tau_j, 0)\}, \{u(\tau_k, 1)\})$$

which respect the above-mentioned $\mathbb{R}$-action and hence are well defined. As before, we denote by $\overline{\mathcal{M}}_1(J_0;\alpha)$ the stable maps of genus 0 with one marked point, and by $\overline{\mathcal{M}}_1(L, J_0;\beta)$ the set of bordered stable maps with one marked point at a boundary of the disc as in Section 9.5. We consider the fiber product

$$\begin{aligned}
&\widetilde{\mathcal{M}}(p, q; B_0; \{\alpha_i\}, \{\beta_{0,j}\}, \{\beta_{1,k}\}) \\
&\quad := \widetilde{\mathcal{M}}_{(m;n_0,n_1)}(p, q; B_0)_{\mathrm{ev}} \\
&\qquad \times_{\mathrm{ev}} \left(\prod_i \mathcal{M}(J_{(\tau_i,t_i)};\alpha_i) \times \prod_j \mathcal{M}(L, J_0;\beta_{0,j}) \times \prod_k \mathcal{M}(L, J_1;\beta_{1,k}) \right)
\end{aligned} \tag{14.3.40}$$

with respect to the obvious evaluation maps. There is the $\mathbb{R}$-action on

$$\widetilde{\mathcal{M}}\left(p, q; B_0; \{\alpha_i\}, \{\beta_{0,j}\}, \{\beta_{1,k}\}\right)$$

of simultaneous translation of the roots of the bubbles attached to the principal component.

Definition 14.3.2 We define the set of *cusp-trajectories* issued at p ending at q to be

$$\mathcal{M}(p, q; B_0; \{\alpha_i\}, \{\beta_{0,j}\}, \{\beta_{1,k}\}) = \widetilde{\mathcal{M}}(p, q; B_0; \{\alpha_i\}, \{\beta_{0,j}\}, \{\beta_{1,k}\})/\mathbb{R}.$$

We then define

$$C\mathcal{M}(p, q; B) = \coprod \mathcal{M}(p, q; B_0; \{\alpha_i\}, \{\beta_{0,j}\}, \{\beta_{1,k}\})$$

for all choices of $B_0, \{\alpha_i\}, \{\beta_{0,j}\}$ and $\{\beta_{1,k}\}$ satisfying

$$B = B_0 + \sum_i \alpha_i + \sum_j \beta_{0,j} + \sum_k \beta_{1,k}$$

and provide it with a topology of stable maps.

We denote the corresponding decomposition of maps by

$$u = u^0 \# \left(\prod_i v_i \right) \# \left(\prod_j w_{0,j} \right) \# \left(\prod_k w_{1,k} \right)$$

and call u^0 the *principal component* and others *bubble components*, and we call such a u a *cusp-trajectory*.

Next we introduce the notion of a *broken configuration*. For each given integer $n \in \mathbb{N}$, we denote

$$\underline{n} = \{1, \ldots, n\}$$

in general.

Definition 14.3.3 We define a *broken configuration* by a pair

$$(\Theta, o),$$

where Θ is a finite connected union of $\Theta_\ell \cong \mathbb{R} \times [0,1]$ and $o = (o^{\text{sphere}}; o^{\text{disc},0}, o^{\text{disc},1})$ and $o^{\text{sphere}} : \underline{m} \to \cup_{\ell=1}^{a} \operatorname{Int} \Theta_\ell$ is an injective map, and

$$o^{\text{disc},0} : \underline{n}_0 \to \bigcup_{\ell=1}^{a} (\partial_0 \Theta_\ell),$$
$$o^{\text{disc},1} : \underline{n}_1 \to \bigcup_{\ell=1}^{a} (\partial_1 \Theta_\ell)$$

are injective maps. Here we denote

$$\Theta_a = \mathbb{R} \times [0,1],$$
$$\partial_i \Theta_\ell = \mathbb{R} \times \{i\}, \quad i = 0,\ 1.$$

We often denote $o^{\text{sphere}}(i) = (\tau_i, t_i)$, $o^{\text{disc},0}(j) = (\tau_j, 0)$ and $o^{\text{disc},1}(k) = (\tau_k, 1)$, and identify o with its image points.

Definition 14.3.4 Let $J = \{J_t\}_{0 \le t \le 1}$ and $x,\ y, \in L_0 \cap L_1$. A *stable broken Floer trajectory* from p to q is a triple

$$u = \Big((u_1, \ldots, u_a); (\sigma_1, \ldots, \sigma_m), (\gamma_1^0, \ldots, \gamma_{n_0}^0), (\gamma_1^1, \ldots, \gamma_{n_1}^1); o\Big)$$

that satisfies the following criteria.

(1) For $i = 1, \ldots, a-1$, $u_i \in \mathcal{M}(x_i, x_{i+1})$ and satisfy

$$u_1(-\infty) = \widehat{p},\ u_a(\infty) = q,$$
$$u_i(\infty) = u_{i+1}(-\infty) \quad \text{for } i = 1, \ldots, a-1. \tag{14.3.41}$$

We call (14.3.41) the matching condition and say that a pair (u, u') of Floer trajectories is *gluable* if it satisfies the matching condition.

(2) $\sigma_i \in \overline{\mathcal{M}}_1(J_{t_i}; \alpha_i)$ for $i = 1, \ldots, m$.

(3) $\gamma_j^0 \in \overline{\mathcal{M}}_1(L_0, J_0; \beta_j^0)$ for $j = 1, \ldots, n_0$ and $\gamma_k^1 \in \overline{\mathcal{M}}_1(L_1, J_1; \beta_k^1)$ for $k = 1, \ldots, n_1$.

(4) For each $\ell = 1, \ldots, a$, either the map u_ℓ is non-stationary or $\Theta_\ell \cap \operatorname{Im} o \neq \emptyset$.

We denote the domain of u simply by Θ_u, which is the product of a broken configuration and the domains of the stable maps σ_i, γ_j^0 and γ_k^1 which are glued to the image points of o at the given marked points. This is the union of a set of finite copies of

$$\mathbb{R} \times [0,1],$$

the *principal components*, and the semi-stable curves of closed or bordered Riemann surfaces of genus 0, the *bubble components* with their roots attached to the principal components of Θ_u *governed by the distribution map o*.

Now we denote by I any of the sets

$$I = (m, n_0, n_1), \quad \underline{I} = \underline{m} \coprod \underline{n_0} \coprod \underline{n_1}$$

and denote by $\rho : \underline{I} \to \underline{I}$ a bijective map preserving the factors of the coproduct $\underline{m} \coprod \underline{n_0} \coprod \underline{n_1}$.

We identify two stable broken trajectories u and u' if there exists a bijection $\rho : \underline{I} \to \underline{I}$ such that

$$u'_i = u_{\rho(i)}, \qquad \sigma'_j = \sigma_{\rho(j)}, \qquad \gamma'_k = \gamma_{\rho(k)}$$

and also denote the corresponding equivalence class by u with an abuse of notation.

Finally, we define

$$\overline{\mathcal{M}}(p, q; B)$$

to be the set of such isomorphism classes. Now we define a topology on $\overline{\mathcal{M}}(p, q; B)$ in a similar way as was done before for the study of the stable map moduli spaces, which makes $\overline{\mathcal{M}}(p, q; B)$ a compact Hausdorff space whose induced topology is the quotient topology on

$$\mathcal{M}(p, q; B) = \widetilde{\mathcal{M}}(p, q; B)/\mathbb{R}$$

of the strong C^∞ topology of $\widetilde{\mathcal{M}}(p, q; B)$.

At this stage, we would like to emphasize that this compactification is defined as a topological space for *any* choice of $J = \{J_t\}_{0 \leq t \leq 1}$ for a transversal pair L_0, L_1 of Lagrangian submanifolds. The topological space $\overline{\mathcal{M}}(p, q; B)$ will not be a smooth manifold in the standard sense, but only in the sense of Kuranishi structures (FOn99), even for a generic choice of J for a general pair (L_0, L_1). However, there is a particular class of Lagrangian submanifolds, that of *monotone* Lagrangian submanifolds, for which $\overline{\mathcal{M}}(p, q; B)$ becomes a smooth orbifold with corners of the expected dimension for a dense set of $J \in j^{\mathrm{reg}}(L_0, L_1) \subset j_\omega$.

14.4 Chain map moduli space and energy estimates

Next we consider an isotopy of $\overline{J} = \{J^s\}$ and a Hamiltonian isotopy $\phi^s_{(i)}(L_0)$ for $0 \leq s \leq 1$ and $i = 0, 1$. The main purpose of this section is to construct a (pre-)chain map

$$h_{(\bar{J},\phi^{(i)};\rho)} : CF(L_0, L_1) \to CF(\phi^1_{(0)}(L_0), \phi^1_{(1)}(L_1)). \tag{14.4.42}$$

We will first briefly summarize the construction given in (Oh93a, FOOO09) using the so-called ‘moving boundary condition’. However, it was noted in (FOOO13) that this construction is not optimal insofar as the study of filtration change is concerned. So we will use the modified construction provided in (FOOO13) for this construction in order to be able to obtain an optimal construction for the study of filtration changes.

We elongate the isotopy by considering a monotone function $\rho : \mathbb{R} \to [0, 1]$ defined by

$$\rho(\tau) = \begin{cases} 0 & \text{for } \tau \le 0, \\ 1 & \text{for } \tau \ge 1. \end{cases} \tag{14.4.43}$$

For a given pair of Hamiltonian isotopies $\phi_{H^{(i)}}$ for $i = 0, 1$, $J^s = \{J^s_t\}$ and a given smooth function ρ as above, we consider the moving Lagrangian boundary-value problem

$$\begin{cases} \partial u/\partial\tau + J^\rho\, \partial u/\partial t = 0, \\ u(\tau, 0) \in \phi^{\rho(\tau)}_{H^{(0)}}(L_0),\ u(\tau, 1) \in \phi^{\rho(\tau)}_{H^{(1)}}(L_1), \end{cases} \tag{14.4.44}$$

where $J^\rho(\tau, t) = J^{\rho(\tau)}_t$. Let $[\ell_p, w] \in \widetilde{\Omega}(L_0, L_1; \ell_a)$ and $[\ell_{q'}, w'] \in \widetilde{\Omega}(L'_0, L'_1; \ell'_a)$. Here we choose

$$\ell'_a(t) := \phi^1_{H^{(1)}}(\phi^{1-t}_{H^{(1)}})^{-1} \circ \phi^1_{H^{(0)}}(\phi^t_{H^{(0)}})^{-1}(\ell_a(t)) \tag{14.4.45}$$

as the base path of $\Omega(L'_0, L'_1)$. We denote by

$$\mathcal{M}^\rho((L_1, \phi^1_{H^{(1)}}), (L_0, \phi^1_{H^{(0)}}); [\ell_p, w], [\ell_{q'}, w']) \tag{14.4.46}$$

the set of solutions of (14.4.44) with

$$[\ell_{q'}, I^\rho_{\phi^1_{H^{(0)}}, \phi^1_{H^{(1)}}}(w\#u)] = [\ell_{q'}, w'],$$

where $w\#u : [0, 1] \times [0, 1] \to X$ is the concatenation in the τ-direction of w and u and

$$I^\rho_{\phi^1_{H^{(0)}}, \phi^1_{H^{(1)}}} v(\tau, t) = \left(\phi^1_{H^{(1)}}(\phi^{\rho(\tau)(1-t)}_{H^{(1)}})^{-1} \circ \phi^1_{H^{(0)}}(\phi^{\rho(\tau)t}_{H^{(0)}})^{-1}\right) v(\tau, t)$$

(see p. 308 in (FOOO09)).

We need to state a topological condition of the trajectory u which is now in order. We choose a two-parameter family $\phi : [0, 1]^2 \to \mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms such that

$$\phi(s, 0) = \phi^s_{(0)}, \quad \phi(s, t) = \phi^s_{(1)}, \quad \phi(0, t) \equiv id. \tag{14.4.47}$$

Such a family can always be chosen, but the choice is not unique even up to homotopy. For each given solution u we consider the map $\widetilde{u} : \mathbb{R} \times [0,1] \to M$ defined by

$$\widetilde{u}(\tau, t) = \phi(\rho(\tau), t)^{-1}(u(\tau, t)).$$

This map satisfies the fixed Lagrangian boundary condition

$$\widetilde{u}(\tau, 0) \in L_0, \quad \widetilde{u}(\tau, 1) \in L_1.$$

Then the topological condition we will impose on (13.9.28) is the following:

$$[\phi(1,t)^{-1}(q'), \phi_\rho^{-1}(w')] = [\phi(1,t)^{-1}(q'), w\#\widetilde{u}]. \tag{14.4.48}$$

We denote by $\pi_2(p, q'; \phi)$ the set of homotopy classes of such maps u. The following explains the dependence of $\pi_2(p, q'; \phi)$ on the ϕ.

Lemma 14.4.1 *Let ϕ and ψ be two-parameter families satisfying* (14.4.47). *Then the map*

$$\psi^{-1}\phi : \phi_\rho^{-1}(u) \mapsto \psi_\rho^{-1}(u)$$

induces one-to-one correspondence

$$\pi_2(p, q'; \phi) \cong \pi_2(p, q'; \psi).$$

For each given $C \in \pi_2(p, q'; \phi)$, we denote by $\mathcal{M}(p, q'; C)$ the set of solutions of (14.4.44) with $[u] = C \in \pi_2(p, q'; \phi)$.

One major difference between (14.4.44) and (13.9.28) is that the former no longer has $\mathbb{R}$-translational symmetry, so we consider the moduli space $\mathcal{M}(p, q'; C)$ of solutions itself in counting the number of trajectories. In particular, the corresponding map will have degree 0.

For the study of the compactness property of $\mathcal{M}(p, q'; C)$ it still remains to check the energy estimate of the solutions. For this energy estimate, we follow the approach taken in (FOOO13).

For simplicity of exposition, we first assume

$$L'_0 = \phi_H^1(L_0), \quad L'_1 = L_1.$$

We will construct the chain map (14.4.42) in two stages by introducing another intermediate chain complex, denoted by $CF(L_0, L_1; H)$,

$$\begin{tikzcd} CF(L_0, L_1) \arrow[rr] \arrow[dr, "(\mathrm{I})"] & & CF(\phi_H^1(L_0), L_1) \\ & CF(L_0, L_1; H) \arrow[ur, "(\mathrm{II})"] & \end{tikzcd} \tag{14.4.49}$$

14.4.1 The geometric version versus the dynamical version

We start with construction of the map (II). This map is essentially some kind of *coordinate change* similar to the one used in (Oh97b). (See Section 11.2 too.) In particular, this map is induced by the bijective map

$$\mathfrak{g}^+_{H;0} : \widetilde{\Omega}(\phi^1_H(L_0), L_1; \ell'_a) \to \widetilde{\Omega}(L_0, L_1; \ell_a); \quad [\ell', w'] \mapsto [\ell, w]$$

whose explanation is in order. In particular, the map (II) preserves the level of the relevant action values modulo a global shift by a constant.

For the given pair (L_0, L_1) of compact Lagrangian submanifolds and a Hamiltonian H, we also consider a family $J^s = \{J^s_t\}_{0\le t\le 1}$ of ω-compatible almost-complex structures. We would like to compare the *geometric* version of Floer theory and the *dynamical* one.

The geometric version of the Floer complex for $(L'_0 = \phi^1_H(L_0), L'_1 = L_1)$ is generated by the intersection points

$$\phi^1_H(L_0) \cap L_1$$

and its Floer boundary map is constructed by the moduli space of the genuine Cauchy–Riemann equation associated with $J' = \{J_t{}'\}$

$$\begin{cases} \partial u'/\partial\tau + J'\,\partial u'/\partial t = 0, \\ u'(\tau, 0) \in \phi^1_H(L_0), \quad u'(\tau, 1) \in L_1. \end{cases} \tag{14.4.50}$$

Here $J'_t = (\phi^1_H(\phi^t_H)^{-1})_* J_t$. We denote by $\mathcal{M}(\phi^1_H(L_0), L_1; J')$ the moduli space of finite-energy solutions of this equation. Owing to the multi-valuedness of the associated action functional, we need to consider these equations on the Novikov covering space of some specified connected component

$$\Omega(\phi^1_H(L_0), L_1; \ell'_a)$$

with the base path $\ell'_a \in \Omega(\phi^1_H(L_0), L_1)$, which is given by

$$\ell'_a(t) = \phi^1_H(\phi^t_H)^{-1}(\ell_a(t)), \tag{14.4.51}$$

and consider the action functional (this is the negative of the action functional defined in Section 13.9 and consistent with the sign of (FOOO09))

$$\mathcal{A}_{\ell'_a}([\ell', w']) = \int (w')^*\omega, \tag{14.4.52}$$

where $[\ell', w'] \in \widetilde{\Omega}(\phi^1_H(L_0), L_1; \ell'_a)$ and $w' : [0,1]^2 \to X$ is a map satisfying

$$w'(0,t) = \ell'_a(t),\ w'(1,t) = \ell'(t),\ w'(s,0) \in \phi^1_H(L_0),\ w'(s,1) \in L_1.$$

On the other hand, the dynamical version of the Floer complex is generated by the solutions of Hamilton's equation

$$\dot{x} = X_H(t, x), \quad x(0) \in L_0,\ x(1) \in L_1 \tag{14.4.53}$$

and its boundary map is constructed by the moduli space of the perturbed Cauchy–Riemann equation

$$\begin{cases} \partial u/\partial\tau + J(\partial u/\partial t - X_H(u)) = 0, \\ u(\tau, 0) \in L_0, \quad u(\tau, 1) \in L_1. \end{cases} \tag{14.4.54}$$

We denote by $\mathcal{M}(L_1, L_0; H; J)$ the moduli space of finite-energy solutions of this equation. The action functional $\mathcal{A}_{H,\ell_a}$ is defined by

$$\mathcal{A}_{H,\ell_a}([\ell, w]) = \int w^*\omega + \int_0^1 H(t, \ell(t))dt \tag{14.4.55}$$

on $\widetilde{\Omega}(L_0, L_1; \ell_a)$.

These two Floer theories are related by the following transformations of the bijective map

$$\mathfrak{g}^+_{H;0} : \widetilde{\Omega}(\phi^1_H(L_0), L_1; \ell'_a) \to \widetilde{\Omega}(L_0, L_1; \ell_a); \quad [\ell', w'] \mapsto [\ell, w]$$

given by the assignment

$$\ell(t) = \phi^t_H(\phi^1_H)^{-1}(\ell'(t)), \quad w(s, t) = \phi^t_H(\phi^1_H)^{-1}(w'(s, t)). \tag{14.4.56}$$

(See Lemma 12.7.1 for a similar coordinate-change map.) This provides a bijective correspondence of the critical points

$$\text{Crit}\, \mathcal{A}_{\ell'_a} \longleftrightarrow \text{Crit}\, \mathcal{A}_{H,\ell_a}; \quad [p', w'] \mapsto [z^H_{p'}, w] \tag{14.4.57}$$

with $p' \in \phi^1_H(L_0) \cap L_1$, $z^H_{p'}(t) := \phi^t_H(\phi^1_H)^{-1}(p')$ and $w = \phi^t_H(\phi^1_H)^{-1}(w'(s, t))$, and of the moduli spaces

$$\mathcal{M}(\phi^1_H(L_0), L_1; J') \mapsto \mathcal{M}(L_0, L_1; H; J)$$

with $J_t = (\phi^t_H(\phi^1_H)^{-1})_* J'_t$, where the map is defined by

$$u(\tau, t) = \phi^t_H(\phi^1_H)^{-1}(u'(\tau, t)).$$

The map $\mathfrak{g}^+_{H;0}$ also preserves the action up to a constant in the following sense.

Lemma 14.4.2 *Denote*

$$c(H; \ell_a) := \int_0^1 H(t, \ell_a(t))dt$$

which is a constant depending only on H and the base path ℓ_a of the connected component $\Omega(L_0, L_1; \ell_a)$. Then

$$\mathcal{A}_{H,\ell_a} \circ \mathfrak{g}^+_{H;0}([\ell', w']) = \mathcal{A}_{\ell'_a}([\ell', w']) + c(H; \ell_a) \tag{14.4.58}$$

on $\widetilde{\Omega}(L_0, L_1; \ell_a)$.

Proof The proof is by a direct calculation. Let $[\ell', w'] \in \widetilde{\Omega}(\phi^1_H(L_0), L_1; \ell'_a)$. Then

$$\mathcal{A}_{H;\ell_a}(\mathfrak{g}^+_{H;0}([\ell', w'])) = \int w^*\omega + \int_0^1 H(t, \ell(t))dt$$

and

$$w^*\omega = \omega\left(\frac{\partial w}{\partial s}, \frac{\partial w}{\partial t}\right) ds \wedge dt.$$

We compute

$$\begin{aligned}
\frac{\partial w}{\partial s} &= d(\phi^t_H(\phi^1_H)^{-1})\left(\frac{\partial w'}{\partial s}\right),\\
\frac{\partial w}{\partial t} &= d(\phi^t_H(\phi^1_H)^{-1})\left(\frac{\partial w'}{\partial t}\right) + X_{H_t}(w(s,t)).
\end{aligned}$$

On substituting this into the above, we obtain

$$\begin{aligned}
\int w^*\omega &= \int_0^1 \int_0^1 \omega\left(\frac{\partial w}{\partial s}, \frac{\partial w}{\partial t}\right) ds\, dt\\
&= \int_0^1 \int_0^1 \omega\left(\frac{\partial w'}{\partial s}, \frac{\partial w'}{\partial t}\right) ds\, dt\\
&\quad + \int_0^1 \int_0^1 \omega\left(d\left(\phi^t_H(\phi^1_H)^{-1}\right)\left(\frac{\partial w'}{\partial s}\right), X_{H_t}(w(s,t))\right) ds\, dt\\
&= \int (w')^*\omega - \int_0^1 \int_0^1 dH_t(w(s,t))\left(d(\phi^t_H(\phi^1_H)^{-1})\frac{\partial w'}{\partial s}\right) ds\, dt\\
&= \int (w')^*\omega - \int_0^1 \int_0^1 \frac{\partial}{\partial s} H_t(w(s,t)) ds\, dt\\
&= \int (w')^*\omega - \int_0^1 H_t(w(1,t))\, dt + \int_0^1 H_t(w(0,t))dt.
\end{aligned}$$

On substituting this into the above definition of $\mathcal{A}_{H;\ell_a}(\mathfrak{g}^+_{H;0}([\ell', w']))$, the proof is finished. □

Remark 14.4.3 If we normalize the Hamiltonians so that $\int_X H_t\omega^n = 0$ for each t, we can take ℓ_a for each connected component of $\Omega(L_0, L_1)$ in such a way that $c(H;\ell_a) = \int_0^1 H(t,\ell_a(t))dt = 0$. It is not essential to choose ℓ_a in the same way as above. In fact, if we take a based path so that $c(H;\ell_a) \neq 0$, it suffices to include an extra term $c(H;\ell_a)$ in the energy estimate on $\Omega(L_0,L_1;\ell_a)$ when we apply the coordinate change $\mathfrak{g}^+_{H;0}$ or its inverse. Since we will consider all connected components of $\Omega(L_0,L_1)$ (see (FOOO09)), we have to add the constant $c(H;\ell_a)$ for each connected component $\Omega(L_0,L_1;\ell_a)$.

We denote by $\mathfrak{g}^-_{H;0}$ the inverse $\mathfrak{g}^-_{H;0} = (\mathfrak{g}^+_{H;0})^{-1}$. The outcome of the above discussion is that the two associated Floer cohomologies are isomorphic to each other, preserving the filtration.

So far we have moved the first argument L_0 in the pair (L_0,L_1). We can also move the second argument L_1 instead. In that case, we define the coordinate change transformation by

$$\mathfrak{g}^+_{\widetilde{H};1} : \widetilde{\Omega}(L_0,\phi^1_H(L_1);\ell'_a) \to \widetilde{\Omega}(L_0,L_1;\ell_a); \quad [\ell',w'] \mapsto [\ell,w]$$

given by the assignment

$$\ell(t) = \phi_H^{1-t}(\phi_H^1)^{-1}(\ell'(t)), \quad w(s,t) = \phi_H^{1-t}(\phi_H^1)^{-1}(w'(s,t)), \tag{14.4.59}$$

where $\widetilde{H}(t,x) := -H(1-t,x)$ is the Hamiltonian generating the latter Hamiltonian path $t \mapsto \phi_H^{1-t}(\phi_H^1)^{-1}$. This provides a bijective correspondence

$$\mathrm{Crit}\,\mathcal{A}_{\ell'_a} \longleftrightarrow \mathrm{Crit}\,\mathcal{A}_{\widetilde{H},\ell_a}$$

and the moduli spaces

$$\mathcal{M}(\phi^1_H(L_1),L_0;J') \mapsto \mathcal{M}(\widetilde{H};L_1,L_0;J)$$

with $\widetilde{J}_t = (\phi_H^{1-t}(\phi_H^1)^{-1})_*J'_t$. Here $\mathcal{M}(\widetilde{H};L_1,L_0;\widetilde{J})$ is the moduli space of solutions of

$$\begin{cases} \partial u/\partial t + \widetilde{J}\big(\partial u/\partial t - X_{\widetilde{H}}(u)\big) = 0, \\ u(\tau,0) \in L_0,\ u(\tau,1) \in L_1. \end{cases} \tag{14.4.60}$$

The action functional $\mathcal{A}_{\widetilde{H},\ell_a}$ is given by

$$\mathcal{A}_{\widetilde{H},\ell_a}([\ell,w]) = \int (\widetilde{w})^*\omega + \int_0^1 \widetilde{H}(t,\ell(t))dt. \tag{14.4.61}$$

The explicit formula of the latter correspondence is given by

$$u(\tau,t) = \phi_H^{1-t}(\phi_H^1)^{-1}(u'(\tau,t)).$$

Again the following can be proved in a similar way.

Lemma 14.4.4 *We have*

$$\mathcal{A}_{\widetilde{H},\ell_a} \circ \mathfrak{g}^{+}_{\widetilde{H};1}([\ell', w']) = \mathcal{A}_{\ell'_a}([\ell', w']) + c(\widetilde{H}; \ell_a), \tag{14.4.62}$$

where $c(\widetilde{H}; \ell_a) := \int_0^1 \widetilde{H}(t, \ell_a(t))dt$ *is a constant depending only on* H *and the base path* ℓ_a.

We denote by $\mathfrak{g}^{-}_{\widetilde{H};1}$ the inverse of $\mathfrak{g}^{+}_{\widetilde{H};1}$. This finishes the description of the map (II) in the diagram (14.4.49).

14.4.2 Deformation of Hamiltonians

Next we consider construction of the map (I) in (14.4.49). This map is nothing but the standard Floer continuation map under the homotopy of Hamiltonians from 0 to the given Hamiltonian H with the fixed Lagrangian boundaries.

We consider the perturbed Cauchy–Riemann equation

$$\begin{cases} \partial u/\partial\tau + J(\partial u/\partial t - \rho(\tau) X_H(u)) = 0, \\ u(\tau, 0) \in L_0,\ u(\tau, 1) \in L_1 \end{cases} \tag{14.4.63}$$

with the finite energy $E_{(J,H,\rho)}(u) < \infty$. The following a-priori energy bound is a key ingredient in relation to the lower bound of the displacement energy.

Lemma 14.4.5 *Let* $\rho = \rho_+$ *as in* (14.4.43). *Let* u *be any finite-energy solution of* (14.4.63). *Then we have*

$$\begin{aligned} E_{(J,H,\rho)}(u) = \int u^*\omega + \int_0^1 H(t, u(\infty, t))dt \\ - \int_{-\infty}^{\infty} \rho'(\tau) \int_0^1 (H_t \circ u)dt\, d\tau. \end{aligned} \tag{14.4.64}$$

Proof The proof will be carried out by an explicit calculation, which is somewhat similar to that of Lemma 14.4.2. We compute

$$\begin{aligned} E_{(J,H,\rho)}(u) &= \int_{-\infty}^{\infty} \int_0^1 \left|\frac{\partial u}{\partial\tau}\right|_J^2 dt\, d\tau \\ &= \int_{-\infty}^{\infty} \int_0^1 \omega\left(\frac{\partial u}{\partial\tau}, J\frac{\partial u}{\partial\tau}\right) dt\, d\tau \\ &= \int_{-\infty}^{\infty} \int_0^1 \omega\left(\frac{\partial u}{\partial\tau}, \frac{\partial u}{\partial t} - \rho(\tau) X_{H_t}(u)\right) dt\, d\tau \\ &= \int_{-\infty}^{\infty} \int_0^1 \omega\left(\frac{\partial u}{\partial\tau}, \frac{\partial u}{\partial t}\right) dt\, d\tau - \int_{-\infty}^{\infty} \rho(\tau) \int_0^1 \omega\left(\frac{\partial u}{\partial\tau}, X_{H_t(u)}\right) dt\, d\tau \end{aligned}$$

$$= \int u^*\omega - \int_{-\infty}^{\infty} \rho(\tau) \int_0^1 \left(-dH_t(u)\frac{\partial u}{\partial \tau}\right) dt\, d\tau$$
$$= \int u^*\omega + \int_{-\infty}^{\infty} \rho(\tau) \int_0^1 \frac{\partial}{\partial \tau}(H_t \circ u) dt\, d\tau$$
$$= \int u^*\omega + \int_0^1 H(t, u(\infty, t)) dt - \int_{-\infty}^{\infty} \rho'(\tau) \int_0^1 (H_t \circ u) dt\, d\tau.$$

Here, in the last equality, we do integration by parts and use the facts that $\rho(\infty) = 1$ and $\rho(-\infty) = 0$. This finishes the proof of (14.4.64). □

This lemma gives rise to the following key formula for the action difference:

$$\mathcal{A}_{H,\ell_a}(u(\infty)) - \mathcal{A}_{\ell_a}(u(-\infty)).$$

Proposition 14.4.6 *Let $p \in L_0 \cap L_1$ and $q' \in \phi_H^1(L_0) \cap L_1$. Denote by $z_{q'}^H \in \Omega(L_0, L_1; \ell_a)$ the Hamiltonian trajectory defined by*

$$z_{q'}^H(t) = \phi_H^t((\phi_H^1)^{-1}(q'))$$

as before and consider $[\ell_p, w] \in \mathrm{Crit}\, \mathcal{A}_{\ell_a}$, $[z_{q'}^H, w'] \in \mathrm{Crit}\, \mathcal{A}_{H,\ell_a}$. Suppose that u is a finite-energy solution of (14.4.63) *with $\rho = \rho_+$ as in* (14.4.43) *satisfying the asymptotic condition and homotopy condition*

$$u(-\infty) = \ell_p, \quad u(\infty) = z_{q'}^H, \quad w\#u \sim w'. \tag{14.4.65}$$

Then we have

$$\mathcal{A}_{H,\ell_a}([z_{q'}^H, w']) - \mathcal{A}_{\ell_a}([w, \ell_p]) = E_{(J,H,\rho)}(u) + \int_{-\infty}^{\infty} \int_0^1 \rho'(\tau) H(t, u(\tau, t)) dt\, d\tau. \tag{14.4.66}$$

Proof By (14.4.65), we obtain

$$\int u^*\omega = \int (w')^*\omega - \int w^*\omega.$$

By substituting this into (14.4.64) and rearranging the resulting formula, we obtain (14.4.66) from the definitions (14.4.52) of $\mathcal{A}_{\ell_a}$ and (14.4.55) of $\mathcal{A}_{H,\ell_a}$. □

An immediate corollary follows.

Corollary 14.4.7 *Let $\rho = \rho_+$ be as in Proposition 20.2.9.*

(1) *For any solution u of* (14.4.63)*, we have the energy bound*

$$E_{(J,H,\rho)}(u) \leq \mathcal{A}_{H,\ell_a}([z_{q'}^H, w']) - \mathcal{A}_{\ell_a}([\ell_p, w]) + \int_0^1 \max H_t\, dt. \tag{14.4.67}$$

(2) *Suppose that there is a solution u of* (14.4.63). *Then we have the estimate of the level changes*

$$\mathcal{A}_{H,\ell_a}([z_{q'}^H, w']) - \mathcal{A}_{\ell_a}([\ell_p, w]) \geq \int_0^1 \min H_t\, dt = -E^-(H). \quad (14.4.68)$$

Similarly, for $\rho = \rho_- = 1 - \rho_+$,

$$E_{(J,H,\rho)}(u) \leq \mathcal{A}_{H,\ell_a}([z_{q'}^H, w']) - \mathcal{A}_{\ell_a}([\ell_p, w]) + \int_0^1 (-\min H_t) dt.$$

for any solution u of (14.4.63) *for* $\rho = \rho_- = 1 - \rho_+$, *and the level change estimate*

$$\mathcal{A}_{\ell_a}([\ell_q, w]) - \mathcal{A}_{H,\ell_a}([z_{p'}^H, w']) \geq \int_0^1 (-\max H_t) dt = -E^+(H), \quad (14.4.69)$$

provided that there exists a solution u of (14.4.63).

Next let us concatenate the two-equation (14.4.63) for ρ_+ as in (14.4.43) and $\rho_- = 1 - \rho_+$ by considering a one-parameter family of elongation functions of the type (this is the same kind of function as (11.2.15))

$$\rho_K = \begin{cases} \rho_+(\cdot + K) & \text{for } \tau \leq -K + 1, \\ \rho_-(\cdot - K + 1) & \text{for } \tau \geq K - 1, \\ 1 & \text{for } |\tau| \leq K - 1 \end{cases} \quad (14.4.70)$$

for $1 \leq K \leq \infty$ and further deforming $\rho_{K=1}$ further down to $\rho_{K=0} \equiv 0$.

Proposition 14.4.8 *Let u be a finite-energy solution for* (14.4.63) *of the elongation* ρ_K *with asymptotic condition*

$$u(-\infty) = \ell_p,\ u(\infty) = \ell_q,\ w_- \# u \sim w_+.$$

Then we have

$$\mathcal{A}_{\ell_a}([\ell_q, w_+]) - \mathcal{A}_{\ell_a}([\ell_p, w_-]) \geq -(E^-(H) + E^+(H)) = -\|H\|, \quad (14.4.71)$$

$$E_{(J,H;\rho_K)}(u) \leq \int u^*\omega + \|H\|. \quad (14.4.72)$$

So far we have moved the first argument L_0 in the pair (L_0, L_1). When we move the second argument L_1 instead, the only difference occurring in the above discussion will be the interchange

$$-\min H \longleftrightarrow \max H.$$

14.4.3 Simultaneous Hamiltonian isotopy of (L_0, L_1)

Now we move L_0 and L_1 simultaneously by Hamiltonian isotopies $\phi^t_{H^{(0)}}$ and $\phi^t_{H^{(1)}}$, respectively:

$$(L_0, L_1) \mapsto (L'_0 = \phi^1_{H^{(0)}}(L_0), L'_1 = \phi^1_{H^{(1)}}(L_1)).$$

Then we have the following bijection:

$$\mathfrak{g}^+_{H^{(0)},H^{(1)}} : (\ell'', w'') \in \widetilde{\Omega}(L'_0, L'_1) \mapsto (\ell, w) \in \widetilde{\Omega}(L_0, L_1),$$

where

$$\ell(t) = \phi^{1-t}_{H^{(1)}} \circ (\phi^1_{H^{(1)}})^{-1} \circ \phi^t_{H^{(0)}} \circ (\phi^1_{H^{(0)}})^{-1}(\ell''(t))$$

and

$$w(s,t) = \phi^{1-t}_{H^{(1)}} \circ (\phi^1_{H^{(1)}})^{-1} \circ \phi^t_{H^{(0)}} \circ (\phi^1_{H^{(0)}})^{-1}(w''(s,t)).$$

We write $\mathfrak{g}^-_{H^{(0)},H^{(1)}} = (\mathfrak{g}^+_{H^{(0)},H^{(1)}})^{-1}$. By an abuse of notation, we also denote by $\mathfrak{g}^\pm_{H^{(0)},H^{(1)}}$ the bijection between the path spaces $\Omega(L_0, L_1)$ and $\Omega(L'_0, L'_1)$. Then we obtain the following improved energy estimate. Here we take the base path ℓ_a in such a way that

$$c(\widehat{H}; \ell_a) = \int_0^1 \widehat{H}(t, \ell_a(t))dt = 0 \tag{14.4.73}$$

as in Remark 14.4.3. Here $\widehat{H}$ is the normalized Hamiltonian generating

$$\phi^{1-t}_{H^{(1)}} \circ (\phi^1_{H^{(1)}})^{-1} \circ \phi^t_{H^{(0)}}.$$

The Hamiltonian $\widehat{H}$ is explicitly written as

$$\widehat{H}(t,x) = -H^{(1)}(1-t,x) + H^{(0)}(t, (\phi^{1-t}_{H^{(1)}} \circ (\phi^1_{H^{(1)}})^{-1})^{-1}(x)). \tag{14.4.74}$$

For $v : [0,1] \times [0,1] \to X$, we put

$$(\Phi_{H^{(0)},H^{(1)}} v)(s,t) = \phi^1_{H^{(0)}} \circ (\phi^t_{H^{(0)}})^{-1} \circ \phi^1_{H^{(1)}} \circ (\phi^{1-t}_{H^{(1)}})^{-1} v(s,t).$$

By using the expression (14.4.74) for $\widehat{H}$, we find that

$$E^-(\widehat{H}) \le E^-(H^{(0)}) + E^+(H^{(1)}), \quad E^+(\widehat{H}) \le E^+(H^{(0)}) + E^-(H^{(1)}). \tag{14.4.75}$$

Recall that we have chosen ℓ_a such that (14.4.73) is satisfied and put $\ell''_a = \mathfrak{g}^-_{H^{(0)},H^{(1)}}(\ell_a)$.

Proposition 14.4.9 *Let (L_0, L_1) be a pair of compact Lagrangian submanifolds and $(L^{(0)\prime}, L^{(1)\prime})$ another pair with*

$$L^{(0)\prime} = \phi^1_{H^{(0)}}(L_0), \quad L^{(1)\prime} = \phi^1_{H^{(1)}}(L_1)$$

and let $H^{(0)}$ *and* $H^{(1)}$ *be the normalized Hamiltonians generating* $\phi^1_{H^{(0)}}$ *and* $\phi^1_{H^{(1)}}$*, respectively. Consider a pair* $[\ell_p, w] \in \widetilde{\Omega}(L_0, L_1; \ell_a)$ *and* $[\ell_{q''}, w''] \in \widetilde{\Omega}(L^{(0)\prime}, L^{(1)\prime}; \ell''_a)$ *for which there exists a solution* u *of* (14.4.63) *with* $\rho = \rho_+$ *as in* (14.4.43) *such that*

$$\lim_{\tau\to-\infty} u(\tau,\cdot) = \ell_p,\ \lim_{\tau\to+\infty} u(\tau,\cdot) = \mathfrak{g}^+_{H^{(0)},H^{(1)}}(\ell_{q''}),\ \Phi_{H^{(0)},H^{(1)}}(w\#u) \sim w''.$$

Then we have

$$\mathcal{A}_{\ell''_a}([\ell_{q''}, w'']) - \mathcal{A}_{\ell_a}([\ell_p, w]) \geq -(E^-(H^{(0)}) + E^+(H^{(1)})). \tag{14.4.76}$$

Similarly, let $[\ell_{p''}, w''] \in \widetilde{\Omega}(L^{(0)\prime}, L^{(1)\prime}; \ell''_a)$ *and* $[\ell_q, w] \in \widetilde{\Omega}(L_0, L_1; \ell_a)$*. If there exists a solution* u *of* (14.4.63) *with* $\rho = \rho_- = 1 - \rho_+$ *such that*

$$\lim_{\tau\to-\infty} u(\tau,\cdot) = \mathfrak{g}^+_{H^{(0)},H^{(1)}}(\ell_{p''}),\ \lim_{\tau\to+\infty} u(\tau,\cdot) = \ell_q,\ \Phi^{-1}_{H^{(0)},H^{(1)}}(w'')\#u \sim w,$$

we have

$$\mathcal{A}_{\ell_a}([\ell_q, w]) - \mathcal{A}_{\ell''_a}([\ell_{p''}, w'']) \geq -(E^+(H^{(0)}) + E^-(H^{(1)})). \tag{14.4.77}$$

The following proposition is parallel to Proposition 14.4.8.

Proposition 14.4.10 *Let* (L_0, L_1) *be a pair of compact Lagrangian submanifolds. If there exists a solution* u *of* (14.4.63) *with* $\rho = \rho_K$ *in* (14.4.70) *satisfying*

$$\lim_{\tau\to-\infty} u(\tau,\cdot) = \ell_p,\ \lim_{\tau\to+\infty} u(\tau,\cdot) = \ell_q,\ w_-\#u \sim w_+.$$

Then we have

$$\mathcal{A}_{\ell_a}([\ell_q, w_+]) - \mathcal{A}_{\ell_a}([\ell_p, w_-]) \geq -\left(\|H^{(0)}\| + \|H^{(1)}\|\right) \tag{14.4.78}$$

and

$$E_{(J,\widehat{H},\rho_K)}(u) \leq \int u^*\omega + \|H^{(0)}\| + \|H^{(1)}\|. \tag{14.4.79}$$

Remark 14.4.11 Estimates of the kind (14.4.79) play an important role in the study of the optimal estimate of the displacement energy of Lagrangian submanifolds. (See (Che94), (Che98), (Oh97c), (FOOO13).)

15

Off-shell analysis of the Floer moduli space

In this chapter, we will provide the *off-shell* descriptions of the Floer moduli space and of its compactification. We compute the dimension of the so-called *virtual tangent space* at a stable broken Floer trajectory $u \in \overline{\mathcal{M}}(x, y; B)$ and explain the orientation of the Floer moduli spaces and other off-shell analysis entering into the construction of Floer operators.

The virtual tangent space at a stable broken Floer trajectory will be defined to be a certain deformation complex of the linearization of the trajectory. This dimension will provide the genuine dimension of the moduli space near u when u is *regular* in a suitable sense, which we will explain in later sections.

15.1 Off-shell framework of smooth Floer moduli spaces

Let L_0, $L_1 \subset (M, \omega)$ be two compact Lagrangian submanifolds intersecting transversely. We fix a pair $(x, y) \subset L_0 \cap L_1$ and a homotopy class $B \in \pi_2(x, y)$. We will define the Banach manifold $\mathcal{F}(x, y; B)$ that hosts the nonlinear $\overline{\partial}_J$-operator. We closely follow Floer's description from (Fl88a) of the weighted Sobolev space setting.

Remark 15.1.1 For the purpose of studying the case of a transversely intersecting pair $L_0 \cap L_1$, one could mostly work with the standard $L^{k,p}$ Sobolev space, especially for the linear analysis. However, to set up the Banach manifold $\mathcal{F}(x, y; B)$, one needs to impose some decay condition at infinity to control the behavior of the map as $\tau \to \pm\infty$. Such a decay will be automatic *on shell*, i.e., for the J-holomorphic map on $\mathbb{R} \times [0, 1]$ with finite energy (see Proposition 14.1.5 for its proof), but should be imposed *in the off-shell level*, i.e., for the general map $\mathcal{F}(x, y; B)$. We refer the reader to (Schw93) for another way of imposing such a decay condition.

For $k > 2/p$ and $\Theta = \mathbb{R} \times [0,1]$, we define

$$\mathcal{F}^{k,p}_{\mathrm{loc}} = \{u \in L^{k,p}_{\mathrm{loc}}(\Theta, M) \mid u(\mathbb{R}, 0) \subset L_0 \quad \text{and} \quad u(\mathbb{R}, 1) \subset L_1\},$$

where $L^{k,p}_{\mathrm{loc}}(\Theta, M)$ is the Sobolev space of maps from Θ to $M \subset \mathbb{R}^N$ defined by

$$L^{k,p}_{\mathrm{loc}}(\Theta, M) = \{u \in L^{k,p}_{\mathrm{loc}}(\Theta, \mathbb{R}^N) \mid \mathrm{Im}\, u \subset M\}.$$

(As before we fix an isometric embedding of M into $\mathbb{R}^N$ and regard a map to M as an $\mathbb{R}^N$-valued function.) Because of the Sobolev embedding $L^{k,p} \hookrightarrow C^0$, the constraint imposed on the image of maps makes sense.

Let $\rho : \mathbb{R} \to [0,1]$ be a smooth function,

$$\rho(\tau) = \begin{cases} 0 & \tau \leq 0, \\ 1 & \tau \geq 1. \end{cases} \tag{15.1.1}$$

For $l > 0$, $q > 1$, and $\sigma = (\sigma_-, \sigma_+) \in \mathbb{R}^2$, we define the Banach spaces

$$L^{l,q}_{\sigma}(\mathbb{R}^{2n}) = \{\xi \in L^{l,q}_{\mathrm{loc}}(\Theta, \mathbb{R}^{2n}) \mid \|\xi\|_{l,q;\sigma} < \infty\}.$$

Here we have used the norms

$$\|\xi\|_{l,q;\sigma} = \|e^{\sigma}\xi\|_{l,q}$$

with $e^{\sigma}(\tau, t) = e^{(\rho(\tau)\sigma_+ + \rho(-\tau)\sigma_-)\tau}$.

Assume that M is equipped with a metric, and denote the corresponding exponential map by $\exp_x : T_x M \to M$. We would like to note that we will use the exponential map with respect to a metric that we allow to vary in a *compact* family of Riemannian metrics.

For a given $u \in \mathcal{F}^{k,p}_{\mathrm{loc}}$, we define $\widetilde{u} \in \mathcal{F}^{k,p}_{\mathrm{loc}}(L_1, L_0)$ by $\widetilde{u}(\tau, t) = u(-\tau, 1-t)$. For $x \in L_0 \cap L_1$, $k \geq 1$, $p \geq 1$ and $a \in \mathbb{R}_+$ such that $k > 2/p$, consider the set

$$\begin{aligned} \mathcal{F}^{k,p}_{a}(\cdot, x) = \{u \in \mathcal{F}^{k,p}_{\mathrm{loc}} \mid & \text{ there exist } \quad R > 0 \text{ and} \\ & \xi \in L^{p}_{k;(0,a)}(T_x M) : u(\tau, t) = \exp_x \xi(\tau, t) \text{ for } \tau > R\}. \end{aligned}$$

Moreover, for $b \in \mathbb{R}_-$, we consider the set

$$\mathcal{F}^{k,p}_{b}(x, \cdot) = \{u \mid \widetilde{u} \in \mathcal{F}^{k,p}_{-b}(L_1, L_0)(\cdot, x)\}.$$

Then, for x, $y \in L_0 \cap L_1$, we define

$$\mathcal{F}^{k,p}_{(b,a)}(x, y) = \mathcal{F}^{k,p}_{b}(x, \cdot) \cap \mathcal{F}^{k,p}_{a}(\cdot, y).$$

Clearly, for the definition of $\mathcal{F}^{p}_{k;(b,a)}(x, y)$ to make sense, it is necessary that $a \geq 0$ and $b \leq 0$. We shall denote such pairs of numbers by δ, whereas arbitrary pairs of real numbers will be denoted by σ. We shall also write

$$\mathcal{F}^{l,p}(x, y) = \mathcal{F}^{l,p}_{(0,0)}(x, y)$$

for $p > 2$. If E is a smooth vector bundle over M and $u \in \mathcal{F}^{k,p}_{\mathrm{loc}}$ then the pull-back bundle u^*E has the structure of an $L^{k,p}$ bundle, i.e., of a locally trivial bundle with transition maps contained in $L^{k,p}(U, GL(\mathbb{R}, N))$ for an open subset $U \subset \Theta$ where $N = \operatorname{rank} E$. We can therefore define the set of sections of u^*E that are locally of type $L^{k,p}$. If $u \in \mathcal{P}^p_{k;\delta}(x,y)$, then we can also consider weighted Sobolev norms $\|\xi\|_{l,q;\sigma}$ for sections, provided that

$$l \le k, \quad q - 2/l \le p - 2/k.$$

Here $\sigma \in \mathbb{R}^2$ is arbitrary. We can therefore define

$$L^{l,q}_\sigma(u^*E) = \{\xi \in L^{l,q}_{\mathrm{loc}}(u^*E) \mid \|\xi\|_{l,q;\sigma} < \infty\}. \tag{15.1.2}$$

In particular, we are interested in the Banach spaces

$$L^{l,q}_\sigma(u) = L^{l,q}_\sigma(u^*TM).$$

In this case, for $l > 2/q$, any element $\xi \in L^{l,q}_{\mathrm{loc}}(u^*E)$ restricts to a continuous section on $\partial\Theta = \mathbb{R} \times \{0,1\}$ by the trace theorem (see pp. 257–261 in (Ev98)). Therefore, if $l > 2/q$, we can consider the subspace

$$W^{l,q}_\sigma(u) := \{\xi \in L^{l,q}_\sigma(u) \mid \forall\tau \in \mathbb{R},\ \xi(\tau,0) \in T_{u(\tau,0)}L_0,\ \xi(\tau,1) \in T_{u(\tau,1)}L_1\}.$$

Furthermore, by exploiting the presence of a global coordinate (τ, t) on Θ we identify $\overline{\partial}_J u$ with the section $\overline{\partial}_J u(\partial/\partial\tau)$ of $L^{l,q}_\sigma(u)$. Then

$$\overline{\partial}_J u\left(\frac{\partial}{\partial\tau}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial\tau} + J(u)\frac{\partial u}{\partial t}\right),$$

where we denote

$$du\left(\frac{\partial}{\partial\tau}\right) = \frac{\partial u}{\partial\tau}, \qquad du\left(\frac{\partial}{\partial t}\right) = \frac{\partial u}{\partial t}.$$

With these understood, we will just denote by $\overline{\partial}_J$ the nonlinear operator

$$u \mapsto \frac{\partial u}{\partial\tau} + J(u)\frac{\partial u}{\partial t}$$

in the discussion of this section and the next, ignoring the coefficient $\frac{1}{2}$.

Theorem 15.1.2 *Suppose that L_0 intersects L_1 transversely. For any $p \ge 1$, $k > 2/p$, and for any $x, y \in L_0 \cap L_1$, there exists a dense set of $\delta = (a,b) \in [0,\infty) \times (-\infty, 0]$, depending only on (x,y) such that the following statements hold.*

(1) *The dense set contains $[0,a_0) \times (b_0, 0]$ for some $a_0, -b_0 > 0$ depending only on x and y, respectively.*

(2) *The set* $\mathcal{F}^{k,p}_\delta(x,y)$ *is a smooth Banach manifold with tangent spaces*

$$T_u\mathcal{F}^{k,p}_\delta(x,y) = W^{k,p}_\delta(u).$$

(3) *For* $l \in [0,k]$ *and* $q \geq 1$ *such that* $l - 2/q \leq k - 2/p$ *and for arbitrary* δ *in the above-mentioned dense subset, the Banach spaces* $W^{l,q}_\delta(u)$ *and* $L^{l,q}_\delta(u)$ *are fibers of smooth Banach space bundles* $\mathcal{W}^{l,q}_\delta$ *and* $\mathcal{L}^{l,q}_\delta$ *over* $\mathcal{F}^{k,p}_\delta(x,y)$.

(4) *The map*

$$\overline{\partial}_J : \mathcal{F}^{k,p}_\delta(x,y) \to \mathcal{L}^{k-1,p}_\delta$$

defined by the assignment $u \mapsto \overline{\partial}_J u$ *is a smooth Fredholm section of the Banach bundle* $\mathcal{L}^{k-1,p}_\delta \to \mathcal{F}^{k,p}_\delta$ *given by the formula*

$$(\overline{\partial}_J u)(\tau,t) = \frac{\partial u}{\partial \tau}(\tau,t) + J(t,u(\tau,t))\frac{\partial u}{\partial t}(\tau,t).$$

The statement and its proof of this theorem are variations of the ones given for the closed curves in Proposition 10.2.3 except for the Fredholm property stated in (4). This Fredholm property can be proved by the arguments used by Lockhart and McOwen in (LM85), explanation of which is now in order.

As long as a, $b > 0$, any element in $\mathcal{F}^{k,p}_\delta$ can be compactified as before and so defines a homotopy class $B \in \pi_2(x,y)$. From now on, we will always assume

$$(a,b) \in (0,a_x) \times (b_y, 0) \tag{15.1.3}$$

for each given pair (x,y). For a given transversal pair L_0, L_1, we choose $a_0 = a_0(L_0,L_1) > 0$ and $b_0 = b_0(L_0,L_1) < 0$ so that

$$0 < a_0 < \min_{x \in L_0 \cap L_1} a_x, \qquad 0 < -b_0 < \max_{y \in L_0 \cap L_1} (-b_y).$$

Then we set $\delta = (b_0, a_0)$ and decompose

$$\mathcal{F}^{k,p}_\delta(x,y) = \bigcup_{B \in \pi_2(x,y)} \mathcal{F}^{k,p}_\delta(x,y;B)$$

in an obvious way. Now we can restrict the above Banach bundle and sections to $\mathcal{F}^{k,p}_\delta(x,y;B)$.

Next let us relate this off-shell setting of weighted Sobolev space to the exponential decay result established for the on-shell moduli space $\mathcal{M}(x,y;B)$ given in Proposition 14.1.5. Obviously we have the inclusion

$$\widetilde{\mathcal{M}}(x,y;B) \subset \left(\overline{\partial}_J\right)^{-1}(\underline{0}),$$

where $\underline{0}$ is the zero section of the bundle $\mathcal{L}_\delta^{k-1,p} \to \mathcal{F}_\delta^{k,p}(x, y; B)$. On the other hand, the elliptic regularity theorem implies that any element in $(\overline{\partial}_J)^{-1}(\underline{0})$ is smooth and so indeed that

$$\mathcal{M}(x, y; B) = \left(\overline{\partial}_J\right)^{-1}(\underline{0}) \cap \mathcal{F}_\delta^{k,p}(x, y; B) \tag{15.1.4}$$

for any p, k and $\delta = (a, b)$ where a, $b > 0$ are sufficiently small whose smallness depends only on the pair (L_0, L_1) and J_0, or more precisely on the *Lagrangian angles* of the tangent spaces $T_x L_0$ and $T_x L_1$ in the Hermitian inner product space $(T_x M, \omega_x, J_{0,x})$ for $x \in L_0 \cap L_1$.

For every $u \in \widetilde{\mathcal{M}}(x, y; B) \subset \mathcal{P}_k^p(x, y; B)$, we can define the linearization of $\overline{\partial}_J$

$$E_u = D\overline{\partial}_J(u) : T_u\mathcal{F}_\delta^{k,p}(x, y) \to L_{k-1;\delta}^p(u),$$

where

$$(D\overline{\partial}(u)\xi)(\tau, t) = (\nabla_\tau + J\,\nabla_t)\xi(\tau, t) + B(\tau, t)\xi(\tau, t).$$

Here, B is a matrix operator, which depends on the choice of the connection ∇.

Owing to the asymptotic conditions, the operator E_u approaches translationally invariant operators of the form

$$E_\infty(\xi) = \frac{\partial}{\partial\tau} + A_\infty,$$

where A_∞ is independent of τ. Moreover, E_u is uniformly elliptic. For such 'asymptotically constant' elliptic operators, we can apply the scheme of (LM85) to $D\overline{\partial}(u)$ and prove that it is Fredholm for any $\delta = (a, b)$ with a, $-b \geq 0$ sufficiently small compared with the above-mentioned Lagrangian angles, whenever x and y are transverse intersections in $L_0 \bigcap L_1$.

Theorem 15.1.3 *Suppose L_0, L_1 are transverse. Let x, $y \in L_0 \cap L_1$ and let $\delta = (a, b)$ be as above. Then for any $u \in \widetilde{\mathcal{M}}(x, y; B)$, the linearization operator*

$$E_u = D\overline{\partial}(u) : T_u\mathcal{F}_\delta^{k,p}(x, y) \to L_{k-1;\delta}^p(u)$$

is a Fredholm operator with its index given by

$$\text{Index } E_u = \mu(x, y; B)$$

independently of $u \in \widetilde{\mathcal{M}}(x, y; B)$, where $\mu(x, y; B)$ is the Maslov–Viterbo index given in Definition 13.6.2 (Vi88).

We will give a proof of the index formula in Section 15.3.

The proof of the following proposition is a variation of that of Theorem 10.4.1, but in the current case the somewhere-injectivity assumption will not be imposed but automatically follows by virtue of the following structure theorem whose proof is rather technical, so the reader is referred to the original article for its proof. (See also (FHS95) for the version for the case of Hamiltonian Floer trajectories.)

Lemma 15.1.4 (Theorem 5.1 (Oh97a)) *Let $u \in \widetilde{\mathcal{M}}(x, y)$. Then the set*

$$\Theta_u := \left\{ (\tau, t) \in \Theta \,\middle|\, \frac{\partial u}{\partial \tau}(\tau, t) \neq 0,\ u(\tau, t) \notin u(\mathbb{R} \setminus \{\tau\}, t) \right\}$$

is open and dense in Θ.

We will just briefly indicate the difference of the proof of the following proposition from that of Theorem 10.4.1, leaving the details of a complete proof to the reader.

Proposition 15.1.5 *Let $J_0, J_1 \in \mathcal{J}_\omega$ be given. Suppose that either $p \neq q$ or $p = q$ and $B \neq 0$. Then there exists a dense subset $j^{\text{reg}}_{(J_0,J_1)} \subset j_{(J_0,J_1)}$ such that $\widetilde{\mathcal{M}}(p, q; B; J)$ is Fredholm-regular and hence a smooth manifold of dimension $\mu(B)$.*

Proof Consider the Cauchy–Riemann section

$$\overline{\partial} : (u, J) \mapsto \overline{\partial}_J u$$

and its linearization $D\overline{\partial}(u, J)$, which has the form

$$D_{(u,J)}\overline{\partial}(\xi, Y) = D_u\overline{\partial}_J(\xi) + \frac{1}{2} Y\, du \circ j,$$

where $\xi \in W^{1,p}(u^*TM)$ with boundary condition

$$\xi(\tau, 0) \in T_{u(\tau,0)}L_0, \quad \xi(\tau, 1) \in T_{u(\tau,1)}L_1$$

and $Y \in C^\infty(\text{End}(TM))$ satisfying

$$Y \cdot J + J \cdot Y = 0.$$

To prove surjectivity, we will prove that $\text{Coker}\, D_{(u,J)}\overline{\partial} = \{0\}$. Using the canonical pairing between L^p and L^q with $1/p + 1/q = 1$, a cokernel element $\eta \in L^q(u^*TM)$ is characterized by the equations

$$\int_{\mathbb{R}\times[0,1]} \left\langle D_{(u,J)}\overline{\partial}(\xi, Y), \eta \right\rangle = 0 \tag{15.1.5}$$

for all $\xi \in W^{1,p}(u^*TM)$ and $Y \in C^\infty(\mathrm{End}(TM))$. Considering the case $Y \equiv 0$, we obtain the equation

$$(D_u\overline{\partial}_J)^\dagger \eta = 0, \tag{15.1.6}$$

where $(D_u\overline{\partial}_J)^\dagger : L^q(u^*TM) \to (W^{1,p}(u^*TM))^*$ is the adjoint to $D_u\overline{\partial}_J$.

In particular, the unique continuation holds for a solution to (15.1.6) and hence it suffices to prove that η vanishes on some open subset of $\mathbb{R} \times [0,1]$. Here enters the part of (15.1.6) with $\xi \equiv 0$, which becomes

$$\int_{\mathbb{R}\times[0,1]} \langle Y \circ j, \eta \rangle = 0$$

for all $Y \in C^\infty(\mathrm{End}(TM))$. By a structure theorem in Lemma 15.1.4 above, we find an immersed point with multiplicity one u in $\mathbb{R} \times [0,1]$. Then the same argument as in the proof of Theorem 10.4.1 proves that there is an open neighborhood of the point on which η must vanish. This finishes the proof. □

15.2 Off-shell description of the cusp-trajectory spaces

Next we give an off-shell description of the cusp-trajectory moduli space

$$\begin{aligned}\widetilde{\mathcal{M}}\Big(p,q;B_0;\{\alpha_i\},\{\beta_{0,j}\},\{\beta_{1,k}\}\Big) &:= \widetilde{\mathcal{M}}_{m;n_0,n_1}(p,q;B_0)_{\mathrm{ev}} \\ &\times_{\mathrm{ev}} \Big(\Pi_i \mathcal{M}_1(J_{(\tau_i,t_i)};\alpha_i) \times \Pi_j \mathcal{M}_1(L,J_0;\beta_{0,j}) \times \Pi_k \mathcal{M}_1(L,J_1;\beta_{1,k})\Big).\end{aligned} \tag{15.2.7}$$

As in Section 15.1, we will find the off-shell analog of this fiber product. For the first factor $\widetilde{\mathcal{M}}_{(m;n_0,n_1)}(p,q;B_0)$, we consider the product

$$\mathcal{F}^{k,p}_{\delta;(m;n_0,n_1)}(x,y;B) = \mathcal{F}^{k,p}_{\delta}(x,y;B) \times \Theta^m \times (\partial_0\Theta)^{n_0} \times (\partial_1\Theta)^{n_1}.$$

Besides the Cauchy–Riemann section $\overline{\partial}_J$, we also have the evaluation map

$$\mathrm{ev}_{\mathcal{F}} : \mathcal{F}^{k,p}_{\delta;(m;n_0,n_1)}(x,y;B) \to M^m \times L_0^{n_0} \times L_1^{n_1}$$

defined by

$$\mathrm{ev}_{\mathcal{F}}(u,\{(\tau_i,t_i)\},\{\tau_{0,j}\},\{\tau_{1,k}\}) = (\{u(\tau_i,t_i)\},\{u(\tau_{0,j},0)\},\{u(\tau_{1,k},1)\}). \tag{15.2.8}$$

For the discussion on the off-shell description, we will consider smooth maps and smooth sections. Appropriate completion of the corresponding spaces will be left to the reader.

For the second factor $\Pi_i \mathcal{M}_1(J_{(\tau_i,t_i)};\alpha_i)$, we consider the parameterized moduli space

$$\mathcal{M}_1(J,\alpha;\text{para}) := \bigcup_{(s,t)\in[0,1]^2} \{(s,t)\} \times \mathcal{M}_1(J_{(s,t)},\alpha).$$

The off-shell counterpart of the moduli space $\widetilde{\mathcal{M}}_1(J,\alpha;\text{para})$ is

$$\mathcal{F}_1(\alpha;\text{para}) := W^{k,p}(\Sigma, M;\alpha)\times\Sigma\times[0,1]^2.$$

We have the evaluation map

$$\mathrm{ev}^S : \mathcal{F}_1(\alpha;\text{para}) \to M\times[0,1]^2$$

defined by

$$\mathrm{ev}^S(v,z,s,t) = (v(z),s,t) \tag{15.2.9}$$

and the parameterized Cauchy–Riemann section

$$\overline{\partial}^{\text{para}} : \mathcal{F}_1(\alpha;\text{para}) \to \mathcal{L}$$

defined by

$$\overline{\partial}^{\text{para}}(v,z,s,t) = \overline{\partial}_{J_{(s,t)}}(v) \in W^{k-1,p}(v^*TM).$$

Here $\mathcal{F}_1^{(k,p)}(\alpha;\text{para})$ is defined to be

$$\mathcal{F}_1(\alpha;\text{para}) = W^{k,p}(\Sigma\times[0,1]^2, M;\alpha)\times\Sigma$$

and $C^\infty(\Sigma\times[0,1]^2, M;\alpha)$ to be the product

$$W^{k,p}(\Sigma, M;\alpha)\times[0,1]^2.$$

For the third and fourth factors of (15.2.7), we consider the moduli space $\widetilde{\mathcal{M}}_1(L,J_0;\beta)$ and $\mathcal{M}_1(L,J_0;\beta)$. The off-shell counterpart of $\widetilde{\mathcal{M}}_1(L,J_0;\beta)$ is given by

$$W^{k,p}((D^2,\partial D^2),(M,L))\times\partial D^2$$

and the evaluation map

$$\mathrm{ev}^D : W^{k,p}((D^2,\partial D^2),(M,L))\times\partial D^2 \to L;\quad \mathrm{ev}^D(w,\theta) = w(\theta).$$

Upon combining the above evaluation maps, we obtain

$$\mathrm{ev} : \Pi_i\mathcal{F}_1(J,\alpha_i)\times\Pi_j\mathcal{F}_1(L_0,J_0;\beta_{0,j})\times\Pi_k\mathcal{F}_1(L_1,J_1;\beta_{1,k}) \to M^m\times L_0^{n_0}\times L_1^{n_1}, \tag{15.2.10}$$

where ev is the combined evaluation map

$$\mathrm{ev} = (\Pi_i\mathrm{ev}_i^S, \Pi_j\mathrm{ev}_j^D, \Pi_k\mathrm{ev}_k^D).$$

On the other hand, we have the combined Cauchy–Riemann section

$$\begin{aligned}\overline{\partial}_J : \Pi_i\mathcal{F}_1(J,\alpha_i) \times \Pi_j\mathcal{F}_1(L_0,J_0;\beta_{0,j}) \times \Pi_k\mathcal{F}_1(L_1,J_1;\beta_{1,k}) \to \\ \mathcal{L}(x,y;B) \oplus \bigoplus_i \mathcal{L}(\alpha_i) \oplus \bigoplus_j \mathcal{L}(L_0,\beta_{0,j}) \oplus \bigoplus_k \mathcal{L}(L_1,\beta_{(1,k)})\end{aligned} \tag{15.2.11}$$

which is the direct sum

$$\overline{\partial}_J = \left(\overline{\partial}_J,\ \oplus\overline{\partial}^{\text{para},J},\ \oplus_j\overline{\partial}_{J_0},\ \oplus_k\overline{\partial}_{J_1}\right).$$

Then we have the following identity:

$$\widetilde{\mathcal{M}}_{(m;n_0,n_1)}(x,y;B_0;\{\alpha_i\},\{\beta_{0,j}\},\{\beta_{1,k}\}) = \left(\overline{\partial}_J\right)^{-1}(0) \cap (\mathrm{ev}_{(m;n_0,n_1)},\mathrm{ev})^{-1}(\Delta). \tag{15.2.12}$$

Now we describe the *virtual tangent space* of

$$\widetilde{\mathcal{M}}_{(m;n_0,n_1)}\left(x,y;B_0;\{\alpha_i\},\{\beta_{0,j}\},\{\beta_{1,k}\}\right).$$

Let

$$\begin{aligned}\mathfrak{u} &= (u,\{v_i\},\{w_{0,j}\},\{w_{1,k}\};\{(\tau_i,t_i)\},\{(\tau_j,0)\},\{(\tau_k,1)\}) \\ &\in \widetilde{\mathcal{M}}_{(m;n_0,n_1)}\left(x,y;B_0;\{\alpha_i\},\{\beta_{0,j}\},\{\beta_{1,k}\}\right).\end{aligned}$$

Motivated by the above fiber-product description thereof, we consider the linearization of the map

$$(\overline{\partial}_J,(\mathrm{ev}_{(m;n_0,n_1)},\mathrm{ev}))$$

at $\mathfrak{u}$ acting on

$$\mathcal{F}^{k,p}_{\delta;(m;n_0,n_1)}(x,y;B) \times \Pi_i\mathcal{F}_1(J,\alpha_i) \times \Pi_j\mathcal{F}_1(L_0,J_0;\beta_{0,j}) \times \Pi_k\mathcal{F}_1(L_1,J_1;\beta_{1,k}).$$

The corresponding tangent space of the latter at $\mathfrak{u}$ is the direct sum

$$\begin{aligned}&W^{k,p}_{\delta,(m;n_0,n_1)}(u^*TM;(\partial_0 u)^*TL_0,(\partial_1 u)^*TL_1) \\ &\quad\oplus (W^{k,p}_1(v_i^*TM))^{\oplus m} \oplus (W^{k,p}_1(w_0^*TM,(\partial w_0)^*TL_0))^{\oplus n_0} \\ &\quad\oplus (W^{k,p}_1(w_1^*TM,(\partial w_1)^*TL_1))^{\oplus n_1}.\end{aligned}$$

Here we have denoted

$$\begin{aligned}&W^{k,p}_{\delta,(m;n_0,n_1)}(u^*TM;(\partial_0 u)^*TL_0,(\partial_1 u)^*TL_1) \\ &\quad= W^{k,p}_{\delta}(u^*TM;(\partial_0 u)^*TL_0,(\partial_1 u)^*TL_1) \oplus ((\mathbb{R}^2)^m \oplus \mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1})\end{aligned}$$

and

$$W^{k,p}_1(w_0^*TM,(\partial w_0)^*TL_0) = W^{k,p}(w_0^*TM,(\partial w_0)^*TL_0) \oplus T\,\partial D^2,$$

and similarly for $W^{k,p}_1(w_1^*TM,(\partial w_1)^*TL_1)$ and more easily for $W^{k,p}_1(v_i^*TM)$.

Then we have the linear map

$$E^1_{\mathfrak{u}} : W^{k,p}_{\delta,(m;n_0,n_1)}(u^*TM; (\partial_0 u)^*TL_0, (\partial_1 u)^*TL_1) \to$$
$$L^{k,p}_\delta(u^*TM) \oplus \left((u^*TM)^{\oplus m} \bigoplus ((\partial_0 u)^*TL_0)^{\oplus n_0} \bigoplus ((\partial_1 u)^*TL_1)^{\oplus n_1}\right)$$

defined by

$$\begin{aligned} &E^1_{\mathfrak{u}}(\xi, \{a_i\}, \{b_{(0,j)}\}, \{b_{(1,k)}\}) \\ &\quad = (D\,\overline{\partial}_J(u)(\xi), \{du(\tau_i, t_i)a_i\}, \{du(\tau_j, 0)b_{0,j}\}, \{du(\tau_k, 0)b_{0,k}\}). \end{aligned}$$

We also have

$$\begin{aligned} E^2_{\mathfrak{u}} &: (W^{k,p}_1(v_i^*TM))^{\oplus m} \oplus (W^{k,p}_1(w_0^*TM, (\partial w_0)^*TL_0))^{n_0} \\ &\quad \oplus (W^{k,p}_1(w_1^*TM, (\partial w_1)^*TL_1))^{n_1} \\ &\to \bigoplus_i (C^\infty(v_i^*TM) \oplus v_i^*TM) \oplus \bigoplus_j (W^{k-1,p}(w_{0,j}^*TM) \oplus (\partial w_{0,j})^*TL_0) \\ &\quad \oplus \bigoplus_k (W^{k-1,p}(w_{1,k}^*TM) \oplus (\partial w_{1,k})^*TL_1) \end{aligned}$$

defined by the combined operator

$$\begin{aligned} E^2_{\mathfrak{u}} = (&\{(D\overline{\partial}_{J_{(\tau,t)}}(v_i), dv_i(z_i))\}, \{(D\overline{\partial}_{J_0}(w_{0,j}), dw_{0,j}(\theta_{0,j}))\}, \\ &\{(D\overline{\partial}_{J_1}(w_{1,k}), dw_{1,k}(\theta_{1,k}))\}). \end{aligned}$$

Now we recall that, $\mathfrak{u}$ being as above, we have the matching condition

$$\begin{aligned} &\mathrm{ev}_{(m;n_0,n_1)}(u, \{(\tau_i, t_i)\}, \{(\tau_j, 0)\}, \{(\tau_k, 1)\}) \\ &\quad = \mathrm{ev}(\{(v_i, z_i)\}, \{(w_{0,j}, \theta_{0,j})\}, \{(w_{1,k}, \theta_{1,k})\}), \end{aligned}$$

which is equivalent to

$$u(\tau_i, t_i) = v(z_i), \quad u(\tau_j, 0) = w_{0,j}(\theta_j), \quad u(\tau_k, 1) = w_{0,k}(\theta_k). \tag{15.2.13}$$

In particular, we have

$$\begin{aligned} T_{u(\tau_i,t_i)}M &= T_{v(z_i)}M, \\ T_{u(\tau_j,0)}L_0 &= T_{w_{0,j}(\theta_j)}L_0, \\ T_{u(\tau_k,1)}L_0 &= T_{w_{1,k}(\theta_k)}L_1. \end{aligned}$$

Therefore we can define the subspace of

$$\begin{aligned} &W^{k,p}_{\delta,(m;n_0,n_1)}(u^*TM; (\partial_0 u)^*TL_0, (\partial_1 u)^*TL_1) \\ &\quad \oplus (W^{k,p}_1(v_i^*TM))^{\oplus m} \oplus (W^{k,p}_1(w_0^*TM, (\partial w_0)^*TL_0))^{\oplus n_0} \\ &\quad \oplus (W^{k,p}_1(w_1^*TM, (\partial w_1)^*TL_1))^{\oplus n_1}, \end{aligned}$$

which consists of those satisfying

$$\begin{aligned} du(\tau_i, t_i)(\xi) - dv_i(z_i) &= 0, \\ du(\tau_j, 0)(\xi) - dw_{0,j}(\theta_j) &= 0, \\ du(\tau_k, 1)(\xi) - dw_{1,k}(\theta_k) &= 0. \end{aligned} \tag{15.2.14}$$

Now we are ready to give the definition of the *virtual tangent space* of

$$\mathcal{M}_{(m;n_0,n_1)}(x, y; B_0, \{\alpha_i\}, \{\beta_{0,j}\}, \{\beta_{1,k}\})$$

at $\mathfrak{u}$.

Definition 15.2.1 We define by

$$\text{Index}\, E_{\mathfrak{u}} = \ker E_{\mathfrak{u}} - \operatorname{coker} E_{\mathfrak{u}}$$

the virtual tangent space of $\mathcal{M}_{(m;n_0,n_1)}(x, y; B_0, \{\alpha_i\}, \{\beta_{0,j}\}, \{\beta_{1,k}\})$ at $\mathfrak{u}$.

15.3 Index calculation

In this section, we do the index calculation for the relevant Riemann–Hilbert problem.

We recall that we fixed a base path $\ell_0 : [0, 1] \to M$ and consider the based path space $\Omega(L_0, L_1; \ell_0)$. We also take a section of $\ell_0^* \Lambda^{\text{ori}}(TM)$, $t \mapsto \lambda_0(t)$, such that

$$\lambda_0(0) = T_{\ell_0(0)} L_0, \quad \lambda_0(1) = T_{\ell_0(1)} L_1.$$

For $p \in L_0 \cap L_1$, we consider a path space of Lagrangian subspaces

$$\mathcal{P}(T_p L_0, T_p L_1) = \{\lambda : [0, 1] \to \Lambda(T_p M) \mid \lambda(0) = T_p L_0,\ \lambda(1) = T_p L_1\}.$$

Using the above path λ_0 and the pair $[\ell_p, w]$, we will define a (homotopy class) of an element of $\mathcal{P}(T_p L_0, T_p L_1)$ as follows. Since $[0, 1]^2$ is contractible, we have an isomorphism $w^* TM \cong [0, 1]^2 \times T_p M$. Then we define a path λ_{w,λ_0} to be the concatenation of the paths

$$\lambda(0, t) = \lambda_0(t), \quad \lambda(s, i) = T_{w(s,i)} L_i.$$

We consider semi-strips

$$\begin{aligned} Z_- &= \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0,\ 0 \leq \operatorname{Im} z \leq 1\}, \\ Z_+ &= \{z \in \mathbb{C} \mid \operatorname{Re} z \leq 0,\ 0 \leq \operatorname{Im} z \leq 1\} \end{aligned}$$

with $z = \tau + \sqrt{-1} t$. For each path $\lambda \in \mathcal{P}(T_p L_0, T_p L_1)$, we define a Fredholm operator as follows. Let $W^{1,p}_\lambda(Z_+; T_p M)$ (respectively $W^{1,p}_\lambda(Z_-; T_p M)$) be the

Banach space consisting of $L^{1,p}_{\rm loc}$ maps $\zeta_+ : Z_+ \to T_pM$ (respectively $\zeta_- : Z_- \to T_pM$) such that

(1) $\zeta_+(\tau, i) \in T_pL_i$ (respectively $\zeta_-(\tau, i) \in T_pL_i$). Here $i = 0, 1$.
(2) $\zeta_+(0, t) \in \lambda(t)$ (respectively $\zeta_-(z_-) \in \lambda(t)$ $(i = 0, 1)$).

Let $L^p(Z_\pm; \Lambda^{0,1}(Z_\pm) \otimes T_pM)$ be the Banach space of $L^p_{\rm loc}$ sections $\zeta_\pm$ of the bundle $\Lambda^{0,1}(Z_\pm) \otimes T_pM$. The Dolbeault operator induces a bounded linear map

$$\overline{\partial}_{\lambda,Z_\pm} : W^{1,p}_\lambda(Z_\pm; T_pM) \to L^p(Z_\pm; \Lambda^{0,1}(Z_\pm) \otimes T_pM),$$

which defines a Fredholm operator.

Proposition 15.3.1 *Let $\mu([\widehat{p}, w]; (\ell_0, \lambda_0))$ be the Maslov–Morse index given in Definition 13.6.1. Then*

$$\operatorname{Index} \overline{\partial}_{(\lambda_w, Z_-)} = \mu([\widehat{p}, w]; (\ell_0, \lambda_0)). \tag{15.3.15}$$

Proof Let λ_w be the loop defined in (13.6.20) and consider the bundle pair (w^*TM, λ_w). For $(s, 0), (s, 1) \in \partial[0, 1]^2$, we denote

$$\lambda_w(s, i) = T_{w(s,i)}L_i, \quad i = 0, 1.$$

We now complete these Lagrangian paths into a bundle pair. For this purpose, we need to take a Lagrangian path α_p in $\Lambda(T_pM)$ such that

$$\alpha_p(0) = T_pL_0, \quad \alpha_p(1) = T_pL_1. \tag{15.3.16}$$

We choose a path $\alpha_p = \alpha_p(t)$ of oriented Lagrangian subspace of T_pM that extends the given orientations on the Lagrangian subspaces (15.3.16). We fix a homotopy class of α_p by putting a condition on the path α_p.

We note that under the trivialization $w^*TM \cong [0, 1]^2 \times T_pM$ the Lagrangian subbundle λ_w defines a Lagrangian loop, which we still denote by λ_w. The Maslov index of the loop is invariant under the symplectic transformations and under the homotopy. We consider homotopy of the loop that is fixed on the boundary

$$\{(1, t) \mid t \in [0, 1]\} \subset \partial[0, 1]^2.$$

We then elongate $[0, 1) \times [0, 1] \cong \mathbb{R}_+ \times [0, 1]$ by an elongation function $\rho : [0, \infty) \to [0, 1]$. Under this elongation, the operator $\overline{\partial}_{(\lambda_w;Z_-)}$ becomes cylindrical as $\tau \to \infty$ and its index is invariant under the continuous deformations of coefficients and of the boundary conditions fixing the asymptotic condition, by the homotopy invariance of the Fredholm index. We refer readers to Appendix C for the details of such a deformation argument.

Therefore it suffices to consider the following special case:

$$T_pM = \mathbb{C}^n, \qquad T_pL_0 = \mathbb{R}^n, \qquad T_pL_1 = \sqrt{-1}\mathbb{R}^n$$

and

$$\alpha_p(t) = e^{\pi\sqrt{-1}t/2}\mathbb{R}^n$$

together with the path $\lambda_w : \partial[0,1]^2 \setminus \{(1,t)\} \to \mathbb{C}^n$ given by

$$\begin{aligned}
&\lambda_w(s,0) \equiv \mathbb{R}^n,\\
&\lambda_w(s,1) \equiv \sqrt{-1}\mathbb{R}^n\\
&\lambda_w(0,t) = e^{-(l_1+1/2)\pi\sqrt{-1}t}\mathbb{R} \oplus \cdots \oplus e^{-(l_n+1/2)\pi\sqrt{-1}t}\mathbb{R}
\end{aligned}$$

with $l_j \in \mathbb{Z}$. Then we have the Lagrangian loop $\lambda_w = \lambda_w \cup \alpha_p$.

Now the index problem splits into one-dimensional problems of the types

$$\begin{cases}
\overline{\partial}\zeta = 0,\\
\zeta(\tau,0) \in \mathbb{R}, \quad \zeta(\tau,1) \in \sqrt{-1}\mathbb{R},\\
\zeta(0,t) \in e^{-(l+1/2)\pi\sqrt{-1}t}\mathbb{R},
\end{cases}$$

The corresponding Lagrangian loop λ_w defined above on $\partial[0,1]^2$ becomes

$$\begin{aligned}
(s,0) &\mapsto \mathbb{R}, \qquad (0,t) \mapsto e^{-(l+1/2)\pi\sqrt{-1}t}\mathbb{R},\\
(s,1) &\mapsto \sqrt{-1}\mathbb{R}, \qquad (1,t) \mapsto e^{\pi\sqrt{-1}t/2}\mathbb{R}.
\end{aligned}$$

This loop which has traveled in the positive direction along $\partial[0,1]^2$ is homotopic to the concatenated Lagrangian path, that is $t \in [0,1] \mapsto e^{(l+1/2)\pi\sqrt{-1}(1-t)}\mathbb{R}$ followed by $t \in [0,1] \mapsto e^{\pi\sqrt{-1}t/2}\mathbb{R}$. This concatenated path defines a loop that has the Maslov index $l+1$. By summing these, we obtain the Maslov index

$$\mu(\lambda_w) = \sum_{j=1}^{n}(l_j + 1).$$

On the other hand, for the corresponding Fredholm index, the one-dimensional problem can be explicitly solved by the complex one-variable Fourier analysis. Again we refer the reader to Proposition C.4.1 in Appendix C for the details. (See also (A.12) (Oh99).)

This proves

$$\text{Index}\,\overline{\partial}_{(\lambda_w, Z_-)} = \sum_{j=1}(l_j + 1). \tag{15.3.17}$$

This finishes the proof. □

With this index formula, we are now ready to compute the virtual dimension of the Floer moduli space.

The following is the main theorem we prove in this section.

Theorem 15.3.2 *Let $B_{ww'} \in \pi_2(p,q)$ be the homotopy class such that $w\#B_{ww'} \sim w'$. Then*

$$\dim \widetilde{\mathcal{M}}(L_0, L_1; [\ell_p, w], [\ell_q, w']) = \mu([\ell_p, w]) - \mu([\ell_q, w']) = \mu(p, q; B_{ww'}). \tag{15.3.18}$$

Proof The proof is based on the gluing formula for the Fredholm index:

$$\mathrm{Index}\, \overline{\partial}_{(\lambda_w, Z_-)} + \mathrm{Index}\, D_u\overline{\partial} = \mathrm{Index}\, \overline{\partial}_{(\lambda_{w'}, Z_-)}. \tag{15.3.19}$$

For the proof of this, we refer the reader to that of Theorem 3.10 in (APS75). Rewrite this into

$$\mathrm{Index}\, D_u\overline{\partial} = \mathrm{Index}\, \overline{\partial}_{(\lambda_{w'}, Z_-)} - \mathrm{Index}\, \overline{\partial}_{(\lambda_w, Z_-)}.$$

Then Proposition 15.3.1 gives rise to (15.3.18) and hence the proof has been completed. □

15.4 Orientation of the moduli space of disc instantons

In this section, we start with the problem of orientation in the tangent space of the *open stratum consisting of smooth pseudoholomorphic discs* with a boundary lying on a Lagrangian submanifold L. The moduli space of holomorphic maps with a Lagrangian boundary in general has no canonical orientation, whereas that of closed holomorphic maps does. The study of orientation involves a family index theory with a boundary. It turns out that the (relative) spin structure of Lagrangian submanifolds is relevant for the orientation problem (FOOO09, Sil98). An elegant exposition of the orientation in general open Gromov–Witten theory is given by Georgieva (Geo13).

In Section 15.6, we will examine the orientation problem for the smooth Floer moduli space for a pair (L_0, L_1) and then the existence of coherent orientations on the compactified moduli spaces. For the latter purpose, we will see that the (relative) spin structure of Lagrangian submanifolds plays a crucial role in the problem.

The following proposition says that a trivialization of the boundary Lagrangian determines an orientation of the tangent space.

Proposition 15.4.1 *Consider the complex bundle pair (E, λ) over $(D^2, \partial D^2)$. Suppose that λ is trivial. Then each trivialization on λ canonically induces an orientation on* $\mathrm{Index}\, \overline{\partial}_{(E,\lambda)}$.

Proof By the general construction, the moduli space $\widetilde{\mathcal{M}}(L;\beta;J)$ carries the determinant line bundle $\det(D\overline{\partial}_J) \to \widetilde{\mathcal{M}}(L;\beta;J)$ whose fiber at w is given by

$$\Lambda^{\max} \ker D_w\overline{\partial}_J \otimes (\Lambda^{\max} \operatorname{coker} D_w\overline{\partial}_J)^*.$$

When $\widetilde{\mathcal{M}}(L;\beta;J)$ is transversal, i.e., $\operatorname{coker} D_w\overline{\partial}_J = \{0\}$, we have

$$\ker D_w\overline{\partial}_J \cong T_w\widetilde{\mathcal{M}}(L;\beta;J)$$

and so an orientation on this determinant bundle provides an orientation on the moduli space. (In general, orientation of $\det(D\overline{\partial}_J)$ provides an orientation on the Kuranishi structure on $\widetilde{\mathcal{M}}(L;\beta;J)$ in the sense of (FOn99).)

We start by explaining how we associate a fiberwise orientation in terms of the geometry of Lagrangian submanifolds, especially in terms of the spin structure.

Consider the complex bundle pair (E,λ) over a compact Riemann surface Σ with nonempty boundary $\partial\Sigma$. We denote by h the number of connected components of $\partial\Sigma$ and by S_i the ith connected component of $\partial\Sigma$ for $i = 1,\ldots,h$.

We fix an annular neighborhood of each S_i and denote it by $A_i((1-\epsilon,1]) \subset D_i$ for any $0 < \epsilon < 1$. We denote the subset

$$C_i = S^1(1-\epsilon) \subset A_i((1-2\epsilon,1]).$$

We consider the contraction along $\cup_i C_i$ of Σ and the quotient space $\Sigma/\!\sim$, where $\sim$ is the identification of C_i with a point. The resulting quotient space carries a structure of the nodal curve so that the quotient map

$$cont : \Sigma \to \Sigma/\!\sim\, \cong \Sigma' \cup \left(\bigcup_i D_i\right)$$

becomes a holomorphic map, where Σ' is a compact Riemann surface *without* a boundary. Denote by $p_i \in \Sigma'$ the ith nodal point at which the center of the disc D_i is attached.

Let S be any one of the S_i. Using the given isomorphism $E|_S \cong \lambda|_S \otimes \mathbb{C}$, *each trivialization of* $\lambda \to S$ induces a complex trivialization

$$\Phi : E|_{T_{2\epsilon}} \to T_{2\epsilon} \times \mathbb{C}^n. \tag{15.4.20}$$

Therefore the bundle E over Σ descends to a bundle pair (E',λ') over the nodal curves

$$\Sigma' \cup \left(\bigcup_i D_i\right).$$

Conversely, we can resolve the nodal point to obtain a family of Riemann surfaces parameterized by $\vec{r} = (r_1, \ldots, r_h)$ with $r_i \in \mathbb{R}$ so that the bundle pair (E', λ') itself can be deformed to a bundle pair (E_r, λ_r) over $(\Sigma_r, \partial\Sigma_r)$ with a canonical identification $\lambda' \cong \lambda_r$. *This is possible because the bundle E' is already trivialized on the D_i and the identification*

$$E'_{ct,p_i} \cong E'_{\mathrm{end}_i,0} \cong \mathbb{R}^n \times \mathbb{C}$$

with $\lambda|_{\partial D_i} \cong \mathbb{R}^n$ is given, and also because we use the real *resolution parameters r.*

A section of this bundle pair is a pair $(\xi_{ct}, \xi_{\mathrm{end}})$, where $\xi_{\mathrm{end}} = (\xi_1, \ldots, \xi_h)$, and each ξ_i is a section of $E' \to D_i$ satisfying

$$\xi_i(z) \in \lambda_{D_i,z}$$

and the matching condition

$$\xi_{ct}(p_i) = \xi_i(0). \tag{15.4.21}$$

In other words, the set of $W^{1,p}$-sections of (E', λ') is given by

$$W^{1,p}(E', \lambda') := \Delta^{-1}(0),$$

where $\Delta : W^{1,p}(E'_{ct}) \oplus W^{1,p}(E'_{\mathrm{end}}, \lambda') \to \oplus_{i=1}^h \mathbb{C}_i^n$ is the map defined by

$$\Delta(\xi_{ct}, \xi_{\mathrm{end}}) = (\xi_{ct}(p_1) - \xi_1(0), \ldots, \xi_{ct}(p_h) - \xi_h(0)).$$

This induces the exact sequence

$$0 \to W^{1,p}(E', \lambda') \to W^{1,p}(E'_{ct}) \oplus W^{1,p}(E'_{\mathrm{end}}, \lambda') \xrightarrow{\Delta} \oplus_{i=1}^h \mathbb{C}_i^n \to 0.$$

The associated index for the bundle pair (E', λ') over $\Sigma/\sim$ is given by the index of the operator

$$\overline{\partial}_{(E',\lambda')} : W^{1,p}(E', \lambda') \to L^p(\Lambda^{(0,1)}(E'_{ct})) \oplus L^p(\Lambda^{(0,1)}(E'_{\mathrm{end}})).$$

Lemma 15.4.2 *Each trivialization of $\lambda \to \partial\Sigma$ canonically determines an orientation on* index $\overline{\partial}_{(E',\lambda')}$ *and hence on* $\det \overline{\partial}_{(E',\lambda')}$.

Proof We first note that $\overline{\partial}_{(E'_{\mathrm{end}}|_{D_i}, \lambda'|_{S_i})} : W^{1,p}(E'_{\mathrm{end}}|_{D_i}, \lambda'|_{S_i}) \to L^p(\Lambda^{(0,1)}(E'_{\mathrm{end}}))$ is surjective and *each trivialization of $\lambda \to \partial\Sigma$ induces an isomorphism*

$$\ker \overline{\partial}_{(E'_{\mathrm{end}}|_{D_i}, \lambda'|_{S_i})} \cong \ker \overline{\partial}_{(D_i \times \mathbb{C}^n, S_i \times \mathbb{R}^n)} \cong \mathbb{R}^n.$$

Therefore we can pick a finite-dimensional *complex* subspace

$$\mathcal{E} \subset L^p(\Lambda^{(0,1)}(E'))$$

so that

$$(\mathcal{E} \oplus 0) + \mathrm{Range}(\overline{\partial}_{(E',\lambda')}) = L^p\left(\Lambda^{(0,1)}(E'_{ct})\right) \oplus L^p(\Lambda^{(0,1)}(E'_{\mathrm{end}})).$$

We fix an isomorphism $\Psi : \mathbb{C}^k \to \mathcal{E} \oplus 0$ and consider the map

$$\overline{\partial}_{(E',\lambda');\Psi} : \mathbb{C}^k \oplus W^{1,p}(E',\lambda') \to \mathbb{C}^k \oplus \left(L^p(\Lambda^{(0,1)}(E'_{ct})) \oplus L^p(\Lambda^{(0,1)}(E'_{\mathrm{end}})\right)$$

defined by

$$\overline{\partial}_{(E',\lambda');\Psi}(h,\xi) = (0, \Psi(h) + \overline{\partial}_{(E',\lambda');\Psi}).$$

We have the exact sequence

$$0 \to \mathrm{Ker}\,\overline{\partial}_{(E',\lambda')} \to \mathrm{Ker}\,\overline{\partial}_{(E',\lambda');\Psi} \to \mathbb{C}^k \to \mathrm{Coker}\,\overline{\partial}_{(E',\lambda')} \to 0,$$

where each of the three maps in the middle are given by

$$d_1(\xi) = (0,\xi), \quad d_2(h,\xi) = h, \quad d_3(h) = \Psi(h) \mod \mathrm{Range}\,\overline{\partial}_{(E',\lambda')}$$

in order. Note also that $\overline{\partial}_{(E',\lambda');\Psi}$ is homotopic to $\mathrm{id}_{\mathbb{C}^k} \oplus \overline{\partial}_{(E',\lambda')}$ through Fredholm operators. Hence we obtain

$$\mathrm{Index}\,\overline{\partial}_{(E',\lambda')} \cong \mathrm{Index}\,\overline{\partial}_{(E',\lambda');\Psi} \cong \mathrm{Ker}\,\widehat{\overline{\partial}}_{(E',\lambda');\Psi} - \mathbb{C}^k. \tag{15.4.22}$$

We note that $\mathrm{Ker}\,\widehat{\overline{\partial}}_{(E',\lambda');\Psi}$ is the set of pairs (h,ξ) satisfying

$$0 = \Psi(h) + \overline{\partial}_{(E',\lambda')}(\xi),$$

i.e.,

$$\mathrm{Ker}\,\widehat{\overline{\partial}}_{(E',\lambda');\Psi} \cong \left(\overline{\partial}_{(E',\lambda')}\right)^{-1}(0 \oplus \mathcal{E}).$$

Then, because $\mathcal{E}$ was chosen so that the evaluation map

$$\mathrm{ev}_p : \left(\overline{\partial}_{(E'|_{\mathbb{C}P^1})}\right)^{-1}(\mathcal{E}) \to E'_p \cong \mathbb{C}^n$$

is surjective, $(\overline{\partial}_{(E',\lambda')})^{-1}(0 \oplus \mathcal{E})$ is nothing but the kernel of the surjective homomorphism

$$\Delta : \left(\overline{\partial}_{(E'|_{\mathbb{C}P^1})}\right)^{-1}(\mathcal{E}) \times \mathrm{Ker}\,\overline{\partial}_{(E'|_D;\lambda'|_S)} \to \mathbb{C}^n; \quad (\xi_{\mathbb{C}P^1}, \xi_D) \mapsto \xi_{\mathbb{C}P^1}(p) - \xi_D(0).$$

Recall that $(\overline{\partial}_{(E'|_{\mathbb{C}P^1})})^{-1}(\mathcal{E})$ and $\mathbb{C}^n$ are complex vector spaces and hence are canonically oriented.

This proves that, if each λ is given a trivialization, it will canonically determine one on $\mathrm{Ker}\,\overline{\partial}_{(E'|_D,\lambda')}$ and hence on $(\overline{\partial}_{(E',\lambda')})^{-1}(0 \oplus \mathcal{E})$. This in turn determines an orientation on $\mathrm{Index}\,\overline{\partial}_{(E',\lambda')}$ by the isomorphism (15.4.22). □

Next we will transfer this orientation to (E,λ) by a gluing process. We can resolve the nodal point to obtain a family of Riemann surfaces parameterized by $r \in \mathbb{R}_+$ so that the bundle pair (E',λ') itself can be deformed to a bundle pair (E_r,λ_r) over $(\Sigma_r, \partial\Sigma_r)$ with a canonical identification $\lambda' \cong \lambda_r$.

More precisely, we choose small coordinate neighborhoods $D_\delta(0)$ and $D^2_\delta(p)$ of $0 \in D$ and p. Here $D_\delta(0)$ is the disc of radius δ with center at 0. For a positive real number $r > 0$ we glue D^2 and $\mathbb{C}P^1$ around O and S by identifying $z \in D^2_\delta(O)$ and $w \in D^2_\delta(S)$ whenever $zw = 1/r$. We denote the resulting bordered Riemann surface by Σ_r, which is biholomorphic to the unit disc. We may identify $\Sigma_1 = D^2$ and $\Sigma_\infty = \mathbb{C}P^1 \cup D$.

We also obtain the vector bundle on Σ_r from E' and denote it by E_r. The totally real subbundle λ' over ∂D induces a totally real subbundle λ_r on $\partial\Sigma_r$.

Now, since we have chosen $\mathcal{E}$ consisting of the elements whose supports are away from the nodal points, it is canonically embedded into $L^p(\Lambda^{(0,1)}\Sigma_r \otimes E_r)$ and hence the map Ψ can be canonically regarded as a map

$$\Psi : \mathbb{C}^k \to L^p(\Lambda^{(0,1)}(\Sigma_r) \otimes E_r)$$

for any sufficiently large r. Then, using the surjectivity of $\overline{\partial}_{(E',\lambda');\Psi}$ and by gluing (SS88), we have

$$\mathrm{Range}\,\widehat{\overline{\partial}}_{(E_r,\lambda_r);\Psi} + \mathcal{E} = L^p(\Lambda^{(0,1)}(\Sigma_r) \otimes E_r)$$

and a canonical isomorphism

$$\mathrm{Index}\,\overline{\partial}_{(E',\lambda');\Psi} \cong \mathrm{Index}\,\overline{\partial}_{(E_r,\lambda_r);\Psi}.$$

Therefore an orientation on $\mathrm{Index}\,\overline{\partial}_{(E',\lambda');\Psi}$ in the limit as $r \to \infty$ canonically gives rise to one on $\mathrm{Index}\,\overline{\partial}_{(E_r,\lambda_r);\Psi}$ for sufficiently large $r > 0$ and hence on $\mathrm{Index}\,\overline{\partial}_{(E_r,\lambda_r)}$ by the first part of the isomorphism (15.4.22). After that, we continuously deform (E_r, λ_r) to the originally given bundle pair $(E_1, \lambda_1) = (E, \lambda)$ as mentioned above, which induces an orientation thereon. This finishes the proof. □

When the moduli space has dimension zero and is regular and orientable, the set of orientations of each element in the moduli space is isomorphic to $\mathbb{Z}_2$. A choice of orientation on the moduli space then defines an isomorphism between this orientation space with $\mathbb{Z}_2$. In dimension 4, we can explicitly describe this *sign of the element in the moduli space* in terms of the geometric intersections of the given Lagrangian boundary and the pseudoholomorphic discs. (See (Cho08).)

Example 15.4.3 (One-point open GW invariant) Let $L \subset M$ be an oriented Lagrangian submanifold. Suppose TL is trivial (e.g., that L is spin) and that a trivialization is given. Proposition 15.4.1 provides an orientation on a moduli space

$$\mathcal{M}_1(L,\beta) \cong \widetilde{\mathcal{M}}(L;\beta) \times \partial D^2/PSL(2,\mathbb{R})$$

that is nothing but the fiber-product orientations of $\widetilde{\mathcal{M}}(L;\beta)$ and of ∂D^2 followed by the quotient. More specifically, we define the orientation of $\mathcal{M}_1(L,\beta)$ via the exact sequence

$$0 \to psl(2,\mathbb{R}) \cdot \{(w,z)\} \to T_w\widetilde{\mathcal{M}}(L;\beta) \oplus T_z\,\partial D^2 \to T_{[w,z]}\mathcal{M}_1(L,\beta) \to 0. \tag{15.4.23}$$

Denote by $\widehat{a}$ the vector field on D^2 generated by the linearized action of the Lie algebra element $a \in psl(2,\mathbb{R})$. We recall that the linearized action $a \in psl(2,\mathbb{R})$ on (w,z) is given by

$$a \mapsto (-dw(\widehat{a}(z)), \widehat{a}(z)). \tag{15.4.24}$$

Let $psl(2,\mathbb{R}) = \mathfrak{h} \oplus \mathfrak{p} \cong \mathbb{R} \oplus \mathbb{R}^2$ be the Cartan decomposition, i.e., where $\mathfrak{h}$ is the Lie subalgebra of diagonal matrices. We *fix an orientation on* $\mathfrak{h}$ *and* $\mathfrak{p}$ and take the direct sum orientation on $psl(2,\mathbb{R})$.

The action (15.4.24) also induces another splitting,

$$psl(2,\mathbb{R}) = psl(2,\mathbb{R})_z \oplus \mathbb{R},$$

where $psl(2,\mathbb{R})_z$ is the isotropy Lie subalgebra of the point $z \in \partial D^2$, which always contains $\mathfrak{h}$. In the upper-half-space model the action $\mathbb{R}$ in the splitting corresponds to the translation generated by $\partial/\partial x$, which induces the counter-clockwise rotation on ∂D^2. We *fix an orientation on* $\mathbb{R}$ *given by* $\partial/\partial x$. Then $psl(2,\mathbb{R})_z$ carries a canonical orientation induced by this splitting. The map

$$psl(2,\mathbb{R}) \cdot \{(w,z)\} \to T_w\widetilde{\mathcal{M}}(L;\beta) \oplus T_z\,\partial D^2$$

in (15.4.23) maps this $\mathbb{R}$-factor to $T_z\,\partial D^2$. These orientations together with (15.4.23) induce an orientation on $T_{[w,z]}\mathcal{M}_1(L,\beta)$ at each $[w,z] \in \mathcal{M}_1(L,\beta)$.

Now we assume $\dim \mathcal{M}_1(L,\beta) = n$ and consider the evaluation map $\mathrm{ev}_1 : \mathcal{M}_1(L,\beta) \to L$. By the dimensional restriction, we can assign a sign to each element $[w]$ of $\mathrm{ev}_1^{-1}(x)$ for a generic choice of $x \in L$. We denote this by $\sigma([w])$. *Assuming the transversality* of the compactified moduli space $\overline{\mathcal{M}}_1(L,\beta)$, we can define the one-point open Gromov–Witten invariant to be the sum

$$\sum_{[w] \in \mathcal{M}(L,\beta)} \sigma([w]).$$

15.5 Gluing of Floer moduli spaces

In Sections 14.2 and 14.3, we gave a precise description of (stable map type) compactification $\overline{\mathcal{M}}(x,y;B;J)$ of $\mathcal{M}(x,y;B;J)$ by describing all possible failures of convergence of sequences in $\mathcal{M}(x, y; B; J)$ and assembling them into the set of broken cusp-trajectories.

In this section, we describe a neighborhood structure of such a broken cusp-trajectory in $\overline{\mathcal{M}}(x, y; B; J)$. In general, $\overline{\mathcal{M}}(x, y; B; J)$ will not carry the standard manifold (or orbifold) structure (with boundary and corners) even if we consider a generic choice of $J = \{J_t\}_{0 \le t \le 1}$ in its construction. This is due to the presence of multiple covered pseudoholomorphic spheres or discs. They only carry the structure of Kuranishi spaces with a boundary and corners in the sense of (FOn99) and Appendix A.1 of (FOOO09). To establish the existence of Kuranishi structure at infinity, i.e., in the neighborhood of a broken cusp-trajectory, we need to prove a gluing theorem that establishes the construction of the collar and corner. Such a result in the setting of Kuranishi structure is established in Chapter 7 of (FOOO09), which, however, goes beyond the level of this book. When all the transversality requirements hold for the maps of all irreducible components and for the relevant evaluation maps of the fiber products, this construction is reduced to construction of collar and corner neighborhoods in the standard manifold with boundary and corners, whose explanation is now in order.

We will always assume that J satisfies the transversality result stated in Proposition 15.1.5. We emphasize that this generic transversality result holds for any transversal pair (L_0, L_1) of any (M, ω).

To make our exposition simple, we will consider only configurations of the following three types.

(1) (Splitting) $u_- \# u_+$, where $u_- \in \mathcal{M}(x, z; B_1) \times \mathcal{M}(z, y; B_2)$ with $B_1 \# B_2 = B$.
(2) (Bubbling-off-disc) $u \# w$, where u is a smooth Floer trajectory and w is a J_i-holomorphic disc with its boundary lying on L_i with $i = 0, 1$ with $B_0 \# \beta = B$.
(3) (Bubbling-off-spheres) $u \# v$, where u is a smooth Floer trajectory and v is a J_{t_0}-holomorphic sphere for some $t_0 \in [0, 1]$ with $B_0 \# \alpha = B$.

The general case is an amalgamation of these three cases that does not include any additional ingredients except that it will involve more complicated notation. Taubes' gluing scheme (Ta82) is a combination of a singular perturbation problem, the Lyapunov–Schmit reduction method and the implicit function theorem in solving nonlinear partial differential equations in general, when an approximate solution is provided from the degenerate limit of the given equations. Typically the limiting picture carries a singular solution that consists of more than one irreducible component. Such an approximate solution can easily be guessed from the given geometric context.

We start with the abstract scheme of the general perturbation scheme.

15.5.1 Outline of the perturbation scheme

Let $F : B_1 \to B_2$ be a smooth map between Banach spaces B_1, B_2. We would like to solve the equation

$$F(x) = 0. \tag{15.5.25}$$

The starting point is the presence of an approximate solution x_0, i.e., $F(x_0)$ is small in B_2, say

$$|F(x_0)| \leq \epsilon. \tag{15.5.26}$$

In applications, an approximate solution is manifest in the given geometric context, and very often comes from some *degenerate* configurations. Then we would like to perturb x_0 to $x_0 + h$ to a solution of the form $x = x_0 + h$. So we Taylor-expand

$$F(x_0 + h) = F(x_0) + dF(x_0)h + N(x_0, h),$$

where N satisfies the following two inequalities:

(1)

$$|N(x_0, h)| \leq o(|h|)|h|,$$
$$|N(x_0, h_1) - N(x_0, h_2)| \leq o(|h_1| + |h_2|)|h_1 - h_2|.$$

(2) Assume that there exists a function $\varepsilon_1 = \varepsilon_1(\delta)$ such that $\varepsilon_1(\delta) \to 0$ as $\delta \to 0$ and

$$\frac{|N(x_0, h)|}{|h|} \leq \varepsilon_1(|h|), \tag{15.5.27}$$

$$\frac{|N(x_0, h_1) - N(x_0, h_2)|}{|h_1 - h_2|} \leq \varepsilon_1(|h_1| + |h_2|). \tag{15.5.28}$$

(3) We also assume that $\varepsilon_1(\delta) \to 0$ uniformly over x_0 when we consider a family of approximate solutions x_0.

Now, setting $F(x_0 + h) = 0$, we obtain $F(x_0) + dF(x_0)h + N(x_0, h) = 0$ or

$$dF(x_0)h = -F(x_0) + N(x_0, h). \tag{15.5.29}$$

(4) Now suppose that $dF(x_0)$ is a surjective linear map and let $Q(x_0)$ be its right inverse so that $dF(x_0) \circ Q(x_0) = id$.
(5) When we consider a family of approximate solutions x_0, then we assume that there exists a uniform constant $C > 0$ with

$$\|Q(x_0)\| \leq C \tag{15.5.30}$$

for x_0.

Then we put an Ansatz $h = Q(x_0)k$ for $k \in B_2$. Then (15.5.29) is reduced to

$$k = -F(x_0) + N(x_0, Q(x_0)k). \tag{15.5.31}$$

Now we regard the right-hand side as a map from B_2 to B_2 and denote this map by

$$G(k) = -F(x_0) + N(x_0, Q(x_0)k).$$

At this point, we note that $G(0) = -F(x_0)$ is assumed to be sufficiently small. To solve the fixed-point problem $k = G(k)$ of the map G, we will apply Picard's fixed-point theorem. To apply the fixed-point theorem, we need to prove the existence of $\delta > 0$ such that

(1) G maps a closed ball $B(\delta) \subset B_1$ to $B(\delta)$ itself; and
(2) G is a contraction map, i.e., it satisfies

$$|G(x_1) - G(x_2)| \leq \lambda |x_1 - x_2|$$

for some $0 < \lambda < 1$.

Let us examine what (1) means in terms of the F, x_0. We have

$$\begin{aligned} |-F(x_0) + N(x_0, Q(x_0)k)| &\leq |F(x_0)| + |N(x_0, Q(x_0)k)| \\ &\leq \epsilon + \varepsilon_1(|Q(x_0)k|)|Q(x_0)k| \leq \epsilon + C\varepsilon_1(C|k|)|k|. \end{aligned}$$

If $|k| \leq \delta$, then we have

$$|-F(x_0) + N(x_0, Q(x_0)k)| \leq \epsilon + C\varepsilon_1(C\delta)\delta.$$

On solving $\epsilon + C\varepsilon_1(C\delta)\delta \leq \delta$, we obtain

$$(1 - C\varepsilon_1(C\delta))\delta \geq \epsilon.$$

Therefore we first choose δ, say, so that

$$|1 - C\varepsilon_1(C\delta)| \geq \frac{1}{2} \text{ (and hence } \delta \geq 2\epsilon).$$

The first inequality is satisfied whenever $\delta \leq \delta_0$ for some $\delta_0 > 0$, which *depends only on N (and in turn on F and x_0.)* Therefore we need to choose δ so that

$$2\epsilon \leq \delta \leq \delta_0. \tag{15.5.32}$$

Note that both ϵ and δ_0 depend on the equation $F(x) = 0$, and the approximate solution x_0. In applications, it happens that the choice of δ_0 can be made uniform on ϵ as long as ϵ is sufficiently small. So, by improving the choice of the approximate solution x_0, we can always find δ that satisfies (15.5.32).

Next we estimate $|G(k_1) - G(k_2)|$,

$$\begin{aligned}|G(k_1) - G(k_2)| &= |N(x_0, Q(x_0)k_1) - N(x_0, Q(x_0)k_2)| \\ &\le \varepsilon_1(|Q(x_0)k_1| + |Q(x_0)k_2|)|Q(x_0)||k_1 - k_2| \\ &\le \varepsilon_1(|Q(x_0)k_1| + |Q(x_0)k_2|)C|k_1 - k_2|.\end{aligned}$$

We choose $\delta_1 > 0$ so that

$$\varepsilon_1(2C\delta_1)C \le \frac{1}{2}.$$

We note that the choice of such δ_1 *depends only on N (and in turn on F and x_0).* This choice is always possible because we assume that $\varepsilon_1(\delta) \to 0$ as $\delta \to 0$, and the choice of δ_1 can be made uniform over x_0 as long as ϵ is sufficiently small. We set

$$\delta_2 = \min\{\delta_0, \delta_1\}.$$

By construction, it follows that, if ϵ is sufficiently small, then we can choose $\delta_2 = 2\epsilon$.

We summarize the above discussion as follows.

Proposition 15.5.1 *Let B_1, B_2 be Banach spaces and let $F : B_1 \to B_2$ be a smooth map. Let $x_0 \in U$ satisfy*

$$|F(x_0)| \le \epsilon,$$

where $U \subset B_1$ is an open subset. Suppose that $dF(x_0)$ is surjective and $Q(x_0)$ is its right inverse and assume that $|Q(x_0)| \le C$ uniformly over $x_0 \in U$. Choose any $\delta_2 = 2\epsilon$ as above. Then $F(x) = 0$ has a unique solution of the form $x = x_0 + Q(x_0)k$ with

$$|k| \le 2\epsilon.$$

In particular, there exists some $\epsilon_0 > 0$ such that whenever $0 < \epsilon \le \epsilon_0$ the perturbation error $|Q(x_0)k|$ can be made of the same order of ϵ so that

$$|Q(x_0)k| \le C\epsilon,$$

i.e., so that the genuine solution x lies in the ball $B_{x_0}(C\epsilon) \subset U$ uniformly over x_0.

In the rest of this section, we will implement this perturbation scheme in constructing smooth Floer trajectories nearby each of the broken trajectories under a suitable transversality hypothesis.

15.5.2 Gluing broken trajectories

Let $u_- \# u_+$, where

$$(u_-, u_+) \in \mathcal{M}(x, z; B_1; J) \times \mathcal{M}(z, y; B_2; J)$$

with $B_1 \# B_2 = B$. Since we assume that $J = \{J_t\}$ satisfies transversality as stated in Proposition 15.1.5, we know that the linearizations

$$D\overline{\partial}_J(u_\pm) : W^{1,p}(u_\pm^* M) \to L^p(u_\pm^* M)$$

are surjective. Since we will not vary such J in the discussion which follows, we will omit the J-dependence in our notation of the moduli spaces.

The following is the main result we will prove in this subsection.

Theorem 15.5.2 *Let $K_- \subset \mathcal{M}(x, z; B_1)$, $K_+ \subset \mathcal{M}(z, y; B_2)$ be given compact subsets.*

(1) *There exists some $R_0 > 0$ and a diffeomorphism*

$$\mathrm{Glue} : K_- \times K_+ \times (R_0, \infty) \to \mathcal{M}(x, y; B)$$

onto its image.

(2) *Furthermore, there exists a homeomorphism*

$$\varphi : [R_0, \infty) \cup \{\infty\} \to (-\epsilon_0, 0]$$

such that the reparameterized diffeomorphism

$$\mathrm{Glue}_\varphi : K_- \times K_+ \times (-\epsilon_0, 0) \to \mathcal{M}(x, y; B)$$

extends to the map

$$\overline{\mathrm{Glue}_\varphi} : K_- \times K_+ \times (-\epsilon_0, 0] \to \overline{\mathcal{M}}(x, y; B)$$

so that the following diagram commutes:

$$\begin{tikzcd}
K_- \times K_+ \times \{0\} \arrow[r, "\cong"] \arrow[d, hook] & K_- \times K_+ \arrow[dr, hook] & \\
K_- \times K_+ \times (-\epsilon_0, 0] \arrow[r, "\overline{\mathrm{Glue}_\varphi}"] \arrow[dr] & \overline{\mathcal{M}}(x, y; B) \arrow[d] & \mathcal{M}(x, z; B_1) \times \mathcal{M}(z, y; B_2) \arrow[l, hook'] \arrow[d] \\
& (-\epsilon_0, 0] & \{0\} \arrow[l, hook']
\end{tikzcd}$$

Here statement (1) is Floer's original form of the gluing theorem (Fl88b), which is, ostensibly, a gluing theorem 'at infinity'. This gluing map Glue itself, however, does not establish the construction of a collar neighborhood of the boundary $\partial\overline{\mathcal{M}}(x, y; B)$ when we regard $\overline{\mathcal{M}}(x, y; B)$ as a manifold with boundary diffeomorphic to

$$\mathcal{M}(x, z; B_1) \times \mathcal{M}(z, y; B_2).$$

The main problem is extending Glue *smoothly* to infinity,

$$\mathcal{M}(x, z; B_1) \times \mathcal{M}(z, y; B_2) \times \{\infty\},$$

by adding the broken trajectory to $\mathcal{M}(x, y; B)$ according to the convergence described in Section 14.2.

Statement (2) then is about establishing this collar neighborhood out of statement (1) by a suitable choice of reparamerization function $\varphi : [R_0, \infty) \to [-\epsilon_0, 0)$.

We start with the gluing theorem of statement (1). The proof consists of four steps:

(1) Construction of approximate solutions
(2) Construction of the right inverse of the linearization
(3) Construction of genuine solutions and hence of Glue
(4) Establishing the diffeomorphism property of Glue

The scheme used in the first three steps is often called *Taubes' gluing method*. Taubes first used it in his construction of self-dual connections on positive definite four-manifolds (Ta82), and it was extensively employed by Donaldson (Do86) in the study of differential topology of four-manifolds. Since then this gluing scheme has become a fundamental ingredient in the analysis of pseudoholomorphic curves and its applications to symplectic geometry as well as in the analysis of Yang–Mills and Seiberg–Witten moduli spaces and its applications to low-dimensional topology.

The last step in turn consists of four steps:

(1) Proof of local injectivity
(2) Proof of properness
(3) Proof of surjectivity of Glue onto

$$V^\epsilon(K_- \times K_+, \delta) = \bigcup_{(u_-, u_+) \in K_- \times K_+} V^\epsilon_\delta(u_-, u_+)$$

for some choice of ϵ, δ, where $V^\epsilon_\delta(u_-, u_+)$ is defined in (14.2.39)
(4) Proof of global injectivity of Glue

Now some detail of each step is in order. We will point out all the main ingredients of the construction, but not attempt to provide complete details of analytic estimates of the construction, for which one can use straightforward integral estimates mainly using the exponential decay estimates established in Proposition 14.1.5. Otherwise, we refer readers to Floer's original paper (Fl88b) or to (FOOO07) for the details.

Pre-gluing and error estimates

We recall that $\mathcal{M}(x, y; B) = \widetilde{\mathcal{M}}(x, y; B)/\mathbb{R}$ is a quotient space and that the gluing map is supposed to have its domain as

$$\mathcal{M}(x, z; B_1) \times \mathcal{M}(z, y; B_2) \times (R_0, \infty)$$

and have as its target $\mathcal{M}(x, y; B)$. However, the actual gluing construction uses *maps*, not their equivalence classes. Therefore we need to take the slices, i.e., good representatives of $\mathcal{M}(x, z; B_1)$, $\mathcal{M}(z, y; B_2)$ from $\widetilde{\mathcal{M}}(x, z; B)$, $\widetilde{\mathcal{M}}(z, y; B)$ first. We again use the normalization condition

$$E_J(u|_{(-\infty,0]\times[0,1]}) = E_J(u|_{[0,\infty)\times[0,1]}) \tag{15.5.33}$$

to represent elements of $\mathcal{M}$ by $\widetilde{\mathcal{M}}_0$ as at the end of Section 14.2.

Now let $u_- \in \widetilde{\mathcal{M}}_0(x, z; B_1)$ and $u_+ \in \widetilde{\mathcal{M}}_0(z, y; B_2)$ be the choices for such a pair. We fix a small geodesic neighborhood U of $z \in L_0 \cap L_1$ so that $U = \exp_z V$ with $V \subset T_z M$ and U does not contain any other intersection points. Then the following lemma is an immediate consequence of Proposition 14.1.5.

Lemma 15.5.3 *For each given compact subset $K_- \subset \widetilde{\mathcal{M}}_0(x, z; B_1)$ and $K_+ \subset \widetilde{\mathcal{M}}_0(z, y; B_2)$, there exists $R_0 = R_0(z, U; K_-, K_+) > 0$ such that*

$$u_-([R_0, \infty) \times [0, 1]),\ u_+((-\infty, -R_0] \times [0, 1]) \subset U. \tag{15.5.34}$$

By virtue of this lemma, we can write

$$\begin{aligned} u_-(\tau, t) &= \exp_z \xi_-(\tau, t) \quad \text{for } \tau \geq R_0, \\ u_+(\tau, t) &= \exp_z \xi_+(\tau, t) \quad \text{for } \tau \leq -R_0. \end{aligned}$$

Then we define the pre-gluing $u_- \#_{(R;\rho)} u_+ =: u_{\text{app}}$ by the formula

$$u_{\text{app}}(\tau, t) = \begin{cases} u_-(\tau + 2R, t) & \text{for } \tau \leq -R, \\ \exp_z((1 - \chi(\tau))\xi_-(\tau + 2R) + \chi(\tau)\xi_+(\tau - 2R, t)) & \text{for } -R \leq \tau \leq R, \\ u_+(\tau - 2R, t) & \text{for } \tau \geq R \end{cases}$$

for a cut-off function $\chi : \mathbb{R} \to [0,1]$ such that $\{1-\chi, \chi\}$ form a partition of unity subordinate to the cover $\{(-R,1),(-1,R)\}$. (To make the construction symmetric with respect to $\pm$, we may also require $\chi(-\tau) = 1-\chi(\tau)$.)

Proposition 15.5.4 *The pregluing map*

$$\begin{aligned}&\mathrm{PreG} : (u_-, u_+, R) \to u_-\#_{R;\chi}u_+;\\ &\widetilde{\mathcal{M}}_0(x,z;B_1)\times\widetilde{\mathcal{M}}_0(z,y;B_2)\times(R_0,\infty)\to\mathcal{F}(x,y;B)\end{aligned}$$

induces a smooth embedding of $K_- \times K_+ \times (R_0,\infty)$ into $\mathcal{F}(x,y;B)$, provided that R_1 is sufficiently large. Furthermore, we have the error estimate

$$\|\overline{\partial}_J(u_-\#_{R;\chi}u_+)\|_p \le Ce^{-\delta_1 R}, \quad R \ge R_1 \tag{15.5.35}$$

for some $C = C(K_-,K_+)$, $\delta_1 = \delta_1(K_-,K_+)$.

The proof of this proposition is a consequence of the exponential decay estimate given in Proposition 14.1.5.

Construction of the right inverse

By the assumption on J, $D\overline{\partial}_J(u_\pm)$ are surjective and each has a right inverse,

$$Q(u_\pm) : L^p(u_\pm^*TM) \to W^{1,p}(u_\pm^*TM).$$

The right inverse is not unique, so we need to choose them smoothly depending on $u_- \in \mathcal{M}_0(x,z;B_1)$, $u_+ \in \mathcal{M}_0(z,y;B_2)$, respectively, so that

$$\|Q(u_\pm))\| \le C, \quad D\overline{\partial}_J(u_\pm)\circ Q(u_\pm) = id \tag{15.5.36}$$

for some $C = C(K_\pm)$ independent of $u_\pm \in K_\pm$.

Exercise 15.5.5 Prove that such a choice is always possible.

Now we consider the pre-glued map

$$\mathrm{PreG}(u_-,u_+,R) := u_-\#_{\chi;R}u_+ = u_{\mathrm{app}}$$

and its covariant linearization $D = D\overline{\partial}_J(u_{\mathrm{app}})$. Similarly to the construction of approximate solutions, we first explicitly construct an approximate right inverse of D as follows.

Let $\eta \in L^p(u_{\mathrm{app}}^*TM)$. We define the families of the elements in $L^p(u_\pm^*TM)$ by

$$\eta_{-,R}(\tau,t) = \begin{cases}\eta(\tau-2R) & \text{for } \tau \le 2R,\\ 0 & \text{for } \tau \ge 2R,\end{cases}$$

$$\eta_{+,R}(\tau,t) = \begin{cases} 0 & \text{for } \tau \le -2R, \\ \eta(\tau+2R) & \text{for } \tau \ge -2R \end{cases}$$

for each $R \ge R_0$, respectively. We note that

$$\eta_{-,R}(\tau,t) \in T_{u_-(\tau,t)}M, \quad \eta_{+,R}(\tau,t) \in T_{u_+(\tau,t)}M$$

by construction and so $\eta_{\pm,R}$ really define sections of $L^p(u_\pm^* TM)$, respectively. By construction, we have

$$\begin{aligned} \operatorname{supp} \eta_{-,R} \subset &\quad (-\infty, 2R] \times [0,1], \\ \operatorname{supp} \eta_{+,R} \subset &\quad [-2R, \infty) \times [0,1]. \end{aligned} \tag{15.5.37}$$

Recall from the definition of u_{app} that

$$u_{\mathrm{app}}([-2R, 2R] \times [0,1]) \subset U.$$

We denote by

$$\Pi_p^q : T_pM \to T_qM$$

the parallel transport along the short geodesic from p to q whenever $d(p,q) \le \iota(g)$, which is the injectivity radius of g. Obviously we have $(\Pi_p^q)^{-1} = \Pi_q^p$.

We denote

$$\xi_{\pm,R} = Q_\pm(u_\pm)(\eta_{\pm,R}) \in W^{1,p}(u_\pm^* TM). \tag{15.5.38}$$

Now we define

$$Q_{\mathrm{app}}(u_\pm, R) : L^p(u_{\mathrm{app}}^* TM) \to W^{1,p}(u_{\mathrm{app}}^* TM)$$

by the following formula: for $\xi_{\mathrm{app}} := Q_{\mathrm{app}}(u_\pm, R)(\eta)$

$$\xi_{\mathrm{app}}(\tau,t) = \begin{cases} \xi_{-,R}(\tau+2R, t) & \text{for } \tau \le -R, \\ \xi_{+,R}(\tau-2R, t) & \text{for } \tau \ge R, \\ \Pi_z^{u_{\mathrm{app}}(\tau,t)} \Big((1-\chi(\tau)) \Pi^z_{u_-(\tau,t)}(\xi_{-,R}(\tau,t)) & \\ \quad + \chi(\tau) \Pi^z_{u_+(\tau,t)}(\xi_{+,R}(\tau,t)) \Big) & \text{for } -R \le \tau \le R. \end{cases} \tag{15.5.39}$$

By construction of this approximate inverse, we easily obtain the following proposition.

Proposition 15.5.6 *We have*

$$\|Q_{\mathrm{app}}(u_\pm, R)\| \le C, \|D\overline{\partial}(u_{\mathrm{app}}) \circ Q_{\mathrm{app}}(u_\pm, R) - id\| \le \frac{1}{2}.$$

In particular, $D\overline{\partial}(u_{\mathrm{app}}) \circ Q_{\mathrm{app}}(u_\pm, R)$ *is invertible.*

Once we have this proposition, we can construct a genuine right inverse by

$$Q(u_\pm, R) = Q_{\rm app}(u_\pm, R) \circ (D\overline{\partial}(u_{\rm app}) \circ Q_{\rm app}(u_\pm, R))^{-1}$$

that satisfies (15.5.36) and varies smoothly on the gluing parameters $(u_\pm, R)$.

At this stage, we apply the Picard contraction mapping theorem as in Proposition 15.5.1 to construct the genuine solution associated with $\mathrm{PreG}(u_-, u_+; R)$ which we denote by $\mathrm{Glue}(u_-, u_+; R) = u_- \widehat{\#}_R u_+$. This defines a smooth map

$$\mathrm{Glue} : K_- \times K_+ \times [R_0, \infty) \to \mathcal{M}(x, y; B).$$

15.5.3 Wrap-up of the proof of Theorem 15.5.2

In this subsection, we prove various properties of the gluing map Glue and wrap up the proof of Theorem 15.5.2.

Local injectivity

We fix the compact subsets $K_1 \subset \mathcal{M}(x, z; B_1)$ and $K_2 \subset \mathcal{M}(z, y; B_2)$. Consider $B = B_1 \# B_2$ and $\mathcal{M}(x, y; B)$.

We start with the local injectivity.

Proposition 15.5.7 Glue $: K_- \times K_+ \times [R_0, \infty) \to \mathcal{M}(x, y; B)$ *is locally injective.*

Proof This follows from the local injectivity of the pre-gluing map PreG and the uniqueness of the fixed point in the application of Picard's fixed-point theorem. □

Properness

Let $K \subset \mathcal{M}(x, y; B)$ be any compact subset.

Proposition 15.5.8 $\mathrm{Glue}^{-1}(K) \subset \mathcal{M}(x, z; B_1) \times \mathcal{M}(z, y; B_2) \times [R_0, \infty)$ *is compact.*

Proof Suppose $u_i = \mathrm{Glue}(u_i^-, u_i^+, R_i)$ with

$$E_{J,(-\infty,0]}(u_i^\pm) = E_{J,[0,\infty)}(u_i^\pm) \tag{15.5.40}$$

and $[u_i] \in K$. Since K is compact, there exists a subsequence, still denoted by is, and τ_i such that $u_i \circ \tau_i$ converges to $u_\infty \in \widetilde{\mathcal{M}}(x, y; B)$. Without loss of generality, we assume $\tau_i = 0$. By the convergence, we have the uniform derivative bound $|du_i| \le C$ for some $C > 0$ and, in particular, bubbling does not occur. It also follows from Proposition 14.2.4 that

$$\lim_{R\to\infty} E_{J,[-R,R]}(u_i) = \omega(B). \tag{15.5.41}$$

It is easy to see from (15.5.40), the definition of Glue and (15.5.41) that $R_i < C$ for some $C > 0$.

It remains to show that there exists a subsequence of i_j such that the $u_{i_j}^-$ and $u_{i_j}^+$ converge. By the definition of $u_i = \mathrm{Glue}(u_i^-, u_i^+, R_i)$ and the derivative bound $|du_i| \le C$, we have the bounds $|du_{i_j}^\pm| \le C'$. Therefore, if $u_{i_j}^-$ (or $u_{i_j}^+$) fails to converge, (15.5.41) implies that there exists a subsequence of i_j, still denoted by i_j, such that

$$\lim_{j\to\infty} E_{J,[-R_{i_j},R_{i_j}]}(u_{i_j}^-) < \omega(B_1) \le \omega(B).$$

Again the definition of $\mathrm{Glue}(u_i^-, u_i^+, R_i)$ and (15.5.41) prevent this from happening. This proves the properness. □

Surjectivity

Consider the neighborhood

$$V^\epsilon(K_-\times K_+, \delta) = \bigcup_{(u_-,u_+)\in K_-\times K_+} V_\delta^\epsilon(u_-, u_+),$$

where we recall that $V_\delta^\epsilon(u_-, u_+)$ is defined in (14.2.39).

Proposition 15.5.9 *Suppose that* Glue *is defined on*

$$K_-\times K_+\times[R_0,\infty) \subset \mathcal{M}(x,z;B_1)\times\mathcal{M}(z,y;B_2)\times[R_0,\infty)$$

for some $R_0 > 0$. *Then there exist* ϵ, $\delta > 0$ *such that*

$$V^\epsilon(K_-\times K_+, \delta)\cap\mathcal{M}(x,y;B) \subset \mathrm{Image\,Glue}.$$

Proof Suppose to the contrary that there exist ϵ_i, $\delta_i \to 0$ and

$$u_i \in V^\epsilon(K_-\times K_+, \delta)\cap\mathcal{M}(x,y;B)\setminus \mathrm{Image\,Glue}.$$

By (14.2.37) and compactness of $K_\pm$, $u_i^\pm$ converge to $u_\pm$ locally for some $u^\pm \in K_\pm$. Furthermore, since $\delta_i \to 0$, (14.2.38) and Proposition 14.2.4 imply that $u_i^\pm \to u_\pm$ globally in $K_\pm$. Therefore, we can write

$$u_i^\pm(\tau,t) = \exp_{u_\pm(\tau,t)}\xi_i^\pm(\tau,t)$$

for some $\xi_i^\pm : \mathbb{R}\times[0,1]\to T_{p_\pm}M$. Then there exists $R_i \to 0$ such that

$$\mathrm{dist}(u_i, \mathrm{PreG}(u_+, u_-, R_i)) \to 0.$$

But we have $\overline{\partial}_J u_i = 0$ and so the uniqueness part of Proposition 15.5.1 implies that $u_i = \mathrm{Glue}(u_+, u_-, R_i)$ if i is sufficiently large. This contradicts the standing hypothesis $u_i \notin \mathrm{Image}\,\mathrm{Glue}$. This finishes the proof. □

Global injectivity

By now, we have proved that $\mathrm{Glue} : K_- \times K_+ \times [R_0, \infty) \to \mathcal{M}(x, y; B)$ is a covering map onto a subset of $\mathcal{M}(x, y; B)$. We now show that the following proposition holds.

Proposition 15.5.10 Glue *is one-to-one if R_0 is sufficiently large.*

Proof We may assume that $K_\pm$ are path-connected and $K_\pm = \overline{U}_\pm$ for some open subsets $U_\pm$.

Suppose to the contrary that

$$\mathrm{Glue}(u_i^-, u_i^+, R_i) = \mathrm{Glue}(u_i^{-\prime}, u_i^{+\prime}, R_i') =: u_i \tag{15.5.42}$$

for some $(u_i^-, u_i^+, R_i) \neq (u_i^{-\prime}, u_i^{+\prime}, R_i') \in U_- \times U_+ \times \mathbb{R}$ with R_i, $R_i' \to \infty$. By construction, we have

$$\lim_{i\to\infty} u_i^\pm = \lim_{i\to\infty} u_i^{\pm\prime} =: u_\infty^\pm$$

after choosing a subsequence, if necessary. Then, again by the uniqueness of the gluing construction, we have

$$\mathrm{Glue}(u_i^-, u_i^+, R_i) = \mathrm{Glue}(u_\infty^-, u_\infty^+, R_{\infty,i}) = \mathrm{Glue}(u_i^{-\prime}, u_i^{+\prime}, R_i')$$

for some $R_{\infty,i} \to \infty$ as $i \to \infty$. As R_i, $R_i' \to \infty$ and $u_i^\pm$, $u_i^{\pm\prime} \to u_\infty^{\pm\prime}$, it follows that

$$(u_i^-, u_i^+, R_i),\ (u_i^{-\prime}, u_i^{+\prime}, R_i') \in V_\delta^\epsilon(u_\infty^-, u_\infty^+)$$

for all sufficiently large i for some ϵ, $\delta > 0$. Then by the same argument as in the surjectivity proof, it follows that

$$\mathrm{dist}\left((u_i^-, u_i^+, R_i), (u_+, u_-, R_{\infty,i})\right),\ \mathrm{dist}\left((u_i^-, u_i^+, R_i), (u_+, u_-, R_{\infty,i})\right) \to 0.$$

Since we assumed $[u_i^-, u_i^+, R_i] \neq [u_i^{-\prime}, u_i^{+\prime}, R_i']$, this violates the local injectivity at $[u_+, u_-, R_{\infty,i}]$ for any $R_{\infty,i} > R_0$ given in Proposition 15.5.7, and hence the proposition follows. □

Construction of a boundary chart

By now, we have proved Theorem 15.5.2 (1). It remains to prove Theorem 15.5.2 (2). For this purpose, we need to choose a smooth function φ :

$(R_0, \infty) \to (-\epsilon_0, 0)$ such that $\lim_{T\to\infty} \varphi(T) = 0$. We define φ by

$$\varphi(T) = \frac{1}{T} =: s.$$

We define the map $\mathrm{Glue}_\varphi : K_- \times K_+ \times (-\epsilon_0, 0) \to \mathcal{M}(x, y; B)$ and

$$\mathrm{Glue}_\varphi(u_-, u_+, s) = \mathrm{Glue}(u_-, u_+, \varphi^{-1}(s)).$$

Then the map Glue_φ continuously extends to a one-to-one map

$$\overline{\mathrm{Glue}}_\varphi : K_- \times K_+ \times (-\epsilon_0, 0] \to \overline{\mathcal{M}}(x, y; B)$$

and hence defines a homeomorphism onto its image. This provides a boundary chart of $\overline{\mathcal{M}}(x, y; B)$.

Exercise 15.5.11 Prove the compatibility of the above-constructed coordinate charts.

15.6 Coherent orientations of Floer moduli spaces

We start with the discussion of orientation for the case of classical Morse theory in the point of view of Witten and Floer's Morse homology approach.

15.6.1 Coherent orientations in Morse homology

It is well known in the classical Morse theory that the negative gradient flow $\dot{x} = -\operatorname{grad} f(x)$ provides the homology and the positive gradient flow provides the cohomology. (See (Mil65).) Considering the (downward) Morse complexes $CM(f)$ and $CM(-f)$, we have the natural isomorphism

$$CM(-f) \cong \mathrm{Hom}(CM(f), \mathbb{Q})$$

induced by the one-to-one correspondence

$$\widetilde{x} \to x^*, \tag{15.6.43}$$

where x^* is the element $\mathrm{Hom}(CM(f), \mathbb{Q})$ defined by $x^*(x) = 1$ and $x^*(y) = 0$ for $x \neq y \in \mathrm{Crit}(f)$. Then this induces a (strongly) nondegenerate pairing

$$CM_k(f) \times CM_{n-k}(-f) \to \mathbb{Q}$$

in the chain level of Morse homology (or more accurately Morse–Smale–Witten–Floer homology).

Following the standard scheme of assigning the coherent orientation (Mil65), we

(1) first choose *orientations* of unstable manifolds $W^u_f(x)$ at each $x \in \mathrm{Crit}\, f$,
(2) then choose *coorientations* of stable manifolds $W^s_f(x)$ so that

$$N_x W^s_f(x) = T_x W^u_f(x)$$

as an oriented vector space.

Denote this orientation of $W^u_f(x)$ by $o(x)$.

For a generic metric g, the stable and unstable manifolds intersect transversally, i.e., the pair (f, g) becomes a *Morse–Smale pair*. Assuming this transversality, we can assign orientations of $S\,W^u_f(x)$ by requiring the equality

$$\mathrm{span}\{\nabla f(p)\} \oplus T_p S\,W^u_f(x) = T_p W^u_f(x)$$

as an oriented vector space. Since the normal bundles of $W^s_f(y) \subset M$ and of $S\,W^s_f(y) \subset f^{-1}(y)$ at p are canonically isomorphic, we give a coorientation on $S\,W^s_f(y)$ in $f^{-1}(y)$ as that of $W^s_f(y)$ in M. Therefore the intersection

$$S\,W^u_f(x) \cap S\,W^s_f(y) \subset f^{-1}(y)$$

carries a natural orientation. We have a natural diffeomorphism

$$S\,W^u_f(x) \cap S\,W^s_f(y) \cong \widetilde{\mathcal{M}}_g(f; x, y)/\mathbb{R},$$

where $\widetilde{\mathcal{M}}_g(f; x, y)$ is the moduli space of *negative* gradient trajectories, i.e., the solutions of

$$\dot{\chi} + \mathrm{grad}\, f(\chi) = 0, \quad \chi(-\infty) = x, \chi(\infty) = y.$$

We denote this orientation by $o(x, y)$.

This way of assigning orientations enjoys the following compatibility or coherence properties under the gluing construction.

Note that the product $\mathcal{M}_g(f; x, z) \times \mathcal{M}_g(f; z, y)$ carries the natural product orientation coming from $o(x, z)$ and $o(z, y)$. Now consider the gluing map

$$\mathrm{Glue} : \mathcal{M}_g(f; x, z) \times \mathcal{M}_g(f; z, y) \times [R, \infty) \to \mathcal{M}_g(f; x, y),$$

which extends to

$$\overline{\mathrm{Glue}} : \mathcal{M}_g(f; x, z) \times \mathcal{M}_g(f; z, y) \times [R, \infty) \cup \{\infty\} \to \overline{\mathcal{M}}_g(f; x, y)$$

smoothly. We denote by

$$o(x, z)\#o(z, y)$$

the push-forward of the product orientation of $\mathcal{M}_g(f; x, z) \times \mathcal{M}_g(f; z, y)$ on

$$\mathcal{M}_g(f; x, z)\#\mathcal{M}_g(f; z, y)$$

under the map $\overline{\mathrm{Glue}}$, which we call the *glued orientation*.

On the other hand, $\mathcal{M}_g(f;x,z)\#\mathcal{M}_g(f;z,y)$ carries another orientation as the boundary component of the manifold with boundary $\overline{\mathcal{M}}_g(f;x,y)$, which we call the *boundary orientation*. We denote this orientation by

$$\partial o(x,y).$$

Proposition 15.6.1 *Denote by $\overline{\mathcal{M}}_g(f;x,y)$ the compactified moduli space of $\mathcal{M}_g(f;x,y)$ and assume*

$$\mathcal{M}_g(f;x,z)\times\mathcal{M}_g(f;z,y)\hookrightarrow\partial\overline{\mathcal{M}}_g(f;x,y)$$

as a boundary component via the gluing map $\overline{\text{Glue}}$ restricted to $R=\infty$. Then we have

$$o(x)=o(x,y)\#o(y) \tag{15.6.44}$$

for all x, $y\in\operatorname{Crit} f$ such that $\mathcal{M}(x,y;f)\neq\emptyset$. In particular, we have

$$o(x,z)\#o(z,y)=\partial o(x,y). \tag{15.6.45}$$

Instead of proving this proposition, we heuristically explain how (15.6.45) follows from (15.6.44). The latter can be formally written as $o(x,y)=o(x)-o(y)$. Then the former follows upon adding $o(x,z)$ and $o(z,y)$.

We call the system $\{o(x,y)\}$ of orientations satisfying (15.6.45) *coherent orientations* on the Morse moduli spaces $\widetilde{\mathcal{M}}(x,y;f)$ for all x, $y\in\operatorname{Crit}(f)$ in general. The above construction then provides a way of providing such a coherent system geometrically. We denote by σ the system of coherent orientations generated by the procedure starting with orienting *unstable* manifolds.

When $\mu_f(x)-\mu_f(y)=1$, we consider the corresponding stable and unstable spheres at a level c,

$$SW_f^u(x):=f^{-1}(c)\cap W_f^u(x),\qquad SW_f^s(y):=f^{-1}(c)\cap W_f^s(y)$$

with $f(y)<c<f(x)$. In general, for a submanifold $T\subset M$ intersecting transversely with $f^{-1}(c)$, the intersection $K\cap f^{-1}(c)$ is oriented by the equation

$$T_pK=\{-\nabla f(p)\}\oplus T_p(K\cap f^{-1}(c))=\left\{\frac{\partial}{\partial\tau}\right\}\oplus T_p(K\cap f^{-1}(c)). \tag{15.6.46}$$

By comparing the two orientations on $T_p(SW_f^u(x))$ and $N_p(SW_f^s(y))$ at each $p\in SW_f^u(x)\cap SW_f^s(y)$, we can define the intersection number $\#(SW_f^u(x)\cap SW_f^s(y))$ in $f^{-1}(c)$. This enables one to define the matrix element $n(f;x,y)=\#(\mathcal{M}_g(f;x,y))$ of the Morse homology boundary map

$$\partial_{(g,f)}:CM_*(f)\to CM_*(f)$$

as an integer and satisfies the boundary property $\partial^2_{(g,f)} = 0$. The coherence property of the system σ is crucial to establish this boundary property. We define the Morse homology of the pair (g, f) by

$$HM^{\sigma}_*(g, f; M)$$

with respect to the coherent system σ of orientations.

Obviously we can replace unstable manifolds by stable manifolds in the above construction and obtain another coherent system of orientations, which we denote by $\widetilde{\sigma}$. For this system, one should note that

(1) we first orient stable manifolds,
(2) we require $N^{\widetilde{\sigma}} W^u_f(x) = T^{\widetilde{\sigma}} W^s_f(x)$ and
(3) for any oriented submanifold $S \subset M$ intersecting $M^c = f^{-1}(c)$ transversely, we give the orientation $S \cap M^c$ by the relation

$$\left\{-\frac{\partial}{\partial \tau}\right\} \oplus T_p^{S\cap M^c} = T_p S.$$

One may regard this construction of $\widetilde{\sigma}$ for f as the construction of σ for $-f$ using the time-reversal involution

$$\mathcal{M}(x, y; f) \to \mathcal{M}(y, x; -f).$$

In general

$$HM^{\widetilde{\sigma}}_k(g, f; M) \ncong HM^{\sigma}_k(g, f; M)$$

but we will show later in Section 19.7.2 the following isomorphism

$$HM^{\widetilde{\sigma}}_{(n-k)}(g, -f; M) \cong HM^k_{\sigma}(g, f; M)$$

holds whether or not M is orientable. When M is orientable, one can also prove

$$HM^{\widetilde{\sigma}}_k(g, f; M) \cong HM^{\sigma}_k(g, f; M).$$

15.6.2 Orientation data on smooth Floer moduli spaces

In this section, we study the orientability of each of the smooth Floer moduli spaces for the given pair (L_0, L_1). It turns out that the geometric discussion of orientation on the Morse moduli spaces can be replaced by the corresponding analytical study of the family index of Fredholm operators of linearizations of the nonlinear map

$$\chi \mapsto \dot{\chi} + \mathrm{grad}_g f(\chi) \tag{15.6.47}$$

even in the Morse homology context. This analytical study can be directly repeated in the Floer-theory context by replacing (15.6.47) by the $\overline{\partial}$-type nonlinear map.

Recall that the Floer homology theory is often regarded as an infinite-dimensional Morse theory on the loop spaces (in the case of closed strings) or on the path spaces (in the case of open strings) with Lagrangian boundary conditions. Equipping an orientation of Floer moduli spaces can be regarded as equipping one on a subset of this loop or path space. It is folklore that *spin structure* on the space M can be regarded as an orientation of the loop space of M. It turns out that a spin structure of Lagrangian submanifold M is what is necessary in order for one to be able to provide the coherent orientations on the associated compactified moduli spaces. More generally we need the following definition.

Definition 15.6.2 A submanifold $L \subset M$ is called *relatively spin* if it is orientable and there exists a class $st \in H^2(M, \mathbb{Z}_2)$ such that $st|_L = w_2(TL)$ for the Stiefel–Whitney class $w_2(TL)$ of TL.

A chain $(L_0, L_1, \ldots, L_k)$ or a pair (L_0, L_1) of Lagrangian submanifolds is said to be relatively spin if there exists a class $st \in H^2(M, \mathbb{Z}_2)$ satisfying $st|_{L_i} = w_2(TL_i)$ for each $i = 0, 1, \ldots, k$.

If N is a spin manifold, then any Lagrangian submanifold $L = \varphi(N) \subset (M, \omega)$ given by the embedding $\varphi : N \to M$ is relatively spin, since we can take $st = 0$ in $H^2(M, \mathbb{Z}_2)$.

15.6.3 Coherent orientations in Floer homology

We now explain the construction of a system of coherent orientations, a counterpart of the coherent orientations of the Morse homology explained in the previous section. We restrict ourselves to the case of the spin pair (L_0, L_1) for simplicity of exposition. (We refer the reader to Section 8.1 (FOOO09) for details in the more general setting of relative spin structure.)

We assume that all Lagrangian submanifolds appearing in the discussion of this section are equipped with orientation and spin structure. We also assume L_0 intersects L_1 transversely.

In this subsection, we prove the following theorem.

Theorem 15.6.3 *If (L_0, L_1) is a spin pair, then $\mathcal{M}(p, q; B)$ is orientable. When L_0, L_1 are spin and a choice of spin structures is made, it provides a natural orientation on $\mathcal{M}(p, q; B)$ for each pair p, $q \in L_0 \cap L_1$ and $B \in \pi_2(p, q)$.*

Orientation $o_{[p,w]}$

Let $[\widehat{p}, w] \subset \Omega(L_0, L_1; \ell_{01})$, i.e., an element such that $p \in L_0 \cap L_1$ and w satisfies

$$w(0,t) = \ell_{01}(t),\ w(1,t) \equiv p,\ w(s,0) \in L_0,\ w(s,1) \in L_1.$$

We denote by $\mathrm{Map}(\ell_{01}; p; L_0, L_1; \alpha)$ the set of such maps $w : [0,1]^2 \to M$ in homotopy class $[w] = \alpha$ of $\pi_2(\ell_{01}; p)$. Let $w \in \mathrm{Map}(\ell_{01}; p; L_0, L_1; \alpha)$. Let $\Phi : w^*TM \to [0,1]^2 \times T_pM$ be a (homotopically unique) symplectic trivialization as before. The trivialization Φ, together with the boundary condition, $w(0,t) = \ell_{01}(t)$ and the Lagrangian path λ_{01} along ℓ_{01}, defines a Lagrangian path

$$\lambda^\Phi = \lambda^\Phi_{([p,w];\lambda_{01})} : [0,1] \to \Lambda^{\mathrm{ori}}(T_pM)$$

satisfying $\lambda^\Phi(0) = T_pL_0$, $\lambda^\Phi(1) = T_pL_1$. The homotopy class of this path does not depend on the trivialization Φ but depends only on $[p,w]$ and the homotopy class of λ_{01} along ℓ_{01}. Unlike the case of Morse homology, each intersection $p \in L_0 \cap L_1$ carries the set $\pi_2(\ell_{01}; p)$ of homotopy classes of maps defined above. We need to find a systematic way of simultaneously choosing orientations of (15.6.49) for various choices of $\alpha \in \pi_2(\ell_{01}; p)$.

For this purpose, we extend the given orientations on $T_{\ell_{01}(0)}L_0$ and $T_{\ell_{01}(1)}L_1$ to a frame on the Lagrangian path λ_{01} along ℓ_{01}. Then we consider the following boundary-value problem for the section ξ of w^*TM on $\mathbb{R}_{\geq 0} \times [0,1]$ of $W^{1,p}$ class such that

$$\begin{cases} D_w\overline{\partial}(\xi) = 0, \\ \xi(0,t) \in \lambda_{01}(t), \quad \xi(\tau,0) \in T_pL_0, \quad \xi(\tau,1) \in T_pL_1. \end{cases} \tag{15.6.48}$$

Here $D_w\overline{\partial}$ is the linearization operator of the Cauchy–Riemann equation.

We denote $Z_- = \mathbb{R}_{\geq 0} \times [0,1]$ and define $W^{1,p}(Z_-; \lambda_{01})$ to be the set of sections ξ of w^*TM on Z_- of $W^{1,p}$ class satisfying the above boundary conditions. Then we obtain a Fredholm operator

$$D_w\overline{\partial} : W^{1,p}(Z_-, T_pM; \lambda) \to L^p(Z_-, T_pM \otimes \Lambda^{0,1}),$$

which we denote by $\overline{\partial}_{([p,w];\lambda_{01})}$. Its index, $\mathrm{Index}\, D_w\overline{\partial}$, as a virtual vector space is given by

$$\ker D_w\overline{\partial} - \mathrm{coker}\, D_w\overline{\partial}$$

and its numerical index is given by the Maslov–Morse index. We denote the determinant line by

$$\det \overline{\partial}_{([p,w];\lambda_{01})}.$$

By varying w in its homotopy class $\alpha \in \pi_2(\ell_{01}; p) = \pi_2(\ell_{01}; p; L_0, L_1)$, these lines define a line bundle

$$\det \overline{\partial}_{([p,w];\lambda_{01})} \to \mathrm{Map}(\ell_{01}; p; L_0, L_1; \alpha). \tag{15.6.49}$$

The following proposition is implicitly proved in the proof of Proposition 15.4.1.

Proposition 15.6.4 *Let (L_0, L_1) be a pair of oriented spin Lagrangian submanifolds. Then for each fixed α the bundle* (15.6.49) *is trivial.*

Exercise 15.6.5 Extract a proof of this proposition from that of Proposition 15.4.1.

We denote by $o_{[p,w]}$ an orientation of the bundle (15.6.49).

The orientation $o_{[p,w]}$ of the bundle (15.6.49) may be regarded as the analog to an orientation of *unstable manifolds of critical points* in Morse homology. As we mentioned before, the unstable manifolds have many connected components parameterized by $\pi_2(p; \ell_{01})$.

Let Φ and λ^Φ be as above. Recall that the data which really enter into the construction of the orientation $o_{[p,w]}$ are

(1) the path $\lambda^\Phi : [0,1] \to T_pM$ connecting T_pL_0 and T_pL_1 (for the definition of $o_{[p,w]}$, the path is the concatenation

$$\left(T_pL_0 \to T_{\ell_{01}(0)}L_0\right)\#\lambda_{01}\#\left(T_{\ell_{01}(1)}L_1 \to T_pL_1\right)$$

in the trivialization $\Phi : w^*TM \to [0,1]^2 \times T_pM$, and the associated Lagrangian subbundle of w^*TM over it) and

(2) the Riemann–Hilbert problem (15.6.48) on the semi-strip $\mathbb{R}_{[0,\infty)} \times [0,1]$ with this path as the boundary condition.

Therefore we need to handle this choice of trivialization Φ over w in a canonical way. This is the first place where the presence of spin structures on L_0, L_1 enters. Recall that a choice of orientations of L_0 and L_1 determines one on $T_pL_0 = \lambda_{01}(0)$ and one on $T_pL_1 = \lambda_{01}(1)$, respectively.

Orientation o_p

We consider the pull-back bundle $\mathrm{pr}^*\, TM \to [0,1] \times M$ of the projection $\mathrm{pr} : [0,1] \times M \to M$. For $p \in L_0 \cap L_1$, we consider a path of oriented Lagrangian subspaces $\lambda_p : [0,1] \to \Lambda^{\mathrm{ori}}(T_pM)$ so that

$$\lambda_p(0) = T_pL_0, \quad \lambda_p(1) = T_pL_1.$$

We denote by λ_p a path connecting T_pL_0 and T_pL_1 in $\Lambda^{\rm ori}(T_pM)$ and regard it as the bundle

$$\lambda_p = \bigcup_{t\in[0,1]} \{t\} \times \lambda_p(t) \to [0,1]$$

by an abuse of notation. This bundle, together with $[0,1] \times T_pL_i$ for $i = 0,\ 1$, defines a Lagrangian subbundle $\mathcal{L} \to K(L_0, L_1)$ of $\mathrm{pr}^* TM|_{K(L_0,L_1)}$ for the subset

$$K(L_0, L_1) = (L_0 \times \{0\}) \cup ((L_0 \cap L_1) \times [0,1]) \cup (L_1 \times \{1\}) \subset [0,1] \times M,$$

which extends the subbundle

$$\{0\} \times TL_0 \cup \{1\} \times TL_1 \subset \mathrm{pr}^* TM|_{\partial[0,1]}.$$

Denote by $P_{\rm spin}(L_i)$ the principal spin bundle of TL_i, which is a fiberwise double cover of its oriented frame bundle $P_{SO}(L_i)$. We may assume that $L_0 \cap L_1$ is contained in the 3-skeleton of M. In order to specify the spin structure on $\mathcal{L}|_{K(L_0,L_1)_{[2]}}$, where $K(L_0, L_1)_{[2]}$ is the 2-skeleton of $K(L_0, L_1)$, we proceed as follows.

Let σ be a section of $P_{SO(n)}(\mathcal{L})$, i.e., an oriented frame bundle of the vector bundle $\mathcal{L} \to K(L_0, L_1)$. Then we consider its restriction to $[0,1] \times \{p\} \subset K(L_0, L_1)$, which we denote by $\sigma_{[0,1]\times\{p\}}$. We assume that the associated orientation restricts to the given orientation of T_pL_i for each $i = 0,\ 1$. Recalling that $P_{{\rm Spin}(n)}(\mathcal{L}) \to P_{SO(n)}(\mathcal{L})$ is a double cover, we let

$$\iota_i^\sigma(p) \in P_{\rm spin}(L_i)|_p,$$

be a lifting of the value $\sigma(1 \times \{p\})$ for each $i = 0,\ 1$. We consider the triple $(\sigma_{[0,1]\times\{p\}}, \iota_0^\sigma(p), \iota_1^\sigma(p))$. Since the bundle

$$P_{{\rm Spin}(n)}(\mathcal{L})|_{[0,1]\times\{p\}} \cong [0,1] \times {\rm Spin}(n)$$

is a trivial double cover of $P_{SO(n)}(\mathcal{L})|_{[0,1]\times\{p\}} \cong [0,1] \times SO(n)$, we can pick a lifting $\iota_{[0,1]\times\{p\}} : [0,1] \times \{p\} \to P_{{\rm Spin}(n)}(\mathcal{L})|_{[0,1]\times\{p\}}$ of $\sigma_{[0,1]\times\{p\}}$ that satisfies

$$\iota_{[0,1]\times\{p\}}(i) = \iota_i^\sigma(p), \quad i = 0,\ 1. \tag{15.6.50}$$

This provides a spin structure of $\mathcal{L}$ covering the section σ of $P_{SO(n)}(\mathcal{L})|_{[0,1]\times\{p\}}$ along $[0,1] \times \{p\}$. Owing to the consistency condition (15.6.50), we can glue the spin bundles $P_{\rm spin}(\mathcal{L} \oplus V)|_{[0,1]\times\{p\}}$ and $P_{\rm spin}(\{i\} \times L_i)$, $i = 0,\ 1$ to obtain a spin structure of the bundle $\mathcal{L} \to K(L_0, L_1)_{[2]}$.

Then, by repeating the same construction as $o_{[p,w]}$ for this canonical path α_p using the above fixed spin structure on the Lagrangian subbundle $\mathcal{L} \to$

$K(L_0, L_1)$ of $\mathrm{pr}^*TM \to K(L_0, L_1)$, we obtain an orientation for the determinant line bundle

$$\det \overline{\partial}_{([p,w];\alpha_p)}.$$

We denote this orientation by o_p for each $p \in L_0 \cap L_1$.

Coherent orientations on $\mathcal{M}(p, q; B)$

Recall that the boundary of $\partial\mathcal{M}(p, r; B)$ consists of the union of the connected components of the type

$$\mathcal{M}(p, q; B_1)\#\mathcal{M}(q, r; B_2), \quad B = B_1\#B_2.$$

Let $\partial o(p, r; B)$ be the induced boundary orientation of the boundary $\partial\mathcal{M}(p, r; B)$. On the other hand, we denote by

$$o(p, q; B_1)\#o(q, r; B_2)$$

the natural orientation on the product $\mathcal{M}(p, q; B_1)\#\mathcal{M}(q, r; B_2)$ arising naturally by gluing the virtual tangent spaces.

Definition 15.6.6 (Coherent system of orientations) We call a collection $\{o(p, q; B)\}$ for $p,\ q \in L_0 \cap L_1$ and $B \in \pi_2(p, q)$ a *coherent system of orientations* if it satisfies the gluing rule

$$\partial o(p, r; B) = o(p, q; B_1)\#o(q, r; B_2) \tag{15.6.51}$$

whenever $\mathcal{M}(p, q; B_1)\#\mathcal{M}(q, r; B_2) \subset \partial\mathcal{M}(p, r; B)$ and whenever the virtual dimension of $\mathcal{M}(p, r; B)$ is 1.

The following is the counterpart of Proposition 15.6.1 in Floer homology.

Theorem 15.6.7 *Let (L_0, L_1) be a pair of oriented spin Lagrangian submanifolds intersecting transversely. Let α_p be the Lagrangian path in T_pM given by* (15.3.16) *for each $p \in L_0 \cap L_1$. Fix a choice of orientations o_p on* Index $\overline{\partial}_{\alpha_p}$ *for all p.*

(1) *It determines orientations on* (15.6.49)*, which we denote by $o_{[p,w]}$.*
(2) *Moreover, o_p, $o_{[p,w]}$ determine the orientations of $\mathcal{M}(p, q; B)$ denoted by $o(p, q; B)$ by the gluing rule*

$$o_{[q,w\#B]} = o_{[p,w]}\#o(p, q; B) \tag{15.6.52}$$

for all $p,\ q \in L_0 \cap L_1$ and $B \in \pi_2(p, q)$.
(3) *The system $\{o(p, q; B)\}$ defined as above is coherent.*

Proof From Proposition 15.6.4, the bundle (15.6.49) is trivial. For each p, we are given the elliptic operators

$$\overline{\partial}_{\alpha_p,Z_+} : W^{1,p}_{\alpha_p}(Z_+; T_pM) \to L^p(Z_+; T_pM \otimes \Lambda^{0,1}(Z_+)).$$

For $R > 0$, we define the space $(\mathbb{R} \times [0,1])\#_R Z_+$ as follows. We consider

$$Z_{+,R} = \{z \in Z_- \mid \operatorname{Re} z \geq -R\}.$$

We glue the two spaces, $[0,R] \times [0,1]$ and $Z_{+,R}$, by identifying $(R,t) \in Z_{-,R}$ with $(-R,t) \in [-R,R] \times [0,1]$ and $(R,t) \in [-R,R] \times [0,1]$ with $(-R,t) \in Z_{+,R}$, respectively.

For (1), we 'glue' operators

$$D_w\overline{\partial}_{(\mathbb{R}_{\geq 0}\times[0,1])}$$

and (15.6.48) in an obvious way to obtain an operator $D_w\overline{\partial}_{(\mathbb{R}_{\geq 0}\times[0,1])\#Z_+}$ on $(\mathbb{R}_{\geq 0} \times [0,1])\#Z_+$. We have an isomorphism of (family of) virtual vector spaces:

$$\operatorname{Index}(D_w\overline{\partial}_{(\mathbb{R}_{\geq 0}\times[0,1])\#Z_+}) \cong \operatorname{Index}\overline{\partial}_{([p,w];\lambda_{01})} \oplus \operatorname{Index}\overline{\partial}_{\alpha_p}. \tag{15.6.53}$$

We would like to remark that the space $\mathbb{R}_{\geq 0}\times[0,1]\#_R Z_+$ is conformally a 2-disc over which the pull-back tangent bundle is given together with a Lagrangian subbundle with spin structure along its boundary.

We fixed positively oriented frames of α_p and λ_{01}, which extend the frames (or trivializations) of TL_0 and TL_1 provided on the two skeletons of L_0 and L_1, respectively by the spin structures thereon. This trivialization induces a canonical orientation of the index bundle $\operatorname{Index}(D_w\overline{\partial}_{(\mathbb{R}_{\geq 0}\times[0,1])\#Z_+})$ by Proposition 15.6.4. Therefore the orientation of $\operatorname{Index}\overline{\partial}_{\alpha_p}$ (together with this canonical orientation) induces an orientation of $\operatorname{Index}\overline{\partial}_{([p,w];\lambda_{01})}$ in a canonical way. This implies statement (1).

Next we prove statement (2). For this, we glue Z_- with $\mathbb{R} \times [0,1]$ similarly to what we did above and derive the statement. We omit the details.

Finally, statement (3) is a consequence of (15.6.52). This finishes the proof. □

Incidentally, Theorem 15.6.7 also implies Theorem 15.6.3.

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Floer homology of monotone Lagrangian submanifolds

Unlike the case of $\pi_2(M, L) = \{e\}$, the moduli space $\mathcal{M}^{\text{reg}}([\ell_p, w], [\ell_q, w']; J)$ will not satisfy the properties required in Definition 13.8.1 in general because of the bubbling phenomenon. There are two different ways in which bubbling affects Lagrangian Floer theory.

(1) One comes from the phenomenon of bubbling off multiple covers of *negative* holomorphic spheres or discs. This is the phenomenon which already exists in the Floer homology theory of Hamiltonian periodic orbits (or in the Gromov–Witten theory).
(2) The other is the phenomenon of *anomaly appearance* $\partial \circ \partial \neq 0$ in the Floer theory of Lagrangian submanifolds.

The second phenomenon is closely related to the fact that the bubbling off of a disc is a phenomenon of *codimension one*, while the bubbling off of a sphere is one of codimension two in general. When ∂ does not satisfy $\partial \circ \partial \neq 0$, we call it a (Floer) *pre-boundary map*.

The problem of bubblings of negative Chern number or of negative Maslov indices can be treated in general by considering the Kuranishi structure and the perturbation theory of multi-valued sections (FOn99). Even when we consider the case where problem (1) is removed, as in the *semi-positive case*, the anomaly appearance (2) cannot be avoided. This is largely responsible for the appearance of A_∞ structure in the Floer theory of Lagrangian intersections (Fu93).

16.1 Primary obstruction and holomorphic discs

In this section, we attempt to extract the structure of the abstract Floer complex formulated in Section 13.8 from the pair (L_0, L_1) of compact Lagrangian submanifolds with $L_0 \pitchfork L_1$, and explain how an anomaly occurs in this open-string case, unlike in the case of closed strings.

Let (L_0, L_1) be a transversal pair. We denote

$$\mathrm{Int}(L_0, L_1; \ell_0) = \{x \in L_0 \cap L_1 \mid \ell_x \text{ lies in } \Omega(L_0, L_1; \ell_0)\},$$
$$CF(L_0, L_1; \ell_0) = \text{the } \Lambda(L_0, L_1; \ell_0)\text{-free module over } \mathrm{Int}(L_0, L_1; \ell_0).$$

Definition 16.1.1 We say (L_0, L_1) is *anomaly-free* or *unobstructed* if it carries $J = \{J_t\}$ that satisfies the following conditions.

(1) For any pair $\{x, y\} \subset \mathrm{Int}(L_0, L_1; \ell_0)$ satisfying

$$\mu(x, y; B) \leq 0,$$

$\widetilde{\mathcal{M}}(x, y; B) = \emptyset$ unless $x = y$ and $B = 0$ in $\pi_2(x, y)$. When $x = y$ and $B = 0$, the only solutions are the stationary solution, i.e., $u(\tau) \equiv x = y$ for all $\tau \in \mathbb{R}$.

(2) For any pair $\{x, y\} \subset \mathrm{Int}(L_0, L_1; \ell_0)$ and a homotopy class $B \in \pi_2(x, y)$ satisfying

$$\mu(x, y; B) = 1,$$

$\mathcal{M}(x, y; B) = \widetilde{\mathcal{M}}(x, y; B)/\mathbb{R}$ is Fredholm-regular and compact, and hence a finite set. We denote by

$$n(x, y; B) = \#\mathcal{M}(x, y; B)$$

the algebraic count of the elements of the space $\mathcal{M}(x, y; B)$. We set $n(x, y; B) = 0$ otherwise.

(3) For any pair $\{x, y\} \subset \mathrm{Int}(L_0, L_1; \ell_0)$ and $B \in \pi_2(x, y)$ satisfying

$$\mu(x, y; B) = 2,$$

$\mathcal{M}(x, y; B)$ can be compactified into a smooth one-manifold either without a boundary or with a boundary comprising the collection of the broken trajectories

$$u_1 \#_\infty u_2,$$

where $u_1 \in \mathcal{M}(x, y : B_1)$ and $u_2 \in \mathcal{M}(y, z; B_2)$ for all possible $y \in \mathrm{Int}(L_0, L_1; \ell_0)$ and $B_1 \in \pi_2(x, y)$, $B_2 \in \pi_2(y, z)$ satisfying

$$B_1 \# B_2 = B; \quad u_1 \in \mathcal{M}(x, y; B_1), \quad [u_2] \in \mathcal{M}(y, z; B_2)$$

and

$$\mu(x, y; B_1) = \mu(y, z; B_2) = 1.$$

We say that any such $J \in j_\omega = C^\infty([0,1], \mathcal{J}_\omega)$ is (L_0, L_1)-*regular* and any such triple $(L_0, L_1; J)$ is *Floer-regular*.

The main hypotheses in this definition concern certain compactness properties of the zero-dimensional (for (2)) and the one-dimensional (for (3)) components of $\mathcal{M}(L_0, L_1; J)$. Another important point to define a complex over the ring $\Lambda(L_0, L_1; \ell_0)$ is the existence of coherent orientations on the moduli spaces. For this purpose, we want the pair (L_0, L_1) to be *relatively spin* in the sense of (FOOO09). We denote by

$$j^{\mathrm{reg}}_{(L_0,L_1)} \subset j_\omega$$

the set of (L_0, L_1)-regular Js. For any given pair (L_0, L_1) intersecting transversely, $j^{\mathrm{reg}}_{(L_0,L_1)}$ is dense in j_ω.

At this stage, we mention that beyond the exact case unobstructed pairs in this strong sense are rather rare. However, there is one distinguished class of unobstructed pairs that was singled out in (Oh93a). This is the case of pairs of *monotone Lagrangian submanifolds* L which plays a significant role in the development of general Lagrangian Floer theory.

We start with a simple example we borrow from (Oh93b), (FOOO09) that illustrates the 'instanton effects' that cause an anomaly on the boundary property of the Floer boundary map.

Example 16.1.2 Consider $(M, \omega) = (\mathbb{C}, \omega_0)$ and

$$L_0 = \mathbb{R} + \sqrt{-1} \cdot 0, \quad L_1 = S^1 = \partial D^2(1).$$

Both L_0 and M are not compact. However, one can conformally compactify $\mathbb{C}$ to $\mathbb{P}^1 \cong S^2$ so that both $\mathbb{R}$ and S^1 become equators. L_0 and L_1 intersect at two points, which we denote by

$$p = (-1, 0), \quad q = (1, 0).$$

We now look at the moduli spaces $\mathcal{M}(p, q)$, $\mathcal{M}(q, p)$ and $\mathcal{M}(p, p)$. It is easy to see that $\pi_2(p, q) = \pi_2(L_0, L_1; p, q)$ is a principal homogeneous space of $\pi_2(\mathbb{C}, S^1) \cong \pi_1(S^1)$. Denote by $B_1 \in \pi_2(p, q)$ the homotopy class represented by the obvious upper semi-disc. Similarly, we denote by $B_2 \in \pi_2(q, p)$ the class represented by the lower semi-disc. Then we denote by $B \in \pi_2(p, p)$ the homotopy class given by

$$B = B_1 \# B_2.$$

By a Maslov index calculation, we derive

$$\mu(p, q; B_1) = \mu(q, p; B_2) = 1$$

and hence $\mu(p, p; B) = 2$.

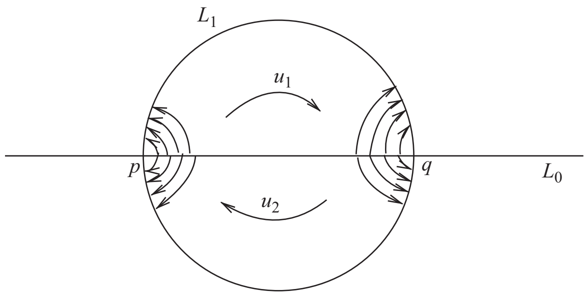

Figure 16.1 Holomorphic maps u_1, u_2.

By a simple application of the Riemann mapping theorem with a boundary, we prove that $\mathcal{M}(p, q; B_1)$ has a unique element that is represented by $u_1 : \mathbb{R} \times [0, 1] \to \mathbb{C}$ whose image is the obvious upper semi-disc (Figure 16.1). Similarly, $\mathcal{M}(q, p; B_2)$ has the unique element whose image is the lower semi-disc. And by a simple linear analysis of the Riemann–Hilbert problem, one can also prove that these maps are also regular in that its linearization map is surjective. By a dimensional consideration, we derive the formula

$$\partial([\ell_p, w]) = [q, w\#B_1], \quad \partial([\ell_q, w']) = [p, w'\#B_2]$$

for all bounding discs w of ℓ_p and w' of ℓ_q.

Exercise 16.1.3 Prove this claim.

Now we analyze the moduli space $\mathcal{M}(p, p; B)$. Considering each element of $\mathcal{M}(p, p; B)$ as the unparameterized curve corresponding to an element of $\widetilde{\mathcal{M}}(p, p; B)$, we can prove by the Riemann mapping theorem that it consists of the holomorphic maps $u_\ell : \mathbb{R} \times [0, 1] \to \mathbb{C}$ for $-1 < \ell < 1$ that satisfy

$$u(\mathbb{R} \times \{0\}) \subset \mathbb{R} \times \{0\}, \qquad u(\mathbb{R} \times \{1\}) \subset \partial D^2,\ [u_\ell] = B,$$
$$u_\ell(-\infty, \cdot) = u_\ell(\infty, \cdot) = \ell_p, \qquad u_\ell(0, 0) = (\ell, 0).$$

By the symmetry consideration, we derive that u_ℓ must also satisfy

$$u_\ell(0, 1) = (1, 0).$$

One can show that the above family does indeed comprise all the elements of $\mathcal{M}(p, p; B)$ and all of these maps are Fredholm-regular. From this description, a natural compactification of $\mathcal{M}(p, p; B)$ is the one obtained by adding the broken trajectory

$$u_1\#u_2$$

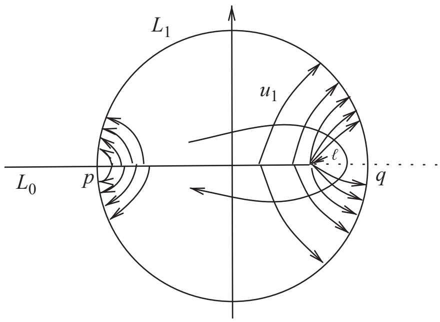

Figure 16.2 Holomorphic maps u_ℓ.

which corresponds to the limit as $\ell \to 1$, and the other end in the limit as $\ell \to -1$. See Figure 16.2. We note that the map u_ℓ satisfies the following.

(1) $u_\ell(\mathbb{R} \times \{0\}) \subset \{(t, 0) \mid -1 < t \le \ell\}$ and, in particular, as $\ell \to -1$ the $u_\ell|_{\mathbb{R}\times\{0\}}$ converges to the constant map $(-1, 0)$.
(2) The image of $u_\ell|_{\mathbb{R}\times\{1\}}$ wraps around the boundary ∂D^2 exactly once.

One can also check that, on any compact subset $K \subset \mathbb{R} \times [0, 1] \setminus \{(0, 1)\}$, we have

$$||du_\ell||_{\infty,K} \to 0 \quad \text{as } \ell \to -1$$

and

$$|du_\ell(0, 1)| \nearrow \infty.$$

From this analysis, we conclude that the real scenario behind the above picture as $\ell \to -1$ is the appearance of the *stable trajectory* corresponding to the following singular curve:

$$(u_\infty, \mathbb{R} \times [0, 1], \{(0, 1)\}) \cup (w, D^2, \{pt\}),$$

where $(u_\infty, \mathbb{R} \times [0, 1], (0, 1))$ is the principal component, $u_\infty : \mathbb{R} \times [0, 1] \to \mathbb{C}$ is the constant map $p = (-1, 0)$ with its domain given by

$$(\mathbb{R} \times [0, 1], \{(0, 1)\}) \cong (D^2 \setminus \{\pm 1\}, \{i\})$$

and $w : (D^2, \partial D^2) \to (\mathbb{C}, L_1)$ is the obvious inclusion map with its domain being the disc D^2 with one marked point on the boundary ∂D^2.

We note that this configuration is an admissible stable trajectory: u_∞ is stable because its domain has three special points on the boundary, and w is stable because it is a non-constant map and so both maps have trivial automorphism groups. This analysis of $\mathcal{M}(p, p; B)$ computes the matrix elements

$$\langle (\partial \circ \partial)([\ell_p, w]), [\ell_p, w\#B] \rangle = 1$$

and

$$\langle (\partial \circ \partial)([\ell_p, w]), [\ell_p, w\#B'] \rangle = 0$$

for any other $B' \in \pi_2(p, p)$ with $B' \neq B$. This clearly shows that

$$\partial \circ \partial \neq 0.$$

This example illustrates that the boundary of $\mathcal{M}(p, p; B)$ with $\mu(B) = 2$ can have a boundary component that is not of the type of broken trajectories. Furthermore, we note that *the Maslov index of the disc w above is two*. We will see later that the latter fact is not coincidental.

In fact, the obstruction described in this example is the only hindrance to defining the Floer homology for embedded Lagrangian submanifolds on a compact Riemann surface without a boundary. On surfaces, any one-dimensional submanifold is Lagrangian and hence a compact embedded Lagrangian submanifold is a finite disjoint union of embedded circles. We summarize this discussion in the following general theorem on surfaces.

Theorem 16.1.4 *Let Σ be an oriented compact surface with an area form. Denote by $\mathcal{E}_\Sigma^{ad}$ the collection of embedded Lagrangian submanifolds none of whose connected components are null-homotopic. Then, for any L_1, $L_2 \in \mathcal{E}_\Sigma^{ad}$, the Floer homology $HF(L_1, L_2; \Lambda)$ with a suitably chosen Novikov ring is well defined.*

In this regard, we give the following definition.

Definition 16.1.5 Let Σ be an oriented compact surface with an area form. An embedded curve $\gamma \subset \Sigma$ (not necessarily connected) is called *unobstructed* if no connected component is null-homotopic.

This definition is a special case of the one introduced in (FOOO09) restricted to the Lagrangian submanifolds on two-dimensional surfaces. We refer readers to (Ab08) for a detailed study of the Fukaya category of higher-genus surfaces, in which Abouzaid computed the Grothendieck group of the derived Fukaya category on Σ.

16.2 Examples of monotone Lagrangian submanifolds

As we indicated by two-dimensional examples, Floer homology for the general pair of Lagrangian submanifolds (L_0, L_1) meets an obstruction. A full study of this obstruction and its deformation theory requires mathematics beyond the level of the current book, which has been thoroughly carried out in (FOOO09). We refer to this book those who would like to learn more about the details.

Floer introduced the notion of monotone symplectic manifolds in (Fl89b) to overcome the transversality difficulty in his study of Hamiltonian Floer homology.

Remark 16.2.1 A general approach towards the study of transversality using perturbations other than almost-complex structures was first proposed by Kontsevich in the moduli problem related to the general construction of Gromov–Witten invariants in his celebrated paper (Kon95).

Nowadays there are essentially two different approaches to this general perturbation transversality theory; one is the *finite-dimensional on-shell approach* and the other is the *infinite-dimensional off-shell approach*. The first is the approach taken by Fukaya and Ono (FOn99), the method of Kuranishi structure and multi-sections, which is an enhancement of the classical Lyapunov reduction method to solve a nonlinear equation. (See also (LT98) for a related construction.) It is also the approach taken by algebraic geometers in (LiT98), (Be97), (BF97). The latter is the approach taken by Siebert (Sie99), and more recently represented by Hofer, Wysocki and Zehnder's polyfold approach (HWZ07), (H08) in which it is attempted to get a global perturbation of the original equation in the off-shell setting. (See also (Ru99) for a related construction.)

Even after 20 years since (FOn99) appeared, this transversality issue still remains an issue requiring further clarification even among the experts in symplectic geometry.

Motivated by Floer's study of monotone symplectic manifolds, the current author singled out the class of *monotone* Lagrangian submanifolds in (Oh93a) in relation to extending Floer's construction of homology theory for the pair (L_0, L_1) beyond the exact cases. This class of Lagrangian submanifolds avoids most of the sophisticated transversality issues of compactified moduli spaces of open Riemann surfaces (of genus zero) and allows one to extend Floer's definition seamlessly and define the map $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ following Floer's original definition as given in Definition 16.1.1.

However, it was already noted in (Oh93a) and later further highlighted in (Oh95a) that, even in this good case, an anomaly can occur, i.e.,

$$\partial^2 \neq 0 \tag{16.2.1}$$

unless the pair (L_0, L_1) satisfies a certain compatibility condition that is closely related to the geometry of holomorphic discs with Maslov index 2.

Remark 16.2.2 The role of monotonicity in general symplectic topology has not been completely clear to the author because it is neither a natural nor a flexible condition from the point of view of intrinsic symplectic topology. However, it is rich enough to illustrate how the holomorphic discs create *anomaly* and *obstruction* in Floer homology, and to test various Floer-theoretic constructions in a *non-exact* context. It also nicely generalizes the discussion on the Floer homology of Lagrangian submanifolds on surfaces when this obstruction is taken care of in one way or another. It is a good testing ground of various algebraic, geometric structures before making an attempt to fully implement them in the context of general Lagrangian submanifolds using the obstruction–deformation theory and the technology of Kuranishi structures. Recently there have been discovered many interesting examples of monotone Lagrangian tori in relation to (almost) toric manifolds and mirror symmetry. See (FOOO12a), (Wu12), (Via13, Via14).

The following is the analog to the definition of $G(L_0, L_1; \ell_0)$ for a single Lagrangian submanifold, which is also an analog to the group

$$\Gamma_\omega = \Gamma(M, \omega) = \frac{\pi_2(M)}{\ker \omega \cap \ker c_1}$$

of the ambient symplectic manifold (M, ω). Denote by $I_\omega : \pi_2(M, L) \to \mathbb{R}$ the evaluations of the symplectic area. We also define another integer-valued homomorphism $I_\mu : \pi_2(M, L) \to \mathbb{Z}$ by

$$I_\mu(\beta) = \mu\,(w^*TM, (\partial w)^*TL)\,,$$

which is the Maslov index of the bundle pair $(w^*TM, (\partial w)^*TL)$ for a particular (and hence any) representative $w : (D^2, \partial D^2) \to (M, L)$ of β.

Definition 16.2.3 We define

$$G(L; \omega) = \frac{\pi_2(M, L)}{\ker I_\omega \cap \ker I_\mu}$$

and define $\Lambda_{(L;\omega)}$ as the associated Novikov ring.

We briefly recall the basic properties on the Novikov ring $\Lambda_{(\omega,L)}(R)$, and its subring $\Lambda_{0,(\omega,L)}(R)$, where R is a commutative ring. For example, R could be $\mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{Q}$. We put

$$q^\beta = T^{\omega(\beta)} e^{\mu_L(\beta)},$$

and

$$\deg(q^\beta) = \mu_L(\beta), \quad E(q^\beta) = \omega(\beta),$$

which makes $\Lambda_{(\omega,L)}$ and $\Lambda_{0,(\omega,L)}$ become a graded ring in general. We have the canonical valuation $\nu : \Lambda_{(\omega,L)} \to \mathbb{R}$ defined by

$$\nu\left(\sum_\beta a_\beta T^{\omega(\beta)} e^{\mu_L(\beta)}\right) = \min\{\omega(\beta) \mid a_\beta \neq 0\}.$$

It induces a valuation on the subring $\Lambda_{0,(\omega,L)} \subset \Lambda_{(\omega,L)}$, which induces a natural filtration on it. This makes $\Lambda_{(\omega,L)}$ a filtered graded ring. For a general Lagrangian submanifold, this ring need not even be Noetherian.

Here is the definition of a monotone Lagrangian submanifold.

Definition 16.2.4 A compact Lagrangian submanifold $L \subset (M, \omega)$ is called *monotone* if

$$I_\omega = \lambda I_\mu \quad \text{for some } \lambda \geq 0.$$

We define the *minimal Maslov number* Σ_L to be the positive generator of the image of I_μ.

Exercise 16.2.5 Prove that monotone Lagrangian submanifolds exist only on monotone symplectic manifolds.

Note that this case includes the so-called *weakly exact* case, i.e., the case of $I_\omega = 0$, in a trivial way. This case can be studied in exactly the same way as in the exact case $\pi_2(M, L) = \{0\}$, if one ignores the grading problem. Therefore we assume that $L \subset M$ is monotone with $\lambda > 0$ and $I_\mu \neq 0$ in our discussion in this subsection.

The following proposition illustrates some special symplectic topology of this class of Lagrangian submanifolds against the general ones.

Proposition 16.2.6 *Let $L \subset (M, \omega)$ be a monotone Lagrangian submanifold. Then $G(L; M, \omega)$ is a free abelian group of rank* 1 *and $\Lambda_{0,(\omega,L)}$ becomes a discrete valuation ring whose residue field is $\Lambda_{(\omega,L)}$.*

The $\mathbb{R}P^n \subset \mathbb{C}P^n$ or the Clifford torus considered in Example 5.19 are examples of monotone Lagrangian submanifolds.

Proposition 16.2.7 *Consider the Clifford torus $T^n \subset \mathbb{P}^n$,*

$$T^n = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n \mid |z_i| = 1\}.$$

This torus is monotone and its associated group $G(T^n; \mathbb{P}^n)$ is isomorphic to $\mathbb{Z}$.

Proof It is easy to check that $\pi_2(\mathbb{P}^n, T^n)$ splits according to

$$\pi_2(\mathbb{P}^n, T^n) \cong \pi_1(T^n) \oplus \pi_2(\mathbb{P}^n).$$

If we denote by β_i the class of the disc

$$D_i = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n \mid z_j \equiv e^{\sqrt{-1}\theta_i^0} \text{ fixed except for } j = i\},$$

then we have the relation

$$\beta_0 + \cdots + \beta_n = \alpha \quad \text{in } \pi_2(\mathbb{P}^n, T^n),$$

where α is the (positive) generator of $\pi_2(\mathbb{P}^n)$. Furthermore, it follows that

$$\begin{aligned} \mu(\beta_i) &= \quad 2 = \frac{c_1(\alpha)}{n+1}, \\ \omega(\beta_i) &= \frac{\omega(\alpha)}{n+1} = 2\pi. \end{aligned} \tag{16.2.2}$$

This finishes the proof of monotonicity.

In addition (16.2.2) also implies that all β_i define the *same* element, which we denote by β, in $G(T^n; \mathbb{P}^n)$ and $\alpha = (n+1)\beta$. Hence we have proved

$$G(T^n; \mathbb{P}^n) = \mathbb{Z} \cdot [\beta] \cong \mathbb{Z}.$$

□

Next we consider other standard tori given by

$$T_{(c_0,\cdots,c_n)} = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid |z_i| = c_i > 0\}.$$

These are not monotone. We denote by β_i and α the classes defined similarly to in the above case where $c_0 = \cdots = c_n$. Depending on the rational dependence of the numbers

$$\{\omega(\beta_0), \ldots, \omega(\beta_n), \omega(\alpha) = 2\pi\}$$

the ranks of $G(T_{(c_0,\cdots,c_n)}; \mathbb{P}^n)$ vary between 1 and $n+1$. This example can be further generalized to the case of Lagrangian torus fibers of toric manifolds, which was carefully analyzed in (CO06), (FOOO10b, FOOO11a).

We remark that the monotonicity condition is devised mainly for the purpose of finding a condition with which the simple-minded Floer homology as defined in (Oh93a) can be constructed. However, the condition is preserved under the Lagrangian suspension construction explained in Section 3.7.2.

Proposition 16.2.8 *For any compact monotone Lagrangian embedding $L \subset (M, \omega)$ and its exact Lagrangian loop (h, ψ) in the sense of Section 3.7.2, their associated suspension*

$$\iota_{(h,\psi)} : S^1 \times L \to T^*S^1 \times M$$

is a monotone Lagrangian embedding with respect to the direct-sum symplectic form $\omega_0 \oplus w$ on $T^*S^1 \times M$ with the same monotonicity constant.

Exercise 16.2.9 Prove this proposition.

We also recall from Proposition 3.7.7 that $\iota_{(h,\psi)}$ is Hamiltonian isotopic to the product embedding $o_{S^1} \times L$ in $T^*S^1 \times M$.

In his study of the Hofer diameter of $\mathrm{Ham}(S^2)$, Polterovich exploited this proposition and applied a version of the Künneth formula for Lagrangian Floer homology to the equator of S^2 and gave an ingenious proof of the following theorem, whose proof will be given in Section 17.6 following the original article (Po98b).

Theorem 16.2.10 (Polterovich) *The Hofer diameter of* $\mathrm{Ham}(S^2)$ *is infinite.*

Next we explain a variant of Chekanov's tori (Che96b) described in Section 3.7.3

Example 16.2.11 (Chekanov torus in $S^2 \times S^2$) Originally Chekanov's torus was constructed in $\mathbb{C}^2$ using a suspension construction as described in Section 3.7.3 applied to the standard circle $S^1 \subset \mathbb{C}$. (See Example 16.3.6(3).) With a suitable choice of parameters appearing in Chekanov's suspension, this torus can be embedded into $S^2 \times S^2$ as an exotic monotone Lagrangian torus in that it is not Hamiltonian isotopic to the product $S^1_{\mathrm{eq}} \times S^1_{\mathrm{eq}} \subset S^2 \times S^2$ (see (EP10), (FOOO12a) for the proof). Its construction can be described as follows (see (AF08) for this description). Identify $S^2 \times S^2 \setminus \overline{\Delta}$ with the unit disc bundle $D^1(TS^2)$. The Lagrangian torus studied in (AF08) is the union of simple closed oriented geodesics lying on $\partial D^r(TS^2)$ passing through a given point, say the north pole N. With a suitable choice of $r > 0$, this torus can be shown to be monotone in $S^2 \times S^2 \supset D^1(TS^2)$. Denote this torus by $T(r)$.

Exercise 16.2.12 Prove that there exists a unique choice of $0 < r < 1$ in the above example with respect to which the torus $T(r)$ becomes monotone, and find the value r.

Chekanov and Schlenk (CS10) constructed many monotone Lagrangian tori in symplectic vector space or in complex projective spaces and studied which of their examples are distinct from one another.

16.3 The one-point open Gromov–Witten invariant

One of the nice properties of monotone Lagrangian submanifolds is the absence of the *negative multiple-cover problem* mentioned in Section 10.6. This is because any non-constant pseudoholomorphic disc has a positive Maslov index. This enables one to define an *open Gromov–Witten invariant* independently of the choice of compatible almost-complex structures.

In this section, we give the construction of this *one-point invariant* and prove its symplectic invariance following the explanation provided in (Oh95a).

For a compatible almost-complex structure J_0, we define

$$\widetilde{\mathcal{M}}(L;\beta;J_0) = \{w : (D^2, \partial D^2) \to (M, L) \mid \overline{\partial}_J w = 0,\ w(\partial D^2) \subset L,\ [w] = \beta\}$$

and its quotient

$$\mathcal{M}(L;\beta;J_0) = \widetilde{\mathcal{M}}(L;\beta;J_0)/PSL(2,\mathbb{R}).$$

Recall the index formula

$$\text{vir.}\dim \widetilde{\mathcal{M}}(L;\beta;J_0) = n + \mu_L(\beta), \tag{16.3.3}$$

where $\mu_L(\beta)$ is the Maslov index $\mu(w^*TM, (\partial w)^*L)$ for the bundle pair $(w^*TM, (\partial w)^*L)$ for a particular (and hence any) element $w \in \widetilde{\mathcal{M}}(L;\beta;J_0)$. We also consider the moduli space $\mathcal{M}_1(L;\beta;J_0)$ of J_0-holomorphic disc with one marked point.

For a given smooth path $J : [0,1] \to \mathcal{J}_\omega$, we define the parameterized moduli spaces

$$\mathcal{M}^{\text{para}}(L;\beta;J) = \bigcup_{t\in[0,1]} \{t\} \times \mathcal{M}(L;\beta;J(t)),$$
$$\mathcal{M}_1^{\text{para}}(L;\beta;J) = \bigcup_{t\in[0,1]} \{t\} \times \mathcal{M}_1(L;\beta;J(t))$$

as a fibration over $[0,1]$. We state the following compactness result.

Proposition 16.3.1 *Suppose $L \subset (M,\omega)$ is monotone with its minimal Maslov number $\Sigma_L \geq 2$. Let $\beta \in \pi_2(M,L)$ satisfy $\mu(\beta) = 2$. Then the following statements hold.*

(1) *For any compatible almost-complex structure J_0, both $\mathcal{M}(L;\beta;J_0)$ and $\mathcal{M}_1(L;\beta;J_0)$ are compact.*
(2) *For a path $J : [0,1] \to \mathcal{J}_\omega$, the parameterized moduli spaces*

$$\mathcal{M}^{\text{para}}(L;\beta;J), \quad \mathcal{M}_1^{\text{para}}(L;\beta;J)$$

are compact.

Proof This is an immediate consequence of the open version of the stable map compactness theorem. We will just prove the first statement. Let $w_i \in \widetilde{\mathcal{M}}(L;\beta;J_0)$ be a sequence. We need to prove that there exists a subsequence of w_i that converges, modulo the action of $PSL(2,\mathbb{R})$, to a stable map.

Since $[w_i] = \beta$ is fixed, we have

$$E_{J_0}(w_i) = \int w_i^*\omega = \omega(\beta)$$

and hence we have the uniform energy bound for the sequence w_i. Then the stable map compactness theorem says that we can choose a subsequence, still denoted by w_i, converging to a stable map

$$w_\infty = \sum_{\ell=1}^{N} v_\ell,$$

where each v_ℓ is either a J_0-holomorphic disc or a sphere. We denote by $\mu(v_\ell)$ either $2c_1(v_\ell)$ for a sphere v_ℓ or the Maslov index $\mu(v_\ell)$ for a disc with a boundary lying on L.

Since $\mu(\beta) = 2$ is minimal and $2 = \mu(w_\infty) = \sum_{\ell=1}^{N} \mu(v_\ell)$, monotonicity implies that there cannot be more than one non-constant component in the above sum. On the other hand, the stability of w_∞ implies that there cannot be any constant component either. Therefore we must have $N = 1$ and hence a suitable reparameterization of w_i indeed converges. This finishes the proof.

A similar argument proves the compactness of $\mathcal{M}_1(L;\beta;J_0)$ and that of parameterized spaces. □

Next we recall a transversality result in terms of the perturbation of almost-complex structures. Let $L \subset (M,\omega)$ be any compact monotone Lagrangian submanifold. Denote by

$$\widetilde{\mathcal{M}}^{\text{inj}}(L;\beta;J_0) \subset \widetilde{\mathcal{M}}(L;\beta;J_0)$$

the subset consisting of somewhere injective J_0-holomorphic discs.

The following basic transverality result was proved in Sections 10.4 and 10.5 for the case of closed surfaces.

Proposition 16.3.2 *Let $x \in L$. There exists a dense subset $\mathcal{J}_\omega(L, x) \subset \mathcal{J}_\omega$ such that, for all $\beta \in \pi_2(M, L)$,*

(1) $\widetilde{\mathcal{M}}^{\mathrm{inj}}(L;\beta;J_0)$ *is transverse*
(2) $\mathrm{ev}_\beta : \mathcal{M}_1^{\mathrm{inj}}(L,\beta;J_0) \to L$ *is transverse to x.*

The same holds for the parameterized moduli spaces for $J = \{J(t)\}_{0\le t\le 1}$.

We call such a pair (J_0, x_0) a transverse pair. For such a pair (J_0, x_0), Propositions 16.3.1 and 16.3.2 imply that $\mathrm{ev}_\beta^{-1}(x_0)$ is a compact zero-dimensional manifold (if nonempty) for any β with $\mu(\beta) = 2$.

Lemma 16.3.3 *Let J_0 be any compatible almost-complex structure, which is not necessarily regular. Let $L \subset (M, \omega)$ be monotone with $\Sigma_L > 0$. Consider $\beta \in \pi_2(M, L)$ with $\mu(\beta) = \Sigma_L$. Then we have*

$$\widetilde{\mathcal{M}}^{\mathrm{inj}}(L;\beta;J_0) = \widetilde{\mathcal{M}}(L;\beta;J_0),$$

i.e., all J_0-holomorphic discs in $\widetilde{\mathcal{M}}(L;\beta;J_0)$ are somewhere injective for any $J_0 \in \mathcal{J}_\omega$. In particular, Proposition 16.3.2 holds for the full moduli spaces.

Proof The first statement follows from the definition of the minimal Maslov number Σ_L, which implies that any J_0-holomorphic disc, irrespective of whether J_0 is regular or not, must be simple by virtue of the structure theorem of the image, Theorem 8.7.3 for the spheres, and (Laz00), (KO00) for the discs.

Then the second follows immediately from the monotonicity hypothesis and $\mu(\beta) = \Sigma_L$, which prevents bubbling from happening. □

We denote

$$\Phi_{J_0}(L; x_0) = \sum_{\beta;\mu(\beta)=2} \#(\mathrm{ev}_\beta^{-1}(x_0)).$$

Here and henceforth, without further mentioning it, the counting will be done with $\mathbb{Z}_2$-coefficients for general L while $\mathbb{Z}$-coefficients will be used for the case of the (relative) spin Lagrangian submanifold L.

Lemma 16.3.4 *Let $L \subset (M, \omega)$ be monotone with $\Sigma_L \ge 2$. Let (J_0, x_0) and (J_1, x_1) be transverse pairs with $[x_0] = [x_1]$ in $\pi_0(L)$. Then*

$$\Phi_{J_0}(L; x_0) = \Phi_{J_1}(L; x_1).$$

Proof We fix an embedded path $\gamma : [0,1] \to L$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$. By the parameterized version of the evaluation transversality and the transversality extension theorems, we can choose a path $J : [0,1] \to \mathcal{J}_\omega$ such that $\widetilde{\mathcal{M}}^{\text{para}}(L;\beta;J)$ is transverse for all β with $\mu(\beta) = 2$ and the evaluation map

$$\mathrm{Ev}_\beta : \mathcal{M}_1^{\text{para}}(L;\beta;J) \to [0,1] \times L; \quad (t,(w,z)) \mapsto (t, w(z))$$

is transverse to $\Lambda = id \times \gamma : [0,1] \to [0,1] \times L$. The index formula gives

$$\dim \mathcal{M}_1^{\text{para}}(L;\beta;J) = n + 1$$

and hence $\mathrm{Ev}_\beta^{-1}(\Lambda)$ is a compact one-dimensional manifold with a boundary given by

$$\{0\} \times \mathrm{ev}_\beta^{-1}(x_0) \coprod \{1\} \times ev_\beta^{-1}(x_1).$$

Hence we have

$$\#(\mathrm{ev}_\beta^{-1}(x_0)) = \#(\mathrm{ev}_\beta^{-1}(x_1))$$

by the classification theorem of compact one-manifolds. By definition, this finishes the proof. □

This lemma enables us to define the one-point GW-invariant as follows.

Definition 16.3.5 Let $L \subset (M,\omega)$ be a compact monotone Lagrangian submanifold with $\Sigma_L \geq 2$, and denote by π an element of $\pi_0(L)$, i.e., a connected component of L. Then the *one-point GW-invariant relative to* π is given by

$$\Phi(L;\pi) = \Phi_{J_0}(L;x_0)$$

for a transversal pair (J_0, x_0) with $[x_0] = \pi$ in $\pi_0(L)$. When L is connected, we just denote $\Phi(L;\{pt\}) = \Phi_{J_0}(L;x_0)$, which we call the one-point GW-invariant of $L \subset (M,\omega)$.

Now we will examine a few examples.

Example 16.3.6

(1) **(Standard torus)** Consider the standard torus

$$T^n = \overbrace{S^1(1) \times \cdots \times S^1(1)}^{n\text{-times}} \subset \mathbb{C}^n \cong \mathbb{R}^{2n}.$$

We have $\pi_2(\mathbb{C}^n, T^n) \cong \pi_1(T^n) \cong \mathbb{Z}^n$. Denote by β_i, $i = 1,\ldots,n$, the obvious generators of $\pi_2(\mathbb{C}^n, T^n)$ represented by the map

$$w_i : (D^2, \partial D^2) \to (\mathbb{C}^n, T^n); \quad w_i(z) = (\underbrace{1,\ldots,z}_{i\text{th}},\ldots,1).$$

By computing their Maslov index and symplectic areas, we derive that this torus is monotone with monotonicity constant $\lambda = \pi$. Obviously these maps are holomorphic passing through the point $x_0 = (1, \ldots, 1)$ and hence we have at least n distinct holomorphic discs with Maslov index 2.

(2) **(Clifford torus)** This time we consider the *Clifford torus* $T^n \subset \mathbb{C}P^n$. This torus is defined by the quotient $T^{n+1}/S^1 \subset S^{2n+1}/S^1 \cong \mathbb{C}P^n$ of the product circle action on $\mathbb{C}^{n+1}$. In projective coordinates we have the following holomorphic discs

$$w_j : (D^2, \partial D^2) \to (\mathbb{C}P^n, T^n); \quad z \mapsto [\underbrace{1; \ldots; z}_{j\text{th}}; \ldots; 1]$$

for $j = 1, \ldots, n+1$. We leave the proof of $\Phi(T^n, \mathbb{C}P^n; \{pt\}) = n + 1$ as an exercise.

(3) **(Chekanov torus)** We apply Chekanov's suspension to the standard circle $S^1(1) \subset \mathbb{R}^2 \cong \mathbb{C}$ and obtain the Chekanov torus

$$\Theta_0(S^1(1)) \subset \mathbb{R}^2; \quad \Theta_0(S^1(1)) = I_1(N_0 \times S^1(1)).$$

Here N_0 is the zero section of T^*S^1 and $I_1 = (i_1^*)^{-1}$ is the symplectic embedding

$$I_1 = (i_1^*)^{-1} : T^*(\mathbb{R} \times S^1) \to T^*\mathbb{R}^2,$$

where $i_1 : S^1 \times \mathbb{R} \to \mathbb{R}^2$ is the map defined by

$$i_1(\theta, q) = (e^q \cos(2\pi\theta), e^q \sin(2\pi\theta)).$$

In (EP97), Eliashberg and Polterovich proved that

$$\Phi(\Theta_0(S^1(1)); \{pt\}) = 1.$$

Quite recently, Chekanov and Schlenk attempted to completely classify the set of monotone Lagrangian submanifolds on various explicit classes of symplectic manifolds (CS10).

Exercise 16.3.7 Prove that the discs given in Example 16.3.6 above for the standard torus in $\mathbb{C}^n$ and the Clifford torus in $\mathbb{C}P^n$ are all transverse and comprise all possible holomorphic discs of Maslov index 2 through a given point x_0 in the torus. (It turns out that, if we take the orientation of these discs into consideration, all of these discs contribute positively and so $\Phi(T^n, \mathbb{C}^n; pt) = n$ and $\Phi(T^n, \mathbb{C}P^n; \{pt\}) = n + 1$. See Section 15.4 for a study of the orientation.)

An immediate consequence of this computation of the one-point invariant gives rise to Chekanov's theorem (Che96b).

Theorem 16.3.8 (Chekanov) *The standard torus $T^2 = S^1(1) \times S^1(1)$ is not symplectically equivalent to the Chekanov torus $\Theta_0(S^1(1))$.*

Chekanov originally proved this theorem by examining the behavior of the sequence of Ekeland–Hofer capacities.

We would like to point out that both the standard torus and the Chekanov torus satisfy the following criteria:

(1) the minimal Maslov number is 2
(2) the monotonicity constant is π
(3) the first Ekeland–Hofer capacity is 2π.

Because of this, there is no simple classical way of distinguishing the standard torus and the Chekanov torus.

16.4 The anomaly of the Floer boundary operator

In this section, we examine the construction of the Floer boundary for pairs (L_0, L_1) of *monotone* Lagrangian submanifolds.

We analyze the structure of Floer moduli spaces of (13.9.28) of low virtual dimension, i.e., those with the Maslov–Viterbo indices 0, 1, 2, in particular.

Denote by $\widetilde{\mathcal{M}}([\ell_p, w], [\ell_q, w'])$ the set of smooth Floer trajectories, i.e., the union

$$\widetilde{\mathcal{M}}([\ell_p, w], [\ell_q, w']) := \bigcup_{w\#B\sim w'} \widetilde{\mathcal{M}}(p, q; B),$$

where $\mathcal{M}(p, q; B)$ is the set of smooth Floer trajectories connecting $p, q \in L_0 \cap L_1$ with $[u] = B$ in $\pi_2(p, q)$ defined before.

16.4.1 Definition of the Floer pre-boundary map

We first note that $\mu(\beta) > 0$ whenever β allows a non-constant pseudoholomorphic disc, provided that the monotonicity constant is positive.

We start with the following lemma.

Lemma 16.4.1 *Assume L is compact and consider any finite subset $F \subset L$. Then there exists a dense subset $\mathcal{J}_L(F)$ of $\mathcal{J}_\omega$ such that, for any $J_0 \in \mathcal{J}_L(F)$, the evaluation chains* $\mathrm{ev}_\beta : \mathcal{M}_1(L, J_0; \beta) \to L$ *do not intersect F for any $\beta \in \pi_2(L)$ with $\mu(\beta) = 1$.*

Proof By the index formula we have

$$\dim \mathcal{M}_1(L, J_0; \beta) = n + \mu(\beta) + 1 - 3 = n - 1 < n.$$

Since F is a subset of finite points, in particular of zero dimension, a dimension-counting argument based on the boundary version of evaluation transversality, Theorem 16.3.2, implies that the evaluation chain $\mathrm{ev}_\beta : \mathcal{M}_1(L, J_0; \beta) \to L$ does not intersect F for a generic choice of J_0. We denote by $\mathcal{J}_L(F)$ the subset consisting of such J_0s. This finishes the proof. □

We now prove the following theorem.

Theorem 16.4.2 *Assume that L_0, L_1 are monotone with non-zero minimal Maslov number and $J_i \in \mathcal{J}_{L_i}(L_0 \cap L_1)$, $i = 0, 1$, where $\mathcal{J}_{L_i}(L_0 \cap L_1)$ is as in Lemma 16.4.1. Then there exists a subset $j^{\mathrm{reg}}_{(L_0,L_1)}(J_0, J_1) \subset j_\omega$ such that for any $J \in j^{\mathrm{reg}}_{(L_0,L_1)}(J_0, J_1)$ the following statements hold.*

(1) *For all p, $q \in L_0 \cap L_1$ and $B \in \pi_2(p, q)$ with $\mu(p, q; B) < 0$, the moduli space $\mathcal{M}(p, q; B; J)$ is empty.*
(2) *For all p, $q \in L_0 \cap L_1$ and $B \in \pi_2(p, q)$ with $\mu(p, q; B) = 0$, the moduli space $\mathcal{M}(p, q; B; J)$ is empty unless $p = q$ and $B = 0$ in $\pi_2(p, q)$.*
(3) *For all p, $q \in L_0 \cap L_1$ and $B \in \pi_2(p, q)$ with $\mu(p, q; B) = 1$, the moduli space $\mathcal{M}(p, q; B; J)$ is a compact manifold of dimension 0 and hence is a finite set.*

Proof The proof is given by the dimension counting and intersection argument based on the index formula (16.3.3) and Propositions 15.1.5 and 16.3.2.

Recall from Lemma 15.1.4 that $\mathbb{R}$ acts freely on any non-constant trajectory u. (See (Oh97a) for the proof.) In particular, whenever $\widetilde{\mathcal{M}}(p, q; B; J) \neq \emptyset$,

$$\dim \widetilde{\mathcal{M}}(p, q; B; J) \geq 1$$

for any Fredholm-regular J, and for any $p \neq q$, or $p = q$ and $B \neq 0$. (We note that, when $p = q$, we may identify each element of $\pi_2(p, p)$ with one from $\pi_2(L_0)$ or $\pi_2(L_1)$.) Therefore (1) and (2) immediately follow for $J \in \mathcal{P}^{\mathrm{reg}}_{(L_0,L_1)}(J_0, J_1)$.

When $\mu(p, q; B) = 1$, $\mathcal{M}(p, q; B; J)$ has dimension 0 for any (L_0, L_1)-regular J. It remains to prove that $\mathcal{M}(p, q; B; J)$ is compact. If not, there is a sequence $u_i \in \mathcal{M}(p, q; B; J)$, which develops either splitting or bubbling as $i \to \infty$.

If u_i splits $(u_{\infty,1}, u_{\infty,2})$ in the limit for $u_{\infty,1} \in \mathcal{M}(p, r; B_1)$, $u_{\infty,2} \in \mathcal{M}(r, q; B_2)$, we have

$$1 = \mu(p, r; B_1) + \mu(r, q; B_2)$$

and hence $\mu(p, r; B_1) = 1$ and $\mu(r, q; B_2) = 0$ or the other way around. But $\mu(r, q; B_2) = 0$ implies that $\dim \widetilde{\mathcal{M}}(r, q; B_2) = 0$. This is impossible unless $r = q$ and $B_2 = 0$. Such a u then must be a constant strip, which is out of the question: such a disc has a continuous automorphism group of $\mathbb{R}$ arising from the τ-translations, so its domain is contracted by the definition of stable map compactfication of the moduli space. Note that $\mathbb{R} \times [0, 1]$ is conformally equivalent to the disc with two marked points, which is not a stable curve.

This proves that $\mathcal{M}(p, q; B; J)$ cannot have splitting ends.

If u_i bubbles off, monotonicity implies that any bubble has a positive index. Therefore a sphere bubble carries at least index $2 > 1$ and hence the dimension counting eliminates this bubbling. Then the only remaining possibility is the limit of the type (u_0, w), where $u_0 \in \mathcal{M}(p, q; B'; J)$ and w is a non-constant disc bubble for one of (L_0, J_0) or (L_1, J_1). We have

$$1 = \mu(p, q; B') + \mu(w).$$

Again monotonicity implies $\mu(w) > 0$, meaning that we must have $\mu(w) = 1$ and $\mu(p, q; B') = 0$. The latter implies $p = q$ and $B' = 0$ and hence u_0 is the constant trajectory $u_0 \equiv p$. Then the connectivity of the image of the Gromov–Floer limit implies that w passes through $p \in L_0 \cap L_1$. However, by the choice of J_i from $\mathcal{J}_{L_i}(L_0 \cap L_1)$, this case has also been ruled out by Lemma 16.4.1. □

Besides this compactness issue, another important issue is that of *coherent orientation* on our moduli space $\mathcal{M}([\ell_p, w], [\ell_q, w']; J)$. It was proven in Section 15.6 (see (FOOO09)) that, if the pair (L_0, L_1) is *relatively spin* and is given a relative spin structure, then there exists a canonical way to put a coherent orientation on the collections $\mathcal{M}([\ell_p, w], [\ell_q, w']; J)$. Therefore, if the pair (L_0, L_1) is also relatively spin, the *integer*

$$n([\ell_p, w], [\ell_q, w']; J) = \#(\mathcal{M}([\ell_p, w], [\ell_q, w']; J)) \quad \text{for } \mu[\ell_q, w'] = \mu([\ell_p, w]) + 1$$

is well defined. We define this as the matrix element of the standard Floer pre-boundary map $\partial : CF_R^k(L_1, L_0; \ell_0) \to CF_R^{k+1}(L_1, L_0; \ell_0)$ as follows:

$$\partial([\ell_p, w]) = \sum_{\mu(\ell_q, w') = \mu(\ell_p, w)+1} n([\ell_p, w], [\ell_q, w']) \cdot [\ell_q, w'].$$

Here $R = \mathbb{Z}$ when (L_0, L_1) is a relatively spin pair and $R = \mathbb{Z}_2$ in general. Gromov–Floer compactness implies that the right-hand side defines an element in $CF_R^*(L_1, L_0; \ell_0)$, i.e., satisfies the Novikov finiteness condition.

Because this map ∂ defined by Floer might not satisfy $\partial^2 = 0$, we will call ∂ the *Floer pre-boundary map*.

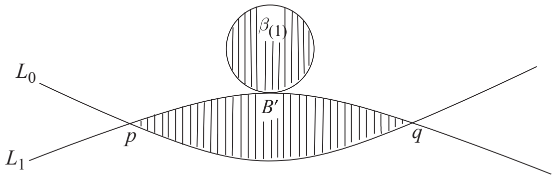

Figure 16.3 Codimension-1 strata $\mathcal{N}(p,q;B';L_1,\beta_{(1)};J)$.

16.4.2 Anomaly and obstruction

Now we study the boundary property of the Floer pre-boundary map ∂. For this purpose, we need to analyze the moduli spaces $\mathcal{M}(p,q;B)$ with virtual dimension 1, and describe singular strata of 'codimension 1' in the compactified moduli $\overline{\mathcal{M}}(p,q;B)$ for each given such B.

Consider the moduli space of Floer trajectories

$$\widetilde{\mathcal{M}}(p,q;B';J),$$

and consider the evaluation maps

$$\begin{aligned} \mathrm{ev}^{L_0} &: \widetilde{\mathcal{M}}(p,q;B';J) \to L_0; \quad (u,\tau) \mapsto u(\tau,0),\\ \mathrm{ev}^{L_1} &: \widetilde{\mathcal{M}}(p,q;B';J) \to L_1; \quad (u,\tau) \mapsto u(\tau,1). \end{aligned}$$

Next we consider $\mathcal{M}_1(L_0;\beta_{(0)};J_0)$, $\mathcal{M}_1(L_1;\beta_{(1)};J_1)$ and the evaluation maps

$$\mathrm{ev}_0 : \mathcal{M}_1(L_0;\beta_{(0)};J_0) \to L_0, \quad \mathrm{ev}_1 : \mathcal{M}_1(L_1;\beta_{(1)};J_1) \to L_1.$$

The following set provides a description of the codimension-1 strata of $\overline{\mathcal{M}}(p,q;B;J)$.

Definition 16.4.3 Let p, $q \in L_0 \cap L_1$ and $B' \in \pi_2(p,q)$ and $\beta_{(0)} \in G(L_0;\omega)$ and $\beta_{(1)} \in G(L_1;\omega)$. For each $i = 0, 1$, we put

$$\begin{aligned} \overset{\circ}{\mathcal{N}}(p,q;B' : L_0,\beta_{(0)};J) &= \widetilde{\mathcal{M}}(p,q;B';J)_{\mathrm{ev}^{L_0}} \times_{\mathrm{ev}_0} \mathcal{M}_1(L_0;\beta_{(0)};J_0)/\mathbb{R},\\ \overset{\circ}{\mathcal{N}}(p,q;B' : L_1,\beta_{(1)};J) &= \widetilde{\mathcal{M}}(p,q;B';J)_{\mathrm{ev}^{L_1}} \times_{\mathrm{ev}_1} \mathcal{M}_1(L_1;\beta_{(1)};J_1)/\mathbb{R}. \end{aligned}$$

See Figure 16.3. We define the moduli space $\mathcal{N}(p,q;B' : L_1,\beta_{(1)};J)$ as the closure of the moduli space $\overset{\circ}{\mathcal{N}}(p,q;B' : L_1,\beta_{(1)})$ in $\overline{\mathcal{M}}(p,q;B;J)$ with $B = B'\#\beta_{(1)}$. The definition of $\mathcal{N}(p,q;B' : L_0,\beta_{(0)};J)$ is similar and hence has been omitted.

Then we define

$$\mathcal{N}([\ell_p, w], [\ell_q, w'] : L_1; J) := \bigcup_{(B',\beta_{(1)});[\ell_p,w]\#(B'\#\beta_{(1)})=[\ell_q,w']} \mathcal{N}(p, q; B' : L_1, \beta_{(1)}; J) \tag{16.4.4}$$

and similarly for $\mathcal{N}([\ell_p, w], [\ell_q, w'] : L_0; J)$.

We would like to say that $\mathcal{N}(p, q, B' : L_i, \beta_{(i)}; J)$ with $B = B'\#\beta_{(i)}$ is of 'codimension 1' in $\overline{\mathcal{M}}(p, q; B; J)$. This statement will not make sense for general Lagrangian pair (L_0, L_1) even for a generic choice of $J = \{J_t\}$. However, this is really the case for a generic choice of J if (L_0, L_1) is monotone (or, more generally, if they are *semi-positive* and $\mu_L(\beta_{(i)}) \neq 0$).

Theorem 16.4.4 *Suppose that L_0, L_1 are monotone, J_0, $J_1 \in \mathcal{J}^0_\omega(L_0, L_1)$ and $J \in j^{\mathrm{reg}}_{(L_0,L_1)}(J_0, J_1)$ are as before. Let p, $q \in L_0 \cap L_1$ and $B \in \pi_2(p, q)$ such that*

$$\mu(p, q; B) = 2. \tag{16.4.5}$$

Then the following statements holds.

(1) *Either $\mathcal{N}(p, q; B' : L_i, \beta_{(i)}; J)$ is empty or $\mathcal{N}(p, q; B' : L_i, \beta_{(i)}; J)$ is a compact zero-dimensional manifold of the type*

$$p = q,\ B' = 0, \quad \mu(\beta_{(i)}) = 2,$$

which consists of the configurations $u\#w$, where $u \equiv p$ and w is a J_i-holomorphic disc passing through p.

(2) *$\overline{\mathcal{M}}(p, q; B; J)$ is a smooth manifold of dimension* 1 *with boundary given by the union*

$$\begin{aligned} &\bigcup_{r\in L_0\cap L_1; B_1\#B_2=B} \mathcal{M}(p, r; B_1; J)\#\mathcal{M}(r, q; B_2; J) \\ \cup &\bigcup_{(B',\beta_{(1)});[\ell_p,w]\#(B'\#\beta_{(0)})=[\ell_q,w']} \mathcal{N}(p, q; B' : L_0, \beta_{(0)}; J) \\ \cup &\bigcup_{(B',\beta_{(1)});[\ell_p,w]\#(B'\#\beta_{(1)})=[\ell_q,w']} \mathcal{N}(p, q; B : L_1, \beta_{(1)}; J), \end{aligned} \tag{16.4.6}$$

where $\mu(p, r; B_1) = \mu(r, q; B_2) = 1$ and $\mu(\beta_{(i)}) = 2$, $i = 0,\ 1$.

Proof This will again follow from a dimension counting that is based on the transversality of the moduli spaces and the evaluation maps.

We start with the proof of (1). We consider only the case $i = 1$ since the case for $i = 0$ will be the same. By definition, $\mathcal{N}(p, q; B' : L_1, \beta_{(1)}; J)$ is the fiber product

$$\frac{\widetilde{\mathcal{M}}(p,q;B';J)_{\mathrm{ev}^{L_1}} \times_{\mathrm{ev}_1} \mathcal{M}_1(L_1;\beta_{(1)};J_1)}{\mathbb{R}} = \frac{(\mathrm{ev}^{L_1} \times \mathrm{ev}_1)^{-1}(\Delta)}{\mathbb{R}}.$$

Here $\Delta \subset L_1 \times L_1$ is the diagonal. We recall that

$$\begin{aligned}\operatorname{vir\,dim} \mathcal{M}_{(0;0,1)}(p,q;B') &= \mu(p,q;B') - 1,\\ \operatorname{vir\,dim} \mathcal{M}_1(L_1,J_1;\beta_{(1)}) &= n + \mu(\beta_{(1)}) - 2\end{aligned}$$

and

$$\mu(p,q;B) = \mu(p,q;B') + \mu(\beta_{(1)}).$$

By the assumption $\mu(B) = 2$, this implies that

$$\mu(p,q;B') = 2 - \mu(\beta_{(1)})$$

and, by the transversality of $\widetilde{\mathcal{M}}(p,q;B';J)$, we have $\mu(p,q;B') \geq 0$.

At this stage, if $\Sigma_{L_1} \geq 3$, this already provides a contradiction and hence $\mathcal{N}(p,q;B' : L_1,\beta_{(1)}) = \emptyset$. A similar argument applies to $i = 0$. Therefore, if we assume that $\Sigma_{L_i} \geq 3$, $\mathcal{M}(p,q;B;J)$ is a compact one-manifold with a boundary only of the type of split trajectories.

Now we consider the case $\Sigma_{L_1} = 2$. In this case, it is possible that we have $\mu(p,q;B') = 0$ and $\mu(\beta_{(1)}) = 2$. Again by the transversality of $\widetilde{\mathcal{M}}(p,q;B';J)$, this happens only when

$$p = q \quad \text{and} \quad B' = 0.$$

Remark 16.4.5 We remark that, in this case, the associated map u is the constant map $u \equiv p$. We emphasize that the nodal curve

$$u\#w; \quad [u] = 0,\ [w] = \beta_{(1)}$$

is a stable map. This is because the component $u \equiv p; \mathbb{R} \times [0,1] \to M$ carries one additional special point $z_0 = (\tau_0, 1) \in \mathbb{R} \times \{1\}$ at which the bubble component w is attached. Since $\mathbb{R} \times [0,1] \setminus \{z_0\}$ is conformally isomorphic to $D^2 \setminus \{1, -1, \sqrt{-1}\}$, the constant component u defines a stable map and hence this configuration is an allowed nodal curve in the compactified Floer moduli space $\overline{\mathcal{M}}(p,q;B)$ with $B = B'\#\beta_{(1)}$.

We would like to point out that up until now we have not used any transversality statement of the disc moduli space $\mathcal{M}(L_i;\beta_{(i)})$ in the above argument but only the transversality of Floer trajectory moduli spaces.

It follows from the assumption $\Sigma_{L_1} = 2$ that every element of $\mathcal{M}(L_1;\beta_{(1)};J_1)$ must be somewhere injective. In fact, if $v \in \mathcal{M}(L_1;\beta_{(1)};J_1)$ is not somewhere injective, then the structure theorem of the image of pseudoholomorphic discs from [KOh00], [Laz00] implies that there exists v_i ($i = 1,\ldots,a$, $a \geq 2$) such that $\sum_{i=1}^{a}[v_i] = [v]$ and $\mathcal{M}(L_1;[v_i];J_1)$ is nonempty. This is impossible since

$\mu_{L_1}(v_i) > 0$. Hence $\mathcal{M}_1(L_1; \beta_{(1)}; J_1)$ is smooth and of dimension $n-1$ for $J_1 \in \mathcal{J}_{L_1}(L_0 \cap L_1)$.

We recall that for a generic choice of J the evaluation map

$$\mathrm{ev}^{L_1} \times \mathrm{ev}_1 : \widetilde{\mathcal{M}}([\ell_p, w], [\ell_q, w_1'']) \times \mathcal{M}_1(L_1; \beta_{(1)}; J_1)/\mathbb{R} \to L_1 \times L_1$$

is transverse to the diagonal $\Delta \subset L_1 \times L_1$ at any pair (u, w) for somewhere-injective J_1-holomorphic w. Then by the gluing theorem in Section 15.5, the moduli space $\overline{\mathcal{M}}(p, q; B; J)$ becomes a compact smooth one-manifold with boundary given by (16.4.6). This finishes the proof. □

By the cobordism argument, Theorem 16.4.4 implies the identity

$$\begin{aligned} 0 = &\Big\langle (\partial \circ \partial)([\ell_p, w]), [\ell_q, w'] \Big\rangle + \#\mathcal{N}\Big([\ell_p, w], [\ell_q, w'] : L_0\Big) \\ &+ \#\mathcal{N}\Big([\ell_p, w], [\ell_q, w'] : L_1\Big), \end{aligned} \tag{16.4.7}$$

where $\langle (\partial \circ \partial)([\ell_p, w]), [\ell_q, w'] \rangle$ denotes the coefficient of $[\ell_q, w']$ in $(\partial \circ \partial)([\ell_p, w])$. Here # is the order with sign.

Now we derive the consequence of (16.4.7). We divide the situation into the following two cases:

(1) the case where L_0, L_1 have $\Sigma_{L_i} > 2$,
(2) the case where one or both of L_0, L_1 have the minimal Maslov number 2.

We start with the case (1). On the basis of the observation that, if $\beta_{(i)} \in \pi_2(M; L_i)$ satisfies $\mu(\beta_{(i)}) > 2$, then $\mathcal{N}([\ell_p, w], [\ell_q, w'] : L_i) = \emptyset$ for a generic choice of $\{J_t\}_t$ when $\mu([\ell_q, w']) - \mu([\ell_p, w]) = 2$, the contribution arising from $\mathcal{N}$ in (16.4.7) becomes zero by a dimension-counting argument. This proves the following theorem.

Theorem 16.4.6 *Let L_0, L_1 be monotone Lagrangian submanifolds with $\Sigma_{L_i} \geq 3$ for $i = 0, 1$. Then the Floer pre-boundary map*

$$\partial : CF_R^*(L_1, L_0; \ell_0) \to CF_R^{*+1}(L_1, L_0; \ell_0)$$

is a boundary map, i.e., satisfies

$$\partial \circ \partial = 0.$$

In particular, we can define the Floer homology

$$HF(L_1, L_0; \ell_0) = \mathrm{Ker}\, \partial / \mathrm{Im}\, \partial.$$

Here we put the underlying coefficient ring $R = \mathbb{Z}$ if L_0, L_1 is relatively spin, and $R = \mathbb{Z}_2$ in general.

Now we study case (2), where the minimal Maslov number of L_1 or of L_0 is 2. In this case, again by a dimension-counting argument, we conclude that the matrix elements of $\partial \circ \partial$ are given by

$$\Big\langle (\partial \circ \partial)([\ell_p, w]), [\ell_q, w'] \Big\rangle = -\#\mathcal{N}\Big([\ell_p, w], [\ell_q, w'] : L_0\Big) \\ - \#\mathcal{N}\Big([\ell_p, w], [\ell_q, w'] : L_1\Big).$$

Here we recall that the pairs $(B', \beta_{(0)})$ or $(B', \beta_{(1)})$ non-trivially contribute to the right-hand side sum only when

$$p = q, \quad B' = 0, \quad \mu(\beta_{(0)}) = 2 = \mu(\beta_{(1)})$$

and the associated configurations of the disc components pass through p.

Lemma 16.4.7 *Let $i = 0$, 1 and $[\ell_p, w]$ be given as above. Then we have*

$$\#\mathcal{N}\Big([\ell_p, w], [\ell_p, w'] : L_i\Big) = \Phi(L_i; \{pt\}). \tag{16.4.8}$$

In particular, it does not depend on $p \in L_0 \cap L_1$ but depends on L_i, $i = 0$, 1. We also have

$$\omega(w') - \omega(w) = \omega(\beta_{(i)}) \tag{16.4.9}$$

if $[q, w'] = [p, w]\#B$ and $\#\mathcal{N}(p, q; B' : L_i, \beta_{(i)}; J) \neq 0$ for $B = B'\#\beta_{(i)}$.

Proof We recall the definition (16.4.4) of $\mathcal{N}\Big([\ell_p, w], [\ell_p, w'] : L_i\Big)$. By definition, we have

$$\mathcal{N}(p, q; B' : L_i, \beta_{(i)}; J) = \mathcal{M}_{(0;1;0)}(p, p; 0; J)_{\mathrm{ev}} \times_{\mathrm{ev}} \mathcal{M}_1(L_i; \beta_{(i)}; J_i)/\mathbb{R}.$$

Since the map component of each element of $\mathcal{M}_{(0;1;0)}(p, p; 0; J)$ is constant, the moduli space $\mathcal{M}_{(0;1;0)}(p, p; 0; J)/\mathbb{R}$ coincides with

$$\{p\} \times \{(\tau, 1) \in \mathbb{R} \times [0, 1] \mid \tau \in \mathbb{R}\}/\mathbb{R} \cong \mathcal{M}_{0;(0,3)},$$

where $\mathcal{M}_{0;(0,3)}$ is the Deligne–Mumford moduli space of genus 0 with three marked points on the boundary and hence is a single point. Furthermore, $\mathbb{R}$ acts trivially on the factor $\mathcal{M}_1(L_i; \beta_{(i)}; J_i)$ and so

$$\widetilde{\mathcal{M}}(p, p; 0; J)_{\mathrm{ev}} \times_{\mathrm{ev}} \overline{\mathcal{M}}_1(L_i; \beta_{(i)}; J_i)/\mathbb{R} \cong \{p\} \times_{\mathrm{ev}_\beta} \mathcal{M}_1(L_i; \beta_{(i)}; J_i).$$

But, by the definition of $\Phi(L_i; \{pt\})$, we have

$$\Phi(L_i; \{p\}) = \sum_{\beta_{(i)} \in \pi_2(L_i)} \#\Big(\mathcal{M}_1(L_i; \beta_{(i)}; J_i)_{\mathrm{ev}_\beta} \times \{p\}\Big),$$

which finishes the proof of (16.4.8) when combined with Lemma 16.3.4.

For the proof of (16.4.9), we first have

$$\omega(B) = \omega(w') - \omega(w)$$

from the homotopy condition $w\#u \cong w'$ for any element $u \in \mathcal{M}([\ell_p, w], [\ell_p, w']; J)$. Since the symplectic area is preserved under the Gromov–Floer limit, (16.4.9) follows. This finishes the proof. □

This lemma concludes

$$\sum_{[w'] \in \pi_2(p;\ell_0)} \langle (\partial \circ \partial)([\ell_p, w]), [\ell_p, w'] \rangle = \pm(\Phi(L_1; \{pt\}) - \Phi(L_0; \{pt\}))$$

for all $p \in L_0 \cap L_1$. (Here the sign can be uniquely determined, the proof of which we leave to the interested reader.) Furthermore, we have $[\ell_p, w'] = g_i \cdot [\ell_p, w]$ for an element $g_i \in G(L_i; \omega)$ depending on for which $i = 0,\ 1$ when $\mathcal{N}(p, p, B' : L_i, \beta_{(i)}) \neq \emptyset$ holds with $B = B' \# \beta_{(i)}$.

Therefore we obtain

$$\partial \circ \partial([\ell_p, w]) = \pm(\Phi(L_1; \{pt\}) g_1 \cdot [\ell_p, w] - \Phi(L_0; \{pt\}) g_0 \cdot [\ell_p, w]).$$

We recall from Proposition 16.2.6 that, for any monotone L, $G(L; \omega)$ is a free abelian group of rank 1. Writing the group $G(L; \omega)$ multiplicatively, we can write the associated Novikov ring as the Laurent polynomial

$$\Lambda(L; \ell_0) \cong R[T^a],$$

where T is a formal parameter and $a > 0$ is the positive generator of the period group $\Gamma(L; \omega) = I_\omega(G(L; \omega))$. Furthermore, each element $\beta \in \pi_2(M, L)$ with $\mu(\beta) = 2$ projects down to the same element $T^a \in G(L; \omega)$.

With this notation, we can write

$$[\ell_p, w'] = T^{\omega(g_i)} [\ell_p, w]$$

if $\mathcal{N}(p, p, B' : L_i, \beta_{(i)}) \neq \emptyset$ with $B = B' \# \beta_{(i)}$) for $i = 0$ or $i = 1$. Since $\mu(\beta_{(i)}) = 2$, any such g_i must be a generator of the group $G(L_i; \omega)$.

Summarizing the above discussion, we have proved the following theorem.

Theorem 16.4.8 *Let L_i be monotone with $\Sigma_{L_i} = 2$. Denote by $a_i = \omega(\beta_{(i)})$, $i = 0, 1$ the above-mentioned generator of the group $\{\omega(\beta) \mid \beta \in \pi(M, L_i)\}$. Then we have*

$$\partial \circ \partial[p, w] = \pm(\Phi(L_0; \{pt\}) T^{a_0} - \Phi(L_1; \{pt\}) T^{a_1})\, [p, w], \tag{16.4.10}$$

where $a_i = \omega(g_i)$ for a generator g_i of $G(L_i; \omega)$. In particular, $\partial \circ \partial$ is a multiplication operator by a scalar element in Λ_{Nov}.

Proof It remains to prove the last statement. But this immediately follows from the fact that the quantity

$$\Phi(L_0; \{pt\})T^{a_0} - \Phi(L_1; \{pt\})T^{a_1}$$

does not depend on the generator $[p, w]$ but depends only on the geometry of the pair (L_0, L_1). □

From this, we see that, unless the equality

$$\Phi(L_0; \{pt\})T^{a_0} = \Phi(L_1; \{pt\})T^{a_1}$$

holds, $\partial \circ \partial \neq 0$.

Corollary 16.4.9 *Let (L_0, L_1) be a pair of monotone Lagrangian submanifolds on (M, ω) and let a_i be as above. Then $\partial \circ \partial = 0$ if and only if $\Phi(L_0; \{pt\}) = \Phi(L_1; \{pt\})$ and $a_0 = a_1$.*

This result is a slight refinement of the result (Oh95a) and a special case of Proposition 3.7.17 (FOOO09) in which a similar criterion was stated in terms of the potential function $\mathfrak{PO}$ in a much greater generality for the deformed Floer (co)homology. The current case corresponds to the *undeformed* monotone case (i.e., the case where the deformation parameters $b_0 = 0 = b_1$ in Proposition 3.7.17 (FOOO09)).

This corollary together with the symplectic invariance of the one-point GW-invariants implies the following result of the present author (Oh93a).

Theorem 16.4.10 *Let L be a compact monotone Lagrangian submanifold with $\Sigma_L \geq 2$. Consider the pair (L_0, L_1) with $L_0 = L$, $L_1 = \phi(L)$ for a Hamiltonian diffeomorphism with $\phi(L)$ intersecting L transversely. Then the Floer homology $HF(L, \phi(L))$ is well defined and is invariant under a change of ϕ.*

The latter invariance property can be proved by constructing a chain map via the *non-autonomous* Cauchy–Riemann equation (14.4.45). It turns out that, because this equation does not embody translational symmetry, we need to look at the associated moduli spaces of dimensions 0 and 1 (not 1 and 2, unlike the moduli space for the Floer boundary map), since the associated moduli space has a trivial automorphism group.

This being said, a similar analysis to that for the boundary map moduli space applies to the chain map moduli space and proves the following theorem.

Theorem 16.4.11 *Let (L_0, L_1) be a monotone pair. Then the moduli space of* (14.4.44) *is compact whenever its dimension is 0 or 1. In particular, whenever* *the Floer pre-boundary map is a boundary, i.e., satisfies* $\partial^2 = 0$ *for the pair* (L_0, L_1)*, there exists a chain isomorphism*

$$h_{\phi_{(0)}\phi_{(1)}} : CF_*(L_0, L_1) \to CF_*(\phi^1_{(0)}(L_0), \phi^1_{(1)}(L_1)).$$

In particular, we have

$$HF_*(L_0, L_1) \cong HF_*(\phi^1_{(0)}(L_0), \phi^1_{(1)}(L_1)).$$

Exercise 16.4.12 Prove the statement of this theorem. (See (Oh93a, Oh96c) for its proof.)

16.5 Product structure; triangle product

We first recall the definition of the pants product described in (Oh99), (FOh97) and put it into a more modern context of the general Lagrangian Floer theory such as in (FOOO09, FOOO10a) and in other more recent literature.

Consider the triple L_0, L_1, L_2 of compact (monotone) Lagrangian submanifolds in (M, ω), and consider the Floer complex

$$CF^*(L_1, L_0), \quad CF^*(L_2, L_1), \quad CF^*(L_2, L_0).$$

To define a product structure which respects the filtrations on the CF^*, we need to use the framework of anchored Lagrangian submanifolds introduced in Section 13.7.

We fix a base point y of the ambient symplectic manifold (M, ω) once and for all. Then we equip y with an anchor γ_i of L_i in the sense of Section13.7 with $\gamma_i : [0, 1] \to M$ satisfying

$$\gamma_i(0) = p_i, \quad \gamma_i(1) = y$$

for each $i = 0, \ldots, 2$. Then we equip a base path $\ell_{ij} \in \Omega(L_i, L_j)$ by concatenating γ_i and γ_j as

$$\ell_{ij} = \gamma_i \# \overline{\gamma}_j$$

as in Section 13.7. In the same spirit, we choose the Lagrangian subbundle $\lambda_i \subset \gamma_i^* TM$ for each $i = 0, \ldots, k$ and define

$$\lambda_{ij} = \lambda_i \# \overline{\lambda}_j$$

in the obvious way.

We consider the anchored Lagrangian submanifolds $(L_i, (\gamma_i, \lambda_i))$ for each $i =$ 0, 1, 2. For simplicity of notation, we just denote the associated filtered Floer complex by $CF^*(L_i, L_j)$, suppressing the given anchors from the notation. We also denote the submodule

$$CF^*_\lambda(L_i, L_j) = \{x \in CF^*(L_i, L_j) \mid \mathfrak{v}(x) \geq \lambda\}.$$

Now we are ready to define the triangle product in the chain level, which we denote by

$$\mathfrak{m}_2 : CF^*(L_1, L_0) \otimes CF^*(L_2, L_1) \to CF^*(L_2, L_0) \tag{16.5.11}$$

following the general notation from (FOOO09), (Se03a). This product is defined by considering all triples

$$p_1 \in L_1 \cap L_0,\ p_2 \in L_2 \cap L_1,\ p_0 \in L_2 \cap L_0$$

with the polygonal Maslov index $\mu(\beta; p_1, p_2; p_0)$ whose associated analytical index, or the virtual dimension of the moduli space

$$\mathcal{M}_3(\beta; p_1, p_2; p_0)$$

of J-holomorphic triangles in class $[w] \in \pi_2(p_1, p_2; p_0)$, becomes zero and by counting the number of elements thereof.

We now give the precise definition of the moduli space $\mathcal{M}_3(\beta; p_1, p_2; p_0)$. Fix a t-dependent family of $J_t^{(i)}$, which are used to define the Floer boundary map ∂ for (L_i, L_{i+1}) for each $i = 0, 1, 2$ (counted mod 3). This will then determine a domain-dependent family J on a given neighborhood of z_0, z_1, z_2 in

$$\dot{\Sigma} = D^2 \setminus \{z_0, z_1, z_2\}$$

in terms of the given analytic coordinate $z = \tau + \sqrt{-1}t$ near the (boundary) punctures z_i. We then extend this to the whole $\dot{\Sigma}$ so that the choice becomes consistent with the already chosen t-dependent family $J_t^{(i)}$ for $i = 0, 1, 2$ at z_i via the given analytic coordinates.

More precisely, let $\varphi^{(i)} : [0, \infty) \times [0, 1] \to \dot{\Sigma}$ be the given analytic coordinate of a punctured neighborhood at each i. We assume that their images are disjoint from one another. We denote $U^{(i)} \setminus \{z_i\} = \mathrm{Image}(\varphi^{(i)}|_{(1,\infty)\times[0,1]})$ and $(\varphi^{(i)})^{-1}(z) = (\tau(z), t(z))$ for $z \in U^{(i)}$. Then we first set

$$J(z) = J^{(i)}_{t(z)}, \qquad \text{for } z \in \bigcup_{i=0}^{2} U^{(i)}. \tag{16.5.12}$$

Using the fact that $\mathcal{J}_\omega$ is contractible, we extend J to the whole $\dot{\Sigma}$ and still denote the extension by J. We then consider the nonlinear operator $w \mapsto \overline{\partial}_J w$ given by

$$\overline{\partial}_J w(z) := \frac{1}{2}(dw(z) + J(z)dw(z)j(z)). \tag{16.5.13}$$

This enables us to define the following finite-energy moduli space.

Definition 16.5.1 Define $\mathcal{M}_3(J; L_0, L_1, L_2)$ to be the set of pairs $(w, \{z_0, z_1, z_2\})$ consisting of a triple $\{z_0, z_1, z_2\} \subset \partial D^2$ and a J-holomorphic map

$$w : D^2 \setminus \{z_0, z_1, z_2\} \to M$$

that satisfy the following:

(1) w extends to a continuous map to D^2;
(2) the map w satisfies the Lagrangian boundary condition

$$w(\partial_0 D^2) \subset L_0,\ w(\partial_1 D^2) \subset L_1,\ w(\partial_2 D^2) \subset L_2,$$

where $\partial_i D^2 \subset \partial D^2$ is the arc segment in between z_i and z_{i+1} ($i \mod 3$);
(3) w has finite energy $E_J(w) = \frac{1}{2} \int |dw|_J^2 = \int w^*\omega < \infty$.

We apply the exponential convergence established in Proposition 14.1.5 to w in any analytic coordinates $(\tau, t) \in [0, \infty) \times [0, 1]$ on a neighborhood $U_i \subset \dot{\Sigma}$ near the puncture z_i and obtain the following lemma.

Lemma 16.5.2 *For any element $w \in \mathcal{M}_3(J_0; L_0, L_1, L_2)$, the following holds:*

(1) *w extends to D^2 continuously and*
(2) *there exists a triple $\{p_0, p_1, p_2\}$ with $p_i \in L_{i-1} \cap L_i$ for $i = 1, 2, 3 \mod 3$ such that $w(z_i) = p_i$.*

In particular, any such w bears a natural homotopy class $[w] \in \pi_2(p_1, p_2; p_0)$.

We summarize the above discussion as follows.

Proposition 16.5.3 *We have the decomposition*

$$\mathcal{M}_3(J_0; L_0, L_1, L_2) = \bigcup_{(p_0,p_1,p_2)} \bigcup_{\beta \in \pi_2(p_1,p_2;p_0)} \mathcal{M}(\beta; p_1, p_2; p_0),$$

where $\mathcal{M}(\beta; p_1, p_2; p_0)$ is the subset

$$\mathcal{M}(\beta; p_1, p_2; p_0) = \{(w, (z_0, z_1, z_2)) \mid \overline{\partial}_J w = 0,\ [w] = \beta, w(z_i) = p_i,\ i = 0,\ 1,\ 2\}.$$

A similar decomposition holds for $\mathcal{M}_3(J_0; L_0, L_1, L_2)$.

The following proposition is a consequence of the way the action functionals are associated with the anchored Lagrangian submanifolds in Section 13.7.

Proposition 16.5.4 *Suppose $w : \dot\Sigma = D^2 \setminus \{z_0, z_1, z_2\} \to M$ is any smooth map that extends continuously to D^2 in class $[w] \in \pi_2(p_1, p_2; p_0)$ that satisfies all the conditions given in Definition 16.5.1, but is not necessarily J-holomorphic. We denote by $\widehat{p} : [0,1] \to M$ the constant path with its value $p \in M$. Then we have*

$$\int w^*\omega = \mathcal{A}_{L_0L_1}([\widehat{p}_1, w_0]) + \mathcal{A}_{L_1L_2}([\widehat{p}_2, w_2]) - \mathcal{A}_{L_0L_2}([\widehat{p}_0, w_0]). \quad (16.5.14)$$

An immediate corollary of this identity is the bound for the energy $E_J(w)$ for any $w \in \mathcal{M}_3(\beta; p_1, p_2; p_0)$ since $E_J(w) = \int w^*\omega$ for such w. Then this energy bound together with the monotonicity of L_0, L_1 and L_2 gives the following compactness. We indicate how the prescription of J given in (16.5.13) enters into the proof. This argument is one of the important points in relation to further development of the recent Floer theory, especially in relation to the Hamiltonian fibration setting of the Floer theory, where one really needs to use the domain-dependent family J.

Corollary 16.5.5 *Whenever* $\dim \mathcal{M}_3(\beta; p_1, p_2; p_0) = 0$, *$\mathcal{M}_3(\beta; p_1, p_2; p_0)$ is compact.*

Proof We fix a Kähler metric h on $\dot\Sigma$ that is cylindrical on $U^{(i)}$, $i = 0,\ 1,\ 2$ and let (τ, t) be the cylindrical coordinates of $U^{(i)}$ such that $h = d\tau^2 + dt^2$.

Suppose w_k is a sequence in $\mathcal{M}_3(\beta; p_1, p_2; p_0)$. We first claim that for any given open neighborhoods $V^{(i)} \subset U^{(i)}$ of z_i, there exists some $C > 0$ depending only on the neighborhoods such that

$$\max_{z \in \dot\Sigma \setminus \cup_i V^{(i)}} |dw_k| \leq C.$$

Otherwise there is a subsequence k_l and $z_l \in \dot\Sigma \setminus \cup_i V^{(i)}$ such that $|dw(z_l)| \to \infty$ and hence creates a bubble on $\dot\Sigma \setminus \cup_i V^{(i)}$. But the dimension-counting argument using the monotonicity of the L_i prevents this bubbling from happening. Therefore we have proved the claim.

Therefore the only possible failure of convergence is through loss of positive energy through the punctures by Proposition 14.2.4. This corresponds to splitting off a J-holomorphic strip from one of the punctures z_i in terms of the analytic cylindrical coordinates (τ, t). This is because, if there is no energy loss, then the same reasoning as that of Proposition 14.2.4 will imply convergence of a subsequence of the sequence.

Then, due to the prescription of J as in (16.5.13), the split-off strip will be a non-constant map $u : \mathbb{R} \times [0,1] \to M$ contained in $\mathcal{M}(L_i, L_{i+1})$.

In particular, its index must be at least 1, which contradicts the hypothesis $\dim \mathcal{M}_3(\beta; p_1, p_2; p_0) = 0$. This finishes the proof. □

Therefore we can define

$$n(\beta; p_1, p_2; p_0) = \#(\mathcal{M}_3(\beta; p_1, p_2; p_0))$$

and

$$\mathfrak{m}_2(p_1, p_2) = \sum_{p_0 \in L_0 \cap L_2} \sum_{\beta} n(\beta; p_1, p_2; p_0) T^{\omega(\beta)} e^{\mu(\beta; p_1, p_2; p_0)} p_0 \tag{16.5.15}$$

extending linearly over the complex CF^* over the universal Novikov ring. Here, strictly speaking, this definition should be applied to $[\widehat{p_i}, w_i]$ with

$$[w_0] = [w_1]\#[w_2]\#\beta$$

in the sense of Lemma 13.7.2, but the matrix coefficient $n(J_0, \beta; p_1, p_2; p_0)$ does not depend on the homotopy classes $[w_i]$ but depends only on $\beta \in \pi(p_1, p_2; p_0)$, as long as they satisfy the last gluing identity. So the precise form of the above formula is

$$\mathfrak{m}_2([\widehat{p}_1, w_1], [\widehat{p}_2, w_2]) = \sum_{\beta} n(\beta; p_1, p_2; p_0)[\widehat{p}_0, w_1\#w_2\#\beta], \tag{16.5.16}$$

extending linearly to the complex CF^* over the universal Novikov ring. To write this into (16.5.15), one has to choose a lifting $\widehat{p_i} \in \widetilde{\Omega}_0(L_{i-1}, L_i)$ and then regard $CF^*(L_{i-1}, L_i)$ as the free module over $L_{i-1} \cap L_i$ with the Novikov ring as its coefficient, instead of $\mathbb{Q}$-vector space. (See Section 13.9 for the explanation.) The product (16.5.11) is nothing but the product whose matrix coefficients are given by $n(J_0, \beta; p_1, p_2; p_0)$.

Another immediate corollary of Proposition 16.5.4 from the definition of $\mathfrak{m}_2$ is that the map (16.5.11) restricts to

$$\mathfrak{m}_2 : CF^*_\lambda(L_1, L_0) \otimes CF^*_\mu(L_2, L_1) \to CF^*_{\lambda+\mu}(L_2, L_0).$$

This is because, if w is J-holomorphic, then $\int w^*\omega \geq 0$. It is straightforward to check.

Proposition 16.5.6 *The product* $\mathfrak{m}_2$ *satisfies*

$$\partial(\mathfrak{m}_2(x, y)) = \mathfrak{m}_2(\partial(x), y) \pm \mathfrak{m}_2(x, \partial(y)) \tag{16.5.17}$$

modulo signs and in turn induces the product map

$$*_F : HF^*(L_1, L_0) \otimes HF^*_\mu(L_2, L_1) \to HF^*_{\lambda+\mu}(L_2, L_0) \tag{16.5.18}$$

in homology.

Proof For the proof of this proposition, we examine the boundary of the compactified moduli space $\overline{\mathcal{M}}_3(\beta; p_1, p_2; p_0)$ for the quadruple $(\beta; p_1, p_2; p_0)$ with

$$\dim \mathcal{M}_3(\beta; p_1, p_2; p_0) = 1.$$

Then the monotonicity of L_0, L_1 and L_2 rules out bubbling by an argument similar to the one used in the proof of Theorem 16.4.4. Again the prescription of J as in (16.5.13) enables us to conclude that the boundary $\mathcal{M}_3(\beta; p_1, p_2; p_0)$ consists of the following three types of fiber products:

$$\begin{gathered}\mathcal{M}_3(\beta_0; p_1, p_2; p_0')\#\mathcal{M}(B_0; p_0', p_0),\\ \mathcal{M}(B_1; p_1, p_1')\#\mathcal{M}_3(\beta_1; p_1', p_2; p_0),\\ \mathcal{M}(B_2; p_2, p_2')\#\mathcal{M}_3(\beta_2; p_1, p_2'; p_0),\end{gathered}$$

with $\beta = \beta_0\#B_0$, $\beta = B_i\#\beta_i$ for $i = 1, 2$, respectively, such that the triangle parts have dimension 0 and the strip parts have dimension 1. By applying the classification theorem of compact 1-manifolds with a boundary, we obtain

$$\begin{aligned}0 = &\sum_{p_0' \in L_0 \cap L_2} \sum_{\beta_0 + B_0 = \beta} n(\beta_0; p_1, p_2; p_0') n(B_0; p_0', p_0)\\ &\pm \sum_{p_1' \in L_0 \cap L_1} \sum_{\beta_1 + B_1 = \beta} n(\beta_1; p_1', p_2; p_0) n(B_1; p_1, p_1')\\ &\pm \sum_{p_2' \in L_1 \cap L_2} \sum_{\beta_2 + B_2 = \beta} n(\beta_0; p_1, p_2'; p_0) n(B_2; p_2, p_2').\end{aligned}$$

This is nothing but the equation for the matrix coefficients of (16.5.17) (modulo the signs). For a precise consideration of the signs, we refer the reader to Chapter 8 of (FOOO09). This finishes the proof. □

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Applications to symplectic topology

The full power of Lagrangian Floer homology theory can be mustered only using the A_∞ machinery introduced by Fukaya (Fu93) and fully developed in the book (FOOO09). This theory is necessary to deal with general Lagrangian submanifolds when the structure of disc-bubbles is not as simple as in the following two special cases:

(1) exact Lagrangian submanifolds in (non-compact) exact symplectic manifolds
(2) monotone Lagrangian submanifolds in (monotone) symplectic manifolds.

Since the A_∞ machinery goes beyond the scope of this book, we will focus on these two cases and use them to illustrate the usage of Floer homology in the study of symplectic topology.

However, even when the Floer homology $HF(L_0, L_1; M)$ is defined it is a highly non-trivial task to explicitly compute this homology as soon as we go beyond the exact case and $L_1 = \phi(L_0)$ for a Hamiltonian diffeomorphism ϕ.

Theorem 16.4.10 or its cousins is the basic starting point of the application of Floer homology to the study of symplectic topology. Most applications so far are related to the construction of nondisplaceable Lagrangian submanifolds or the study of the symplectic topology of displaceable Lagrangian submanifolds such as the Hofer displacment energy and the Maslov class obstruction. In this study, it is also crucial to analyze the structure of the Floer moduli spaces when $\phi \to \mathrm{id}$, more precisely under the adiabatic limit of $\phi(L) \to L$, which gives rise to thick–thin decomposition of Floer trajectories.

We refer those who are interested in learning more about Lagrangian Floer theory beyond the above two cases and its application to mirror symmetry and symplectic topology to (CO06), (FOOO10b)–(FOOO13).

17.1 Nearby Lagrangian pairs: thick–thin dichotomy

In this section, we study some convergence results of $\mathcal{M}(L, \phi(L); p, q)$ as $\phi \to id$ in C^1. We will also explain how this study of degeneration gives rise to a spectral sequence introduced in (Oh96b). This spectral sequence in this form has been further explored in (Bu10), (BCo09), (D09). A more general version of spectral sequences of this kind is presented in (FOOO09), which also handles non-monotone Lagrangian submanifolds with an additional *unobstructedness* hypothesis.

Here is the precise setting of the study of this degeneration. Let L be a given compact Lagrangian submanifold. The following notion was introduced in (Oh05d, Spa08).

Definition 17.1.1 Let V be a given Darboux–Weinstein neighborhood of L. A Hamiltonian isotopy $L^t = \phi_H^t(L)$ is called V-engulfed if $\phi_H^t(L) \subset V$ for all $t \in [0, 1]$.

We fix a Darboux neighborhood U of L and cotangent chart of U

$$\Phi : U \to V \subset T^*L,$$

where V is a neighborhood of the zero section of T^*L. If ϕ is C^1-close to id, then we can write $\Phi(\phi(L)) = \operatorname{Graph} df$ for a smooth function $f : L \to \mathbb{R}$. We assume $\|df\|_{C^1}$ is sufficiently small.

When $\|df\|_{C^1}$ becomes small, there occurs the thick and thin dichotomy of the moduli space of Floer trajectories. This dichotomy was analyzed in detail in (Oh96b). It turns out to be a crucial ingredient in the later applications of Lagrangian Floer homology in symplectic topology. Such a dichotomy arises largely because there is an 'energy gap' between the thin trajectories and the thick (or non-thin) trajectories.

To give the precise definition of this gap, we use the geometric invariant $A(\omega, L; J_0)$ introduced in (8.4.32) for each time-independent compatible almost complex structure $J_0 \in \mathcal{J}_\omega$. Recall

$$A(\omega, J_0 : L) = \min\{A_S(\omega, J_0), A_D(\omega, J_0; L)\} > 0,$$

where

$$A_S(\omega, J_0) := \inf\left\{ \int v^*\omega > 0 \mid v : S^2 \to M,\ \overline{\partial}_{J_0} v = 0 \right\}$$

and for a given Lagrangian submanifold $L \subset (M, \omega)$

$$A_D(\omega, J_0; L) := \inf\left\{ \int w^*\omega > 0 \mid w : (D^2, \partial D^2) \to (M, L),\ \overline{\partial}_{J_0} w = 0 \right\}.$$

One can generalize this invariant for any compact family $K \subset \mathcal{J}_\omega$ of compatible almost-complex structures. We start with $A_S(\omega; K)$ for $K : [0,1]^k \to \mathcal{J}_\omega$, which is a continuous k-parameter family in the C^1 topology, and define $A_S(\omega; K)$ to be the constant

$$A_S(\omega; K) = \sup_{\kappa \in [0,1]^k} \{A_S(\omega, J(\kappa))\}.$$

This is also positive and enjoys the following lower semi-continuity property.

Proposition 17.1.2 *The constant $A_S(\omega; K)$ is lower semi-continuous in K. In other words, for any given K and $0 < \epsilon < A_S(\omega; K)$, there exists some $\delta = \delta(K, \epsilon) > 0$ such that for any K' with $\|K' - K\|_{C^1} \le \delta$ we have*

$$A_S(\omega; K') \ge A_S(\omega; K) - \epsilon.$$

Proof We prove this by contradiction. Suppose to the contrary that there exists $\epsilon > 0$ and a sequence $K_i \to K$ as $i \to \infty$ such that

$$A(\omega; K_i) < A_S(\omega; K) - \epsilon \tag{17.1.1}$$

for all i. For each i, let v_i be a non-constant J_{t_i}-holomorphic sphere for $t_i \in [0,1]^n$ such that

$$A(\omega; K_i) \le \omega([v_i]) < A_S(\omega; K) - \frac{\epsilon}{3},$$

which exists by virtue of the definition of $A(\omega; K_i)$ and (17.1.1). By the Gromov compactness, v_i converges to a J_{t_∞}-cusp curve

$$v_\infty = \sum_a v_{\infty,a},$$

where $J_{t_\infty} \in K$. If v_∞ is constant, then v_i will be homologous to zero and so $w([v_i]) = 0$ for sufficiently large i, which in turn implies that v_i is a constant map, which is a contradiction with the standing hypothesis. If some $v_{\infty,a}$ is a non-constant J_{t_∞}-holomorphic sphere, then

$$\omega([v_{\infty,a}]) \le A_S(\omega; K) - \frac{\epsilon}{3}.$$

But this contradicts the definition of $A_S(\omega; K)$. This finishes the proof. □

A similar argument proves the semi-continuity of the invariants $A(\omega, K; L)$.

Now let U be a Darboux neighborhood of L in (M, ω), let H be a Hamiltonian such that

$$\phi_H^1(L) \subset U \tag{17.1.2}$$

and let $J = \{J_t\}_{t\in[0,1]}$ be a t-dependent family of almost-complex structures. We study the moduli space, denoted by $\mathcal{M}(J; L, \phi_H^1(L))$, of solutions of

$$\frac{\partial u}{\partial \tau} + J_t \frac{\partial u}{\partial t} = 0,\ u(\tau, 0) \in L,\ u(\tau, 1) \in \phi_H^1(L),\ E_J(u) < \infty.$$

Denote by $d_{C^1}(L_0, L_1)$ the C^1-distance between L_0 and L_1.

Proposition 17.1.3 *For any given $0 < \alpha < A(\omega, L; J_0)$ and for any given time-independent J_0, there exists a constant $\epsilon > 0$ such that if $d_{C^1}(\phi_H^1(L), L) < \epsilon$ and $\max_{t\in[0,1]} |J_t - J_0|_{C^0} < \epsilon$, the following hold.*

(1) *We have*

$$\int u^*\omega < A(\omega, L; J_0) - \alpha \tag{17.1.3}$$

for all $u \in \mathcal{M}(J; L, \phi_H^1(L))$ satisfying $\operatorname{Im} u \subset U$, and vice versa.

(2) *All other $u \in \mathcal{M}(J; L, \phi_H^1(L))$ are not contained in U and satisfy*

$$\int u^*\omega > A(\omega, L; J_0) - \delta \tag{17.1.4}$$

for a sufficiently small δ depending on $\epsilon > 0$ but not on α.

Proof We start with the following simple lemma.

Lemma 17.1.4 *Let L_0 be the zero section of T^*L and $L_1 = \operatorname{Graph} df$ for some function f on L. Suppose that L_0 intersects L_1 transversely and that $x, y \in L_0 \cap L_1 = \operatorname{Crit}(f)$. Suppose $u : [0,1] \times [0,1] \to T^*L$ is a C^1-map with*

$$u(s, 0) \subset L_0,\ u(s, 1) \subset L, \quad u(0, t) \equiv y,\ u(1, t) \equiv x.$$

Then we have $\int u^*\omega = f(y) - f(x)$.

Proof Denote $u(1, t) = (q(1, t), p(1, t))$ and then $p(1, t) = df(q(1, t))$ since $u(1, t) \in \operatorname{Graph} df$ by hypothesis. Then we define a map $u_f : [0,1]^2 \to T^*L$ by

$$u_f(s, t) = (q(1, t), s\, df(q(1, t))).$$

We join u with the map u_f and obtain a map $w := u\#u_f$ whose boundary is contained in L_0. Therefore we obtain

$$0 = \int w^*(-d\theta) = \int w^*\omega_0 = \int u^*\omega - \int u_f^*\omega.$$

But we evaluate

$$\int u_f^*\omega = -\int_{\partial u_f} \theta = -\int_{u_f|_{t=1}} \theta = -\int (u_f|_{t=1})^*\theta = -\int_{s\mapsto q(s,1)} df = f(y) - f(x).$$

This finishes the proof. □

Applying the lemma to $u \in \mathcal{M}_J(L, \phi_H^1(L); U)$ when $\phi_H^1(L)$ is C^1-close to L, we can write $\Phi(\phi_H^1(L)) = \mathrm{Graph}\, df$ for some f. Therefore we obtain

$$\int u^*\omega = \int (\Phi \circ u)^*\omega_0 = f(y) - f(x)$$

since Φ is symplectic, i.e., $\Phi^*\omega_0 = \omega$. Furthermore we have

$$\frac{1}{C}|df|_{C^0} \le d_{C^1}(L, \phi_H^1(L)) \le C|df|_{C^0}$$

for $C > 0$ independent of H as long as $d_{C^1}(L, \phi_H^1(L))$ is small enough, say, less than ϵ. Obviously we have

$$|f(y) - f(x)| \le \max f - \min f \le |df|_{C^0}\, \mathrm{diam}(L).$$

Now let $0 < \alpha < A(\omega, L; J_0)$ be given. We first prove statement (1). By the above discussion, if $d_{C^1}(L, \phi_H^1(L)) < \epsilon$ with ϵ small enough, we will have

$$|df|_{C^0} < C\epsilon \le \frac{A(\omega, L; J_0) - \alpha}{\mathrm{diam}(L)}.$$

Then we have

$$\begin{aligned}
\int u^*\omega &= f(y) - f(x) \le \max f - \min f \le |df|_{C^0}\, \mathrm{diam}(L) \\
&\le \frac{A(\omega, L; J_0) - \alpha}{\mathrm{diam}(L)} \times \mathrm{diam}(L) \\
&= A(\omega, L; J_0) - \alpha.
\end{aligned}$$

We now prove the converse statement. Suppose to the contrary that there exists $0 < \alpha < A(\omega, L; J_0)$ such that we have sequences (J^i, H^i) that satisfy $d_{C^1}(\phi_{H^i}^1(L), L) \to 0$ and $\max_{t\in[0,1]} |J_t^i - J_0|_{C^1} \to 0$ as $i \to \infty$ and bear an element $u_i \in \mathcal{M}(\phi_H^1(L), L; J^i)$ satisfying

$$\mathrm{Im}\, u_i \not\subset U, \qquad \int u_i^*\omega \le A(\omega, L; J_0) - \alpha \tag{17.1.5}$$

for all i. By definition of $A(\omega, L; J_0)$, this sequence u_i can bubble off neither a J_0-holomorphic sphere nor a disc. Since $\phi_{H^i}^1(L) \to L$ in C^1 topology and L is compact, we may assume by choice a subsequence such that there exist pairs of x_i, $y_i \in \phi_{H^i}^1(L) \cap L$ with $x_i \to x_\infty$ and $y_i \to y_\infty$, with $x_\infty, y_\infty \in L$ and

$$u_i \in \mathcal{M}(\phi_{H^i}^1(L), L; x_i, y_i; J^i).$$

Moreover, by (17.1.5), we may assume, by translating u^i in the τ-direction if necessary, that

$$u_i(0, t_i) \to p_\infty \in M \setminus U$$

as $i \to \infty$. Since the sequence u_i does not bubble off, we can apply local convergence and we may find a J_0-holomorphic map $u_\infty : \mathbb{R} \times [0,1] \to M$ as a local limit of u_i that satisfies

$$u_\infty(0, t_\infty) = p_\infty \notin U, \qquad u_\infty(\tau, 0),\ u_\infty(\tau, 1) \subset L,$$
$$\int u^*\omega \le A(\omega, L; J_0) - \alpha.$$

By the conformal equivalence between $\mathbb{R} \times i[0,1]$ and $D^2 \setminus \{-1,1\}$, we may consider u_∞ as a non-constant map from $D^2 \setminus \{-1,1\}$ with

$$\overline{\partial}_{J_0} u_\infty = 0\,, \quad u_\infty(\partial D^2 \setminus \{-1,1\}) \subset L, \ \int u_\infty^*\omega < A(\omega, L; J_0) - \alpha.$$

By the removal of singularity, we can extend u_∞ to a non-constant J_0-holomorphic map $u_\infty : (D^2, \partial D^2) \to (M, L)$ with

$$0 < \int u_\infty^*\omega \le A(\omega, L; J_0) - \alpha.$$

This contradicts the definition of $A(\omega, L; J_0)$ and finishes the proof of statement (1).

For the proof of statement (2), again suppose to the contrary that there exist $\delta > 0$ and sequences H^k with $\phi_{H^k}(L) \to L$ and $J^k \to J_0$ in C^1 and $u_k \in \mathcal{M}(J_k; \phi^1_{H^k}(L), L)$ whose image is not contained in U but

$$\int u_k^*\omega < A(\omega, L; J_0) - \delta. \tag{17.1.6}$$

By expressing $\phi^1_{H^k}(L)$ as $\mathrm{Graph}(df_k)$ in a Darboux chart, it will suffice to consider those H_k such that $H_k = f_k \circ \pi$ in the Darboux neighborhood U where f is a smooth function on L with $|df_k|_{C^0} \to 0$ as $k \to \infty$. By refining the argument above in the proof of the first statement, under the uniform-area bound (17.1.6), we derive a subsequence, again denoted by u_k, that locally converges to u_∞, the image of which passes through a point in $M \setminus U$, because we assume that the image of u^k is not contained in U. Then by virtue of the removable singularity, u_∞ produces a non-constant disc with a boundary lying on L or a sphere. Therefore, by the definition of $A(\omega, L; J_0)$,

$$\int u_\infty^*\omega \ge A(\omega, L; J_0)$$

and hence

$$\liminf_{k\to\infty} \int u_k^*\omega \ge \int u_\infty^*\omega \ge A(\omega, L; J_0).$$

However, this gives rise to a contradiction to (17.1.6) if k is sufficiently large. This finishes the proof. □

17.2 Local Floer homology

In this section, following the idea in (Fl89b), (Oh96b), we define a local version of the Floer cohomology $HF(L, L)$ that singles out the contribution from the Floer trajectories whose images are contained in a given Darboux–Weinstein neighborhood U of L in M. We denote this local Floer homology by $HF(L, L; U)$. We show that this local contribution depends only on the pair (L, U) which enables us to perform its computation for the pair (o_L, V) in a neighborhood $V \subset T^*L$ of the zero section $o_L \cong L$. Furthermore, by definition, $HF(L, L; U)$ is always well defined without any unobstructedness assumption of $L \subset M$ such as exactness of ω or monotonicity of the pair (L, M).

As immediate applications, we prove two celebrated theorems of Gromov on exact Lagrangian embedding, following the scheme from (Oh96b).

17.2.1 Definition of local Floer homology

We start with the following flexible notion that Floer introduced in (Fl89b) in the Hamiltonian setting. This is in turn the Floer-theoretic analog of the notion of an *isolating block* introduced by Conley (Co78) in dynamical systems. The definition can be formulated in a more general abstract setting, but we will focus on the current geometric context.

Let $U \subset M$ be a Darboux neighborhood of L. We introduce the following definition.

Definition 17.2.1 Let $L' \subset U$ be any compact Lagrangian submanifold and $J \in j_\omega$. Consider the set $\mathcal{M}_1(L, L'; J)$ of Floer trajectories with one marked point and its evaluation map $\mathcal{M}_1(L, L'; J) \to M$. Define the subset

$$\mathcal{M}(L, L'; J; U) := \{u \in \mathcal{M}(L, L'; J) \mid \text{Image}\, u \subset U\}$$

and its evaluation image

$$\mathcal{S}(L, L'; J; U) = \text{ev}(\mathcal{M}_1(L, L'; J; U)).$$

We call $\mathcal{S}(L, L'; J; U)$ an *invariant set* in U and say that $\mathcal{S}(L, L'; J; U)$ is *isolated* in U (under the Cauchy–Riemann flow) if $\overline{\mathcal{S}}(L, L'; J; U) \subset U$, where $\overline{\mathcal{S}}(L, L'; J; U)$ is the closure of $\mathcal{S}(L, L'; J; U)$ in M. When $\mathcal{S}(L, L'; J; U)$ is isolated, we call $\overline{\mathcal{S}}(L, L'; J; U)$ the *maximal invariant set* in U.

Now we define the notion of *continuation* of maximal invariant sets.

Definition 17.2.2 Let $\{L^s\}_{s \in [0,1]}$ be an isotopy of compact Lagrangian submanifold L_0. Consider

$$(\overline{J}, \overline{H}) \in \mathcal{P}([0, 1]^2, \mathcal{J}_\omega) \times C^\infty([0, 1]^2 \times M, \mathbb{R})$$

and an open subset $\mathcal{U} \subset [0, 1] \times M$.

Denote by $\mathcal{S}_0 := \mathcal{S}(J^0, (L, L^0); U^0)$ and $\mathcal{S}_1 := \mathcal{S}(J^1, (L, L^1); U^1)$ the maximal invariant sets defined as above. We call $(\overline{J}, \overline{H}, \mathcal{U})$ a *continuation* between the maximal invariant sets $\mathcal{S}_0 \subset U^0$ and $\mathcal{S}_1 \subset U^1$ if it satisfies the following.

(1) For each $s \in [0, 1]$ and all $t \in [0, 1]$,

$$L^s \subset U^s := \{x \in P \mid (x, s) \in \mathcal{U}\}.$$

(2) Each invariant set

$$\mathcal{S}_s := \mathcal{S}(J^s, (L, L^s); U^s)$$

is isolated in U^s for all $s \in [0, 1]$.

Now we consider a family of Lagrangian submanifolds that are close to L in the following sense.

Definition 17.2.3 We call a Lagrangian submanifold $L' \subset (M, \omega)$ *exact relative to* L if there is a Darboux neighborhood $U \supset L$ such that L' is exact in $U \cong V \subset T^*L$.

Once we have set up these definitions, the following is easy to prove.

Lemma 17.2.4 *Let $L \subset M$. Let L' be exact relative to L and intersect L transversally. Suppose that $\mathcal{S}(L, L'; J; U)$ is isolated in U. Then there exists a C^∞ perturbation J' of J for which $\mathcal{M}(J', L, L'; U)$ is Fredholm-regular and $\mathcal{S}(J', L, L'; U)$ remains isolated in U. In particular, for any pair x, $y \in L \cap L'$ with $\mu(x; U) - \mu(y; U) = 1$, $\mathcal{M}(x, y; J'; U) \subset \mathcal{M}(L, L'; J'; U)$ has finite cardinality.*

Proof The isolatedness is stable under C^∞-perturbation by definition. Then the existence of such a perturbation J' follows immediately. The finiteness follows from the compactness since bubbling cannot occur for (L, L') in U. □

Now suppose that $(L, L'; J; U)$ and J' are as in Lemma 17.2.4. We define $n_U(x, y; J')$ by

$$n_U(x, y : J') := \#\text{of isolated trajectories in } \mathcal{M}((x, y); J; U) \pmod 2.$$

If $L' = \phi_H^1(L)$, then we can define an integer $n_U(x, y : J')$ using the coherent orientation established in Section 15.6.

Theorem 17.2.5 *Suppose $(L, L'; J; U)$ is as in Lemma 17.2.4. Then, for any small perturbation J' of J for which $\mathcal{M}(L, L'; J'; U)$ is Fredholm regular, the homomorphism*

$$\partial_U : CF(L, L'; J'; U) \to CF(L, L'; J'; U), \quad \partial_U x = \sum_{y \in L \cap \phi_H^1(L)} n_U(x, y : J')y$$

satisfies $\partial_U \circ \partial_U = 0$*. Furthermore, the corresponding quotients*

$$HF^*(L, L'; J'; U) = \ker \partial_U / \operatorname{im} \partial_U$$

are isomorphic under the continuation $(\overline{\mathcal{S}}, \overline{J}, \overline{H}, \mathcal{U})$ *as long as the continuation is Floer-regular at the ends* $s = 0, 1$.

Proof The statement on the boundary follows from the isolatedness hypothesis since the triple $(L, L'; U)$ does not carry any bubble. For the statement on the continuation invariance, we construct a chain map between the two ends by taking the composition of an 'adiabatic sequence' of chain maps associated with a partition $0 = t_0 < t_1 < \ldots < t_N = 1$ of the unit interval $[0, 1]$. Therefore we assume that the C^1-distance between the two ends is as small as we want in the remaining proof.

Consider the 'non-autonomous' Floer trajectory equation

$$\begin{cases} \partial u/\partial\tau + J_t^{\rho(\tau)} \partial u/\partial t = 0, \\ u(\tau, 0) \in L^{\rho(\tau)}\ u(\tau, 1) \in L, \end{cases} \tag{17.2.7}$$

where $\overline{J} = \{J^s\}$ is a homotopy between two Fredholm-regular J^0, J^1 and $\mathcal{L} = \{L^s\}$ is a Hamiltonian isotopy between L^0 and L^1. We denote by $\mathcal{M}(L, L^\rho; J^\rho)$ the moduli space of solutions of (17.2.7) and by $\overline{\mathcal{M}}(L, L^\rho; J^\rho)$ its closure consisting of stable broken Floer trajectories as defined in Section 14.3.

Define

$$d_{C^1}(L', \mathcal{L}) := \max_{s \in [0,1]} d_{C^1}(L', L^s), \quad d_{C^0}(J_0, \overline{J}) := \max_{(s,t) \in [0,1]} |J_0 - J_t^s|_{C^0}. \tag{17.2.8}$$

Let $L^s = \phi_{H_i}^s(L')$. We need only show that the following holds.

Lemma 17.2.6 *Consider the evaluation map* $\overline{\mathrm{ev}}^\rho : \overline{\mathcal{M}}(L, L^\rho; J^\rho) \to M$*. There exists some* $\epsilon > 0$ *such that, if* $d_{C^1}(L', \mathcal{L}) < \epsilon$ *and* $d_{C^0}(J, \overline{J}) < \epsilon$*, then we have*

$$\text{Image}\, \overline{\mathrm{ev}}^\rho \subset U.$$

Proof We know that

$$\mathcal{M}(L, L^\rho; J^\rho) = \bigcup_{x \in L \cap L^0; y' \in L \cap L1} \overline{\mathcal{M}}(x, y'; J^\rho).$$

Suppose to the contrary that there exists a sequence $\overline{J}_i = \{J_i^s\}$ and H_i with $L^s = \phi_{H_i}^s(L')$ and $d_{C^1}(L, \phi_{H_i}(L')) \to 0$ and $d_{C^0}(J, \overline{J}) \to 0$ as $i \to \infty$ such that there exists for $u_i \in \mathcal{M}(x, y; L, L^\rho; J^\rho)$ a point (τ_i, t_i) with

$$u_i(\tau_i, t_i) \in M \setminus U. \tag{17.2.9}$$

By choosing a subsequence, we may assume $\rho(\tau_i) \to s_0$, $t_i \to t_0$. Using the energy bound and the condition

$$d_{C^1}(L, \mathcal{L}) \to 0,\ d_{C^0}(J, \overline{J}) \to 0$$

and suitably translating the sequence in the τ-direction, we can extract a local limit u_∞ that satisfies (17.2.7) for J_0,

$$u_\infty(-\infty) \to x_0,\ u_\infty(\infty) \to y_0,\ u_\infty(0, t_0) \in M \setminus U,$$

for some $x_0, y_0 \in L \cap L'$. This contradicts the hypothesis that

$$\mathcal{S}(L, L; J; U)$$

is isolated in U. This finishes the proof. □

For such a homotopy $(\overline{J}, \{L^s\})$ as in this lemma, we can define the number

$$n_U(p, q'; J^\rho) = \#\mathcal{M}(p, q'; J^\rho; U).$$

Then the homomorphism

$$h_{(J^\rho, L^\rho; U)}; CF(L, L^0; U) \to CF(L, L^1; U)$$

is well defined and satisfies

$$h_{(J^\rho, L^\rho; U)} \partial_{(L, L^0; J^0; U)} + \partial_{(L, L^1; J^1; U)} h_{(J^\rho, L^\rho; U)} = 0$$

for any Floer regular $\overline{J}$. This establishes the chain property of $h_{(J^\rho, L^\rho; U)}$. By the same argument as in Section 12.5.2, we can prove that it induces an isomorphism in homology and hence the proof of Theorem 17.2.7. □

This finishes the construction of the local Floer cohomology $HF(L, L'; U)$ for any L' that is exact relative to L.

17.2.2 Engulfed Hamiltonian isotopy

In the remaining section, we consider the natural context in which the local Floer homology of a given compact Lagrangian submanifold $L \subset (M, \omega)$ is defined. Let $V \subset \overline{V} \subset U$ be a pair of given Darboux neighborhoods of L.

We first deform the maximal invariant set of $(L, \phi_H^1(L); J; U)$ with respect to the pair $(L, \phi_H^1(L); J_0; U)$ by an isolated continuation in U, where J_0 is a time-independent almost-complex structure that restricts to the canonical almost-complex structure J_g on $U \subset T^*L$ associated with the Riemannian metric g on L. (See Definition 12.2.1 for the definition of J_g.) Such a deformation is possible by what we established in Lemma 17.2.6.

This completes the following computation by the arguments used in (Fl89a).

Theorem 17.2.7 *Let $L \subset M$ be as above and let U be a Darboux neighborhood of L. Then, if $d_{C^1}(L, \phi_H(L)) \leq \epsilon_3$ and $d_{C^0}(J_0, J) < \epsilon_3$ for some time independent J_0 and if J is $(L, \phi_H^1(L))$-regular, we have*

$$HF((L, \phi_H^1(L)); J; U) \cong H_*(L; \mathbb{Z}_2).$$

Proof We will choose a special regular pair (J, ϕ_H) for which we compute $HF(J, (L, \phi_H^1(L); U)$ explicitly. For this purpose, we implant the graph $\mathrm{Graph}\, df \subset T^*L$ of a C^2-small Morse function f into U using the fact that U is symplectomorphic to a neighborhood of zero section of T^*L.

Let V be a neighborhood of the zero section with

$$V \subset \overline{V} \subset U.$$

We consider the canonical almost-complex structure J_g associated with g and the Hamiltonian isotopy $\phi_{f\circ\pi} : T^*L \to T^*L$ induced by the autonomous Hamiltonian $H = f \circ \pi$, where f is a Morse function on L and $\pi : T^*L \to L$ is the canonical projection. In general $\phi_{f\circ\pi}$ does not necessarily map U to U, but if f is sufficiently C^1-small then

$$\phi_{f\circ\pi}^t(\overline{V}) \subset U$$

for all $t \in [0, 1]$. Now, for each smooth map $\chi : \mathbb{R} \to L$, we define a map $u_\chi : \mathbb{R} \times [0, 1] \to P$ by

$$u_\chi(\tau, t) = \phi_{f\circ\pi}^t(\chi(\tau))$$

for $\tau \in \mathbb{R}$ and $t \in [0, 1]$. Again, if f is C^1-small, then

$$\phi_{f\circ\pi}^t(L) \subset V \quad \text{for all } t \in [0, 1]$$

and so

$$\mathrm{Im}\, u_\chi \subset V \subset U.$$

We define a family $J^f = \{J_t^f\}$ of almost-complex structures on T^*L by

$$J_t^f = (\phi_{f\circ\pi}^t)_* J_g (\phi_{f\circ\pi}^t)_*^{-1},$$

which can be made arbitrarily close to the time-independent J_g if we make f C^1-small.

Now we apply Corollary 12.5.7 for $L = o_N$ in the cotangent bundle T^*N to finish the proof. □

We would like to remark that $A(\omega, J; L) = \infty$ if L is a compact exact Lagrangian submanifold in an exact (M, ω). This holds even for a *non-compact* exact symplectic manifold M that is tame or bounded at infinity. This immediately implies the following.

Theorem 17.2.8 *Let (M, ω) be a tame exact symplectic manifold. Let $L \subset (M, \omega)$ be a compact exact Lagrangian submanifold. Then we have*

$$HF_*(L, L) \cong H_*(L).$$

Proof We fix a Darboux–Weinstein neighborhood U of L. Then consider a Morse function $f : L \to \mathbb{R}$ and let $\epsilon > 0$ be a sufficiently small constant. Since $A(\omega, J; L) = \infty$, only case (1) in Proposition 17.1.3 holds, i.e., all the trajectories in $\mathcal{M}(L, \phi_{\epsilon f}(L); J)$ are contained in U if $\epsilon > 0$ is sufficiently small and $|J - J_0|_{C^0} \leq \epsilon$. Therefore we have proven that $HF_*(L, \phi_{\epsilon f}(L)) \cong HF_*(L, \phi_{\epsilon f}(L); U) \cong H_*(L)$. □

This theorem immediately gives rise to the following two theorems of Gromov (Gr85) as demonstrated in (Oh96b), which reveals two different characteristics of the ambient symplectic manifolds.

Theorem 17.2.9 (Gromov) *There exists no exact Lagrangian embedding in $\mathbb{C}^n$.*

Proof If there were such an embedding $L \subset \mathbb{C}^n$, its Floer homology, say, over $\mathbb{Z}_2$ coefficients, would be isomorphic to $H_*(L; \mathbb{Z}_2)$. Since L is compact, we know that at least $H_0(L; \mathbb{Z}_2) \neq 0$ and so $HF_*(L, \phi_H^1(L); J; \mathbb{Z}_2) \neq 0$ for any Hamiltonian isotopy ϕ_H. In particular, we conclude that

$$L \cap \phi_H^1(L) \neq \emptyset$$

for *any*, not necessarily transversal, Hamiltonian diffeomorphism ϕ_H^1.

On the other hand, obviously we can displace L away from itself by a translation that is a Hamiltonian isotopy, which amounts to a contradiction. This finishes the proof. □

This time we consider the cotangent bundle T^*N of any compact manifold N.

Theorem 17.2.10 (Gromov) *Let $L \subset T^*N$ be any compact exact Lagrangian embedding. Then $L \cap o_N \neq \emptyset$.*

Proof Again we prove this by contradiction. Suppose that $L \cap o_N = \emptyset$. We consider the radial multiplication map $R_\lambda : T^*N \to T^*N$, $(q, p) \mapsto (q, \lambda p)$ for $\lambda > 0$. Since $L \cap o_N = \emptyset$, it follows that

$$R_\lambda(L) \cap L = \emptyset \tag{17.2.10}$$

for sufficiently large $\lambda > 0$. The map R_λ is not quite symplectic but only conformally symplectic, i.e., $R_\lambda^*\omega_0 = \lambda\omega_0$.

However, on combining this conformally symplectic property of R_λ and the exactness of L, it follows that the isotopy

$$\lambda \mapsto R_\lambda(L)$$

is an exact Lagrangian isotopy. Then, by Theorem 3.6.7, we can find a Hamiltonian H such that

$$\phi_H^\lambda(L) = R_\lambda(L)$$

and hence $R_\lambda(L) \cap L \neq \emptyset$ for any λ, which is again a contradiction. □

Recently, Theorem 17.2.10 has been greatly strengthened by Nadler (Na09) and Fukaya, Seidel and Smith (FSS08) by the homological machinery of the Fukaya category.

Remark 17.2.11 In relation to the study of spectral invariants of topological Hamiltonian paths in (Oh11b), it is important to understand the localization of the Floer complex of the V-engulfed Hamiltonian H whose time-one map is sufficiently C^0-small. For such a Hamiltonian, the way in which the thick–thin dichotomy was established in Section 17.1 by considering the areas and actions is not appropriate because we have no control of the areas or the actions for the pair $(\phi_H^1(L), L)$ when ϕ_H^1 is only C^0-small. It turns out that the correct way of obtaining the thick–thin decomposition of the Floer moduli space is by exploiting the *maximum principle* for the almost-complex structure J_g. When H is C^2-small, this decomposition coincides with the one given in Section 17.1. We refer the reader to (Oh11b) for more details.

17.3 Construction of the spectral sequence

In this section, we present an important tool, the spectral sequence, which is useful for the computation of Floer homology or for the application of Floer homology in practical problems. This particular spectral sequence, which relates the intrinsic topology of a Lagrangian submanifold L and its extrinsic symplectic topology of the Lagrangian embedding $L \subset (M, \omega)$, was formulated by the author (Oh96b) in terms of the Floer complex

$$CF(\phi_{-\epsilon f}^1(L), L)$$

for small $\epsilon > 0$ (over $\mathbb{Z}_2$ coefficients). The construction has been further strengthened by Buhovsky (Bu10), Biran and Cornea (BCo09) and Damian

(D09), still in its original form in terms of Morse homology of small Morse functions. However, the optimal setting of this spectral sequence is the one in the Bott–Morse setting presented in (FOOO09) by directly considering the clean pair (L, L). The mathematics involved in the latter construction goes beyond the framework of this book, so we will restrict the discussion to the setting of (Oh96b) but we at least explain an important ingredient, namely the compatibility of the product structure of the spectral sequence, borrowing the exposition from Theorem D (FOOO09) with some simplification adapted to the monotone context.

Because of the usage of product structure in Floer homology, we will use the cohomological version of Floer homology in this section, which means the switch from $-\epsilon f$ to ϵf.

On the basis of our analyses of thick–thin dichotomy and of local Floer cohomology, we fix a Morse function $f : L \to \mathbb{R}$ and a sufficiently small $\epsilon > 0$ such that the dichotomy in Proposition 17.1.3 and Theorem 17.2.7 hold for the pair $(L, \phi^1_{\epsilon f}(L))$. We denote by $\Sigma_L \geq 2$ the minimal positive Maslov number of L.

We put $N_L = [(n + 1)/\Sigma_L]$ and define a formal parameter

$$q^\beta = T^{\omega(\beta)} e^{\mu_L(\beta)},$$

with

$$\deg(q^\beta) = \mu_L(\beta), \quad E(q^\beta) = \omega(\beta)$$

as before and consider the Novikov ring $\Lambda_{(\omega,L)}$ and its subring $\Lambda_{0,(\omega,L)}$.

Consider the Floer complex $CF(L, \phi^1_{\epsilon f}(L))$ with the structure of a $\Lambda_{(L,\omega)}$ module as formulated in Section 13.9.3. Recall that $CF^*(L_1, L_0) \cong CF_*(L_0, L_1)^*$ was associated with the full path space $\Omega(L_0, L_1)$ as a $\Lambda(L_0, L_1)$ module in Section 13.9.3.

In the current context of $(L, \phi^1_{\epsilon f}(L))$, we provide the filtration level and grading of $q^\beta \cdot [p]$ by

$$\ell(q^\beta \cdot [p]) = \omega(\beta) + f(p), \tag{17.3.11}$$

$$\deg(q^\beta \cdot [p]) = n - \mu_f(p) - \mu_L(\beta). \tag{17.3.12}$$

The way these are set up is motivated by our wish that $\overline{\partial}_{(0)}$ coincides with the Morse boundary operator of $-f$ (or the Morse coboundary operator of f). In other words, we provide the $\Lambda_{(\omega,L)}$-module structure on $CF(L, \phi^1_{\epsilon f}(L))$ by extending the relation

$$[\beta] \cdot [p] = T^{-\omega(\beta)} e^{-\mu_L(\beta)} \cdot [p] \tag{17.3.13}$$

in $\Lambda_{(\omega,L)}$. By definition, we have

$$CF(L, \phi^1_{\epsilon f}(L)) = \Lambda_{\omega,L}\{\mathrm{Crit}\, f\}.$$

Then the coboundary map

$$\delta : CF(L, \phi^1_{\epsilon f}(L)) \to CF(L, \phi^1_{\epsilon f}(L))$$

respects this $\Lambda_{(\omega,L)}$-module structure.

By the thick–thin dichotomy, the boundary operator $\delta : C^* \to C^*$ has the form

$$\delta = \delta_{(0)} + \delta',$$

where $\delta_{(0)}$ comes from the coboundary operator associated with the local Floer complex and δ' is the contribution of thick trajectories.

To analyze δ' further, let $u \in \mathcal{M}(L, \phi^1_{\epsilon f}(L); x, y)$ be any thick trajectory and $w : ([0,1]^2, \delta[0,1]^2) \to (M, L)$ the associated map used in the proof of Lemma 17.1.4. If we choose $\epsilon > 0$ sufficiently small, then we have

$$\int w^*\omega \geq \Sigma_L - \frac{1}{4}\Sigma_L = \frac{3}{4}\Sigma_L > 0,$$
$$\mu_L(w) = 1 - \mu_f(x) + \mu_f(y) \leq n + 1.$$

By the monotonicity of L and $\int w^*\omega > 0$, we have $\mu_L(w) > 0$.

It follows from this that

$$\mu_f(x) - \mu_f(y) = 1 - \mu_L(w)$$

and

$$\mu_L(w) = l\Sigma \quad \text{for} \quad 1 \leq l \leq \left[\frac{n+1}{\Sigma_L}\right].$$

Therefore each thick trajectory changes the Morse grading by $-l\Sigma_L + 1$ for some $1 \leq l \leq [(n+1)/\Sigma_L]$ and so δ is decomposed into

$$\delta' = \delta_{(0)} + \overline{\delta}_{(1)} \otimes q + \cdots + \overline{\delta}_{(N_L)} \otimes q^{N_L}, \tag{17.3.14}$$

where $\overline{\delta}_{(\ell)} : C^{(*)}(f) \to C^{(*-1+\Sigma_L)}(f)$ is the linear map induced from the trajectories connecting critical points of f with indices $*$ and $* - l\Sigma + 1$. Recall that $\delta_{(0)}$ maps C^* to C^{*+1}.

Theorem 17.3.1 *There exists a spectral sequence with the following properties.*

(1) $E_2^{p,q} = \bigoplus_k H^k(L; \mathbb{Q}) \otimes \left(T^{q\lambda}\Lambda_{(\omega,L)}/T^{(q+1)\lambda}\Lambda_{(\omega,L)}\right)^{(p-k)}$. *Here* $\lambda > 0$.

(2) *There exists a filtration* $F^*HF(L, \phi'_{\epsilon f}(L); \Lambda_{(\omega,L)})$ *on the Floer cohomology* $HF(L, \phi'_{\epsilon f}(L); \Lambda_{(\omega,L)})$ *such that*

$$E_\infty^{p,q} \cong \frac{F^q HF^p(L, \phi'_{\epsilon f}(L); \Lambda_{(\omega,L)})}{F^{q+1}HF^p(L, \phi'_{\epsilon f}(L); \Lambda_{(\omega,L)})}.$$

(We refer the reader to Theorem D in (FOOO09) for a more general statement of this theorem, especially in relation to the version deformed by *bulk deformations* or by ambient cohomology classes of M.)

The rest of this subsection will be occupied by the proof of this theorem.

Since we have $\Sigma_L \geq 2$, $\overline{\delta}_{(l)}$ has degree less than or equal to -1. By comparing the degrees of each summand of the equation

$$0 = \delta \circ \delta = (\delta_{(0)} + \delta')^2$$

we immediately obtain

$$\begin{aligned} \delta_{(0)} \circ \delta_{(0)} &= 0, \\ \delta_{(0)} \circ \delta' + \delta' \circ \delta_{(0)} + \delta' \circ \delta' &= 0. \end{aligned}$$

Now the first equation enables us to define $H(C_*, \delta_{(0)}) \cong H^*(L; f)$. Using the definition (17.3.14), we expand the second equation as

$$\begin{aligned} 0 &= \delta_{(0)} \circ \delta' + \delta' \circ \delta_{(0)} + \delta' \circ \delta' \\ &= \left(\delta_{(0)}\overline{\delta}_{(1)} + \overline{\delta}_{(1)}\delta_{(0)}\right) q + \left(\overline{\delta}_{(1)}\overline{\delta}_{(1)} + \delta_{(0)}\overline{\delta}_{(2)} + \overline{\delta}_{(2)}\delta_{(0)}\right) q^2 + \cdots . \end{aligned}$$

By collecting the terms of q-degree 1 and 2, we obtain two equations:

$$\begin{aligned} \delta_{(0)}\overline{\delta}_{(1)} + \overline{\delta}_{(1)}\delta_{(0)} &= 0, \\ \overline{\delta}_{(1)}\overline{\delta}_{(1)} + \delta_{(0)}\overline{\delta}_{(2)} + \overline{\delta}_{(2)}\delta_{(0)} &= 0. \end{aligned}$$

The first equation implies that $\overline{\delta}_{(1)}$ descends to $H(C_*, \delta_{(0)}) = H^*(L; \epsilon f)$ and the second implies that it defines a differential there.

The action filtration on $C(L, \phi'_{\epsilon f}(L))$ induces the spectral sequence we are defining. We first note that there is an isomorphism

$$\mathrm{gr}_*(CF(L, \phi^1_{\epsilon f}(L))) \cong C(L; \epsilon f) \otimes \mathrm{gr}_*(F^*\Lambda_{(\omega,L)})$$

as $\mathrm{gr}_*(F^*\Lambda_{(\omega,L)})$ modules. (See Definition 16.2.3 and afterwards for the filtration on $\Lambda_{(\omega,L)}$.) Then we obtain the spectral sequence $E_r^{p,q}$, where

$$E_2^{p,q} = \bigoplus_{k=0}^{n} H^k(L; R) \otimes \mathrm{gr}_q\left(F\Lambda^{(p-k)}_{(\omega,L)}\right). \tag{17.3.15}$$

Here $\Lambda^{(p-k)}_{(\omega,L)}$ denotes the part of degree $p-k$. We can also show that there exists a filtration on $HF^p(L, \phi^1_{\epsilon f}(L); \Lambda_{(\omega,L)})$ such that

$$E_\infty^{p,q} \cong F^q HF^p(L, \phi^1_{\epsilon f}(L); \Lambda_{(\omega,L)}) / F^{q+1} HF^p(L, \phi^1_{\epsilon f}(L); \Lambda_{(\omega,L)}).$$

We remark that the proof of convergence $E_2^{p,q} \Longrightarrow E_\infty^{p,q}$ of the above spectral sequence works for an arbitrary commutative ring R.

Now consider a graded $\Lambda_{(\omega,L)}$ module D that has a filtration F^*D compatible with that on $\Lambda_{(\omega,L)}$ in that for $D = CF(L, \phi'_{\epsilon f}(L))$

$$F^\ell \Lambda_{(\omega,L)} \cdot F^k D \subseteq F^{k+\ell} D.$$

Let $\Lambda^{(0)}_{(\omega,L)}$ be the degree-0 part of $\Lambda_{(\omega,L)}$. We have

$$\Lambda^{(0)}_{(\omega,L)} \cong R[[T^{A_{(\omega,L)}}]][T^{-A_{(\omega,L)}}],$$

which shows that, if R is a field, then $\Lambda^{(0)}_{(\omega,L)}$ is a field and hence the degree-p part D^p of D is a vector space over $\Lambda^{(0)}_{(\omega,L)}$. We use this filtration to define our spectral sequence following the standard procedure applied to the filtered complex (McC85).

Note that multiplications by $T^{A_{(\omega,L)}}$ and e give the following isomorphisms, respectively:

$$T^{A_{(\omega,L)}} \cdot : F^q HF^p(L, \phi^1_{\epsilon f}(L); \Lambda_{(\omega,L)}) \longrightarrow F^{q+1} HF^p(L, \phi^1_{\epsilon f}(L); \Lambda_{(\omega,L)}),$$
$$e \cdot : F^q HF^p(L, \phi^1_{\epsilon f}(L); \Lambda_{(\omega,L)}) \longrightarrow F^q HF^{p+2}(L, \phi^1_{\epsilon f}(L); \Lambda_{(\omega,L)}).$$

Denote

$$\Lambda^{(0)}(\lambda) = \Lambda^{(0)}_{(\omega,L)} / F^\lambda \Lambda^{(0)}_{(\omega,L)}.$$

We define a filtration of $\Lambda^{(0)}_{(\omega,L)}$ by $F^n \Lambda^{(0)}_{(\omega,L)} = F^{nA_{(\omega,L)}} \Lambda^{(0)}_{(\omega,L)}$. Then its associated graded module is given by

$$\mathrm{gr}_* \left(F\Lambda^{(0)}_{(\omega,L)} \right) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathrm{gr}_n \left(F\Lambda^{(0)}_{(\omega,L)} \right),$$

where each $\mathrm{gr}_n \left(F\Lambda^{(0)}_{(\omega,L)} \right)$ is naturally isomorphic to $\Lambda^{(0)}(A_{(\omega,L)})$. We also have

$$\mathrm{gr}_* \left(F\Lambda_{(\omega,L)} \right) = \mathrm{gr}_* \left(F\Lambda^{(0)}_{(\omega,L)} \right) [e, e^{-1}] = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathrm{gr}_n \left(F\Lambda^{(0)}_{(\omega,L)} \right) [e, e^{-1}].$$

We denote by $\widehat{C}$ the $\Lambda^{(0)}_{(\omega,L)}$ module over $\mathrm{Crit}\, f$ and define

$$Z_r^{p,q}(\widehat{C}) = \{x \in F^q \widehat{C}^p \mid \delta x \in F^{q+r-1} \widehat{C}^{p+1}\} + F^{q+1} \widehat{C}^p,$$
$$B_r^{p,q}(\widehat{C}) = (\delta(F^{q-r+2} \widehat{C}^{p-1}) \cap F^q \widehat{C}^p) + F^{q+1} \widehat{C}^p,$$
$$E_r^{p,q}(\widehat{C}) = Z_r^{p,q}(\widehat{C}) / B_r^{p,q}(\widehat{C}).$$

Recall that $E_r^{p,q}$ has the natural structure of a $\Lambda^{(0)}_{(\omega,L)}$ module. The multiplication by $e^{\pm 1} \in \Lambda_{(\omega,L)}$ defines a map

$$e^{\pm 1} : E_r^{p,q}(\widehat{C}) \to E_r^{p\pm 2,q}(\widehat{C}),$$

which turns $E_r(\widehat{C}) := \bigoplus_{p,q} E_r^{p,q}(\widehat{C})$ into a $\mathrm{gr}_*(F\Lambda_{(\omega,L)})$ module and

$$\bigoplus_{p,q} \delta_r^{p,q} : \bigoplus_{p,q} E_r^{p,q}(\widehat{C}) \to \bigoplus_{p,q} E_r^{p+1,q+r-1}(\widehat{C})$$

is a $\mathrm{gr}(F^*\Lambda_{(\omega,L)})$-module homomorphism.

Lemma 17.3.2 *Let $E_r^{p,q}\widehat{(C)}$ be the $\Lambda_{(\omega,L)}^{(0)}$ module defined above. There exists a $\Lambda_{(\omega,L)}^{(0)}$-module homomorphism*

$$\delta_r^{p,q} : E_r^{p,q}(\widehat{C}) \to E_r^{p+1,q+r-1}(\widehat{C})$$

such that

(i) $\delta_r^{p+1,q+r-1} \circ \delta_r^{p,q} = 0$,
(ii) $\mathrm{Ker}(\delta_r^{p,q})/\mathrm{Im}(\delta_r^{p-1,q-r+1}) \cong E_{r+1}^{p,q}(\widehat{C})$,
(iii) $e^{\pm 1} \circ \delta_r^{p,q} = \delta_r^{p\pm 2,q} \circ e^{\pm 1}$.

Proof We define $\delta_r^{p,q}[x] = [\delta x] \in E_r^{p+1,q+r-1}(\widehat{C})$. Once we have this, the proof immediately follows. □

The following lemma reflects the fact that we are working with the monotone case.

Lemma 17.3.3 *The spectral sequence collapses at the E_{N_L} page for*

$$N_L = \left[\frac{n+1}{\Sigma_L}\right].$$

In particular the spectral sequence stabilizes after at most N steps.

Proof This is an immediate consequence of the structure of δ given in (17.3.14). □

This lemma enables us to prove

$$E_\infty^{p,q}(\widehat{C}) = E_N^{p,q}(\widehat{C}).$$

Remark 17.3.4 The corresponding spectral sequence in the generality of (FOOO09) beyond the monotone case does not collapse in the finite stage, but it was proved that the projective limit

$$\varprojlim E_r^{p,q}(\widehat{C})$$

exists and so the spectral sequence still converges. We refer the reader to (FOOO09) for the proof of this convergence in general.

Now let $(\overline{C}, \overline{\delta})$ be the local Floer cohomology complex of $(L, \phi^1_{\epsilon f}(L))$, which is isomorphic to the Morse complex $(C^*(f), \delta_{\epsilon f})$ by definition. Then we have the following lemma.

Lemma 17.3.5 *There exists an isomorphism*

$$E_2^{*,*}(\widehat{C}) \cong H(\overline{C}; \overline{\delta}) \otimes_R \mathrm{gr}_*(F\Lambda_{(\omega,L)})$$

as $\mathrm{gr}_*(F\Lambda_{(\omega,L)})$ *modules.*

Proof By definition,

$$E_1^{*,*}(\widehat{C}) \cong \overline{C} \otimes_R \mathrm{gr}_* \left(F\Lambda_{(\omega,L)}\right).$$

It follows from Proposition 17.1.3 that $\delta_{(1)} = \overline{\delta}$. This finishes the proof. □

Definition 17.3.6 We define $F^q H(\widehat{C}, \delta)$ to be the image of $H(F^q\widehat{C}, \delta)$ in $H(\widehat{C}, \delta)$.

Then, by the general result on the spectral sequence or by an explicit identification in the current case, which stabilizes in a finite page, we obtain the following theorem.

Theorem 17.3.7

$$E_N^{p,q}(\widehat{C}) \cong F^q H^p(\widehat{C}, \delta)/F^{q+1} H(\widehat{C}, \delta)$$

as a $\Lambda^{(0)}(\lambda_0) := \Lambda^{(0)}_{(\omega,L)}/F^{\lambda_0}\Lambda^{(0)}_{(\omega,L)}$ *module.*

By definition

$$H^p(\widehat{C}, \delta) = HF^p(L, \phi^1_{\epsilon f}(L); \Lambda_{(\omega,L)}).$$

Remark 17.3.8 Recall that $\Lambda_{0;(\omega,L)}$ is a discrete valuation ring in the monotone case and its residue field is $\Lambda_{(\omega,L)}$. Therefore we have the isomorphism

$$HF(L, \phi^1_{\epsilon f}(L); \Lambda_{(\omega,L)}) \cong HF(L, \phi^1_{\epsilon f}(L); \Lambda_{0;(\omega,L)}) \otimes_{\Lambda_{0;(\omega,L)}} \Lambda_{(\omega,L)}. \qquad (17.3.16)$$

The Floer complex of the pair $(L, \phi^1_{\epsilon f}(L))$ over the subring $\Lambda_{0;(\omega,L)}$ can always be constructed for the non-monotone case in the same way as we did over the full Novikov ring $\Lambda_{(\omega,L)}$, but the associated homology $HF(L, L'; \Lambda_{0;(\omega,L)})$ need not be invariant under the Hamiltonian isotopy. On the other hand, we have shown that the homology $HF(L, L'; \Lambda_{(\omega,L)})$ is invariant under the Hamiltonian isotopy of the monotone pair (L, L') for the cases

(1) where both L and L' have minimal Maslov number greater than 2 (Theorem 16.4.6) or
(2) where L' is a Hamiltonian deformation of L and L has a minimal Maslov number that is greater than or equal to 2 (Theorem 16.4.10).

Consideration of δ for the complex $CF(L, \phi^1_{\epsilon f}(L))$ and its associated spectral sequence constructed here gives rise to the following theorem proved by the author in (Oh96b). In the statement of the theorem, we will just denote $HF^*(L, \phi^1_{\epsilon f}(L))$ as $HF^*(L, L)$ or $HF\left(L; \Lambda^{\mathbb{Z}_2}_{(\omega,L)}\right)$ over the coefficient ring $\mathbb{Z}_2$.

Theorem 17.3.9 (Compare this with (Oh96b)) *Let $L \subset (M, \omega)$ be monotone with $\Sigma_L \geq n + 1$. Consider the Novikov ring over $\mathbb{Z}_2$, $\Lambda^{\mathbb{Z}_2}_{(\omega,L)}$ and the associated Floer cohomology $HF^*(L, L)$. Then the following statements hold.*

(1) *If $\Sigma_L > n + 1$, then $HF^*(L, L) \cong H^*(L; \mathbb{Z}_2) \otimes \Lambda^{\mathbb{Z}_2}_{(\omega,L)}$.*
(2) *If $\Sigma_L = n + 1$, then $HF^\ell(L, L) \cong H^\ell(L; \mathbb{Z}_2)$ for $1 < \ell < n$ and either $HF^0(L, L) = HF^n(L, L) = \{0\}$ or*

$$\begin{aligned} HF^n(L, L) &= H^n(L; \mathbb{Z}_2), \\ HF^0(L, L) &= H^0(L; \mathbb{Z}_2). \end{aligned} \tag{17.3.17}$$

Proof Since $n \geq 1$, we have $\Sigma_L \geq 2$ and so $HF(L, \phi(L))$ is well defined and invariant under the Hamiltonian of ϕ. From now on, we will just denote this as $HF(L, L)$.

If $\Sigma_L > n + 1$, we have $N_L = 0$, i.e., $\delta = \delta_{(0)}$ and so Lemma 17.3.3 and Theorem 17.3.7 imply

$$\begin{aligned} HF^*\left(L, L; \Lambda^{\mathbb{Z}_2}_{(\omega,L)}\right) &\cong E_2^{*,*}(\widehat{C}) \cong H(C(f); \delta_f) \otimes_{\mathbb{Z}_2} \mathrm{gr}_*(F\Lambda^{\mathbb{Z}_2}_{(\omega,L)}) \\ &\cong H^*(L; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Lambda^{\mathbb{Z}_2}_{(\omega,L)} \end{aligned}$$

as a $\Lambda^{\mathbb{Z}_2}_{(\omega,L)}$ module, which finishes the proof.

On the other hand, if $\Sigma_L = n + 1$, then $N_L = 1$ and so $\delta = \delta_{(0)} + \delta_{(1)} \otimes q$, where $\delta_{(1)} : C^*(f) \to C^{*+n}(f)$. Therefore the only non-trivial contribution occurs on $C^0(f)$. In the E_2 page, it induces a coboundary map

$$\delta_{(1)} \otimes \Lambda^{(0);\mathbb{Z}_2}_{(\omega,L)} : H^0(f; \mathbb{Z}_2) \otimes \Lambda^{(0);\mathbb{Z}_2}_{(\omega,L)} \to H^0(f; \mathbb{Z}_2) \otimes \Lambda^{(0);\mathbb{Z}_2}_{(\omega,L)} \tag{17.3.18}$$

and zero in all other degrees. Therefore the first statement of (2) holds. For the second statement, there are two cases to consider, depending on whether $\delta_{(1)}$ is zero or not.

When it is not zero, it induces an isomorphism and so

$$HF^0(L, L) = HF^n(L, L) = \{0\}.$$

On the other hand, if it is zero, then (17.3.17) follows.

The above theorem can be improved over $\mathbb{Z}$ coefficients for the case of (relative) spin L. □

Finally, we briefly mention the product structure

$$\mathfrak{m}_2 : HF(L, L; \Lambda_{(\omega,L)}) \otimes HF(L, L; \Lambda_{(\omega,L)}) \to HF(L, L; \Lambda_{(\omega,L)})$$

on the Floer cohomology defined in (FOOO09). One may regard the Morse–Bott version $HF(L, L; \Lambda_{(\omega,L)})$ of the Floer cohomology as $HF(L, \phi^1_{\epsilon f}(L))$ when $\epsilon > 0$ is sufficiently small. Then the triangle product

$$\mathfrak{m}_2 : HF(L, \phi^1_{\epsilon f}(L)) \times HF(\phi^1_{\epsilon f}(L), L) \times HF(L, L)$$

constructed in Section 16.5 gives rise to a product $\mathfrak{m}_2 : \widehat{C}(L) \otimes \widehat{C}(L) \to \widehat{C}(L)$ on letting $\epsilon \to 0$. It is easy to see from the construction that

$$\mathfrak{m}_2(F^{q_1}\widehat{C}(L) \otimes F^{q_2}\widehat{C}(L)) \subseteq F^{q_1+q_2}\widehat{C}(L).$$

This implies that the filtration defined above on $HF(L, L; \Lambda_{(\omega,L)})$ is preserved by $\mathfrak{m}_2$. The following theorem was proved in Section 6.4.5 (FOOO09). We refer readers there for its proof. (See also (Bu10).)

Theorem 17.3.10 *The spectral sequence constructed in Theorem 17.3.1 is compatible with the ring structure in that we have the following. Each page E_r of the spectral sequence has a ring structure $\mathfrak{m}_2$ that satisfies*

$$\delta_r(\mathfrak{m}_2(x, y)) = -\mathfrak{m}_2(\delta_r(x), y) + (-1)^{\deg x}\mathfrak{m}_2(x, \delta_r(y)).$$

The filtration F is compatible with the ring structure induced by $\mathfrak{m}_2$. The isomorphisms present in Theorem 17.3.1 are ring isomorphisms.

We also refer the reader to (BCo09), (D09) for further enhancement of this spectral sequence and its other applications.

17.4 Biran and Cieliebak's theorem

One geometrically obvious fact from its definition, which plays a crucial role in application of the Floer homology invariants, is the following vanishing result.

Theorem 17.4.1 *Suppose that $L \subset (M, \omega)$ carries a 'Floer homology' $HF(L; M)$ that is invariant under the Hamiltonian isotopy. If L is displaceable, i.e., if there exists a Hamiltonian diffeomorphism ϕ with $\phi(L) \cap L = \emptyset$, then $HF(L; M) = \{0\}$.*

Proof Choose any Hamiltonian diffeomorphism ϕ such that $L \cap \phi(L) = \emptyset$. Then by virtue of the definition of $CF(L, \phi(L))$, $CF(L, \phi(L)) = \{0\}$ and hence $HF(L, \phi(L)) = \{0\}$. On the other hand, since $HF(L, \phi(L))$ is isomorphic under the Hamiltonian isotopy, we also have $HF(L, \phi(L)) \cong HF(L; M)$. This finishes the proof. □

Recall that we have shown in Chapter 16 that, if one of the following conditions holds, then the Floer homology is well defined and invariant under the Hamiltonian isotopy:

(1) L_0, L_1 satisfy $\Sigma_{L_i} > 2$,
(2) L_1 is Hamiltonian isotopic to $L_0 = L$ and L satisfies $\Sigma_{L_i} \geq 2$.

In particular, for case (2), the following holds.

(1) $HF(L, \phi(L))$ is well defined for any Hamiltonian diffeomorphism ϕ with $L \pitchfork \phi(L)$.
(2) There exists a canonical isomorphism between $HF(L, \phi(L))$ and $HF(L, \phi'(L))$, provided that a Hamiltonian isotopy between them is given, and the isomorphism depends only on the (Hamiltonian) isotopy class thereof.

Corollary 17.4.2 *For any compact monotone Lagrangian submanifold of $\mathbb{R}^{2n}$ with $\Sigma_L \geq 2$, $HF(L, \phi(L))$ satisfies the above properties and hence $HF(L, \phi(L)) = \{0\}$.*

In (BCi02), Biran and Cieliebak observed that a similar automatic displacement property of compact Lagrangian embedding holds on any symplectic manifold of the type

$$V \times M, \tag{17.4.19}$$

where V is a sub-critical Weinstein manifold and M is any compact symplectic manifold. In particular, they applied this observation with Theorem 17.3.9 and Corollary 17.4.2 in an interesting way to the study of a topological property of compact Lagrangian submanifolds $L \subset V \times \mathbb{C}P^n$. In the remaining section, we describe their theorem, closely following the arguments from (BCi02).

We first introduce a few standard notions in the study of non-compact symplectic manifolds from (EG91).

Definition 17.4.3 A symplectic manifold (M, ω) is called *convex at infinity* if it carries a vector field X that is *completely symplectically dilating* at infinity. A vector field X is completely symplectically dilating if the flow $\{\phi^t\}$ of X is complete and satisfies $(\phi^t)^*\omega = e^t\omega$. We assume that (M, ω) allows a plurisubharmonic exhaustion function at infinity.

Let (Q, ξ) be a contact manifold and λ a contact form of ξ, i.e., a one-form with $\ker\lambda = \xi$. The Reeb vector field X_λ associated with the contact form λ is the unique vector field that satisfies

$$X \rfloor \lambda = 1, \quad X \rfloor d\lambda = 0. \tag{17.4.20}$$

Therefore the contact form naturally provides a splitting

$$TQ = \mathbb{R}\{X_\lambda\} \oplus \xi.$$

We denote by $\Pi = \Pi_\lambda : TQ \to TQ$ the idempotent satisfying Image $\Pi = \xi$ and $\ker\Pi = \mathbb{R}\{X_\lambda\}$, and by $\pi = \pi_\lambda : TQ \to \xi$ the corresponding projection.

The following is a standard definition in the study of a contact manifold.

Definition 17.4.4 We say that a compact contact manifold (Q, ξ) is *strongly symplectically fillable* if there exists a compact symplectic manifold (W, ω) such that

(a) $\partial W = Q$
(b) $\omega = d\lambda$ for some one-form λ near the boundary and $\xi = \ker\lambda$ on ∂W
(c) the orientation of Q defined by $\lambda \wedge (d\lambda)^{n-1}$ coincides with the boundary orientation of $Q = \partial W$.

We call (W, ω) a strong symplectic filling of (Q, ξ).

According to the symplectic neighborhood theorem, one can choose a function r in a collar neighborhood U_δ of $\partial W = Q$ in W such that

$$\begin{aligned} U_\delta \cong & \quad (1-\delta, 1] \times Q, \\ \omega = & \quad d(r\pi^*\lambda) \quad \text{on } U_\delta \end{aligned} \tag{17.4.21}$$

for the projection $\pi : M^{\geq R} \to Q$. We then consider the cylinder $(1-\delta, \infty) \times Q$ and form the union

$$\widehat{W} = W\#((1-\delta, \infty) \times Q) \tag{17.4.22}$$

along the strip $(1-\delta, 1] \times Q \cong U_\delta$. The symplectic form ω naturally extends to $\widehat{W} := M$ by gluing it with $d(r\pi^*\lambda)$ on $(1-\delta, \infty) \times Q$. By definition, every symplectic manifold that is convex at infinity has this decomposition upon identifying $\varphi = \log r$, or equivalently $r = e^\varphi$, in terms of the identification

$M^{\geq R} = M^{r\geq R} \cong [0,\infty) \times \varphi^{-1}(\ln R)$. In terms of the chosen contact form λ on Q and the projection $\pi : M^{\geq R} \to Q$, the symplectic form ω has the expression

$$\widehat{\omega} = d(r\pi^*\lambda) = d(e^s \pi^*\lambda). \tag{17.4.23}$$

We call (s, y) the cylindrical coordinates and $(r = e^s, y)$ the cone coordinates.

Remark 17.4.5 In the case of $M = \mathbb{C}^n \setminus \{0\} \cong (0,\infty) \times S^{2n-1}$, $(\sqrt{r}, y)$ with $y \in S^{2n-1}$ is nothing but the standard polar coordinates of $\mathbb{C}^n \setminus \{0\}$. On the other hand, if we write

$$T^*N \setminus \{0\} \cong S^1(T^*N) \times \mathbb{R}_+$$

then $r = |p|$ for the canonical coordinates (q, p) of $T^*N \setminus \{0\} \subset T^*N$.

On the cylinder $(-\infty, 0] \times Q \subset (-\infty,\infty) \times Q$, we have the natural splitting

$$T_xM \cong \mathbb{R}\left\{\frac{1}{r}\frac{\partial}{\partial r}\right\} \oplus T_qQ \cong \operatorname{span}\left\{\frac{\partial}{\partial s}, X_\lambda(q)\right\} \oplus \xi_q \cong \mathbb{R}^2 \oplus \xi_q.$$

We denote by $\widetilde{X}_\lambda$ the unique vector field on M which is invariant under the translation, tangent to the level sets of r and projected to X_λ. When there is no danger of confusion, we sometimes just denote it by X_λ.

Now we describe a special family of almost-complex structure adapted to the given cylindrical structure of M of contact type (Q, ξ).

We first introduce the following definition of CR-almost-complex structure on a contact manifold (Q, ξ) equipped with a contact form λ.

Definition 17.4.6 (Contact triad and triad metric) We call an endomorphism $J_Q : TQ \to TQ$ satisfying $J^2 = -\Pi$ a λ-compatible CR-almost-complex structure, if the bilinear form $d\lambda(\cdot, \Pi\cdot)$ is nondegenerate on ξ. We call the triple (Q, λ, J_Q) a contact triad. We call a *triad metric* the metric on Q defined by

$$g_Q(h,k) = g_{(\lambda,J_Q)} := \lambda(h)\lambda(k) + d\lambda(h, J_Qk)$$

for h, $k \in TQ$.

Definition 17.4.7 Let (M, ω) be a symplectic manifold with a cylindrical end as above. An almost-complex structure J on M is said to be of *λ-contact type* if it is split into

$$J = j \oplus J_Q|_\xi : TM \cong \mathbb{R}^2 \oplus \xi \to TM \cong \mathbb{R}^2 \oplus \xi$$

on the end, where $J_Q|_\xi$ is compatible with $d\lambda|_\xi$ and $j : \mathbb{R}^2 \to \mathbb{R}^2$ maps $\partial/\partial s$ to X_λ.

We choose a pluri-subharmonic exhaustion function φ with respect to a tame almost-complex structure J on M. We also assume that J is invariant under the flow of X outside a compact set. Then the level set $\varphi^{-1}(S)$ for sufficiently large S carries the induced contact structure on it, and the flow map

$$\phi : (0,\infty)\times\varphi^{-1}(S) \to M \cap \varphi^{-1}([S,\infty)); \ (s,y) \mapsto \phi^s(y)$$

defines a diffeomorphism. In these coordinates, we have $\varphi = s$ and ϕ^s is nothing but the translation map $\phi^s(S,y) = (S+s,y)$. With this having been said, the definition of a symplectic manifold that is convex at infinity can be given in terms of this contact manifold $N = \varphi^{-1}(S)$.

Definition 17.4.8 A non-compact symplectic manifold (V,ω) is called *Weinstein* if there exists an exhausting Morse function $\varphi : V \to \mathbb{R}$ and a complete Liouville vector field X (i.e., one satisfying $L_X\omega = \omega$) that is gradient-like. We say that V is *sub-critical* if the Morse index of φ at any critical point is strictly less than $n = \frac{1}{2}\dim V$ and *critical* otherwise.

Exercise 17.4.9 Let (V,ω,φ) be as in Definition 17.4.8. Prove that any unstable manifold of $p \in \operatorname{Crit}\varphi$ is isotropic.

Corollary 17.4.10 *The Morse index of any critical point of a pluri-subharmonic function is less than or equal to $n = \frac{1}{2}\dim V$.*

We start with the following general property of sub-critical Weinstein manifolds.

Proposition 17.4.11 (Lemma 3.2 of (BCi02)) *Let (V,ω,X,φ) be a complete sub-critical Weinstein manifold. Let $A \subset V$ be any compact subset. Then there exists a compactly supported Hamiltonian H such that $\phi_H^1(A)\cap A = \emptyset$.*

Proof Consider the (positively) invariant set, denoted by Δ_X, of the flow of X. By definition, it is the union of unstable manifolds, which is compact and is a finite union of smooth manifolds of dimension less than or equal to $n-1$. Therefore we can find a Hamiltonian K that is supported in a small neighborhood of Δ_X such that $\phi_K^1(\Delta_X)\cap\Delta_X = \emptyset$. Since Δ_X is compact, we can find a small neighborhood $U \subset \Delta_X$ such that $\phi_K^1(U)\cap\overline{U} = \emptyset$.

Denote by ψ^t the flow of X. Since $L_X\omega = \omega$, ψ^t is conformally symplectic and hence the conjugation $\psi^t\circ\phi_K^1\circ(\psi^t)^{-1}$ is symplectic. Denote $h_t = \psi^t\circ\phi_K^1\circ(\psi^t)^{-1}$. We fix $T > 0$ so large that $\psi^{-T}(A) \subset U$. Then, since ϕ_K^1 moves U away from itself, we derive $h_T(A)\cap A = \emptyset$. Finally we check that h_T is generated by the Hamiltonian $H = H(t,x)$ defined by

$$H(t,x) = e^{tT}(\psi^{-tT})^*(K_t + G_t),$$

where G_t is the unique compactly supported Hamiltonian determined by the equation (see the proof of Proposition 3.4.8)

$$dG_t = \lambda_V - (\phi_K^t)_*\lambda_V,$$

which can be explicitly written as

$$G(t,x) = \int_0^t (K_s - dK_s(X))((\phi_K^s)^{-1}(x))ds.$$

From this, it follows that H has compact support, since K does. □

Lemma 17.4.12 *Let (V, ω_V) be a sub-critical Weinstein manifold with $\omega_V = d\lambda_V$ and (M, ω_M) a closed symplectic manifold. Then any compact Lagrangian submanifold $L \subset V \times M$ is displaceable by a compactly supported Hamiltonian isotopy.*

Proof We will displace L by a Hamiltonian vector field that is parallel to $TV \oplus \{0\}$. Since L is compact, its projection to V is also compact. Therefore the proof is an immediate consequence of Proposition 17.4.11. □

The following homological sphericality result is then proved by Biran and Cieliebak (BCi02).

Theorem 17.4.13 (Biran and Cieliebak) *Suppose that V is a connected sub-critical Weinstein manifold of $2 \leq \dim V \leq 2n$ with $c_1(V) = 0$. Consider any compact Lagrangian $L \subset V \times \mathbb{C}P^n$ with $H_1(L,\mathbb{Z}) = \{0\}$. Then L is cohomologically spherical in that*

$$H^\ell(L, \mathbb{Z}_2) = \{0\}$$

for all $1 \leq \ell \leq \dim L - 1$.

Proof First we note that $V \times \mathbb{C}P^n$ is monotone with the same monotonicity constant as $\mathbb{C}P^n$ by the hypothesis that $c_1(V) = 0$ and V is Weinstein, in particular, it is exact. We know that $\mathbb{C}P^n$ is a monotone symplectic manifold, i.e., $I_\omega = \lambda_0 I_{c_1}$ on $\pi_2(M) \cong H_2(M;\mathbb{Z})$ with $\lambda_0 = \omega(L)/c_1(L) > 0$.

Next we prove that any compact Lagrangian embedding $L \subset V \times \mathbb{C}P^n =: M$ with $H_1(L,\mathbb{Z}) = \{0\}$ is monotone with $\Sigma_L \geq \dim V = 2n+1 > 2$. Since $H_1(L,\mathbb{Z}) = \{0\}$, the natural map

$$H_2(V \times \mathbb{C}P^n, \mathbb{Z}) \to H_2(V \times \mathbb{C}P^n, L; \mathbb{Z})$$

is surjective and so any element in $H_2(V \times \mathbb{C}P^n, L; \mathbb{Z})$ comes from $H_2(V \times \mathbb{C}P^n, \mathbb{Z})$.

Now consider the homomorphisms $I_{(\omega;L)}$, $I_{(\mu;L)}$ on $\pi_2(M, L)$. Note that both homomorphisms are factored through $H_2(M, L; \mathbb{Z})$ and vanish on torsion elements thereof. On the other hand, we have

$$H_2(V \times \mathbb{C}P^n, \mathbb{Z}) \cong H_2(V, \mathbb{Z}) \oplus H_2(\mathbb{C}P^n, \mathbb{Z}) \quad \text{mod torsion.}$$

Therefore it suffices to show that

$$I_{(\omega,L)} = \frac{\lambda_0}{2} I_{(\mu;L)} \tag{17.4.24}$$

holds for the generators of $H_2(V, \mathbb{Z})$ and $H_2(\mathbb{C}P^n, \mathbb{Z})$. For $H_2(V, \mathbb{Z})$, this trivially holds since ω is exact and $c_1(V) = 0$ by assumption.

Let $[C_0] \subset \mathbb{C}P^n$ be the homology class of the complex line $[C_0] \in H_2(\mathbb{C}P^n, \mathbb{Z})$, regarded as an element of $H_2(M, \mathbb{Z})$, and denote by $i_*[C_0]$ its push-forward to $H_2(M, L; \mathbb{Z})$. Obviously we have

$$I_{(\omega,L)}(i_*[C_0]) = \omega([C_0]) = I_\omega([C_0]),$$
$$I_{(\mu,L)}(i_*[C_0]) = 2c_1([C_0]) = 2I_{c_1}([C_0]).$$

Recall that $2c_1([C_0]) = 2(n+1) > 2$. Furthermore, we have $\Sigma_L = 2(n+1) = \dim L + 1$, and so L satisfies the hypothesis of Theorem 17.3.9 (2). This implies that

$$H^\ell(L; \Lambda_{(\omega,L)}) \cong HF^\ell(L, L; \Lambda_{(\omega,L)})$$

for $1 \le \ell \le \dim L - 1$.

On the other hand, *since V is sub-critical*, there is a compactly supported Hamiltonian diffeomorphism ϕ on $V \times \mathbb{C}P^n$ such that $L \cap \phi(L) = \emptyset$, i.e., L is displaceable. On combining Theorem 17.3.9 (2) and Corollary 17.4.2, we have finished the proof of Theorem 17.4.13. □

In the next section, we will derive various consequences of Theorem 17.4.1 combined with the spectral sequence constructed in Section 17.3.

17.5 Audin's question for monotone Lagrangian submanifolds

One of the very first insightful questions on the symplectic topology of compact Lagrangian embedding in $\mathbb{C}^n$ next to Gromov's non-exactness theorem is the following question posed by Audin (Au88).

Question 17.5.1 Is the minimal Maslov number 2 for any compact Lagrangian torus?

The first positive answer was obtained by Viterbo (Vi90) in general and by Polterovich (Po91a) in $\mathbb{R}^4$. Viterbo used the classical critical-point theory of the action functional and Polterovich uses the method of pseudoholomorphic curves. For the case of monotone Lagrangian tori $L \subset \mathbb{C}^n$, the author exploited the well-definedness of Lagrangian Floer homology and the idea of thick–thin dichotomy and answered the question affirmatively when $n \leq 24$ (Oh96b). The same theorem holds for any displaceable monotone Lagrangian tori in general (M, ω).

This dimensional restriction $n \leq 24$ has been removed by Fukaya, Oh, Ohta and Ono (FOOO09) and by L. Buhovsky (Bu10) independently using the compatibility of the spectral sequence with the product induced by $\mathfrak{m}_2$. The idea of using the multiplicative property of the spectral sequence is due to P. Seidel (personal communication, 2002). The outcome is the following, whose proof here is borrowed from (FOOO09).

Theorem 17.5.2 *Suppose that $L \subset (M, \omega)$ is a displaceable monotone Lagrangian torus. Then its minimal Maslov number Σ_L satisfies*

$$\Sigma_L = 2.$$

Proof We first remark that T^n is orientable and spin, and the monotonicity implies $\Sigma_L > 0$ by definition. Since T^n is orientable, the non-zero integer Σ_L must be even and hence $\Sigma_L \geq 2$.

We will prove the theorem by contradiction. Suppose to the contrary that $\Sigma_L \geq 3$. (Since T^n is orientable, this in fact implies $\Sigma_L \geq 4$. This fact will not be used in the proof.) The monotonicity then implies that $HF(L, L; \Lambda_{(\omega,L)})$ is well defined and invariant under the Hamiltonian isotopy.

We have the decomposition of the boundary map into

$$\delta = \delta_{(0)} + \delta_{(1)} + \cdots + \delta_{(N)} =: \delta_{(0)} + \delta',$$

where $N \leq \left[(n+1)/\Sigma_L\right]$ and $\delta_{(k)}$ has the form

$$\delta_{(k)} = \overline{\delta}_{(k)} \otimes T^{k\lambda_0} e^{k\Sigma_L}. \tag{17.5.25}$$

Here $\overline{\delta}_{(k)} : E_k \to E_k$ has the degree $1 - k\Sigma_L$. Since $\Sigma_L \geq 3$, this degree of $\overline{\delta}_{(k)}$ is smaller than or equal to -2 for $k \geq 1$.

Now we consider the action of δ on the E_2 term

$$E_2 = H^*(T^n; \mathbb{Q}) \otimes \Lambda_{0,\mathrm{Nov}}.$$

We recall that the cohomology ring $H^*(T^n;\mathbb{Q})$ is generated by the one-dimensional cohomology classes $\alpha_1,\dots,\alpha_n \in H^1(L;\mathbb{Q})$. By simple degree counting, we derive

$$\delta_2(\alpha_k) = \overline{\delta}_{(k)}(\alpha_k)\cdot T^{k\lambda_0}e^{k\Sigma_L} = 0 \quad \text{for } k = 1,\dots,n.$$

Therefore we have

$$\delta_2(\mathfrak{m}_2(\alpha_i,\alpha_j)) = -\mathfrak{m}_2(\delta_2(\alpha_i),\alpha_j) - \mathfrak{m}_2(\alpha_i,\delta_2(\alpha_j)) = 0.$$

On the other hand, from the energy consideration, we have

$$\mathfrak{m}_2(\alpha_i,\alpha_j) \equiv \alpha_i \cup \alpha_j \mod T^{\lambda_0}.$$

Therefore $\delta_2(\alpha_i \cup \alpha_j) = 0$ for all $i,\ j = 1,\dots,n$. By inductively applying the above arguments to all possible products of the α_i, we derive that the spectral sequence degenerates at the E^2 term and hence we conclude that

$$HF^*(L;\Lambda_{0,\mathrm{Nov}}) \cong H^*(T^n;\mathbb{Q}) \otimes \Lambda_{0,\mathrm{Nov}},$$

which is in particular a free module over $\Lambda_{0,\mathrm{Nov}}$. Therefore

$$HF^*(L;\Lambda_{\mathrm{Nov}}) \cong H^*(T^n;\mathbb{Q}) \otimes \Lambda_{\mathrm{Nov}} \neq 0. \tag{17.5.26}$$

(So far we have not used the assumption that $L \subset M$ is displaceable, which enters in the following last stage of the proof.) On the other hand, it follows from the invariance of $HF^*(L;\Lambda_{\mathrm{Nov}})$ and the existence of a Hamiltonian diffeomorphism ϕ with $\phi(L) \cap L = \emptyset$ that $HF^*(L;\Lambda_{\mathrm{Nov}}) = \{0\}$. This contradicts (17.5.26) and hence we have obtained the proof of $\Sigma_L \leq 2$. □

Remark 17.5.3 In the modern perspective of Floer theory, Viterbo's proof in (Vi90) can be recast as an application of Hamiltonian Floer homology on the loop space of symplectic manifolds (Vi99), while Polterovich's can be regarded as one of Lagrangian Floer homology. It had been quite a mystery how these two versions of Floer homology give rise to the same kind of results, until Fukaya (Fu06) invoked Chas and Sullivan's bracket on the loop space of Lagrangian submanifolds and reconstructed the A_∞-algebra of $L \subset (M,\omega)$ as a deformation of the DGA of the de Rham complex, thereby answering Audin's question affirmatively in complete generality. There is also a symplectic-field-theory approach that was advocated by Y. Eliashberg (personal communication, 2001). As far as the author knows, the details of these proofs have not appeared yet.

17.6 Polterovich's theorem on Ham(S^2)

In relation to the Hofer distance, it is a natural question to ask whether the diameter of Ham(M, ω) for a given compact symplectic manifold is finite or infinite.

Definition 17.6.1 (Hofer's diameter) Let (M, ω) be a symplectic manifold and Ham(M, ω) be its Hamiltonian diffeomorphism group. (If (M, ω) is open, we take $\mathrm{Ham}_c(M, \omega)$ instead.) Define

$$\mathrm{diam}(\mathrm{Ham}(M, \omega)) = \sup_{\phi \in \mathrm{Ham}(M,\omega)} \|\phi\|.$$

It was proved by Ostrover (Os03) and Polterovich (Po98b), respectively, that the Hofer diameter is infinite for the case with $\pi_2(M) = \{0\}$, i.e., the aspherical case, and for S^2.

Theorem 17.6.2 (Ostrover) *Let (M, ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Then the Hofer diameter of* Ham(M, ω) *is infinite.*

Usher has obtained various generalizations of this result in (Ush10a). On the other hand, Polterovich proved the following.

Theorem 17.6.3 (Polterovich) *The Hofer diameter of* Ham(S^2) *is infinite.*

Ostrover's theorem is proved using the Hamiltonian Floer homology, while Polterovich uses the Lagrangian intersection Floer homology. We will come back to Ostrover's proof later, in Part 4. In this section, we will present Polterovich's proof, closely following his original argument.

We start with a Lagrangian suspension construction applied to Hamiltonian loops. Consider a Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$ satisfying

$$\phi_H^1 = id.$$

Consider an arbitrary closed symplectic manifold (M, ω). Let $\phi \in$ Ham(M, ω) and let F be a normalized Hamiltonian with $F \mapsto \phi$.

Definition 17.6.4 Denote by $\mathcal{H}_{\mathrm{loop}}$ the set of one-periodic Hamiltonians H (i.e., $H(t, x) = H(t + 1, x)$) generating a Hamiltonian loop, i.e., ϕ_H satisfying $\phi_H^1 = id$.

Recall that, for any given Hamiltonian loop, we can always make the associated Hamiltonian H one-periodic, by reparameterizing the Hamiltonian loop so that $\phi_H^t \equiv id$ near $t = 0, 1$. In particular, $H \equiv c$ on $[0,\delta] \cup [1-\delta, 1] \times M$ for some constant c.

We associate the suspension of the equator $L \subset (M,\omega)$ by the Hamiltonian isotopy ϕ_H defined in Definition 3.7.6. We denote the suspension by

$$N(L,\phi_H) \subset T^*L \times M$$

and $N_0 = N(L, id)$. Note that N_0 is nothing but the product $o_L \times L$, and that Proposition 3.7.7 implies that $L = N(L,\phi_H)$ is Hamiltonian isotopic to N_0. Furthermore, since $L \subset (M,\omega)$ is monotone with its minimal Maslov number being 2, so is N_0. Therefore its Floer homology $HF(N_0, N(L,\phi_H))$ is well defined over the Novikov ring $\Lambda^{\mathbb{Z}_2}$ and isomorphic to $HF(N_0,N_0)$ by Theorem 16.4.10. On the other hand, we compute

Lemma 17.6.5 *Suppose $L \subset (M,\omega)$ is monotone with $\Sigma_L \geq 2$. Then we have*

$$HF(N_0,N_0) \cong HF(o_L,o_L) \otimes HF(L,L;M).$$

Proof Since N_0 is monotone and has minimal Maslov number 2, the associated Floer homology of N_0 and its Hamiltonian deformation does not depend on the choice of isotopy and of almost-complex structures (Theorem 16.4.10).

For the proof we observe that the Lagrangian submanifold $N_0 = o_L \times L$ is a direct product in the product symplectic manifold $T^*L \times M$. Using the invariance property of HF under the Hamiltonian isotopy (Theorem 16.4.10 again), we can consider the product almost-complex structure J_0 on $T^*L \times M$ and a generic product Hamiltonian isotopy $\phi_{H'} = \phi_{H_1'} \times \phi_{H_2'}$ of N_0 therein. Then the associated Floer trajectory equation splits and hence the corresponding Floer complex $(CF(N_0, \phi_{H'}^1(N_0)), \partial)$ becomes

$$CF(N_0, L') = CF(o_L, \phi_{H_1'}^1(o_L)) \otimes CF(L, \phi_{H_2'}^1(L))$$

with its boundary map

$$\partial(x \otimes y) = \partial(x) \otimes y + x \otimes \partial(y).$$

The isomorphism $HF(N_0,N_0) \cong HF(o_L,o_L) \oplus HF(L,L;M)$ immediately follows. □

An immediate corollary is the following *stable intersection property* of $L \subset M$ with $HF(L,L;M) \neq 0$.

Corollary 17.6.6 *Suppose $L \subset (M,\omega)$ is monotone with $\Sigma_L \geq 2$ so that the Floer homology $HF(L,L;M)$ is defined in particular. Assume $HF(L,L;M) \neq 0$. Then for any Hamiltonian loop ϕ_H of (M,ω), we have*

$$(o_L \times L) \cap N(L,\phi_H) \neq \emptyset.$$

From the point of view of the following definition, such a Lagrangian submanifold L has a stable Lagrangian intersection property.

Definition 17.6.7 We say that a Lagrangian submanifold $L \subset (M, \omega)$ has a *stable Lagrangian intersection property* if $(o_L \times L) \cap N(L, \phi_H) \neq \emptyset$ for any Hamiltonian loop ϕ_H, i.e., ϕ_H with $\phi_H^1 = id$.

The following non-vanishing result of Floer homology is essentially proved in (Oh93b).

Theorem 17.6.8 *Consider* $S^1_{\mathrm{eq}} \subset S^2$. *Then we have*

$$HF(S^1_{\mathrm{eq}}, S^1_{\mathrm{eq}}; S^2) \cong H(S^1, \Lambda^{\mathbb{Z}_2}).$$

In particular, S^1_{eq} *has a stable Lagrangian intersection property.*

Proof Take a C^2-small Morse function f on S^1_{eq} with exactly two critical points at the south and the north poles that generates a small rotation of the given equator. Applying Riemann mapping theorem, one checks that the thick–thin decomposition explicitly provides

$$\partial = \partial_f^{\mathrm{Morse}} + \partial'$$

in this case, where ∂' is contributed by two Floer trajectories that are obtained by slicing away the 'thin' trajectories between S^1_{eq} and $\phi_f^1(S^1_{\mathrm{eq}})$ from the upper and lower hemispheres. There are exactly two such 'thick' trajectories. Therefore the matrix coefficient is given by

$$T^{\omega(u_1)} + T^{\omega(u_2)},$$

where $\omega(u_i)$ is the area of the corresponding Floer trajectory u. On the other hand, it follows from the monotonicity of S^1_{eq} that the areas are the same and hence the matrix coefficient becomes $2T^\lambda$, where λ is the common area. Therefore the matrix coefficient becomes zero in $\Lambda^{\mathbb{Z}_2}$. This shows that $\partial = \partial_f^{\mathrm{Morse}}$, which finishes the proof. □

Next we recall the explicit definition of a Lagrangian suspension,

$$\iota_{\phi_H, L} : (t, x) \mapsto (t, -H(t, \phi_H^t(x)), \phi_H^t(x)),$$

whose image is nothing but $N(L, \phi_H)$. An intersection point

$$(t, a, y) \in N_0 \cap N(L, \phi_H)$$

is characterized by the equation

$$0 = -H(t, y) = a, \quad y \in \mathrm{Fix}\, \phi_H^t \cap L. \tag{17.6.27}$$

We also recall that $H \equiv c$ on $[-\delta, \delta] \times M$, where we regard $[-\delta, \delta]$ as a subset of $S^1 = \mathbb{R}/\mathbb{Z}$. To remove the trivial intersections, we choose $c \neq 0$, which can be made arbitrarily small, and renormalize H_t so that it satisfies the normalization condition. This can be done with the norm $\|H\|$ as small as we want.

With these preparations, we are ready to complete the proof of Theorem 17.6.3.

Proof of Theorem 17.6.3 We start with the following description of the Hofer norm $\|\phi\|$ for $\phi \in \mathrm{Ham}(M, \omega)$ in terms of the set $\mathcal{H}_\phi$ of periodic Hamiltonians (see Definition 17.6.4).

Lemma 17.6.9 *Let* $\phi \in \mathrm{Ham}(M, \omega)$ *and* $\phi = \phi_F^1$. *Then we have*

$$\|\phi\| = \inf_{H \in \mathcal{H}_{\mathrm{loop}}} \|F - H\|.$$

Proof The proof is left as an exercise. □

We associate the suspension of the equator $L = S^1_{\mathrm{eq}} \subset S^2$ by the Hamiltonian isotopy ϕ_H. We denote the suspension

$$N(S^1_{\mathrm{eq}}, \phi_H) \subset T^*S^1 \times S^2,$$

where $S^1 = S^1(2) = \mathbb{R}/\subset 2\mathbb{Z}$ and $N_0 = N(S^1_{\mathrm{eq}}, id)$. Note that N_0 is nothing but the product $o_{T^*S^1_{\mathrm{eq}}} \times S^1_{\mathrm{eq}}$ and Proposition 3.7.7 implies that $L = N(S^1_{\mathrm{eq}}, \phi_H)$ is Hamiltonian isotopic to N_0 as long as the Hamiltonian loop is contractible in $\mathrm{Ham}(S^2, \omega)$.

On the other hand, we know that $\pi_1(\mathrm{Ham}(S^2)) \cong \mathbb{Z}_2$ and that full rotation along the axis orthogonal to the given equator S^1_{eq} generates a non-trivial loop. Note that this loop maps S^1_{eq} to itself and the associated Hamiltonian vanishes on S^1_{eq}. Therefore the associated suspension is again given by N_0. Therefore the suspension $N(S^1_{\mathrm{eq}}, \phi_H)$ is Hamiltonian isotopic to N_0 for all Hamiltonian loops ϕ_H.

Furthermore, N_0 is monotone, with its minimal Maslov number being 2, since so is $S^1_{\mathrm{eq}} \subset S^2$. Therefore the Floer homology $HF(N_0, N(S^1, \phi_H))$ over the Novikov ring $\Lambda^{\mathbb{Z}_2}$ is well defined and isomorphic to $HF(N_0, N_0)$ by Theorem 16.4.11. Then we apply the cotangent bundle isomorphism for $HF(o_{T^*S^1}, o_{T^*S^1}) \cong H(S^1) \cong \Lambda^{\mathbb{Z}_2} \oplus \Lambda^{\mathbb{Z}_2}$ and Theorem 17.6.8 for $HF(S^1_{\mathrm{eq}}, S^1_{\mathrm{eq}}; S^2) \cong H(S^1_{\mathrm{eq}}) \cong \Lambda^{\mathbb{Z}_2} \oplus \Lambda^{\mathbb{Z}_2} \neq 0$. By Corollary 17.6.6, we conclude that S^1_{eq} has a stable Lagrangian intersection property.

Then, by the characterization (17.6.27), we derive that there exists a point $(t, y) \in S^1 \times L$ such that $H(t, y) = 0$. We note that the above proof is based on the fact that $\mathrm{Ham}(S^2)$ has a finite fundamental group.

An immediate corollary of this and Lemma 17.6.9 is the following.

Corollary 17.6.10 *Consider any normalized Hamiltonian F such that $F|_{S^1_{\mathrm{eq}}} \equiv c > 0$, and its time-one map $\phi = \phi^1_F$. Then we have $\|\phi\| \geq c$.*

Finally, we note that we can choose a normalized Hamiltonian F in this corollary so that c can be arbitrarily large, which finishes the proof. □

Indeed, the above proof applies to any closed (M, ω) as long as it contains a compact Lagrangian submanifold that has a stable intersection property.

Theorem 17.6.11 *Suppose that (M, ω) contains a compact Lagrangian submanifold that has a stable Lagrangian intersection property. Then* $\operatorname{diam} \mathrm{Ham}(M, \omega)$ *is infinite.*

In this regard, we have shown in the above proof that $S^1_{\mathrm{eq}} \subset S^2$ has a stable Lagrangian intersection property. By the same argument, one can conclude that also $\mathbb{R}P^n$ and the Clifford torus T^n in $\mathbb{C}P^n$ have a stable Lagrangian intersection property.

Remark 17.6.12 Theorem 17.6.3 is quite a contrast to the finiteness theorem of the *spectral diameter*, which we shall introduce in Section 22.3. We will show that

$$\operatorname{diam}_\rho(\mathrm{Ham}(S^2)) \leq 4\pi \ (= \text{area of } S^2)$$

in Section 22.5.

PART 4

Hamiltonian fixed-point Floer homology

18

The action functional and the Conley–Zehnder index

A popular item of folklore on the Hamiltonian Floer homology is that it is an $(\infty/2)$-dimensional homology theory on the infinite-dimensional space $\mathcal{L}(M)$, the free loop space of a symplectic manifold (M, ω): the symplectic form ω naturally induces a differential form Ω on $\mathcal{L}(M)$ by taking the average over the loop. The natural S^1-action on $\mathcal{L}(M)$ induced by the domain rotation defines an iterated integral $\mathbb{X} \rfloor \Omega$, where $\mathbb{X}(\gamma) = \dot{\gamma}$ is the vector field which generates this action. By virtue of the S^1-invariance of Ω, this form $\mathbb{X} \rfloor \Omega$ is closed in the sense of iterated integrals of Chen (Chen73), (GJP91). This closed one-form is called the *action one-form* in symplectic geometry, which we denote by α.

When a one-periodic Hamiltonian $H : S^1 \times M \to \mathbb{R}$ is turned on, its differential $\{dH_t\}_{t \in S^1}$ induces an exact one-form on $\mathcal{L}(M)$, whose anti-derivative is the averaging function $\mathbb{H} : \mathcal{L}(M) \to \mathbb{R}$ defined by

$$\mathbb{H}(\gamma) := \int_0^1 H(t, \gamma(t))dt$$

on $\mathcal{L}(M)$. The sum $\alpha + d\mathbb{H}$ is again a closed one-form (in the sense of (Chen73)). While α is S^1-invariant $d\mathbb{H}$ is not, unless $\mathbb{H}$ is an autonomous Hamiltonian. Novikov (No81, No82) developed a Morse theory of such closed one-forms in the finite-dimensional context. In this sense, one may regard Floer homology as the Novikov homology of the one-form $\alpha + d\mathbb{H}$. Replacing the critical point theory of the classical action functional on $\mathbb{C}^n$ developed, e.g., in (Bn82, BnR79) by the elliptic approach of Novikov–Floer theory has been a powerful locomotive in the development of symplectic topology and Hamiltonian dynamics for the last three decades.

18.1 Free loop space and its S^1 action

Let M be a general smooth manifold, not necessarily symplectic. We denote by $\mathcal{L}(M) := \mathrm{Map}(\mathrm{S}^1, \mathrm{M})$ the free loop space, i.e., the set of smooth maps

$$\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to M.$$

We emphasize that the loops have a marked point $0 \in \mathbb{R}/\mathbb{Z}$ and often parameterize them by the unit interval $[0,1]$ with the periodic boundary condition $\gamma(0) = \gamma(1)$. $\mathcal{L}(M)$ has the distinguished connected component of contractible loops, which we denote by $\mathcal{L}_0(M)$. The universal covering space of $\mathcal{L}_0(M)$, denoted by $\widetilde{\mathcal{L}}_0(M)$, can be expressed as the bundle $\pi : \widetilde{\mathcal{L}}_0(M) \to \mathcal{L}_0(M)$ with its fiber at $\gamma \in \mathcal{L}_0(M)$ given by

$$\{[\gamma, w] \mid w : D^2 \to M \ \partial w =: w|_{\partial D^2} = \gamma\},$$

where $[\gamma, w]$ is the set of homotopy classes of the disc $w : D^2 \to M$ relative to $\partial w =: w|_{\partial D^2} = \gamma$. Here we identify ∂D^2 with S^1 under the multiplication by 2π. We call such a w a bounding disc of γ. We derive

$$\pi_1(\mathcal{L}(M)) \cong \pi_1(\mathcal{L}_0(M)) \times \pi_1(M)$$

from the fibration $\mathcal{L}_0(M) \to \mathcal{L}(M) \to M$, where $\mathcal{L}_0(M)$ is the based loop space, based say, at the constant loop $\widehat{y_0} \in \mathcal{L}_0(M)$ with $y_0 \in M$. Recalling that $\pi_1(\mathcal{L}_0(M)) = \pi_1(\mathcal{L}_0(M), \widehat{y_0}) \cong \pi_2(M)$, we obtain the description of $\pi_1(\mathcal{L}_0(M))$ as

$$\pi_1(\mathcal{L}_0(M), \widehat{y_0}) \cong \pi_2(M, y_0) \times \pi_1(M, y_0). \tag{18.1.1}$$

The deck transformation of the universal covering space $\widetilde{\mathcal{L}}_0(M)$ associated with an element $(A, a) \in \pi_2(M) \times \pi_1(M)$ can be realized as follows. Represent A by a sphere $v : S^2 \to M$ with $v(N) = y_0$. Let $[\gamma, w] \in \widetilde{\mathcal{L}}_0(M)$. Regard the interval $[0,1]$ as the subset $[0,1] \times i \cdot \{0\} \subset D^2$ and denote by 0 the origin of D^2. We connect $w(0)$ to y_0 by any path $\ell_1 : [0,1] \to M$ with $\ell_1(0) = y_0$, $\ell_1(1) = w(0)$. Then we choose another path $\ell_2 : [0,1] \to M$ with

$$\ell_2(0) = y_0, \qquad \ell_2(1) = \gamma(1)$$

and consider the concatenation

$$\ell_1 \# w|_{[0,1]} \# \widetilde{\ell_2},$$

which defines a loop based at y_0. We require the homotopy class of this loop to be the given $a \in \pi_1(M)$. In particular, the action of $A \in \pi_2(M)$ on $[\gamma, w]$ is realized by the operation of 'gluing a sphere',

$$(\gamma, w) \mapsto (\gamma, w \# v) \tag{18.1.2}$$

(modulo the action of $\pi_1(M)$ described as above), by a particular (and hence any) sphere $v : S^2 \to M$ representing $A \in \pi_2(M)$.

There is a natural circle action on $\mathcal{L}(M)$ induced by the domain translation

$$\gamma \mapsto \gamma \circ R_\varphi = \gamma(\cdot + \varphi), \tag{18.1.3}$$

where $R_\varphi : S^1 \to S^1$ is the map given by

$$R_\varphi(t) = t + \varphi, \quad \varphi \in S^1.$$

The infinitesimal generator of this action is the vector field $\mathbb{X}$ on $\mathcal{L}(M)$ provided by

$$\mathbb{X}(\gamma) = \dot{\gamma}. \tag{18.1.4}$$

The fixed-point set of this S^1 action is the set of constant loops

$$M \hookrightarrow \mathcal{L}_0(M).$$

This action lifts to an action on the set of pairs

$$(\gamma, w) \mapsto (\gamma \circ R_\varphi, w \circ R_\varphi) \tag{18.1.5}$$

induced by the complex multiplication, which we again denote by

$$R_\varphi : z \in D^2 \subset \mathbb{C} \mapsto e^{2\pi i \varphi} z.$$

Lemma 18.1.1 *The fixed-point set of the induced S^1 action on $\widetilde{\mathcal{L}}_0(M)$ consists of the pairs $[\gamma_x, w_x]$, where γ_x and w_x are the constant loop and the constant disc with image at x, respectively.*

Proof The domain rotation R_φ acts trivially on the constant loops $\gamma \equiv x \in M$. On the other hand, it acts trivially on the homotopy class of the pairs (γ_x, w_x) because the pair (γ_x, w_x) and $(\gamma_x \circ R_\varphi, w_x \circ R_\varphi)$ are homotopic to each other since R_φ is homotopic to the identity. □

18.2 The free loop space of a symplectic manifold

We specialize the discussion of the previous section to the case of symplectic manifolds (M, ω).

18.2.1 S^1 action and its moment map

In this case, $\mathcal{L}(M)$ carries a canonical (weak) symplectic form Ω defined by

$$\Omega(\xi_1, \xi_2) := \int_0^1 \omega(\xi_1(t), \xi_2(t))dt. \tag{18.2.6}$$

We summarize the basic folklore properties of this two-form without attempting to give a completely rigorous proof. We refer the reader to, e.g., (Chen73), (GJP91) for a precise exposition on the differential forms on the loop space in general.

Proposition 18.2.1
(1) *The form Ω is weakly nondegenerate. In other words, at any $\gamma \in \mathcal{L}(M)$, $\Omega(\xi, \eta) = 0$ for all $\eta \in T_\gamma \mathcal{L}(M)$ if and only if $\xi = 0$.*
(2) *Ω is closed.*
(3) *The S^1 action (18.1.5) is symplectic, or preserves Ω, i.e., $L_{\mathbb{X}}\Omega = 0$.*

The (weak) nondegeneracy follows from the nondegeneracy of ω and the closedness of Ω is a consequence of the closedness of ω *together with the fact that S^1 has no boundary*. We leave the relevant calculations to the reader, or one can refer to (Wn78) for the verification of these statements.

Then the form Ω induces a symplectic form on the covering space $\widetilde{\mathcal{L}}_0(M)$ by the pull-back under the projection $\widetilde{\mathcal{L}}_0(M) \to \mathcal{L}_0(M)$, which we denote by $\widetilde{\Omega}$.

To explain the meaning of statement (3) above, we first recall the useful notion of a *canonical thin cylinder* between two nearby loops. We denote by exp the exponential map of the metric

$$g := \omega(\cdot, J_{\mathrm{ref}}\, \cdot),$$

where J_{ref} is any given fixed reference compatible almost-complex structure. Let $\iota(g)$ be the injectivity radius of the metric g. As long as $d(x, y) < \iota(g)$ for the given two points of M, we can write

$$y = \exp_x(E(x, y))$$

for a unique vector

$$E(x, y) := (\exp_x)^{-1}(y).$$

Therefore, if the C^0 distance $d_{C^0}(\gamma, \gamma')$ between the two loops

$$\gamma,\ \gamma' : S^1 \to M$$

is smaller than $\iota(g)$, we can define the canonical map

$$u^{\mathrm{can}}_{\gamma\gamma'} : [0, 1] \times S^1 \to M$$

by

$$u^{\mathrm{can}}_{\gamma\gamma'}(s,t) = \exp_{\gamma(t)}(sE(\gamma,\gamma')(t)), \tag{18.2.7}$$

where $E(\gamma,\gamma')(t) := (\exp_{\gamma(t)})^{-1}(\gamma'(t))$.

Lemma 18.2.2 *Let $\mathbb{X}$ be the vector field given in* (18.1.4). *The form $\mathbb{X}\rfloor\Omega$ is a closed one-form on $\mathcal{L}(M)$ in that there is a function $\mathcal{A} = \mathcal{A}_0$ defined on a C^1-neighborhood of any given loop γ_0 that satisfies*

$$d\mathcal{A} = \mathbb{X}\rfloor\Omega. \tag{18.2.8}$$

Proof For any path γ that is sufficiently C^∞ close to a given γ_0, we consider

$$u^{\mathrm{can}}_{\gamma_0\gamma} : s \in [0,1] \mapsto \exp_{\gamma_0}(sE(\gamma_0,\gamma)),$$

which defines a distinguished homotopy class of paths $[u_{\gamma_0\gamma}]$ with fixed ends,

$$u(0) = \gamma_0,\ u(1) = \gamma.$$

We also denote the associated parameterized cylinder in M

$$[0,1]\times S^1 \to M; \quad (s,t) \mapsto \exp_{\gamma_0}(sE(\gamma_0(t),\gamma(t)))$$

also by $u^{\mathrm{can}}_{\gamma_0\gamma}$. It follows from the explicit expression $u^{\mathrm{can}}_{\gamma_0\gamma}(s,t) = \exp_{\gamma_0}(sE(\gamma_0(t),\gamma(t)))$ that we can define a function $\mathcal{A}$ by the formula

$$\mathcal{A}(\gamma;\gamma_0) = 0 - \int u^*_{\gamma_0\gamma}\omega \tag{18.2.9}$$

in a C^1-small neighborhood of $\gamma_0 \in \mathcal{L}(M)$. Here '0' should be regarded as the value $\mathcal{A}(\gamma_0;\gamma_0)$, which can be chosen arbitrarily.

Now we verify (18.2.8). Let γ be a loop in a given C^1-small neighborhood of γ_0 and and consider a tangent vector $\xi \in T_\gamma\mathcal{L}(M)$. Represent ξ by a germ of path $\{\gamma_s\}_{-\epsilon<s<\epsilon}$. Then we consider the family of pairs (γ_s, u_s) for

$$u_s := u^{\mathrm{can}}_{\gamma_0\gamma}\#\{\gamma_s\}_{[0,s]}$$

for $-\epsilon < s < \epsilon$. We compute

$$\begin{aligned}
d\mathcal{A}(\gamma)(\xi) &= \frac{d}{ds}\bigg|_{s=0}\left(-\int u_s^*\omega\right)\\
&= \frac{d}{ds}\bigg|_{s=0} - \left(\int (u^{\mathrm{can}}_{\gamma_0\gamma})^*\omega + \int_0^s\int_0^1 \omega\left(\frac{\partial\gamma_u(t)}{\partial u}, \frac{\partial\gamma_u(t)}{\partial t}\right)dt\,du\right)\\
&= \int_0^1 -\omega(\xi(t),\dot\gamma(t))dt = (\mathbb{X}(\gamma)\rfloor\Omega)(\xi).
\end{aligned} \tag{18.2.10}$$

This combined with (18.2.10) proves (18.2.8) and hence the lemma. $\square$

Remark 18.2.3 From the point of view of de Rham theory of the loop space (Chen73), (GJP91), a symplectic form ω on M induces a canonical cohomology class of degree one induced by the closed one-form $\mathbb{X}\rfloor\Omega$. This cohomology class is obtained by *the iterated integrals* $\widetilde{P}_1(\omega)$ in the notation on p. 344 of (GJP91). This one-form is *not exact* in general. The exactness of this one-form is precisely the so-called *weak exactness* of the symplectic form ω.

If we restrict this closed one-form to $\mathcal{L}_0(M)$ and consider its lifting to the universal covering space $\widetilde{\mathcal{L}}_0(M)$, the formula (18.2.9) can be extended to a *global* lifting induced by the function of the pairs (γ, w), again denoted by $\mathcal{A} = \mathcal{A}_0$

$$\mathcal{A}_0(\gamma, w) = -\int w^*\omega$$

if we regard w as a path from a constant path $w(0)$ to $\gamma = \partial w$. Here we put the negative sign in front of the integral to be consistent with the conventions employed in the author's series of papers on spectral invariants (e.g., in (Oh06a)). We call $\mathcal{A}_0$ the *unperturbed action functional*. It satisfies

$$d\mathcal{A}_0 = \mathbb{X}\rfloor\Omega \tag{18.2.11}$$

on $\widetilde{\mathcal{L}}_0(M)$. In other words, the S^1 action on $\widetilde{\mathcal{L}}_0(M)$ is Hamiltonian and its associated moment map is nothing but the function $\mathcal{A}_0 : \widetilde{\mathcal{L}}_0(M) \to \mathbb{R}$ (see (Wn78) for a more detailed discussion).

18.2.2 The Novikov covering

Although the universal covering space provides a model on which the lifting of the action one-form becomes exact, it is too big to carry out a meaningful critical-point theory in that the symmetry group $\pi_1(\mathcal{L}_0(M)) \cong \pi_2(M) \times \pi_1(M)$ of the corresponding function could be highly *noncommutative* unless M is simply connected. This leads one to find a smaller covering space for which the deck transformation group is small and abelian. Novikov (No81) was the first to develop a Morse theory of a closed one-form on a compact manifold by considering a *cyclic* covering space on which the form pulls back to an exact one-form, and estimated the number of zeros of a closed one-form in terms of the topological data of M.

Following (Fl89b) and (HS95), we now introduce the notion of the *Novikov covering space* of $\mathcal{L}_0(M)$. This is an abelian covering space of $\mathcal{L}_0(M)$ that is an analog to the cyclic covering in Novikov Morse theory.

Following the terminology used by Seidel (Se97), we introduce the following definition.

Definition 18.2.4 Let (γ, w) be a pair of $\gamma \in \mathcal{L}_0(M)$ and w be a disc bounding γ. We say that (γ, w) is Γ-*equivalent* to (γ, w') if and only if

$$\omega([w'\#\overline{w}]) = 0 \quad \text{and} \quad c_1([w'\#\overline{w}]) = 0,$$

where $\overline{w}$ is the map with opposite orientation on the domain and $w'\#\overline{w}$ is the obvious glued sphere. Here Γ stands for the group

$$\Gamma = \frac{\pi_2(M)}{\ker(\omega|_{\pi_2(M)}) \cap \ker(c_1|_{\pi_2(M)})}.$$

We denote

$$\Gamma_\omega := \omega(\Gamma) = \omega(\pi_2(M)) \subset \mathbb{R}$$

and call it the (spherical) *period group* of (M, ω).

Definition 18.2.5 We call (M, ω) *rational* if $\Gamma_\omega \subset \mathbb{R}$ is a discrete subgroup, and *irrational* otherwise.

Example 18.2.6 The product $S^2(r_1) \times S^2(r_2)$ with the product symplectic form $\omega_1 \oplus \omega_2$ is rational if and only if the ratio r_2^2/r_1^2 is rational.

Remark 18.2.7 Note that, for an irrational (M, ω), the period group is a countable dense subset of $\mathbb{R}$. In general, the dynamical behavior of the Hamiltonian flow on an irrational symplectic manifold is expected to become much more complicated than that on a rational one. The period group Γ_ω is the simplest indicator of this distinct dynamical behavior between them.

From now on, we will exclusively denote by $[\gamma, w]$ the Γ-equivalence class of (γ, w) by $\widetilde{\mathcal{L}}_0(M)$. We denote the canonical projection by $\pi : \widetilde{\mathcal{L}}_0(M) \to \mathcal{L}_0(M)$ and call $\widetilde{\mathcal{L}}_0(M)$ the Γ-*covering space* of $\mathcal{L}_0(M)$. We denote by A or q^A the image of $A \in \pi_2(M)$ under the projection $\pi_2(M) \to \Gamma$. There are two natural invariants associated with A: the *valuation* $\nu(A)$,

$$\nu : \Gamma \to \mathbb{R}; \quad \nu(A) = \omega(A), \tag{18.2.12}$$

and the *degree* $d(A)$,

$$d : \Gamma \to \mathbb{Z}; \quad d(A) = 2c_1(A). \tag{18.2.13}$$

In general these two invariants are independent (see (Gom98)) and hence q^A is a formal parameter depending on two variables. In that sense, we may also denote

$$q^A = T^{\omega(A)} e^{c_1(A)/2}$$

with two different formal parameters T and e. We put the degree of T to be 0 and the degree of e to be 2.

The (unperturbed) action functional $\mathcal{A}_0$ defined above obviously projects down to the Γ-covering space by the same formula

$$\mathcal{A}_0([\gamma, w]) = -\int w^*\omega$$

as in Section 18.2. This functional provides a natural increasing filtration on the space $\widetilde{\mathcal{L}}_0(M)$: for each $\lambda \in \mathbb{R}$, we define

$$\widetilde{\mathcal{L}}_0^\lambda(M) := \{[z, w] \in \widetilde{\mathcal{L}}_0(M) \mid \mathcal{A}_0([z, w]) \le \lambda\}.$$

We note that

$$\widetilde{\mathcal{L}}_0^\lambda(M) \subset \widetilde{\mathcal{L}}_0^{\lambda'}(M) \quad \text{if } \lambda \le \lambda'.$$

It follows from (18.2.11) that the critical set, denoted by $\operatorname{Crit}\mathcal{A}_0$, of $\mathcal{A}_0 : \widetilde{\mathcal{L}}_0(M) \to \mathbb{R}$ is the disjoint union of copies of M

$$\operatorname{Crit}\mathcal{A}_0 = \bigcup_{g\in\Gamma} g \cdot M,$$

where $M \hookrightarrow \mathcal{L}_0(M); x \mapsto [x, \widehat{x}]$ is the canonical inclusion, where $[x, \widehat{x}]$ is the pair of constant loop x and constant disc $\widehat{x}$. We have the natural commutative diagram

$$\begin{tikzcd}
\operatorname{Crit}\mathcal{A}_0 \arrow[r, hook] \arrow[d, "\pi"] & \widetilde{\mathcal{L}}_0(M) \arrow[d, "\pi"] \\
M \arrow[r, hook] & \mathcal{L}_0(M)
\end{tikzcd}$$

with the common fiber isomorphic to Γ.

18.2.3 Second variation

We now compute the second variation $d^2\mathcal{A}_0$ of $\mathcal{A}_0$ at each critical point $[x, \widehat{x}\#A]$. Here we use the canonical identification

$$T_{[x,\widehat{x}]}\widetilde{\mathcal{L}}(M) \cong T_x\mathcal{L}(M) \tag{18.2.14}$$

induced by the tangent map of the covering projection, and regard $d^2\mathcal{A}_0([x, \widehat{x}\#A])$ as a quadratic form defined on $T_x\mathcal{L}(M)$.

Proposition 18.2.8 *At each* $[x, \widehat{x}\#A] \in \operatorname{Crit}\mathcal{A}_0$*, the Hessian* $d^2\mathcal{A}_0$ *defines a bilinear form on*

$$T_{[x,\widehat{x}\#A]}\widetilde{\mathcal{L}}_0(M) \cong T_x\mathcal{L}_0(M),$$

which is (weakly) nondegenerate in the direction normal to $\operatorname{Crit}\mathcal{A}_0$ *on* $\mathcal{L}_0(M)$.

Proof For convenience of notation, we denote

$$[x, \widehat{x}] = \widehat{x}, \quad [x, \widehat{x}\#A] = \widehat{x} \otimes q^A.$$

Since the action $\#A$ is the deck transformation of the covering space $\widetilde{\mathcal{L}}_0(M) \to \mathcal{L}_0(M)$, we will just consider the point $[x, \widehat{x}]$. Let $V : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to \mathcal{L}(M)$ be a two-parameter family of loops such that

$$\begin{cases} V(0,0,t) \equiv x, \\ \partial V/\partial s\,|_{s=u=0}\,(t) = \xi_1(t), \quad \partial V/\partial u\,|_{s=u=0}\,(t) = \xi_2(t). \end{cases}$$

Denote by $\gamma_{u,s} : S^1 \to M$ the loop defined by

$$\gamma_{u,s}(t) := V(u, s, t)$$

and let $w_{u,s} : D^2 \to M$ be the 'small' disc with $w_{u,s}\big|_{\partial D^2} = \gamma_{u,s}$. Such a disc is homotopically unique.

Then, by virtue of the definition of the second variation, we have

$$d^2\mathcal{A}_0([x, \widehat{x}])(\xi_1, \xi_2) = \left.\frac{\partial^2}{\partial u\,\partial s}\right|_{s=u=0} \mathcal{A}_0([\gamma_{u,s}, w_{u,s}]).$$

Here we regard ξ_i as tangent vectors in $T_x\mathcal{L}(M)$ via the identification (18.2.14). From (18.2.11), we have

$$\frac{\partial}{\partial s}\mathcal{A}_0([\gamma_{u,s}, w_{u,s}]) = -\int \omega\left(\frac{\partial V}{\partial t}, \frac{\partial V}{\partial s}\right) dt.$$

Therefore using any torsion-free connection ∇, we obtain

$$\begin{aligned} &\left.\frac{\partial^2}{\partial u\,\partial s}\right|_{s=u=0} \mathcal{A}_0([\gamma_{u,s}, w_{u,s}]) \\ &\quad = -\int_0^1 \nabla\omega(x)\left(\frac{\partial V}{\partial t}, \frac{\partial V}{\partial s}\right)\bigg|_{u=s=0} dt - \int_0^1 \omega\left(\nabla_u \frac{\partial V}{\partial t}\bigg|_{u=s=0}, \frac{\partial V}{\partial s}\bigg|_{u=s=0}\right) dt \\ &\qquad - \int_0^1 \omega\left(\frac{\partial V}{\partial t}\bigg|_{u=s=0}, \nabla_u \frac{\partial V}{\partial s}\bigg|_{u=s=0}\right) dt. \end{aligned}$$

Since $V(0,0,t) \equiv$ x, $\partial V/\partial t|_{u=s=0} \equiv 0$, both the first term and the third term will vanish. For the second term, since ∇ is torsion-free, we have

$$\nabla_u \frac{\partial V}{\partial t} = \nabla_t \frac{\partial V}{\partial u}.$$

Therefore we obtain

$$\nabla_u \frac{\partial V}{\partial t}\bigg|_{u=s=0} = \nabla_t \frac{\partial V}{\partial u}\bigg|_{u=s=0} = \nabla_t \xi_2|_{u=s=0} = \frac{d\xi_2}{dt}(t)\bigg|_{u=s=0},$$

where the last identity follows since ξ_2 is a variation along the constant loop x. Altogether we derive the formula

$$d^2\mathcal{A}_0([x,\widehat{x}])(\xi_1,\xi_2) = -\int_0^1 \omega\left(\frac{d\xi_2}{dt}(t),\ \ \xi_1(t)\right)dt.$$

In particular, the kernel of $d^2\mathcal{A}_0([x,\widehat{x}])$ consists of vector fields along the constant path $\widehat{x}$,

$$\left\{\xi_2 \in C^\infty(S^1, T_xM) \,\middle|\, \frac{d\xi_2}{dt} = 0\right\}.$$

We note that, via the embedding $M \hookrightarrow C^\infty(S^1, M)$, we can identify T_xM with the image T_xM in $C^\infty(S^1, T_xM)$. Then this implies

$$\ker d^2\mathcal{A}_0([x,\widehat{x}]) \cong T_xM,$$

i.e., the kernel coincides with the tangent space of the critical manifold $\mathrm{Crit}\,\mathcal{A}_0$ of $\mathcal{A}_0$ at any point $[x,\widehat{x}] \in \mathrm{Crit}\,\mathcal{A}_0$. The same holds at other critical points $[x,\widehat{x}\#A]$ for any $A \in \Gamma$. This finishes the proof. □

According to this proposition, $\mathcal{A}_0$ can be regarded as a Bott–Morse function (Bo54). We will see that, by adding a generic time-dependent Hamiltonian function $H : [0,1] \times M \to \mathbb{R}$, one can kill most of these critical points except for a finite number of *nondegenerate* one-periodic Hamiltonian orbits of H. Arnol'd's celebrated conjecture (Ar65) states that there are at least $SB(M)$ (= the rank of $H^*(M)$) (contractible) one-periodic orbits of the associated Hamiltonian flow.

18.3 Perturbed action functionals and their action spectrum

When a one-periodic Hamiltonian $H : (\mathbb{R}/\mathbb{Z}) \times M \to \mathbb{R}$ is given, we consider the perturbed functional $\mathcal{A}_H : \widetilde{\mathcal{L}}(M) \to \mathbb{R}$ defined by

$$\mathcal{A}_H([\gamma,w]) = \mathcal{A}_0([\gamma,w]) - \int H(t,\gamma(t))dt = -\int w^*\omega - \int H(t,\gamma(t))dt. \tag{18.3.15}$$

Unless stated otherwise, we will always consider one-periodic mean-normalized Hamiltonian functions, i.e., those which satisfy

$$\int_M H_t\,\omega^n = 0 \quad \text{for all } t \in S^1. \tag{18.3.16}$$

We denote by $\mathcal{H}_0(M) = C_0^\infty(S^1 \times M)$ the set of such mean-normalized Hamiltonians.

Lemma 18.3.1 *The set of critical points of $\mathcal{A}_H$ is given by*

$$\mathrm{Crit}(\mathcal{A}_H) = \{[z,w] \mid z \in \mathrm{Per}(H),\ \partial w = z\}$$

to which the Γ *action on* $\widetilde{\mathcal{L}}_0(M)$ *canonically restricts.*

Definition 18.3.2 We define the *action spectrum* of H by

$$\mathrm{Spec}(H) := \{\mathcal{A}_H([z,w]) \in \mathbb{R} \mid [z,w] \in \widetilde{\mathcal{L}}_0(M),\ z \in \mathrm{Per}(H)\},$$

i.e., the set of critical values of $\mathcal{A}_H : \widetilde{\mathcal{L}}(M) \to \mathbb{R}$. For each given $z \in \mathrm{Per}(H)$, we denote

$$\mathrm{Spec}(H;z) = \{\mathcal{A}_H([z,w]) \in \mathbb{R} \mid [z,w] \in \pi^{-1}(z)\}.$$

Note that $\mathrm{Spec}(H;z)$ is a Γ_ω principal bundle over $\mathrm{Per}(H)$. It follows that

$$\mathrm{Spec}(H) = \bigcup_{z \in \mathrm{Per}(H)} \mathrm{Spec}(H;z).$$

Since $\Gamma_\omega = \omega(\pi_2(M))$ is a countable subgroup of $\mathbb{R}$, it is either a discrete or a dense subgroup of $\mathbb{R}$. The following is an important ingredient in the study of the behavior of critical values of $\mathcal{A}_H$ under a change of the Hamiltonian H. The proof here is taken from (Oh02).

Lemma 18.3.3 *Let H be any periodic Hamiltonian.* $\mathrm{Spec}(H)$ *is a measure-zero subset of* $\mathbb{R}$ *for any H.*

Proof First note that $\mathrm{Spec}(H;z) \subset \mathbb{R}$ is a countable subset of $\mathbb{R}$ for each given periodic orbit z since Γ_ω is so. We consider the Poincaré return map in a tubular neighborhood of each $z \in \mathrm{Per}(H)$. More precisely, we choose a small neighborhood $V \subset M$ of $z(0)$. We identify V with the $2n$-ball $B^{2n}(\delta)$ with the point $z(0)$ at the center of the ball. Choose another ball neighborhood $V' = B^{2n}(\delta')$ with $\overline{V} \subset V'$ such that the (first) Poincaré return map denoted by

$$R_z : V \to V';\ p \mapsto \phi_H^1(p)$$

is well defined. We now define a continuous map from V to the space of piecewise smooth maps from $S^1 \cong \mathbb{R}/\mathbb{Z}$ to M as follows: for each $p \in V$, we first follow the flow of X_H and then follow from $R_z(p)$ to p by the straight line under the identification of V' with $B^{2n}(\delta')$. We reparameterize the domain of the loop by rescaling it to be $[0,1]$.

Denote by z_p the loop corresponding to $p \in V$ constructed as above, and by $\mathcal{V}_z \subset \mathcal{L}_0(M)$ the image of the assignment $p \mapsto z_p$. Obviously z_p is homotopic to z and so any given disc w bounding z can be naturally continued to bound the loop z_p. We denote by w_p the disc continued from w and corresponding to $p \in V$. It can be easily checked that the function

$$h : \pi^{-1}(\mathcal{V}_z) \to \mathbb{R}; \quad h([z_p, w_p]) := \mathcal{A}_H([z_p, w_p])$$

defines a smooth function on $\pi^{-1}(\mathcal{V}_z)$ and its critical values comprise those of $\mathcal{A}_H$ near $\mathrm{Spec}(H; z)$. This can be proved by writing $\mathcal{A}_H([z_p, w_p])$ explicitly and by a simple local calculation. Noting that $\pi^{-1}(\mathcal{V}_z)$ is a finite-dimensional (in fact, $2n$-dimensional) manifold, Sard's theorem implies that the set of critical values is a measure-zero subset in $\mathbb{R}$. Note that the set of initial points $z(0)$ with $z \in \mathrm{Per}(H)$ is compact. Therefore a finite number of such tubular neighborhoods together with their complement covers M and hence $\mathrm{Spec}(H) \subset \mathbb{R}$ is contained in a finite union of measure-zero subsets of $\mathbb{R}$ and thus itself has measure zero. □

Exercise 18.3.4 Give the proof of the claim that the function h defined in the above proof is indeed smooth.

Note that, when $H = 0$, we have

$$\mathrm{Spec}(H) = \Gamma_\omega.$$

Definition 18.3.5 We say that two Hamiltonians H and F are *homotopic* if $\phi_H^1 = \phi_F^1$ and their associated Hamiltonian paths ϕ_H, $\phi_K \in \mathcal{P}(\mathrm{Ham(M}, \omega), \mathrm{id})$ are path-homotopic relative to the ends. In this case we denote $H \sim F$ and denote the set of equivalence classes by $\widetilde{\mathrm{Ham}}(M, \omega)$.

The following lemma is another important ingredient in the study of critical values of the action functional under the deformation of Hamiltonians. It was proven in the aspherical case in (Schw00), (Po01) and in the general case in (Oh05a).

Proposition 18.3.6 *Suppose that F, G are mean-normalized. If $F \sim G$, we have*

$$\mathrm{Spec}\, F = \mathrm{Spec}\, G$$

as a subset of $\mathbb{R}$.

Before giving its proof, we need some preparation. Let ϕ be a given Hamiltonian diffeomorphism. Let $\{F^s\}_{s\in[0,1]}$ be a path of Hamiltonians such that $F^0 = G$ and $F^1 = F$ and $\phi^1_{F^s} = \phi$ for all $s \in [0,1]$. We consider the two-parameter family of Hamiltonian diffeomorphisms $\phi(s,t) = \phi^t_{F^s} \circ (\phi^t_G)^{-1}$. We denote $F(s,t,x) = F^s(t,x)$ and let $K = K(s,t,x)$ be the Hamiltonian function in the s-direction, i.e., the Hamiltonian associated with the vector fields

$$\frac{\partial \phi^t_{F^s}}{\partial s} \circ (\phi^t_{F^s})^{-1}.$$

This K is unique if we require that $K(s,t,\cdot)$ is normalized, which can always be achieved. Furthermore, we have

$$\frac{\partial X_F}{\partial s} - \frac{\partial X_K}{\partial t} + [X_K, X_F] = 0$$

from (2.4.26), which in turn implies that (see (2.2.9) and Lemma 2.3.4)

$$\frac{\partial F}{\partial s} - \frac{\partial K}{\partial t} - \{K, F\} = c(s,t)$$

for some function c depending only on (s,t). Since both F and K are assumed to be normalized and the Poisson bracket $\{K, F\}$ is automatically normalized by Liouville's lemma, we conclude that $c \equiv 0$. In other words, F and K satisfy

$$\frac{\partial F}{\partial s} - \frac{\partial K}{\partial t} - \{K, F\} = 0. \tag{18.3.17}$$

It is worthwhile to state separately the following equivalent identity.

Lemma 18.3.7 *The equality* (18.3.17) *is equivalent to*

$$\frac{\partial F}{\partial s}(s,t,\phi^t_{F^s}(p)) = \frac{\partial}{\partial t}(K(s,t,\phi^t_{F^s}(p))). \tag{18.3.18}$$

Proof Assume (18.3.17). Then we just compute

$$\begin{aligned} \frac{\partial F}{\partial s}(s,t,\phi^t_{F^s}(p)) &= \frac{\partial K}{\partial t}(s,t,\phi^t_{F^s}(p)) + \{K,F\}(s,t,\phi^t_{F^s}(p)) \\ &= \frac{\partial K}{\partial t}(s,t,\phi^t_{F^s}(p)) + dK(X_F)(\phi^t_{F^s}(p)) \\ &= \frac{\partial}{\partial t}(K(s,t,\phi^t_{F^s}(p))), \end{aligned}$$

which proves (18.3.18). For the converse, we just read the above computations from the bottom to the first line of the right-hand side. This finishes the proof. □

Now we are ready to give the proof of Proposition 18.3.6.

Proof of Proposition 18.3.6 Let ϕ be a given Hamiltonian diffeomorphism. Let $\{F^s\}_{s\in[0,1]}$ be a path of Hamiltonians such that $F^0 = G$ and $F^1 = F$ and $\phi^1_{F^s} = \phi$ for all $s \in [0,1]$. Consider the two-parameter family of Hamiltonian diffeomorphisms $\phi(s,t) = \phi^t_{F^s} \circ (\phi^t_G)^{-1}$. Denote $F(s,t,x) = F^s(t,x)$, and denote by $K = K(s,t,x)$ the Hamiltonian function in the s-direction mentioned above. Note that for each $s \in [0,1]$ $t \mapsto \phi(s,t)$ defines a Hamiltonian loop that is contractible with $\phi(s,0) \equiv id$ by the hypothesis on $\{F^s\}$. We denote this loop by h^s and denote by $\widetilde{h}^s$ the lifted loop satisfying $\widetilde{h}^s(0) = id$ on $\widetilde{\mathrm{Ham}}(M,\omega)$. The loop $\widetilde{h}^s$ defines a one-to-one correspondence

$$(\widetilde{h}^s)_* : \mathrm{Crit}\,\mathcal{A}_G \to \mathrm{Crit}\,\mathcal{A}_F$$

by the formula

$$\widetilde{h}^s \cdot [z,w] = [h^s \cdot z, \widetilde{h}^s \cdot w]$$

for $[z,w] \in \mathrm{Crit}\,\mathcal{A}_G$. Therefore it will suffice to prove that the function $\chi : [0,1] \to \mathbb{R}$ defined by

$$\chi(s) = A_{F^s}(\widetilde{h}^s \cdot [z,w])$$

is constant.

Since $\phi^1_{F^s} \equiv \phi^1_G$ and $\phi^0_{F^s} \equiv id$ for all $s \in [0,1]$, $K(s,1,x)$ and $K(s,0,x)$ must be constant for each s and in turn

$$K(s,1,\cdot) = K(s,0,\cdot) \equiv 0 \tag{18.3.19}$$

by the normalization condition. We now differentiate $\chi(s)$:

$$\chi'(s) = d\mathcal{A}_{F^s}(\widetilde{h}^s \cdot [z,w])\left(\frac{\partial}{\partial s}(\widetilde{h}^s \cdot [z,w])\right) - \int_0^1 \frac{\partial F}{\partial s}(s,t,(h^s \cdot z)(t))dt.$$

The first term vanishes since $\widetilde{h}^s \cdot [z,w]$ is a critical point of $\mathcal{A}_{F^s}$. For the second term we note that $z(t) = \phi^t_G(p)$ for a fixed point $p \in M$ of ϕ^1_G since z is a periodic solution of $\dot{x} = X_G(t,x)$. Therefore

$$(h^s \cdot z)(t) = h^s_t(z(t)) = (\phi^t_{F^s} \circ (\phi^t_G)^{-1}) \circ \phi^t_G(p) = \phi^t_{F^s}(p).$$

Hence we obtain

$$\int_0^1 \frac{\partial F}{\partial s}(s,t,(h^s \cdot z)(t))dt = \int_0^1 \frac{\partial F}{\partial s}(s,t,\phi^t_{F^s}(p))dt.$$

By substituting (18.3.18) into this, integrating the total derivative over $[0,1]$ and using (18.3.19), we derive

$$\int_0^1 \frac{\partial F}{\partial s}(s,t,(h^s \cdot z)(t))dt = K(s,1,f^s_1(p)) - K(s,0,f^s_0(p)) = 0.$$

This proves that χ must be constant, which finishes the proof. $\square$

This lemma shows that the action spectrum $\mathrm{Spec}\, H$ of a normalized Hamiltonian H depends only on its homotopy class.

Definition 18.3.8 Let $h \in \widetilde{\mathrm{Ham}}(M, \omega)$. We define the spectrum of h by

$$\mathrm{Spec}(h) := \mathrm{Spec}\, F$$

for a particular (and hence any) normalized Hamiltonian F with $h = [\phi, F]$.

Obviously the action functional $\mathcal{A}_H$ can be defined irrespective of whether H is normalized or not and hence is its action spectrum. If we denote the corresponding normalized Hamiltonian by $\underline{H}$, i.e.,

$$\underline{H}(t, x) = H(t, x) - \frac{1}{\mathrm{vol}_\omega(M)} \int_M H_t(x) d\mu_\omega, \tag{18.3.20}$$

then

$$\mathrm{Spec}(\underline{H}) = \mathrm{Spec}(H) - \mathrm{Cal}(H), \tag{18.3.21}$$

where

$$\mathrm{Cal}(H) := \frac{1}{\mathrm{vol}_\omega(M)} \int_0^1 \int_M H_t(x) d\mu_\omega dt.$$

18.4 The Conley–Zehnder index of $[z, w]$

First, we note that the symplectic group $Sp(S, \Omega) \cong Sp(\mathbb{R}^{2n})$ has a maximal compact subgroup $U(n)$ to which $Sp(S, \omega)$ deformation retracts, and hence the determinant map $\det : U(n) \to S^1$ induces a natural homomorphism

$$\mu : \pi_1(Sp(S, \Omega), id) \to \mathbb{Z}.$$

Next, we further amplify the discussion on the structure of $Sp(S, \Omega)$ in Section 2.2.2. Using the notation from (SZ92), we denote

$$Sp(S, \Omega)^* = \{A \in Sp(S, \Omega) \mid \det(I - A) \neq 0\}.$$

Note that, when $p \in \mathrm{Fix}\, \phi_H^1$ is nondegenerate, we have $d\phi_H^1(p) \in Sp(T_pM, \omega_p)$.

The following lemma proved in (CZ84) is an important one in our further discussion. We leave its proof to the original article or to (SZ92).

Lemma 18.4.1 *$Sp(S, \Omega)^*$ has two connected components*

$$Sp^\pm(S, \Omega) = \{A \in Sp(S, \Omega)^* \mid \pm \det(I - A) > 0\}.$$

Moreover, every loop in $Sp(S, \Omega)^*$ is contractible in $Sp(S, \Omega)$.

It follows that $Sp(S,\Omega)^*$ and $Sp(S,\Omega)^\pm$ are preserved under the conjugate action of $Sp(S,\omega)$. We consider the set of paths

$$\mathcal{SP}^*(1) = \{\alpha : [0,1] \to Sp(S,\Omega) \mid \alpha(0) = id, \alpha(1) \in Sp^*(S,\Omega)\}. \quad (18.4.22)$$

Without loss of generality, we assume that $\alpha(1) \in Sp^+(2n;\mathbb{R})$. The case with $Sp^-(2n;\mathbb{R})$ is the same. It follows from Lemma 18.4.1 that we can choose a (homotopically) unique path $\alpha_+^{\mathrm{can}} : [0,1] \to Sp(2n;\mathbb{R})$ such that

$$\begin{aligned} &\alpha_+^{\mathrm{can}}(1) = \alpha(1),\ \alpha_+^{\mathrm{can}}(2) = W_+, \\ &\alpha_+^{\mathrm{can}}(t) \in Sp^+(2n;\mathbb{R})\ \forall t \in [1,2], \end{aligned} \quad (18.4.23)$$

where $W^+ = -id$. For the path with $\alpha(1) \in Sp^-(2n)$, we consider the matrix

$$W^- = \mathrm{diag}(2,-1,\ldots,-1,1/2,\ldots,-1) \in Sp^-(2n)$$

and define α_-^{can} similarly. Then the following continuous map $\rho : Sp(V,\omega) \to S^1$ was constructed in (CZ84). (See also Theorem 3.1 of (SZ92) for its proof.)

Theorem 18.4.2 *Let (V,ω) be a symplectic vector space. Then there is a unique collection of continuous mappings*

$$\rho : Sp(V,\omega) \to S^1$$

(one for every (V,ω)) satisfying the following conditions.

(1) **(Naturality)** *If $T : (V_1,\omega_1) \to (V_2,\omega_2)$ is a symplectic isomorphism, then $\rho(TAT^{-1}) = \rho(A)$ for all $A \in Sp(V_1,\omega_1)$.*
(2) **(Product)** *If $(V,\omega) = (V_1 \times V_2, \omega_1 \times \omega_2)$, then $\rho(A) = \rho(A_1)\rho(A_2)$ for $A \in Sp(V,\omega)$ with $A = A_1 \oplus A_2$.*
(3) **(Determinant)** *If $A \in Sp(2n,\mathbb{R}) \cap O(2n) = U_{\mathbb{R}}(n)$, then $\rho(A) = \det(X+iY)$, where*

$$A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

with $X + iY \in U(n)$.
(4) **(Normalization)** *If A has no eigenvalue on the unit circle, then $\rho(A) = \pm 1$.*

By concatenating α and α_+^{can} and considering the composition

$$\rho \circ (\alpha \# \alpha_+^{\mathrm{can}}) : [0,2] \to S^1,$$

we get a map such that $\rho \circ (\alpha \# \alpha_+^{\mathrm{can}})(2) = \pm 1$ and hence carries the winding number induced from the map $\det^2 : U(n) \to S^1$.

Definition 18.4.3 Let $\alpha \in \mathcal{SP}^*(1)$. We define the Conley–Zehnder index of α, denoted by $\mu_{CZ}(\alpha)$, by

$$\mu_{CZ}(\alpha) = \deg\left(\det{}^2(\rho \circ (\alpha \# \alpha_+^{\text{can}}))\right).$$

Our definition of μ_{CZ} in Definition 18.4.3 coincides with that of (SZ92).

On the other hand, for a given Lie group G, we denote by

$$\Omega(G) = \{g : S^1 \to G \mid g(0) = id\}$$

the group of based loops in G. Then we have the obvious index map $\mu : \Omega(Sp(2n, \mathbb{R})) \to \mathbb{Z}$ defined by the degree of the composition map

$$\pi_1(\Omega(Sp(2n, \mathbb{R})), id) \cong \pi_1(\Omega(U(n)), id) \cong \mathbb{Z},$$

where the first map is the isomorphism arising from the deformation retraction of $Sp(2n, \mathbb{R})$ to $U(n)$ and the second is induced by the map

$$g \mapsto \frac{1}{2\pi} \int_{S^1} (\det \circ g)^* \, d\theta$$

for $g \in \Omega(U(n))$. The Conley–Zehnder index μ_{CZ} is then characterized by the following properties.

Proposition 18.4.4 (Proposition 5 in (FH93)) *There exists a unique map*

$$\mu_{CZ} : \mathcal{SP}^*(1) \to \mathbb{Z}$$

satisfying the following criteria.

(1) **(Normalization)**

$$\mu_{CZ}(\tau \mapsto \{e^{-\tau} + ie^{\tau}\} \oplus \{e^{\pi i \tau}\} \oplus \cdots \oplus \{e^{\pi i \tau}\}) = n - 1,$$
$$\mu_{CZ}(\tau \mapsto e^{\pi i \tau} Id) = n.$$

(2) **(Action by based loops)** *Under the action of the group $\Omega(Sp(2n, \mathbb{R}), id)$ of based loops on $\mathcal{SP}^*(1)$,*

$$\mu_{CZ}(g\alpha) = 2\mu(g) + \mu_{CZ}(\alpha)$$

for every loop $g : S^1 \to Sp(2n, \mathbb{R})$.

Let $[z, w] \in \operatorname{Crit} \mathcal{A}_H$. We choose a trivialization

$$\Psi : w^*TM \to D^2 \times \mathbb{R}^{2n}.$$

Recalling that $z(t) = \phi_H^t(p)$ for some fixed point $p \in \operatorname{Fix} \phi_H^1$, we can write

$$\Psi \circ d\phi_H^t \circ \Psi^{-1}(t, v) = (t, \alpha_\Psi(t)v).$$

By definition, we have

$$\alpha_\Psi(0) = id,\ \alpha_\Psi(1) \in Sp^*(2n;\mathbb{R}).$$

Definition 18.4.5 We define the Conley–Zehnder index of $[z, w]$, denoted by $\mu_H([z, w])$, to be

$$\mu_H([z, w]) = \mu_{CZ}(\alpha_\Psi).$$

We state an important identity relating the Conley–Zehnder index and the first Chern number $c_1(A)$ under the action 'gluing a sphere' $[z, w] \mapsto [z, w\#A]$.

We follow the exposition of (Oh06a) in the proof of the following index formula.

Theorem 18.4.6 *Let $z : S^1 = \mathbb{R}/\mathbb{Z} \to M$ be a given one-periodic solution of $\dot{x} = X_H(x)$ and w, w' two given bounding discs. Then we have the identity*

$$\mu_H([z, w']) = \mu_H([z, w]) - 2c_1([w'\#\overline{w}]). \tag{18.4.24}$$

In particular, we have

$$\mu_H([z, w\#A]) = \mu_H([z, w]) - 2c_1(A). \tag{18.4.25}$$

Remark 18.4.7 We would like to emphasize that, in our convention, the sign in front of the first Chern number term in the formula is '−'. The difference of the sign from the formula in (HS95) is due to the different convention of the canonical symplectic form on $\mathbb{C}^n$: when we identify $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ and denote the corresponding canonical coordinates by $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, then the canonical symplectic form is given by

$$\omega_0 = \sum dq_i \wedge dp_i \tag{18.4.26}$$

in our convention, while it is given by

$$\omega_0' = -\omega_0 = \sum dp_i \wedge dq_i.$$

according to the convention of (HS95), (SZ92), or (Po01).

Proof We give the proof of the index formula in several steps.

1. **(Canonical symplectic form)** Our convention for the canonical symplectic form on $T^*\mathbb{R}^n = \mathbb{R}^{2n} \cong \mathbb{C}^n$ in the coordinates $z_j = q_j + ip_j$ is given by (18.4.26).
2. **(Canonical complex structure)** Let J_0 be the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with $z_j = q_j + ip_j$ obtained by multiplication by the complex

number i. In our convention of the canonical symplectic form ω_0 on $\mathbb{C}^n$, the associated Hermitian structure

$$\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$$

becomes *complex linear in the first argument, but anti-linear in the second argument*. In other words, the Hermitian inner product is given by

$$\langle u, v \rangle = g(u, v) - i\omega_0(u, v), \tag{18.4.27}$$

where g is the standard Euclidean inner product on $\mathbb{R}^{2n}$. We would like to note that this Hermitian structure on $\mathbb{C}^n$ is the conjugate to that of (HS95), (SZ92), (Po01), which corresponds to

$$\langle u, v \rangle = g(u, v) + i\omega_0(u, v). \tag{18.4.28}$$

(See the remark right before Lemma 5.1 in (SZ92).) Equivalently, the latter Hermitian structure is associated with the almost-Kähler structure

$$(g, \omega_0', J_0'),$$

where J_0' is the almost-complex structure conjugate to J_0. *This change of complex structure on $\mathbb{C}^n$ affects the sign of the first Chern number of general complex vector bundles E*: we recall the following general formula for the Chern classes of the complex vector bundle E (see Lemma 14.9 (MSt74)):

$$c_k(\overline{E}) = (-1)^k c_k(E).$$

3. **(The Conley–Zehnder index on $\mathcal{SP}^*(1)$)** Note that the definitions of $Sp(2n, \mathbb{R})$ in the two conventions are the same.
4. When we are given two maps

$$w,\ w' : D^2 \to M$$

with $w|_{\partial D^2} = w'|_{\partial D^2}$, we define the glued map $u = w \# \overline{w}' : S^2 \to M$ in the following way:

$$u(z) = \begin{cases} w(z), & z \in D^+, \\ w'(1/\overline{z}), & z \in D^-. \end{cases}$$

Here D^+ is D^2 with the same orientation, and D^- with the opposite orientation. This is a priori only continuous, but we can deform to a smooth one without changing its homotopy class by 'flattening' the maps near the boundary. In other words, we may assume that

$$w(z) = w(z/|z|) \quad \text{for } |z| \geq 1 - \epsilon$$

for sufficiently small $\epsilon > 0$. The bounding disc will be assumed to be flat in this sense. With this adjustment, u defines a smooth map from S^2.

5. **(The marking condition)** For the given $[z, w]$, $[z, w']$ with a periodic orbit $z(t) = \phi_H^t(z(0))$, we impose the additional *marking* condition

$$\Phi_w(z(0)) = \Phi_{w'}(z(0)) \tag{18.4.29}$$

as a map from $T_{z(0)}M$ to $\mathbb{R}^{2n}$ for the trivialization

$$\Phi_w, \Phi_{w'} : w^*TM \to D^2 \times (\mathbb{R}^{2n}, \omega_0),$$

which is always possible. *With this additional condition*, we can write

$$\alpha_{[z,w']}(t) = S_{w'w}(t) \cdot \alpha_{[z,w]}(t), \tag{18.4.30}$$

where $S_{w'w} : S^1 = \mathbb{R}/\mathbb{Z} \to Sp(2n, \mathbb{R})$ is the *loop* defined by the relation (18.4.30). Note that this really does define a loop because we have

$$\alpha_{[z,w']}(0) = \alpha_{[z,w]}(0)\,(= id), \tag{18.4.31}$$

$$\alpha_{[z,w']}(1) = \alpha_{[z,w]}(1). \tag{18.4.32}$$

In fact, it follows from the definition of (18.4.30) and (18.4.29) that we have the identity

$$\begin{aligned} S_{w'w}(t) &= \left(\Phi_{w'}(z(t)) \circ d\phi_H^t(z(0)) \circ \Phi_{w'}(z(0))^{-1}\right) \\ &\quad \circ \left(\Phi_w(z(t)) \circ d\phi_H^t(z(0)) \circ \Phi_w(z(0))^{-1}\right)^{-1} \\ &= \Phi_{w'}(z(t)) \circ \left(d\phi_H^t(z(0)) \circ \Phi_{w'}(z(0))^{-1} \circ \Phi_w(z(0)) \circ (d\phi_H^t)^{-1}(z(0))\right) \\ &\quad \circ (\Phi_w(z(t)))^{-1}. \end{aligned} \tag{18.4.33}$$

Then the marking condition (18.4.29) implies that the middle terms in (18.4.33) are canceled out and hence we have proved

$$S_{w'w}(t) = \Phi_{w'}(z(t)) \circ \Phi_w(z(t))^{-1}. \tag{18.4.34}$$

Then the formula

$$\mathrm{ind}_{CZ}(\overline{\alpha}_{[z,w']}) = 2\,\mathrm{wind}(\overline{\widehat{S}}_{w'w}) + \mathrm{ind}_{CZ}(\overline{\alpha}_{[z,w]}) \tag{18.4.35}$$

follows from Proposition 18.4.4 μ_{CZ} in (CZ84) and from (18.4.34). Here $\widehat{S}_{w'w} : S^1 \to U(n)$ is a loop in $U(n)$ that is homotopic to $S_{w'w}$ inside $Sp(2n, \mathbb{R})$. Such a homotopy always exists and is unique up to homotopy because $U(n)$ is a deformation retraction to $Sp(2n, \mathbb{R})$.

6. **(Normalization of c_1)** Finally, we recall the definition of the first Chern class c_1 of the symplectic vector bundle $E \to S^2$. We normalize the Chern class so that the tangent bundle of $S^2 \cong \mathbb{C}P^1$ has the first Chern number 2,

which also coincides with the standard convention in the literature. *We like to note that this normalization is compatible with the Hermitian structure on* $\mathbb{C}^n$ *given by (18.4.27) in our convention*. (See p. 167 in (Mil65).)

We decompose $S^2 = D^+ \cup D^-$ and consider the symplectic trivializations $\Phi_+ : E|_{D^+} \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$ and $\Phi_- : E|_{D^-} \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$. Note that under the Hermitian structure on $\mathbb{C}^n$ in our convention, these are homotopic to a unitary trivialization, while in the other convention they are homotopic to a conjugate unitary trivialization. Denote the transition matrix loop by

$$\phi_{+-} : S^1 \to Sp(2n, \mathbb{R}),$$

which is the loop determined by the equation

$$\Phi_+|_{S^1} \circ (\Phi_-|_{S^1})^{-1}(t, \xi) = (t, \phi_{+-}(t)\xi)$$

for $(t, \xi) \in E|_{S^1}$, where $S^1 = \partial D^+ = \partial D^-$. Then, by definition,

$$c_1(E) = \mu(\widehat{\phi}_{+-}) \tag{18.4.36}$$

in our convention. Equivalently, we have

$$c_1(E) = -\mu(\overline{\widehat{\phi}}_{+-}). \tag{18.4.37}$$

Now we apply this to $u^*(TM)$, where $u = w\#\overline{w}'$ and Φ_w and $\Phi_{w'}$ are the trivializations given in step 4. It follows from (18.4.34) that $S_{w'w}$ is the transition matrix loop between Φ_w and $\Phi_{w'}$. Then, by definition, the first Chern number $c_1(u^*TM)$ is provided by the winding number $\mu(\widehat{S}_{w'w})$ of the loop of unitary matrices

$$\widehat{S}_{w'w} : t \mapsto \widehat{S}_{w'w}(t); \quad S^1 \to U(n)$$

in the Hermitian structure of $\mathbb{C}^n$ in our convention. One can easily check that this winding number is indeed 2 when applied to the tangent bundle of S^2 and thus is consistent with the convention of the Chern class which we are adopting.

7. **(Wrap-up of the proof)** With these steps, especially steps 2, 5 and 6, combined, the formulae (18.4.35) and (18.4.37) turn into the index formula we want to prove.

□

18.5 The Hamiltonian-perturbed Cauchy–Riemann equation

To do the Morse theory of $\mathcal{A}_H$, we need to provide a metric on $\widetilde{\mathcal{L}}_0(M)$. We do this by first defining a metric on $\mathcal{L}_0(M)$ and then pulling it back to $\widetilde{\mathcal{L}}_0(M)$. Note that any S^1-family $\{g_t\}_{t\in S^1}$ of Riemannian metrics on M induces an L^2-type metric on $\mathcal{L}(M)$ by the formula

$$\langle\langle \xi_1, \xi_2 \rangle\rangle = \int_0^1 g_t(\xi_1(t), \xi_2(t))dt \tag{18.5.38}$$

for ξ_1, $\xi_2 \in T_\gamma\mathcal{L}(M)$. On the symplectic manifold (M, ω), we will particularly consider the family of *almost-Kähler metrics* induced by a one-parameter family J of almost-complex structures compatible with the symplectic form ω

$$g_J = \omega(\cdot, J\cdot)$$

and its associated norm by $|\cdot|_J$. For a given $J = \{J_t\}_{t\in S^1}$, it induces the associated L^2-metric on $\mathcal{L}(M)$ by $\langle\langle\cdot,\cdot\rangle\rangle_J$. This can be also written as

$$\langle\langle \xi_1, \xi_2 \rangle\rangle_J = \int_0^1 \omega(\xi_1(t), J_t\,\xi_2(t))dt \tag{18.5.39}$$

for ξ_1, $\xi_2 \in T_\gamma\mathcal{L}(M)$. From now on, we will always denote by J an S^1*-family* of compatible almost-complex structures unless stated otherwise, and denote

$$j_\omega := C^\infty(S^1, \mathcal{J}_\omega).$$

If we denote by $\mathrm{grad}_J\mathcal{A}_H$ the associated L^2-gradient vector field, a straightforward computation using (18.2.11) and (18.5.39) gives rise to

$$\mathrm{grad}_J\mathcal{A}_H([\gamma, w])(t) = J_t(\dot\gamma(t) - X_H(t, \gamma(t))), \tag{18.5.40}$$

which we will simply write $J(\dot\gamma - X_H(\gamma))$.

Exercise 18.5.1 Prove this formula.

It follows from this formula that the gradient is projectable to $\mathcal{L}_0(M)$. Therefore, when we project the *negative* gradient flow equation of a path $u : \mathbb{R} \to \widetilde{\mathcal{L}}_0(M)$ to $\mathcal{L}_0(M)$, it has the form

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0 \tag{18.5.41}$$

if we regard u as a map $u : \mathbb{R}\times S^1 \to M$. We call this equation *Floer's perturbed Cauchy–Riemann equation* or simply the perturbed Cauchy–Riemann equation associated with the pair (H, J).

The Floer theory largely relies on the study of the moduli spaces of *finite-energy* solutions $u : \mathbb{R} \times S^1 \to M$ of the kind (18.5.41) of perturbed Cauchy–Riemann equations. The relevant off-shell energy function is given by the following definition.

Definition 18.5.2 (Energy) For a given smooth map $u : \mathbb{R} \times S^1 \to M$, we define the energy, denoted by $E_{(H,J)}(u)$, of u by

$$E_{(H,J)}(u) = \frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|^2_{J_t} + \left| \frac{\partial u}{\partial t} - X_H(u) \right|^2_{J_t} \right) dt\, d\tau.$$

The following proposition is the closed analog to Lemma 12.2.4, whose proof is the same as before and hence has been omitted.

Proposition 18.5.3 *Let (M, ω) be any symplectic manifold, not necessarily compact. Let $H : S^1 \times M \to \mathbb{R}$ be any Hamiltonian. Suppose that $u : \mathbb{R} \times S^1 \to M$ is a finite-energy solution of (18.5.41) with its image having compact closure in* $\operatorname{int} M$*. Then there exists a sequence $\tau_k \to \infty$ (respectively $\tau_k \to -\infty$) such that the loop $z_k := u(\tau_k) = u(\tau_k, \cdot)$ converges in C^∞ to a one-periodic solution $z : S^1 \to M$ of the Hamilton equation $\dot{x} = X_H(x)$.*

We denote by

$$\mathcal{M}(H, J) = \mathcal{M}(H, J; \omega)$$

the set of finite-energy solutions of (18.5.41) for a general H that is not necessarily nondegenerate.

A similar discussion can be carried out for the *non-autonomous* version of (18.5.41), which we now describe. We first recall that

$$\mathcal{H}_0(M) = \{H : S^1 \times M \to \mathbb{R} \mid H \text{ is mean-normalized}\}.$$

Consider the $\mathbb{R}$-family

$$\begin{aligned} \mathcal{H}_{\mathbb{R}} &: \mathbb{R} \to \mathcal{H}_0; \quad \tau \mapsto H(\tau), \\ j_{\mathbb{R}} &: \mathbb{R} \to \mathcal{J}_\omega; \quad \tau \mapsto J(\tau), \end{aligned}$$

which are asymptotically constant, i.e.,

$$H(\tau) = H^{\pm\infty}, \qquad J(\tau) = j^{\pm\infty}$$

for some $H^{\pm\infty} \in \mathcal{H}_0$ and $j^{\pm\infty} \in j_\omega$ if $|\tau| > R$ for a sufficiently large constant R. With any such pair there is associated the following *non-autonomous* version of (18.5.41):

$$\frac{\partial u}{\partial \tau} + J(\tau)\left(\frac{\partial u}{\partial t} - X_{H(\tau)}(u)\right) = 0. \tag{18.5.42}$$

The associated energy function is given by

$$E_{(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}})}(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \left(\left|\frac{\partial u}{\partial \tau}\right|^2_{J(\tau)} + \left|\frac{\partial u}{\partial t} - X_{H(\tau)}(u)\right|^2_{J(\tau)}\right) dt\, d\tau.$$

We denote by

$$\mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}) = \mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; \omega)$$

the set of finite-energy solutions of (18.5.42).

Here is the analog to Proposition 18.5.3, whose proof is essentially the same as that for Proposition 18.5.3 due to the asymptotically constant condition on $(\mathcal{H}, j)$.

Proposition 18.5.4 *Let $\mathcal{H}_{\mathbb{R}}$ and $j_{\mathbb{R}}$ be as above. Suppose that $u : \mathbb{R}\times S^1 \to M$ is a finite-energy solution of (18.5.42). Then there exists a sequence $\tau_k \to \infty$ such that the loop $z_k := u(\tau_k) = u(\tau_k, \cdot)$ in C^∞ converges to a one-periodic solution $z : S^1 \to M$ of the Hamilton equation $\dot{x} = X_{H^{\pm\infty}}(x)$. The same applies at $-\infty$.*

A typical way in which such an asymptotically constant family appears is through an *elongation* of a given smooth one-parameter family over $[0, 1]$.

Definition 18.5.5 A continuous map $f : [0, 1] \to T$ for any topological space T is said to be *boundary flat* if the map is constant near the boundary $\partial[0, 1] = \{0, 1\}$.

Let $\mathcal{H} : [0, 1] \to \mathcal{H}_0(M)$ be a homotopy connecting two Hamiltonians $H_\alpha, H_\beta \in \mathcal{H}_0(M)$, and let $j : [0, 1] \to \mathcal{J}_\omega$ connect $J_\alpha, J_\beta \in \mathcal{J}_\omega$. We denote

$$\mathcal{P}(j_\omega) := C^\infty([0, 1], j_\omega),$$
$$\mathcal{P}(\mathcal{H}_0(M)) := C^\infty([0, 1], \mathcal{H}_0(M)).$$

Define a function $\rho : \mathbb{R} \to [0, 1]$ of the type

$$\rho(\tau) = \begin{cases} 0 & \text{for } \tau \le 0, \\ 1 & \text{for } \tau \ge 1 \end{cases} \tag{18.5.43}$$

and $\rho'(\tau) \ge 0$. We call ρ an elongation function.

Each such pair $(\mathcal{H}, j)$, combined with an elongation function ρ, defines a pair $(\mathcal{H}^\rho, j^\rho)$ of asymptotically constant $\mathbb{R}$-families

$$\mathcal{H}_{\mathbb{R}} = \mathcal{H}^\rho, \quad j_{\mathbb{R}} = j^\rho,$$

where $\mathcal{H}^\rho = \{H^\rho(\tau)\}_{\tau\in\mathbb{R}}$ is the reparameterized homotopy defined by

$$\tau \mapsto H^\rho(\tau, t, x) = H(\rho(\tau), t, x).$$

We call $\mathcal{H}^\rho$ the *ρ-elongation* of $\mathcal{H}$ or the *ρ-elongated homotopy* of $\mathcal{H}$. The same definition applies to j. Therefore such a triple $(\mathcal{H}, j; \rho)$ gives rise to the *non-autonomous* equation

$$\frac{\partial u}{\partial \tau} + J^\rho\left(\frac{\partial u}{\partial t} - X_{H^\rho}(u)\right) = 0. \tag{18.5.44}$$

We denote by

$$\mathcal{M}(\mathcal{H}, j; \rho)$$

the set of finite-energy solutions of (18.5.44).

19

Hamiltonian Floer homology

Floer (Fl89b) introduced Hamiltonian Floer homology in his attempt to prove Arnol'd's conjecture. To handle the transversality issue, he restricted his consideration to the case of monotone symplectic manifolds, the notion of which he also introduced in the same paper. This construction was subsequently generalized by Hofer and Salamon (HS95) and Ono (On95) to the semi-positive case.

The construction of a Floer complex in complete generality using the idea of obstruction bundles was first indicated by Kontsevich (Kon95). This idea was materialized by Fukaya and Ono (FOn99), Liu and Tian (LT98) and Ruan (Ru99). In Fukaya and Ono's terminology, the Kuranishi structure and the usage of multi-valued sections play crucial roles in achieving the abovementioned transversality to prove the existence of a fundamental moduli cycle.

In this section we closely follow the exposition given in (Oh05c), restricting the discussion to the semi-positive case. We will briefly indicate how the construction is generalized using the Kuranishi structure at the end of the section.

Although one can work with the $\mathbb{Z}$ coefficients for the semi-positive case, we will use the $\mathbb{Q}$ coefficients instead, since that is needed in the general context using the Kuranishi structure.

19.1 Novikov Floer chains and the Novikov ring

Suppose that $\phi = \phi_H^1 \in \mathrm{Ham}(M, \omega)$ is nondegenerate. For each such $H : S^1 \times M \to \mathbb{R}$, we know that the cardinality of $\mathrm{Per}(H)$ is finite. We consider the free $\mathbb{Q}$ vector space generated by the critical set of $\mathcal{A}_H$,

$$\mathrm{Crit}\, \mathcal{A}_H = \{[z, w] \in \widetilde{\mathcal{L}}_0(M) \mid z \in \mathrm{Per}(H)\}.$$

To be able to define the Floer boundary operator correctly, we need to complete this vector space downward with respect to the real filtration provided by the action $\mathcal{A}_H([z,w])$ of the element $[z,w]$.

Definition 19.1.1 Consider the formal sum

$$\beta = \sum_{[z,w]\in \mathrm{Crit}\mathcal{A}_H} a_{[z,w]}[z,w],\ a_{[z,w]} \in \mathbb{Q}. \tag{19.1.1}$$

(1) We define the *support* of β by

$$\mathrm{supp}(\beta) := \{[z,w] \in \mathrm{Crit}\,\mathcal{A}_H \mid a_{[z,w]} \neq 0 \text{ in the sum (19.1.1)}\}.$$

(2) We call those $[z,w]$ with $a_{[z,w]} \neq 0$ *generators* of the sum β and write

$$[z,w] \in \mathrm{supp}\,\beta.$$

We also say that $[z,w]$ *contributes* to β in that case.

(3) We call the formal sum β a *Novikov Floer chain* (or simply a *Floer chain*) if it satisfies

$$\#\Big(\mathrm{supp}(\beta) \cap \{[z,w] \mid \mathcal{A}_H([z,w]) \geq \lambda\}\Big) < \infty \tag{19.1.2}$$

for any $\lambda \in \mathbb{R}$. We call this the Novikov finiteness condition and denote by $CF_*(H)$ the set of Floer chains.

Note that according to this definition $CF_*(H)$ is a $\mathbb{Q}$-vector space, which is always infinite-dimensional in general. Only in the case where (M,ω) is symplectically aspherical does this become finite-dimensional. We associate the Conley–Zehnder index $\mu_H([z,w])$ with each generator $[z,w] \in \mathrm{Crit}\,\mathcal{A}_H$, which provides the structure of a graded vector space with $CF_*(H)$.

Now consider a Floer chain

$$\beta = \sum a_{[z,w]}[z,w], \quad a_{[z,w]} \in \mathbb{Q}.$$

The following notion is a crucial concept for the mini-max critical-point theory via Floer homology.

Definition 19.1.2 Let $\beta \neq 0$ be a Floer chain in $CF_*(H)$. We define the *level* of the chain β and denote it by

$$\lambda_H(\beta) = \max_{[z,w]}\{\mathcal{A}_H([z,w]) \mid [z,w] \in \mathrm{supp}(\beta)\},$$

and set $\lambda_H(0) = -\infty$. We call a generator $[z,w] \in \beta$ satisfying $\mathcal{A}_H([z,w]) = \lambda_H(\beta)$ a *peak* of β, and denote it by $\mathrm{peak}(\beta)$.

We emphasize that it is the Novikov finiteness condition (19.1.2) of Definition 19.1.1 which guarantees that $\lambda_H(\beta)$ is well defined. The following lemma illustrates optimality of the definition of the Novikov covering space.

Lemma 19.1.3 *Let $\beta \neq 0$ be a homogeneous Floer chain. Then the peak of β over a fixed periodic orbit is unique.*

Proof Let $[z, w]$ and $[z, w']$ be two such peaks of β. Then we have

$$\mathcal{A}_H([z, w]) = \lambda_H(\beta) = \mathcal{A}_H([z, w']),$$

which in turn implies $\omega([w]) = \omega([w'])$. By the homogeneity assumption, we also have

$$\mu_H([z, w]) = \mu_H([z, w']).$$

It follows from the definition of Γ-equivalence classes that $[z, w] = [z, w']$, which finishes the proof. □

So far we have defined $CF_*(H)$ as a $\mathbb{Z}$-graded $\mathbb{Q}$-vector space with $\operatorname{Crit}\mathcal{A}_H$ as its generating set. We now explain the description of $CF(H)$ as a module over the *Novikov ring* as in (Fl89b), (HS95).

Recalling that Γ is an abelian group, we consider the group ring $\mathbb{Q}[\Gamma]$ consisting of the finite sum

$$R = \sum_{i=1}^{k} r_i q^{A_i} \in \mathbb{Q}[\Gamma]$$

and define its support by

$$\operatorname{supp} R = \{A \in \Gamma \mid A = A_i \text{ in this sum with } r_i \neq 0\}.$$

We recall the valuation $v : \Gamma \to \mathbb{R}$ and the degree map $d : \Gamma \to \mathbb{R}$. The valuation $v : \mathbb{Q}[\Gamma] \to \mathbb{R}$ is given by

$$v(R) = v\left(\sum_{i=1}^{k} r_i q^{A_i}\right) := \min\{\omega(A) \mid A \in \operatorname{supp} R\}.$$

This satisfies the following non-Archimedean triangle inequality

$$v(R_1 + R_2) \geq \min\{v(R_1), v(R_2)\} \tag{19.1.3}$$

and so induces a natural metric topology on $\mathbb{Q}[\Gamma]$ induced by the metric

$$d(R_1, R_2) := e^{v(R_1 - R_2)}.$$

Definition 19.1.4 The Novikov ring is the *upward* completion $\mathbb{Q}[\Gamma]]$ of $\mathbb{Q}[\Gamma]$ with respect to the valuation $v : \mathbb{Q}[\Gamma] \to \mathbb{R}$. We denote it by $\Lambda_\omega = \Lambda_\omega^\uparrow$.

More concretely, we have

$$\Lambda_\omega = \left\{ \sum_{A \in \Gamma} r_A q^A \mid \forall \lambda \in \mathbb{R}, \#\{A \in \Gamma \mid r_A \neq 0, \omega(A) < \lambda\} < \infty \right\}.$$

We define a Γ action on $\mathrm{Crit}\,\mathcal{A}_H$ by 'gluing a sphere',

$$[z, w] \mapsto [z, w\#(-A)],$$

which in turn induces the Λ_ω-module structure on $CF(H)$ by the convolution product with Λ_ω by

$$\begin{aligned}\left(\sum_{A \in \Gamma} r_A q^A\right) \cdot \left(\sum a_{[z,w]}[z, w]\right)\\ := \sum_{[z,w']} \left(\sum_{A,[z,w];[z,(-A)\#w]=[z,w']} r_A a_{[z,(-A)\#w]}\right)[z, w'].\end{aligned} \tag{19.1.4}$$

We note that $q^A[z, w] = [z, (-A)\#w]$ according to this definition, i.e., we change the sign because we are looking at *homology*, not the cohomology.

We will try to consistently denote a Λ_ω module by $CF(H)$, and a graded $\mathbb{Q}$ vector space by $CF_*(H)$.

The action functional provides a natural filtration on $CF_*(H)$: for any given $\lambda \in \mathbb{R} \setminus \mathrm{Spec}(H)$, we define

$$CF_*^\lambda(H) = \{\alpha \in CF_*(H) \mid \lambda_H(\alpha) \leq \lambda\}$$

and denote the natural inclusion homomorphism by

$$i_\lambda : CF_*^\lambda(H) \to CF_*(H).$$

We will see later that it is sometimes useful to separate the formal variable T and e in the expression $q^A = T^{\omega(A)} e^{c_1(A)}$. Denote by $\Lambda_\omega^{(0)}$ the degree-zero part of Λ_ω, i.e., the set of formal series

$$\sum_{A \in \Gamma; c_1(A)=0} a_A q^A = \sum_{A \in \Gamma; c_1(A)=0} a_A T^{\omega(A)}$$

with the Novikov finiteness condition. It is immediately evident that we need to check the following.

Proposition 19.1.5 *$\Lambda_\omega^{(0)}$ is a field and $\Lambda_\omega = \Lambda_\omega^{(0)}[e, e^{-1}]$, which is the ring of polynomials of e and e^{-1} over the field $\Lambda_\omega^{(0)}$.*

Exercise 19.1.6 Prove this proposition.

19.2 Definition of the Floer boundary map

Suppose H is a nondegenerate one-periodic Hamiltonian and J a one-periodic family of compatible almost-complex structures. We first give the definition of the Floer boundary map, and point out the transversality conditions needed to define the Floer homology $HF_*(H, J)$ of the pair. We will fully discuss this transversality issue under the semi-positivity hypothesis of (M, ω).

Definition 19.2.1 Let z, $z' \in \mathrm{Per}(H)$. We denote by $\pi_2(z, z')$ the set of homotopy classes of smooth maps $u : [0, 1] \times S^1 := T \to M$ relative to the boundary

$$u(0, t) = z(t), \quad u(1, t) = z'(t).$$

We denote by $[u] \in \pi_2(z, z')$ its homotopy class.

Similarly, we define by $\pi_2(z)$ the set of relative homotopy classes of the maps

$$w : D^2 \to M; \quad w|_{\partial D^2} = z.$$

Note that there is a natural operation of $\pi_2(M)$ on $\pi_2(z)$ and $\pi_2(z, z')$ (modulo the action of $\pi_1(M)$) by 'gluing a sphere'. Furthermore, there is a natural map of $C \in \pi_2(z, z')$

$$(\cdot)\#C : \pi_2(z) \to \pi_2(z')$$

induced by the gluing map

$$w \mapsto w\#u.$$

More specifically, we will define the map $w\#u : D^2 \to M$ in the polar coordinates (r, θ) of D^2 by the formula

$$w\#u : (r, \theta) = \begin{cases} w(2r, \theta) & \text{for } 0 \le r \le \frac{1}{2}, \\ u(2r - 1, \theta) & \text{for } \frac{1}{2} \le r \le 1 \end{cases} \tag{19.2.5}$$

once and for all. There is also the natural gluing map

$$\pi_2(z_0, z_1) \times \pi_2(z_1, z_2) \to \pi_2(z_0, z_2),$$
$$(u_1, u_2) \mapsto u_1\#u_2.$$

Now we define the Conley–Zehnder index $\mu_H : \pi_2(z, z') \to \mathbb{Z}$. This can be defined without assuming z_0, z_1 to be contractible, as long as z_0 and z_1 lie in the same component of $\Omega(M)$.

Let $T = [0,1] \times S^1$. For any given map $u : T \to M$, choose a symplectic trivialization

$$\Phi : u^* TM \to T \times \mathbb{R}^{2n}.$$

We require the marking condition

$$\Phi(t,1) \equiv \Phi(0,1), \quad t \in [0,1] \tag{19.2.6}$$

We know that $z_0(t) = \phi_H^t(p_0)$ and $z_1(t) = \phi_H^t(p_1)$ for some $p_0,\ p_1 \in \mathrm{Fix}\,\phi_H^1$. Then we have two maps

$$\alpha_{\Phi,i} : [0,1] \to Sp(2n), \quad i = 0,\ 1$$

such that

$$\Phi \circ d\phi_H^t(p_i) \circ \Phi^{-1}(i,t,v) = (i,t,\alpha_{\Phi,i}(t)v)$$

for $v \in \mathbb{R}^{2n}$ and $t \in [0,1]$. Owing to the condition (19.2.6) and the nondegeneracy of H, we have

$$\alpha_{\Phi,i}(0) = id, \quad \alpha_{\Phi,i}(1) \in Sp^*(2n)$$

for both $i = 0,\ 1$ and thus the Conley–Zehnder indices $\mu_{CZ}(\alpha_{\Phi,i})$ for $i = 0,\ 1$ are defined.

Definition 19.2.2 For each $C \in \pi_2(z,z')$, we define the *relative Conley–Zehnder index* of $C \in \pi_2(z,z')$ by

$$\mu_H(z,z';C) := \mu_{CZ}(\alpha_{\Phi,0}) - \mu_{CZ}(\alpha_{\Phi,1}). \tag{19.2.7}$$

We will also write $\mu_H(C)$, when there is no danger of confusion on the boundary condition.

Exercise 19.2.3 Prove that the definition does not depend on the trivialization Φ satisfying (19.2.6).

The following proposition relates this relative index to the absolute Conley–Zehnder index $\mu_H([z,w])$ for $[z,w] \in \mathrm{Crit}\,\mathcal{A}_H$.

Proposition 19.2.4 *We have*

$$\mu_H(z,z';C) = \mu_H([z,w]) - \mu_H([z',w\#u]) \tag{19.2.8}$$

for a (and so any) representative $u : [0,1] \times S^1 \times M$ *of the class* $C \in \pi_2(z,z')$.

Proof We choose a trivialization

$$\Psi_w : w^*TM \to D^2 \times \mathbb{R}^{2n}$$

and a trivialization $\Phi : u^*TM \to T \times \mathbb{R}^{2n}$ that satisfies (19.2.6) and

$$\Psi_w|_{\partial D^2} = \Phi|_{\{0\}\times S^1} \tag{19.2.9}$$

after identifying $\partial D^2 \cong \{0\} \times S^1$. The gluing $\Psi_w \# \Phi$ then induces a natural trivialization of $(w\#u)^*TM$. We now compute the index $\mu_H(\alpha_{\Phi_w\#\Phi})$. But, by construction and the requirement (19.2.9), we have

$$\mu_H(\alpha_{\Psi_w\#\Phi}) = \mu_H(\alpha_{\Phi,1}),$$
$$\mu_H(\alpha_{\Psi_w}) = \mu_H(\alpha_{\Phi,0}).$$

Therefore we have

$$\begin{aligned}\mu_H([z', w\#u] &= \mu_H(\alpha_{\Psi_w\#\Phi}) = \mu_H(\alpha_{\Phi,1}) - \mu_H(\alpha_{\Phi,0}) + \mu_H(\alpha_{\Phi,0})\\ &= -\mu_H(z, z'; [u]) + \mu_H([z, w])\end{aligned}$$

which finishes the proof. □

We now denote by

$$\widetilde{\mathcal{M}}(H, J; z, z'; C)$$

the set of finite-energy solutions of (18.5.41) with the asymptotic condition and the homotopy condition

$$u(-\infty) = z, \quad u(\infty) = z'; \quad [u] = C \tag{19.2.10}$$

and set

$$\mathcal{M}(H, J; z, z'; C) = \widetilde{\mathcal{M}}(H, J; z, z'; C)/\mathbb{R}.$$

Here we remark that, although u is a map defined on $\mathbb{R} \times S^1$, it can be compactified into a continuous map $\overline{u} : [0, 1] \times S^1 \to M$ with the corresponding boundary condition

$$\overline{u}(0) = z, \quad \overline{u}(1) = z'$$

due to the exponential decay property of solutions u of (18.5.41), recalling that we assume that H is nondegenerate. We will call $\overline{u}$ the *compactified map* of u. By an abuse of notation, we will also denote by $[u]$ the class $[\overline{u}] \in \pi_2(z, z')$ of the compactified map $\overline{u}$.

We have the following basic formula for the Fredholm index. (See (Schw95) for the proof.)

Proposition 19.2.5 *Let $u \in \mathcal{M}(H, J; z, z'; C)$. Then*

$$\operatorname{Index} u = \mu_H(z, z'; C). \tag{19.2.11}$$

In particular, if J is H-regular, then $\dim \widetilde{\mathcal{M}}(H, J; z, z'; C) = \mu_H(z, z'; C)$.

Exercise 19.2.6 Translate the proof of Proposition 15.3.1 into the closed string context and prove this proposition.

The Floer boundary map

$$\partial_{(H,J)}; CF_{k+1}(H) \to CF_k(H)$$

is defined by counting the number of elements in $\mathcal{M}(H, J; z, z'; C)$ for the triple $(z, z'; C)$ with $\mu_H(z, z'; C) = 1$ under the following conditions.

Definition 19.2.7 (The boundary map) Let H be nondegenerate. Suppose that J satisfies the following conditions.

(1) For any pair $(z_0, z_1) \subset \operatorname{Per}(H)$ satisfying

$$\mu_H(z_0, z_1; C) = \mu_H([z_0, w_0]) - \mu_H([z_1, w_0 \# C]) \leq 0,$$

$\widetilde{\mathcal{M}}(H, J; z_0, z_1; C) = \emptyset$ unless $z_0 = z_1$ and $C = 0$. When $z_0 = z_1$ and $C = 0$, the only solutions are the stationary solution, i.e., $u(\tau) \equiv z_0 = z_1$ for all $\tau \in \mathbb{R}$.

(2) For any pair $(z_0, z_1) \subset \operatorname{Per}(H)$ and a homotopy class $C \in \pi_2(z_0, z_1)$ satisfying

$$\mu_H(z_0, z_1; C) = 1,$$

$\widetilde{\mathcal{M}}(H, J; z_0, z_1; C)$ is transverse and the quotient

$$\mathcal{M}(H, J; z_0, z_1; C) = \widetilde{\mathcal{M}}(H, J; z_0, z_1; C)/\mathbb{R}$$

is compact and hence a finite set. We denote

$$n(H, J; z_0, z_1; C) = \#(\mathcal{M}(H, J; z_0, z_1; C))$$

the algebraic count of the elements of the space $\mathcal{M}(H, J; z_0, z_1; C)$. We set $n(H, J; z_0, z_1; C) = 0$ if the triple $(z, z'; C)$ does not satisfy the index requirement.

(3) For any pair $(z_0, z_2) \subset \operatorname{Per}(H)$ and $C \in \pi_2(z_0, z_2)$ satisfying

$$\mu_H(z_0, z_2; C) = 2,$$

$\mathcal{M}(H, J; z_0, z_2; C)$ can be compactified into a smooth one-manifold with a boundary comprising the collection of the broken trajectories

$$[u_1]\#_\infty[u_2],$$

where $u_1 \in \widetilde{\mathcal{M}}(H, J; z_0, y : C_1)$ and $u_2 \in \widetilde{\mathcal{M}}(H, J; y, z_2 : C_2)$ for all possible $y \in \mathrm{Per}(H)$ and $C_1 \in \pi_2(z_0, y)$, $C_2 \in \pi_2(y, z_2)$ satisfying

$$C_1\#C_2 = C; \quad [u_1] = C_1, \ [u_2] = C_2$$

and

$$\mu_H(z_0, y; C_1) = \mu_H(y, z_2; C_2) = 1.$$

Here we denote by $[u]$ the equivalence class represented by u.

We call any such J *H-regular* and call such a pair (H, J) *Floer-regular*.

The upshot of this definition is that for a Floer-regular pair (H, J) the Floer boundary map

$$\partial = \partial_{(H,J)} : CF_*(H) \to CF_*(H)$$

is defined and satisfies $\partial\partial = 0$, which enables one to take its homology.

Since $CF_*(H)$ is generated by the pairs $[z, w] \in \mathrm{Crit}\,\mathcal{A}_H$, we need to explain this construction in more detail.

Let $[z, w]$ be given. For each given $[z', w'] \in \mathrm{Crit}\,\mathcal{A}_H$, we consider all $C \in \pi_2(z, z')$ that satisfy

$$[z', w\#C] = [z', w'], \quad \mu_H(z, z'; C) = 1, \tag{19.2.12}$$

and define the moduli space

$$\mathcal{M}(H, J; [z, w], [z', w']) := \bigcup_C \{\mathcal{M}(H, J; z, z'; C) \mid C \text{ satisfies (19.2.12)}\}.$$

We would like to note that there could be more than one $C \in \pi_2(z^-, z^+)$ that satisfies (19.2.12) according to the definition of the Γ-covering space $\widetilde{\mathcal{L}}_0(M)$. However, we have the following finiteness property which is a consequence of Gromov–Floer compactness.

Lemma 19.2.8 *This union is a finite union. In other words, for any given pair* $([z, w], [z', w'])$*, there are only finitely many* $C \in \pi_2(z, z')$ *that satisfy (19.2.12) and* $\mathcal{M}(H, J; z, z'; C) \neq \emptyset$.

Proof Suppose to the contrary that there exists an infinite sequence $u_i \in \mathcal{M}(H, J; [z, w], [z', w'])$ such that $[u_i] \in \pi_2(z, z')$ are all distinct. We first note that

$$E_{(H,J)}(u_i) = \mathcal{A}_H([z, w]) - \mathcal{A}_H([z', w'])$$

independently of the is. By virtue of the Gromov–Floer compactness, there exists a subsequence, again denoted by u_i, such that u_i converges to

$$u_\infty = u^1 \# u^2 \# \cdots \# u^N,$$

where each u^k has the form

$$u^k = u_{k,0} + \sum_{\ell=1}^{L_k} v_\ell^k,$$

where $\{u_{k,0}\}_k$ defines a connected chain of Floer trajectories starting from z and ending at z', and v_ℓ^k are sphere bubbles attached to $u_{k,0}$. Here each $u_{k,0}$ is a smooth Floer trajectory connecting a pair of periodic orbits z_{k-1}, z_k with $z_0 = z$, $z_N = z'$.

Furthermore, Gromov–Floer compactness also implies the equality

$$[u_i] = [u_\infty] + \sum_{\ell=1}^{L_k} [v_\ell^k]$$

independently of i for all sufficiently large i, which is in contradiction to the hypothesis that all $[u_i] \in \pi_2(z, z')$ are distinct. $\square$

Now considering u as a path in the covering space $\widetilde{\mathcal{L}}_0(M)$, we write the asymptotic condition of $u \in \mathcal{M}(H, J; [z, w], [z', w'])$ as

$$u(-\infty) = [z, w], \quad u(\infty) = [z', w']. \tag{19.2.13}$$

The Floer boundary map $\partial = \partial_{(H,J)} : CF_*(H) \to CF_*(H)$ is defined by its matrix coefficient

$$\langle \partial([z, w]), [z', w'] \rangle := \sum_C n_{(H,J)}(z, z'; C),$$

where C is as in (19.2.12) and the Conley–Zehnder indices of $[z, w]$ and $[z', w']$ satisfy

$$\mu_H([z, w]) - \mu_H([z', w']) = \mu(z, z'; C) = 1.$$

By definition and the hypotheses given in Definition 19.2.7, $\partial = \partial_{(H,J)}$ has degree -1 and satisfies $\partial \circ \partial = 0$.

Definition 19.2.9 We say that a Floer chain $\beta \in CF(H)$ is a *Floer cycle* of (H, J) if $\partial\beta = 0$, i.e., if $\beta \in \ker \partial_{(H,J)}$, and a *Floer boundary* if $\beta \in \operatorname{Image} \partial_{(H,J)}$. Two Floer chains β, β' are said to be *homologous* if $\beta' - \beta$ is a boundary.

We define the Floer homology of (H, J) by

$$HF_*(H, J) := \ker \partial_{(H,J)} / \operatorname{Image} \partial_{(H,J)}.$$

One may regard this either as a graded $\mathbb{Q}$ vector space or as a Λ_ω module.

19.3 Definition of a Floer chain map

Suppose we are given a family $(\mathcal{H}, j)$ with $\mathcal{H} = \{H^s\}_{0\le s\le 1}$ and $j = \{J^s\}_{0\le s\le 1}$ and an elongation function $\rho : \mathbb{R} \to [0, 1]$. For each such pair $(\mathcal{H}, j)$ and ρ, $(\mathcal{H}, j; \rho)$ bears the *non-autonomous* equation (18.5.42) with the asymptotic condition

$$u(-\infty) = z_0, \quad u(\infty) = z_1 \tag{19.3.14}$$

and the homotopy condition $[u] = C \in \pi_2(z_0, z_1)$ for a fixed C. We denote by

$$\mathcal{M}((\mathcal{H}, j; \rho); z_0, z_1; C)$$

the set of finite-energy solutions of (18.5.44) with these asymptotic boundary and homotopy conditions.

The chain homomorphism

$$h_{\mathcal{H}} = h_{(\mathcal{H},j;\rho)} : CF_*(H^0) \to CF_*(H^1)$$

is defined by considering these moduli spaces $\mathcal{M}((\mathcal{H}, j; \rho); z_0, z_1; C)$ over the set of pairs of periodic orbits z_0, z_1 and the homotopy classes C. One can easily adapt the definition of the Conley–Zehnder index, Definition 19.2.2, to define the relevant index for the chain map.

Exercise 19.3.1 Adapt Definition 19.2.2 to define the Conley–Zehnder index for the triple $(z_0, z_1; C)$ with $z_0 \in \operatorname{Per}(H^0)$, $z_1 \subset \operatorname{Per}(H^1)$ and $C \in \pi_2(z_0, z_1)$ for the homotopy $\mathcal{H}$ connecting H^0 and H^1.

Definition 19.3.2 (The chain map) We say that $(\mathcal{H}^\rho, j^\rho)$ *is Floer-regular* if the following hold.

(1) $\mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; z_0, z_1; C) = \emptyset$ if $\mu_{\mathcal{H}^\rho}(z_0, z_1; C) \le -1$.
(2) For any pair $z_0 \in \mathrm{Per}(H^0)$ and $z_1 \subset \mathrm{Per}(H^1)$ satisfying

$$\mu_{\mathcal{H}^\rho}(z_0, z_1; C) = 0,$$

$\mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; z_0, z_1; C)$ is transverse and compact, and hence a finite set. We denote by

$$n(\mathcal{H}^\rho, j^\rho; z_0, z_1; C) := \#(\mathcal{M}(\mathcal{H}^\rho, j^\rho; z_0, z_1; C))$$

the algebraic count of the elements in $\mathcal{M}(\mathcal{H}^\rho, j^\rho; z_0, z_1; C)$. Otherwise, we set

$$n(\mathcal{H}^\rho, j^\rho; z_0, z_1 : C) = 0.$$

(3) For any pair $z_0 \in \mathrm{Per}(H_0)$ and $z_2 \in \mathrm{Per}(H_1)$ satisfying

$$\mu_{\mathcal{H}^\rho}(z_0, z_2; C) = 1,$$

$\mathcal{M}(H, J; z_0, z_2; C)$ is transverse and can be compactified into a smooth one-manifold with a boundary comprising the collection of broken trajectories

$$u_1 \#_\infty u_2,$$

where

$$\begin{aligned}(u_1, u_2) \in\ & \mathcal{M}(\mathcal{H}^\rho, j^\rho; z_0, y : C_1) \times \mathcal{M}(H^1, J^1; y, z_2 : C^2);\\ & \mu_{\mathcal{H}^\rho}(z_0, y; C_1) = 0,\ \mu_H(y, z_2; C_2) = 1\end{aligned}$$

or

$$\begin{aligned}(u_1, u_2) \in\ & \mathcal{M}(H^0, J^0; z_0, y : C^1) \times \mathcal{M}(\mathcal{H}^\rho, j^\rho; y, z_2 : C_1);\\ & \mu_{\mathcal{H}^\rho}(z_0, y; C_1) = 1,\ \mu_H(y, z_2; C_2) = 0\end{aligned}$$

and $C_1 \# C_2 = C$ for all possible such $y \in \mathrm{Per}(H)$ and $C_1 \in \pi_2(z_0, y)$, $C_2 \in \pi_2(y, z_2)$.

Now for each given pair of $[z_0, w_0] \in \mathrm{Crit}\,\mathcal{A}_{H_0}$ and $[z_1, w_1] \in \mathrm{Crit}\,\mathcal{A}_{H_1}$, we define

$$\mathcal{M}((\mathcal{H}, j; \rho); [z_0, w_0], [z_1, w_1]) := \bigcup_C \mathcal{M}((\mathcal{H}, j; \rho); z_0, z_1; C),$$

where $C \in \pi_2(z_0, z_1)$ are the elements satisfying

$$[z_1, w_1] = [z_1, w_0 \# C] \tag{19.3.15}$$

similarly to in (19.2.12). We say that $(\mathcal{H}, j; \rho)$ is *Floer-regular* if the ρ-elongation $(\mathcal{H}^\rho, j^\rho)$ is Floer-regular in the sense of Definition 19.3.2.

Under the condition in Definition 19.3.2, we can define a map of degree 0

$$h_{(\mathcal{H},j;\rho)} : CF(H^0) \to CF(H^1)$$

by the matrix element $n_{(\mathcal{H},j;\rho)}([z_0, w_0], [z_1, w_1])$ similarly to what we did for the boundary map. The conditions in Definition 19.3.2 then also imply that $h_{(\mathcal{H},j)}$ has degree 0 and satisfies the identity

$$h_{(\mathcal{H},j;\rho)} \circ \partial_{(H_0,J_0)} = \partial_{(H_1,J_1)} \circ h_{(\mathcal{H},j;\rho)}.$$

Two such chain maps $h_{(j^1,\mathcal{H}^1)}$, $h_{(j^2,\mathcal{H}^2)}$ are also chain homotopic, the proof of which follows.

19.4 Construction of a chain homotopy map

Although the above isomorphism *in homology* depends only on the end Hamiltonians H_α and H_β, the corresponding chain map depends not just on the homotopy $\mathcal{H} = \{H(\eta)\}_{0\le\eta\le1}$ between H_α and H_β, but also on the homotopy $j = \{J(\eta)\}_{0\le\eta\le1}$. Let us fix nondegenerate Hamiltonians H_α, H_β and a homotopy $\mathcal{H}$ between them. We then fix a homotopy $j = \{J(\eta)\}_{0\le\eta\le1}$ of compatible almost-complex structures and a cut-off function $\rho : \mathbb{R} \to [0, 1]$.

We recall that the homotopy condition

$$[z^+, w^+] = [z^+, w^-\#u]; \quad [u] = C \quad \text{in} \quad \pi_2(z^-, z^+) \tag{19.4.16}$$

was imposed when the definitions of the moduli spaces, $\mathcal{M}(H, J; [z^-, w^-], [z^+, w^+])$ and $\mathcal{M}((\mathcal{H}, j; \rho); [z^-, w^-], [z^+, w^+])$, were given in the previous sections. One consequence of (19.4.16) is

$$[z^+, w^+] = [z^+, w^-\#u] \quad \text{in} \quad \Gamma,$$

but the latter is a weaker condition than the former. In other words, there could be more than one pair of distinct elements C_1, $C_2 \in \pi_2(z^-, z^+)$ such that

$$\mu(z^-, z^+; C_1) = \mu(z^-, z^+; C_2), \quad \omega(C_1) = \omega(C_2).$$

When we are given a homotopy $(\overline{\mathcal{H}}, \overline{j})$ of homotopies with $\overline{j} = \{j_\kappa\}$, $\overline{\mathcal{H}} = \{\mathcal{H}_\kappa\}$, we also define the elongations $\overline{\mathcal{H}}^\rho$ consisting of $\mathcal{H}^\rho_\kappa$: we have

$$\overline{\mathcal{H}}^\rho = \{\mathcal{H}^\rho_\kappa\}_{0\le\kappa\le1}.$$

We consider the parameterized version of (18.5.44) by solving

$$\begin{cases} \partial u/\partial\tau + J^\rho_\kappa(\partial u/\partial t - X_{H^\rho_\kappa}(u)) = 0, \\ u(-\infty) = [z^-, w^-],\ u(\infty) = [z^+, w^+] \end{cases} \tag{19.4.17}$$

for each $0 \le \kappa \le 1$, and define the parameterized moduli space

$$\begin{aligned} &\mathcal{M}^{\mathrm{para}}((\overline{\mathcal{H}}, \overline{j}; \rho); [z^-, w^-], [z^+, w^+]) \\ &\quad = \bigcup_{\kappa \in [0,1]} \{\kappa\} \times \mathcal{M}((\mathcal{H}_\kappa, j_\kappa; \rho); [z^-, w^-], [z^+, w^+]). \end{aligned}$$

We would like to emphasize in this consideration of parameterized moduli space that the asymptotic conditions are fixed independently of the parameter $\kappa \in [0, 1]$.

Consideration of the pairs $[z^-, w^-]$, $[z^+, w^+]$ whose associated moduli space $\mathcal{M}((\mathcal{H}_\kappa, j_\kappa; \rho); [z^-, w^-], [z^+, w^+])$ has (virtual) dimension 0 defines the chain homotopy map

$$\Upsilon_{\overline{\mathcal{H}}^\rho} : CF_*(H_\alpha) \to CF_*(H_\beta)$$

by its matrix element given by $\#(\mathcal{M}((\mathcal{H}_\kappa, j_\kappa; \rho); [z^-, w^-], [z^+, w^+]))$. (Here we again use the transversality–compactness argument using dimension counting under the semi-positivity hypothesis as before.)

Owing to the additional dimension arising from the parameter space, we derive

$$\mu_{H_\alpha}([z^-, w^-]) - \mu_{H_\beta}([z^+, w^+]) = -1,$$

which implies that the map $\Upsilon_{\overline{\mathcal{H}}^\rho}$ has degree 1.

The following structure theorem of the boundary of $\overline{\mathcal{M}}^{\mathrm{para}}(\overline{\mathcal{H}}, \overline{j}; \rho)$ can be derived again by using the Gromov–Floer compactness result.

Proposition 19.4.1 *Assume $(\mathcal{H}_0, j_0)$ and $(\mathcal{H}_1, j_1)$ are Floer-regular in the sense of Definition 19.3.2 and $(\overline{\mathcal{H}}, \overline{j})$ is Floer-regular in the parameterized sense thereof. Let $[z^-, w^-] \in \mathrm{Crit}\,\mathcal{A}_{H_\alpha}$ and $[z^+, w^+] \in \mathrm{Crit}\,\mathcal{A}_\beta$ be a pair satisfying*

$$\mu_{H_\alpha}([z^-, w^-]) - \mu_{H_\beta}([z^+, w^+]) = 0$$

and hence the associated moduli space $\mathcal{M}^{\mathrm{para}}((\overline{\mathcal{H}}, \overline{j}; \rho); [z^-, w^-], [z^+, w^+])$ has dimension 1. Then there are finitely many points $0 < \kappa_1, \ldots, \kappa_\ell < 1$ for some integer $\ell \ge 0$ such that

$$\begin{aligned}
&\partial\overline{\mathcal{M}}^{\text{para}}((\overline{\mathcal{H}},\overline{j};\rho);[z^-,w^-],[z^+,w^+])\\
&\quad = \bigcup_{[z'^-,w'^-]} (\{0\}\times\mathcal{M}(H_\alpha,J_\alpha;[z^-,w^-],[z'^-,w'^-]))\\
&\qquad\qquad\qquad \times\mathcal{M}^{\text{para}}((\overline{\mathcal{H}},\overline{j};\rho);[z'^-,w'^-],[z^+,w^+])\\
&\quad\cup \bigcup_{[z'^+,w'^+]} \mathcal{M}^{\text{para}}((\overline{\mathcal{H}},\overline{j};\rho);[z^-,w^-],[z'^+,w'^+])\\
&\qquad\qquad\qquad \times\Big(\{1\}\times\mathcal{M}(H_\beta,J_\beta;[z'^+,w'^+],[z^+,w^+])\Big)\\
&\quad\cup \bigcup_{i=1}^{\ell}\{\kappa_j\}\times\mathcal{M}((\mathcal{H}_{\kappa_i},j_{\kappa_i};\rho);[z^-,w^-],[z^+,w^+]).
\end{aligned}$$

This proposition then immediately gives rise to the identity (modulo the sign)

$$h_{(j_1,\mathcal{H}_1;\rho_1)} - h_{(j_0,\mathcal{H}_0;\rho_0)} = \partial_{(J^1,H^1)}\circ\Upsilon_{\overline{\mathcal{H}}^\rho} + \Upsilon_{\overline{\mathcal{H}}^\rho}\circ\partial_{(J^0,H^0)}. \tag{19.4.18}$$

Again the map $\Upsilon_{\overline{\mathcal{H}}^\rho}$ depends on the choice of a homotopy $\overline{j}$ and ρ. Therefore we will denote

$$\Upsilon_{\overline{\mathcal{H}}^\rho} = \Upsilon_{(\overline{\mathcal{H}},\overline{j};\rho)}$$

as well. Equation (19.4.18) in particular proves that two chain maps for different homotopies $(j_0,\mathcal{H}_0;\rho)$ and $(j_1,\mathcal{H}_1;\rho)$ connecting the same end points are chain homotopic and hence proves that the chain map (19.5.19) induces the same map in homology independently of the homotopies $(\overline{\mathcal{H}},\overline{j})$ or of ρ.

Remark 19.4.2 In the above discussion, we do not vary the elongation function ρ over κ. This is just for the sake of simplicity of the notation. It would be most natural to vary the elongation function depending on κ.

19.5 The composition law of Floer chain maps

In this section, we examine the composition law:

$$h_{\alpha\gamma} = h_{\beta\gamma}\circ h_{\alpha\beta}$$

of the Floer homomorphism

$$h_{\alpha\beta} : HF_*(H_\alpha)\to HF_*(H_\beta). \tag{19.5.19}$$

Now we re-examine the equation (18.5.42). One key analytic fact in the study of the Floer moduli spaces is that there is an a-priori upper bound of the energy, which we will explain in the next section.

Next, we consider the triple

$$(H_\alpha,\ H_\beta,\ H_\gamma)$$

of Hamiltonians and homotopies $\mathcal{H}_1 = \{H_1(s)\}_{0\le s\le 1}$, $\mathcal{H}_2 = \{H_2(s)\}_{0\le s\le 1}$ satisfying

$$H_1(0) = H_\alpha,\ H_1(1) = H_\beta = H_2(0),\ H_2(1) = H_\gamma.$$

We define their concatenation $\mathcal{H}_1\#\mathcal{H}_2 = \{H_3(s)\}_{0\le s\le 1}$ by

$$H_3(s) = \begin{cases} H_1(2s), & 0 \le s \le \frac{1}{2}, \\ H_2(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

We note that, due to the choice of the cut-off function ρ, the continuity equation (18.5.42) is *autonomous* for the region $|\tau| > R$, i.e., it is invariant under translation by τ. When we are given a triple $(H_\alpha,\ H_\beta,\ H_\gamma)$, this fact enables one to glue solutions of two such equations corresponding to the pairs (H_α, H_β) and (H_β, H_γ), respectively.

Now a more precise explanation is in order. For a given pair of elongation functions

$$\rho = (\rho_1, \rho_2)$$

and a positive number $R > 0$, we define an elongated homotopy of $\mathcal{H}_1\#\mathcal{H}_2$

$$\mathcal{H}_1\#_{(\rho;R)}\mathcal{H}_2 = \{H_{(\rho;R)}(\tau)\}_{-\infty<\tau<\infty}$$

by

$$H_{(\rho;R)}(\tau,t,x) = \begin{cases} H_1(\rho_1(\tau+2R),t,x), & \tau \le 0, \\ H_2(\rho_2(\tau-2R),t,x), & \tau \ge 0. \end{cases}$$

Note that

$$H_{(\rho;R)} \equiv \begin{cases} H_\alpha & \text{for } \tau \le -(R_1+2R), \\ H_\beta & \text{for } -R \le \tau \le R, \\ H_\gamma & \text{for } \tau \ge R_2 + 2R \end{cases}$$

for some sufficiently large R_1, $R_2 > 0$ depending on the elongation functions ρ_1, ρ_2 and the homotopies $\mathcal{H}_1$, $\mathcal{H}_2$, respectively. *In particular, this elongated homotopy is always smooth, even when the usual glued homotopy $\mathcal{H}_1\#\mathcal{H}_2$ need not be so.* We define the elongated homotopy $j_1\#_{(\rho;R)}j_2$ of $j_1\#j_2$ in a similar way.

For an elongated homotopy $(j_1\#_{(\rho;R)}j_2, \mathcal{H}_1\#_{(\rho,R)}\mathcal{H}_2)$, we consider the associated perturbed Cauchy–Riemann equation

$$\begin{cases}\partial u/\partial\tau + J_3^{\rho(\tau)}(\partial u/\partial t - X_{H_3^{\rho(\tau)}}(u)) = 0,\\ \lim_{\tau\to-\infty} u(\tau) = z^-,\ \lim_{\tau\to\infty} u(\tau) = z^+\end{cases}$$

with the condition (19.4.16).

Let u_1 and u_2 be given solutions of (18.5.44) associated with ρ_1 and ρ_2, respectively. If we define the pre-gluing map $u_1\#_R u_2$ by the formula

$$u_1\#_R u_2(\tau,t) = \begin{cases} u_1(\tau+2R,t) & \text{for } \tau \le -R,\\ u_2(\tau-2R,t) & \text{for } \tau \ge R\end{cases}$$

and a suitable fixed interpolation between them by a partition of unity on the region $-R \le \tau \le R$, the assignment defines a diffeomorphism

$$(u_1,u_2,R) \to u_1\#_R u_2$$

from

$$\mathcal{M}\big(j_1,\mathcal{H}_1;[z_1,w_1],[z_2,w_2]\big)\times\mathcal{M}\big(j_2,\mathcal{H}_2;[z_2,w_2],[z_3,w_3]\big)\times(R_0,\infty)$$

onto its image, provided that R_0 is sufficiently large. Denote by $\overline{\partial}_{(\mathcal{H},j;\rho)}$ the corresponding perturbed Cauchy–Riemann operator

$$u \mapsto \frac{\partial u}{\partial\tau} + J_3^{\rho(\tau)}\left(\frac{\partial u}{\partial t} - X_{H_3^{\rho(\tau)}}(u)\right)$$

acting on the maps u satisfying the asymptotic condition $u(\pm\infty) = z^\pm$ and fixed homotopy condition $[u] = C \in \pi_2(z^-,z^+)$. By perturbing $u_1\#_R u_2$ by an amount that is smaller than the error for $u_1\#_R u_2$ to be a genuine solution, i.e., less than a weighted L^p-norm, for $p > 2$,

$$\|\overline{\partial}_{(\mathcal{H},j;\rho)}(u_1\#_{(\rho;R)}u_2)\|_p$$

in a suitable $W^{1,p}$ space of us, one constructs a unique genuine solution near $u_1\#_R u_2$. By an abuse of notation, we will denote this genuine solution by $u_1\#_R u_2$ using the gluing theorem in Section 15.5. Then the corresponding map defines an embedding

$$\begin{aligned}\mathcal{M}\big(j_1,\mathcal{H}_1;[z_1,w_1],[z_2,w_2]\big)&\times\mathcal{M}\big(j_2,\mathcal{H}_2;[z_2,w_2],[z_3,w_3]\big)\times(R_0,\infty)\\ &\to \mathcal{M}\big(j_1\#_{(\rho;R)}j_2,\mathcal{H}_1\#_{(\rho;R)}\mathcal{H}_2;[z_1,w_1],[z_3,w_3]\big).\end{aligned}$$

In particular, when we have

$$\mu_{H_\beta}([z_2,w_2]) - \mu_{H_\alpha}([z_1,w_1]) = \mu_{H_\gamma}([z_3,w_3]) - \mu_{H_\beta}([z_2,w_2]) = 0$$

both $\mathcal{M}(j_1, \mathcal{H}_1; [z_1, w_1], [z_2, w_2])$ and $\mathcal{M}(j_2, \mathcal{H}_2; [z_2, w_2], [z_3, w_3])$ are compact, and hence consist of a finite number of points. Furthermore, the image of the above-mentioned embedding exhausts the 'end' of the moduli space

$$\mathcal{M}\Big(j_1 \#_{(\rho;R)} j_2, \mathcal{H}_1 \#_{(\rho;R)} \mathcal{H}_2; [z_1, w_1], [z_3, w_3]\Big)$$

and the boundary of its compactification consists of the broken trajectories

$$u_1 \#_{(\rho;\infty)} u_2 = u_1 \#_\infty u_2.$$

This then proves the following gluing identity.

Proposition 19.5.1 *There exists $R_0 > 0$ such that for any $R \geq R_0$ we have*

$$h_{(\mathcal{H}_1, j_1)\#_{(\rho;R)}(\mathcal{H}_2, j_2)} = h_{(\mathcal{H}_1, j_1;\rho_1)} \circ h_{(\mathcal{H}_2, j_2;\rho_2)}$$

as a chain map from $CF_*(H_\alpha)$ to $CF_*(H_\gamma)$.

Here we remind the reader that the homotopy $\mathcal{H}_1 \#_{(\rho;R)} \mathcal{H}_2$ itself is an elongated homotopy of the glued homotopy $\mathcal{H}_1 \# \mathcal{H}_2$. This proposition then gives rise to the composition law $h_{\alpha\gamma} = h_{\beta\gamma} \circ h_{\alpha\beta}$ in homology.

19.6 Transversality

In general, we can always achieve transversality for the *smooth* Floer moduli space for a generic choice of Hamiltonians H or J. However, this is not possible for the compactified Floer moduli space because of the bubbling of negative multiple cover spheres. In general, one must use the technique of virtual fundamental chains and multi-valued perturbation in the abstract setting of Kuranishi structures to achieve this general transversality which is used to define the various Floer maps.

We will restrict ourselves to the semi-positive case and explain how the hypotheses imposed in Definition 19.2.7 and 19.3.2 can be achieved for the semi-positive case following the exposition by Hofer and Salamon (HS95). We leave the general case via the Kuranishi structures to (FOn99).

We start with the definition of semi-positive symplectic manifolds from (HS95), (On95).

Definition 19.6.1 A symplectic manifold (M, ω) is called *semi-positive* if there is no $A \in \pi_2(M)$ such that

$$\omega(A) > 0 \quad \text{and} \quad 3 - n \leq c_1(A) < 0.$$

For the case of the boundary map $\partial_{(H,J)}$, we have the following theorem.

Theorem 19.6.2 *Suppose that there is no $A \in \pi_2(M)$ such that*

$$\omega(A) > 0 \quad \textit{and} \quad 3 - n \le c_1(A) < 0.$$

Consider a nondegenerate $H : S^1 \times M \to \mathbb{R}$. Then the hypotheses stated in Definition 19.2.7 hold for H-regular J.

Proof The matrix elements of $\partial_{(H,J)}$ are determined by the moduli space $\widetilde{\mathcal{M}}(H, J; [z^-, w^-], [z^+, w^+])$ with the relative Conley–Zehnder index

$$\mu_H([z^-, w^-], [z^+, w^+]) = 1. \tag{19.6.20}$$

We have shown that $\widetilde{\mathcal{M}}(H, J; [z^-, w^-], [z^+, w^+])$ is a union of $\widetilde{\mathcal{M}}(H, J; z^-, z^+; C)$ over a finite number of $C \in \pi_2(z^-, z^+)$.

Our goal is then to prove that there is a dense subset of H-regular Js in $j_\omega = C^\infty([0, 1], \mathcal{J}_\omega)$. We first choose a generic such J so that $\widetilde{\mathcal{M}}(H, J; z^-, z^+; C)$ is empty if $\mu_{(H,J)}(C) \le 0$ and it has the expected dimension if $\mu_{(H,J)}(C) > 0$. This follows from the generic transversality, whose proof follows the same lines as the one given in Section 10.4.

Exercise 19.6.3 Prove this transversality.

For such a generic choice of J, $\widetilde{\mathcal{M}}(H, J; z^-, z^+; C)$ is a smooth manifold of one dimension. The index condition (19.6.20) and the index formula (18.4.25),

$$\mu_H([z, w\#A]) = \mu_H([z, w]) - 2c_1(A),$$

imply that $z^- \ne z^+$ and hence $\mathbb{R}$ acts on $\widetilde{\mathcal{M}}(H, J; z^-, z^+; C)$ freely for each such C. Therefore the quotient

$$\mathcal{M}(H, J; z^-, z^+; C) = \widetilde{\mathcal{M}}(H, J; z^-, z^+; C)/\mathbb{R}$$

is a zero-dimensional manifold. The matrix element $n_{(H,J)}([z^-, w^-], [z^+, w^+])$ is supposed to be given by the sum of the cardinality $\#\mathcal{M}(H, J; z^-, z^+; C)$ over such C. *To be able to define this number, we need to prove that $\widetilde{\mathcal{M}}(H, J; z^-, z^+; C)/\mathbb{R}$ is compact.*

Proposition 19.6.4 *Suppose that $\mu_H([z^-, w^-], [z^+, w^+]) = 1$. Then, for a generic choice of J, $\mathcal{M}(H, J; [z^-, w^-], [z^+, w^+])$ is compact.*

Proof According to the Gromov–Floer compactness, any divergent subsequence in $\widetilde{\mathcal{M}}(H, J; z^-, z^+; C)/\mathbb{R}$ in general has a subsequence that converges to

$$u_\infty = u_1 \# \cdots \# u_N; \quad u_1(-\infty) = z^-, \; u_N(\infty) = z^+,$$

which is a connected chain of cusp Floer trajectories u_ℓ such that

$$u_\ell^0(-\infty) = z_\ell^-,\ u_\ell^0(\infty) = z_\ell^+; \quad z_\ell^+ = z_{\ell+1}^-$$

for $1 \leq \ell \leq N$, where each u_ℓ is of the form

$$u_\ell = u_\ell^0 + \sum_{k=1}^{L} v_\ell^k, \tag{19.6.21}$$

and $\{v_\ell^k\}_k$ is a finite set of sphere bubbles v_ℓ^k. Denote $C_\ell = [u_\ell] \in \pi_2(z_\ell^-, z_\ell^+)$.

To show that $\mathcal{M}(H, J; z^-, z^+; C)$ is compact, we would like to show that for a generic choice of J it holds that $N = 1$ and $\{v_\ell^k\} = \emptyset$ for any (z^-, z^+) and a homotopy class $C \in \pi_2(z^-, z^+)$ with $\mu_{(H,J)}(C) = 1$.

We recall that each v_ℓ^k is a J_{t^k}-holomorphic sphere attached to u_ℓ at (τ_k, t_k) for some point (τ_k, t_k), where $\tau_k = \infty$ is also a priori allowed. We examine the case $\tau_k = \infty$. This is the case where the bubble is attached to the locus of a periodic orbit.

Consider the parameterized moduli space

$$\mathcal{M}^{\text{para}}(J; \alpha) = \bigcup_{t \in [0,1]} \{t\} \times \mathcal{M}(J_t; \alpha).$$

The virtual dimension of $\mathcal{M}^{\text{para}}(J; \alpha)$ is given by

$$2c_1(\alpha) + 2n - 6 + 1 = 2c_1(\alpha) + 2n - 5.$$

For a generic J, the open subset of somewhere-injective curves in $\mathcal{M}^{\text{para}}(J; \alpha)$ are smooth manifolds of dimension $2c_1(\alpha) + 2n - 5$. Therefore we must have $2c_1(\alpha) + 2n - 5 \geq 0$, i.e.,

$$c_1(\alpha) \geq 3 - n \tag{19.6.22}$$

whenever $\mathcal{M}^{\text{para}}(J; \alpha) \neq \emptyset$. We now assume that J satisfies this property for all somewhere-injective curves.

On the other hand, for bubbles of multiple covers, especially those with negative Chern numbers $3 - n \leq c_1(\alpha) < 0$, we cannot achieve the required transversality by a perturbation of Js in general. *Here the semi-positivity plays a crucial role.* Under the assumption of semi-positivity, (19.6.22) automatically implies that $c_1(\alpha) \geq 0$. So, if we choose a generic J so that (19.6.22) holds for all somewhere-injective curves, any other not somewhere-injective curves u will also have non-negative c_1 since the structure theorem, Theorem 8.7.3, implies that u must be a multiple cover, i.e., $u = \widetilde{u} \circ \phi$, where $\phi : S^2 \to S^2$ is a branched covering of S^2 of degree d greater than 1 and $\widetilde{u}$ is somewhere injective. In particular,

$$c_1(u) = d\, c_1(\widetilde{u}) \geq c_1(\widetilde{u})$$

for some positive integer d (and equality holds only when $c_1(\widetilde{u}) = 0$). Therefore

$$\text{vir.dim}\mathcal{M}(J;[u]) \geq \text{vir.dim}\mathcal{M}(J;[\widetilde{u}]).$$

Obviously the locus of the image of $\widetilde{u}$ is the same as that of u and hence, if $\widetilde{u}$ does not intersect $\mathrm{Per}(H)$, neither does the locus of the image of u. Here, with slight abuse of the notation, $\mathrm{Per}(H) \subset M$ denotes the image locus of the periodic orbits of H. This enables us to focus solely on the bubbles that are somewhere injective.

We now consider the evaluation map $\mathrm{ev}_0 : \mathcal{M}_1^{\mathrm{para}}(J;\alpha) \to M$ defined by $\mathrm{ev}_0(v,z) = v(z)$. The virtual dimension of $\mathcal{M}_1^{\mathrm{para}}(J;\alpha)$ is two bigger than that of $\mathcal{M}^{\mathrm{para}}(J;\alpha)$, which is $2c_1(\alpha) + 2n - 3$.

This evaluation map ev restricted to the open subset of somewhere-injective curves is transversal to the given one-dimensional submanifold $\mathrm{Per}(H) \subset M$ after we further perturb J if necessary. Therefore, if $3 - 2c_1(\alpha) > 1$ or if

$$c_1(\alpha) < 1, \tag{19.6.23}$$

the image of $\mathrm{ev}_0 : \mathcal{M}_1^{\mathrm{para}}(J;\alpha) \to M$ will not intersect $\mathrm{Per}(H)$. The explanation for why it is harder for more bubbles to happen is left to the reader.

By combining (19.6.22) and (19.6.23), we can conclude that, if $3 - n \leq c_1(\alpha) \leq 0$, bubbling cannot occur on the image of any periodic orbit of X_H.

Because each u_k should have an index of at least 1, we have $\mu_{(H,J)}([u_k]) \geq 1$. Since bubbling cannot occur at $(\pm\infty, *)$, it can occur only at some point $(z_\ell^k, t_\ell^k) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$. Then we consider the evaluation map

$$\mathrm{ev}_C : \mathcal{M}_L(H,J;z^-,z^+;C) \to M^L,$$

where L is the number of marked points and $\mathcal{M}_L(H,J;z^-,z^+;C)$ is the moduli space of Floer trajectories with L marked points. More precisely, it is defined by

$$\mathcal{M}_L(H,J;z^-,z^+;C) = \widetilde{\mathcal{M}}(H,J;z^-,z^+;C) \times (\mathbb{R} \times S^1)^L/\sim,$$

where the equivalence relation $\sim$ is given by

$$(u;(z_1,\ldots,z_L)) \sim (u';(z_1',\ldots,z_L'))$$

if and only if

$$u'(\tau,t) = u(\tau - \tau^0, t),\ (\tau_i', t_i') = (\tau_i + \tau^0, t_i)$$

for some τ^0.

By virtue of the definition of $\sim$, the map ev_C is well defined. Then each configuration appearing in (19.6.21) is an element of the fiber product

$$\mathcal{M}_{L_j}(H, J; z_j, z_{j+1}; C_j)_{\mathrm{ev}_{C_j}} \times_{(\mathrm{ev}_{\alpha_1},\ldots,\mathrm{ev}_{\alpha_L})} (\mathcal{M}_1^{\mathrm{para}}(\alpha_1^j), \ldots, \mathcal{M}_1^{\mathrm{para}}(\alpha_L^j))$$

with $C = C^0 \# \sum_{j=1}^N \sum_{k=1}^{L_j} \alpha_k^j$. This has the virtual dimension given by

$$\sum_{j=1}^{N} (\mu_H(C_j) + 2L_j) - (N-1) + \sum_{j=1}^{N} \sum_{k=1}^{L_j} (2c_1(\alpha_k^j) + 2n - 3) - 2nL, \quad (19.6.24)$$

where $L_j \geq 0$ and $L := \sum_{j=1}^N L_j$, provided that $N \geq 1$. (If $N = 0$, we must have $z^- = z^+$ and hence there cannot be any bubble attached by the choice of J. In this case, we must have

$$[z^+, w^+] = [z^-, w^- \# A], \; C = C^0 = -A$$

for some $A \in \pi_2(M)$. Then, by virtue of the index formula, we have $\mu_H(z^-, z^+; C^0) = 2c_1(A)$, which cannot be 1 as required.)

Furthermore,

$$\mu_H(C) = \sum_{j=1}^{N} \left(\mu_H(C_j) + \sum_{k=1}^{L_j} 2c_1(\alpha_k^j) \right). \quad (19.6.25)$$

Lemma 19.6.5 *Suppose that $\mu_H(C) = 1$. Then the above virtual dimension in (19.6.24) is given by $2 - L - N$ with $N \geq 1$.*

Proof By substituting $\mu_H(C) = 1$ into (19.6.25), we obtain

$$\sum_{j=1}^{N} \left(\mu_H(C_j) + \sum_{k=1}^{L_j} 2c_1(\alpha_k^j) \right) = 1.$$

We substitute this into (19.6.24), use the identity $L = \sum_{j=1}^N L_j$ and prove that the virtual dimension becomes

$$1 + 2L - (N-1) + L(2n-3) - 2nL = 2 - L - N.$$

This finishes the proof. □

Now this dimension must be non-negative if the moduli space is transversal. Therefore $0 \leq 2 - N - L$. Since $N \geq 1$, this implies L is either 0 or 1.

When $L = 0$, we have $1 = \sum_{j=1}^N \mu_H(C_j)$. But, since $\mu_H(C_j) \geq 0$, this implies that all $\mu_H(C_j) = 0$ except for one j, say $j = 1$. Then, except for $j = 1$, we have $z_j^- = z_j^+$ for all j, and at least one bubble must be attached to $z_j^- = z_j^+$ for $j = 2, \ldots, N$ which cannot happen because of the choice of J. It also implies that $N = 1$ in this case.

If it were the case that $L = 1$, the above inequality would become $N \leq 1$ and then we must have $N = 1$. This would mean that the bubble is attached to a periodic orbit, which was, however, ruled out by the choice of J we made above. Therefore we must have $N = 1$ and $L = 0$, i.e., the sequence can neither produce a non-trivial bubble nor split into a broken trajectory. This finishes the proof of the compactness of each $\mathcal{M}(H, J; z^-, z^+; C)$ whenever $\mu_H(z^-, z^+; C) = 1$. □

By this proposition, the matrix element of the boundary map $\overline{\partial}_{(H,J)}$ is well defined and hence so is the boundary map $\overline{\partial}_{(H,J)}$. By a similar consideration, one proves that $\mathcal{M}(H, J; z^-, z^+; C)$ is compact modulo the splitting of $N = 2$ with $L = 0$ in the above discussion when $\mu_H(z^-, z^+; C) = 2$. This then finishes the proof of the hypotheses given in Definition 19.2.7, and hence the proof of Theorem 19.6.2 follows. □

Exercise 19.6.6 Fill in the details of the proof of the claim regarding the case of $\mu_H(z^-, z^+; C) = 2$ in the last paragraph of the above proof.

Next we consider the case of the chain map $h_{(\mathcal{H},j;\rho)}$ for which the only difference is that we do not consider any quotient by $\mathbb{R}$ but just consider the moduli space

$$\mathcal{M}((\mathcal{H}, j; \rho); [z_0, w_0], [z_1, w_1]) = \bigcup_C \mathcal{M}((\mathcal{H}, j; \rho); z_0, z_1; C)$$

itself. This one has as its dimension $\mu_{(\mathcal{H},j;\rho)}(C)$, and the case with $\mu_{(\mathcal{H},j;\rho)}(C) = 0$ is relevant for the definition of the chain map.

By a similar proof, we can prove the following theorem.

Theorem 19.6.7 *Suppose that (M, ω) satisfies the statement that there is no $A \in \pi_2(M)$ such that*

$$\omega(A) > 0 \quad \textit{and} \quad 3 - n \leq c_1(A) < 0.$$

Then the hypotheses stated in Definition 19.3.2 hold.

Proof By the same kind of reasoning, we will obtain the virtual dimension

$$\sum_{j=1}^{N}(\mu_{\mathcal{H}}(C_j) + 2L_j) - (N-1) + \sum_{j=1}^{N}\sum_{k=1}^{L_j}(2c_1(\alpha_k^j) + 2n - 3) - 2nL$$

for the limit configuration with $N \geq 1$ and

$$\mu_{\mathcal{H}}(C) = \sum_{j=1}^{N} \mu_{\mathcal{H}}(C_j) + \sum_{j=1}^{N} \sum_{k=1}^{L_j} 2c_1(\alpha_k^j).$$

If $N = 0$, we must have

$$[z^-, w^-] = [z^+, w^+], \quad C^0 = 0,$$

since there cannot be any bubbles attached in the limit configurations. Therefore the corresponding moduli space $\mathcal{M}_{(\mathcal{H},j;\rho)}([z^-, w^-], [z^+, w^+])$ is compact.

So assume $N \geq 1$. This time we consider $\mu_{\mathcal{H}}(C) = 0$ and thus we obtain

$$0 = 1 - N - L.$$

This implies that $L = 0$ and $N = 1$, which in turn implies that $\mathcal{M}_{(\mathcal{H},j;\rho)}([z^-, w^-], [z^+, w^+])$ is compact. This finishes the proof. □

Next we study the chain property of $h_{(\mathcal{H},j;\rho)}$, i.e., the identity

$$h_{(\mathcal{H},j;\rho)} \circ \partial_{(H^0,J^0)} = \partial_{(H^1,J^1)} \circ h_{(\mathcal{H},j;\rho)}. \tag{19.6.26}$$

For this purpose, we consider the pairs $[z^-, w^-]$, $[z^+, w^+]$ for which the chain map moduli space $\mathcal{M}_{(\mathcal{H},j;\rho)}([z^-, w^-], [z^+, w^+])$ has virtual dimension 1, and analyze the boundary of its compactification.

For the chain homotopy map, we consider the parameterized moduli space

$$\mathcal{M}^{\text{para}}(\overline{\mathcal{H}}, \overline{j}; \rho; [z^-, w^-], [z^+, w^+]),$$

where $\overline{\mathcal{H}} = \{\mathcal{H}_\kappa\}_{0 \leq \kappa \leq 1}$ and $\overline{j} = \{j_\kappa\}_{0 \leq \kappa \leq 1}$ are one-parameter families with

$$\mathcal{H}_\kappa = \{H^{s,\kappa}\},\ j_\kappa = \{J^{s,\kappa}\}; \quad H^{0,\kappa} = H^\alpha,\ H^{1,\kappa} = H^\beta \quad \text{for all } 0 \leq \kappa \leq 1,$$

and consider the family of perturbed Cauchy–Riemann equations

$$\frac{\partial u}{\partial \tau} + J_t^{\rho(\tau),\kappa}\left(\frac{\partial u}{\partial t} - X_{H_t^{\rho(\tau),\kappa}}\right) = 0$$

for $0 \leq \kappa \leq 1$. We consider the pairs of $([z^-, w^-], [z^+, w^+])$ such that

$$\mathcal{M}^{\text{para}}((\overline{\mathcal{H}}, \overline{j}; \rho); [z^-, w^-], [z^+, w^+])$$

has virtual dimension 0 and 1, i.e.,

$$\mu_{H^\alpha}([z^-, w^-]) - \mu_{H^\beta}([z^+, w^+]) + 1 = 0, \quad \text{or } 1.$$

For the case where the dimension is 0, we have

$$\mu_{H^\alpha}([z^-, w^-]) - \mu_{H^\beta}([z^+, w^+]) = -1.$$

Lemma 19.6.8 *For a generic family of* $(\overline{\mathcal{H}}, \overline{j})$*,* $\mathcal{M}^{\mathrm{para}}(\overline{\mathcal{H}}, \overline{j}; \rho; [z^-, w^-], [z^+, w^+])$ *is compact whenever* $\mu_{H^\alpha}([z^-, w^-]) - \mu_{H^\beta}([z^+, w^+]) = -1$.

Exercise 19.6.9 Formulate the correct Fredholm set-up and prove this lemma.

By considering the family $\overline{j}$ and $\overline{\mathcal{H}}$ associated with this homotopy, the chain homotopy map

$$\Upsilon_{(\overline{j}, \overline{\mathcal{H}}; \overline{\rho})} : CF_*(J^0, H^0) \to CF_{*+1}(J^1, H^1)$$

is constructed by counting the elements of

$$\mathcal{M}^{\mathrm{para}}(\overline{\mathcal{H}}, \overline{j}; \rho; [z^-, w^-], [z^+, w^+])$$

for

$$\mu_{H^\alpha}([z^-, w^-]) - \mu_{H^\beta}([z^+, w^+]) = -1.$$

In particular the map $\Upsilon_{(\overline{j}, \overline{\mathcal{H}}; \overline{\rho})}$ has degree -1.

Now we consider $\overline{\mathcal{M}}^{\mathrm{para}}(\overline{\mathcal{H}}, \overline{j}; \rho; [z^-, w^-], [z^+, w^+])$ for the pair $([z^-, w^-], [z^+, w^+])$ with

$$\mu_{H^\alpha}([z^-, w^-]) - \mu_{H^\beta}([z^+, w^+]) = 0.$$

In this case, $\overline{\mathcal{M}}^{\mathrm{para}}(\overline{\mathcal{H}}, \overline{j}; \rho; [z^-, w^-], [z^+, w^+])$ has dimension 1. The boundary of this moduli space is as described in Proposition 19.4.1.

Remark 19.6.10 For the general symplectic manifolds, one needs to use the concept of a virtual moduli cycle and abstract multi-valued perturbations in the context of the Kuranishi structure (FOn99).

19.7 Time-reversal flow and duality

In this section, we study certain duality pairings present in the Floer complex, namely one induced from the time-reversal of the flows, which does not use the orientation of the ambient manifold, and the other induced from the Poincaré duality, which essentially uses the orientation of the ambient manifold.

19.7.1 Time-reversal duality

In this section, we compare two different systems of coherent orientations on the classical Morse complex by explicitly examining the dependence on the

Morse function and on the system of orientations. Here we do not use the orientability of M.

Let $\sigma(f) = \{o_\sigma(x; f); o_\sigma(x, y; f)\}_{x,y\in \mathrm{Crit}\, f}$ be a system of coherent orientations of the Morse complex of f defined in Section 15.6.1 *starting with orienting unstable manifolds and then associating coorientations on unstable manifolds*. This induces the Morse boundary map $\partial^\sigma_f : CM_k(f) \to CM_{k-1}(f)$. We denote by

$$\delta_f : CM^*(f) \to CM^{*+1}(f)$$

the associated coboundary map for the dual complex

$$CM^\ell(f) = \mathrm{Hom}(CM_\ell(f), \mathbb{Z}).$$

Recall that we have $\mathrm{Crit}\, f = \mathrm{Crit}(-f)$ as a set and that stable and unstable manifolds of f and $-f$ are switched. So, at each $y \in \mathrm{Crit}(-f)$, we assign an orientation σ on $T_y W^u_f(y) = T_y W^s_{-f}(y)$. Note that $W^s_{-f}(y) = W^u_f(y)$ as a set.

Next we denote by $\widetilde{\sigma}(g)$ the system of orientations, which *starts by orienting stable manifolds of g and then proceeds by associating coorientations on unstable manifolds.*

Lemma 19.7.1 *The system of orientations*

$$\widetilde{\sigma}(-f) = \{o_{\widetilde{\sigma}}(y; -f)\}_{y\in \mathrm{Crit}(-f)}$$

is coherent if $\sigma(f)$ is.

Exercise 19.7.2 Prove this lemma.

Using this system, one also induces an orientation on $\mathcal{M}_g(y, x; -f)$ for each pair $y,\ x \subset \mathrm{Crit}(-f)$ and hence the associated boundary map

$$\partial^{\widetilde{\sigma}}_{-f} : CM_\ell(-f) \to CM_{\ell-1}(-f).$$

By definition, we have the canonical one-to-one correspondence

$$\mathcal{M}_g(x, y; f) \leftrightarrow \mathcal{M}_g(y, x; -f)$$

of 'reversing the flow'. In particular, it follows that $\dim \mathcal{M}_g(x, y; f) = 0$ if and only if $\dim \mathcal{M}_g(y, x; -f) = 0$. In this case, we can compare the two integers $n^\sigma(x, y; f)$ and $n^{\widetilde{\sigma}}(y, x; -f)$. We recall that we have a canonical isomorphism

$$CM^\ell(f) \to CM_{n-\ell}(-f),$$

which is the linear extension of the assignment of the basis element $x^* \to \widetilde{x}$, where $x \in \mathrm{Crit}\, f$ with $\mu_f(x) = \ell$, x^* is its dual and $\widetilde{x}$ is x regarded as a critical point of $-f$.

Proposition 19.7.3 *Suppose that* $\mu_f(x) - \mu_f(y) = 1$ *and* $\mu_f(y) = k$. *Then we have*

$$n^{\widetilde{\sigma}}(y, x; -f) = (-1)^k n^{\sigma}(x, y; f).$$

In particular, we have the commutative diagram

$$\begin{tikzcd} CM^k(f) \arrow[r, "(-1)^k\delta_f"] \arrow[d] & CM^{k+1}(f) \arrow[d] \\ CM_{n-k}(-f) \arrow[r, "\partial^{\widetilde{\sigma}}_{-f}"] & CM_{n-k-1}(-f) \end{tikzcd} \tag{19.7.27}$$

Proof Let $p \in SW^u_{-f}(y) \cap SW^s_{-f}(x)$. The sign of p with respect to the coherent orientation $\widetilde{\sigma}$, which corresponds to $o_{\widetilde{\sigma}}(y, x; -f)$, is determined by the equation of oriented vector space

$$T_p^{\widetilde{\sigma}}(SW^u_{-f}(y)) = o_{\widetilde{\sigma}}(x, y; -f) \cdot N_p^{\widetilde{\sigma}}(SW^s_{-f}(x)) \tag{19.7.28}$$

with respect to $\widetilde{\sigma}$. Here we refer readers to the discussion around (15.6.46).

On the other hand, we have

$$N_p^{\sigma}(SW^s_f(y)) = o_{\sigma}(x, y; f) T_p^{\sigma}(SW^u_f(x)) \tag{19.7.29}$$

by definition of $o_\sigma(x, y; f)$. Now we need to compare the equations (19.7.28) and (19.7.29).

It follows that (19.7.28) is equivalent to

$$\begin{aligned} \{\nabla f(p)\} \oplus N_p^{\widetilde{\sigma}}(W^s_{-f}(x)) &= o_{\widetilde{\sigma}}(x, y; -f)\{\nabla f(p)\} \oplus T_p^{\widetilde{\sigma}}(SW^u_{-f}(y)) \\ &= o_{\widetilde{\sigma}}(x, y; -f) T_p^{\widetilde{\sigma}} W^u_{-f}(y). \end{aligned} \tag{19.7.30}$$

On the other hand, we have the following lemma.

Lemma 19.7.4

$$T_p^{\widetilde{\sigma}} W^s_{-f}(x) = T_p^{\sigma} W^u_f(x) \tag{19.7.31}$$

as an oriented vector space.

Proof By definition of the coorientation of $W^s_{-f}(y)$,

$$N_p^{\widetilde{\sigma}} W^s_{-f}(x) \oplus T_p^{\widetilde{\sigma}} W^s_{-f}(x) = T_p M$$

as an oriented vector space. On the other hand, by virtue of the definition of $\widetilde{\sigma}$, we also have

$$N_p^{\sigma} W^u_f(x) = N_p^{\widetilde{\sigma}} W^s_{-f}(x)$$

and so

$$N_p^{\widetilde{\sigma}} W^s_{-f}(x) \oplus T_p^{\sigma} W^u_f(x) = T_p M.$$

By comparing the two, we have obtained the lemma. □

By taking the direct sum of (19.7.30) with $T_p^{\widetilde{\sigma}}(S W^s_{-f}(x))$ from the right, we get

$$\begin{aligned}(-1)^{k+1} T_p M &= o_{\widetilde{\sigma}}(x, y; -f) T_p^{\widetilde{\sigma}} W^u_{-f}(y) \oplus T_p^{\widetilde{\sigma}}(S W^s_{-f}(x)) \\ &= o_{\widetilde{\sigma}}(x, y; -f)\{\nabla f(p)\} \oplus T_p^{\widetilde{\sigma}}(S W^u_{-f}(y)) \oplus T_p^{\widetilde{\sigma}}(S W^s_{-f}(x)) \\ &= o_{\widetilde{\sigma}}(x, y; -f)(-1)^{n-k-1} T_p^{\widetilde{\sigma}}(S W^u_{-f}(y)) \oplus T_p^{\widetilde{\sigma}} W^s_{-f}(x).\end{aligned}$$

Here we use the identity

$$\begin{aligned}&o_{\widetilde{\sigma}}(x, y; -f) T_p^{\widetilde{\sigma}} W^u_{-f}(y) \oplus T_p^{\widetilde{\sigma}}(S W^s_{-f}(x)) \\ &\quad = (-1)^{k+1} T_p^{\widetilde{\sigma}} W^u_{-f}(y) \oplus o_{\widetilde{\sigma}}(x, y; -f) \oplus T_p^{\widetilde{\sigma}}(S W^s_{-f}(x)) \\ &\quad = (-1)^{k+1} T_p M\end{aligned}$$

for the left-hand side and similarly for the right-hand side.

By taking the direct sum with $T_p^{\sigma} W^s_f(y)$ from the right of (19.7.29) and noting $N_p^{\sigma}(S W^s_f(y)) \cong N_p^{\sigma} W^s_f(y)$, we get

$$\begin{aligned}T_p M &= o_{\sigma}(x, y; f) T_p^{\sigma}(S W^u_f(x)) \oplus T_p^{\sigma} W^s_f(y) \\ &= o_{\sigma}(x, y; f)(-1)^k T_p^{\sigma} W^u_f(x) \oplus T_p^{\sigma}(S W^s_f(y)) \\ &= o_{\sigma}(x, y; f)(-1)^k (-1)^{(k+1)(n-k-1)} T_p^{\sigma}(S W^s_f(y)) \oplus T_p^{\sigma} W^u_f(x).\end{aligned}$$

Therefore, from (19.7.31), we have obtained

$$\begin{aligned}&(-1)^{k+1+n-k-1} o_{\widetilde{\sigma}}(x, y; -f) T_p^{\widetilde{\sigma}}(S W^u_{-f}(y)) \\ &\quad = (-1)^{k+(k+1)(n-k-1)} o_{\sigma}(x, y; f) T_p^{\sigma}(S W^s_f(y)).\end{aligned}$$

Finally we note that

$$\{\nabla f(p)\} \oplus T_p^{\widetilde{\sigma}}(S W^u_{-f}(y)) = T_p^{\widetilde{\sigma}} W^u_{-f}(y)$$

and

$$\{-\nabla f(p)\} \oplus T_p^{\sigma}(S W^s_f(y)) = T_p^{\sigma} W^s_f(y).$$

Therefore, transferring the above relations to y, we derive

$$(-1)^{k+1+n-k-1} o_{\widetilde{\sigma}}(x, y; -f) T_y^{\widetilde{\sigma}} W^u_{-f}(y) = (-1)^{k+(k+1)(n-k-1)+1} o_{\sigma}(x, y; f) T_y^{\sigma} W^s_f(y). \tag{19.7.32}$$

Also we have

$$N_y^{\widetilde{\sigma}} W^u_{-f}(y) \oplus T_y^{\widetilde{\sigma}} W^u_{-f}(y) = T_y M$$

and

$$N_y^{\sigma} W_f^s(y) \oplus T_y^{\sigma} W_f^s(y) = T_y M.$$

Therefore it remains to compare $N_y^{\widetilde{\sigma}} W_{-f}^u(y)$ and $N_y^{\sigma} W_f^s(y)$. For this purpose, we take the direct sum with $T_y^{\widetilde{\sigma}} W_{-f}^u(y)$ and compute

$$\begin{aligned} N_y^{\sigma} W_f^s(y) \oplus T_y^{\widetilde{\sigma}} W_{-f}^u(y) &= N_y^{\sigma} W_f^s(y) \oplus N_y^{\widetilde{\sigma}} W_{-f}^s(y) \\ &= T_y^{\sigma} W_f^u(y) \oplus N_y^{\sigma} W_f^u(y) = (-1)^{k(n-k)} T_p M, \end{aligned}$$

where the penultimate equality follows from the definition of $\widetilde{\sigma}$. This proves that

$$T_y^{\widetilde{\sigma}} W_{-f}^u(y) = (-1)^{k(n-k)} T_y^{\sigma} W_f^s(y).$$

By substituting this into (19.7.32), we obtain

$$(-1)^{k+1+n-k-1} o_{\widetilde{\sigma}}(x, y; -f) = (-1)^{k+(k+1)(n-k-1)+1+k(n-k)} o_{\sigma}(x, y; f).$$

On simplifying the expression, we obtain

$$o_{\widetilde{\sigma}}(x, y; -f) = (-1)^k o_{\sigma}(x, y; f),$$

which finishes the proof. □

Therefore we have constructed a canonical isomorphism.

Corollary 19.7.5 (Time reversal)

$$HM_{n-k}(-f; \widetilde{\sigma}) \cong HM^k(f; \sigma) \tag{19.7.33}$$

for all $k \in \mathbb{N}$ on any manifold M, which is not necessarily orientable.

19.7.2 Poincaré duality in Morse homology

Now suppose that M is orientable and let o_M be a given orientation thereof. Recall that the orientation of the normal bundle of an oriented submanifold $S \subset M$ is determined by the relation

$$N_p S \oplus T_p S = T_p M \tag{19.7.34}$$

as an oriented vector space at $p \in S$. In particular, if we provide an orientation on $T_y W_y^u(f)$ according to the orientation scheme of σ laid out in Section 15.6.1, then the relation

$$N_y W_y^u(f) \oplus T_y^{\sigma} W_y^u(f) = T_y M$$

canonically equips the normal space $N_y W_y^u(f)$ itself with an orientation. Recall that *this was not the case without ambient orientation.* In this case the *stable* *manifold* $W^s_f(y)$ is also equipped with the natural orientation obtained by the standard relation

$$N^\sigma_y W^s_f(y) \oplus T_y W^s_f(y) = T_y M \tag{19.7.35}$$

as an oriented vector space, because $N_y W^s_f(y) = T_y W^u_f(y)$, which is already oriented by the orientation scheme of σ.

We denote the corresponding oriented vector space $T_y W^s_f(y)$ by $T^{\check\sigma}_y W^s_f(y)$. We then associate coorientations of *unstable manifolds* by requiring that

$$N^{\check\sigma} W^u_f(y) = T^{\check\sigma}_y W^s_f(y). \tag{19.7.36}$$

Therefore, if M is orientable, $\sigma(f)$ also give rise to another system, $\check\sigma(f)$, by the relations (19.7.35) and (19.7.36), which provides orientations of stable manifolds and coorientations on unstable manifolds of f itself, rather than of $-f$ as we did to define the system $\widetilde\sigma(-f)$ in the study of time-reversal. Now we would like to compare the two systems $\sigma(f)$ and $\check\sigma(f)$ on a given Morse function f.

The system $\widetilde\sigma$ provides an orientation on stable manifolds of y of f or on unstable manifolds of $-f$. On requiring the relation

$$N^{\widetilde\sigma}_y W^u_{-f}(y) = T^{\check\sigma}_y W^s_{-f}(y)$$

as an oriented vector space, an orientation is induced on $N^{\check\sigma}_y W^u_{-f}(y)$ at each y.

Proposition 19.7.6 *Let M be orientable and let $n = \dim M$. Then*

$$o_{\check\sigma}(x, y; f) = (-1)^n o_\sigma(x, y; f).$$

In particular, the sign depends only on $\dim M$ *and does not depend on the individual critical points of the same index.*

Proof Let σ be the coherent system as above. Owing to the orientability and by definition of $\check\sigma$, we have

$$T^{\check\sigma}_x W^s_f(x) \oplus T^\sigma_x W^u_f(x) = T_x M$$

with $T^{\check\sigma}_x W^s_f(x) = N^{\check\sigma}_x W^u_f(x)$.

The sign $o_{\check\sigma}(x, y; f)$ is determined by the relation

$$N^{\check\sigma}_p(S\,W^u_f(x)) = o_{\check\sigma}(x, y; f) T^{\check\sigma}_p(S\,W^s_f(y))$$

by definition of $\check\sigma$ and $o_{\check\sigma}(x, y; f)$. By summing with $T^{\check\sigma}_p(S\,W^u_f(x))$ from the right, we obtain

$$T^{\check\sigma}_p M^c = o_{\check\sigma}(x, y; f) T^{\check\sigma}_p(S\,W^s_f(y)) \oplus T^{\check\sigma}_p(S\,W^u_f(x)). \tag{19.7.37}$$

On the other hand, $o_\sigma(x, y; f)$ is determined by

$$N_p^\sigma(SW_f^s(y)) = o_\sigma(x, y; f) T_p^\sigma(SW_f^u(x)).$$

By adding $T_p^\sigma SW_f^s(y)$ from the right, we obtain

$$T_p^\sigma M^c = o_\sigma(x, y; f) T_p^\sigma(SW_f^u(x)) \oplus T_p^\sigma(SW_f^s(y)). \tag{19.7.38}$$

We derive from the definition of $\check{\sigma}$ the following lemma, whose proof we leave to the reader, since it involves a straightforward calculation.

Lemma 19.7.7

$$T^{\check{\sigma}} W_f^s(y) = (-1)^{k(n-k)} T^{\check{\sigma}} W_f^s(y),$$
$$T^{\check{\sigma}} W_f^u(x) = (-1)^{n-k-1} T_p^\sigma W_f^u(x).$$

Note that the second identity in this lemma can be rewritten as

$$\{\nabla f(p)\} \oplus T^{\check{\sigma}} SW_f^u(x) = (-1)^{n-k-1}\{-\nabla f(p)\} \oplus T_p^\sigma SW_f^u(x)$$

and hence

$$T^{\check{\sigma}} SW_f^u(x) = (-1)^{n-k} T_p^\sigma SW_f^u(x).$$

Substituting these into (19.7.37), we obtain

$$\begin{aligned} T_p M &= \{\nabla f(p)\} \oplus T_p^{\check{\sigma}} M^c = o_{\check{\sigma}}(x, y; f)\{\nabla f(p)\} \oplus T_p^{\check{\sigma}}(SW_f^s(y)) \oplus T_p^{\check{\sigma}}(SW_f^u(x)) \\ &= o_{\check{\sigma}}(x, y; f) T_p^{\check{\sigma}} W_f^s(y) \oplus T_p^{\check{\sigma}}(SW_f^u(x)) \\ &= (-1)^{k(n-k)+n-k} T_p^\sigma W_f^s(y) \oplus T_p^\sigma(SW_f^u(x)) \\ &= (-1)^{k(n-k)+n-k+k(n-k)} T_p^\sigma(SW_f^u(x)) \oplus T_p^\sigma W_f^s(y). \end{aligned} \tag{19.7.39}$$

On the other hand, (19.7.38) gives rise to

$$\begin{aligned} T_p M &= \{-\nabla f\} \oplus T_p^\sigma M^c = o_\sigma(x, y; f)\{-\nabla f\} \oplus T_p^\sigma(SW_f^u(x)) \oplus T_p^\sigma(SW_f^s(y)) \\ &= o_\sigma(x, y; f)(-1)^k T_p^\sigma(SW_f^u(x)) \oplus \{-\nabla f\} \oplus T_p^\sigma(SW_f^s(y)) \\ &= o_\sigma(x, y; f)(-1)^k T_p^\sigma(SW_f^u(x)) \oplus T_p^\sigma W_f^s(y). \end{aligned} \tag{19.7.40}$$

By comparing (19.7.39) and (19.7.40) and simplifying the exponent, we obtain

$$o_{\check{\sigma}}(x, y; f) = (-1)^n o_\sigma(x, y; f),$$

which finishes the proof. □

In particular, we have

$$\partial_f^{\check{\sigma}}|_{C_k(f)} = (-1)^n \, \partial_f^\sigma|_{C_k(f)} \tag{19.7.41}$$

for any $f : M \to \mathbb{R}$ and so

$$H^{\check{\sigma}}_{n-k}(f;\mathbb{Z}) \cong H^{\sigma}_{n-k}(f;\mathbb{Z}). \tag{19.7.42}$$

Obviously the same isomorphism holds for their cohomology. Then, by combining (19.7.33) and (19.7.42), we obtain the isomorphism

$$H^k(f;\sigma) \cong H^{\sigma}_{n-k}(f;\mathbb{Z}), \tag{19.7.43}$$

which precisely coincides with the classical Poincaré duality isomorphism

$$a \mapsto a \cap [M]$$

under the isomorphism

$$H^{\sigma}_*(f;M) \cong H^{\text{sing}}_*(M;\mathbb{Z}). \tag{19.7.44}$$

(See (Mil65) for this proof (with sign) in the classical Morse-theory context, which is essentially the same proof as the above.) We would like to emphasize that the time-reversal isomorphism (19.7.33) does not always require orientation of M. On the other hand, the Poincaré duality isomorphism (19.7.43) requires orientability of M.

In fact, (19.7.43) together with the isomorphism (19.7.44) can be used to find an example of M for which

$$H^{\sigma}_*(f;M) \not\cong H^{\check{\sigma}}_*(f;M) \tag{19.7.45}$$

holds. (This discussion is borrowed from (Oh99).)

Example 19.7.8 Consider the real projective space $\mathbb{R}P^{2m}$ of even dimension. We know that it is not orientable and

$$H_0^{\text{sing}}(\mathbb{R}P^{2m};\mathbb{Z}) \cong \mathbb{Z}, \qquad H_{2m}^{\text{sing}}(\mathbb{R}P^{2m};\mathbb{Z}) = 0,$$
$$H^0_{\text{sing}}(\mathbb{R}P^{2m};\mathbb{Z}) \cong \mathbb{Z}, \qquad H^{2m}_{\text{sing}}(\mathbb{R}P^{2m};\mathbb{Z}) = 0.$$

Therefore, by (19.7.44), we obtain

$$H_0^{\sigma}(f;\mathbb{R}P^{2m}) \cong \mathbb{Z}, \quad H_{2m}^{\sigma}(f;\mathbb{R}P^{2m}) = 0.$$

On the other hand, (19.7.33) implies that

$$H_0^{\widetilde{\sigma}}(-f;\mathbb{R}P^{2m}) \cong H^{2m}_{\text{sing}}(\mathbb{R}P^{2m};\mathbb{Z}) = 0,$$
$$H_{2m}^{\widetilde{\sigma}}(-f;\mathbb{R}P^{2m}) \cong H^{0}_{\text{sing}}(\mathbb{R}P^{2m};\mathbb{Z}) = \mathbb{Z}.$$

By the continuity isomorphism, this isomorphism can be transferred to f itself as long as we fix the coherent system of orientations and hence

$$H_0^{\widetilde{\sigma}}(f;\mathbb{R}P^{2m}) \cong H^{2m}_{\mathrm{sing}}(\mathbb{R}P^{2m};\mathbb{Z}) = 0,$$
$$H_{2m}^{\widetilde{\sigma}}(f;\mathbb{R}P^{2m}) \cong H^{0}_{\mathrm{sing}}(\mathbb{R}P^{2m};\mathbb{Z}) = \mathbb{Z}.$$

This example illustrates that the two Morse homologies $H^{\sigma}(f;M)$ and $H^{\widetilde{\sigma}}(f;M)$ could indeed be different.

19.7.3 Duality in a Floer complex

Motivated by the discussion on the Morse complex in the previous section, we examine the effect of a similar time-reversal map in Floer theory. One outstanding difference between the two is that we need to study the duality in the *infinite-dimensional* situation.

We start with the $\mathbb{Z}_2$-symmetry in the free loop space $\mathcal{L}_0(M)$ and its Novikov covering $\widetilde{\mathcal{L}}_0(M)$ provided by

$$[z,w] \mapsto [\widetilde{z},\widetilde{w}], \tag{19.7.46}$$

where $\widetilde{z}(t) = z(1-t)$ and $\widetilde{w}(x) = w(\overline{x})$, where $\overline{x}$ is the complex conjugate of $x \in D^2 \subset \mathbb{C}$. We also consider the operations on H, J and the system of coherent orientations

$$H \mapsto \widetilde{H}, \quad J \mapsto \widetilde{J}, \quad \sigma \mapsto \widetilde{\sigma}$$

defined by $\widetilde{H}(t,x) = -H(1-t,x)$, $\widetilde{J}_t = J_{1-t}$. One can easily check from the Floer equation that this symmetry induces a one-to-one correspondence between the solution sets of the associated Floer equations, and induces the identification of matrix coefficients

$$n^{\sigma}_{(H,J)}([z,w],[z',w']) = n^{\widetilde{\sigma}}_{(\widetilde{H},\widetilde{J})}([\tilde{z}',\tilde{w}'],[\widetilde{z},\widetilde{w}]).$$

This induces an injective chain homomorphism

$$CF^{\widetilde{\sigma}}(\widetilde{H};\mathbb{Q}) \hookrightarrow \mathrm{Hom}(CF^{\sigma}(H);\mathbb{Q}).$$

We define a natural nondegenerate pairing by setting

$$\langle[\widetilde{z},\widetilde{w}],[z',w']\rangle = \begin{cases} q^{-(\int \widetilde{w}^*\omega + \int (w')^*\omega)} & \text{if } z = z', \\ 0 & \text{if } z \neq z' \end{cases}$$

for the generators and extending bilinearly. Here we would like to mention that the exponent does not look as if it would depend on the Hamiltonian H. But this is misleading, because the universal formula

$$\mathcal{A}_{\widetilde{H}}([\widetilde{z},\widetilde{w}]) + \mathcal{A}_H([z',w']) = -\int \widetilde{w}^*\omega - \int (w')^*\omega \tag{19.7.47}$$

holds for any H, when $z' = z$. This is because the Hamiltonian terms are canceled out by the following change of variables:

$$\int_0^1 \widetilde{H}(t,\widetilde{z}(t))dt = \int_0^1 -H(1-t,z(1-t))dt = -\int_0^1 H(t,z(t))dt$$
$$= -\int_0^1 H(t,z'(t))dt.$$

We have the explicit formula

$$\langle\widetilde{\alpha},\beta\rangle = \sum_{A;c_1(A)=0}\left(\sum_{[z,w]\in \mathrm{Crit}\,\mathcal{A}_H}(a_{[\widetilde{z},\widetilde{w}]}b_{[z,w\#(-A)]})\right)q^A,$$

where

$$\widetilde{\alpha} = \sum_{[\widetilde{z},\widetilde{w}]\in \mathrm{Crit}\,\mathcal{A}_{\widetilde{H}}} a_{[\widetilde{z},\widetilde{w}]}[\widetilde{z},\widetilde{w}]$$

and $\beta = \sum_{[z,w]\in\mathrm{Crit}\,\mathcal{A}_H} b_{[z,w]}[z,w]$.

Furthermore, we have the relation between the Conley–Zehnder indices

$$\mu_H([z,w]) = 2n - \mu_{\widetilde{H}}([\widetilde{z},\widetilde{w}])$$

and hence the pairing induces an injective homomorphism

$$CF_k^{\widetilde{\sigma}}(\widetilde{H}) \hookrightarrow \mathrm{Hom}_{\Lambda_\omega^{(0)}}(CF_{2n-k}^{\sigma}(H);\Lambda_\omega^{(0)}),$$

where

$$\Lambda_\omega^{(0)} = \left\{\sum_A a_A q^A = \sum_A a_A T^{\omega(A)}\,\middle|\, c_1(A)=0\right\}.$$

We note that $\Lambda_\omega^{(0)}$ is a field and both $CF_k^{\widetilde{\sigma}}(\widetilde{H})$ and $CF_{2n-k}^{\sigma}(H)$ are finite-dimensional vector spaces over $\Lambda_\omega^{(0)}$ with the same number of generators and hence this homomorphism induces an isomorphism

$$HF_k(\widetilde{H},\widetilde{J};\widetilde{\sigma}) \cong \mathrm{Hom}_{\Lambda_\omega^{(0)}}(HF_{2n-k}(H,J;\sigma),\Lambda_\omega^{(0)}) \tag{19.7.48}$$

by the universal coefficient theorem.

Here we recall that the coherent system $\widetilde{\sigma}$ of orientations on the Floer moduli space is canonically induced by the map (19.7.46) on $\mathcal{L}_0(M)$.

We summarize the above discussion as follows.

Theorem 19.7.9 *Let $\Lambda_\omega^{(0)}$ be as above. Then there exists a strongly nondegenerate pairing*

$$L = \langle \cdot, \cdot \rangle : HF_k(\widetilde{H}, \widetilde{J}; \tilde{\sigma}) \times HF_{2n-k}(H, J; \sigma) \to \Lambda_\omega^{(0)}. \tag{19.7.49}$$

Next we define the projection

$$\pi_0 : \Lambda_\omega^{(0)} \to \mathbb{Q}; \quad \pi_0(\lambda) = \lambda_0, \tag{19.7.50}$$

where $\lambda_0 \in \mathbb{Q}$ is the coefficient with $A = 0$ of $\lambda = \sum_A \lambda_{A \in \Gamma} q^A$. We denote

$$L' = \pi_0 \circ L : HF_k(\widetilde{H}, \widetilde{J}; \tilde{\sigma}) \times HF_{2n-k}(H, J; \sigma) \to \mathbb{Q} \tag{19.7.51}$$

following the notation used in (Os03).

At this point, it is not obvious whether L' is nondegenerate. In Section 20.4, we will prove the nondegeneracy of L' for the rational symplectic manifolds by following the scheme given by Entov and Polterovich (EnP03), (Os03) which relates this pairing to the Frobenius pairing present in the quantum cohomology ring. The proof of this nondegeneracy for the irrational case requires a rather non-trivial argument using non-Archimedean analysis and was proved by Usher (Ush10b).

19.8 The Floer complex of a small Morse function

Since $HF_*(H, J)$ are isomorphic over the change of (H, J), we need only choose a special pair to determine the isomorphism type of the Λ_ω module.

In (Fl89a), (Fl89b), Floer used the special case

$$H = \epsilon f, \quad J \equiv J_0$$

for a Morse function f and time-independent almost-complex J_0 when $\epsilon > 0$ is sufficiently small. In this section, we analyze the structure of the Floer complex

$$\left(CF(\epsilon f), \partial_{(\epsilon f, J_0)} \right).$$

The following theorem was essentially proven by Floer in (Fl89b). We refer to (HS95) for the semi-positive case and (FOn99), (LT98) in general for complete details.

Theorem 19.8.1 *Let f be any small Morse function on M and let J_0 be a compatible almost-complex structure such that f is Morse–Smale with respect* *to the metric g_{J_0}. Then there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \le \epsilon_0$ we have the chain isomorphism*

$$\left(CF_*(\epsilon f), \partial_{(\epsilon f, J_0)}\right) \cong \left(CM_*(-\epsilon f), \partial^{\text{Morse}}_{(-\epsilon f, g_{J_0})}\right) \otimes \Lambda^{\downarrow}_{\omega}.$$

Proof For the distinguished pair $(\epsilon f, J_0)$, each gradient trajectory χ satisfying $\dot{\chi} - \text{grad}_g(\epsilon f) = 0$ gives rise to the t-independent solution $u(\tau, t) \equiv \chi(\tau)$ of the equation

$$\frac{\partial u}{\partial \tau} + J_0 \left(\frac{\partial u}{\partial t} - X_{\epsilon f}(u) \right) = 0.$$

It is proved in Mainlemma 22.4 of (FOn99) (or see the appendix of (Oh96b) for the open-string version) that all these trajectories are transversal as long as ϵ is sufficiently small. Then, using the fact that any t-dependent family occurs as an at least two-dimensional family for a generic t-independent $J \equiv J_0$, we conclude that only the above t-independent trajectory contributes to the Floer boundary map and hence the proof has been completed. □

Once we have this theorem, applying the Poincaré duality using the canonical orientation of (M, ω)

$$\left(CM^*(-\epsilon f), \delta^{\text{Morse}}_{(-\epsilon f, g_{J_0})}\right) \cong \left(CM_{2n-*}(-\epsilon f), \partial^{\text{Morse}}_{(-\epsilon f, g_{J_0})}\right),$$

where the grading $*$ in HF_* stands for the degree of the Floer cycle α which is provided by the Conley–Zehnder index of its generators.

Our convention of the grading of $CF_*(H)$ comes from (Oh05d), which is

$$\deg([z, w]) = \mu_H([z, w]) \tag{19.8.52}$$

for $[z, w] \in \text{Crit}\,\mathcal{A}_H$. This convention is the analog to the one we use in (Oh99) in the context of Lagrangian submanifolds.

We next compare this grading and the Morse grading of the Morse complex of the *negative* gradient flow equation of $-f$, (i.e., of the *positive* gradient flow of f

$$\dot{\chi} - \text{grad}\, f(\chi) = 0.$$

This corresponds to the *negative* gradient flow of the action functional $\mathcal{A}_{\epsilon f}$). This gives rise to the relation between the Morse indices $\mu^{\text{Morse}}_{-\epsilon f}(p)$ and the Conley–Zehnder indices $\mu_{\epsilon f}([p, \widehat{p}])$ in our convention. (See Lemma 7.2 of (SZ92), but take some care about the different convention of the Hamiltonian vector field. Their definition of X_H is $-X_H$ in the convention of the present book.) The relation is as follows:

$$\mu_{\epsilon f}([p, \widehat{p}]) = \mu^{\text{Morse}}_{-\epsilon f}(p) - n$$

or

$$\mu^{\text{Morse}}_{-\epsilon f}(p) = \mu_{\epsilon f}([p, \widehat{p}]) + n.$$

Corollary 19.8.2 *We have the natural canonical graded isomorphism*

$$H^*(M) \otimes \Lambda_\omega \cong HF_*(\epsilon f, J_0) =: HF_*(\epsilon f)$$

as a $\mathbb{Q}$*-vector space or as a* $\Lambda^{(0)}_\omega$ *module.*

We also recall that $H^*(M) \otimes \Lambda_\omega$ is isomorphic to the quantum cohomology $QH^*(M)$ as a Λ_ω module by taking the model of $QH^*(M)$ as the cohomology of the Morse complex

$$CM_*(-\epsilon f) \otimes \Lambda^{\downarrow}$$

for the quantum chain complex $CQ_*(-\epsilon f)$. Then we have the following grading-preserving isomorphism:

$$QH^{n-k}(M) \to QH_{n+k}(M) \cong HQ_{n+k}(-\epsilon f) \to HF_k(\epsilon f, J_0) \to HF_k(H, J)$$

as a graded $\mathbb{Q}$-vector space.

20

The pants product and quantum cohomology

Floer introduced an action of $H^*(M)$ on $HF^*(H)$ in (Fl89b) in the context of Hamiltonian fixed points and used it to estimate the number of fixed points for general Hamiltonian diffeomorphisms for the case of $\mathbb{C}P^n$. He conjectured the presence of such an action under the assumption that a certain transversality is available for a generic choice of almost-complex structures, and, as a consequence, he implicitly introduced a ring structure on $HF^*(H)$. Piunikhin (Pi94) used the interpretation of the quantum cohomology $QH^*(M)$ as a Bott–Morse generalization of the Floer cohomology for the zero Hamiltonian H and proposed the isomorphism between $QH^*(M)$ and $HF^*(H)$ by interpolating the given H and the 'zero' Hamiltonian in the level of rings. (See also (RT95b) for further elaboration.) This would give rise to an isomorphism between $QH^*(M)$ and $HF^*(H)$. Piunikhin, Salamon and Schwarz (PSS96) later made a proposal for the construction of the ring isomorphism that is somewhat different from (Pi94, RT95b), being based on the construction of the so-called PSS map between $HF^*(H)$ and $QH^*(M)$. The isomorphism property, especially at the level of a ring, plays an important role in applications of Floer homology to problems in symplectic topology (Se97), (Schw00), (Oh05c, Oh05d). However, the proposals both of Piunikhin (Pi94) and of Piunikhin, Salamon and Schwarz (PSS96) lacked some non-trivial gluing analysis, which was not available at the time of the proposals but has been developed later. For the construction in (Pi94), one needs the kind of gluing analysis developed in Chapter 7 (FOOO09), whereas for the construction in (PSS96), one needs the analysis presented in (OhZ11a) or (OhZ11b).

In this chapter, we explain the structure of a quantum cohomology ring and its chain-level description via the Floer complex of small Morse functions. Then a complete proof of the PSS isomorphism property will be given modulo the key gluing analysis entering into the proof of its isomorphism property, which we leave to (OhZ11b).

20.1 The structure of a quantum cohomology ring

In this section, we describe the algebro-geometric structure of a quantum cohomology ring without delving into the technical details of the rigorous construction of the ring structure. This structure was systematically explained first by the physicists Witten (Wi91) and Vafa (Va92). In the next section we will explain what is actually involved in the rigorous construction of those structures, which is carried out in (RT95a), (MSa04) in the semi-positive case and in (FOn99), (Ru99) in the general case. In the setting of algebraic geometry, the rigorous construction is given in (Be97), (LiT98).

As in the previous chapter, we denote by Λ_ω *the* upward *Novikov ring in this section.*

The quantum cohomology ring, as a module, is indeed defined by

$$QH^*(M) = H^*(M,\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_{0,\omega}$$

over the subring $\Lambda_{0,\omega} = \Lambda^{\mathbb{Q}}_{0,\omega}$,

$$\Lambda^{\mathbb{Q}}_{0,\omega} = \left\{ \sum_{A\in\Gamma_\omega} c_A T^{\omega(A)} e^{c_1(A)} \,\Big|\, c_A \in \mathbb{Q},\ \omega(A) \geq 0 \right\},$$

of the Novikov ring Λ_ω. This is because the symplectic area of any J-holomorphic curve will have non-negative areas. However, when one tries to compare the quantum cohomology and the Floer homology, especially when one studies the invariance property and the duality pairing of Floer homology, one needs to extend the ring to the full Novikov ring Λ_ω. From now on, we will use the full Novikov ring unless stated otherwise.

We also consider the homological version of the quantum cohomology. Consider the Λ^{op}_ω module

$$QH_*(M) = H_*(M) \otimes_{\mathbb{Q}} \Lambda^{op}_\omega,$$

where $\Lambda^{op}_{0,\omega}$ is the 'downward' completion of the group ring $\mathbb{Q}[\Gamma_\omega]$ with respect to the (downward) valuation defined by

$$v\left(\sum_{B\in\Gamma} a_B q^B\right) = \max\{\omega(B) : a_B \neq 0\}, \tag{20.1.1}$$

which satisfies the criterion that for $f,\ g \in QH_*(M)$

$$v(f+g) \leq \max\{v(f), v(g)\}.$$

By definition, we have a canonical isomorphism

$$\flat : QH^*(M) \to QH_*(M); \quad \sum a_i T^{\omega(A_i)} e^{c_1(A_i)} \to \sum PD(a_i) T^{\omega(-A_i)} e^{n+c_1(-A_i)} \tag{20.1.2}$$

with $\omega(A_i) \geq 0$ and its inverse

$$\sharp : QH_*(M) \to QH^*(M); \quad \sum b_j T^{\omega(B_j)} e^{n+c_1(B_j)} \to \sum PD(b_j) T^{\omega(-B_j)} e^{c_1(-B_j)} \tag{20.1.3}$$

with $\omega(B_j) \leq 0$. We denote by $a^\flat$ and $b^\#$ the images under these maps, respectively.

Let J_0 be a compatible almost-complex structure. Consider the moduli space $\mathcal{M}_3(J_0; \alpha)$ of J_0-holomorphic spheres with three marked points $\{z_0, z_1, z_2\} \subset S^2$ in class α and the evaluation maps

$$\mathrm{ev}_i : \mathcal{M}_3(J_0; \alpha) \to M, \quad i = 0,\ 1,\ 2.$$

20.1.1 The quantum intersection product

Let us first start with the homological version of the quantum product. The finiteness assumption in the definition of the Novikov ring $\Lambda^{op}_{0,\omega}$ allows one to enumerate the elements $B_j \in \mathrm{supp}(f)$ for $f = \sum_{B \in \Gamma} a_B q^B$ so that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_j > \cdots \to -\infty$$

with $\lambda_j = -\omega(B_j)$.

For any given two f_0, $g_0 \in H_*(M; \mathbb{Q})$, we take singular cycles C_1, C_2 of M with their homology classes

$$[C_1] = f_0, \ [C_2] = g_0.$$

Here we recall the definition of the singular chain C of M.

Definition 20.1.1 A singular k-chain is a finite linear combination over $\mathbb{Z}$ of k-simplices $c_\ell : \Delta^k \to M$ that are continuous maps on the k standard simplex $\Delta^k \subset \mathbb{R}^k$. It is called a cycle if its boundary satisfies

$$\partial C = \sum_{\ell=1}^m \partial c_\ell = 0$$

as a singular chain. We say that C is C^k if each simplex c_ℓ is C^k.

We denote by Q_1, Q_2 the union of the domain simplices of C_1, C_2, respectively, regarding them as a disjoint union of piecewise smooth manifolds. By slightly abusing the notation, we regard a smooth singular chain C as a piecewise smooth map $C : Q \to M$ that satisfies the obvious compatibility condition between the maps on the simplices comprising Q, which arises from the structure of the given singular chain C.

We take the fiber product

$$\overline{\mathcal{M}}_3(J_0;\alpha)_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times_{C_1\times C_2} (Q_1 \times Q_2). \tag{20.1.4}$$

By definition, this is compact if Q_1, Q_2 are so. The virtual dimension of $\overline{\mathcal{M}}_3(J_0;\alpha)$ is

$$(2n + 2c_1(\alpha)) + 6 - 6 + \dim Q_1 + \dim Q_2 - 4n = 2c_1(\alpha) + \dim Q_1 + \dim Q_2 - 2n.$$

The following is the key proposition which we will prove in the next section.

Proposition 20.1.2 *Suppose that (M,ω) is semi-positive. Then the fiber product* (20.1.4) *defines an integral cycle for any given integral cycles Q_1, Q_2 for a generic choice of J_0. Furthermore, its associated homology class does not depend on the choice of cycles Q_1, Q_2 as long as they represent the same homology classes f_0, $g_0 \in H_*(M,\mathbb{Z})$, respectively.*

Remark 20.1.3 At this point, one might hope that $\mathcal{M}_3(J_0;\alpha)$ is an open dense subset of $\overline{\mathcal{M}}_3(J_0;\alpha)$ and that the complement $\overline{\mathcal{M}}_3(J_0;\alpha) \setminus \mathcal{M}_3(J_0;\alpha)$ is the union of a finite number of strata, each of which has codimension at least 2. In particular, $\partial\overline{\mathcal{M}}_3(J_0;\alpha) = 0$. This would immediately give the proof of this proposition by a generic transversality–intersection argument. However, one subtle point to handle in taking the fiber product is the presence of multiple covers in the moduli space which obstructs such a statement. The upshot of the semi-positivity hypothesis is then that such multiple covers can be avoided in taking this fiber product to produce a homology class out of it, as we will indicate in the proof of the proposition.

Now we take the summation over $\alpha \in H_2(M;\mathbb{Z})/\mathrm{T}$ or of

$$\sum_\alpha \left[\overline{\mathcal{M}}_3(J_0;\alpha)_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times_{(C_1,C_2)} (Q_1 \times Q_2), ev_0\right] T^{-\omega(\alpha)} e^{n-c_1(\alpha)},$$

which defines a cycle of dimension

$$\dim Q_1 + \dim Q_2 - 2n \tag{20.1.5}$$

with $\Lambda_{0,\omega}^{op}$ as its coefficients.

Definition 20.1.4 (Quantum intersection product) We define the quantum intersection product of f, $g \in H_*(M)$ to be the homology class, again denoted, as above, by

$$f * g = \sum_\alpha [\mathcal{M}_3(J_0;\alpha)_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times_{(C_1,C_2)} (Q_1 \times Q_2), \mathrm{ev}_0]\, T^{-\omega(\alpha)} e^{n-c_1(\alpha)}. \tag{20.1.6}$$

Extend this product linearly over $\Lambda_{0,\omega}^{op}$ to arbitrary elements $f, g \in H_*(M) \otimes \Lambda_{0,\omega}^{op}$. We denote by $QH_*(M)$ the module $H_*(M) \otimes \Lambda_{0,\omega}^{op}$ equipped with this product, and call it a *quantum homology ring*.

The product has degree $-2n$ on $QH_*(M)$, i.e., we define maps

$$QH_k(M) \times QH_\ell(M) \to QH_{k+\ell-2n}(M) \tag{20.1.7}$$

for all k, ℓ. Equivalently, it has degree 0 on the degree shift $QH_*(M)[n]$, where $QH_*(M)[n]$ is the degree shift by n of $QH_*(M)$, i.e., defined by $QH_k(M)[n] := QH_{k+n}(M)$.

20.1.2 The quantum cup product and quantum cohomology

We assume Proposition 20.1.2 for the following discussion.

For two singular cohomology classes $a, b \in H^*(M)$, we choose cycles Q_1, Q_2 representing $PD(a)$, $PD(b)$, respectively.

Definition 20.1.5 (Quantum cup product) We define

$$\begin{aligned} a \cup_Q b &= \left[\sum_\alpha \left[\overline{\mathcal{M}}_3(J_0; \alpha)_{(\mathrm{ev}_1, \mathrm{ev}_2)} \times_{(C_1, C_2)} (Q_1 \times Q_2), \mathrm{ev}_0 \right] T^{-\omega(\beta)} e^{n - c_1(\alpha)} \right]^\# \\ &= \sum_\alpha PD \left[\overline{\mathcal{M}}_3(J_0; \alpha)_{(\mathrm{ev}_1, \mathrm{ev}_2)} \times_{(C_1, C_2)} (Q_1 \times Q_2), \mathrm{ev}_0 \right] T^{\omega(\alpha)} e^{c_1(\alpha)} \end{aligned} \tag{20.1.8}$$

and extend the definition to $H^*(M) \otimes \Lambda_{0,\omega}$ linearly over $\Lambda_{0,\omega}$. We denote by $QH^*(M)$ the $\Lambda_{0,\omega}$ module $H^*(M) \otimes \Lambda_{0,\omega}$ equipped with this product.

It is easy to check that (20.1.5) implies that

$$\deg a \cup_Q b = \deg a + \deg b$$

and so the product $\cup_Q$ has degree 0 on $QH^*(M)$,

$$* : QH^k(M) \times QH^\ell(M) \to QH^{k+\ell}(M).$$

There is also the de Rham version of the quantum cup product. The de Rham version is more useful for the exposition purpose of the theory because it allows a more direct intuitive formulation of the relevant Gromov–Witten invariants, especially in the algebraic geometric context. We briefly explain this construction without delving into the details of the proof.

Let $a,\ b \in H^*(M;\mathbb{Z})$. We represent them by closed differential forms $\omega_1,\ \omega_2$. Then consider the differential form

$$(\mathrm{ev}_0^\alpha)_!(\mathrm{ev}_1^\alpha, \mathrm{ev}_2^\alpha)^*(a \times b),$$

where $\mathrm{ev}_i^\alpha : \overline{\mathcal{M}}_3(J_0;\alpha) \to M$ is the evaluation map and $(\mathrm{ev}_0^\alpha)_!$ is the integration over the fiber. The following proposition is the key proposition to prove in this de Rham context. Its proof is beyond the scope of this book and thus omitted. We refer the reader to (Ru99), (Fu06) for the application of this approach.

Proposition 20.1.6 *Suppose that (M,ω) is semi-positive. Let $\eta_1,\ \eta_2$ be closed differential forms representing $a,\ b \in H^*(M;\mathbb{Z})$. Then, for a generic choice of J_0, the form $(\mathrm{ev}_0^\alpha)_!(\mathrm{ev}_1^\alpha, \mathrm{ev}_2^\alpha)^*(\eta_1 \times \eta_2)$ defines a smooth differential form on M that is closed. Again its de Rham cohomology class depends only on the de Rham cohomology classes of $\omega,\ \eta$.*

Then the quantum cup product is represented by the sum

$$a \cup_Q b = \sum_{\alpha \in H_2(M)} [(\mathrm{ev}_0^\alpha)_!(\mathrm{ev}_1^\alpha, \mathrm{ev}_2^\alpha)^*(\eta_1 \times \eta_2)] T^{\omega(\alpha)} e^{c_1(\alpha)} \tag{20.1.9}$$

in this de Rham formulation. As pointed out by Ruan [Ru96], there is a considerable technical difficulty in this cohomological approach.

20.1.3 Construction of the quantum product

In this section, we prove Proposition 20.1.2. We refer to [Ru96], [MSa04] for complete discussion on its proof.

Let $C_i : Q_i \to M$, $i = 1, 2$ be the given cycles. Consider the map

$$\begin{aligned} \Upsilon &: \mathcal{F}_3^{k,p}(S^2, M;\alpha) \times \mathcal{J}_\omega \times (Q_1 \times Q_2) \to \mathcal{H}''(\alpha) \times (M \times M) \times (M \times M) \\ \Upsilon&(v, (z_0, z_1, z_2), J) = \left(\overline{\partial}_J v, (v(z_1), v(z_2)), (C_1(q_1), C_2(q_2))\right), \end{aligned} \tag{20.1.10}$$

where $\mathcal{F}_3^{k,p}(S^2, M;\alpha)$ is the set of $L^{k,p}$-maps from S^2 to M in class α and $\mathcal{H}''(\alpha)$ is the fiber bundle

$$\mathcal{H}''(\alpha) = \bigcup_{(v,J)} \mathcal{H}''_{v,J}(\alpha) \to \mathcal{F}^{k,p}(S^2, M;\alpha)$$

with $\mathcal{H}''_{v,J}(\alpha) = L^{k-1,p}(\Lambda_J^{(0,1)}(v^*TM))$. We recall that the map $v \mapsto \overline{\partial}_J(v)$ is a smooth map, but the evaluation map $\mathrm{ev}_i(v) = v(z_i)$ is only a C^{k-2}-map. We choose the integer $k = k(\alpha)$ so large that the evaluation map is in the range where we can apply the Sard–Smale theorem, Theorem 10.1.7, i.e., we choose

$$k - 2 > 2c_1(\alpha) + 2n + 6 - 6 + 4n, \quad \text{i.e., } k > 2c_1(\alpha) + 6n + 2.$$

By the mapping and evaluation transversality, Theorems 10.4.1 and 10.5.2, Υ is stratawise transversal to

$$o_{\mathcal{H}''(\alpha)} \times \Delta_{13} \times \Delta_{24},$$

when restricted to

$$\mathcal{F}_3^{\mathrm{inj}}(\alpha)_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times_{(C_1,C_2)} (Q_1 \times Q_2),$$

at each point

$$(v, (z_0, z_1, z_3), J, q_1, q_2) \in \mathcal{M}_3^{\mathrm{inj}}(\alpha) \times Q_1 \times Q_2,$$

i.e., at those satisfying

$$\overline{\partial}_J v = 0,\ v(z_1) = C_1(q_1),\ v(z_2) = C_2(q_2) \tag{20.1.11}$$

and $(v, J) \in \mathcal{M}^{\mathrm{inj}}(\alpha)$. Here Δ_{ij} is the diagonal of M^4 for which the ith and jth factors coincide. More precisely speaking, Υ is transversal to the latter on

$$\mathcal{F}_3^{\mathrm{inj}}(\alpha)_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times_{(C_1,C_2)} (\Delta_1 \times \Delta_2),$$

for each pair (Δ_1, Δ_2) of simplices $\Delta_1 \subset Q_1$ and $\Delta_2 \subset Q_2$.

Now it remains to handle those points satisfying (20.1.11) but for which $v \in \mathcal{M}_3(J; \alpha) \setminus \mathcal{M}_3^{\mathrm{inj}}(J; \alpha)$. By the structure theorem, Theorem 8.7.3, any such v is factorized into

$$v = \widetilde{v} \circ \phi,$$

where $\widetilde{v}$ is somewhere injective and $\phi : S^2 \to S^2$ is a (branched) covering of degree $\ell \geq 2$. Obviously we have $\alpha = \ell\widetilde{\alpha}$ for $\widetilde{\alpha} = [\widetilde{v}]$ and so $c_1(\alpha) = \ell c_1(\widetilde{\alpha})$. By the semi-positive assumption, we may assume $c_1(\widetilde{\alpha}) > 0$, which in turn implies

$$\mathrm{vir.dim}\, \mathcal{M}_3(J; \alpha) = 2n + 2c_1(\alpha) \geq 2n + 2c_1(\widetilde{\alpha}) + 2 \geq \mathrm{vir.dim} \mathcal{M}_3(J; \widetilde{\alpha}) + 2. \tag{20.1.12}$$

This in particular implies that the image of the map

$$(\mathcal{M}_3(\alpha) \setminus \mathcal{M}_3^{\mathrm{inj}}(\alpha))_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times_{(C_1,C_2)} (Q_1 \times Q_2)$$

is contained in that of

$$\mathcal{M}_3(\widetilde{\alpha})_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times_{(C_1,C_2)} (Q_1 \times Q_2),$$

which has its dimension at least 2 smaller than that of

$$\mathcal{M}_3^{\text{inj}}(\alpha)_{(\text{ev}_1,\text{ev}_2)} \times_{(C_1,C_2)} (Q_1 \times Q_2).$$

Now we are ready to prove the following triangulability, which will finish the proof of the proposition.

Lemma 20.1.7 *Let $\Pi_\alpha : \mathcal{M}_3(\alpha) \to \mathcal{J}_\omega$ be the projection given in Proposition 10.4.7. There exists a dense subset of $\mathcal{J}_\omega$ such that for any J_0 therein the intersection*

$$\Upsilon^{-1}(o_{\mathcal{H}''}(\alpha) \times \Delta_{13} \times \Delta_{24}) \cap \Pi_\alpha^{-1}(J_0) \cong \mathcal{M}_3(J_0;\alpha)_{(\text{ev}_1,\text{ev}_2)} \times_{(C_1,C_2)} (Q_1, Q_2)$$

is triangulable. Denote this intersection by Q_0 and define a map $C_0 : Q_0 \to M$ by

$$C_0 = \text{ev}_0 \circ \pi_{\mathcal{M}_3}|_{Q_0} : Q_0 \to M,$$

where

$$\pi_{\mathcal{M}_3} : \mathcal{M}_3(J_0;\alpha)_{(\text{ev}_1,\text{ev}_2)} \times_{(C_1,C_2)} (Q_1, Q_2) \to \mathcal{M}_3(J_0;\alpha)$$

is the natural projection. Then C_0 defines a piecewise smooth cycle in M.

Proof We consider the map

$$\overline{C}_0 : \overline{\mathcal{M}}_3(J;\alpha)_{(\text{ev}_1,\text{ev}_2)} \times_{(C_1,C_2)} (Q_1 \times Q_2) \to M,$$

where $\overline{C}_0 := \text{ev}_0 \circ \pi_{\overline{\mathcal{M}}_3}|_{Q_0}$. From the discussion above, we have

$$\text{Image}\,\overline{C}_0 = \text{Image}\,C_0|_{\mathcal{M}_3^{\text{inj}}(J;\alpha)\times Q_1 \times Q_2} \tag{20.1.13}$$

modulo chains of codimension at least two.

Once we have proved this equality, the piecewise smoothness of the fiber product

$$\mathcal{M}_3(J;\alpha)_{(ev_1,ev_2)} \times_{(C_1,C_2)} (Q_1 \times Q_2)$$

implies the triangulability thereof.

Finally we compute the boundary of the chain (Q_0, C_0). Again, by the discussion above,

$$\partial C_0 = \partial[\mathcal{M}_3^{\text{inj}}(\alpha)_{(\text{ev}_1,\text{ev}_2)} \times_{(C_1,C_2)} (Q_1 \times Q_2), \text{ev}_0].$$

But, since all the factors in this fiber product are piecewise smooth, the general boundary formula for the fiber product gives rise to

$$\begin{aligned}\partial C_0 = {} & (\partial \mathcal{M}_3^{\text{inj}}(\alpha))_{(\text{ev}_1,\text{ev}_2)} \times_{(C_1,C_2)} (Q_1 \times Q_2) \\ & \pm \mathcal{M}_3^{\text{inj}}(\alpha)_{(\text{ev}_1,\text{ev}_2)} \times_{(C_1,C_2)} (\partial Q_1 \times Q_2) \\ & \pm \mathcal{M}_3^{\text{inj}}(\alpha)_{(\text{ev}_1,\text{ev}_2)} \times_{(C_1,C_2)} (Q_1 \times \partial Q_2) = 0,\end{aligned}$$

where the vanishing follows since $\partial Q_i = 0$ and $\partial \mathcal{M}_3^{\mathrm{inj}}(\alpha) = 0$. This proves the cycle property of C_0. □

This now finishes the proof of Proposition 20.1.2. □

Remark 20.1.8 As indicated by the above proof of Proposition 20.1.2, the construction of the quantum product on general symplectic manifolds other than the semi-positive ones will have to deal with the negative multiple cover problem since the above argument crucially uses the inequality (20.1.12) which is no longer true in general. To overcome this difficulty, one again has to involve the machinery of Kuranishi structure and virtual fundamental cycle techniques. We refer the reader to (FOn99), (LT99) for the details of such constructions.

20.2 Hamiltonian fibrations with prescribed monodromy

In this section, we explain the so-called *pants product* in Floer homology.

Recall the definition of positive and negative *punctures with analytical coordinates* of a compact Riemann surface Σ as defined in Section 4.3.1. By definition, each puncture p carries an analytic chart

$$\varphi^+ : D \setminus \{p\} \to (-\infty, 0] \times S^1$$

or

$$\varphi^- : D \setminus \{p\} \to [0, \infty) \times S^1$$

depending on the parity of the puncture p. We denote by (τ, t) the standard cylindrical coordinates on the cylinder. We fix a cut-off function $\rho^+ : (-\infty, 0] \to [0, 1]$ defined by

$$\rho^+ = \begin{cases} 1, & \tau \le -2, \\ 0, & \tau \ge -1 \end{cases}$$

and $\rho^- : [0, \infty) \to [0, 1]$ by $\rho^-(\tau) = \rho^+(-\tau)$. We will just denote by ρ either of these two cut-off functions when there is no danger of confusion. Let $(\mathfrak{p}, \mathfrak{q})$ be a given set of positive punctures $\mathfrak{p} = \{\mathfrak{p}_g, \ldots, \mathfrak{p}_{\mathfrak{k}}\}$ and negative punctures $\mathfrak{q} = \{\mathfrak{q}_g, \ldots, \mathfrak{q}_\ell\}$ on Σ.

On completing $\dot\Sigma = \Sigma \setminus (\mathfrak{p} \cup \mathfrak{q})$ by the *real blow-up* at each puncture, we obtain a compact Riemann surface with boundary $\overline{\Sigma}$

$$\partial \overline{\Sigma} = \coprod_{* \in \mathfrak{p} \cup \mathfrak{q}} S^1_*$$

as defined in Section 4.3.1. Each boundary component S^1_* carries an identification $\theta : S^1_* \to \mathbb{R}/\mathbb{Z}$ and thus a distinguished marked point corresponding to the image $\theta_*(0) \in S^1_*$.

Definition 20.2.1 Let $\dot\Sigma$ be a punctured Riemann surface with $h \geq 1$ and $\overline{\Sigma}$ its real blow-up. We define the *holonomy at the punctures* of the marked Hamiltonian fibration $E \to \dot\Sigma$ by the holonomy $C = (C_{H_1}, \ldots, C_{H_h})$ of a marked Hamiltonian fibration with connection (E, Ω, C).

Definition 20.2.2 We call a one-form $K \in \Omega^1(\dot\Sigma, C_0^\infty(M))$ *cylindrical* at the puncture $p \in \Sigma$ with analytic chart (D, z) if it has the form

$$K(\tau, t, x) = H(t, x)dt$$

in some punctured neighborhood $D' \setminus \{p\} \subset D \setminus \{p\}$ with respect to the given analytic coordinate $z = e^{2\pi(\tau+it)}$ or $z = e^{-2\pi(\tau+it)}$ depending on the parity of the puncture. We denote by $\mathcal{K}_{\dot\Sigma}$ the set of such Ks.

Every such K give rise to a Hamiltonian fibration on the trivial bundle $E = \dot\Sigma \times (M, \omega)$ with the coupling form given by

$$\Omega = \pi_2^*\omega + d(\pi_1^*K).$$

The horizontal distribution of the associated Hamiltonian connection has the explicit form

$$\Gamma_\Omega(z, x) = \{(\eta, X_{K(\eta)}) \in T_z\dot\Sigma \oplus T_xM\}. \tag{20.2.14}$$

One important quantity associated with the one-form K is the curvature two-form which is denoted by R_K and defined by

$$R_K(\xi_1, \xi_2) = \xi_1[K(\xi_2)] - \xi_2[K(\xi_1)] - \{K(\xi_2), K(\xi_1)\} \tag{20.2.15}$$

for two vector fields ξ_1, ξ_2, where $\xi_1[K(\xi_2)]$ denotes the directional derivative of the function $K(\xi_2)(z, x)$ with respect to the vector field ξ_1 as a function on Σ, holding the variable $x \in M$ fixed.

The following topological energy is defined in the off-shell level which will control the geometric energy in the on-shell level later.

Definition 20.2.3 (Topological energy) Let Ω_K be the coupling form associated with the Hamiltonian fibration over $\dot\Sigma$ associated with K. Then we define the topological energy by the integral

$$E_K(u) = \int_{\dot\Sigma} v^*\Omega_K$$

for a map $u : \dot{\Sigma} \to M$ if the integral converges, where $v : \dot{\Sigma} \to E$ is the section lifted from the map u.

Now we relate this topological energy to the action integrals of the asymptotic orbits at the punctures or along $\partial\overline{\Sigma}$.

Note that, for a given set of asymptotic orbits $\vec{z}$, one can define the space of maps $u : \dot{\Sigma} \to M$ which can be extended to $\overline{\Sigma}$ such that $u \circ \theta_* = z_*(t)$ for $* \in \mathfrak{p} \cup \mathfrak{q}$. Each such map defines a natural homotopy class B relative to the boundary. We denote the corresponding set of homotopy classes by $\pi_2(\vec{z})$. When we are given the additional data of bounding discs w_* for each z_*, we can form a natural homology class (in fact a homotopy class), denoted by $B\#\left(\coprod_{*\in\mathfrak{p}\cup\mathfrak{q}}[w_*]\right) \in H_2(M)$, by 'capping-off' the boundary components of B using the corresponding discs w_*. As usual, we denote such a lifting of a periodic orbit z by $[z, w]$, where $w : D^2 \to M$ is a disc bounding the loop z.

Definition 20.2.4 Let $\{[z_*, w_*]\}_{*\in\mathfrak{p}\cup\mathfrak{q}}$ be given. We say $B \in \pi_2(\vec{z})$ is *admissible* to $\{[z_*, w_*]\}$ if it satisfies

$$B\#\left(\coprod_{*\in\mathfrak{p}\cup\mathfrak{q}}[w_*]\right) = 0 \quad \text{in } H_2(M, \mathbb{Z}), \tag{20.2.16}$$

where

$$\# : \pi_2(\vec{z}) \times \prod_{*\in\mathfrak{p}\cup\mathfrak{q}} \pi_2(z_*) \to H_2(M, \mathbb{Z})$$

is the natural gluing operation of the homotopy class from $\pi_2(\vec{z})$ and those from $\pi_2(z_*)$ for $* \in \mathfrak{p} \cup \mathfrak{q}$.

The following formula is a direct generalization of Lemma 4.1 (Schw00).

Lemma 20.2.5 *Let $u : \dot{\Sigma} \to M$ be a map satisfying*

$$[u]\#\left(\coprod_{*\in\mathfrak{p}\cup\mathfrak{q}}[w_*]\right) = 0. \tag{20.2.17}$$

Then

$$-\int v^*\Omega_E = \sum_{p\in\mathfrak{p}} \mathcal{A}_{H_p}([z_p, w_p]) - \sum_{q\in\mathfrak{q}} \mathcal{A}_{H_q}([z_q, w_q]), \tag{20.2.18}$$

with $v(z) = (z, u(z))$, where H_p, H_q are the Hamiltonians attached to the respective punctures.

Proof By Lemma 4.3.4, we have

$$\int_{D^2} v_*^*\Omega_* = -\mathcal{A}_{H_*}([z_*, w_*]) \tag{20.2.19}$$

for any $* \in \mathfrak{p} \cup \mathfrak{q}$. On the other hand, by the topological condition

$$[u]\# \coprod_{*\in\mathfrak{p}\cup\mathfrak{q}} [w_*] = 0$$

we obtain

$$\int v^*\Omega_E + \sum_{p\in\mathfrak{p}} \mathcal{A}_{H_p}([z_p, w_p]) - \sum_{q\in\mathfrak{q}} \mathcal{A}_{H_q}([z_q, w_q]) = 0, \tag{20.2.20}$$

which is equivalent to (20.2.18). □

20.2.1 Perturbed Cauchy–Riemann equations

Now we describe the Hamiltonian perturbations in a coordinate-free fashion. Denote by

$$\mathcal{P}^{\mathrm{cyl}}(\dot\Sigma, \mathcal{J}_\omega)$$

the set of maps from $\dot\Sigma$ to $\mathcal{J}_\omega$ that are cylindrical near the punctures in terms of the given analytic charts $z = e^{\pm 2\pi(\tau+it)}$. In other words, we require

$$J(z, x) \equiv J(t, x), \quad z \equiv e^{\pm 2\pi(\tau+it)} \tag{20.2.21}$$

independently of τ with $|\tau| \geq R$ for some R that is sufficiently large.

For each given such pair (K, J), which we call a Floer datum, one defines a perturbed Cauchy–Riemann operator by

$$\overline\partial_{(K,J)} u := (du + P_K(u))^{(0,1)}_{(j,J)} = \frac{1}{2}(du + P_K(u) + J \circ (du + P_K(u)) \circ j),$$

where the vector-valued one-form $P_K \in \Omega^1(\Sigma \times M, TM)$ is the unique form determined by the equation

$$d^{\mathrm{vert}} K(z, x) = P_K(z, x) \rfloor \Omega\Big|_{T^v_{(z,x)}E}.$$

In particular, $P_K(u)$ is a section $\Omega^1(v^*T^vE) = \Omega^1(u^*TM)$. The associated Hamiltonian-perturbed Cauchy–Riemann equation has the form

$$(du + P_K(u))^{(0,1)}_{(j,J)} = 0 \quad \text{or equivalently } \overline\partial_{(j,J)} u + (P_K(u))^{(0,1)}_{(j,J)} = 0 \tag{20.2.22}$$

on (Σ, j) in general.

For each given Floer datum (K, J) and a collection $\vec{z} = \{z_*\}_{*\in\mathfrak{p}\cup\mathfrak{q}}$ of asymptotic periodic orbits z_* associated with the punctures $* = p_i$ or $* = q_j$, we consider the perturbed Cauchy–Riemann equation

$$\begin{cases} \overline{\partial}_{(K,J)}(u) = 0, \\ u(\infty_*, t) = z_*(t). \end{cases} \tag{20.2.23}$$

To provide the definition of the Hamiltonian-perturbed moduli space, one more ingredient we need is the choice of an appropriate energy of the map u. For this purpose, we fix a Kähler metric $h_{\dot\Sigma}$ of $(\dot\Sigma, j)$ that has the cylindrical ends with respect to the given cylindrical coordinates near the punctures, i.e., h_Σ has the form

$$h_\Sigma = d\tau^2 + dt^2 \tag{20.2.24}$$

on $D_* \setminus \{*\}$. We denote by $dA_{\dot\Sigma}$ the corresponding area element on $\dot\Sigma$.

Definition 20.2.6 An almost complex structure $\widetilde{J}$ on $\pi : E \to \dot\Sigma$ is called (π, Ω)-compatible if the following statements hold.

(1) $\widetilde{J}$ preserves the vertical tangent space and is Ω-compatible on each fiber,
(2) The projection $\pi : E \to \dot\Sigma$ is pseudoholomorphic, i.e., $d\pi \circ j = \widetilde{J} \circ d\pi$.

When we are given t-periodic Hamiltonian $H = (H_1, \ldots, H_h)$, we say that $\widetilde{J}$ is (H, J)-compatible, if $\widetilde{J}$ additionally satisfies the following statement.

(3) For each i, $(\Phi_i)_*\widetilde{J} = j \oplus J_{H_i}$, where

$$J_{H_i}(\tau, t, x) = (\phi^t_{H_i})^* J$$

for some t-periodic family of almost-complex structure $J = \{J_t\}_{0\le t\le 1}$ on M over a disc $D'_i \subset D_i$ in terms of the cylindrical coordinates.

The condition (3) implies that a $\widetilde{J}$-holomorphic section v over D'_i is precisely a solution of the equation

$$\frac{\partial u}{\partial \tau} + J_t\left(\frac{\partial u}{\partial t} - X_{H_i}(u)\right) = 0 \tag{20.2.25}$$

if we write $v(\tau, t) = (\tau, t, u(\tau, t))$ in the trivialization with respect to the cylindrical coordinates (τ, t) on D'_i induced by $\phi_i^\pm$ above.

The following geometric energy controls the analysis of Floer moduli spaces, especially their compactness properties.

Definition 20.2.7 (Geometric energy) For a given asymptotically cylindrical pair (K, J), we define

$$E_{(K,J)}(u) = \frac{1}{2}\int_\Sigma |du + P_K(u)|_J^2 \, dA_\Sigma,$$

where $|\cdot|_{J(\sigma,u(\sigma))}$ is the norm of $\Lambda^{(0,1)}(u^*TM) \to \dot\Sigma$ induced by the metrics $h_{\dot\Sigma}$ and $g_J := \omega(\cdot, J\cdot)$.

Note that this energy depends only on the conformal class of h_Σ, i.e., it depends only on the complex structure j of Σ and is restricted to the standard energy for the usual Floer trajectory moduli space given by

$$E_{(K,J)}(u) = \frac{1}{2}\int_{C_*}\left(\left|\frac{\partial u}{\partial \tau}\right|_J^2 + \left|\frac{\partial u}{\partial t} - X_{H_*}(u)\right|_J^2\right) dt\, d\tau$$

in the cylindrical coordinates (τ, t) on the cylinder C_* corresponding to the puncture $*$.

Now we are ready to give the definition of the Floer moduli spaces. Consider the triple

$$\mathfrak{p} = \{p_1, \ldots, p_{s_+}\}, \qquad \mathfrak{q} = \{q_1, \ldots, q_{s_-}\}, \qquad \mathfrak{r} = \{r_1, \ldots, r_{s_0}\}$$

of positive punctures, negative punctures and marked points, respectively. We assume that they are all distinct points and that each puncture is equipped with an analytic chart. We denote by

$$\widetilde{\mathcal{M}}_{(s_0;s_+,s_-)}$$

the set of all such triples and by $\mathcal{M}_{(s_0;s_+,s_-)}$ the quotient space defined by the action of automorphisms of the punctured Riemann surface with marked points. We call a triple $(\mathfrak{r}; \mathfrak{p}, \mathfrak{q})$ *stable* if it has a finite automorphism group. The space $\mathcal{M}_{(s_0;s_+,s_-)}$ is nonempty as long as $s_0 + s_+ + s_- \geq 3$.

Definition 20.2.8 Let (K, J) be a Floer datum over Σ with punctures $\mathfrak{p}$, $\mathfrak{q}$, and let $\{[z_*, w_*]\}_{*\in\mathfrak{p}\cup\mathfrak{q}}$ be the given asymptotic orbits and $\vec{z} = \{z_*\}_{*\in\mathfrak{p}\cup\mathfrak{q}}$ a given set of asymptotic periodic orbits. Let $B \in \pi_2(\vec{z})$ be a homotopy class admissible to $\{[z_*, w_*]\}_{*\in\mathfrak{p}\cup\mathfrak{q}}$. We define the moduli space

$$\begin{aligned}\mathcal{M}_{(s_0;s_+,s_-)}(K, J; \vec{z}; B) = &\{(u; \mathfrak{r}; \mathfrak{p}, \mathfrak{q}) \mid u \text{ satisfies (20.2.23) and} \\ &E_{(K,J)}(u) < \infty\ [u] = B\}. \end{aligned} \tag{20.2.26}$$

To avoid having continuous automorphisms, we will always assume that the asymptotic Hamiltonian H at the puncture is *not* time-independent when we consider the moduli space corresponding to

$$(s_0; s_+, s_-) = (1; 1, 0) \text{ or } (1; 0, 1).$$

In addition, we will also assume that K and J satisfy

$$\begin{aligned} K &\equiv H(t, x)dt \\ J &\equiv J(t, x) \end{aligned} \tag{20.2.27}$$

independently of τ in the cylindrical coordinates (τ, t) near each puncture.

Now we define the moduli space $\mathcal{M}_{(s_0;s_+,s_-)}(K, J; \{[z_*, w_*]\}_*)$ to be the finite union of the moduli spaces $\mathcal{M}_{(s_0;s_+,s_-)}(K, J; \vec{z}; B)$ over $B \in \pi_2(\vec{z})$ satisfying

$$B\#\left(\coprod_{*\in\mathfrak{p}\cup\mathfrak{q}} [w_*]\right) = 0.$$

The following energy formula is an important ingredient not only for the analysis of Floer moduli spaces but also for the application of the Floer theory to the symplectic topology. The related identity has played an important role in much of the recent literature on modern symplectic topology. See (Schw00), (Oh05d), (Se03a) for a few earlier appearances of such an identity.

Proposition 20.2.9 *Assume that* (K, J) *satisfies* (20.2.27). *Let* $\{[z_*, w_*]\}_{*\in\mathfrak{p}\cup\mathfrak{q}}$ *be a given collection of asymptotic periodic orbits and let* u *have finite energy. Then we have the identity*

$$\begin{aligned} E_{(K,J)}(u) &= \int v^*\Omega_E + \int_\Sigma R_K(v) \\ &= \sum_{i=1}^{k} \mathcal{A}_{H_{p_i}}([z_i^+, w_i^+]) - \sum_{j=1}^{\ell} \mathcal{A}_{H_{q_j}}([z_j^-, w_j^-]) + \int_\Sigma R_K(v). \end{aligned} \tag{20.2.28}$$

Proof It suffices to prove the first identity. The second then is a consequence of Lemma 20.2.5. Denote by $v : \Sigma \to E$ the section associated with the map u. Then we compute

$$E_{(K,J)}(u) = \int_{\dot\Sigma} |du + P_K(u)|_J^2 \, dA_\Sigma.$$

But we have

$$|du + P_K(u)|_J^2 = |du(e_1) + P_K(u)(e_1)|_J^2 + |du(e_2) + P_K(u)(e_2)|_J^2$$

for an orthonormal frame $\{e_1, e_2\}$ for TM with $e_2 = je_1$. Then $e_1 = -je_2$. We also have

$$\begin{aligned}
|(du + P_K(u))(e_1)|_J^2 &= \omega(du(e_1) + P_K(u)(e_1), J(du(e_1) + P_K(u)(e_1))) \\
&= -\omega(du(e_1) + P_K(u)(e_1), J(du + P_K(u))(je_2)) \\
&= \omega(du(e_1) + P_K(u)(e_1), (du + P_K(u))(e_2)) \\
&= \omega(du(e_1), du(e_2)) + \omega(du(e_1), P_K(u)(e_2)) \\
&\quad + \omega(P_K(u)(e_1), du(e_2)) + \omega(P_K(u)(e_1), P_K(u)(e_2)) \\
&= \omega(du(e_1), du(e_2)) + R_K(e_1, e_2).
\end{aligned}$$

Similarly, we obtain

$$|(du + P_K(u))(e_2)|_J^2 = |(du + P_K(u))(e_1)|_J^2 = \omega(du(e_1), du(e_2)) + R_K(u)(e_1, e_2)$$

and so

$$\frac{1}{2}|du + P_K(u)|_J^2 \, dA_\Sigma = v^*\omega + R_K(v).$$

By integrating this, we have finished the proof of the first identity. The second identity follows from the first by Lemma 20.2.5. □

Remark 20.2.10 Here we remark that the last curvature integral converges since $R_K(u)$ has compact support by the hypothesis (20.2.27) that K is cylindrical near the ends of $\dot{\Sigma}$.

The following special case is important later in relation to the proof of homotopy invariance of spectral invariants.

Corollary 20.2.11 *Suppose that H is the family of functions $H = H(s,t,x)$, where $H^s := H(s,\cdot,\cdot) : [0,1] \times M \to \mathbb{R}$ is the Hamiltonian generating the vector field*

$$\frac{\partial \phi}{\partial t} \circ \phi^{-1}$$

of a two-parameter family $\phi = \phi(s,t)$ of Hamiltonian diffeomorphisms. Consider the one-form K on $\mathbb{R} \times S^1 \cong S^2 \setminus \{N, S\}$ defined by

$$K(\tau, t, x) = H(\rho(\tau), t, x)dt$$

defined on the trivial bundle $E = \mathbb{R} \times S^1 \times (M, \omega)$. Then we have $E_{(K,J)}(v) = \int v^*\Omega_E$ where $v(\tau, t) = (\tau, t, u(\tau, t))$.

Proof By Proposition 20.2.9, we need only prove that $R_K \equiv 0$. Denote by F the family of functions $F = F(s,t,x)$, where $F_s := F(\cdot, s, \cdot) : [0,1] \times M \to \mathbb{R}$ is the Hamiltonian generating the vector field

$$\frac{\partial \phi}{\partial s} \circ \phi^{-1}.$$

Then we have

$$R_K\left(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial t}\right) = \frac{\partial F^\rho}{\partial\tau} - \frac{\partial H^\rho}{\partial t} - \{H^\rho, F^\rho\},$$

where $H^\rho(\tau, t, x) = H(\rho(\tau), t, x)$, $F^\rho(\tau, t, x) = F(\rho(\tau), t, x)$. But H^ρ generates the two-parameter family

$$\phi^\rho(\tau, t) = \phi(\rho(\tau), t)$$

of Hamiltonian diffeomorphisms and so the identity (18.3.17) finishes the proof of $R_K \equiv 0$. □

20.2.2 The pants product

Consider a compact Riemann surface Σ of genus 0 with three marked points. Denote the neighborhoods of the three by D_i, $i = 1, 2, 3$. We assume that the associated punctures for $i = 1, 2$ are positive with analytic coordinates given by

$$\varphi_i^+ : D_i \setminus \{z_i\} \to (-\infty, 0] \times S^1 \quad \text{for } i = 1,\ 2$$

and that the puncture for $i = 0$ is negative with analytic coordinate given by

$$\varphi_3^- : D_3 \to [0, \infty) \times S^1.$$

Denote by (τ, t) the standard cylindrical coordinates on the cylinder

$$[0, \infty) \times S^1 = [0, \infty) \times \mathbb{R}/\mathbb{Z}$$

and fix a cut-off function $\rho^+ : (-\infty, 0] \to [0, 1]$ and $\rho^- : [0, \infty) \to [0, 1]$ by $\rho^-(\tau) = \rho^+(-\tau)$ as before.

Let $[z_i, w_i]$ be a critical point of $\mathcal{A}_{H_i}$ for $i = 1, 2$. To simplify the notation, we denote

$$\widehat{z} = [z, w]$$

in general and $\widehat{z} = (\widehat{z}_1, \widehat{z}_2, \widehat{z}_3)$. The matrix element of the pants product $\widehat{z}_1 * \widehat{z}_2$ is provided by a certain moduli space of perturbed pseudoholomorphic maps from a sphere with three directed punctures that satisfy suitable asymptotic conditions. We now give the precise definition of this moduli space.

We consider a compact connected surface Σ of genus 0 with three directed punctures, with the first two being positive and the third negative. Then we take its real blow-up $\overline{\Sigma}$, where $\vec{z} = \{z_1, z_2, z_3\}$. By virtue of the definition of the directed puncture, $\dot{\Sigma}$ carries analytic coordinates (τ, t) in a neighborhood of the puncture z_i for each $i = 1, 2, 3$. Now we *fix* a Hamiltonian fibration over S with a prescribed Hamiltonian connection on $\partial\overline{\Sigma}$ induced by normalized

Hamiltonians H_i, $i = 1, 2, 3$. We fix the trivial Hamiltonian fibration $E_i = D^2 \times (M, \omega)$ equipped with the Hamiltonian connection represented by the coupling two-form

$$\Omega_i := \pi_2^* \omega + \pi_1^* d(H_i^\rho \, dt).$$

In particular the holonomies along ∂S with respect to the connection lie in the conjugacy class $C = ([\phi_{H_1}^1], [\phi_{H_2}^1], [\phi_{H_3}^1])$.

Now we are ready to define the moduli space relevant to the definition of the pants product that we need to use. Let $\widehat{z} = (\widehat{z}_1, \widehat{z}_2, \widehat{z}_3)$ be given, where $\widehat{z}_i = [z_i, w_i] \in \mathrm{Crit}\, \mathcal{A}_{H_i}$ for $i = 1, 2, 3$.

We consider $\widetilde{J}$ to be a (H, J)-compatible almost-complex structure. We denote by $\mathcal{M}(K, \widetilde{J}; \widehat{z})$ the space of all $u : \Sigma \to M$ that satisfy the following.

(1) $(du + P_K(u))^{(0,1)}_{\widetilde{J}} = 0$ for one-form K as in Definition 20.2.2.
(2) The maps $u_i := u \circ (\varphi_i^{-1}) : (-\infty, K_i] \times S^1 \to M$, which are solutions of (20.2.25), satisfy

$$\lim_{\tau \to -\infty} u_i(\tau, \cdot) = z_i, \quad i = 1, 2$$

and similarly for $i = 3$ changing $-\infty$ to $+\infty$.
(3) The closed surface obtained by capping off $u(\Sigma)$ with the discs w_i taken with the same orientation for $i = 1, 2$ and the opposite one for $i = 3$ represents zero in $\pi_2(M)/\sim$.

Definition 20.2.12 Consider two sets $\widehat{z} = (\widehat{z}_1, \ldots, \widehat{z}_k)$ and $\widehat{z}' = (\widehat{z}_1', \ldots, \widehat{z}_k')$. We say that $\widehat{z}' \sim \widehat{z}$ if they satisfy

$$z_i' = z_i, \quad w_i' = w_i \# A_i$$

for $A_i \in \pi_2(M)$ and $\sum_{i=1}^k A_i$ represents zero (mod) Γ.

Note that $\mathcal{M}(K, \widetilde{J}; \widehat{z})$ depends only on the equivalence class of the $\widehat{z}$s. We postpone the proof of the following index formula until Appendix C.2.

Proposition 20.2.13 *The (virtual) dimension of* $\mathcal{M}(K, \widetilde{J}; \widehat{z})$ *is given by*

$$\begin{aligned} \dim \mathcal{M}(K, \widetilde{J}; \widehat{z}) &= 2n - (-\mu_{H_1}(\widehat{z}_1) + n) - (-\mu_{H_2}(\widehat{z}_2) + n) - (\mu_{H_3}(\widehat{z}_3) + n) \\ &= -n + (\mu_{H_1}(\widehat{z}_1) + \mu_{H_2}(\widehat{z}_2) - \mu_{H_3}(\widehat{z}_3)). \end{aligned}$$

Note that, when $\dim \mathcal{M}(K, \widetilde{J}; \widehat{z}) = 0$, we have

$$n = -\mu_{H_3}(\widehat{z}_3) + \mu_{H_1}(\widehat{z}_1) + \mu_{H_2}(\widehat{z}_2),$$

which is equivalent to

$$\mu_{H_3}(\widehat{z}_3) = (\mu_{H_1}(\widehat{z}_1) + \mu_{H_2}(\widehat{z}_2)) - n$$

and provides the degree of the pants product in our convention of the grading of the Floer complex adopted in the present book. Now the pair-of-pants product $*$ for the chains is defined by

$$\widehat{z}_1 * \widehat{z}_2 = \sum_{\widehat{z}_3} \#(\mathcal{M}(K, \widetilde{J}; \widehat{z}))\widehat{z}_3 \tag{20.2.29}$$

for the generators $\widehat{z}_i$ and then by linearly extending over the chains in $CF_*(H_1) \otimes CF_*(H_2)$. Our grading convention makes this product of degree $-n$.

20.3 The PSS map and its isomorphism property

Let $f : M \to \mathbb{R}$ be a background Morse function on M and $H = H(t, x)$ and $J = J(t, x)$. The goal of the PSS map is to establish an isomorphism between the Morse homology of f and the Floer homology of (H, J).

One of the moduli spaces entering into the construction of the PSS map is the space of solutions of (20.2.23) on the domain of genus 0 with one puncture, which can be either positive or negative, and with one marked point playing the role of the origin of $\dot{\Sigma} \cong \mathbb{C}$. We first introduce the general moduli space of this type with an arbitrary number of positive, negative and marked points.

20.3.1 Definition of the PSS-map moduli spaces

In this subsection, we recall the definitions of the two PSS maps Φ and Ψ from (PSS96) except that we use Morse cycles of $-f$, instead of f, to represent the homology of M. In particular, the grading of Morse cycles is given by

$$\mathrm{Index}_{(-f)}(p) = 2n - \mathrm{Index}_f(p),$$

which is consistent with the author's convention used in papers such as (Oh05c)–(Oh06a) and (OhZ11b). As a result, we will construct an isomorphism $QH^*(M) \to HF_*(H, J)$ between quantum cohomology and Floer homology.

Let (C_*, ∂) be any chain complex on M whose homology is the singular homology $H_*(M)$. We take for C_* the Morse homology complex

$$\left(CM_*(-f), \partial^{\mathrm{Morse}}_{(-f, g_{J_0})}\right) \cong \left(CF_*(f), \partial^{\mathrm{Floer}}_{(f, J_0)}\right)$$

for $(-f, g_{J_0})$ with a sufficiently small C^2-norm $\|f\|_{C^2}$. In this convention, solutions with a negative gradient of $-f$ correspond to ones for the negative gradient flow of the action functional $\mathcal{A}_f$.

Let $\dot\Sigma_+$ be the Riemann sphere with one marked point o_+ and one positive puncture e_+. We identify $\dot\Sigma_+ \setminus \{o_+\} \cong \mathbb{R} \times S^1$ and denote by (τ, t) the corresponding coordinates so that $\{+\infty\} \times S^1$ corresponds to e_+. We note that the coordinate (τ, t) is defined modulo the action of $\mathbb{R} \times S^1 \cong \mathbb{C}^*$,

$$(\tau, t) \mapsto (\tau + a, t + b).$$

We consider the one-form $K_+ \in \Omega^1(\dot\Sigma, \mathrm{ham}(M, \omega))$ which can be written as

$$K_+ = \begin{cases} 0 & \text{near } o_+, \\ H_+(t, x)dt & \text{near } e_+ \end{cases} \tag{20.3.30}$$

for some H_+. Let $z_+ = z_+(t)$ $(t \in S^1)$ be a nondegenerate periodic orbit of H_+ and $[z_+, w_+]$ a lifting of z_+ to $\widetilde{\mathcal{L}}_0(M)$. We consider the moduli space

$$\begin{aligned}\mathcal{M}(K_+, J_+; [z_+, w_+]; A_+) = \Big\{u : \dot\Sigma \to M \mid \overline{\partial}_{(K_+,J_+)}u = 0,& \\ u(+\infty, t) = z_+(t), [u\#w_+] = A_+\Big\},&\end{aligned}$$

where $A_+ \in H_2(M, \mathbb{Z})$ is in the image of the Hurwitz map $\pi_2(M) \to H_2(M, \mathbb{Z})$.

For generic J_+ or K_+, the moduli space is regular and its dimension is equal to

$$\text{Index}\, D_u\overline{\partial}_{(K_+,J_+)} = n - \mu_{H_+}([z_+, w_+]) + 2c_1(A_+).$$

Similarly, we define the moduli space

$$\begin{aligned}\mathcal{M}(K_-, J_-; [z_-, w_-]; A_-) = \Big\{u : \dot\Sigma \to M \mid \overline{\partial}_{(K_-,J_-)}u = 0,& \\ u(-\infty, t) = z_-(t), [\overline{w}_-\#u] = A_-\Big\},&\end{aligned}$$

where A_- is similar to A_+, and have

$$\text{Index}\, D_u\overline{\partial}_{(K_-,J_-)} = n + \mu_{H_-}([z_-, w_-]) + 2c_1(A_-).$$

If we use the Morse homology of $-f$ to represent $H_*(M)$, then we can represent the quantum homology $QH_*(M)$ as the homology of $C_*(-f) \otimes \Lambda_\omega$, where $C_*(-f)$ is the chain complex of the Morse homology of $-f$ generated by the critical points of f. The grading of $[p]q^{-A}$ is $\mu_{(-f)}(p) - 2c_1(A)$, where $[p] \in C_*(-f)$, and $\mu_{(-f)}(p)$ is the Morse index of f at p.

We are going to define an isomorphism

$$\Phi_* : QH_k(M)[n]\ (= QH_{k+n}(M)) \to HF_k(H, J)$$

by first defining the corresponding chain map $\Phi : C_*(-f) \otimes \Lambda_\omega \to CF_*(H)$. This chain map is defined by describing how it acts on the generators $[p]$ of $C_*(-f)$ as

$$\Phi : [p] \to \sum_{[z_+,w_+]\in\widetilde{\mathrm{Per}}(H_+)} \#(\mathcal{M}(p,[z_+,w_+];A_+))[z_+,w_+]q^{-A_+}$$

and then linearly extending over the ring Λ_ω. Here, roughly speaking, the moduli space $\mathcal{M}(p,[z_+,w_+];A_+)$ consists of 'spike discs' emerging from the critical point p and ending on the periodic orbit z_+ in class $[u\#w_+] = A_+$ in Γ. More precisely, we have the definition

$$\begin{aligned}\mathcal{M}(p,[z_+,w_+];A_+) = \{(\chi_+,u_+) \mid u_+ : \dot\Sigma_+ \to M, [u_+\#w_+] = A_+,\\ u(+\infty,t) = z_+(t), \overline{\partial}_{(K_+,J_+)}u_+ = 0,\\ \dot\chi_+ = \nabla f(\chi_+), \chi_+(-\infty) = p, \chi_+(0) = u_+(o_+)\}.\end{aligned}$$

We put an index condition so that $\mathcal{M}(p,[z_+,w_+];A_+)$ is a zero-dimensional oriented manifold so that we can do an algebraic count '#'. The index condition is

$$n - \mu([z_+,w_+] + 2c_1(A_+)) + (2n - \mu(p)) - 2n = 0,$$

i.e.,

$$\mu([z_+,w_+]) = n - (\mu(p) - 2c_1(A_+)).$$

The gluing argument shows that ϕ is a chain map, so it passes to homology, which is the PSS map $\Phi_* : QH_*(M) \to HF_*(H,J)$ introduced in (Pi94), (PSS96).

Next we define the inverse of Φ

$$\Psi_* : HF_*(H,J) \to QH_*(M)$$

in the chain level. For any $[z_-,w_-] \in \widetilde{\mathrm{Per}}(K_-)$, define $\Psi : CF_*(M) \to C_*(-f) \otimes \Lambda_\omega$,

$$\Psi : [z_-,w_-] \to \sum_{p\in\mathrm{Crit}(-f);A_-\in\pi_2(M)} \#\mathcal{M}([z_-,w_-],p;A_-)p \otimes q^{-A_-}$$

(see Figure 20.1). Here $\mathcal{M}([z_-,w_-],p;A_-)$ consists of 'spiked discs' emerging from the periodic orbit z_- and ending on the critical point p, namely

$$\begin{aligned}\mathcal{M}([z_-,w_-],p;A_-) = \{(u_-,\chi_-) \mid u_- : \dot\Sigma_- \to M, [\overline{w}_-\#u_-] = A_-,\\ u(-\infty,t) = z_-(t),\ \overline{\partial}_{(K_-,J_-)}u_- = 0,\\ \dot\chi_- = \nabla f(\chi_-),\ \chi_-(+\infty) = p, \chi_-(0) = u_-(o_-)\}.\end{aligned}$$

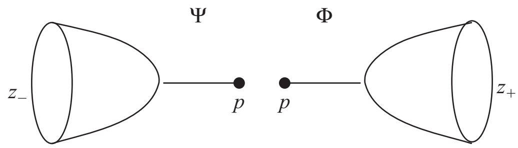

Figure 20.1 The PSS maps Ψ and Φ.

Here we also impose the index condition

$$\mu_{H_-}([z_-, w_-]) = n - (\mu_{(-f)}(p) - 2c_1(A_-))$$

so that $\mathcal{M}([z_-, w_-], p; A_-)$ becomes a zero-dimensional (orientable) manifold. The same continuation map argument shows that Ψ is a chain map and thus induces a homomorphism in homology.

20.3.2 Adiabatic degeneration of Floer moduli spaces

In this subsection, we describe the main adiabatic gluing result established in (OhZ11b) without delving into the highly non-trivial analytic details. We refer the reader to the original article for the complete details.

On $U\pm$, using the given analytic coordinates $z = e^{2\pi(\tau+it)}$, we fix a function

$$\kappa^+(\tau) = \begin{cases} 0, & \text{if } |\tau| \le 1, \\ 1, & \text{if } |\tau| \ge 2 \end{cases} \tag{20.3.31}$$

and let $\kappa^-(\tau) = \kappa^+(-\tau)$. We set $\kappa^+_\epsilon(\tau) = \kappa^+(\tau - R(\epsilon) + 1)$ and $\kappa^-_\epsilon(\tau) = \kappa^+_\epsilon(-\tau)$. It is easy to see that

$$\kappa^+_\epsilon(\tau) = \begin{cases} 1 & \text{for } \tau \ge R(\epsilon) + 1, \\ 0 & \text{for } \tau \le R(\epsilon), \end{cases} \qquad \kappa^-_\epsilon(\tau) = \begin{cases} 1 & \text{for } \tau \le -R(\epsilon) - 1, \\ 0 & \text{for } \tau \ge -R(\epsilon). \end{cases} \tag{20.3.32}$$

We then extend these outside the charts $U\pm$ by zero. We choose $R = R(\epsilon)$ so that

$$\epsilon R(\epsilon) \to 0 \quad \text{as } \epsilon \to 0. \tag{20.3.33}$$

We remark that the choice of $R(\epsilon)$ made in (20.3.33) will be needed for a normalization procedure in the adiabatic degeneration argument.

We also consider a cut-off function $\kappa^0_\epsilon : \mathbb{R} \to [0,1]$ so that

$$\kappa^0_\epsilon(\tau) = \begin{cases} 1 & \text{for } |\tau| \le R(\epsilon) - 1, \\ 0 & \text{for } |\tau| \ge R(\epsilon), \end{cases} \tag{20.3.34}$$

$$|(\kappa_\epsilon^0)'(\tau)| \leq 2 \quad \text{for } R(\epsilon) - 1 \leq |\tau| \leq R(\epsilon) \tag{20.3.35}$$

and consider a one-parameter family of pairs (K_ϵ, J_ϵ) with their cylindrical ends given by

$$\mathrm{End}_\pm(K_\epsilon, J_\epsilon) = (H, J), \quad R_0 \leq R(\epsilon) < \infty$$

for a given Floer-regular pair (H, J) with the following form. For J_ϵ, we define

$$J_\epsilon(\tau, t, x) = \begin{cases} J^{\kappa_\epsilon^+(\tau)}(t, x) & \text{for } \tau \geq R(\epsilon), \\ J_0(x) & \text{for } |\tau| \leq R(\epsilon) - 1, \\ J^{\kappa_\epsilon^-(\tau)}(t, x) & \text{for } \tau \leq -R(\epsilon). \end{cases} \tag{20.3.36}$$

Thanks to the above cut-off functions $\kappa_\pm$, this defines a smooth $\mathbb{R} \times S^1$ family of almost-complex structures J on M.

Similarly, we define the family $K_\epsilon : \mathbb{R} \times S^1 \times M \to \mathbb{R}$ by

$$K_\epsilon(\tau, t, x) = \begin{cases} \kappa_\epsilon^+(\tau) \cdot H(t, x) & \text{for } \tau \geq R(\epsilon), \\ \kappa_\epsilon^0(\tau) \cdot \epsilon f(x) & \text{for } |\tau| \leq R(\epsilon), \\ \kappa_\epsilon^-(\tau) \cdot H(t, x) & \text{for } \tau \leq -R(\epsilon). \end{cases} \tag{20.3.37}$$

Remark 20.3.1

(1) We note that, when we define K_ϵ, we use the linear homotopy $s \mapsto sH$ which is *canonically* associated with the given Hamiltonian $H = H(t, x)$. A similar map can be defined for any homotopy $\mathcal{H} : s \mapsto H(s)$, where $H(s) = H(s, t, x)$ is a two-parameter family satisfying $H(0) = \epsilon f$ and $H(1) = H$. The resulting maps in homology will not depend on the choice of such a homotopy.

(2) We would like to compare our choice of K_ϵ above with that of (PSS96, MSa04): they use a family of K_ϵ with $K_\epsilon \equiv 0$ in the neck region of Σ_ϵ, while we use one obtained by putting a small Morse function ϵf in the neck and take the adiabatic limit as $\epsilon \to 0$. With the choice $K_\epsilon \equiv 0$ in the neck region, this process of degenerating Floer trajectories to nodal ones as $\epsilon \to 0$ is somewhat hard to recover from its limit under the gluing construction outlined in (PSS96, MSa04). On the other hand, we provide a one-jet datum at the node in its limit that remembers the presence of the background Morse function.

Using this particular one-parameter family (K_R, J_R) for a given cut-off function $\kappa = \{\kappa_+, \kappa_-\}$, we consider the corresponding parameterized moduli space

$$\begin{aligned}&\mathcal{M}^{\rm para}([z_-, w_-], [z_+, w_+]; \{(K, J; \kappa)\})\\ &\quad = \bigcup_{0<\epsilon\le\epsilon_0} \{\epsilon\} \times \mathcal{M}([z_-, w_-], [z_+, w_+]; K_\epsilon, J_\epsilon).\end{aligned}$$

For the sake of simplicity of notation, we will also write $\mathcal{M}^{\rm para} = \bigcup_{0<\epsilon\le\epsilon_0} \mathcal{M}^\epsilon$. To study the map $\Psi \circ \Phi$ in homology, we need to analyze the compactification of $\mathcal{M}^{\rm para}$.

The following is the key analytic theorem that is needed to complete the isomorphism property of the PSS map stated in (Pi94), (PSS96).

Theorem 20.3.2 (Theorem 10.19 of (OhZ11b)) *Let (K_ϵ, J_ϵ) be the family of Floer data defined in (20.3.37). Then, the following statements hold.*

(1) There exists a topology on $\mathcal{M}^{\rm para}_{(0;1,1)}([z_-, w_-], [z_+, w_+]; \{(K_\epsilon, J_\epsilon)\})$ with respect to which the gluing construction defines a proper embedding

$$\begin{aligned}\mathrm{Glue} \ :\ &(0, \epsilon_0) \times \mathcal{M}^{\rm nodal}_{(0;1,1)}([z_-, w_-], [z_+, w_+]; (H, J), (f, J_0))\\ &\to \mathcal{M}^{\rm para}_{(0;1,1)}([z_-, w_-], [z_+, w_+]; \{(K, J)\})\end{aligned}$$

for sufficiently small ϵ_0.

(2) The above-mentioned topology can be compactified into

$$\overline{\mathcal{M}}^{\rm para}_{(0;1,1)}([z_-, w_-], [z_+, w_+]; \{(K, J)\}),$$

where $\overline{\mathcal{M}}^{\rm para}_{(0;1,1)}([z_-, w_-], [z_+, w_+]; \{(K, J)\})$ is given by

$$\begin{aligned}&\overline{\mathcal{M}}^{\rm para}_{(0;1,1)}([z_-, w_-], [z_+, w_+]; \{(K, J)\})\\ &\quad = \bigcup_{0<\epsilon\le\epsilon_0} \{\epsilon\} \times \mathcal{M}_{(0;1,1)}([z_-, w_-], [z_+, w_+]; \{(K_\epsilon, J_\epsilon)\})\\ &\quad\quad \cup\ \mathcal{M}^{\rm nodal}_{(0;1,1)}([z_-, w_-], [z_+, w_+]; (H, J), (f, J_0))\end{aligned}$$

as a set.

(3) The embedding Glue *smoothly extends to the embedding*

$$\begin{aligned}\overline{\mathrm{Glue}} \ :\ &[0, \epsilon_0) \times \mathcal{M}^{\rm nodal}_{(0;1,1)}([z_-, w_-], [z_+, w_+]; (H, J), (f, J_0))\\ &\to \overline{\mathcal{M}}^{\rm para}_{(0;1,1)}([z_-, w_-], [z_+, w_+]; \{(K, J)\})\end{aligned}$$

which satisfies

$$\overline{\mathrm{Glue}}(u_+, u_-, u_0; 0) = \mathrm{Glue}(u_+, u_-, u_0).$$

We refer the reader to (OhZ11b) for the complete details of the proof of this theorem.

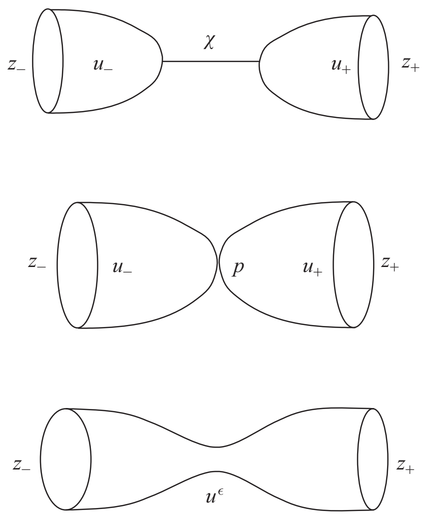

Figure 20.2 The PSS scheme.

20.3.3 The PSS scheme of the proof

In this section, we sketch the argument of Piunikhin, Salamon and Schwarz towards a proof of the isomorphism property of PSS maps that is based on a deformation leading to the chain isomorphism between the composition

$$\Phi \circ \Psi : CF_*(H_+) \to CF_*(H_+)$$

and the identity map. The deformation involves moduli spaces of three different types in the course of deformations (see Figure 20.2):

(1) Disk-flow-disk
(2) Nodal Floer trajectories
(3) Chain map Floer trajectories

For the sake of the following discussion, we denote the deformation parameter by $\lambda \in [-1, 1]$ so that the nodal configuration occurs at $\lambda = 0$. As long as $\lambda > 0$ or $\lambda < 0$, the deformation involves the same type of moduli spaces and hence the standard argument can be applied to construct cobordisms over $[-1, -\epsilon_0]$ and $[\epsilon_0, 1]$, respectively, for $\epsilon_0 > 0$. To complete the cobordism over the whole interval $[-1, 1]$, one needs to connect the two cobordisms with the one over $[-\epsilon_0, \epsilon_0]$. However, there occurs a 'phase change' in the moduli spaces at $\lambda = 0$. Owing to this 'phase change' at $\lambda = 0$, one can a priori expect only a *piecewise smooth* cobordism and needs to prove a *bi-collar theorem* of $\mathcal{M}_0 \subset \mathcal{M}^{\text{para}}$ to materialize the PSS scheme. From $-\epsilon_0$ to 0, one can construct the left one-sided collar by finite-dimensional construction. (See Section 9 of (OhZ11b).) For the right one-sided collar over $[0, \epsilon_0]$, the construction of the collar is non-trivial and is based on Theorem 20.3.2, whose proof is given by the method of adiabatic degeneration in (OhZ11b).

Using Theorem 20.3.2, we prove the following isomorphism property stated in (PSS96) by repeating verbatim the argument from (OhZ11b), which in turn largely follows the one from (PSS96).

Theorem 20.3.3 *Let $(-f; g)$ be a generic Morse–Smale pair of a Morse function f and a metric g on M and $H^{\rm Morse}(-f; g)$ the Morse homology of $(-f; g)$, and let (H, J) be a generic time-periodic Hamiltonian function H and a family of compatible almost-complex structure $J = \{J_t\}$ on M. Let Ψ, Φ be the PSS maps given in (PSS96). Then there exists a homomorphism*

$$\Upsilon_{\rm PSS} : CF_*(H, J) \to CF_{*+1}(H, J)$$

that satisfies

$$\Phi \circ \Psi - id = \partial_{(H,J)} \circ \Upsilon_{\rm PSS} - \Upsilon_{\rm PSS} \circ \partial^{\rm Morse}_{(-f,g)}. \tag{20.3.38}$$

In particular, we have $\Phi_* \circ \Psi_* = id$ in homology.

This shows that $\Phi_* \circ \Psi_* = id$. The other identity, $\Psi_* \circ \Phi_* = id$, is much easier to prove. Details of the proof are given in Section 9 of (OhZ11b).

Remark 20.3.4 The adiabatic degeneration of the moduli space of solutions of the Floer trajectory equation with a Morse function ϵf in the middle does not produce just nodal Floer trajectories as used in the PSS scheme but produces the nodal Floer trajectories with a one-jet datum that reflects the presence of the background Morse function f. This datum is crucial in the process of scale dependent gluing in (OhZ11b).

20.3.4 $\Phi_* \circ \Psi_* = id$; Floer via Morse back to Floer

Consider the PSS deformation defined over $\kappa \in [-\infty, 1]$. We fix a homotopy (K^κ, J^κ) with $K^\kappa = K^\kappa(\tau, t, x)$, $J^\kappa = J^\kappa(\tau, t, x)$ as any generic homotopy from $(K^{\epsilon_0}, J^{\epsilon_0})$ to $(K^1(\tau, t, x), J^1(\tau, t, x)) \equiv (H(t, x), J(t, x))$.

Fix a sufficiently small $\epsilon_0 > 0$ and a sufficiently large $\ell_0 > 0$. We divide the deformation into the following four pieces:

$$(K^\kappa, J^\kappa) \quad \text{for } \epsilon_0 \le \kappa \le 1,$$

$$(K_\epsilon, J_\epsilon) \quad \text{for } 0 < \kappa = \epsilon \le \epsilon_0,$$

$$(H^{\rho_-}, J^{\rho_-})_{o_-} * (-f, g_{J_0}; [-\ell, \ell]) *_{o_+} (H^{\rho_+}, J^{\rho_+}) \quad \text{for } -\ell_0 \le \ell < 0$$

and

$$(H^{\rho_-}, J^{\rho_-})_{o_-} * (-f, g_{J_0}; [-\ell, \ell]) *_{o_+} (H^{\rho_+}, J^{\rho_+}) \quad \text{for } -\infty \le \ell < -\ell_0.$$

Here $(f, J_0; [-\ell, \ell])$ stands for the deformation

$$\ell \in (-\infty, 0) \mapsto (-f, g_{J_0}; [-\ell, \ell]),$$

where f is a Morse function with respect to the metric g_{J_0} and we consider its gradient trajectories over the interval $[-\ell, \ell]$.

We denote by $\mathcal{M}_\kappa^{\Psi\Phi}([z_-, w_-], [z_+, w_+])$ the moduli space of the configuration corresponding to κ and form the parameterized moduli space

$$\overline{\mathcal{M}}^{\text{para}}_{\Psi\Phi}([z_-, w_-], [z_+, w_+]; f) = \bigcup_{\kappa \in [-1, \infty]} \mathcal{M}_\kappa^{\Psi\Phi}([z_-, w_-], [z_+, w_+]).$$

By the nondegeneracy hypothesis and the index condition, $\mathcal{M}_\kappa^{\Psi\Phi}$ is empty except at a finite number of points

$$\kappa \in (-\ell_0, -\ell_1) \cup (\epsilon_0, 1),$$

but a priori those κ could be accumulated in $[-\ell_1, \epsilon_0]$ as $\kappa \to 0$. The one-jet transversality of the enhanced nodal Floer trajectory moduli space, which corresponds to $\kappa = 0$ and Theorem 20.3.2 *rules out this accumulation*. As a consequence, we derive

$$\mathcal{M}_\kappa^{\Psi\Phi}([z_-, w_-], [z_+, w_+]) = \emptyset$$

for all $\kappa \in [-\ell_1, \epsilon_0]$ if we choose ℓ_1, ϵ_0 sufficiently small. Together with the main gluing compactness result, Theorem 20.3.2, the above discussion proves the following proposition.

Proposition 20.3.5 *There exist constants ℓ_0, ℓ_1, ϵ_0 and ϵ_1 such that the following statements hold.*

(1) *If $\mu_H([z_-, w_-]) - \mu_H([z_+, w_+]) = -1$, then $\overline{\mathcal{M}}^{\text{para}}_{\Psi\Phi}([z_-, w_-]), [z_+, w_+])$ is a compact zero-dimensional manifold such that*

$$\mathcal{M}_\kappa^{\Psi\Phi}([z_-, w_-], [z_+, w_+]) = \emptyset$$

for $\kappa \in [-\infty, -\ell_0] \cup [-\ell_1, \epsilon_0] \cup [1 - \epsilon_1, 1]$.

(2) *If* $\mu_H([z_-, w_-]) - \mu_H([z_+, w_+]) = 0$*, then* $\overline{\mathcal{M}}^{\text{para}}_{\Psi\Phi}([z_-, w_-], [z_+, w_+]; f)$ *is a compact one-dimensional manifold with boundary given by*

$$\begin{aligned}
&\partial\overline{\mathcal{M}}^{\text{para}}_{\Psi\Phi}([z_-, w_-], [z_+, w_+]; f) = \\
&\qquad \mathcal{M}_{\kappa=1}([z_-, w_-], [z_+, w_+]) \cup \mathcal{M}_{-\infty}([z_-, w_-], [z_+, w_+]) \\
&\qquad \cup \left(\bigcup_{[z,w]} \overline{\mathcal{M}}^{\text{para}}_{\Psi\Phi}([z_-, w_-], [z, w]) \# \mathcal{M}_{\kappa=1}([z, w], [z_+, w_+]) \right) \\
&\qquad \cup \left(\bigcup_{[z,w]} \mathcal{M}_{\kappa=1}([z_-, w_-], [z, w]) \# \overline{\mathcal{M}}^{\text{para}}_{\Psi\Phi}([z, w], [z_+, w_+]) \right),
\end{aligned}$$

where the union is taken over all $[z, w]$ *with* $\mu_H([z_-, w_-]) - \mu_H([z, w]) = -1$ *for the first and* $\mu_H([z, w]) - \mu_H([z_+, w_+]) = -1$ *for the second.*

Statement (1) in this proposition allows one to define the matrix coefficients, which are of the order

$$\#\overline{\mathcal{M}}^{\text{para}}_{\Psi\Phi}([z_-, w_-], [z_+, w_+]; f).$$

We then define the map

$$\Upsilon^{\Psi\Phi}_{\text{PSS}} : CF_*(H) \to CF_{*+1}(H)$$

by the matrix coefficients

$$\langle \Upsilon^{\Psi\Phi}_{\text{PSS}}([z_-, w_-], [z_+, w_+]) \rangle := \#\overline{\mathcal{M}}^{\text{para}}_{\Psi\Phi}([z_-, w_-], [z_+, w_+]; f).$$

Then statement (2) concerning the description of the boundary of the one-dimensional moduli space $\overline{\mathcal{M}}^{\text{para}}_{\Psi\Phi}([z_-, w_-], [z_+, w_+]; f)$ is translated into the equation

$$\Phi \circ \Psi - id = \partial \circ \Upsilon^{\Psi\Phi}_{\text{PSS}} + \Upsilon^{\Psi\Phi}_{\text{PSS}} \circ \partial.$$

This finishes the proof $\Phi_* \circ \Psi_* = id$ in homology.

20.3.5 $\Psi_* \circ \Phi_* = id$; Morse via Floer back to Morse

In this section, for each given pair $p, q \in \operatorname{Crit} f$, we consider the parameterized moduli space

$$\overline{\mathcal{M}}^{\text{para}}_{\Phi\Psi}(p, q) = \bigcup_{0 \le R \le \infty} \mathcal{M}^{\Phi\Psi}_R(p, q).$$

We define $\mathcal{M}^{\Phi\Psi}_R(p, q)$ in the following way.

First, for each $0 < R < \infty$, we introduce the moduli space $\mathcal{M}_{(2;0,0)}(K^R, J^R)$ of finite-energy solutions of

$$\overline{\partial}_{(K^R,J^R)} u = 0 \tag{20.3.39}$$

on Σ, which is a Riemann surface with two marked points $\{o_-, o_+\}$ so that $\Sigma \setminus \{o_\pm\} \cong \mathbb{R} \times S^1$ conformally. We first define a family of Riemann surface (Σ, j_R) by the connected sum

$$(D^-, o_-) \cup C_R \cup (D^+, o_+), \quad j_R = j_{D^-} \# j_{C_R} \# j_{D^+},$$

where C_R is the cylinder $[-R, R] \times S^1$, j_{C_R} is the standard conformal structure and j_R is the obvious glued conformal structure on $D^- \cup C_R \cup D^+$. We denote by (τ, t) the conformal coordinates on $D^- \cup C_R \cup D^+ \setminus \{o_-, o_+\}$ extending the standard coordinates on $\mathbb{C}_R$.

In these conformal coordinates, we fix a family of cut-off functions χ^R by

$$\chi^R(\tau) = \begin{cases} 1 - \kappa^+(\tau - R) & \text{for } \tau \geq 0, \\ 1 - \kappa^-(\tau + R) & \text{for } \tau \leq 0 \end{cases}$$

for $1 \leq R < \infty$, and $\chi^R = R\chi^1$ for $0 \leq R \leq 1$. We note that $\chi^0 \equiv 0$ and that χ^R has compact support and $\chi^R \equiv 1$ on any given compact subset if R is sufficiently large. Therefore, Equation (20.3.39) is reduced to $\overline{\partial}_{J_0} u = 0$ near the marked points $o_\pm$. Then we define (K^R, J^R) as before.

We have two evaluations,

$$\mathrm{ev}_{o_\pm} : \mathcal{M}_{(2;0,0)}(K^R, J^R) \to M; \quad \mathrm{ev}_{o_\pm}(u) = u(o_\pm).$$

We denote

$$\widetilde{\mathcal{M}}^-(p; f) = \{\chi : \mathbb{R} \times M \mid \dot{\chi} + \nabla(-f)(\chi) = 0, \chi(-\infty) = p\},$$
$$\widetilde{\mathcal{M}}^+(q; f) = \{\chi : \mathbb{R} \times M \mid \dot{\chi} + \nabla(-f)(\chi) = 0, \chi(+\infty) = q\}$$

and define

$$\widetilde{\mathcal{M}}_1^-(p; -f) = \widetilde{\mathcal{M}}^-(p; -f) \times \mathbb{R},$$
$$\widetilde{\mathcal{M}}_1^+(q; -f) = \widetilde{\mathcal{M}}^+(q; -f) \times \mathbb{R}.$$

$\tau_0 \in \mathbb{R}$ acts on both by the action

$$(\tau_0, (\chi, \tau)) \mapsto (\chi(* - \tau_0), \tau + \tau_0).$$

This action is free and so their quotients

$$\mathcal{M}_1^-(p; -f) = \widetilde{\mathcal{M}}_1^-(p; -f)/\mathbb{R}, \quad \mathcal{M}_1^+(q; -f) = \widetilde{\mathcal{M}}_1^+(q; -f)/\mathbb{R}$$

become smooth manifold of dimension $\mu_{\mathrm{Morse}}(p; f)$ and $2n - \mu_{\mathrm{Morse}}(q; f)$, respectively. We have the asymptotic evaluation maps

$$\mathrm{ev}_+ : \mathcal{M}_1^+(q; -f) \to M, \quad \mathrm{ev}_- : \mathcal{M}_1^-(p; -f) \to M$$

whose images are in one-to-one correspondence with the unstable manifold $W^u(p; -f)$ and the stable manifold $W^s(q; -f)$, respectively.

Now we define the moduli space $\mathcal{M}_R^{\Phi\Psi}(p, q; A)$ to be the fiber product

$$\begin{aligned}\mathcal{M}_R^{\Phi\Psi}(p, q; A) &= \mathcal{M}_1^-(p; -f)_{\mathrm{ev}_-} \times_{\mathrm{ev}_{o_-}} \mathcal{M}_{(2;0,0)}(K^R, J^R; A)_{\mathrm{ev}_{o_+}} \times_{\mathrm{ev}_+} \mathcal{M}_1^+(q; -f)\\ &= \{((\chi_-, \tau_-), u, (\chi_+, \tau_+)) \mid \chi_-(\tau_-) = u(o_-), \chi_+(\tau_+) = u(o_+)\}\end{aligned}$$

and

$$\overline{\mathcal{M}}^{\Phi\Psi,\mathrm{para}}(p, q; A) = \bigcup_{0 \le R \le \infty} \mathcal{M}_R^{\Phi\Psi}(p, q; A).$$

A straightforward calculation shows that

$$\dim^{\mathrm{virt}} \mathcal{M}_R^{\Phi\Psi}(p, q; A) = \mu_{\mathrm{Morse}}(p) - \mu_{\mathrm{Morse}}(q) + 2c_1(A).$$

Proposition 20.3.6 *Choose a generic pair (f, J_0) and let $0 < \epsilon_1 < R_1$.*

(1) *Suppose that $\mu_{\mathrm{Morse}}(p) - \mu_{\mathrm{Morse}}(q) + 2c_1(A) = -1$. Then there exist some $\epsilon_1 > 0$ and $R_1 > 0$ such that $\overline{\mathcal{M}}_R^{\Phi\Psi,\mathrm{para}}(p, q; A)$ is a compact zero-dimensional manifold such that*

$$\mathcal{M}_R^{\Phi\Psi}(p, q; A) = \emptyset$$

if $0 \le R \le \epsilon_1$ or $R \ge R_1$.

(2) *Suppose that $\mu_{\mathrm{Morse}}(p) - \mu_{\mathrm{Morse}}(q) + 2c_1(A) = 0$. Then $\overline{\mathcal{M}}^{\Phi\Psi,\mathrm{para}}(p, q; A)$ is a compact one-manifold with boundary given by*

$$\partial\overline{\mathcal{M}}^{\Phi\Psi,\mathrm{para}}(p, q; A) = \mathcal{M}_0^{\Phi\Psi}(p, q; A) \cup \mathcal{M}_\infty^{\Phi\Psi}(p, q; A) \cup \bigcup_r \overline{\mathcal{M}}^{\Phi\Psi}(p, r; A),$$

where the union $\bigcup_r$ is taken over $r \in \mathrm{Crit}(-f)$ such that

$$\mu_{\mathrm{Morse}}(p) - \mu_{\mathrm{Morse}}(r) + 2c_1(A) = -1.$$

Proof We recall that, when $R = 0$, Equation (20.3.39) is reduced to $\overline{\partial}_{J_0} u = 0$. Since $\mu_{\mathrm{Morse}}(p) - \mu_{\mathrm{Morse}}(q) + 2c_1(A) = -1$ represents the virtual dimension of $\mathcal{M}_0^{\Phi\Psi}(p, q; A)$, $\mathcal{M}_0^{\Phi\Psi}(p, q; A)$ must be empty for a generic choice of (f, J_0). Here we emphasize the fact that this moduli space depends only on (f, J_0) for which the genericity argument can be applied independently of the parameter R. Therefore the same must be the case when $R_1 \le \epsilon_1$ for a sufficiently small $\epsilon_1 > 0$. This finishes the proof.

We leave the proof of statement (2) to the reader. □

Using statement (1), we define the chain homotopy map

$$\Upsilon^{\Phi\Psi}_{\mathrm{PSS}} : CM_*(-f, g_{J_0}; \Lambda_\omega) \to CM_{*+1}(-f, g_{J_0}; \Lambda_\omega)$$

by the matrix element

$$\langle \Upsilon^{\Phi\Psi}_{\mathrm{PSS}}(p), q\#(-A)\rangle = \sum_{(r,A)} \#\left(\bigcup_r \overline{\mathcal{M}}^{\Phi\Psi,para}(p, r; A)\right).$$

Next we prove the following lemma.

Lemma 20.3.7 *Suppose* $\mu_{\mathrm{Morse}}(p) - \mu_{\mathrm{Morse}}(q) + 2c_1(A) = 0$. *Then, if* $A \neq 0$,

$$\dim \mathcal{M}^{\Phi\Psi}_0(p, q; A) \geq 2$$

unless $\mathcal{M}^{\Phi\Psi}_0(p, q; A) = \emptyset$. *When* $A = 0$, *we have*

$$\dim \mathcal{M}^{\Phi\Psi}_0(p, q; A) \geq 1$$

unless $p = q$.

Proof If $A \neq 0$, u is non-constant in $(u; o_-, o_+) \in \mathcal{M}_{(2;0,0)}(K^R, J^R)$. Then the conformal automorphism on the domain $(\Sigma; o_-, o_+)$ produces at least a real two-dimensional family, which contradicts the index hypothesis. (See (Fl89b), (FHS95) for the semi-positive case and (FOn99), (LT98) in general.)

On the other hand, if $A = 0$, any J_0-holomorphic sphere must be constant and hence the corresponding configuration $(\chi_-, \text{constant}, \chi_+)$ becomes a full gradient trajectory $\chi = \chi_-\#\chi_+$. Unless χ is constant, i.e., unless $p = q$, $\mathbb{R}$-translation produces at least a one-dimensional family, which again contradicts the index hypothesis. This finishes the proof. □

Now we are ready to finish the proof of the identity

$$\Psi \circ \Phi - \mathrm{id} = \Upsilon^{\Phi\Psi}_{\mathrm{PSS}} \partial^{\mathrm{Morse}}_{(-f, g_{J_0})} + \partial^{\mathrm{Morse}}_{(-f, g_{J_0})} \Upsilon^{\Phi\Psi}_{\mathrm{PSS}}. \tag{20.3.40}$$

A priori, Proposition 20.3.6 implies merely that

$$\sum_{q,A} \langle (\Psi \circ \Phi - id)(p), q\#(-A)\rangle = \sum_{q,A} \left\langle \Upsilon^{\Phi\Psi}_{\mathrm{PSS}} \partial^{\mathrm{Morse}}_{(-f, g_{J_0})}(p) + \partial^{\mathrm{Morse}}_{(-f, g_{J_0})} \Upsilon^{\Phi\Psi}_{\mathrm{PSS}}(p), \right.$$
$$\left. q\#(-A) \right\rangle.$$

But the above lemma implies that

$$\langle (p), q\#(-A)\rangle = 0$$

unless $A = 0$ and $p = q$. This finishes the proof of (20.3.40). □

20.4 Frobenius pairing and duality

Recall the isomorphisms $\flat : QH^*(M) \to QH_*(M)$ and its inverse $\sharp : QH_*(M) \to QH^*(M)$ given in (20.1.2) and (20.1.3). We denote by $a^\flat$ and $b^\#$ the images under these maps as before.

Definition 20.4.1 (Frobenius pairing) We define the nondegenerate pairing

$$\Delta = \langle \cdot, \cdot \rangle : QH^k(M) \otimes QH^{n-k}(M) \to \Lambda_\omega^{(0)} \tag{20.4.41}$$

by

$$\left\langle \sum a_A q^A, \sum b_B q^B \right\rangle = \sum_{C; c_1(C)=0} \left(\sum_A (a_A, b_{C-A}) \right) q^C, \tag{20.4.42}$$

where (a_A, b_{C-A}) is the canonical Poincaré pairing between $H^k(M,\mathbb{Q})$ and $H_k(M,\mathbb{Q})$

$$(a_A, b_B) = a_A \cap PD(b_B).$$

Note that this sum in (20.4.42) is always finite by the finiteness condition in the definition of $QH^*(M)$ and hence is well defined.

Proposition 20.4.2 *Denote by $[M]$ the fundamental class and by $1 = PD[M]$ the identity in $QH^*(M)$. Then*

$$\langle a, b \rangle = (a \cup_Q b) \cap [M] = \langle (a \cup_Q b), 1 \rangle.$$

Proof We have only to prove the first equality for a cohomology class a, $b \in H^*(M)$ for which the left-hand side of the identity becomes nothing but the classical Poincaré duality pairing

$$(a_A, b_{C-A}) = a_A \cap PD(b_{C-A}) = (a_A \cup b_{C-A}) \cap [M].$$

Next we evaluate the right-hand side of the identity. We represent $PD(a)$, $PD(b)$ by cycles P_1, P_2. Then, by definition, $(a \cup_Q b) \cap [M]$ lies in $H_0(M; \Lambda_\omega) \cong \Lambda_\omega$ given by the formula

$$\sum_{\alpha \in \pi_2(M)} \mathcal{M}_3(J_0; \alpha) \times_{(\mathrm{ev}_1, \mathrm{ev}_2, \mathrm{ev}_0)} (P_1 \times P_2 \times M) T^{\omega(\alpha)} e^{c_1(\alpha)}$$

with $c_1(\alpha) = 0$. Therefore $(a \cup_Q b) \cap [M] \in \Lambda_\omega^{(0)} \subset \Lambda_\omega$. On the other hand, the fiber product $\mathcal{M}_3(J_0; \alpha) \times_{(\mathrm{ev}_1, \mathrm{ev}_2, \mathrm{ev}_0)} (P_1 \times P_2 \times M)$ defines a cycle contained in $C_0(M; \mathbb{Q})$ and hence must have dimension zero. But this can be possible only when $\alpha = 0$ and hence all the elements in $\mathcal{M}_3(J_0; \alpha)$ are constant maps, for otherwise the dimension of the moduli space must have at least two real dimensions corresponding to the variation of the zeroth marked point z_0. Note that, since M is a full space, there is no restriction on the zeroth marked point z_0 and so we can freely move z_0 without affecting the map and the given intersection with P_1 and P_2 at the marked points z_1, z_2, respectively. This proves that the above fiber product becomes precisely the classical cap product $(a \cup_Q b) \cap [M] = (a, b)$. This finishes the proof. □

We define the $\mathbb{Q}$-bilinear pairing Π by

$$\Pi(a, b) = \pi_0 \circ \Delta(a, b) = \pi_0(\langle a \cdot b, 1\rangle), \tag{20.4.43}$$

where $\pi_0 : \Lambda_{0,\omega} \to \mathbb{Q}$ is the obvious projection given by $\pi_0(\sum a_A q^A) = a_{A=0}$.

Remark 20.4.3 We would like to note that the dual vector space $(QH_*(M))^*$ of $QH_*(M)$ is *not* isomorphic to the standard quantum cohomology $QH^*(M)$ even as a $\mathbb{Q}$-vector space. Rather the above pairing induces an injection

$$QH^*(M) \hookrightarrow (QH_*(M))^*$$

whose images lie in the set of *continuous* linear functionals on $QH_*(M)$ with respect to the topology induced by the valuation v in (20.1.1) on $QH_*(M)$.

Proposition 20.4.4 *Π is nondegenerate. In other words, if $\Pi(a, b) = 0$ for all $b \in QH^{n-k}(M)$, then $a = 0$.*

Proof If $\Pi(a, b) = 0$, we have $\pi_0(\Delta(a, b)) = 0$ and hence

$$\sum_{C; c_1(C)=\omega(C)=0} \sum_A (a_A, b_{C-A}) = 0.$$

But $c_1(C) = \omega(C) = 0$ implies $C = 0$ in Γ and hence we have $\sum_{A\in\Gamma}(a_A, b_{-A}) = 0$ for all $b_{-A} \in H^{n-k}(M)$. By considering $b = a_{-A}q^{-A}$, which lies in $QH^{n-k}(M)$ if a is in $QH^k(M)$, we prove that $a_A = 0$ for all $A \in \Gamma$ and hence that $a = 0$. This finishes the proof. □

In subsection 22.5.1, we will come back to further study of the Frobenius pairing (20.4.41) and the pairing (19.7.49) of the Floer complex, and their relationship, using a natural isomorphism between $QH^*(M)$ and $HF_*(H)$ regarding $QH^*(M)$ as the Floer cohomology of a small Morse function.

21

Spectral invariants: construction

Motivated by an attempt to provide a systematic explanation of Eliashberg's C^0 symplectic rigidity result, Viterbo (Vi92) constructed a set of symplectic invariants of compactly supported Hamiltonian diffeomorphisms ϕ on $\mathbb{R}^{2n}$ by considering the graph of such Hamiltonian diffeomorphisms. He did this by compactifying the graph in the diagonal direction in $\mathbb{R}^{4n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ into T^*S^{2n}, and then applying the critical-point theory of generating functions of the Lagrangian submanifold graph $\phi \subset T^*S^{2n}$, and then by applying the 'stable Morse theory' on the cotangent bundle T^*N of arbitrary closed smooth manifolds N. When a Lagrangian submanifold L carries a generating function, e.g., when L is Hamiltonian isotopic to the zero section of T^*N, Viterbo associated a homologically essential critical value of the given generating function with each cohomology class $a \in H^*(N)$, and proved that they depend on the Lagrangian submanifold but not on the generating functions, at least up to normalization.

Utilizing Weinstein's observation that the action functional is a generating function defined on the path space of the time-one image $\phi_H^1(o_N)$ of the zero section under the given Hamiltonian flow associated with the Hamiltonian H, the present author (Oh97b, Oh99) developed a Floer-theoretic approach to the mini-max theory of the action functional and constructed a set of symplectic invariants again parameterized by $H^*(N)$, called the *Lagrangian spectral invariants*. This approach is canonical *including normalization* and provides a direct link between Hofer's geometry and Viterbo's invariants in a transparent way. Then Milinković and the author proved that this set of spectral invariants does indeed coincide with Viterbo's invariants modulo the normalization (MiO95, Mi00).

Schwartz (Schw00) performed a similar construction by applying the scheme to the Hamiltonian fixed-point Floer theory on *symplectically* *aspherical* (M, ω), i.e., for (M, ω) with $c_1|_{\pi_2(M)} = \omega|_{\pi_2(M)} = 0$ and associated similar invariants parameterized by the cohomology $H^*(M)$. Among other things, he proved the important optimal triangle inequality for the spectral invariants which was missing for the above-mentioned Lagrangian spectral invariants at the time of writing (Oh97b, Oh99). Quite recently Monzner, Vichery and Zapolsky established the similar optimal form of the triangle inequality for the Lagrangian spectral invariants (MVZ12).

On non-exact symplectic manifolds, the action functional is not single-valued, and the Floer homology theory was developed as a circle-valued Morse theory or a Morse theory on a covering space $\widetilde{\mathcal{L}}_0(M)$ of the space $\mathcal{L}_0(M)$ of contractible (free) loops on M. The Floer theory now involves quantum effects and uses the Novikov ring in an essential way. The presence of quantum effects, and the *denseness of the action spectrum* in $\mathbb{R}$ as in non-rational symplectic manifolds, had been the most serious obstacles that plagued the study of the *family of Hamiltonian diffeomorphisms*. The present author then developed a general mini-max theory via the chain-level Floer theory in (Oh02, Oh05c) on the covering space $\widetilde{\mathcal{L}}_0(M)$ and associated a set of spectral invariants with each given time-dependent Hamiltonian H. This set is parameterized this time by the quantum cohomology $QH^*(M)$, the contents of which are now called the *Hamiltonian spectral invariants*.

21.1 Energy estimates and Hofer's geometry

In this section, we consider various Floer operators and how they change the filtration levels of the complex CF^λ.

Let us fix the Hamiltonians H_α, H_β and a homotopy $\mathcal{H}$ between them. We emphasize that H_α and H_β are *not* necessarily nondegenerate for the discussion of this section. We choose a homotopy $j = \{J(\eta)\}_{0\le\eta\le 1}$ of compatible almost-complex structures and a cut-off function $\rho : \mathbb{R} \to [0, 1]$. Then we consider the associated Floer equation

$$\begin{cases} \partial u/\partial\tau + J^\rho\,(\partial u/\partial t - X_{H^\rho}(u)) = 0, \\ u(-\infty) = z^-,\ u(\infty) = z^+. \end{cases} \tag{21.1.1}$$

The following energy estimate is a key estimate that provides all the relations between the action integrals, the energy of Floer connecting trajectories and the Hofer norm of the associated Hamiltonians. This is a special case of Proposition 20.2.9. Here we give a direct proof, which is also instructive in this special case. It is a variation of the proof of Lemma 14.4.5.

Proposition 21.1.1 *Let H_α, H_β be any, not necessarily nondegenerate, Hamiltonians and consider a homotopy $\mathcal{H} = \{H^s\}_{0\le s\le 1}$ between them. Then consider the pair $(\mathcal{H}, j)$ and let ρ be any cut-off function as before. Suppose that u satisfies (21.1.1) with $w^-\#u \sim w^+$ (19.4.16), has finite energy and satisfies*

$$\lim_{j\to\infty} u(\tau_j^-) = z^-, \quad \lim_{j\to\infty} u(\tau_j^+) = z^+$$

for some sequences $\tau_j^\pm$ with $\tau_j^- \to -\infty$ and $\tau_j^+ \to \infty$. Then we have

$$\begin{aligned}\mathcal{A}_F([z^+, w^+]) - \mathcal{A}_H([z^-, w^-]) = &-\int_{-\infty}^{\infty}\int_0^1 \left|\frac{\partial u}{\partial \tau}\right|^2_{J^{\rho(\tau)}} dt\,d\tau \\ &-\int_{-\infty}^{\infty}\rho'(\tau)\int_0^1 \left(\left.\frac{\partial H^s}{\partial s}\right|_{s=\rho(\tau)}(t, u(\tau,t))\right) dt\,d\tau.\end{aligned} \tag{21.1.2}$$

Proof With the homotopy condition $w^-\#u \sim w^+$, we have

$$\int u^*\omega = \int (w^+)^*\omega - \int (w^-)^*\omega.$$

More generally, we obtain

$$\int_{w^-\#(u|_{-\infty<\tau'\le\tau})} \omega = \int (w^-)^*\omega - \int_{-\infty}^{\tau}\int_0^1 u^*\omega\,d\tau.$$

For the sake of simplicity of notation, we denote by $u(\tau)$ the loop defined by $u(\tau)(t) = u(\tau, t)$ for $t \in S^1$. In particular, if we denote $\widetilde{u}(\tau) = [u(\tau), w^-\#(u|_{-\infty<\tau'\le\tau})]$, then

$$\mathcal{A}_F([z^+, w^+]) - \mathcal{A}_H([z^-, w^-]) = \int_{-\infty}^{\infty} \frac{d}{d\tau}(\mathcal{A}_{H^\rho}(\widetilde{u}(\tau)))d\tau$$

and then compute

$$\frac{d}{d\tau}(\mathcal{A}_{H^\rho}(\widetilde{u}(\tau))) = d\mathcal{A}_{H^{\rho(\tau)}}(u(\tau))\left(\frac{\partial u}{\partial \tau}\right) - \int_0^1 \frac{\partial H^{\rho(\tau)}}{\partial \tau}(\widetilde{u}(\tau, t))dt,$$

where $d\mathcal{A}_{H^{\rho(\tau)}}(\widetilde{u}(\tau))$ is the differential at $\widetilde{u}(\tau)$ of the action functional $\mathcal{A}_{H^{\rho(\tau)}}$ associated with the Hamiltonian $H^{\rho(\tau)} = H^{\rho(\tau)}(t, x)$ for given fixed τ. Since u satisfies (21.1.1), which is also the (τ-dependent) negative gradient flow of $\mathcal{A}_{H^\tau(\tau)}$, the right-hand side becomes

$$-\int_0^1 \left|\frac{\partial u}{\partial \tau}\right|_{J_t^{\rho(\tau)}} dt - \rho'(\tau)\int_0^1 \left.\frac{\partial H^{\rho(\tau)}}{\partial s}\right|_{s=\rho(\tau)} (u(\tau, t))dt$$

and hence

$$\frac{d}{d\tau}(\mathcal{A}_{H^\rho}(\widetilde{u}(\tau))) = -\int_0^1 \left|\frac{\partial u}{\partial \tau}\right|_{J_t^{\rho(\tau)}} dt - \rho'(\tau)\int_0^1 \left.\frac{\partial H^{\rho(\tau)}}{\partial s}\right|_{s=\rho(\tau)} (u(\tau,t))dt.$$

Upon integrating this over $\tau \in \mathbb{R}$, we obtain (21.1.2), which finishes the proof. □

Now we derive various consequences of (21.1.2) for the action difference between the two asymptotic orbits.

We start with the boundary map equation (18.5.41), i.e., when $H_\alpha = H_\beta$ and $H^s \equiv H_\alpha$ for all $s \in [0,1]$.

Corollary 21.1.2 *For any finite-energy solution u of (18.5.41) with (19.4.16), we have*

$$\mathcal{A}_H([z^+,w^+]) - \mathcal{A}_H([z^-,w^-]) \le -\int\int \left|\frac{\partial u}{\partial \tau}\right|_J^2 \le 0. \tag{21.1.3}$$

In particular, when (H,J) is Floer-regular, the associated boundary map $\partial_{(H,J)}$ satisfies

$$\partial_{(H,J)}(CF_*^\lambda(H)) \subset CF_*^\lambda(H)$$

and hence canonically restricts to a boundary map

$$\partial_{(H,J)} : \left(CF_*^\lambda(H), \partial_{(H,J)}\right) \to \left(CF_*^\lambda(H), \partial_{(H,J)}\right)$$

for any real number $\lambda \in \mathbb{R}$.

Next we consider the chain map. The following is an immediate consequence of (21.1.2).

Corollary 21.1.3 *Let $(\mathcal{H}, j;\rho)$ and u be as in Proposition 21.1.1. Suppose that ρ is monotone. Then we have*

$$\mathcal{A}_F([z^+,w^+]) - \mathcal{A}_H([z^-,w^-]) \le -\int\int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^{\rho_1(\tau)}} + \int_0^1 -\min_{x,s}\left(\frac{\partial H_t^s}{\partial s}\right) dt \tag{21.1.4}$$

$$\le \int_0^1 -\min_{x,s}\left(\frac{\partial H_t^s}{\partial s}\right) dt. \tag{21.1.5}$$

Furthermore, (21.1.4) can be rewritten as the upper bound for the energy

$$\int\int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^{\rho_1(\tau)}} \le \mathcal{A}_H([z^+,w^+]) - \mathcal{A}_F([z^-,w^-]) + \int_{-\infty}^{\infty}\int_0^1 -\min_{x,s}\left(\frac{\partial H_t^s}{\partial s}\right) dt. \tag{21.1.6}$$

Here we would like to emphasize that *the various energy upper bounds above do not depend on u or on the choice of j, ρ or J, but depend only on the homotopy $\mathcal{H}$ itself* and the asymptotic condition of u.

Motivated by the upper estimate (21.1.5), we introduce the following definition.

Definition 21.1.4 Let $\mathcal{H} = \{H(s)\}_{0\le s\le 1}$ be a homotopy of Hamiltonians. We define the *negative part of the variation* and the *positive part of the variation* of $\mathcal{H}$ by

$$E^-(\mathcal{H}) := \int_0^1 -\min_{x,s}\left(\frac{\partial H_t^s}{\partial s}\right) dt$$
$$E^+(\mathcal{H}) := \int_0^1 \max_{x,s}\left(\frac{\partial H_t^s}{\partial s}\right) dt.$$

Also we define the *total variation* $E(\mathcal{H})$ of $\mathcal{H}$ by

$$E(\mathcal{H}) = E^-(\mathcal{H}) + E^+(\mathcal{H}).$$

If we denote by $\mathcal{H}^{-1}$ the time reversal of $\mathcal{H}$, i.e., the homotopy given by

$$\mathcal{H}^{-1} : s \in [0,1] \mapsto H^{1-s}$$

then we have the identity

$$E^\pm(\mathcal{H}^{-1}) = E^\mp(\mathcal{H}) \quad \text{and } E(\mathcal{H}^{-1}) = E(\mathcal{H}).$$

With these definitions, applied to a pair $(\mathcal{H}, j)$ such that their ends $H(0)$ and $H(1)$ are nondegenerate, the a-priori energy estimate (21.1.5) can be written as

$$\int\int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^{\rho(\tau)}} \le -\mathcal{A}_F(u(\infty)) + \mathcal{A}_H(u(-\infty)) + E^-(\mathcal{H})$$

for a monotone ρ. Here we recall that, when the Hamiltonian is nondegenerate, Proposition 18.5.3 implies that any finite-energy solution has well-defined asymptotic limits as $\tau \to \pm\infty$.

We denote by $HF_*^\lambda(H, J)$ the associated filtered homology and call it the filtered Floer homology group.

Corollary 21.1.5 *Suppose that (H^0, J^0) and (H^1, J^1) are Floer-regular, $(\mathcal{H}, j)$ is a Floer-regular path between them, and ρ is as before. Then the chain map $h_{(\mathcal{H},j;\rho)}$ satisfies*

$$h_{(\mathcal{H},j;\rho)}(CF_*^\lambda(H^0)) \subset CF_*^{\lambda+E^-(\mathcal{H})}(H^1)$$

and hence canonically restricts to a chain map

$$h_{(\mathcal{H},j;\rho)} : (CF_*^\lambda(H^0), \partial_{(H^0,J^0)}) \to (CF_*^{\lambda+E^-(\mathcal{H})}(H^1), \partial_{(H^1,J^1)}).$$

It is worthwhile to mention separately one particular case of Corollaries 21.1.3 and 21.1.5 that will be used in the construction of the spectral invariants $\rho(H; a)$ later. Recall that the same result was used in Section 12.5.2 for the spectral invariants of Lagrangian submanifolds on the cotangent bundle.

Corollary 21.1.6 *Let H be given. Consider two J^0 and J^1, an elongation function ρ and the homotopy $(\mathcal{H}, j)$ between (H, J^0) and (H, J^1) satisfying $\mathcal{H} \equiv H$. Then, for any finite-energy solution u of (18.5.42) with (19.4.16), we have*

$$\mathcal{A}_H([z^+, w^+]) - \mathcal{A}_H([z^-, w^-]) \le -\int\int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^{\rho(\tau)}} \le 0. \tag{21.1.7}$$

In particular, when H is nondegenerate, J^0, J^1 are H-regular and (H, j) is generic, then the associate chain map $h_{(H,j);\rho}$ satisfies

$$h_{(\mathcal{H},j);\rho}(CF^\lambda_*(H)) \subset CF^\lambda_*(H)$$

and hence canonically restricts to a chain map

$$h^\lambda_{(H,j);\rho} : (CF^\lambda_*(H), \partial_{(H,J^0)}) \to (CF^\lambda_*(H), \partial_{(H,J^1)})$$

and induces an isomorphism in homology for any real number $\lambda \in \mathbb{R}$.

Proof It remains to prove that $h^\lambda_{(H,j);\rho}$ induces an isomorphism in homology. For this, we choose any homotopy j' connecting from J^1 to J^0 such that (H, j) is Floer-regular, and a cut-off function ρ. Then we consider the $j\#j'$ which connects from J^0 to J^0. Now we deform $j\#j'$ to the constant homotopy $j_{\mathrm{const}} \equiv J^0$. We denote the homotopy of homotopy by $\overline{j}$ connecting from j_{const} to $j\#j'$. Then by (21.1.7), $(H, \overline{j})$ provides a chain homotopy from $h_{(H,j_{\mathrm{const}};\rho)}$ and $h_{(H,j)\#_{(\rho;R)}(H,j')}$. We note that since $j_{\mathrm{const}} \equiv J^0$, the elongated homotopy of $(H^\rho, j^\rho_{\mathrm{const}})$ becomes the constant homotopy (H, J^0). Therefore, by the Floer-regularity hypothesis of (H, J^0) *as a family*, we derive $h_{(H,j_{\mathrm{const}};\rho)} = id$. On the other hand, by choosing $R > 0$ sufficiently large, we have the gluing identity

$$h_{(H,j)\#_{(\rho;R)}(H,j')} = h_{(H,j;\rho)} \circ h_{(H,j';\rho)}.$$

Therefore we have proved that $h_{(H,j;\rho)} \circ h_{(H,j';\rho)} = id$ on $HF^\lambda_*(H, J^0)$. By the same argument, we also have $h_{(H,j';\rho)} \circ h_{(H,j;\rho)} = id$ on $HF^\lambda_*(H, J^1)$. □

Remark 21.1.7 This corollary can be rephrased into the statement that the assignment $J \in \mathcal{J}_\omega \mapsto HF^\lambda_*(H, J)$ defines a local system on $\mathcal{J}_\omega$ for each given Hamiltonian H and the level $\lambda \in \mathbb{R}$.

Now we relate the above study of filtration changes to the Hofer norm of the Hamiltonian path. We introduce the useful definitions

$$E^-(H) = \int_0^1 -\min_x H_t\,dt, \quad E^+(H) = \int_0^1 \max_x H_t\,dt,$$
$$\|H\| = E^+(H) + E^-(H) = \int_0^1 (\max_x H_t - \min_x H_t)dt$$

in Hofer's geometry. (See (Po01), (Oh05d) for example.)

Note that, when $\mathcal{H}$ is the linear homotopy

$$\mathcal{H}^{\mathrm{lin}} : s \mapsto (1-s)H_1 + sH_2$$

between H_1 and H_2, $E^\pm(\mathcal{H}^{\mathrm{lin}})$ and $E(\mathcal{H}^{\mathrm{lin}})$ just become $E^\pm(H_2 - H_1)$, and $\|H_2 - H_1\|$, respectively. In fact, $E^\pm(H)$ or $E(H)$ can be seen to correspond to the variations of the *linear path*

$$s \in [0,1] \mapsto sH$$

in the sense of Definition 21.1.4. However, when H is non-autonomous, this linear path does not have much intrinsic meaning in terms of the geometry of $\mathrm{Ham}(M,\omega)$ itself.

Remark 21.1.8 We would like to mention that, even when $H \sim F$,

$$\|H\| \neq \|F\|.$$

Therefore the map $H \mapsto \|H\|$ does not push down to the universal covering space $\pi : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathrm{Ham}(M,\omega)$. One standard way of defining an invariant for the elements $h \in \widetilde{\mathrm{Ham}}(M,\omega)$ is by taking the infimum

$$\|h\| := \inf_{[H]=h} \|H\| = \inf_{[H]=h} \mathrm{leng}(\phi_H). \tag{21.1.8}$$

This function

$$h \in \widetilde{\mathrm{Ham}}(M,\omega) \mapsto \|h\| \in \mathbb{R}_+$$

is *not* a priori continuous with respect to the natural topology on $\widetilde{\mathrm{Ham}}(M,\omega)$.

Taking the infimum of $E(\mathcal{H})$ over all $\mathcal{H}$ with fixed end points $H(0) = H^0$ and $H(1) = H^1$, we have the inequality

$$\inf_{\mathcal{H}} \left\{ E(\mathcal{H}) \mid H(0) = H^0,\ H(1) = H^1 \right\} \leq \|H^1 - H^0\|,$$

which is a strict inequality in general. The geometric meaning of the quantity on the left-hand side is precisely the size of Hamiltonian fibrations with asymptotic holonomy $([\phi_{H^0}], [\phi_{H^1}])$ introduced in Definition 4.3.3.

Now we study the Floer chain homotopy map. For this, we consider nondegenerate Hamiltonians H and Floer regular pairs (H, J). Similarly, we will consider only the Floer-regular homotopy $(\mathcal{H}, j)$ connecting those Floer-regular pairs. We also consider homotopy of homotopies, $(\overline{\mathcal{H}}, \overline{j})$ with $\overline{\mathcal{H}} = \{\mathcal{H}_\kappa\}_{0\le\kappa\le 1}$ and $\overline{j} = \{j_\kappa\}_{0\le\kappa\le 1}$, and the induced chain homotopy map $H_{\mathcal{H}} = \mathcal{H}_{(\overline{\mathcal{H}},\overline{j};\rho)}$.

The following proposition shows how the level of a Floer chain α changes under the various Floer operators.

Proposition 21.1.9 *Suppose that ρ is a (positively) monotone cut-off function.*

(1) $\lambda_H(\partial_{(H,J)}(\alpha)) < \lambda_H(\alpha)$ *for an arbitrary Floer chain α.*
(2) $\lambda_{H^1}(h_{(\mathcal{H},j;\rho)}(\alpha)) \le \lambda_{H^0}(\alpha) + E^-(\mathcal{H})$ *for an arbitrary choice of ρ.*
(3) $\lambda_{H^1}(\Upsilon_{\overline{\mathcal{H}}}(\alpha)) \le \lambda_{H^0}(\alpha) + \max_{\kappa\in[0,1]} E^-(\mathcal{H}_\kappa)$.

Proof Statements (1) and (2) are immediate consequences of Corollary 21.1.3.

For the proof of (3), let $[z', w'] \in \operatorname{Crit}\mathcal{A}_{H^1}$ be the peak of the chain. By the definition of the chain map $H_{\mathcal{H}}(\alpha)$, there exists a generator $[z, w] \in \operatorname{Crit}\mathcal{A}_{H^0}$ and a parameter $\kappa \in (0, 1)$ such that the equation

$$\frac{\partial u}{\partial \tau} + J^{\kappa,\rho}\left(\frac{\partial u}{\partial t} - X_{H^{\kappa,\rho}}(u)\right) = 0 \tag{21.1.9}$$

with the asymptotic condition

$$u(-\infty) = [z, w], \quad u(\infty) = [z', w']$$

has a solution for some generator $[z, w]$ of α. Then, by using (21.1.5), we derive

$$\mathcal{A}_{H^1}([z', w']) \le \mathcal{A}_{H^0}([z, w]) + E^-(\mathcal{H}_\kappa). \tag{21.1.10}$$

Since we have chosen $[z', w']$ to be the peak of $\Upsilon_{\mathcal{H}}(\alpha)$, applying (21.1.5) for the pair $(\mathcal{H}_\kappa, j_\kappa)$ using arguments similar to that above and the definition of the level λ_H, we prove

$$\lambda_{H^1}(\Upsilon_{\overline{\mathcal{H}}}(\alpha)) \le \lambda_{H^0}(\alpha) + E^-(\mathcal{H}_\kappa).$$

By taking the supremum of the right-hand side of this inequality over $\kappa \in (0, 1)$, we have proved (3). □

We denote

$$E^-(\overline{\mathcal{H}}) := \max_{\kappa\in[0,1]} E^-(\mathcal{H}_\kappa).$$

Then we have the following corollary of Proposition 21.1.9 (3).

Corollary 21.1.10 *Let (H^0, J^0) and (H^1, J^1) be two Floer-regular pairs. Consider a generic homotopy of homotopies, $(\overline{\mathcal{H}}, \overline{j})$ with*

$$\overline{\mathcal{H}} = \{\mathcal{H}_\kappa\}_{0\le\kappa\le1}, \qquad \overline{j} = \{j_\kappa\}_{0\le\kappa\le1},$$

where each $\mathcal{H}_\kappa$ is a homotopy connecting (H^0, J^0) and (H^1, J^1). Then the induced chain homotopy map $H_{\mathcal{H}} = \mathcal{H}_{(\overline{\mathcal{H}},\overline{j};\rho)}$ satisfies

$$\Upsilon_{\mathcal{H}}(CF^\lambda(H^0)) \subset CF^{\lambda+E^-(\overline{\mathcal{H}})}(H^1).$$

Finally, we relate the filtration changes to the ϵ-regularity type invariants associated with the perturbed Cauchy–Riemann equations. We first remark that our family $J = \{J_t\}_{0\le t\le1}$ is a compact family in that $[0, 1]$ is a compact set.

Let H be a given nondegenerate Hamiltonian function and consider the perturbed Cauchy–Riemann equation

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0$$

for each H-regular J. We call a solution *u stationary* if it is τ-independent. We define

$$A_{(H,J)} := \inf\left\{\int \left|\frac{\partial u}{\partial\tau}\right|_J^2 \,\middle|\, u \text{ satisfies (18.5.41) and is not stationary}\right\} > 0$$

and

$$A^\mu_{(H,J)} := \inf\left\{\int \left|\frac{\partial u}{\partial\tau}\right|_J^2 \,\middle|\, u \text{ satisfies (18.5.41) and } \mu_H(u) = 1\right\}.$$

Obviously we have $A^\mu_{(H,J)} \ge A_{(H,J)} > 0$.

Then we can strengthen the statement (1) of Proposition 21.1.9 to the inequality

$$\lambda_H(\partial_{(H,J)}(\alpha)) \le \lambda_H(\alpha) - A^\mu_{(H,J)} \tag{21.1.11}$$

for an arbitrary Floer chain α.

21.2 The boundary depth of the Hamiltonian *H*

Before we launch into the construction of spectral invariants $\rho(H; a)$, we first define a more simple-minded invariant associated with the Hamiltonian H, namely the boundary depth $\beta(H)$, which is a special case of the one introduced in Definition 13.8.5 of Section 13.8. We will show that $\beta(H)$ is an invariant of the isomorphism type of the filtered chain complex $CF_*(H)$.

Definition 21.2.1 (Boundary depth) Let H be a given nondegenerate Hamiltonian and assume (H,J) to be Floer-regular. Let $(CF_*(H,J), \partial_{(H,J)})$ be the associated Floer complex. The boundary depth $\beta(H,J)$ is defined to be $\beta(H,J) := \beta(\partial_{(H,J)})$, i.e.,

$$\beta(H,J) = \inf\{\beta \in \mathbb{R} \mid \forall \lambda \in \mathbb{R},\ CF_*^\lambda(H,J) \cap \partial CF_*(H,J) \subset \partial(CF_*^{\lambda+\beta}(H,J))\}.$$

The following lemma is an immediate consequence of Corollary 21.1.6

Lemma 21.2.2 *Let H be nondegnenerate and J^0, J^1 be a pair for which both (H,J^0) and (H,J^1) are Floer-regular. Then*

$$\beta(H,J^0) = \beta(H,J^1).$$

Proof We will prove that $\beta(H,J^0) \le \beta(H,J^1)$ and vice versa. It will suffice to prove that, for any $\beta \ge 0$ satisfying

$$CF_*^\lambda(H,J^0) \cap \partial CF_*(H,J^0) \subset \partial(CF_*^{\lambda+\beta}(H,J^0)),$$

we have

$$CF_*^\lambda(H,J^1) \cap \partial CF_*(H,J^1) \subset \partial(CF_*^{\lambda+\beta+\epsilon}(H,J^1)) \tag{21.2.12}$$

for any $\epsilon > 0$. Choose a path j from J^0 and J^1 that is regular relative to H and denote by $h_{(H,j)}$ the chain map from $CF_*(H,J^0)$ to $CF_*(H,J^1)$. Then $h_{(H,j)}$ restricts to the chain isomorphism $h^\lambda_{(H,j)} : CF_*^\lambda(H,J^0) \to CF_*^\lambda(H,J^1)$ for any $\lambda \in \mathbb{R} \setminus \mathrm{Spec}(H)$ by Corollary 21.1.6. But we also have

$$h^\lambda_{(H,j)} \circ \partial_{(H,J^0)} = \partial_{(H,J^1)} \circ h^\lambda_{(H,j)}.$$

From this, (21.2.12) holds even with $\epsilon = 0$ if $\lambda+\beta \in \mathbb{R}\setminus\mathrm{Spec}(H)$ and otherwise it holds for any $\epsilon > 0$ such that $\lambda + \beta + \epsilon \in \mathbb{R} \setminus \mathrm{Spec}(H)$. Since $\mathbb{R} \setminus \mathrm{Spec}(H)$ is dense in $\mathbb{R}$, we can choose $\epsilon > 0$ arbitrarily small, which finishes the proof of (21.2.12). By changing the role of J^0 and J^1, we have finished the proof. □

On the basis of this lemma, we just denote $\beta(H) = \beta(H,J)$ for a particular (and hence any) J such that (H,J) is Floer-regular. The following basic properties of the boundary depth were proved by Usher in (Ush11). (See Proposition 8.8 in (Oh09a) for a similar statement to (2) in this theorem.)

Theorem 21.2.3 (Usher) *The boundary depth β satisfies the following properties.*

(1) *If H and K are two nondegenerate Hamiltonians we have*

$$|\beta(H) - \beta(K)| \le \|H - K\|.$$

In particular, β is a Lipschitz continuous function with respect to the $L^{(1,\infty)}$ topology and thus continuously extends to a nonnegative function in $L^{(1,\infty)}$ topology.

(2) $\beta(H) \leq \|H\|$.

(3) *If H and K are two nondegenerate Hamiltonians such that $H \sim K$, then $\beta(H) = \beta(K)$.*

Proof Consider the linear homotopy $\mathcal{H}^{\rm lin} : s \mapsto (1-s)H + sK$. We may assume that $(\mathcal{H}^{\rm lin}, J)$ is Floer-regular for a suitable choice of J. Again we will prove the inequality by proving

$$\beta(H) \leq \beta(K) + \|H - K\| \tag{21.2.13}$$

and the corresponding version with H and K reversed. But the same kind of argument as the proof of Lemma 21.2.2, this time using Corollary 21.1.5 for $\mathcal{H} = \mathcal{H}^{\rm lin}$, proves (21.2.13). This finishes the proof of (1).

The statement (2) follows from (1) by letting $K \to 0$.

Finally, the same proof as Lemma 21.2.2 applies to the proof of (3), this time using Corollary 21.1.5 in Section 21.6.2. □

As mentioned by Usher (Ush11), the above theorem shows that the boundary depth $\beta(H)$ is an invariant of the isomorphism type of the filtered chain complex $CF_*(H, J)$. An interesting fact regarding the boundary depth that differentiates it from the spectral invariants $\rho(H; a)$ which we will construct in the next chapter is that it is really an invariant of the complex $CF_*(H, J)$ that does not arise from a homological reason, unlike for the spectral invariants.

We refer readers to (Ush11), (Ush12) and (Ush13) for interesting applications of the boundary depth to various problems in symplectic topology and Hamiltonian dynamics.

21.3 Definition of spectral invariants and their axioms

In this section, we do *not* assume that H is normalized, but for any given Hamiltonian we denote its normalization by $\underline{H}$ defined by

$$\underline{H}(t, x) = H(t, x) - \frac{1}{\mathrm{vol}_\omega(M)} \int_M H(t, x)\, \mu_\omega. \tag{21.3.14}$$

21.3.1 Definition of spectral invariants

For each given (homogeneous) quantum cohomology class $a \in QH^*(M)$, we denote by $a^\flat = a^\flat_H \in HF_*(H, J)$ the image under the isomorphism Φ_H. We denote by

$$i_\lambda : HF^\lambda_*(H, J) \to HF_*(H, J)$$

the canonical inclusion-induced homomorphism.

Definition 21.3.1 Let H be a nondegenerate one-periodic Hamiltonian and J be H-regular. For any given $0 \neq a \in QH^*(M)$, we consider Floer cycles $\alpha \in \ker \overline{\partial}_{(H,J)} \subset CF_*(H)$ of the pair (H, J) representing $a^\flat$. Then we define

$$\rho((H, J); a) := \inf_{\alpha;[\alpha]=a^\flat} \lambda_H(\alpha),$$

or equivalently

$$\rho((H, J); a) := \inf\{\lambda \in \mathbb{R} \mid a^\flat \in \operatorname{Im} i_\lambda \subset HF_*(H, J)\}.$$

The following theorem is the basic fundamental theorem concerning nontriviality of spectral numbers. Usher (Ush08) gave a purely algebraic proof of the finiteness of $\rho((H, J); a)$ in his abstract formulation of Floer homology. The proof here is an adaptation of the proofs of the corresponding statements in (Oh05c), (Oh06a) and (Ush08).

Theorem 21.3.2 *Suppose that H is nondegenerate and let $0 \neq a \in QH^*(M)$.*

(1) *We have $\rho((H, J); a) > -\infty$ for any H-regular J.*
(2) *The definition of $\rho((H, J); a)$ does not depend on the choice of H-regular Js. We denote by $\rho(H; a)$ the common value.*

Proof By Theorem 21.2.3 (2) and by the definition of the boundary depth, we have

$$\partial(CF_*(H, J)) \cap CF^\lambda_*(H, J) \subset \partial(CF^{\lambda+\|H\|}_*(H, J)). \tag{21.3.15}$$

The next proposition is an important ingredient in the Floer-theoretic mini-max theory which corresponds to the crucial *linking property* of the 'mini-maxing sets' in classical critical-point theory (see, e.g., (Bn82) for such a rationale).

Proposition 21.3.3 *Let α be a ∂-cycle in $CF_*(H, J)$ with non-zero homology class $[\alpha]$ for $\partial = \partial_{(H,J)}$. Then we have*

$$\inf_{\beta \in CF_*(H,J)} \{\lambda_H(\alpha - \partial\beta)\} > -\infty. \tag{21.3.16}$$

Proof Suppose to the contrary that there exists a sequence $\beta_j \in CF_*(H, J)$ such that $\lambda_H(\alpha - \partial\beta_j) \to -\infty$. Denote

$$\lambda_j = \lambda_H(\alpha - \partial\beta_j). \tag{21.3.17}$$

By choosing a subsequence, we may assume $\lambda_j > \lambda_{j+1}$ for all j. Now we will inductively construct a sequence β'_k such that they satisfy

$$\lambda_H(\alpha - \partial\beta'_k) \leq \lambda_k \tag{21.3.18}$$

$$\lambda_H(\beta'_{k+1} - \beta'_k) \leq \lambda_k + \|H\|. \tag{21.3.19}$$

Once we have constructed such a sequence, we consider the sequence of chains

$$\alpha_N := \alpha - \partial\beta'_N.$$

By definition of the chains α_N, it follows from (21.3.18) and the assumption $\lambda_j \to -\infty$ that $[\alpha_N] = [\alpha]$, and $\alpha_N \to 0$ in the λ_H non-Archimedean topology. On the other hand, (21.3.19) implies β'_j converges in the λ_H non-Archimedean topology of $CF_*(H, J)$. Denote the limit by β'_∞. Since ∂ is a continuous operator (with respect to the non-Archimedean topology on $CF_*(H, J)$), it follows $\partial\beta'_N \to \partial\beta'_\infty$ and so $\alpha = \partial\beta'_\infty$, a contradiction to $[\alpha] \neq 0$.

Therefore it remains to construct such a sequence β'_j under the standing hypothesis $\lambda_H(\alpha - \partial\beta_j) \to -\infty$.

For $j = 1$, we just define $\beta'_1 = \beta_1$. We will prove that under the assumption that β'_k satisfying (21.3.18) is given for k, there exists β'_{k+1} so that both (21.3.19) and (21.3.18) hold for $k + 1$.

Suppose we have constructed such β'_j for $j = k$ satisfying (21.3.18). Since $\lambda_H(\alpha - \partial\beta'_k) \leq \lambda_k$ and $\lambda_H(\alpha - \partial\beta_{k+1}) = \lambda_{k+1}$ and $\lambda_{k+1} < \lambda_k$, it follows

$$\lambda_H(\partial(\beta_{k+1} - \beta'_k)) \leq \lambda_k.$$

Therefore it follows from (21.3.15) that there exists $\gamma'_k \in CF_*^{\lambda_k + \|H\|}(H, J)$ such that

$$\partial(\beta_{k+1} - \beta'_k) = \partial\gamma'_k.$$

If we define $\beta'_{k+1} = \beta'_k + \gamma'_k$,

$$\lambda_H(\beta'_{k+1} - \beta'_k) = \lambda_H(\gamma'_k) \leq \lambda_k + \|H\|$$

as required. This proves (21.3.19). Next we prove

Lemma 21.3.4 *The chain $\beta'_{k+1} = \beta'_k + \gamma'_k$ satisfies* (21.3.18) *with k replaced by $k + 1$ and* (21.3.19).

Proof We compute

$$\alpha - \partial\beta'_{k+1} = \alpha - \partial(\beta'_k + \gamma'_k) = \alpha - \partial\beta_{k+1} \tag{21.3.20}$$

by definition of γ'_k above. Therefore $\lambda_H(\alpha - \partial\beta'_{k+1}) \leq \lambda_{k+1}$ as required in (21.3.18) with k replaced by $k + 1$. This finishes the proof. □

Combining the above discussion, we have finished the proof of the proposition. □

Proposition 21.3.3 immediately implies finiteness of $\rho((H, J); a)$ for any $0 \neq a \in QH^*(M)$ by definition. Now we prove independence of $\rho((H, J); a)$ on J. Let J' be another H-regular one, and $j = \{J^s\}_{0\leq s\leq 1}$ be a path with $J^0 = J$, $J^1 = J'$. We will prove

$$\rho((H, J'); a) \leq \rho((H, J); a). \tag{21.3.21}$$

By changing the role of J' and J, we then get the opposite inequality which will finish the proof.

For any given $\epsilon > 0$, we choose a Floer cycle α of (H, J) satisfying

$$\lambda_H(\alpha) < \rho((H, J); a) + \epsilon.$$

We consider the transfer map

$$h_{(H,j)} : CF(H, J) \to CF(H, J')$$

considered in Corollary 21.1.6. Then the transferred cycle $\alpha' := h_{(H,j)}(\alpha)$ of (H, J') satisfies $\lambda_H(\alpha') \leq \lambda_H(\alpha)$ by Corollary 21.1.6. By definition of $\rho((H, J'); a)$, we also have $\rho((H, J'); a) \leq \lambda_H(\alpha')$. Combining the above inequalities, we obtain

$$\rho((H, J'); a) \leq \rho((H, J); a) + \epsilon.$$

Since this holds for all $\epsilon > 0$, we have proved (21.3.21). This finishes the proof of Theorem 21.3.2. □

Remark 21.3.5 An examination of the above proof of Proposition 21.3.3 shows that the only place where we use the particular Floer chain complex $CF_*(H, J)$ is for the finiteness of the boundary depth given in Theorem 21.2.3 (2). Indeed the proof of finiteness of the spectral invariants stated in Theorem 13.8.4 in the abstract context of Floer theory can be proved by verbatim following the above proof once the finiteness of the boundary depth of the general differential ∂ is established.

Now the following proposition can be proven by considering the homotopy connecting H and K that is arbitrarily close to the linear homotopy

$$s \mapsto (1-s)H + sK.$$

Proposition 21.3.6 *For any nondegenerate H, K, we have*

$$-E^+(H-K) \le \rho(H;a) - \rho(K;a) \le E^-(H-K). \tag{21.3.22}$$

In particular, $\rho_a : H \mapsto \rho(H;a)$ is continuous in the C^0 topology (or in the $L^{(1,\infty)}$ topology) and hence can be continuously extended to $C^0([0,1]\times M;\mathbb{R})$.

Proof Let $\delta > 0$ be any given number. We choose a cycle α of H so that $[\alpha] = a^\flat$ and

$$\lambda_H(\alpha) \le \rho(H;a) + \delta. \tag{21.3.23}$$

We would like to emphasize that *this is possible, because we have already shown that* $\rho(H;a) > -\infty$.

By taking a generic approximation of the linear homotopy h^{lin}_{HK} from H to K and using Corollary 21.1.5, we can find some $\mathcal{H}$ connecting H and K such that

$$\lambda_K(h_{\mathcal{H}}(\alpha)) \le \lambda_H(\alpha) + \int -\min_x (K_t - H_t)\,dt + \delta.$$

On the other hand (21.3.23) implies

$$\lambda_H(\alpha) + \int -\min_x (K_t - H_t)\,dt + \delta \le \rho(H;a) + 2\delta + \int -\min_x (K_t - H_t)\,dt.$$

Since $[h^{\mathrm{lin}}_{HK}(\alpha)] = a^\flat$, we have

$$\lambda_K(h^{\mathrm{lin}}_{HK}(\alpha)) \ge \rho(K;a)$$

by the definition of $\rho(K;a)$. Combining these inequalities, we have derived

$$\rho(K;a) - \rho(H;a) \le 2\delta + \int_0^1 -\min_x (K_t - H_t)\,dt.$$

Since this holds for arbitrary δ, we obtain

$$\rho(K;a) - \rho(H;a) \le \int_0^1 -\min_x (K_t - H_t)\,dt.$$

By changing the role of H and K, we also prove that

$$\rho(H;a) - \rho(K;a) \le \int_0^1 -\min_x (H_t - K_t)\,dt = \int_0^1 \max_x (K_t - H_t)\,dt.$$

Hence,

$$\int_0^1 -\max_x(K_t - H_t)dt \le \rho(K;a) - \rho(H;a) \le \int_0^1 -\min_x(K_t - H_t)dt,$$

which finishes the proof of (21.3.22). Obviously, (21.3.22) enables us to extend the definition of ρ by continuity to arbitrary $L^{1,\infty}$ Hamiltonians. This finishes the proof. □

21.3.2 Axioms of spectral invariants

In this subsection, we state basic properties of the function ρ in a list of axioms.

Theorem 21.3.7 *Let (M, ω) be an arbitrary closed symplectic manifold. For any given quantum cohomology class $0 \neq a \in QH^*(M)$, we have a continuous function denoted by*

$$\rho : C^\infty(S^1 \times M, \mathbb{R}) \times QH^*(M) \to \mathbb{R}$$

such that the following axioms are satisfied. Let H, $F \in \mathcal{H}_m$ be smooth (time-dependent) Hamiltonian functions and $a \neq 0 \in QH^*(M)$.

(1) *If $r : [0,1] \to \mathbb{R}$ is smooth, then*

$$\rho(H + r; a) = \rho(H; a) - \int_0^1 r(t)dt.$$

(2) *(Projective invariance) $\rho(H; \lambda a) = \rho(H; a)$ for any $0 \neq \lambda \in \mathbb{Q}$.*
(3) *(Normalization) For $a = \sum_{A\in\Gamma} a_A q^{-A}$, we have $\rho(\underline{0}; a) = v(a)$, where $\underline{0}$ is the zero function and*

$$v(a) := \min\{\omega(-A) \mid a_A \neq 0\} = -\max\{\omega(A) \mid a_A \neq 0\}$$

is the (upward) valuation of a.
(4) *(Symplectic invariance) $\rho(\eta_* H; \eta_* a) = \rho(H; a)$ for any symplectic diffeomorphism η. In particular, if $\eta \in \mathrm{Symp}_0(M, \omega)$ then we have $\rho(\eta_* H; a) = \rho(H; a)$.*
(5) *(Triangle inequality) $\rho(H \# F; a \cdot b) \le \rho(H; a) + \rho(F; b)$.*
(6) *(Hamiltonian continuity) $|\rho(H; a) - \rho(F; a)| \le \|H \# \overline{F}\| = \|H - F\|$, where $\|\cdot\|$ is the $L^{(1,\infty)}$-norm on $C^\infty(S^1 \times M, \mathbb{R})$. In particular, the function $\rho_a : H \mapsto \rho(H; a)$ is C^0-continuous.*
(7) *(Additive triangle inequality) $\rho(H; a + b) \le \max\{\rho(H; a), \rho(H; b)\}$.*

Proof For the property (1), we observe that the two Hamiltonians $H + r$ and H carry exactly the same Hamiltonian flows, the same Floer equations and the corresponding off-shell setting. The only difference between the two cases is the shift of the action functional by

$$-\int_0^1 r(t)dt$$

from which (1) follows by the construction of the spectral invariant.

The projective invariance (2) is also obvious from the construction. The C^0-continuity is an immediate consequence of Proposition 21.3.6. We postpone the proof of the triangle inequality until Section 21.4. For the proof of symplectic invariance, we consider the symplectic conjugation

$$\phi \mapsto \eta\phi\eta^{-1}; \quad \mathrm{Ham}(M,\omega) \to \mathrm{Ham}(M,\omega)$$

for any symplectic diffeomorphism $\eta : (M,\omega) \to (M,\omega)$. Recall that the push-forward function $\eta_* H$ given by

$$\eta_* H(t,x) = H(t, \eta^{-1}(x))$$

generates the conjugation $\eta\phi\eta^{-1}$ when $H \mapsto \phi$. We summarize some basic facts on this conjugation that are relevant to the filtered Floer homology here.

(1) When $H \mapsto \phi$, $\eta_* H \mapsto \eta\phi\eta^{-1}$.
(2) If H is nondegenerate, $\eta_* H$ is also nondegenerate.
(3) If (H, J) is Floer-regular, then so is $(\eta_* J, \eta_* H)$.
(4) There exists a natural bijection $\eta_* : \mathcal{L}_0(M) \to \mathcal{L}_0(M)$ defined by

$$\eta_*([z,w]) = ([\eta \circ z, \eta \circ w]),$$

under which we have the identity

$$\mathcal{A}_H([z,w]) = \mathcal{A}_{\eta_* H}(\eta_*[z,w]).$$

(5) $\mathrm{Symp}(M,\omega)$ naturally acts on $QH^*(M)$ so that, if $\alpha = \sum_{[z,w]} a_{[z,w]}[z,w]$ represents $a^\flat$, then $\eta_*(\alpha) := \sum_{[z,w]} a_{[z,w]}\eta_*([z,w])$ represents $(\eta^* a)^\flat$.
(6) The L^2-gradients of the corresponding action functionals satisfy

$$\eta_*(\mathrm{grad}_J \mathcal{A}_H)([z,w]) = \mathrm{grad}_{\eta_* J}(\mathcal{A}_{\eta_* H})(\eta_*([z,w])).$$

(7) If $u : \mathbb{R}\times S^1 \to M$ is a solution of the perturbed Cauchy–Riemann equation for (H, J), then $\eta_* u = \eta \circ u$ is a solution for the pair $(\eta_* J, \eta_* H)$. In addition, all the Fredholm properties of (J, H, u) and $(\eta_* J, \eta_* H, \eta_* u)$ are the same.
(8) $\eta \in \mathrm{Symp}(M,\omega)$ acts on $QH^*(M)$ by the pull-back $a \mapsto \eta^* a$.

These facts imply that the conjugation by η induces a canonical filtration-preserving chain isomorphism,

$$\eta_* : (CF_*^\lambda(H), \partial_{(H,J)}) \to (CF_*^\lambda(\eta_* H), \partial_{(\eta_* H, \eta_* J)}),$$

for any $\lambda \in \mathbb{R} \setminus \mathrm{Spec}(H) = \mathbb{R} \setminus \mathrm{Spec}(\eta_* H)$. In particular, it induces a filtration preserving isomorphism

$$\eta_* : HF_*^\lambda(H, J) \to HF_*^\lambda(\eta_* H, \eta_* J)$$

in homology. The symplectic invariance is then an immediate consequence of the Floer mini-max procedure used in the construction of $\rho(H; a)$. □

Recall that until now in the construction of $\rho(H; a)$ we have assumed that the Hamiltonian H is one-periodic, which is an artifact of the construction of Hamiltonian Floer homology. However, this artifact can be removed by applying a simple soft trick of reparameterizing of Hamiltonian paths. This extension makes the usage of spectral invariants in applications very flexible and turns out to be important in various applications to continuous Hamiltonian dynamics. (See (Oh10) for example.) Now we explain how to dispose the one-periodicity of the Hamiltonian and extend the definition of $\rho(H; a)$ for arbitrary time-dependent Hamiltonians $H : [0, 1] \times M \to \mathbb{R}$. Note that it is obvious that the Hofer norm $\|H\|$ is defined without assuming the periodicity.

Out of the given Hamiltonian H, which is *not necessarily periodic*, we consider the time-periodic Hamiltonian of the type H^ζ, where ζ is a reparameterization of $[0, 1]$ of the type

$$\zeta(t) = \begin{cases} 0 & \text{for } 0 \le t \le \epsilon_0/2, \\ 1 & \text{for } 1 - \epsilon_0/2 \le t \le 1 \end{cases} \tag{21.3.24}$$

and

$$\zeta'(t) \ge 0 \quad \text{for all} \quad t \in [0, 1],$$

and the reparameterized Hamiltonian by H^ζ is given by

$$H^\zeta(t, x) = \zeta'(t) H(\zeta(t), x),$$

which generates the Hamiltonian isotopy $t \mapsto \phi_H^{\zeta(t)}$ in general. Since $\phi_{H^{\zeta_0}}$ and $\phi_{H^{\zeta_1}}$ are homotopic to each other for any two such reparameterization functions ζ_0 and ζ_1, we have $\rho(H^{\zeta_0}; a) = \rho(H^{\zeta_1}; a)$. This enables us to write the following definition.

Definition 21.3.8 Let H be any time-dependent, not necessarily one-periodic, Hamiltonian. Then we define

$$\rho(H; a) := \rho(H^\zeta; a) \tag{21.3.25}$$

for a particular (and hence any) reparameterization ζ.

Using the one-to-one correspondence between normalized H and its associated Hamiltonian path $\phi_H : t \mapsto \phi_H^t$, we define the spectral function as a function on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), id)$

$$\rho_a : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), id) \to \mathbb{R}.$$

Definition 21.3.9 (Spectral invariants of a Hamiltonian path) Let λ be a Hamiltonian path. We define the spectral invariant $\rho(\lambda; a)$ by

$$\rho(\lambda; a) := \rho(\mathrm{Dev}(\lambda); a),$$

i.e., if $\lambda = \phi_H$, $\rho(\lambda; a) = \rho(\underline{H}; a)$.

We denote by $\widetilde{\mathrm{Ham}}(M, \omega)$ the set of path homotopy classes on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), id)$, i.e., the universal covering space of $\mathrm{Ham}(M, \omega)$. We equip $\widetilde{\mathrm{Ham}}(M, \omega)$ with the quotient topology. An important question to ask is whether we have the equality

$$\rho(\lambda; a) = \rho(\mu; a)$$

when λ is path-homotopic to μ relative to the ends, and descends to $\widetilde{\mathrm{Ham}}(M, \omega)$. We will discuss this later in Section 21.5.

21.4 Proof of the triangle inequality

To start with the proof of the triangle inequality, we need to recall the definition of the 'pants product'

$$HF_*(H_1, J^1) \otimes HF_*(H_2, J^2) \to HF_*(H_3, J^3).$$

For the purpose of studying the effect on the filtration under the product, we need to define this product in the chain level in an optimal way as in (Oh99), (Schw00).

The structure constants of the pants product in Floer homology in the chain level are provided by the 'number' of rigid solutions of perturbed pseudoholomorphic equations that intertwine the Floer equation

$$\frac{\partial u}{\partial \tau} + J_i \left(\frac{\partial u}{\partial t} - X_{H_i}(u) \right) = 0$$

for $i = 1, 2, 3$ near the punctures. Since the Floer equation specifically uses the cylindrical nature of the domain $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and the coordinates (τ, t) while the general Riemann surface S does not carry global coordinates, one needs to use an invariant form of the Floer equation

$$(du + P_K(u))_J^{(0,1)} = 0$$

in its construction. For the proof of the triangle inequality, the relevant triple of Hamiltonians $(H_1, H_2 : H_3)$ must be such that the associated conjugacy classes satisfy

$$[\phi_{H_1}]^{-1}[\phi_{H_2}]^{-1}[\phi_{H_3}] = id.$$

For example, we can consider the choice of Hamiltonians

$$H_3 = H_1 \# H_2$$

noting that the triple $(\overline{H}_1, \overline{H}_2, H_1 \# H_2)$ is the triple whose holonomy class defined in Section 4.3.1 is trivial.

The following lemma is an important lemma in the study of the triangle inequality of the spectral invariants. Here we regard $\mathrm{curv}(\Gamma)$ or R_K as the $\dot{\Sigma}$-family of Hamiltonians.

Definition 21.4.1 Let $K \in C^\infty(\dot{\Sigma}, C_0^\infty(M))$. We define the L^∞ Hofer norm of K by

$$\|K\|_\infty := \sup_{z \in \dot{\Sigma}} \operatorname{osc} K_z = \sup_{z \in \dot{\Sigma}} (\max K_z - \min K_z) \tag{21.4.26}$$

and $L^{(1,\infty)}$ Hofer norm of K by

$$\|K\|_{1,\infty} := \int_{\dot{\Sigma}} \operatorname{osc} K_z \, dA_\Sigma = \int_{\dot{\Sigma}} (\max K_z - \min K_z) dA_\Sigma. \tag{21.4.27}$$

Lemma 21.4.2 *Let the H_i be a triple with $H_1 \# H_2 = H_3$ and consider the holonomy $(\overline{H}_1, \overline{H}_2, H_1 \# H_2)$. Then, for any given $\delta > 0$, we can choose a Hamiltonian connection Γ with its holonomy class given by $C = ([\phi_{H_1}^{-1}], [\phi_{H_2}^{-1}], [\phi_{H_1}\phi_{H_2}])$ so that its curvature has a norm satisfying*

$$\|\mathrm{curv}(\Gamma)\|_\infty = \|R_K\|_\infty \leq \delta.$$

Proof We first recall the description of conformal structures on a compact Riemann surface Σ of genus 0 with three punctures in terms of the *minimal-area metric* (Z93) whose explanation is now in order.

We conformally identify Σ with the union of three half-cylinders, which we denote by Σ_1, Σ_2 and Σ_3 in the following way: the conformal structure is realized by the union of three half-cylinders Σ_i with flat metrics, with each meridian circle having length 2π. The metric is singular only at two points

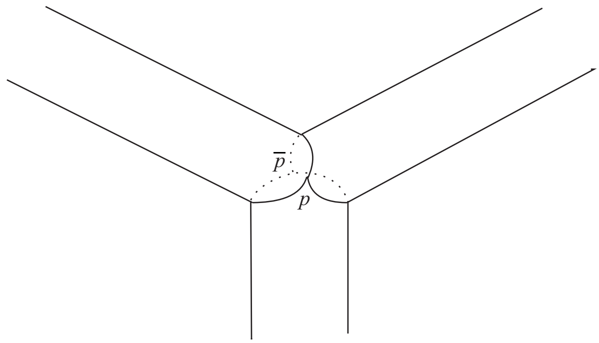

Figure 21.1 A minimal-area metric.

p, $\overline{p} \in \Sigma$ that lie on the boundary circles of Σ_i. Therefore the conformal structure induced from the metric naturally extends over the two points $p, \overline{p}$. The resulting conformal structure is the standard unique conformal structure on $\Sigma = S^2 \setminus \{z_1, z_2, z_3\}$. One important property of this *singular* metric is the flatness everywhere except at the two points $p, \overline{p}$, where the metric is singular but Lipschitz. Such a metric is an example of the so-called minimal area metric, which was extensively studied by Zwiebach (Z93) in relation to the construction of *string vertices* in the string field theory. See Figure 21.1.

We identify Σ as the union of Σ_is

$$\Sigma = \cup_{i=1}^3 \Sigma_i$$

in the following way: if we identify Σ_i with $(-\infty, 0] \times S^1$, then there are three paths θ_i of length $\frac{1}{2}$ for $i = 1, 2, 3$ in Σ connecting p to $\overline{p}$ such that

$$\begin{aligned}
\partial_1\Sigma &= \theta_1 \circ \theta_3^{-1}, \\
\partial_2\Sigma &= \theta_2 \circ \theta_1^{-1}, \\
\partial_3\Sigma &= \theta_3 \circ \theta_2^{-1} = (\theta_1 \circ \theta_3^{-1})^{-1} \circ (\theta_2 \circ \theta_1^{-1})^{-1}.
\end{aligned} \tag{21.4.28}$$

We note that, in these coordinates, the points p, $\overline{p}$ correspond to

$$(s,t) = (0,0), \quad (s,t) = (0, \tfrac{1}{2}).$$

We fix a holomorphic identification of each Σ_i, $i = 1, 2$, with $(-\infty, 0] \times S^1$ with $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and Σ_3 with $[0, \infty) \times S^1$ considering the decomposition (21.4.28). We denote the identification by

$$\varphi_i^- : \Sigma_i \to (-\infty, 0] \times S^1, \quad i = 1, 2$$

for positive punctures and

$$\varphi_3^+ : \Sigma_3 \to [0, \infty) \times S^1$$

for the negative puncture. We denote by (τ, t) the standard cylindrical coordinates on the cylinders.

Now we construct an explicit two-parameter family of Hamiltonian diffeomorphisms on the three copies of $[0, 1]^2$. After reparameterizing the isotopies in time t using the kind of function (21.3.24), we may assume that ϕ_i is constant in time near $t = 0,\ 1$.

Here are the formulae for the ϕ_i:

$$\phi_1(s,t) = \begin{cases} \phi_{H_1}^{(1-s)t/(1-s/2)}, & 0 \le t \le 1 - s/2, \\ \phi_{H_1}^{(1-s)} \phi_{H_2}^{2(t-1+s/2)/s}, & 1 - s/2 \le t \le 1, \end{cases}$$

$$\phi_2(s,t) = \begin{cases} \phi_{H_1}^{2t}, & 0 \le t \le s/2, \\ \phi_{H_1}^{s} \phi_{H_2}^{(t-s/2)/(1-s/2)}, & s/2 \le t \le 1, \end{cases}$$

$$\phi_3(s,t) = \begin{cases} \phi_{H_1}^{t(1+s)} \phi_{H_2}^{t(1-s)}, & 0 \le t \le 1 - s/2, \\ \phi_{H_1}^{(1+s)(1-s/2)} \phi_{H_2}^{2(t-1+s/2)/s}, & 1 - s/2 \le t \le 1. \end{cases}$$

Exercise 21.4.3 Check the compatibility with the gluing rule (21.4.28) and the smoothness of the family away from p, $\overline{p}$.

Furthermore,

$$\phi_1(0,t) = \phi_{H_1}^t, \qquad \phi_2(0,t) = \phi_{H_2}^t, \qquad \phi_3(1,t) = \phi_{H_1}^t \phi_{H_2}^t$$

can also be seen from the formula. The flatness away from p, $\overline{p}$ of the associated connection obviously follows from the fact that it comes from the above smooth two-parameter family of Hamiltonian diffeomorphisms.

Now we elongate the pants in the direction of s using elongation functions ρ, which gives rise to a flat connection on Σ that is smooth everywhere except at two points p, $\overline{p}$. We denote by

$$K : \dot{\Sigma} \to \Omega^1(\dot{\Sigma}, C_0^\infty(M))$$

the connection one-form on the trivial bundle $\dot{\Sigma} \times M \to \dot{\Sigma}$. This is smooth everywhere including p, $\overline{p}$ thanks to the boundary flatness of H_1, H_2 and has curvature 0. This finishes the proof. □

Remark 21.4.4 There is another realization of the homotopy class $[H_1 \# H_2]$ by the concatenation Hamiltonian $H_1 * H_2$ defined by

$$H_1 * H_2(t,x) = \begin{cases} 2H_1(2t, x), & t \le 1/2, \\ 2H_2(2t-1, x), & t \ge 1/2 \end{cases}$$

instead of $H_1 \# H_2$. Here we assume that both H_1 and H_2 are boundary flat near $t = 0, 1$ in order to make the associated connection smooth. This choice brings a somewhat different Hamiltonian fibration associated with the triple $(H_1, H_2; H_1 * H_2)$. However, since $H_1 \# H_2$ is homotopic to $H_1 * H_2$, one can use this as the third Hamiltonian H_3, with which the construction of the flat connection is more apparent than with $H_1 \# H_2$. To produce such a Hamiltonian fibration, we need only consider $\phi_3(s,t) \equiv \phi_3(0,t)$ by noting that

$$\phi_3(0,t) = \begin{cases} \phi_{H_1}^{2t}, & 0 \le t \le \frac{1}{2}, \\ \phi_{H_1}^1 \phi_{H_2}^{2t-1}, & \frac{1}{2} \le t \le 1, \end{cases}$$

which is exactly the path $\phi_{H_1} * \phi_{H_2} : t \mapsto \phi_{H_1}^t * \phi_{H_2}^t$. Indeed, the formula for ϕ_3 above defines an explicit homotopy between $\phi_{H_1 * H_2}$ and $\phi_{H_1}\phi_{H_2}$. (We refer the reader to (FOOO11b) for more details.)

Now, with this preparation, we are ready to prove the triangle inequality.

Let $\alpha \in CF_*(H)$ and $\beta \in CF_*(F)$ be Floer cycles with $[\alpha] = a^\flat$, $[\beta] = b^\flat$ and consider their pants-product cycle $\alpha * \beta := \gamma \in CF_*(H \# F)$. Then we have

$$[\alpha * \beta] = (a \cdot b)^\flat$$

and so

$$\rho(H \# F; a \cdot b) \le \lambda_{H \# F}(\alpha * \beta). \tag{21.4.29}$$

Let $\delta > 0$ be any given number and choose $\alpha \in CF_*(H)$ and $\beta \in CF_*(F)$ so that

$$\lambda_H(\alpha) \le \rho(H; a) + \frac{\delta}{2}, \tag{21.4.30}$$

$$\lambda_F(\beta) \le \rho(F; b) + \frac{\delta}{2}. \tag{21.4.31}$$

Then we have the expressions

$$\alpha = \sum_i a_i [z_i, w_i] \text{ with } \mathcal{A}_H([z_i, w_i]) \le \rho(H; a) + \frac{\delta}{2},$$

$$\beta = \sum_j a_j [z_j, w_j] \text{ with } \mathcal{A}_F([z_j, w_j]) \le \rho(F; b) + \frac{\delta}{2}.$$

Now using the pants product (20.2.29), we would like to estimate the level of the chain $\alpha * \beta \in CF_*(H \# F)$.

We recall the identities

$$\int v^* \Omega_E = -\mathcal{A}_{H_1 \# H_2}([z_3, w_3]) + \mathcal{A}_{H_1}([z_1, w_1]) + \mathcal{A}_{H_2}([z_2, w_2])$$

and

$$E_{(K,J)}(u) = \int v^*\Omega_E + R_K(u)$$

from (4.3.16) and Proposition 20.2.9, where $v(z) = (z, u(z))$ for any $v \in \mathcal{M}(H, \widetilde{J}; \widehat{z})$.

From the definition of $*$ we have that for any $[z_3, w_3] \in \alpha * \beta$ there exist $[z_1, w_1] \in \alpha$ and $[z_2, w_2] \in \beta$ such that $\mathcal{M}(K, J; \widehat{z})$ is nonempty with the asymptotic condition

$$\widehat{z} = ([z_1, w_1], [z_2, w_2]; [z_3, w_3]).$$

Applying this and Lemma 21.4.2, and also applying (21.4.29)–(21.4.31), we immediately derive

$$\begin{aligned}\mathcal{A}_{H\#F}([z_3, w_3]) &\leq \mathcal{A}_H([z_1, w_1]) + \mathcal{A}_F([z_2, w_2]) + \delta \\ &\leq \lambda_H(\alpha) + \lambda_F(\beta) + \delta \\ &\leq \rho(H; a) + \rho(F; b) + 2\delta \end{aligned} \tag{21.4.32}$$

for any $[z_3, w_3] \in \alpha * \beta$. By combining (21.4.29)–(21.4.32), we derive

$$\rho(H\#F; a \cdot b) \leq \rho(H; a) + \rho(F; b) + 2\delta.$$

Since this holds for any δ, we have proven

$$\rho(H\#F; a \cdot b) \leq \rho(H; a) + \rho(F; b).$$

This finishes the proof of the triangle inequality. □

21.5 The spectrality axiom

One of the most non-trivial properties of the spectral invariants $\rho(H; a)$ on general, especially irrational (M, ω) is the following property.

(Spectrality) *For any H and $a \in QH^*(M)$, we have*

$$\rho(H; a) \in \mathrm{Spec}(H). \tag{21.5.33}$$

This property is relatively easy to prove on any *rational* symplectic manifold even for degenerate Hamiltonians H. See (Oh05c).

On the other hand, only the following weaker version has been proved so far on *irrational* symplectic manifolds and it remains unknown whether this spectrality holds for degenerate Hamiltonians.

(Nondegenerate spectrality) *For any nondegenerate H and $a \in QH^*(M)$, (21.5.33) holds.*

Problem 21.5.1 Does the spectrality axiom hold for degenerate Hamiltonians on irrational symplectic manifolds? Otherwise construct a Hamiltonian that violates the spectrality axiom.

However, this nondegenerate spectrality will be sufficient to prove the homotopy invariance of $\rho(H; a)$ in general by combination with a soft approximation argument. See Section 21.6.

The scheme of the proof of this spectrality axiom given in (Oh05c, Oh09a) relies on the existence of *tight Floer cycles*.

Definition 21.5.2 We call a Floer cycle $\alpha \in CF(H)$ *tight* if, for any Floer cycle $\alpha' \in CF(H)$ homologous to α, i.e., $\alpha' - \alpha = \partial_{(H,J)}\gamma$, it satisfies

$$\lambda_H(\alpha') \geq \lambda_H(\alpha).$$

In this section, we attempt to prove the spectrality axiom without initially assuming the rationality of symplectic manifolds, explain what could go wrong with this geometric argument for the irrational case, and then give the proof of the spectrality axiom for the rational case *for any* Hamiltonian H irrespective of whether it is nondegenerate or not. We refer readers to (Oh09a) or (Ush08) for a complete proof of the nondegenerate spectrality axiom for the irrational case. The proof in (Oh09a) is geometric and proves many other properties of spectral invariants, especially under the generic one-parameter family of Hamiltonians called a *Cerf homotopy*, while the proof in (Ush08) is purely algebraic in that it applies in the context of any abstract Floer complex in the sense of Section 13.8.

Recall the definition of the canonical thin cylinder (18.2.7). It is important to note that the image of $u^{\mathrm{can}}_{zz'}$ is contained in a small neighborhood of z (or z'), and uniformly converges to z_∞ when z and z' converge to a loop z_∞ in the C^1 topology. Therefore $u^{\mathrm{can}}_{zz'}$ also picks out a canonical homotopy class, denoted by $[u^{\mathrm{can}}_{zz'}]$, among the set of homotopy classes of the maps $u : [0,1] \times S^1 \to M$ satisfying the given boundary condition

$$u(0,t) = z(t), \quad u(1,t) = z'(t).$$

Lemma 21.5.3 *Let $z, z' : S^1 \to M$ be two smooth loops and let u^{can} be the above canonical cylinder given in* (18.2.7). *For any bounding disc w of z and nearby loop z', we associate the bounding disc*

$$w' := w \# u^{\mathrm{can}}_{zz'}$$

with z'. Then, as $d_{C^1}(z, z') \to 0$, the map $u^{\mathrm{can}}_{zz'}$ converges in the C^1 topology, and its geometric area $\mathrm{Area}(u^{\mathrm{can}})$ *converges to zero.*

In particular, as $d_{C^1}(z, z') \to 0$ as $z' \to z$, the following statements also hold.

(1) *$w' \to w$ in C^1 topology of the maps from the unit disc.*
(2) *The symplectic area*

$$\int_{u^{\mathrm{can}}_{zz'}} \omega \to 0. \tag{21.5.34}$$

Proof Statement (1) is an immediate consequence of the explicit form of $u^{\mathrm{can}}_{zz'}$ above and from the standard property of the exponential map.

On the other hand, from the explicit expression of the canonical thin cylinder and from the property of the exponential map, it follows that the geometric area $\mathrm{Area}(u^{\mathrm{can}}_{zz'})$ converges to zero as $d_{C^1}(z, z') \to 0$ by an easy area estimate. Since z, z' are assumed to be C^1, it follows $u^{\mathrm{can}}_{zz'}$ is also C^1 and hence we have the inequality

$$\mathrm{Area}(u^{\mathrm{can}}_{zz'}) \geq \left| \int_{u^{\mathrm{can}}_{zz'}} \omega \right|.$$

This implies that

$$\lim_{j \to \infty} \int_{u^{\mathrm{can}}_{zz'}} \omega = 0$$

as $z' \to z$ in C^1 topology, which finishes the proof. □

Our goal is to show that the mini-max value $\rho(H; a)$ is a critical value, or that there exists $[z, w] \in \widetilde{\mathcal{L}}_0(M)$ such that

$$\mathcal{A}_H([z, w]) = \rho(H; a),$$
$$d\mathcal{A}_H([z, w]) = 0, \quad \text{i.e.,} \quad \dot{z} = X_H(z).$$

The finiteness of the value $\rho(H; a)$ was proved before.

Theorem 21.5.4 *Suppose that (M, ω) is rational. Then the spectrality axiom holds for any Hamiltonian H, not necessarily nondegenerate.*

Proof We can prove the spectrality axiom for an arbitrary smooth Hamiltonian by contradiction as follows.

If H is nondegenerate, fix H. Otherwise, we approximate H by a sequence of nondegenerate Hamiltonians H_i in C^2 topology. Then choose a sequence of Floer cycles $\alpha_i \in CF_*(H_i)$ such that $\lambda_{H_i}(\alpha_i) \to \rho(H; a)$ as $i \to \infty$. Let $\mathrm{peak}(\alpha_i) = [z_i, w_i] \in \mathrm{Crit}\,\mathcal{A}_{H_i}$ be the peak of the Floer cycle $\alpha_i \in CF_*(H_i)$. Then we will have

$$\lim_{i \to \infty} \mathcal{A}_{H_i}([z_i, w_i]) = \rho(H; a). \tag{21.5.35}$$

Such a sequence can be chosen by the definition of $\rho(\cdot; a)$ and its finiteness property.

Since M is compact and $H_i \to H$ in the C^2 topology, and $\dot z_i = X_{H_i}(z_i)$ for all i, it follows from the standard boot-strap argument that z_i has a subsequence, which we still denote by z_i, converging to some loop $z_\infty : S^1 \to M$ satisfying $\dot z = X_H(z)$. Now we show that the sequences $[z_i, w_i]$ are pre-compact on $\widetilde{\mathcal{L}}_0(M)$. Since we fix the non-zero quantum cohomology class $a \in QH^*(M)$ (or, more specifically, since we fix its degree) and since the Floer cycle is assumed to satisfy $[\alpha_i] = a^\flat$, we have

$$\mu_{H_i}([z_i, w_i]) = \mu_{H_j}([z_j, w_j]).$$

Now consider the sequence of bounding discs of z_∞ given by

$$w_i' = w_i \# u_{i\infty}^{\mathrm{can}}$$

for all sufficiently large i, where $u_{i\infty}^{\mathrm{can}} = u_{z_i z_\infty}^{\mathrm{can}}$ is the canonical thin cylinder between z_i and z_∞. We note that as $i \to \infty$ the geometric area of $u_{i\infty}^{\mathrm{can}}$ converges to 0.

We compute the action of the critical points $[z_\infty, w_i'] \in \operatorname{Crit} \mathcal{A}_H$,

$$\begin{aligned}
\mathcal{A}_H([z_\infty, w_i']) &= -\int_{w_i'} \omega - \int_0^1 H(t, z_\infty(t))\,dt \\
&= -\int_{w_i} \omega - \int_{u_{i\infty}^{can}} \omega - \int_0^1 H(t, z_\infty(t))\,dt \\
&= \left(-\int_{w_i} \omega - \int_0^1 H_i(t, z_i(t))\,dt\right) - \int_{u_{i\infty}^{\mathrm{can}}} \omega \\
&\quad -\left(\int_0^1 H(t, z_\infty(t))\,dt - \int_0^1 H_i(t, z_i(t))\,dt\right) \\
&= \mathcal{A}_{H_i}([z_i, w_i]) - \int_{u_{i\infty}^{\mathrm{can}}} \omega \\
&\quad -\left(\int_0^1 H(t, z_\infty(t))\,dt - \int_0^1 H_i(t, z_i(t))\,dt\right).
\end{aligned} \tag{21.5.36}$$

Since z_i converges to z_∞ uniformly and $H_i \to H$, we have

$$-\left(\int_0^1 H(t, z_\infty(t))\,dt - \int_0^1 H(t, z_i(t))\,dt\right) \to 0. \tag{21.5.37}$$

Therefore, by combining (21.5.35), (21.5.36) and (21.5.37), we derive

$$\lim_{i \to \infty} \mathcal{A}_H([z_\infty, w_i']) = \rho(H; a).$$

In particular, $\mathcal{A}_H([z_\infty, w'_i])$ is a Cauchy sequence, which implies that

$$\left| \int_{w'_i} \omega - \int_{w'_j} \omega \right| = \left| \mathcal{A}_H([z_\infty, w'_i]) - \mathcal{A}_H([z_\infty, w'_j]) \right| \to 0,$$

i.e.,

$$\int_{w'_i \# \overline{w}'_j} \omega \to 0.$$

Since Γ is discrete and $\int_{w'_i \# \overline{w}'_j} \omega \in \Gamma$, this indeed implies that

$$\int_{w'_i \# \overline{w}'_j} \omega = 0 \tag{21.5.38}$$

for all sufficiently large $i,\ j \in \mathbb{Z}_+$. In particular the set $\left\{ \int_{w'_i} \omega \right\}_{i \in \mathbb{Z}_+}$ is bounded. Therefore we conclude that the sequences $\int_{w'_i} \omega$ eventually stabilize, by choosing a subsequence if necessary. By virtue of the definition of $\mathcal{A}_H$, we have $\mathcal{A}_H([z_\infty, w'_N]) = \lim_{i \to \infty} \mathcal{A}_H([z_\infty, w'_i])$ for any sufficiently large N. Then, returning to (21.5.36), we derive that the actions $\mathcal{A}_H([z_\infty, w'_i])$ themselves stabilize. Therefore we obtain

$$\mathcal{A}_H([z_\infty, w'_N]) = \lim_{i \to \infty} \mathcal{A}_H([z_\infty, w'_i]) = \rho(H; a)$$

for a fixed sufficiently large $N \in \mathbb{Z}_+$. This proves that $\rho(H; a)$ is indeed the value of $\mathcal{A}_H$ at the critical point $[z_\infty, w'_N]$. This finishes the proof of the rational case. □

In the proof given above for the rational case, Lemma 21.5.5 below plays a crucial role. Partially because this fails for the irrational case, the proof of spectrality, even for the nondegenerate case, is much more non-trivial than for the rational case.

Here is the key point that fails to hold for the irrational case.

Lemma 21.5.5 *When (M, ω) is rational,* $\mathrm{Crit}\, \mathcal{A}_K \subset \widetilde{\mathcal{L}}_0(M)$ *is a closed subset of $\mathbb{R}$ for any smooth Hamiltonian K, and is locally compact in the subspace topology of the covering space*

$$\pi : \widetilde{\mathcal{L}}_0(M) \to \mathcal{L}_0(M).$$

Proof First note that, when (M, ω) is rational, the covering group Γ of π above is discrete. Together with the fact that the set of solutions of $\dot{z} = X_K(z)$ is compact (on compact M), it follows that

$$\mathrm{Crit}(\mathcal{A}_K) = \{[z, w] \in \widetilde{\mathcal{L}}_0(M) \mid \dot{z} = X_K(z)\}$$

is a closed subset that is also locally compact. □

A proof of the nondegenerate spectrality axiom for the irrational case was first given in (Oh09a) (for the semi-positive case). This proof uses a continuity method starting from a small Morse function under a *Cerf homotopy*, and follows a very intricate analysis of the change of the critical values corresponding to the spectral invariant. Later Usher (Ush08) provided an elegant algebraic proof based on a beautiful approximation theorem in non-Archimedean analysis in the axiomatic setting of the Floer homology explained in Section 13.8. We just state Usher's abstract formulation of the nondegenerate spectrality axiom in the setting of Section 13.8, and refer readers to (Ush08) for the details of his proof, which is based on the existence of tight cycles. We recall the definition of $\mathrm{Spec}(\mathfrak{c})$ from Definition 13.8.1.

Theorem 21.5.6 (Theorem 1.4 (Ush08)) *Let $\mathfrak{c}$ be a filtered Floer–Novikov complex over a Noetherian ring R. Then, for every $a \in H_*(\mathfrak{c})$, there is an $\alpha \in C_*(\mathfrak{c})$ such that $[\alpha] = a$ and $\rho(a) = \lambda(\alpha)$. In particular, $\rho(a) \in \mathrm{Spec}(\mathfrak{c})$.*

Regarding the Floer complex $(CF_*(H), \partial_{(H,J)})$ for any nondegenerate Hamiltonian H, this theorem implies that the spectral number $\rho(H; a)$ is realized by the level of a tight Floer cycle α, i.e., $\lambda_H(\alpha) = \rho(H; a)$. In particular, this implies that $\rho(H; a) \in \mathrm{Spec}(H)$, which proves the nondegenerate spectrality for $\rho(H; a)$.

21.6 Homotopy invariance

The following proposition shows that the function $\rho_a = \rho(\cdot, a)$ descends to $\widetilde{\mathrm{Ham}}(M, \omega)$ as a continuous function.

Theorem 21.6.1 (Homotopy invariance) *Let (M, ω) be an arbitrary closed symplectic manifold. Suppose that the nondegenerate spectrality axiom holds for (M, ω). Then we have*

$$\rho(\underline{H}; a) = \rho(\underline{K}; a) \tag{21.6.39}$$

for any smooth functions $H \sim K$, i.e., satisfying that $\phi_H^1 = \phi_K^1$ and ϕ_H, ϕ_K are path-homotopic relative to the ends.

We will provide two different proofs of this theorem in this section, each of which contains some important ingredients in the study of Floer theory. Leaving the proof thereto, we first state a consequence thereof.

By homotopy invariance, we can define the function $\rho_a : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$ by setting

$$\rho_a(h) := \rho(\underline{H}; a) \tag{21.6.40}$$

for a particular (and hence any) H representing $h = [\phi_H]$, irrespective of whether h is nondegenerate or not. This defines a well-defined function

$$\rho_a : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}.$$

Theorem 21.6.2 *The function ρ_a is a continuous function to $\widetilde{\mathrm{Ham}}(M, \omega)$ in the quotient topology of $\widetilde{\mathrm{Ham}}(M, \omega)$ induced from $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), id)$.*

Proof Recall the definition of the quotient topology under the projection

$$\pi : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), id) \to \widetilde{\mathrm{Ham}}(M, \omega).$$

We have proved that the assignment

$$\lambda \mapsto \rho(\lambda; a) \tag{21.6.41}$$

is continuous on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), id)$. By the definition of the quotient topology, the function

$$\rho_a : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$$

is continuous, because the composition

$$\rho_a \circ \pi : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), id) \to \mathbb{R},$$

which is nothing but (21.6.41), is continuous. □

21.6.1 Proof using nondegenerate spectrality

We first consider nondegenerate Hamiltonians H, K with $H \sim K$. We recall the following basic facts.

(1) The nondegeneracy of a Hamiltonian function H depends only on its time one map $\phi = \phi_H^1$.
(2) The set $\mathrm{Spec}(H) \subset \mathbb{R}$, which is the set of critical values of the action functional $\mathcal{A}_H$ is a set of measure zero (see Lemma 2.2 of (Oh02)).
(3) For any two Hamiltonian functions H, $H' \mapsto \phi$ such that $H \sim H'$, we have

$$\mathrm{Spec}(H) = \mathrm{Spec}(H')$$

as a subset of $\mathbb{R}$, provided that H, H' are normalized (Proposition 18.3.6).
(4) The function $H \mapsto \rho(H; a)$ is continuous with respect to the smooth topology on $C_0^\infty(S^1 \times M)$ (Proposition 21.3.6).

(5) The only continuous functions on a connected space (e.g., the interval $[0, 1]$) to $\mathbb{R}$, whose values lie in a measure-zero subset, are constant functions.

Since $H \sim K$, we have a smooth family $\mathcal{H} = \{H(s)\}_{0\leq s\leq 1}$ with $H(0) = H$ and $H(1) = K$. We define a function $\rho : [0, 1] \to \mathbb{R}$ by

$$\rho(s) = \rho(H(s); a).$$

Note that $H(s)$ is nondegenerate, since the time-one map for all $s \in [0, 1]$ is $\phi^1_{H(s)} = \phi^1_H$, and that its image is contained in the *fixed* subset

$$\mathrm{Spec}(h) \subset \mathbb{R}$$

which is independent of s, where h is the path homotopy class $[H] = [K]$. This subset has measure zero by (1) above and hence is totally disconnected. Therefore, since the function ρ is continuous by the C^0-continuity axiom, ρ must be constant and thus $\rho(H; a) = \rho(0) = \rho(1) = \rho(K; a)$, which finishes the proof for the nondegenerate Hamiltonians.

Now consider the case of a degenerate Hamiltonian H. We would like to emphasize that at present, because we do not know whether the spectrality axiom is valid for degenerate Hamiltonians, the scheme of the above proof used for the nondegenerate case cannot be applied to degenerate Hamiltonians. Therefore, we will use an approximation argument based on the following lemma.

Lemma 21.6.3 *Suppose $H \sim K$ that are not necessarily nondegenerate. Then there exists a sequence of Morse function $f_i \to 0$ in C^2 such that $H\#f_i \sim K\#f_i$ and both $H\#f_i$ and $K\#f_i$ are nondegenerate.*

Proof The proof is left as an exercise. □

Once this lemma has been proved, we obtain

$$\rho(H; a) = \lim_{i\to\infty} \rho(H\#f_i; a) = \lim_{i\to\infty} \rho(K\#f_i; a) = \rho(K; a).$$

This finishes the proof. □

21.6.2 Proof using an optimal continuation

We recall the context of the statement of homotopy invariance. We are given two Hamiltonians H_0, H_1 with $\phi^1_{H_0} = \phi^1_{H_1}$ and $[H_0] = [H_1]$. By definition, we are given a two-parameter family $\{\phi(s,t)\} \subset \mathrm{Ham}(M,\omega)$ for $(s,t) \in [0,1]^2$. Let $\psi \in \mathrm{Ham}(M,\omega)$ such that

$$\begin{aligned} &\phi(s,0) \equiv id, && \phi(s,1) \equiv \psi && \text{for all } s \in [0,1], \\ &\phi(0,t) = \phi^t_{H_0}, && \phi(1,t) = \phi^t_{H_1} && \text{for all } t \in [0,1]. \end{aligned} \tag{21.6.42}$$

We denote by H the family of functions $H = H(s,t,x)$, where $H^s := H(s,\cdot,\cdot) : [0,1] \times M \to \mathbb{R}$ is the Hamiltonian generating the vector field

$$\frac{\partial \phi}{\partial t} \circ \phi^{-1}$$

and denote by $F = F(s,t,x)$ the corresponding s-Hamiltonian generating

$$\frac{\partial \phi}{\partial s} \circ \phi^{-1}.$$

Owing to (21.6.42), by reparameterizing the family $\phi : [0,1]^2 \to \mathrm{Ham}(M,\omega)$ in the directions of $t,\ s$ near $t = 0,\ 1$ and $s = 0,\ 1$ respectively, we may assume that

$$F \equiv 0 \quad \text{for all } (s,t) \in ([0,\epsilon] \times [0,1]) \cup ([1-\epsilon,1] \times [0,1])$$

and in particular that the s-Hamiltonian vector field X_F is one-periodic in t for all s. Now, for each given pair $[z^-,w^-]$ and $[z^+,w^+]$ with $[w^- \# u] = [w^+] \in \pi_2(z^+)$, we consider the perturbed Cauchy–Riemann equation

$$\begin{cases} \partial u/\partial \tau - X_{F^\rho}(u) + J\,(\partial u/\partial t - X_{H^\rho}(u)) = 0, \\ u(-\infty) = z^-,\ u(\infty) = z^+, \end{cases} \tag{21.6.43}$$

and seek a solution with finite energy,

$$E_{(\phi^\rho,J)}(u) := \frac{1}{2}\left(\int_{\mathbb{R}\times[0,1]} \left|\frac{\partial u}{\partial \tau} - X_{F^\rho}(u)\right|_J^2 + \left|\frac{\partial u}{\partial t} - X_{H^\rho}(u)\right|_J^2\right).$$

This equation is equivalent to $(du + P_{(K,J)}(u))^{0,1} = 0$, where $K = H^\rho\,dt + F^\rho\,d\tau$. Since (H,F) is the pair associated with the two-parameter family ϕ, we also have $R_K \equiv 0$. Owing to the fact that $F \equiv 0$ for s near 0, 1, $F^\rho \equiv 0$ for all τ with $|\tau| > R$ for a sufficiently large $R > 0$. In particular, if $\tau < -R$, (21.6.43) becomes

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_{H_0}(u)\right) = 0$$

and if $\tau > R$, it becomes

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_{H_1}(u)\right) = 0.$$

We would like to define a map

$$h_{(\phi^\rho;J)} : CF(H_0, J_0) \to CF(H_1, J_1)$$

by considering the zero-dimensional moduli space, and to show that $h_{(\phi^\rho;J)}$ is a chain map by considering the one-dimensional moduli space as before. We also want to analyze how the chain map affects the filtration of the Floer complex. For this purpose, we need to study the action difference

$$\mathcal{A}_{H_0}([z^-, w^-]) - \mathcal{A}_{H_1}([z^+, w^-\#u])$$

whenever (21.6.43) has a *finite*-energy solution.

First we recall the following basic identity:

$$R_K = \frac{\partial F^\rho}{\partial t} - \frac{\partial H^\rho}{\partial \tau} + \{F^\rho, H^\rho\} = 0. \tag{21.6.44}$$

This is because the pair (H^ρ, F^ρ) is associated with the two-parameter isotopy $\phi = \phi(s,t) \in \mathrm{Ham}(M, \omega)$. Now we can apply Proposition 20.2.9 with $R_K = 0$ to (21.6.43) and obtain

$$\mathcal{A}_{H_1}([z^+, w^-\#u]) - \mathcal{A}_{H_0}([z^-, w^-]) = -E_{(\phi^\rho,J)}(u) \le 0.$$

We summarize the above calculation in the following proposition.

Proposition 21.6.4 *Let $\phi = \{\phi(s,t)\}$ and (H, F) be the pair associated with ϕ as before. Let $\rho : \mathbb{R} \to [0,1]$ be any smooth function with $\rho \equiv 0$ for $\tau \le -R$ and $\rho \equiv 1$ for $\tau \ge R$ for some $R > 0$. Let $[z^-, w^-] \in \mathrm{Crit}\,\mathcal{A}_{H_0}$ be given and let $u : \mathbb{R}\times[0,1] \to M$ be a solution of* (21.6.43) *such that $u(-\infty) = z^-$, $u(\infty) = z^+$. Then we have*

$$\mathcal{A}_{H_1}([z^+, w^-\#u]) \le \mathcal{A}_{H_0}([z^-, w^-]). \tag{21.6.45}$$

An immediate corollary of this proposition is the following.

Corollary 21.6.5 *The chain map $h_{(\phi;J)} : CF(H_0, J_0) \to CF(H_1, J_1)$ preserves the filtration, i.e., it induces a chain map*

$$h^\lambda_{(\phi;J)} : CF^\lambda(H_0, J_0) \to CF^\lambda(H_1, J_1)$$

for all $\lambda \in \mathbb{R}$.

Obviously the s-time reversal isotopy

$$\widetilde{\phi}(s,t) := \phi(1-s, t)$$

induces a filtration-preserving map $h_{(\widetilde{\phi};\widetilde{J})} : CF^\lambda(H_1, J_1) \to CF^\lambda(H_0, J_0)$.

Now we are ready to wrap up the proof of $\rho(H_0; a) = \rho(H_1; a)$. It suffices to prove $\rho(H_0; a) \le \rho(H_1; a)$ since the other direction can be proved in the same way by considering the s-time reversal.

Let $\epsilon > 0$ be given. Then there exists a Floer chain α_{H_1} such that $[\alpha_{H_1}] = a^\flat$ and $\lambda_{H_1}(\alpha_{H_1}) \le \rho(H_1; a) + \epsilon$. Then we have

$$\lambda_{H_0}\left(h_{\widetilde{\phi};\widetilde{J}}(\alpha_{H_1})\right) \le \lambda_{H_1}(\alpha_{H_1}) \le \rho(H_1; a) + \epsilon.$$

Therefore, since $h_{\widetilde{\phi};\widetilde{J}}(\alpha_{H_1}) \in CF_*(H_0)$ and $[h_{\widetilde{\phi};\widetilde{J}}(\alpha_{H_1})] = a^\flat$, we have

$$\rho(H_0; a) \le \lambda_{H_0}\left(h_{\widetilde{\phi};\widetilde{J}}(\alpha_{H_1})\right) \le \rho(H_1; a) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this implies that $\rho(H_0; a) \le \rho(H_1; a)$. This finishes the proof when the Hamiltonian $\psi = \phi^1_{H_0} = \phi^1_{H_1}$ is nondegenerate. The general case follows from this by application of the C^0 continuity of $H \mapsto \rho(H; a)$. $\square$

22

Spectral invariants: applications

Although the construction of spectral invariants heavily relies on the analytic theory of pseudoholomorphic curves, and in particular depends on the smooth structure of symplectic manifolds, the invariants themselves are C^0-type invariants which are continuous in the *C^0-Hamiltonian topology* introduced in (OhM07) (see Section 6.2 of this book too), and hence can be extended to the topological Hamiltonian flows defined in Section 6.2.

In this chapter, we illustrate several applications of the spectral invariants to the study of symplectic topology. One of the important advantages of spectral invariants over other more direct dynamical invariants of Hofer type is their *homotopy-invariance*, which enables one to naturally push forward the spectral invariants to the universal covering space of the Hamiltonian diffeomorphism group and sometimes even down to the group itself. This point is highlighted by the striking construction of partial symplectic quasi-states by Entov and Polterovich (EnP06) which is based on the purely axiomatic properties of spectral invariants $\rho(H; 1)$ and the natural operation of taking the asymptotic average in dynamical systems. Their construction was carried out for the monotone case and later extended to the arbitrary compact symplectic manifolds by Usher (Ush10b).

Firstly, we explain the construction of an invariant spectral norm performed in (Oh05d) and its application to problems of symplectic rigidity and of minimality of geodesics in Hofer's geometry. We also explain Usher's applications to Polterovich's and Lalonde and McDuff's minimality conjecture and to the sharp energy–capacity inequality. Secondly, we give a self-contained presentation of Entov and Polterovich's partial symplectic quasi-states and quasimorphisms constructed out of spectral invariants and their applications to symplectic intersection problems. Finally, we return to the study of the group of Hamiltonian homeomorphisms and explain how one can extend all these constructions to the realm of a continuous Hamiltonian category in the sense of Chapter 6.

It appears that these constructions as a whole bear much importance in the development of symplectic topology which is starting to unveil the mystery around what the true meaning of Gromov's pseudoholomorphic curves and Floer homology is from the point of view of pure symplectic topology.

22.1 The spectral norm of Hamiltonian diffeomorphisms

In this section, we explain the construction of an invariant norm of Hamiltonian diffeomorphisms following (Oh05d), which is called the *spectral norm*. This involves a careful usage of the spectral invariant $\rho(H;1)$ corresponding to the quantum cohomology class $1 \in QH^*(M)$.

Recall that there is a natural involution on $C^\infty([0,1]\times M)$ given by

$$H \mapsto \overline{H}; \quad \overline{H}(t,x) = -H(t, \phi_H^t(x)),$$

which is also equivalent to the involution of taking the inverse

$$\lambda \mapsto \lambda^{-1}; \qquad \mathcal{P}^{\text{ham}}(\text{Symp}(M,\omega), id) \to \mathcal{P}^{\text{ham}}(\text{Symp}(M,\omega), id).$$

We symmetrize $\rho(\cdot;1)$ over this involution and define a function

$$\gamma : C^\infty([0,1]\times M) \to \mathbb{R}$$

by

$$\gamma(H) = \rho(H;1) + \rho(\overline{H};1), \tag{22.1.1}$$

on $C^\infty([0,1]\times M)$. Obviously we have $\gamma(H) = \gamma(\overline{H})$ for any H. The general triangle inequality

$$\rho(H;a) + \rho(F;b) \geq \rho(H\#F; a\cdot b)$$

restricted to $a = b = 1$ and the normalization axiom $\rho(id;1) = 0$ from Theorem 21.3.7 imply

$$\gamma(H) = \rho(H;1) + \rho(\overline{H};1) \geq \rho(\underline{0};1) = 0. \tag{22.1.2}$$

Here $a \cdot b$ is the quantum product of the quantum cohomology classes a, $b \in QH^*(M)$ and $\underline{0}$ is the zero function.

We next compare $\gamma(H)$ with the $L^{(1,\infty)}$-norm $\|H\| = \int_0^1 \text{osc}(H_t)dt$. We recall the definitions

$$E^-(H) = \int_0^1 -\min H\,dt, \quad E^+(H) = \int_0^1 \max H\,dt \tag{22.1.3}$$

and call them the negative and positive Hofer lengths, respectively. Obviously we have $E^+(H) = E^-(\overline{H})$. We prove the following lemma.

Lemma 22.1.1 *For any H and $0 \neq a \in QH^*(M)$, we have*

$$-E^+(H) + v(a) \leq \rho(H; a) \leq E^-(H) + v(a), \tag{22.1.4}$$

where $v(a)$ is the valuation of a. In particular, for any classical cohomology class $b \in H^*(M) \hookrightarrow QH^*(M)$, we have

$$-E^+(H) \leq \rho(H; b) \leq E^-(H) \tag{22.1.5}$$

for any Hamiltonian H.

Proof We start with the inequality (21.3.22), which is

$$\int_0^1 - \max(H - K)dt \leq \rho(H, a) - \rho(K, a) \leq \int_0^1 - \min(H - K)dt.$$

This can be rewritten as

$$\rho(K; a) + \int_0^1 - \max(H - K)dt \leq \rho(H; a) \leq \rho(K; a) + \int_0^1 - \min(H - K)dt.$$

Now, by letting $K \to 0$, we obtain

$$\rho(0; a) + \int_0^1 - \max(H)dt \leq \rho(H; a) \leq \rho(0; a) + \int_0^1 - \min(H)dt. \tag{22.1.6}$$

By the normalization axiom, we have $\rho(0; a) = v(a)$, which turns (21.3.22) into

$$v(a) - E^+(H) \leq \rho(H; a) \leq v(a) + E^-(H)$$

for any H. The inequality (22.1.5) immediately follows from the definitions and the identity $v(b) = 0$ for a classical cohomology class b. This finishes the proof. □

Applying the right-hand side of (22.1.5) to $b = 1$, we derive $\rho(H; 1) \leq E^-(H)$ and $\rho(\overline{H}; 1) \leq E^-(\overline{H})$ for arbitrary H. On the other hand, we also have $E^-(\overline{H}) = E^+(H)$ for arbitrary Hs and hence

$$\gamma(H) \leq \|H\|. \tag{22.1.7}$$

The non-negativity (22.1.2) leads us to the following definition.

Definition 22.1.2 We define $\gamma : \mathrm{Ham}(M, \omega) \to \mathbb{R}_+$ by

$$\gamma(\phi) := \inf_{H \mapsto \phi} \gamma(H) = \inf_{H \mapsto \phi} (\rho(H; 1) + \rho(\overline{H}; 1)).$$

We call $\gamma(\phi)$ the spectral norm of ϕ.

The following theorem summarizes the basic properties of the function γ. In the remaining subsection, we will give the proofs of these statements, postponing the most non-trivial statement, nondegeneracy, to the next subsection.

Theorem 22.1.3 *Let γ be as above. Then $\gamma : \mathrm{Ham}(M, \omega) \to \mathbb{R}_+$ defines an invariant norm i.e., it enjoys the following properties.*

(1) *(**Nondegeneracy**) $\phi = id$ if and only if $\gamma(\phi) = 0$*
(2) *(**Symplectic invariance**) $\gamma(\eta^{-1}\phi\eta) = \gamma(\phi)$ for any symplectic diffeomorphism η*
(3) *(**Triangle inequality**) $\gamma(\phi\psi) \le \gamma(\phi) + \gamma(\psi)$*
(4) *(**Inversion invariance**) $\gamma(\phi^{-1}) = \gamma(\phi)$*
(5) $\gamma(\phi) \le \|\phi\|$

Proof of the symplectic invariance We recall the symplectic invariance of spectral invariants $\rho(H; a) = \rho(\eta_* H; \eta_* a)$. Applying this to $a = 1$ and using the fact $\eta_* 1 = 1$, we derive the identity

$$\begin{aligned}\gamma(\phi) &= \inf_{H \mapsto \phi} \Big(\rho(H; 1) + \rho(\overline{H}; 1)\Big)\\ &= \inf_{H \mapsto \phi} \Big(\rho(\eta_* H; 1) + \rho(\overline{\eta_* H}; 1)\Big) = \gamma(\eta^{-1}\phi\eta),\end{aligned}$$

which finishes the proof. □

Proof of the triangle inequality We first recall the triangle inequality

$$\rho(H\#K; 1) \le \rho(H; 1) + \rho(K; 1) \tag{22.1.8}$$

and

$$\rho(\overline{K}\#\overline{H}; 1) \le \rho(\overline{K}; 1) + \rho(\overline{H}; 1). \tag{22.1.9}$$

Adding up (22.1.8) and (22.1.9), we have

$$\begin{aligned}\rho(H\#K; 1) + \rho(\overline{H\#K}; 1) &= \rho(H\#K; 1) + \rho(\overline{K}\#\overline{H}; 1)\\ &\le \Big(\rho(H; 1) + \rho(\overline{H}; 1)\Big) + \Big(\rho(K; 1) + \rho(\overline{K}; 1)\Big).\end{aligned} \tag{22.1.10}$$

Now let $H \mapsto \phi$ and $K \mapsto \psi$. Because $H\#K$ generates $\phi\psi$, we have

$$\gamma(\phi\psi) \le \gamma(H\#K) = \rho(H\#K; 1) + \rho(\overline{H\#K}; 1)$$

and hence

$$\gamma(\phi\psi) \le \Big(\rho(H; 1) + \rho(\overline{H}; 1)\Big) + \Big(\rho(K; 1) + \rho(\overline{K}; 1)\Big)$$

from (22.1.10). By taking the infimum of the right-hand side over all $H \mapsto \phi$ and $K \mapsto \psi$, property (3) is proved. □

Proof of inversion invariance The proof immediately follows from the observation that the definition of γ is symmetric over the map $\phi \mapsto \phi^{-1}$. □

Proof of (5) By taking the infimum of (22.1.7) over $H \mapsto \phi$, we have proved $\gamma(\phi) \leq \|\phi\|$. □

It now remains to prove the nondegeneracy of γ. The original proof in (Oh05d) uses a quite delicate existence theorem of particular pseudoholomorphic curves. Although this existence theorem itself has significant applications to rigidity problems, we will provide a different proof as a corollary of a more familiar energy–capacity-type inequality, which was given by Usher (Ush10a). This proof is somewhat indirect and requires some digression to the study of Hofer's geometry in $\mathrm{Ham}(M, \omega)$. (Another proof of nondegeneracy was given by McDuff and Salamon in (MSa04).) □

Remark 22.1.4 In fact, if one incorporates the homotopy invariance of $\rho(\cdot; 1)$ into the proof of $\gamma(\phi) \leq \|\phi\|$, the inequality can be improved in the following way. Define the *medium Hofer norm* by

$$\|\phi\|_{\mathrm{med}} := \inf\{\|h\| \mid \pi(h) = \phi\},$$

where we define for $h \in \widetilde{\mathrm{Ham}}(M, \omega)$

$$\|h\| := \inf\{\|H\| \mid h = [\phi_H] \in \widetilde{\mathrm{Ham}}(M, \omega)\}.$$

Then the above proof shows $\gamma(H) \leq \|[\phi_H]\|$ and so

$$\gamma(\phi) \leq \|\phi\|_{\mathrm{med}}.$$

See Section 11 of (Oh09a) for further relevant discussion.

22.2 Hofer's geodesics and periodic orbits

In this section, we study the length-minimizing property of a Hamiltonian path with respect to the Hofer distance. After introducing a necessary criterion following Bialy and Polterovich (BP94) and Lalonde and McDuff (LM95b), we present a proof of the so-called minimality conjecture given by Usher (Ush10a). In the meantime, we will also give a proof of the nondegeneracy of the γ-norm on $\mathrm{Ham}(M, \omega)$ following Usher's argument.

22.2.1 Quasi-autonomous Hamiltonians

The $L^{(1,\infty)}$-norm of H $\|H\| = \int_0^1 (\max H_t - \min H_t) dt$ can be identified with the Finsler length

$$\mathrm{leng}(\phi_H) = \int_0^1 \Big(\max_x H(t, \phi_H^t(x)) - \min_x H(t, \phi_H^t(x) \Big) dt$$

of the path $\phi_H : t \mapsto \phi_H^t$, where the associated Banach norm on

$$T_{id} \operatorname{Ham}(M, \omega) \cong C^\infty(M)/\mathbb{R}$$

is nothing but

$$\|h\| = \operatorname{osc}(h) = \max h - \min h.$$

The map $t \mapsto H(t, \phi_H^t(\cdot))$ can be regarded as the tangent vector of the path $t \mapsto \phi_H^t$ on $\operatorname{Ham}(M, \omega)$.

Definition 22.2.1 Consider the metric $d : \mathcal{P}^{\mathrm{ham}}(\operatorname{Symp}(M, \omega), id) \to \mathbb{R}_+$ defined by

$$d(\lambda, \mu) := \mathrm{leng}(\lambda^{-1} \circ \mu),$$

where $\lambda^{-1} \circ \mu$ is the Hamiltonian path $t \in [0, 1] \mapsto \lambda(t)^{-1}\mu(t)$. The Hofer topology on $\mathcal{P}^{\mathrm{ham}}(\operatorname{Symp}(M, \omega), id)$ is induced by this metric.

Exercise 22.2.2 Consider the evaluation map

$$ev_1 : \mathcal{P}^{\mathrm{ham}}(\operatorname{Symp}(M, \omega), id) \to \operatorname{Ham}(M, \omega).$$

Assume that M is compact. Prove that the above Hofer topology on $\operatorname{Ham}(M, \omega)$ is the strongest topology for which the evaluation map $\lambda \mapsto \lambda(1)$ is continuous.

The non-triviality of this topology is a consequence of the nondegeneracy of Hofer's norm function $\| \cdot \| : \operatorname{Ham}(M, \omega) \to \mathbb{R}_+$. Hofer (H93) proved that the path of any compactly supported *autonomous* Hamiltonian on $\mathbb{C}^n$ is length-minimizing as long as the corresponding Hamilton equation has no non-constant periodic orbit *of period less than or equal to one*. This result is generalized in (En00), (MSl01) and (Oh02)–(Oh05b) under the additional hypothesis that the linearized flow at each fixed point is not over-twisted, i.e., it has no closed trajectory of period less than one.

In (BP94) and (LM95b), Bialy and Polterovich and Lalonde and McDuff proved that any length-minimizing (respectively, locally length-minimizing) Hamiltonian path is generated by *quasi-autonomous* (respectively, *locally quasi-autonomous*) Hamiltonian paths.

Definition 22.2.3 A Hamiltonian H is called *quasi-autonomous* if there exist two points $x_-, x_+ \in M$ fixed over the time $t \in [0,1]$ such that

$$H(t, x_-) = \min_x H(t,x), \quad H(t, x_+) = \max_x H(t,x)$$

for all $t \in [0,1]$.

The following is the definition which has been used for the study of the length-minimizing property of Hamiltonian paths.

Definition 22.2.4 Let $H : [0,1] \times M \to \mathbb{R}$ be a Hamiltonian, which is not necessarily time-periodic, and let ϕ_H^t be its Hamiltonian flow.

(1) A point $p \in M$ is called a *time-T-periodic point* if $\phi_H^T(p) = p$. We call $t \in [0,T] \mapsto \phi_H^t(p)$ a *contractible time-T-periodic orbit* if it is contractible.
(2) When H has a fixed critical point p over $t \in [0,T]$, we call p *over-twisted* as a time-T-periodic orbit if its linearized flow $d\phi_H^t(p); T_pM \to T_pM$ for $t \in [0,T]$ has a closed trajectory of period less than or equal to T. Otherwise we call it *under-twisted*. If in addition the linearized flow has only the origin as the fixed point, then we call the fixed point *generically under-twisted*.

Note that generic under-twistedness is a C^2-stable property of the Hamiltonian H.

We now recall the Ustilovsky–Lalonde–McDuff necessary condition on the stability of geodesics to motivate further discussion, but without giving its proof.

Theorem 22.2.5 (Ustilovsky (Ust96), Lalonde and McDuff (LM95b)) *For a generic ϕ in the sense that all its fixed points are isolated, any stable geodesic ϕ_t, $0 \le t \le 1$ from the identity to ϕ must have at least two fixed points that are under-twisted.*

The following conjecture, which was raised by Polterovich (Conjecture 12.6.D in (Po01)), is sometimes called the minimality conjecture. (See also (Po01), (LM95b) and (MSl01).)

Conjecture 22.2.6 (Minimality conjecture) *Any autonomous Hamiltonian path that has no contractible periodic orbits of period less than or equal to one is Hofer-length minimizing in its path-homotopy class relative to the end points.*

The following definition of Entov (En00) simplifies the terminology.

Definition 22.2.7 We call a Hamiltonian H that has no contractible periodic orbits of period less than or equal to one *slow*.

The minimality conjecture has been proved by various authors under the various conditions on the linearized flows at the fixed points of the associated autonomous Hamiltonian paths. A complete proof of this conjecture without any additional restriction on the linearized flows is provided by Usher (Ush10a). He uses a length-minimizing criterion provided in (Oh05b) together with a certain approximation theorem of Hamiltonians. (See also (Sch06) for a related result.)

In the next two subsections, we present Usher's proof.

22.2.2 The length-minimizing criterion via $\rho(H; 1)$

We describe a simple criterion of the length-minimizing property of Hamiltonian paths in terms of the spectral invariant $\rho(H; 1)$ from (Oh05b). The criterion is similar to the one used in (H93) and in (BP94) for the case of $\mathbb{C}^n$.

To describe this criterion, it is useful to decompose the norm $\|H\|$ into

$$\|H\| = E^-(H) + E^+(H)$$

as defined in (22.1.3).

Proposition 22.2.8 *Let $G : [0,1] \times M \to \mathbb{R}$ be any Hamiltonian that satisfies*

$$\rho(G; 1) = E^-(G). \tag{22.2.11}$$

Then H is negative Hofer-length minimizing in its homotopy class with fixed ends. In particular, if $\rho(\overline{G}) = E^-(\overline{G}) = E^+(G)$ in addition, G must be quasi-autonomous.

Proof Let F be any Hamiltonian with $F \sim G$. Then we have a string of equalities and an inequality

$$E^-(G) = \rho(G; 1) = \rho(F; 1) \leq E^-(F)$$

from (22.2.11), (21.6.39) for $a = 1$ and (22.1.5), respectively. This proves the first statement. The second statement then follows from the criterion for the minimality of Bialy and Polterovich, Ustilovsky and Lalonde and McDuff. This finishes the proof. □

On the other hand, *if G is one-periodic*, we can consider the associated action functional $\mathcal{A}_G$. $\mathcal{A}_G$ has two obvious critical values of $\mathcal{A}_G$ for a quasi-autonomous periodic Hamiltonian G, which are given by

$$\mathcal{A}_G([x^-, \widehat{x^-}]) = \int_0^1 -G(t, x^-)dt,$$
$$\mathcal{A}_G([x^+, \widehat{x^+}]) = \int_0^1 -G(t, x^+)dt.$$

The upshot is that these action values coincide with the negative and positive parts of $\|H\|$,

$$E^-(G) = \int_0^1 -\min G_t\, dt, \quad E^+(G) = \int_0^1 \max G_t\, dt,$$

respectively.

22.2.3 Construction of fundamental Floer cycles

Now we introduce a fundamental concept of homological essentialness, namely the *no-pushing-down property under Cauchy–Riemann flow*. This is the heart of matter in *the chain-level Floer theory*, especially in the construction of spectral invariants (Oh05c) and its applications (Oh05b, Oh05d, Oh09a). This notion was first implicitly introduced into the Floer theory in (Oh97b) in the context of exact Lagrangian submanifolds in the cotangent bundle, and appeared explicitly as the notion of tight cycles in (Oh02, Oh09a) in the context of Hamiltonian diffeomorphisms in general.

By the definition of tight Floer cycles given in Definition 21.5.2, if $[\alpha] = a^\flat$ for a quantum homology class, any tight Floer cycle of (H, J) realizes the value $\rho(H; a)$, i.e., $\lambda_{(H,J)}(\alpha) = \rho(H; a)$. It is a non-trivial matter to prove the existence of such a cycle. The following theorem is proven in (Oh09a) by a geometric continuity method, and in (Ush08) by a purely algebraic method.

Theorem 22.2.9 (Oh (Oh09a), Usher (Ush08)) *For any $0 \neq a \in QH^*(M)$, there exists a tight Floer cycle associated with a.*

The proof of this theorem in general is quite subtle (unless (M, ω) is rational), and goes beyond the scope of this book. Thus the reader is referred to the original articles for its proof.

In relation to the length-minimizing criterion in Theorem 22.2.8, it turns out to be crucial to construct a tight Floer cycle of G associated with $a = 1$ with level $\lambda_{(G,J)}(\alpha) = E^-(G)$ for a quasi-autonomous Hamiltonian G under the slowness condition.

However, the definition of Floer cycles of a given Hamiltonian in general is rather abstract and complicated due to how it is formulated. As a result, it is in general not easy to use them in applications. On the other hand, the Floer cycles realizing the *fundamental cycle* of a given (M, ω) can be systematically studied through transferring the Morse fundamental cycles via the Floer chain map.

Definition 22.2.10 We call a Floer cycle α a fundamental cycle if its homology class satisfies $[\alpha] = 1^\flat$.

Similarly, the Morse fundamental cycle is defined in Morse theory.

When the Hamiltonian path satisfies some particular property like the quasi-autonomous property, one would like to exploit that geometric property in the study of such transferred cycles. We would like to emphasize that in general *this process is well defined only at the level of Floer chains, not at the level of critical points.* This is because a critical point is transferred to a Floer chain, not just to a critical point, under the Floer chain map.

Choose a Morse function f and the fundamental Morse cycle α of $-\epsilon f$ *for a sufficiently small* $\epsilon > 0$ *that Theorem 19.8.1 holds*. Then we may regard α also as a Floer cycle of ϵf. We then transfer α via the Floer chain map $h_{\mathcal{L}}$ and define a fundamental Floer cycle of H as

$$\alpha_H := h_{\mathcal{L}}(\alpha) \in CF(H),$$

where $h_{\mathcal{L}}$ is the Floer chain map over any given homotopy $\mathcal{L}$ connecting ϵf to H. In relation to the minimality conjecture the canonically given linear path

$$\mathcal{L} : \ s \mapsto (1 - s)\epsilon f + sH$$

plays an important role. We also note that this cycle depends on the choice of the Morse function f. In general, we do not expect that this cycle will be tight even when H is quasi-autonomous.

In this regard, Kerman and Lalonde (KL03) proved that, under the following two requirements,

(1) the quasi-autonomous Hamiltonian H *has the unique nondegenerate global minimum* x^- *that is under-twisted for all* $t \in [0, 1]$, and
(2) the Morse function f appearing in the above linear path $\mathcal{L}$
 - has a global minimum point at x^-
 - and satisfies

$$f(x^-) = 0, \quad f(x^-) < f(x_j) \tag{22.2.12}$$

 for all other critical points x_j,

the corresponding fundamental Floer cycle transferred by $\mathcal{L}$ from the Morse fundamental cycle of f enjoys the following nice property. This result is proved for the aspherical manifolds in (KL03) and extended to general symplectic manifolds in (Oh05b).

Proposition 22.2.11 *Suppose that H is a generic one-periodic Hamiltonian such that H_t has the unique nondegenerate global minimum at x^- which is fixed and* under-twisted *for all $t \in [0,1]$. Suppose that $f : M \to \mathbb{R}$ is a Morse function that has the unique global minimum point x^- and satisfies $f(x^-) = 0$. Then the canonical fundamental cycle has the expression*

$$\alpha_H = [x^-, \widehat{x^-}] + \beta \in CF(H) \tag{22.2.13}$$

for some Floer chain $\beta \in CF(H)$ with its level

$$\lambda_H(\beta) < \lambda_H([x^-, \widehat{x^-}]) = \int_0^1 -H(t, x^-)dt. \tag{22.2.14}$$

In particular, the level of α_H satisfies

$$\begin{aligned} \lambda_H(\alpha_H) &= \lambda_H([x^-, \widehat{x^-}]) \\ &= \int_0^1 -H(t, x^-)dt = \int_0^1 -\min H_t\, dt. \end{aligned} \tag{22.2.15}$$

Proof The proof is based on the following lemma (see Proposition 5.2 in (KL03)). We warn readers that our sign convention is different from the one in (KL03).

Lemma 22.2.12 *Let H and f be as above. Then, for all sufficiently small $\epsilon > 0$, the function G^H defined by*

$$G^H(t, x) = H(t, x^-) + \epsilon f$$

satisfies

$$G^H(t, x^-) = H(t, x^-), \quad G^H(t, x) \le H(t, x)$$

for all (t, x) and equality holds only at x^-.

Exercise 22.2.13 Prove this lemma.

We now return to the proof of the proposition. Since x^- is an under-twisted fixed minimum of both H and f, we have the Conley–Zehnder index

$$\mu_H([x^-, w_{x^-}]) = \mu_{\epsilon f}([x^-, w_{x^-}])(= -n)$$

and thus the moduli space $\mathcal{M}^{\mathcal{L}}([x^-, w_{x^-}], [x^-, w_{x^-}])$ has dimension zero. Since the constant map $u(x, t) \equiv x^-$ is always contained therein, $\mathcal{M}^{\mathcal{L}}([x^-, w_{x^-}], [x^-, w_{x^-}]) \neq \emptyset$.

Let u be any element of $\mathcal{M}^{\mathcal{L}}([x^-, w_{x^-}], [x^-, w_{x^-}])$. We note that the Floer continuity equation for the linear homotopy

$$\mathcal{L} : s \to (1-s)\epsilon f + sH$$

is unchanged even if we replace the homotopy by the homotopy

$$\mathcal{L}' : s \to (1-s)G^H + sH.$$

This is because the added term $H(t, x^-)$ in G^H to ϵf does not depend on $x \in M$ and hence

$$X_{\epsilon f} \equiv X_{G^H}.$$

Therefore u is also a solution for the continuity equation under the linear homotopy $\mathcal{L}'$. Using this, we derive the identity

$$\begin{aligned}\int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^\rho} dt\, d\tau &= \mathcal{A}_{G^H}([x^-, w_{x^-}]) - \mathcal{A}_H([x^-, w_{x^-}]) \\ &\quad - \int_{-\infty}^{\infty} \rho'(\tau)\Big(H(t, u(\tau, t))dt\, d\tau - G^H(t, u(\tau, t))\Big)dt\, d\tau.\end{aligned} \tag{22.2.16}$$

Since we have

$$\mathcal{A}_H([x^-, w_{x^-}]) = \mathcal{A}_{G^H}([x^-, w_{x^-}]) = \int_0^1 -\min H\, dt$$

and $G^H \leq H$, the right-hand side of (22.2.16) is non-positive. Therefore we derive that $\mathcal{M}^{\mathcal{L}}([x^-, w_{x^-}], [x^-, w_{x^-}])$ consists only of the constant solution $u \equiv x^-$. This in particular gives rise to the matrix coefficient of $h_{\mathcal{L}}$ satisfying

$$\langle [x^-, w_{x^-}], h_{\mathcal{L}}([x^-, w_{x^-}])\rangle = \#(\mathcal{M}^{\mathcal{L}}([x^-, w_{x^-}], [x^-, w_{x^-}])) = 1.$$

Now consider any other generator of α_H

$$[z, w] \in \operatorname{supp} \alpha_H \quad \text{with } [z, w] \neq [x^-, w_{x^-}]. \tag{22.2.17}$$

By the definitions of $h_{\mathcal{L}}$ and $\alpha_H = h_{\mathcal{L}}(\alpha)$, there is a generator $[x, w_x] \in \operatorname{supp} \alpha$ such that

$$\mathcal{M}^{\mathcal{L}}([x, w_x], [z, w]) \neq \emptyset.$$

Then, for any $u \in \mathcal{M}^{\mathcal{L}}([x, w_x], [z, w])$, we have the identity

$$\begin{aligned}\mathcal{A}_H([z, w]) - \mathcal{A}_{G^H}([x, w_x]) &= -\int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^\rho} dt\, d\tau \\ &\quad - \int_{-\infty}^{\infty} \rho'(\tau)\Big(H(t, u(\tau, t)) - G^H(t, u(\tau, t))\Big)dt\, d\tau.\end{aligned}$$

Since $-\int |\partial u/\partial \tau|^2_{J^\rho} \leq 0$ and $G^H \leq H$, we have

$$\mathcal{A}_H([z,w]) \leq \mathcal{A}_{G^H}([x,w_x]), \tag{22.2.18}$$

with equality holding only when u is stationary. There are two cases to consider, one for the case of $x = x^-$ and the other for $x = x_j$ for $x_j \neq x^-$ for $[x_j, w_{x_j}] \in \operatorname{supp} \alpha$.

For the first case, *since we assume* $[z,w] \neq [x^-, w_{x^-}]$, u cannot be constant and thus the strict inequality holds in (22.2.18), i.e.,

$$\mathcal{A}_H([z,w]) < \mathcal{A}_{G^H}([x^-,w_{x^-}]). \tag{22.2.19}$$

For the second case, (22.2.18) holds for some $x_j \neq x^-$ with $[x_j, w_{x_j}] \in \alpha$. We note that the inequality $f(x_-) < f(x_j)$ in (22.2.12) is equivalent to

$$\mathcal{A}_{G^H}([x_j, w_{x_j}]) < \mathcal{A}_{G^H}([x^-, w_{x^-}]).$$

This together with (22.2.18) again gives rise to (22.2.19). On the other hand, we also have

$$\mathcal{A}_{G^H}([x^-, w_{x^-}]) = \mathcal{A}_H([x^-, w_{x^-}])$$

because $G^H(t,x^-) = H(t,x^-)$. Altogether, we have proved that

$$\mathcal{A}_H([z,w]) < \mathcal{A}_H([x^-, w_{x^-}]) = \int_0^1 -H(t,x^-)dt$$

for any $[z,w] \in \operatorname{supp} \alpha_H$ with $[z,w] \neq [x^-, w_{x^-}]$ too. This finishes the proof of the proposition. □

22.2.4 The case of autonomous Hamiltonians

To prove the minimality conjecture, according to the criterion provided in Theorem 22.2.8, it will be enough to prove that the value $\mathcal{A}_G([x^-, \widehat{x^-}]) = E^-(G)$ coincides with the mini-max value $\rho(G;1)$ (and also similarly that $\mathcal{A}_{\overline{G}}([x^+, \widehat{x^+}]) = E^-(\overline{G})$ with $\rho(\overline{G};1)$).

This latter fact is an immediate consequence of the following theorem.

Theorem 22.2.14 *Suppose that G is any slow autonomous Hamiltonian. Then the canonical fundamental cycle α_G constructed in Proposition 22.2.11 is tight, i.e.,*

$$\rho(G;1) = \lambda_G(\alpha_G).$$

In particular, we have $\rho(G;1) = E^-(G)$. Similarly we have $\rho(\overline{G};1) = E^-(\overline{G}) = E^+(G)$.

In this section, we give the proof of this theorem. Before launching into its proof, some remarks on the history of this theorem are in order. In various previous literature, the minimality conjecture was studied with the following conditions or their variations on G:

(1) G is slow
(2) it has a maximum and a minimum that are generically under-twisted
(3) all of its critical points are nondegenerate in the Floer-theoretic sense (i.e., the linearized flow of X_G at each critical point has only the zero as a periodic orbit).

See (MSl01), (En00), (Oh02), (KL03), (Oh05b) for example. The existence of conditions (2) and (3) turns out to be an artifact of the method of study, in which one uses the moduli space of some form of pseudoholomorphic curve. This requires study of the transversality issue of the moduli space. The strongest possible result that can be obtained through this study can be summarized as follows.

Theorem 22.2.15 (Theorem (Oh05b)) *Let (M, ω) be an arbitrary closed symplectic manifold. Suppose that G is an autonomous Hamiltonian such that*

(1) *it has no non-constant contractible periodic orbits 'of period one'*
(2) *it has a maximum and a minimum that are generically under-twisted*
(3) *all of its critical points are nondegenerate in the Floer-theoretic sense (i.e., the linearized flow of X_G at each critical point has only the zero as a periodic orbit).*

Then the one-parameter subgroup ϕ_G^t is length-minimizing in its homotopy class with fixed ends for $0 \le t \le 1$.

This theorem in turn is an immediate consequence of the following.

Theorem 22.2.16 *Suppose that G is any slow autonomous Hamiltonian that satisfies the conditions given in Theorem 22.2.15. Then the canonical fundamental cycle α_G constructed in Proposition 22.2.11 is tight, i.e.,*

$$\rho(G; 1) = \lambda_G(\alpha_G) \, (= -G(x^-) = E^-(G)).$$

Proof Note that the conditions in Theorem 22.2.15 in particular imply that G is nondegenerate. We fix a *time-independent* J_0 that is G-regular.

Suppose that α is homologous to the canonical fundamental Floer cycle α_G, i.e.,

$$\alpha = \alpha_G + \partial_G(\gamma) \tag{22.2.20}$$

for some Floer chain $\gamma \in CF_*(G)$. When G is autonomous and $J \equiv J_0$ is t-independent, there is no non-stationary t-independent trajectory of $\mathcal{A}_G$ landing at $[x^-, \widehat{x^-}]$, because any such trajectory comes from the negative Morse gradient flow of G, but x^- is the minimum point of G. Therefore any non-stationary Floer trajectory u landing at $[x^-, \widehat{x^-}]$ must be t-dependent. Because of the assumption that G has no non-constant contractible periodic orbits of period one, any critical point of $\mathcal{A}_G$ has the form

$$[x, w] \quad \text{with } x \in \operatorname{Crit} G.$$

Let u be a trajectory starting at $[x, w]$, $x \in \operatorname{Crit} G$ with

$$\mu([x, w]) - \mu([x^-, \widehat{x}^-]) = 1, \tag{22.2.21}$$

and denote by $\widetilde{\mathcal{M}}_{(G,J_0)}([x, w], [x^-, \widehat{x^-}])$ the corresponding Floer moduli space of connecting trajectories. The general index formula (18.4.24) shows that

$$\mu([x, w]) = \mu([x, w_x]) - 2c_1([w]). \tag{22.2.22}$$

We consider two cases separately: the cases of $c_1([w]) = 0$ and $c_1([w]) \neq 0$. If $c_1([w]) \neq 0$, we derive from (22.2.21) and (22.2.22) that $x \neq x^-$. This implies that any such trajectory must come with (locally) free S^1-action (see Section 19.8), i.e., the moduli space

$$\mathcal{M}_{(G,J_0)}([x, w], [x^-, \widehat{x}^-]) = \widetilde{\mathcal{M}}_{(G,J_0)}([x, w], [x^-, \widehat{x}^-])/\mathbb{R}$$

and its stable map compactification $\overline{\mathcal{M}}_{(G,J_0)}([x, w], [x^-, \widehat{x^-}])$ have a locally free S^1-action *without fixed points*. Then it follows from the S^1-equivariant transversality theorem from (FHS95) that $\mathcal{M}_{(G,J_0)}([x, w], [x^-, \widehat{x^-}])$ becomes empty for a suitable choice of an autonomous J_0. This is because the quotient has the virtual dimension -1 by the assumption (22.2.21). We refer the reader to (FHS95) for more explanation on this S^1-invariant regularization process. Now consider the case $c_1([w]) = 0$. First note that (22.2.21) and (22.2.22) imply that $x \neq x^-$. On the other hand, if $x \neq x^-$, the same argument as above shows that the perturbed moduli space becomes empty.

It now follows that there is no trajectory of index 1 that lands at $[x^-, \widehat{x^-}]$. Therefore $\partial_G(\gamma)$ cannot kill the term $[x^-, \widehat{x^-}]$ in (22.2.20) away from the cycle

$$\alpha_G = [x^-, \widehat{x^-}] + \partial_G(\gamma)$$

for any choice of γ and hence we have

$$\lambda_G(\alpha) \geq \lambda_G([x^-, \widehat{x^-}])$$

by the definition of the level λ_G. Combined with (22.2.15), this finishes the proof. □

In order to generalize Theorem 22.2.15 to general slow Hamiltonians, which are not necessarily under-twisted, as in the minimality conjecture, we use the following approximation result together with C^0-continuity of $\rho(\cdot; 1)$. This is proved by Usher in Theorem 4.5 of (Ush10a), which is in turn an improvement of Theorem 1.3 (Sch06) augmented with nondegeneracy in the approximation process.

Theorem 22.2.17 *Let $H : M \to \mathbb{R}$ be any slow Hamiltonian. Then, for any $\epsilon > 0$, there is a smooth function $K : M \to \mathbb{R}$ such that $\|K - H\|_{C^0} < \epsilon$ and that K is a slow, under-twisted, Morse function.*

Proof See the next section, Section 22.2.5, for the proof. □

Finally we wrap up the proof of the minimality conjecture.

Theorem 22.2.18 (Usher (Ush10a)) *Let (M, ω) be an arbitrary closed symplectic manifold. Suppose that G is any slow autonomous Hamiltonian. Then the one-parameter group ϕ_G^t is length-minimizing in its homotopy class with fixed ends for $0 \leq t \leq 1$.*

Proof This is an immediate consequence of Theorem 22.2.8 and Theorem 22.2.14. More precisely, Theorem 22.2.14 implies that $\rho(G; 1) = E^-(G)$. Then Theorem 22.2.8 implies that G is negative-Hofer-length-minimizing. Since $\overline{G}$ is also slow when G is slow, we also have $\rho(\overline{G}; 1) = E^-(\overline{G}) = E^+(G)$. Therefore $\overline{G}$ is also negative-Hofer-length-minimizing, which is equivalent to the statement that G is positive-length-minimizing. Obviously, the combination of the two implies that G is Hofer-length-minimizing. This finishes the proof. □

22.2.5 The approximation theorem of slow Hamiltonians

The purpose of this section is to provide a proof of Theorem 22.2.17 following verbatim Usher's proof presented in (Ush10a).

We start with the following estimates of the period of a general ordinary differential equation obtained by Yorke (Yo69).

Let $F : \Omega \to \mathbb{R}^n$ be a Lipschitz vector field on a domain $\Omega \subset \mathbb{R}^n$. We denote the Euclidean norm by $\|\cdot\|$. Then Yorke (Yo69) proves the following. (We refer readers to the original article for the details of the proof.)

Theorem 22.2.19 *Suppose that F is Lipschitz with Lipschitz constant $L > 0$, i.e., satisfies*

$$\|F(x_1) - F(x_2)\| \leq L\|x_1 - x_2\| \quad \textit{for each } x_1,\ x_2 \in \Omega. \tag{22.2.23}$$

Let $x = x(t)$ be a non-constant periodic solution with a period p of $\dot{x} = F(x)$. Then $p \geq 2\pi/L$.

A similar theorem was first proved by Sibuya (unpublished work) with a weaker result $p \geq 2/L$, which is indeed enough for this purpose.

We recall the statement of Theorem 22.2.17 again here.

Theorem 22.2.20 *Let $H : M \to \mathbb{R}$ be any slow Hamiltonian. Then, for any $\epsilon > 0$, there is a smooth function $K : M \to \mathbb{R}$ such that $\|K - H\|_{C^0} \leq \epsilon$ and such that K is a slow, generically under-twisted and Morse function.*

Proof Choose a background Riemannian metric on M induced by the almost-complex structure compatible with ω so that $X_H = J\nabla H$ and in particular $|X_H(p)| = |\nabla H(p)|$ for all p. Let

$$F_n = \{p \in M| \mid |\nabla H(p)| \leq 1/n\}.$$

Note that, if $s \in \cap_{n=1}^\infty H(F_n)$, then there are $p_n \in M$ such that $H(p_n) = s$ and $|\nabla H(p_n)| \leq 1/n$, so the compactness of M and the continuity of H and ∇H show that there exists $p \in M$ with $H(p) = s$ and $\nabla H(p) = 0$. Thus $\cap_{n=1}^\infty H(F_n)$ is precisely the set of critical values of H, and hence has Lebesgue measure zero by Sard's theorem. Since the image of H has finite Lebesgue measure, we have $m(H(F_n)) \to 0$ as $n \to \infty$. Let $\zeta > 0$ be given. There is then N such that $m(H(F_N)) < \zeta/2$. Choose a smooth function $\psi : M \to \mathbb{R}$ such that $0 \leq \psi \leq 1$, $\psi|_{F_{2N}} = 1$ and $\mathrm{supp}(\psi) \subset F_N$.

Now, using that F_N and hence $H(F_N) \subset \mathbb{R}$ are compact, there are open intervals $I_1, \ldots, I_k$ with disjoint closures such that $H(F_N) \subset \bigcup_{j=1}^k I_j$ and $\sum_{j=1}^k m(I_j) < 3\zeta/4$. Choose open intervals $I'_1, \ldots, I'_k$ with disjoint closures such that, for each j, $I_j \subset I'_j$ and $\sum_{j=1}^k m(I'_j) < \zeta$. Write $S = \bigcup_{j=1}^k I_j$, $S' = \bigcup_{j=1}^k I'_j$. Denote

$$B = \sup_{p \in M} |\nabla(\nabla H)(p)|$$

and let $\eta > 0$ be a small number that will be further specified later; in particular it should be smaller than 1. Let $f : [\min H, \max H] \to [\min H, \max H]$ be a C^∞ function satisfying the following properties:

(1) $f(\min H) = \min H$
(2) $0 < f'(s) \leq 1$ for all s
(3) $f'(s) = 1$ if $s \notin S'$
(4) $f'(s) = \eta/2$ if $s \in S$.

Since $m(S') < \zeta$, we have $0 \leq s - f(s) < \zeta$ for all $s \in [\min H, \max H]$. In particular, $\|f \circ H - H\|_{C^0} < \zeta$, independently of η. If $q \in F_N$ (which in particular implies that $f''(H(q)) = 0$ by the construction of f) one has

$$|\nabla(f \circ H)(q)| = f'(H(q))|\nabla H(q)| \leq \frac{\eta}{2N}$$

and

$$|\nabla(\nabla(f \circ H))(q)| = \frac{\eta}{2}|\nabla(\nabla H)(q)| \leq \frac{B\eta}{2}.$$

Hence we have

$$\|X_{f\circ H}\|_{C^1} \leq \frac{\eta}{2} + \frac{B\eta}{2}.$$

So, on recalling our above function $\psi : M \to \mathbb{R}$ (which is supported in F_N), we obtain

$$\|\psi X_{f\circ H}\|_{C^1} \leq \left(\frac{\eta}{2} + \frac{B\eta}{2}\right)\|\psi\|_{C^1}.$$

By the above estimate given in Theorem 22.2.19 (and the Whitney embedding theorem), there exists $\eta_0 > 0$ such that any vector field V on M with $\|V\|_{C^1} \leq \eta_0(B+1)\|\psi\|_{C^1}$ will have no non-constant periodic orbits of period at most one. So take $0 < \eta < \eta_0$ and let $\{K_m\}_{m=1}^\infty$ be a sequence of Morse functions on M such that $K_m \to f \circ H$ in the C^2-norm. Then $X_{K_m} \to X_{f\circ H}$ in the C^1-norm, and thus $\psi X_{K_m} \to \psi X_{f\circ H}$ in the C^1-norm. Hence, for m large enough that

$$\|\psi \circ X_{f\circ H} - \psi X_{K_m}\|_{C^1} < \left(\frac{(B+1)\eta}{2}\right)\|\psi\|_{C^1},$$

the vector field ψX_{K_m} has no non-constant periodic orbits of period at most one. Now, since H is slow and $0 < f' \leq 1$, it quickly follows that $f \circ H$ is also slow. Thus all contractible periodic orbits of $X_{f\circ H}$ of period at most one are constant orbits at critical points of $f \circ H$; in particular all such orbits are contained in the interior of F_{2N}. An easy application of the Arzelà–Ascoli theorem then shows that, if m is sufficiently large, any contractible periodic orbit of period at most one of X_{K_m} must be contained in F_{2N}. So (since $\psi|_{F_2 N} = 1$) for m sufficiently large the only contractible periodic orbits of X_{K_m} with period at most one are in fact periodic orbits of ψX_{K_m} and hence are constant by virtue of what we have already shown. Thus K_m is slow for m sufficiently large. Further, the fact that $K_m \to f \circ H$ in C^2 topology shows that all critical points p of K_m are, for m large enough, contained in F_N, and at any such p we have $|\nabla(\nabla(f \circ H))(p)| < B\eta/2$ and therefore $|\nabla(\nabla K_m)(p)| < B\eta$ for large enough m. So (as long as η has been taken smaller than $2\pi/B$) the linearized flow $\phi^t_{K_m}$ at any such critical point will have no non-constant periodic orbits of period at most 1. Thus, for sufficiently large m_0, K_{m_0} is a slow, flat, Morse function with $\|K_{m_0} - f \circ H\|_{C^2} < \eta$. Since $\|f \circ H - H\|_{C^0} < \eta$, this proves the theorem on setting $\zeta = \epsilon/2$ and $K = K_{m_0}$. □

22.3 Spectral capacities and sharp energy–capacity inequality

In this section, we study various spectral analogs of several known construction of symplectic capacities. We take Hofer's diameter, Hofer's displacement energy and the π_1-sensitive Hofer–Zehnder capacity as examples. The spectral displacement energy was introduced in (Oh05d) and the π_1-sensitive Hofer–Zehnder capacity was introduced by G. Lu (LuG96) and Schwarz (Schw00) and further studied by Usher (Ush10a) in terms of the spectral invariants.

Definition 22.3.1 (Spectral diameter) Let (M, ω) be a compact symplectic manifold, not necessarily without a boundary. Consider the group $\mathrm{Ham}^c(M, \omega)$ of Hamiltonian diffeomorphisms supported in $M \setminus \partial M$. We define the *spectral diameter* of $\mathrm{Ham}^c(M, \omega)$ by

$$\begin{aligned}\mathrm{diam}_\rho(\mathrm{Ham}^c(M, \omega)) &:= \sup_H \{\gamma(H) \mid \mathrm{supp}\, H \subset M \setminus \partial M\} \\ &= \sup_{\phi \in \mathrm{Ham}^c(M,\omega)} \{\gamma(\phi)\}.\end{aligned}$$

It is not known in general for which symplectic manifolds (M, ω) we have $\mathrm{diam}_\rho(M, \omega) < \infty$. We will show later, in Section 22.5, that $\mathbb{C}P^n$ has a finite spectral diameter.

However, if one considers an open subset $U \subset M$ that is displaceable, we can get a bound of $\mathrm{diam}_\rho(\mathrm{Ham}^c(U, \omega))$ in terms of the *spectral displacement energy* (and hence also in terms of Hofer's displacement energy) which we now introduce.

Definition 22.3.2 Let A be any closed subset of M. We define the *spectral displacement energy* $e_\rho(A)$ by

$$e_\rho(A) = \inf_H \{\gamma(H) \mid \phi^1_H(A) \cap A = \emptyset\}. \tag{22.3.24}$$

For the simplicity of notation, we denote by $\mathcal{P}^{\rm ham}_U = \mathcal{P}^{\rm ham}(\mathrm{Symp}_U(M,\omega), id)$ the set of Hamiltonian paths ϕ_F with $\operatorname{supp} F \subset U$.

With these definitions, we state the following.

Proposition 22.3.3 (Compare with Lemma 7.2 (EnP06)) *Suppose that U is an open subset such that $\overline{U}$ is displaceable. For any $\lambda = \phi_F \in \mathcal{P}^{\rm ham}_U$,*

$$\gamma(\phi_F) \leq 2e_\rho(\overline{U}).$$

In particular,

$$\mathrm{diam}_\rho(\mathrm{Ham}^c(U)) \leq 2e_\rho(\overline{U}).$$

Proof Let F be any Hamiltonian on M such that $\phi_H^1(U) \cap \overline{U} = \emptyset$, and let F be a Hamiltonian with $\operatorname{supp} F \subset U$. We note that the value $\gamma(\lambda)$ does not depend on the normalization of the Hamiltonian F representing $\lambda = \phi_F$.

Therefore, to prove the proposition, it will suffice to prove

$$\gamma(F) = \rho(\phi_F; 1) + \rho(\phi_F^{-1}; 1) \leq 2\gamma(H) \tag{22.3.25}$$

for the Hamiltonian F with $\operatorname{supp} F \subset U$, not the mean-normalized one.

We start with the following lemma.

Lemma 22.3.4 *We have*

$$\rho(\phi_H\phi_F; 1) = \rho(\phi_H; 1) + \mathrm{Cal}_U(F),$$
$$\rho(\phi_H^{-1}(\phi_F)^{-1}; 1) = \rho(\phi_H^{-1}; 1) - \mathrm{Cal}_U(F).$$

Proof It will suffice to prove the first identity, since $(\phi_F)^{-1} = \phi_{\overline{F}}$ and $\operatorname{supp} \overline{F} \subset U$ and $\mathrm{Cal}^{\rm path}(\phi_{\overline{F}}) = -\mathrm{Cal}^{\rm path}(\phi_F)$ if $\operatorname{supp} F \subset U$. We follow the scheme used in the proof of Lemma 7.2 of (EnP06), which in turn is based on a trick of Ostrover (Os03).

Consider the family λ_s for $0 \leq s \leq 1$ defined by

$$\lambda_s(t) = \begin{cases} \phi_H^{2t}, & 0 \leq t \leq \frac{1}{2}, \\ \phi_H^1 \phi_F^{2s(t-\frac{1}{2})}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

We have

$$\mathrm{Dev}(\lambda_s)(t) = \begin{cases} 2H(2t, x), & 0 \leq t \leq \frac{1}{2}, \\ 2sF(2s(t-\frac{1}{2}), x), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Denote $F^s := \mathrm{Dev}(\lambda_s)$. We note that $\mathrm{Fix}(\phi_H^1\lambda_s) = \mathrm{Fix}(\phi_H^1)$. For each fixed point $x \in \mathrm{Fix}(\phi_H^1\lambda_s) \equiv \mathrm{Fix}\,\phi_H^1$, we denote by z_x^s the corresponding periodic orbit of F^s. Then $z_x^s(t) = \phi_H^{2t}(x)$ for $0 \leq t \leq \frac{1}{2}$ and $z_x^s(t) \equiv \phi_H^1(x) = \phi_H^1(x)$ for $\frac{1}{2} \leq t \leq 1$.

Therefore we can take a smooth lift $[z_x^s, w] \in \widetilde{\Omega}_0(M)$, where the bounding disc w is independent of s. Then

$$\begin{aligned}\mathcal{A}_{F^s}([z_x^s, w]) &= -\int w^*\omega - \int_0^1 F^s(t, z_x^s(t))dt \\ &= -\int w^*\omega - \int_0^{\frac{1}{2}} 2H(2t, \phi_H^{2t}(x))dt \\ &\quad - \int_{\frac{1}{2}}^1 2sF(2s(t-1/2), \phi_H^1(x))dt\end{aligned}$$

for each $s \in [0,1]$. We note that

$$\int_{\frac{1}{2}}^1 F^s(t, z_x^s(t))dt = \int_{\frac{1}{2}}^1 2sF(2s(t-1/2), \phi_H^1(x))dt = \int_0^s F(t,x)dt,$$

where the second equality arises since $x \in \mathrm{Fix}\,\phi_H^1$, and

$$\int_0^{\frac{1}{2}} 2H(2t, \phi_H^{2t}(x))dt = \int_0^1 H(t, \phi_H^t(x))dt.$$

Using this, we obtain

$$\mathcal{A}_{F^s}([z_x^s, w]) = \mathcal{A}_H([z_x^0, w]) - \int_0^s F(t,x)dt.$$

But, for any $x \in M \setminus U$,

$$F(t,x) \equiv \{\mathrm{vol}(M)\}^{-1} \int_M F_t\, d\mu. \tag{22.3.26}$$

Therefore we obtain

$$\mathcal{A}_{F^s}([z_x^s, w]) = \mathcal{A}_H([z_x^0, w]) - \int_0^s \{\mathrm{vol}(M)\}^{-1} \int_M F_t\, d\mu\, dt$$

and so

$$\mathcal{A}_{F^s}([z_x^s, w]) \in \mathrm{Spec}\, H + c(s); \quad c(s) = \int_0^s \mathrm{vol}_\omega(M)^{-1} \int_M F_t\, d\mu\, dt$$

for all $x \in \mathrm{Fix}\,\phi_H^1$ and $s \in [0,1]$. In particular, we have

$$\mathrm{Spec}\, F^s = \mathrm{Spec}\, H - c(s) \tag{22.3.27}$$

as a subset of $\mathbb{R}$.

Now we consider the function $s \mapsto \rho(\lambda^s; 1) = \rho(F^s; 1)$, which is continuous. Since F^s are all nondegenerate, the nondegenerate spectrality axiom implies that $\rho(F^s; 1) \in \mathrm{Spec}\, F^s$ and hence

$$\rho(F^s; 1) + c(s) \in \mathrm{Spec}\, H$$

by (22.3.27). Since $s \mapsto \rho(F^s; 1) + c(s)$ is continuous and $\operatorname{Spec} H$ is nowhere dense, $\rho(F^s; 1) + c(s)$ must be constant. In particular, we have

$$\rho(F^0; 1) + c(0) = \rho(F^1; 1) + c(1).$$

But we have $c(0) = 0$ and

$$c(1) = \frac{1}{\mathrm{vol}_\omega(M)} \int_0^1 \int_M F \, d\mu \, dt = \mathrm{Cal}_U(F)$$

and also $F^0 = H$ and $F^1 = H * F$. Hence we obtain

$$\rho(\phi_H; 1) = \rho(\phi_H \phi_F; 1) - \mathrm{Cal}_U(F),$$

or equivalently

$$\rho(\phi_H \phi_F; 1) = \rho(\phi_H; 1) + \mathrm{Cal}_U(F), \tag{22.3.28}$$

which finishes the proof. □

We return to the proof of (22.3.25) and of Proposition 22.3.3. Upon adding the two equalities in Lemma 22.3.4, we obtain

$$\rho(\phi_H \phi_F; 1) + \rho(\phi_H^{-1} \phi_F^{-1}; 1) = \rho(\phi_H; 1) + \rho(\phi_H^{-1}; 1). \tag{22.3.29}$$

By the triangle inequality, we also have

$$\rho(\phi_F; 1) \le \rho(\phi_H \phi_F; 1) + \rho(\phi_H^{-1}; 1),$$
$$\rho(\phi_F^{-1}; 1) \le \rho(\phi_F^{-1} \phi_H; 1) + \rho(\phi_H; 1).$$

By adding the two inequalities and substituting the sum into (22.3.29), we obtain

$$\gamma(F) \le 2\rho(\phi_H; 1) + 2\rho(\phi_H^{-1}; 1) = 2\gamma(H)$$

and hence the proof has been completed. □

At this point, it is not obvious whether $\mathrm{diam}_\rho(\mathrm{Ham}^c(U)) > 0$. We will prove this by comparing $\rho(\cdot; 1)$ with another capacity called the π_1-sensitive Hofer–Zehnder capacity, following the arguments from (LuG96), (Schw00), (Ush10a). Originally this notion was introduced by Guangcun Lu (LuG96). We adopt Schwarz's terminology used in (Schw00) here.

To motivate Lu's definition of the π_1-sensitive Hofer–Zehnder capacity, we first recall the construction of Hofer–Zehnder capacity $c_{HZ}(A)$. For the definition of $c_{HZ}(A)$, we recall that *M is assumed to have a nonempty boundary ∂M.* We denote

$$\mathcal{H}_{HZ}(A) = \{H \in C^\infty(M) \mid \mathrm{supp}(H) \subset A \setminus \partial M,\ 0 \le H \le \max H,$$
$$H^{-1}(0) \text{ and } H^{-1}(\max H) \text{ both contain nonempty open sets}\}.$$

Definition 22.3.5 Call $H \in \mathcal{H}_{HZ}(A)$ *HZ-admissible* if the Hamiltonian flow ϕ_H of H regarded as a one-periodic function has no non-constant periodic orbits of period at most 1. Similarly, we call $H \in \mathcal{H}_{HZ}(A)$ HZ°-admissible if ϕ_H has no non-constant periodic orbits of period at most 1 that are contractible in M. We denote by $\mathcal{H}^{\mathrm{ad}}_{HZ}(A)$ (respectively by $\mathcal{H}^{\circ,\mathrm{ad}}_{HZ}(A, M)$) the set of HZ-admissible (respectively HZ°-admissible) Hamiltonians. (Lu used the term C-admissibility instead of HZ°-admissibility.)

The Hofer–Zehnder capacity (HZ94) is defined as follows.

Definition 22.3.6 (Hofer–Zehnder capacity) Let $A \subset M \subset \partial M$. Define

$$c_{HZ}(A) = \sup\{\max H \mid H \in \mathcal{H}_{HZ}(A) \text{ is } HZ\text{-admissible}\},$$

which is called the Hofer–Zehnder capacity.

Incorporating the effect of $\pi_1(M)$ on the Hamiltonian flow, Lu (LuG96) and Schwarz (Schw00) introduced the following variation of c_{HZ}:

$$c^\circ_{HZ}(A, M) := \sup\{\max H \mid H \in \mathcal{H}_{HZ}(A) \ \text{ is } HZ^\circ\text{-admissible}\}.$$

They called it the π_1-sensitive Hofer–Zehnder capacity. For its definition, M *now could be closed.*

Here the two notations are consistent with the fact that $c_{HZ}(A)$ depends only on $(A, \phi^t_H|_A)$ (and on $\dim M$), whereas whether or not an orbit of ϕ^t_H is contractible in M depends on M, and therefore $c^\circ_{HZ}(A, M)$ depends on the ambient manifold M and not just on A. Obviously any HZ-admissible Hamiltonian is HZ°-admissible, so

$$c_{HZ}(A) \le c^\circ_{HZ}(A, M).$$

Hofer and Zehnder proved in (HZ94) that c_{HZ} is a symplectic capacity and satisfies $c_{HZ}(B^{2n}(r)) = \pi r^2$ when $B^{2n}(r)$ is the ball of radius r in $\mathbb{R}^{2n}$.

The following result was proved by Usher (Ush10a).

Proposition 22.3.7 *Suppose that F, $H : (\mathbb{R}/\mathbb{Z}) \times M \to \mathbb{R}$ are two Hamiltonians, not necessarily normalized, on compact (M, ω) without a boundary, and satisfying the property*

$$\phi^1_H(\operatorname{supp} F) \cap \operatorname{supp} F = \emptyset,$$

where

$$\operatorname{supp} F := \bigcup_{t \in \mathbb{R}/\mathbb{Z}} \operatorname{supp} F_t \subset M.$$

Then

$$\rho(F;1) \leq \gamma(H)(= \rho(H;1) + \rho(\overline{H};1)).$$

Proof Choose a displaceable open subset $U \subset M$ with $\operatorname{supp} F \subset U$. We denote by $\underline{F}$ the normalization of F given by

$$(\underline{F})_t = F_t - \frac{1}{\operatorname{vol}(M)} \int_M F_t \omega^n$$

for each t. Then, by definition, we have

$$\begin{aligned} \rho(F;1) &= \rho(\underline{F};1) + \frac{1}{\operatorname{vol}(M)} \int_0^1 \int F_t \, dt = \rho(\underline{F};1) + \mathrm{Cal}_U(F) \\ &\leq \rho(\underline{F}\#H;1) + \rho(\overline{H};1) + \mathrm{Cal}_U(F). \end{aligned} \tag{22.3.30}$$

On the other hand, (22.3.28) implies that

$$\mathrm{Cal}_U^{\mathrm{path}}(\phi_F) = \rho(H;1) - \rho(\underline{F}\#H;1)$$

for every H with $\phi_H^1(U) \cap \overline{U} = \emptyset$. Therefore, on substituting this into (22.3.30), we obtain

$$\rho(F;1) \leq \rho(H;1) + \rho(\overline{H};1),$$

which finishes the proof. □

On combining Theorem 22.2.14 and this proposition, we obtain the following theorem.

Theorem 22.3.8 (Usher) *If (M,ω) is closed, then, for any $A \subset M$, we have*

$$c_{HZ}^{\circ}(A,M) \leq e(A,M).$$

Proof We approximate any admissible F by a sequence of slow and undertwisted nondegenerate autonomous Hamiltonians F_i. Then Theorem 22.2.14 implies that $\rho(-F_i;1) = E^+(F_i)$. By taking the limit using the C^0-continuity of $\rho(\cdot;1)$ and E^+ we obtain $\rho(-F;1) = E^+(F) = \max F$. Therefore Proposition 22.3.7 implies that

$$\max F \leq \rho(H;1) + \rho(\overline{H};1).$$

By taking the infimum of the right-hand side over all H with $\phi_H^1(U) \cap \overline{U} = \emptyset$ and the supremum of $\max F$ for all HZ°-admissible F, we obtain the theorem. □

One immediate corollary of Theorem 22.3.8 is the nondegeneracy of the spectral norm.

Corollary 22.3.9 *The spectral norm* $\gamma : \mathrm{Ham}(M,\omega) \to \mathbb{R}_+$ *satisfies that* $\gamma(\phi) = 0$ *if and only if* $\phi = id$.

Proof Suppose that $\phi \neq id$. Then we can find a sufficiently small symplectic ball $B \cong B^{2n}(r)$ that is displaceable. Then we have $\gamma(\phi) \geq c^{\circ}_{HZ}(B, M)$. On the other hand, we have $c^{o}_{HZ}(B, M) \geq c_{HZ}(B, M) = \pi r^2 > 0$. On combining the two inequalities, we have $\gamma(\phi) > 0$, which finishes the proof. □

Another corollary is the following sharp energy–capacity inequality between the Gromov area and the Hofer displacement energy which is proven in (Ush10a), (Sch06).

Corollary 22.3.10 (Sharp energy–capacity inequality) *Let* $A \subset M$ *be any compact closed subset. Denote by* $c(A)$ *the Gromov area and* $e(A)$ *the Hofer displacement energy. Then*

$$c(A) \leq e(A, M).$$

Proof It is well known (see Proposition 6.1.5) that $c(A) \leq c_{HZ}(A)$ as the Gromov area is the smallest possible symplectic capacity. Therefore Theorem 22.3.8 and the inequality $c_{HZ}(A) \leq c^{\circ}_{HZ}(A, M)$ immediately imply the corollary. □

22.4 Entov and Polterovich's partial symplectic quasi-states

In this section, we extract some symplectic topological properties from the spectral invariant $\rho(\cdot; 1)$ with $1 = PD[M]$. For this purpose, we closely follow the procedure employed by Entov and Polterovich (EnP06).

Consider the asymptotic average

$$\mu_1^{\mathrm{path}}(\phi_H) = \lim_{k\to\infty} \frac{\rho(\phi_H^k; 1)}{k}. \tag{22.4.31}$$

By definition, μ_1^{path} satisfies

$$\mu_1^{\mathrm{path}}(\phi_H^m) = m\mu_1^{\mathrm{path}}(\phi_H) \tag{22.4.32}$$

once the definition has been proved to be well defined. We will prove this well-definedness in Proposition 22.4.10 below.

Let $F \in C^\infty(M)$ be any autonomous function, not necessarily normalized. We consider its normalization

$$\underline{F} = F - \frac{1}{\mathrm{vol}_\omega(M)} \int_M F \, d\mu_\omega.$$

Then Entov and Polterovich considered the limit

$$\zeta_{\mathrm{sp}}(F) := \lim_{k\to\infty} \frac{\rho(kF;1)}{k}, \tag{22.4.33}$$

where the subscript 'sp' stands for 'spectral'. We recall the definition of $\rho(\phi_H;1) := \rho(\underline{H};1)$ from Definition 21.3.9. Since $F = \underline{F} + (1/\mathrm{vol}(M))\int_M F\,d\mu_\omega$, the definition (22.4.33) can be rephrased as

$$\begin{aligned}\zeta_{\mathrm{sp}}(F) &= \lim_{k\to\infty} \frac{\rho(k\underline{F};1)}{k} + \frac{\int_M F\,d\mu_\omega}{\mathrm{vol}(M)}\\ &= \lim_{k\to\infty} \frac{\rho(\phi_F^k;1)}{k} + \frac{\int_M F\,d\mu}{\mathrm{vol}(M)},\end{aligned}$$

where the second equality follows since F is autonomous.

One may regard ζ_{sp} as the 'infinitesimal' version of μ_1^{path}. Recall that the germ of curves at the identity of $\mathrm{Ham}(M,\omega)$ corresponds to the tangent space of $\mathrm{Ham}(M,\omega)$ at the identity, which is isomorphic to the set of normalized autonomous Hamiltonians. By the C^0-continuity property of ρ it follows that ζ_{sp} continuously extends to $C^0(M)$.

In (EnP06), Entov and Polterovich observed that ζ_{sp} behaves like a *quasi-state* as introduced by Aarnes (Aa91) and proposed the above definition of ζ_{sp}. According to the terminology of (EnP06), ζ_{sp} is an example of a *partial symplectic quasi-state*.

We begin with Aarnes' general notion of quasi-states.

Definition 22.4.1 (Aarnes) Let M be a topological space. Denote by $C^0(M)$ the set of continuous real-valued functions on M. A (nonlinear) functional $\zeta : C^0(M) \to \mathbb{R}$ is called a *quasi-state* if it satisfies the following properties.

1. (Linearity) The function ζ is linear on $\mathcal{A}_F$, which is the $\mathbb{R}$-algebra generated by $F \in C(M)$.
2. (Monotonicity) $\zeta(F) \le \zeta(G)$ if $F \le G$ pointwise.
3. (Normalization) $\zeta(1) = 1$.

Here is the symplectic version introduced in (EnP06).

Definition 22.4.2 (Entov and Polterovich) Let (M,ω) be a symplectic manifold. A (nonlinear) functional $\zeta : C^0(M) \to \mathbb{R}$ is called a *symplectic quasi-state* if it is a quasi-state in the sense of Aarnes on M and satisfies the following additional symplectic properties.

4. (Strong quasi-additivity) Suppose F_1, F_2 are smooth and $\{F_1, F_2\} = 0$, where $\{F_1, F_2\}$ is the Poisson bracket associated with ω. Then $\zeta(F_1 + F_2) = \zeta(F_1) + \zeta(F_2)$.
5. (Vanishing) $\zeta(F) = 0$, provided that $\operatorname{supp} F$ is displaceable.
6. (Symplectic invariance) $\zeta(F) = \zeta(F \circ \psi)$ for any $\psi \in \mathrm{Symp}_0(M, \omega)$, where $\mathrm{Symp}_0(M, \omega)$ is the identity component of $\mathrm{Symp}(M, \omega)$.

It turns out that the above ζ_{sp} does not quite satisfy all the properties of a (symplectic) quasi-state of this definition for general (M, ω) but satisfies weaker properties than that of a (symplectic) quasi-state.

Definition 22.4.3 (Entov–Polterovich) A *partial symplectic quasi-state* is defined to be a function $\zeta : C^0(M) \to \mathbb{R}$ that exhibits the following properties.

1. (Lipschitz continuity) $|\zeta(F) - \zeta(H)| \leq \|F - H\|_{C^0}$
2. (Semi-homogeneity) $\zeta(\lambda F) = \lambda \zeta(F)$ for any $F \in C^0(M)$ and $\lambda \in \mathbb{R}_{\geq 0}$
3. (Monotonicity) $\zeta(F) \leq \zeta(G)$ for $F \leq G$
4. (Normalization) $\zeta(1) = 1$
5. (Partial additivity) If two F_1, $F_2 \in C^\infty(M)$ satisfy $\{F_1, F_2\} = 0$ and $\operatorname{supp} F_2$ is displaceable, then $\zeta(F_1 + F_2) = \zeta(F_1)$
6. (Symplectic invariance) $\zeta(F) = \zeta(F \circ \psi)$ for all $\psi \in \mathrm{Symp}_0(M, \omega)$

We would like to point out that the vanishing axiom in Definition 22.4.2 is an immediate consequence of the partial-additivity axiom. With this definition, we state the following theorem.

Theorem 22.4.4 (Entov and Polterovich (EnP03)) *Let (M, ω) be any compact symplectic manifold. The function $\zeta_{\mathrm{sp}} : C^\infty(M) \to \mathbb{R}$ extends to a function, again denoted by $\zeta_{\mathrm{sp}} : C^0(M) \to \mathbb{R}$, which becomes a partial symplectic quasi-state. We call ζ_{sp} a* spectral partial quasi-state.

Entov and Polterovich stated this theorem under the assumption that (M, ω) is strongly semi-positive and *rational*. However, these restrictions are unnecessary once the nondegenerate spectrality axiom has been shown to hold, which was proved in (Oh09a) and (Ush08) in complete generality on the basis of the construction of virtual fundamental classes via the Kuranishi structure. We will give its proof at the end of this section following Entov and Polterovich's argument borrowed from (EnP06).

We first explain some striking consequences on symplectic intersections of the mere presence of partial symplectic quasi-states, which were proved by Entov and Polterovich (EnP06).

Definition 22.4.5 (Entov–Polterovich) A linear subspace $\mathcal{A} \subset C^\infty(M)$ is *Poisson-commutative* (or *in involution*) if $\{F, G\} = 0$ for all F, $G \in \mathcal{A}$. Given a finite-dimensional Poisson-commutative subspace $\mathcal{A} \subset C^\infty(M)$, we define *the generalized moment map* $\Phi_{\mathcal{A}} : M \to \mathcal{A}^*$ by the equation

$$\langle \Phi_{\mathcal{A}}(x), F \rangle = F(x).$$

Nonempty subsets of the form $\Phi_{\mathcal{A}}^{-1}(p)$, $p \in \mathcal{A}^*$ are called *fibers* of $\mathcal{A}$.

The standard moment map associated with the Lie-group action of the torus is a special case of this definition. It follows that any fiber of the moment map $\Phi_{\mathcal{A}}$ is *coisotropic* whenever it is smooth.

Theorem 22.4.6 (Entov and Polterovich (EnP06)) *Any finite-dimensional Poisson-commutative subspace of $C^\infty(M)$ has at least one nondisplaceable fiber. Moreover, if every fiber has a finite number of connected components, there exists a nondisplaceable connected component of the fiber.*

Proof Suppose to the contrary that all fibers of $\mathcal{A}$ are displaceable. Since M is compact, the image Δ of $\Phi_{\mathcal{A}}$ is compact. Therefore we can choose a finite open covering $\mathcal{U} = \{U_1, \ldots, U_d\}$ of Δ such that the $\Phi_{\mathcal{A}}^{-1}(U_i)$ are displaceable for $i = 1, \ldots, d$. We choose a partition of unity $\{\chi_i\}_{1 \le i \le d}$ subordinate to $\mathcal{U}$. Then $\Phi_{\mathcal{A}}^* \chi_i = \chi_i \circ \Phi_{\mathcal{A}}$ has its support displaceable. Then $\zeta(\Phi_{\mathcal{A}}^* \chi_i) = 0$ by the vanishing axiom. We also have $\{\Phi_{\mathcal{A}}^* \chi_i, \Phi_{\mathcal{A}}^* \chi_j\} = 0$. Applying the normalization and the partial additivity, we obtain

$$1 = \zeta(1) = \zeta\left(\sum_{i=1}^{d} \Phi_{\mathcal{A}}^* \chi_i\right) = \sum_{i=1}^{d} \zeta(\Phi_{\mathcal{A}}^* \chi_i) = 0,$$

which is a contradiction. This finishes the proof.

For the second statement, we need only further refine the open set $\Phi_{\mathcal{A}}^{-1}(U_i)$ by separately covering the connected components of the fiber. We leave the details as an exercise. □

This theorem also provides a simple way of obtaining a *coisotropic* object that has the symplectic intersection property, since the fiber $\Phi_{\mathcal{A}}^{-1}(p)$ is a coisotropic submanifold whenever it is smooth.

Theorem 22.4.7 (Entov–Polterovich) *Consider a triangulation Σ of M. The $2n-1$ skeleton Σ^{2n-1} of M is nondisplaceable, provided that the mesh of the triangulation Σ is sufficiently small.*

Proof To prove this, we first express Σ^{2n-1} as the level set of some continuous function F. Define the function in the following way. First we set $F \equiv 0$ on Σ^{2n-1}. Then, on each simplex Δ, we define

$$F = 1 - \mathrm{dist}(\cdot, x_\Delta) \geq 0,$$

where x_Δ is the barycenter of Δ and $\mathrm{dist}(\cdot, x_\Delta)$ is the affine distance function from x_Δ on Δ. We can choose piecewise linear coordinates of Δ so that $F(x_\Delta) = 1$ for all barycenters. By definition, we have $F^{-1}(0) = \Sigma^{2n-1}$. By making the mesh of Σ sufficiently small, we can make each simplex in Σ displaceable. Then, by definition, $F^{-1}(\epsilon)$ for each $0 < \epsilon \leq 1$ is a disjoint union of boundaries of 'small' simplices, so each connected component thereof is contained in some simplex of Σ and hence is displaceable.

Now suppose that $F^{-1}(0) = \Sigma^{2n-1}$ is displaceable, i.e., there exists $\phi \in \mathrm{Ham}(M, \omega)$ such that $\phi(F^{-1}(0)) \cap F^{-1}(0)$. Since $F^{-1}(0)$ is compact, there exists $\epsilon_0 > 0$ such that

$$\phi(F^{-1}(0)) \subset F^{-1}((\epsilon_0, \infty)).$$

Then, by continuity, $F^{-1}([0, \epsilon_1])$ is displaceable if $0 < \epsilon_1 < \epsilon_0$ is small enough. Now we C^0-approximate F by a smooth function G. If $\|G - F\|_{C^0}$ is sufficiently small, it follows that there exist some $\delta = \delta(\|G - F\|_{C^0}) > 0$ and $0 < \epsilon_2 < \epsilon_1$ such that

(1) $\min G \geq -\delta$
(2) $G^{-1}([-\delta, \delta]) \subset F^{-1}([0, \epsilon_0])$ (in particular $G^{-1}([-\delta, \delta])$ is displaceable)
(3) $G^{-1}((-\delta, \delta)) \supset F^{-1}([0, \epsilon_2])$.

We consider $G^{-1}(\epsilon)$ with $\epsilon > \delta$. If $\|G - F\|_{C^0}$ is sufficiently small then the Hausdorff distance between $F^{-1}(\epsilon)$ and $G^{-1}(\epsilon)$ can be made arbitrarily small and hence each connected component of $G^{-1}(\epsilon)$ must be contained in a simplex of Σ and thus displaceable for all $\epsilon > \delta$. On the other hand, for any $\epsilon \leq \delta$, $G^{-1}(\epsilon) \subset G^{-1}([-\delta, \delta])$ and thus must be also displaceable. Obviously this contradicts Theorem 22.4.6 applied to $\mathcal{A} = \{G\}$ for the smooth function G. □

Now the remaining section will be occupied by the proof of Theorem 22.4.4. For the sake of notational simplicity, we denote

$$\mathcal{P}_U^{\text{ham}} = \mathcal{P}^{\text{ham}}(\mathrm{Symp}_U(M,\omega), id)$$

as before. We start with (22.3.25) and the following fragmentation lemma of Banyaga (Ba78).

Lemma 22.4.8 (Fragmentation lemma (Ba78)) *Let (M,ω) be a closed symplectic manifold. Let $\{U_1,\ldots,U_m\}$ be an open covering consisting of U_i such that $\omega|_{U_i}$* is exact. *Then any element $\lambda \in \mathcal{P}^{\text{ham}}(\mathrm{Symp}_0(M,\omega), id)$ can be written as*

$$\lambda = \prod_{i=1}^{m} \lambda_i, \quad \lambda_i \in \mathcal{P}_{U_i}^{\text{ham}},\ \mathrm{Cal}_{U_i}^{\text{path}}(\lambda_i) = 0.$$

Proof By an obvious induction argument it suffices to consider the case $N = 2$, namely $U = U_1 \cup U_2$. Let $\lambda = \phi_H \in \mathcal{P}^{\text{ham}}(\mathrm{Symp}_0(M,\omega), id)$. By taking a partition of the unit interval into a finite number of subintervals and reparameterizing them into the unit interval, we may assume without loss of generality that $\|H\|_{C^1} < \epsilon$ for a sufficiently small positive number depending only on U_1, U_2 and U, whose smallness is to be determined later.

We take a pair of open subsets $V_1'' \subset V_1'$ so that $V_1'' \subset V_1' \subset U_1$, $\overline{V}_1'' \subset V_1' \subset \overline{V}_1' \subset U_1$ and $V_1'' \cup U_2 \supset \mathrm{supp}\, H$. Let $\chi : M \to [0,1]$ be a smooth cut-off function such that $\mathrm{supp}\,\chi \subset U_1$ and $\chi = 1$ on V_1', and consider the path $\lambda_1 = \phi_{\chi H}$. It is easy to see that, if ϵ is sufficiently small, then $\lambda_1 = \lambda$ on V_1''. Moreover, we may assume that $\lambda(t,x) \equiv x$ for all $t \in [0,1]$ and for $x \notin V_1'' \cup U_2$, using the fact that they are C^1-close to the identity path.

In particular, the support of $\lambda_2 := \lambda_1^{-1}\lambda$ is contained in U_2 and the support of λ_1 in U_1. By definition, $\lambda = \lambda_1\lambda_2$, which finishes the proof for $m = 2$. This finishes the proof. □

This lemma motivates the following definition.

Definition 22.4.9 For a given open subset $U \subset M$ and $\lambda \in \mathcal{P}_U^{\text{ham}}$, we define the *fragmentation length* of λ on U as the minimal possible number of factors in the product of the lemma applied to (U,ω). We denote it by $\|\lambda\|_U$.

The following proposition is nothing but the path version of Theorem 7.1 (EnP06).

Proposition 22.4.10 *The function μ_1^{path}* (22.4.31) *is well defined and satisfies the following properties.*

1. *(Controlled quasi-additivity) Given a displaceable open subset U of M, there exists a constant K, depending only on U, such that*

$$|\mu_1^{\text{path}}(\lambda\delta) - \mu_1^{\text{path}}(\lambda) - \mu_1^{\text{path}}(\delta)| \le K \min\{\|\lambda\|_U, \|\delta\|_U\} \tag{22.4.34}$$

for any λ, $\delta \in \mathcal{P}^{\text{ham}}(\mathrm{Symp}(M, \omega), id)$.

2. *(Calabi property)* $\mu_1^{\text{path}}(\lambda) = \mathrm{Cal}_U^{\text{path}}(\lambda)$ *for any* $\lambda \in \mathcal{P}_U^{\text{ham}}$.

Proof We start with Proposition 22.3.3, which implies that

$$\rho(\lambda; 1) + \rho(\lambda^{-1}; 1) \le 2e_\rho(U; M) \tag{22.4.35}$$

for any $\lambda \in \mathcal{P}_U^{\text{ham}}$. Denote $C = 2e_\rho(U; M)$.

Next let δ be any Hamiltonian path on M. By the triangle inequality, we have

$$\begin{aligned}\rho(\lambda\delta; 1) &\ge \rho(\delta; 1) - \rho(\lambda^{-1}; 1)\\ &\ge \rho(\delta; 1) - (C - \rho(\lambda; 1))\\ &\ge \rho(\delta; 1) + \rho(\lambda; 1) - C.\end{aligned} \tag{22.4.36}$$

We also have $\rho(\lambda\delta; 1) \le \rho(\lambda) + \rho(\delta)$ by the triangle inequality.

Then we apply an induction argument for $\lambda_1, \ldots, \lambda_m \in \mathcal{P}_U^{\text{ham}}$ starting from (22.4.36) and obtain

$$|\rho(\lambda_1 \cdots \lambda_m \delta) - \sum_{i=1}^m \rho(\lambda_i; 1) - \rho(\delta; 1)| \le mC. \tag{22.4.37}$$

In particular, by putting $\delta \equiv id$ and taking the ℓth power, we have

$$|\rho((\lambda_1 \cdots \lambda_m)^\ell) - \ell \sum_{i=1}^m \rho(\lambda_i; 1)| \le \ell mC. \tag{22.4.38}$$

Now take any Hamiltonian path λ and express it as $\lambda = \lambda_1 \cdots \lambda_m$ with $\|\lambda_i\| = 1$ for all i, applying the fragmentation lemma. Then we obtain

$$\rho(\lambda^\ell; 1) \ge \ell \sum_{i=1}^m \rho(\lambda_i; 1) - \ell mC = \ell \left(\sum_{i=1}^m \rho(\lambda_i; 1) - mC \right)$$

and so

$$\frac{\rho(\lambda^\ell; 1)}{\ell} \ge \sum_{i=1}^m \rho(\lambda_i; 1) - mC =: E.$$

We note that m and the decomposition depend only on λ and hence the term on the right-hand side of this inequality depends only on λ. Therefore we have

$$E \le \frac{\rho(\lambda^\ell; 1)}{\ell} \le \rho(\lambda; 1) \tag{22.4.39}$$

for all ℓ, where the second inequality follows from the triangle inequality. The following lemma then will finish the proof of well-definedness of the limit and hence that of $\mu_1^{\text{path}}(\lambda)$.

Lemma 22.4.11 *The limit*

$$\lim_{\ell\to\infty} \frac{\rho(\lambda^\ell; 1)}{\ell}$$

exists as $\ell \to \infty$.

Proof We imitate the argument of the proof of Lemma 2.21 in (C09) for the following proof. By (22.4.39), we have

$$0 \le \rho(\lambda^\ell; 1) - E\ell.$$

Substitute the dyadic number $\ell = 2^i$ for positive integer i and then apply the triangle inequality to obtain

$$0 \le \rho(\lambda^{2^i}; 1) - 2^i E \le 2\rho(\lambda^{2^{i-1}}; 1) - 2^i E.$$

Dividing this by 2^i, we obtain

$$0 \le \frac{\rho(\lambda^{2^i}; 1) - 2^i E}{2^i} \le \frac{\rho(\lambda^{2^{i-1}}; 1) - 2^{i-1} E}{2^{i-1}}$$

and so the sequence $(\rho(\lambda^{2^i}; 1) - 2^i E)/2^i$ is non-negative and monotonically decreasing as $i \to \infty$. Therefore the sequence converges. Obviously this convergence implies convergence of $\rho(\lambda^{2^i}; 1)/2^i$ as $i \to \infty$, and in turn convergence of $\rho(\lambda^l; 1)/l$ as $l \to \infty$. □

Next we prove (22.4.34), controlled quasi-additivity. Consider any $\lambda, \delta \in \mathcal{P}^{\text{ham}}(\text{Symp}_0(M,\omega), id)$ that are not the identity. Then $\|\lambda\|_U, \|\delta\|_U \ge 1$. We claim that

$$|\mu_1^{\text{path}}(\lambda\delta) - \mu_1^{\text{path}}(\lambda) - \mu_1^{\text{path}}(\delta)| \le 2C(2\|\lambda\|_U - 1, 2\|\delta\|_U - 1). \tag{22.4.40}$$

We prove this by induction over $m := \min\{\|\lambda\|_U, \|\delta\|_U\}$. We note that, when $m = 0$, i.e., when either λ or δ is the identity path, this is trivial. Therefore we start with $m = 1$ as the first step of the induction, i.e., either $\|\lambda\|_U = 1$ or $\|\delta\|_U = 1$. Without any loss of generality, we may assume that $\|\lambda\|_U = 1$. We write

$$(\lambda\delta)^k = \left(\prod_{i=0}^{k-1} \delta^i \lambda \delta^{-i}\right) \cdot \delta^k. \tag{22.4.41}$$

By the conjugation invariance of $\|\cdot\|_U$, we have $\|\delta^i\lambda\delta^{-i}\|_U = \|\lambda\|_U = 1$. Applying (22.4.37) and the conjugation invariance of $\rho(\cdot; 1)$, we obtain

$$|\rho((\lambda\delta)^k; 1) - k\rho(\lambda; 1) - \rho(\delta^k; 1)| \le Ck.$$

On combining this with the inequality

$$|\rho(\lambda^k; 1) - k\rho(\lambda; 1)| \le Ck,$$

which corresponds to (22.4.36) for $m = 1$, $\ell = k$, we obtain

$$\frac{1}{k}\left|(\rho((\lambda\delta)^k; 1) - k\rho(\lambda; 1) - \rho(\delta^k; 1)) - (\rho(\lambda^k; 1) - k\rho(\lambda; 1))\right| \le 2C,$$

i.e.,

$$\frac{1}{k}\left|(\rho((\lambda\delta)^k; 1) - \rho(\delta^k; 1)) - \rho(\lambda^k; 1)\right| \le 2C.$$

Taking the limit of this as $k \to \infty$, we obtain

$$\left|\mu_1^{\mathrm{path}}(\lambda\delta) - \mu_1^{\mathrm{path}}(\lambda) - \mu_1^{\mathrm{path}}(\delta)\right| \le 2C,$$

which finishes the proof with $m = 1$ and $K = 2C$.

Now suppose (22.4.40) holds for all $m \le N$ and consider $m = N + 1$. Again we may assume $\|\lambda\|_U = N + 1$. Then we can decompose $\lambda = \lambda'\lambda_1$ with $\|\lambda_1\|_U = 1$ and $\|\lambda'\|_U = N$. Using the induction hypothesis, we obtain

$$\begin{aligned}
|\mu_1^{\mathrm{path}}(\lambda'\lambda_1\delta) - \mu_1^{\mathrm{path}}(\lambda') - \mu_1^{\mathrm{path}}(\lambda_1\delta)| &\le 2C(2m-1),\\
|\mu_1^{\mathrm{path}}(\lambda_1\delta) - \mu_1^{\mathrm{path}}(\lambda_1) - \mu_1^{\mathrm{path}}(\delta)| &\le 2C,\\
|\mu_1^{\mathrm{path}}(\lambda_1) + \mu_1^{\mathrm{path}}(\lambda') - \mu_1^{\mathrm{path}}(\lambda'\lambda_1)| &\le 2C.
\end{aligned}$$

Adding up these inequalities, we obtain

$$|\mu_1^{\mathrm{path}}(\lambda\delta) - \mu_1^{\mathrm{path}}(\lambda) - \mu_1^{\mathrm{path}}(\delta)| \le 2C(2m+1).$$

Setting $K = 4C = 8e_\rho(U; M)$, we have finished the proof of controlled quasi-additivity.

Finally we prove the Calabi property of μ_1^{path}. Let $\lambda = \phi_F \in \mathcal{P}_U^{\mathrm{ham}}$ and choose a Hamiltonian path $\delta = \phi_H$ with $\phi_H^1(U) \cap \overline{U} = \emptyset$. Proposition 22.3.3 implies

$$\rho(\phi_H\phi_F^m; 1) = \rho(\phi_H; 1) + \mathrm{Cal}_U^{\mathrm{path}}(\phi_F^m; 1) = \rho(\phi_H; 1) + m\,\mathrm{Cal}_U^{\mathrm{path}}(\phi_F)$$

for all m. On the other hand, we also have

$$|\rho(\phi_H\phi_F^m; 1) - \rho(\phi_H; 1) - \rho(\phi_F^m; 1)| \le 2e_\rho(U; M)$$

from (22.4.36). By dividing this inequality by m and letting $m \to \infty$, we obtain

$$\begin{aligned}
\lim_{m\to\infty}\frac{\rho(\phi_F^m; 1)}{m} &= \lim_{m\to\infty}\frac{\rho(\phi_H\phi_F^m; 1)}{m}\\
&= \lim_{m\to\infty}\frac{\rho(\phi_H; 1)}{m} + \mathrm{Cal}_U^{\mathrm{path}}(\phi_F) = \mathrm{Cal}_U^{\mathrm{path}}(\phi_F).
\end{aligned}$$

Therefore, by definition of μ_1^{path}, we obtain

$$\mu_1^{\text{path}}(\phi_F) = \lim_{m\to\infty} \frac{\rho(\phi_F^m; 1)}{m} = \mathrm{Cal}_U^{\text{path}}(\phi_F).$$

This finishes the proof. □

Now we are ready to wrap up the proof of Theorem 22.4.4.

Monotonicity, normalization, Lipschitz continuity and symplectic invariance immediately follow from the axioms of $\rho(\cdot; 1)$, whose proofs we leave to the reader. Semi-homogeneity immediately follows from the homogeneity of μ_1^{path} and the Lipschitz continuity of μ_1^{path}.

It remains to prove partial additivity and vanishing. Let $F, H \in C^\infty(M)$ with $\{F, H\} = 0$ and $\operatorname{supp} H \in U$. Note that $\|\phi_H^k\|_U = 1$ for all $k \in \mathbb{N}$. Because $\{F, H\} = 0$, the flows ϕ_H and ϕ_F commute. Therefore we have

$$\mu_1^{\text{path}}(\phi_F\phi_H) = \frac{1}{k}\mu_1^{\text{path}}((\phi_F\phi_H)^k) = \frac{1}{k}\mu_1^{\text{path}}(\phi_F^k\phi_H^k).$$

But we have

$$|\mu_1^{\text{path}}(\phi_F^k\phi_H^k) - \mu_1^{\text{path}}(\phi_F^k) - \mu_1^{\text{path}}(\phi_H^k)| \le K$$

for all k by (22.4.34). By combining the two expressions and dividing by k, we obtain

$$|\mu_1^{\text{path}}(\phi_F\phi_H) - \mu_1^{\text{path}}(\phi_F) - \mu_1^{\text{path}}(\phi_H)| \le \frac{K}{k}$$

for all k and so

$$\mu_1^{\text{path}}(\phi_F\phi_H) = \mu_1^{\text{path}}(\phi_F) + \mu_1^{\text{path}}(\phi_H).$$

By the Calabi property, we have

$$\mu_1^{\text{path}}(\phi_H) = \mathrm{Cal}_U^{\text{path}}(\phi_H) = (\mathrm{vol}_\omega(M))^{-1} \int_M H\, d\mu_\omega.$$

On substituting this into the definition of ζ_{sp}, we get $\zeta_{\text{sp}}(F + H) = \zeta_{\text{sp}}(F)$.

22.5 Entov and Polterovich's Calabi quasimorphism

Since the group $\mathrm{Ham}(M, \omega)$ for closed M is simple by Banyaga's theorem, there is no non-trivial surjective homomorphism from $\mathrm{Ham}(M, \omega)$ to $\mathbb{R}$ (or to any non-trivial abelian group). In other words, for any *surjective* map $\mu : \mathrm{Ham}(M, \omega) \to \mathbb{R}$, the defect

$$\sup_{\phi_1,\phi_2\in \mathrm{Ham}(M,\omega)} |\mu(\phi_1\phi_2) - \mu(\phi_1) - \mu(\phi_2)|$$

is positive. If μ is a homomorphism, then its defect is certainly zero. When there is no non-trivial surjective homomorphism around, the next best thing to expect is a map μ whose defect is *bounded*. It turns out that $\mathrm{Ham}(M, \omega)$ carries some interesting quasimorphisms arising from the dynamics of Hamiltonian flows and their fixed points. This construction was performed by Entov and Polterovich (EnP03) on the basis of the axiomatic properties of spectral invariants. This is another manifestation of the fact that the spectral invariants encode the symplectic topological properties of (M, ω) and dynamical properties of Hamiltonian flows and their periodic orbits on (M, ω), and decode the symplectic invariants embedded in the analytic theory of Floer homology.

In Entov and Polterovich's construction it is essential to prove that the assignment $\phi_H \mapsto \rho(\phi_H; e)$ is a quasimorphism for a suitable choice of *idempotent* element $e \in QH^*(M)$. They prove this quasimorphism property under the assumption that the quantum cohomology ring of the underlying symplectic manifold (M, ω) satisfies some algebraic property (e.g., semi-simpleness thereof) and that the Frobenius pairing defines a nondegenerate pairing. The latter nondegeneracy property is not hard to prove for *rational* symplectic manifolds, but is much harder to prove for the irrational case (see (Ush10b)). The semi-simpleness of the quantum cohomology ring is satisfied by S^2, or more generally by $\mathbb{C}P^n$. This is a remarkable amalgamation of the algebraic properties of the quantum cohomology ring with the study of Hamiltonian dynamics via the axiomatic properties of spectral invariants. It leads to a new construction of quasimorphisms on the area-preserving diffeomorphism even for the case of S^2. Such a study has been of much interest in the area of dynamical systems. See (GG97), (GG04), (Gh06), (C09), for example.

22.5.1 Poincaré duality revisited

We first collect some basic facts on the duality in Floer homology.

Consider the filtered Floer complex

$$CF_k^{\lambda}(H) = CF_k^{(-\infty,\lambda]}(H) = \{\alpha \in CF_k(H) \mid \lambda_H(\alpha) \le \lambda\}$$

and its quotient

$$CF_k^{(\mu,\lambda]}(H) = CF_k^{(-\infty,\lambda]}(H)/CF_k^{(-\infty,\mu]}(H)$$

for any general nondegenerate H and for $k \in \mathbb{Z}_+$. We denote by

$$i_\lambda : CF_*^{(-\infty,\lambda]}(H) \to CF_*(H)$$

the canonical inclusion and by

$$\pi_\lambda : CF_*(H) \to CF_*^{(\lambda,\infty)}(H) = CF_*(H)/CF_*^{(-\infty,\lambda]}(H)$$

the canonical projection.

For each fixed λ_1, $\lambda_2 < \lambda$, we have natural projections

$$\pi^\lambda_{\lambda_2\lambda_1} : CF_k^{(\lambda_1,\lambda]}(H) \to CF_k^{(\lambda_2,\lambda]}(H), \quad \lambda_1 \le \lambda_2,$$

defining an inverse system. We have the following lemma.

Lemma 22.5.1 *Let (M,ω) be an arbitrary compact symplectic manifold and let (H,J) be Floer-regular. Then the canonical limit map*

$$\pi^\lambda : CF_k^{(-\infty,\lambda]}(H) \to \varprojlim CF_k^{(\mu,\lambda]}(H)$$

is an isomorphism for all $\lambda \in \mathbb{R} \cup \{+\infty\}$.

Proof Let $\alpha \in CF_k^{(-\infty,\lambda]}(H)$. By definition, we can write

$$\alpha = \sum_{i=1} a_i[z_i,w_i] = \lim_{N\to\infty} \sum_{i=1}^N a_i[z_i,w_i], \quad \mathcal{A}_H([z_i,w_i]) \ge \mathcal{A}_H([z_{i+1},w_{i+1}]),$$

where either only finitely many of the a_i are non-zero or $\lim_{i\to\infty} \mathcal{A}_H([z_i,w_i]) \to -\infty$ otherwise. From this it follows that $\alpha = 0$ if and only if $\sum_{i=1}^N a_i[z_i,w_i] = 0$ for all N, i.e., $\pi^\lambda_{-\infty\mu}(\alpha) = 0$ for all μ. Therefore π^λ is injective. On the other hand, any element $\{\alpha_\mu\} \in \varprojlim CF_k^{[\mu,\lambda)}(H)$ can be represented by a finite sum

$$\sum_{i=1}^{N_\mu} a_i[z_i,w_i]$$

for some $N_\mu \in \mathbb{Z}_+$. Suppose we are given an element $\{\alpha_\mu\}_{\mu\in\mathbb{R}}$ satisfying $\pi^\lambda_{\mu'\mu}(\alpha_{\mu'}) = \alpha_\mu$. Then, for any sequence $\mu_k \to -\infty$ as $k \to \infty$, we can write

$$\alpha_{\mu_k} = \sum_{i=1}^{N_{\mu_k}} a_i[z_i,w_i],$$

which converges to, say, α in $CF^\lambda(H)$. It is easy to check that this limit does not depend on the choice of the sequences μ_k, which proves that π^λ is an isomorphism. This finishes the proof. □

Remark 22.5.2 Now suppose that (M,ω) is rational. In this case, it follows that $CF_k^{(\mu,\lambda]}(H)$ is a finite-dimensional $\mathbb{Q}$-vector space. We would like to emphasize that this is not the case for an irrational (M,ω).

We consider H and $\widetilde{H} = -H(1-t,x)$ and define a $\mathbb{Q}$-valued pairing

$$\langle\cdot,\cdot\rangle : CF_n^{(-\infty,\lambda]}(H) \times CF_n^{(-\lambda,\infty)}(\widetilde{H}) \to \mathbb{Q} \tag{22.5.42}$$

by

$$\langle\alpha,\widetilde{\beta}\rangle = \sum_{[z,w]\in \operatorname{Crit}\mathcal{A}_H} a_{[z,w]} b_{[z,w]}$$

for $\alpha = \sum_{[z,w]\in\operatorname{Crit}\mathcal{A}_H} a_{[z,w]}[z,w]$ and $\widetilde{\beta} = \sum_{[\widetilde{z},\widetilde{w}]\in\operatorname{Crit}\mathcal{A}_{\widetilde{H}}} b_{[\widetilde{z},\widetilde{w}]}[\widetilde{z},\widetilde{w}]$. By the Novikov finiteness condition, any element in

$$CF_n^{(-\lambda,\infty)}(\widetilde{H}) = CF_n(\widetilde{H})/CF_n^{(-\infty,-\lambda]}(\widetilde{H})$$

can be represented by a finite sum and hence the above pairing is well defined and nondegenerate. This pairing then induces an injective chain homomorphism

$$(CF_n^{(-\infty,\lambda]}(H), \partial_{(H,J)}) \to (\operatorname{Hom}_{\mathbb{Q}}(CF_n^{(-\lambda,\infty)}(\widetilde{H}),\mathbb{Q}), (\partial_{\widetilde{H},\widetilde{J}})^*). \tag{22.5.43}$$

Recall that we have the canonical identification

$$\operatorname{Hom}(CF_n^{(-\lambda,\infty)}(\widetilde{H}),\mathbb{Q}) = \{\varphi\in\operatorname{Hom}(CF_n(\widetilde{H}),\mathbb{Q}) \mid \varphi|_{CF_n^{(-\infty,\lambda]}(\widetilde{H})} \equiv 0\}.$$

We denote $\operatorname{Hom}(CF_n^{(-\lambda,\infty)}(\widetilde{H}),\mathbb{Q}) = CF_n^{(-\lambda,\infty)}(\widetilde{H})^*$. Then, via this identification, we have the canonical inclusion map

$$CF_n^{(-\mu,\infty)}(\widetilde{H})^* \hookrightarrow CF_n^{(-\lambda,\infty)}(\widetilde{H})^*$$

when $\lambda > \mu$. We also have the natural identification

$$CF_n^{(-\lambda,-\mu]}(\widetilde{H})^* \cong CF_n^{(-\lambda,\infty)}(\widetilde{H})^*/CF_n^{(-\mu,\infty)}(\widetilde{H})^*.$$

The following theorem is a crucial fact that was used in the construction of quasimorphisms by Entov and Polterovich for the rational (M,ω). The corresponding theorem for the irrational case is much harder to prove. It was proved by Usher (Ush10b) using arguments from non-Archimedean analysis. Here we give the full details of the proof in the rational case following Entov and Polterovich and leave readers to consult the original article of Usher for the irrational case.

Theorem 22.5.3 *Let (M,ω) be any compact rational symplectic manifold. Then the canonical $\mathbb{Q}$-homomorphism* (22.5.43) *is a chain isomorphism. In particular the pairing* (22.5.42) *descends to a nondegenerate pairing*

$$HF_n^{(-\infty,\lambda]}(H) \times HF_n^{(-\lambda,\infty)}(\widetilde{H}) \to \mathbb{Q}. \tag{22.5.44}$$

Proof We recall from Remark 22.5.2 that both $CF_k^{(\mu,\lambda]}(H)$ and $CF_n^{(-\lambda,-\mu]}(\widetilde{H})^*$ are *finite*-dimensional $\mathbb{Q}$-vector spaces. (In the irrational case, they are infinite-dimensional, which is the main reason why the irrational case cannot be covered by this proof.)

We have the commutative diagram of chain complexes

$$
\begin{tikzcd}
CF_*^{(-\infty,\lambda]}(H) \arrow[r, "\pi_\mu^\lambda"] \arrow[d] & CF_*^{(\mu,\lambda]}(H) \arrow[d] \\
CF_*^{(-\lambda,\infty)}(\widetilde{H})^* \arrow[r] & CF_n^{(-\lambda,-\mu)}(\widetilde{H})^*
\end{tikzcd}
\tag{22.5.45}
$$

and the map $CF_*^{(\mu,\lambda]}(H) \to CF_*^{(-\lambda,-\mu)}(\widetilde{H})^*$ is a chain isomorphism, since for each degree the map is injective, and both spaces are finite-dimensional by the *rationality* assumption on (M,ω). Therefore the above diagram induces a commutative diagram

$$
\begin{tikzcd}
CF_*^{(-\infty,\lambda]}(H) \arrow[r, "\pi^\lambda"] \arrow[d] & \varprojlim CF_*^{(\mu,\lambda]}(H) \arrow[d] \\
CF_*^{(-\lambda,\infty)}(\widetilde{H})^* \arrow[r] & \varprojlim CF_*^{(-\lambda,-\mu)}(\widetilde{H})^*
\end{tikzcd}
$$

whose right downward map is a chain isomorphism. Since the horizontal maps are isomorphisms, the left downward map is also a chain isomorphism. Therefore it induces an isomorphism

$$
H\left(CF_*^{(-\infty,\lambda]}(H), \partial_{(H,J)}\right) \cong H\left(CF_n^{(-\lambda,\infty)}(\widetilde{H})^*, \partial^*_{(\widetilde{H},\widetilde{J})}\right).
$$

Here the left-hand side is precisely $HF^{(-\infty,\lambda]}(H)$ and, by the universal coefficient theorem (over the field $\mathbb{Q}$), the right-hand side is isomorphic to

$$
\mathrm{Hom}_{\mathbb{Q}}(H(CF_n^{(-\lambda,\infty)}(\widetilde{H}), \partial_{(\widetilde{H},\widetilde{J})}), \mathbb{Q}) = HF^{(-\lambda,\infty)}(\widetilde{H}, \widetilde{J}).
$$

This isomorphism naturally induces a (strongly) nondegenerate pairing

$$
HF_n^{(-\infty,\lambda]}(H) \times HF_n^{(-\lambda,\infty)}(\widetilde{H}) \to \mathbb{Q}
$$

for all λ. Hence we obtain the nondegenerate pairing (22.5.44). □

When (M,ω) is irrational, the above map

$$
CF_*^{(\mu,\lambda]}(H) \to CF_*^{(-\lambda,-\mu]}(\widetilde{H})^*
$$

is not an isomorphism. In fact, the latter space is considerably bigger in that the map is injective but not surjective.

Example 22.5.4 Let (M, ω) be irrational so that the period group $\{\omega(A) \mid A \in \pi_2(M)\}$ is a dense subgroup of $\mathbb{R}$. Then, for any Hamiltonian H, Spec(H) is a dense subset of $\mathbb{R}$. Consider a formal sum

$$\gamma = \sum_{i=1}^{\infty} a_i[z_i, w_i], \quad a_i \neq 0$$

such that $\mathcal{A}([z_i, w_i]) \in (\mu, \lambda)$ for all i and $[z_i, w_i] \neq [z_j, w_j]$ whenever $i \neq j$. Since this infinite sum does not satisfy the Novikov finiteness condition, γ is not an element of $CF_*^{(\mu,\lambda]}(H)$. But one can easily check that the pairing

$$\gamma \mapsto \langle \gamma, \cdot \rangle$$

defines a linear functional on $CF_*^{(-\lambda,-\mu)}(\widetilde{H})$ that does not lie in the image of $CF_*^{(\mu,\lambda]}(H)$.

We now state the following general theorem on the relation between the spectral invariants of $\overline{H}$ and those of H. This relation plays a crucial role in Entov and Polterovich's construction of quasimorphisms.

Lemma 22.5.5 (Lemma 2.2 of (EnP03)) *Let* Π *be the pairing given in* (20.4.43). *For* $b \in QH^*(M) \setminus \{0\}$, *denote by* $\Upsilon(b)$ *the set of* $a \in QH^*(M)$ *with* $\Pi(b, a) \neq 0$. *Then we have*

$$\rho(\overline{H}; b) = -\inf_{a \in \Upsilon(b)} \rho(H; a). \tag{22.5.46}$$

Proof First, we prove

$$\rho(\overline{H}; b) \geq -\inf_{a \in \Upsilon(b)} \rho(H; a). \tag{22.5.47}$$

The *triangle inequality* for ρ is the crucial ingredient of the proof of this inequality.

Suppose to the contrary that $\rho(\overline{H}; b) + \inf_{a \in \Upsilon(b)} \rho(H; a) < 0$ and hence there exists a with $\Pi(a, b) \neq 0$ such that

$$\rho(\overline{H}; b) + \rho(H; a) < 0.$$

On the other hand, for any a with $\Pi(a, b) \neq 0$, we have $(a * b)_0 \neq 0$ by the Frobenius property (20.4.43) of $QH^*(M)$ and hence $\rho(0; a*b) = -v(a*b) \geq 0$. But then we have

$$0 \leq \rho(0; a * b) = \rho(H\#\overline{H}; a * b) \leq \rho(H; a) + \rho(\overline{H}; b) < 0,$$

which is a contradiction. This finishes the proof of (22.5.47).

Next we prove $\rho(\overline{H}; b) \leq -\inf_{a \in \Upsilon(b)} \rho(H; a)$, i.e.,

$$\inf_{a \in \Upsilon(b)} \rho(H; a) \leq -\rho(\overline{H}; b). \tag{22.5.48}$$

The *nondegeneracy of the duality pairing* above plays a key role in this proof.

Let $c = \rho(\overline{H}; b) > -\infty$. Let $\Phi_{\widetilde{H}} : QH^*(M) \to FH_*(\widetilde{H})$ be the PSS map. Then $\Phi_{\widetilde{H}}(b) \notin \mathrm{Im}(i_{c-\epsilon})_* \subset HF_*(\overline{H})$ for any $\epsilon > 0$. By the exactness of

$$HF_*^{c-\epsilon}(H) \overset{(i_{c-\epsilon})_*}{\to} HF_*(H) \overset{(\pi_{c-\epsilon})_*}{\to} HF_*^{(c-\epsilon,\infty)}(H)$$

at the middle, we have $\ker(\pi_{c-\epsilon})_* = \mathrm{Im}(i_{c-\epsilon})_*$ and hence

$$(\pi_{c-\epsilon})_* \left(\Phi_{\widetilde{H}}(b)\right) \neq 0.$$

Then, by the nondegeneracy of the duality pairing, we can find some Floer homology class $0 \neq a_H \in HF_*^{(-\infty,-c+\epsilon]}(H)$ represented by a cycle $\alpha_H \in CF_*(H)$ with $\lambda_H(\alpha_H) \leq -c + \epsilon$ such that

$$L'([\alpha_H], \Phi_{\widetilde{H}}(b)) \neq 0.$$

Now we write the Floer homology class $[\alpha_H] \in FH_*(H)$ as $[\alpha_H] = \Phi_H(a')$ for $a' \in QH^*(M)$. Then we have

$$\Pi(a', b) = L'\left(\Phi_H(a'), \Phi_{\widetilde{H}}(b)\right) = L'([\alpha_H], \Phi_{\widetilde{H}}(b)) \neq 0$$

and hence $a' \in \Upsilon(b)$. Here, for the last equality, we use Theorem 22.5.3 and (22.5.42).

Since $\rho(H; a') \leq \lambda_H(\alpha_H)$, we have $\rho(H; a') \leq -c + \epsilon$. Therefore

$$\inf_{a \in \Upsilon(b)} \rho(H; a) \leq \rho(H; a') \leq -c + \epsilon.$$

Since this holds for all $\epsilon > 0$, we have finished the proof of (22.5.48) and hence the proof of the lemma. □

22.5.2 Some generalities of quasimorphisms

We first summarize basic facts on general quasimorphisms. A good up-to-date reference on the study of quasimorphisms in the general context is the book (C09) by Calegari.

Consider any group G and a surjective map $\mu : G \to \mathbb{R}$.

Definition 22.5.6 (Quasimorphism) Consider a function $\mu : G \to \mathbb{R}$. We define its (homomorphism) *defect* $\mathrm{Def}_\mu : G \times G \to \mathbb{R}$ by the formula

$$\mathrm{Def}_\mu(g_1, g_2) = \mu(g_1 g_2) - \mu(g_1) - \mu(g_2)$$

and its norm by

$$D(\mu) := |\operatorname{Def}_\mu| = \sup_{g_1,g_2\in G} |\mu(g_1g_2) - \mu(g_1) - \mu(g_2)|.$$

We say μ is a *quasimorphism* if $D(\mu) < \infty$, i.e., if there exists $R > 0$ such that

$$|\mu(g_1g_2) - \mu(g_1) - \mu(g_2)| < R. \tag{22.5.49}$$

The map Def measures the defect of the homomorphism property of the quasimorphism $\mu : G \to \mathbb{R}$.

We recall the following standard definition of group cohomology. (See (Wb94) for example.)

Definition 22.5.7 Let M be a left G-module. Let $(C^*(G, M), \delta)$ be the complex, where $C^n(G, M)$ is the module of functions $\varphi : G^n \to R$ and the boundary map $\delta : C^n(G, M) \to C^{n+1}(G, M)$ is given by

$$\begin{aligned}(\delta\varphi)\,(g_1, \ldots, g_{n+1}) = g_1 \cdot \varphi(g_2, \ldots, g_{n+1})& \\ + \sum_{i=1}^{n}(-1)^i\varphi(g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_{n+1})& \\ +(-1)^{n+1}\varphi(g_1, \ldots, g_n).&\end{aligned} \tag{22.5.50}$$

A function $\varphi : G^k \to R$ is called a group k-cocycle if $\delta\varphi = 0$ and a coboundary if $\varphi = \delta\psi$ for some cochain $\psi : G^{k-1} \to R$. The cohomology groups of G with coefficients in the left G-module M are the homology groups, denoted by $H^*(G, M)$, of the complex $(C^*(G, M), \delta)$.

Let us consider the special case of $R = \mathbb{R}$ with trivial G-action. Then (22.5.50) is reduced to

$$(\delta\varphi)\,(g_1, g_2, g_3) = \varphi(g_1g_2, g_3) - \varphi(g_1, g_2g_3) - \varphi(g_1, g_2)$$

for a two-chain φ, and

$$\delta\varphi(g_1, g_2) = \varphi(g_1g_2) - \varphi(g_1) - \varphi(g_2)$$

for a one-chain φ. From a straightforward computation or the formal expression $\operatorname{Def}_\mu = \delta\mu$, it follows that Def_μ is a two-cocycle.

However, when the group G is non-compact, as in the case of G being $\widetilde{\operatorname{Ham}}(M, \omega)$ or its like, there is a sub-complex $(C^*_{bd}(G, \mathbb{R}), \delta)$ of $(C^*(G, \mathbb{R}), \delta)$ consisting of the *bounded cochains*. When the defect is bounded, it defines a two-cocycle of $(C^*_{bd}(G, \mathbb{R}), \delta)$. Since μ need not be bounded, this expression $\operatorname{Def}_\mu = \delta\mu$ itself does not mean that Def_μ defines a trivial bounded 2-cohomology class, although formally it has the form of coboundary.

Definition 22.5.8 A quasimorphism is called *homogeneous* if it satisfies

$$\mu(g^n) = n\mu(g) \tag{22.5.51}$$

for all $n \in \mathbb{Z}$ and $g \in G$.

When G is given a quasimorphism μ, there is a natural procedure of homogenizing μ by taking its asymptotic average. We would like to note that this way of taking the asymptotic average is a common practice in many areas of mathematics, especially in dynamical systems and ergodic theory.

Definition 22.5.9 (Homogenization) The *homogenization* μ^{homo} of a quasimorphism $\mu : G \to \mathbb{R}$ is defined by the limit

$$\mu^{\text{homo}}(g) = \lim_{n\to\infty} \frac{\mu(g^n)}{n} \tag{22.5.52}$$

for $g \in G$.

Proposition 22.5.10 *Suppose that $\mu : G \to \mathbb{R}$ is a quasimorphism. Then μ^{homo} is again a quasimorphism that is homogeneous. In addition, we have*

$$|\mu^{\text{homo}}(g) - \mu(g)| \le D(\mu). \tag{22.5.53}$$

Proof The proof of well-definedness is essentially the same as that of Lemma 22.4.11 and hence has been omitted. Once well-definedness has been established, homogeneity immediately follows. Since any bounded perturbation of a quasimorphism is again a quasimorphism, the quasimorphism property will follow from (22.5.53). Hence it remains to prove the inequality (22.5.53).

By definition, we have

$$\mu^{\text{homo}}(g) - \mu(g) = \lim_{n\to\infty} \frac{\mu(g^n)}{n} - \mu(g) = \lim_{n\to\infty} \frac{\mu(g^n) - n\mu(g)}{n}.$$

We can rewrite

$$\mu(g^n) - n\mu(g) = \sum_{j=1}^{n} \left(\mu(g^j) - \mu(g^{j-1}) - \mu(g)\right).$$

For each summand in this sum, $|\mu(g^j) - \mu(g^{j-1}) - \mu(g)| \le D(\mu)$, so

$$|\mu(g^n) - n\mu(g)| \le nD(\mu).$$

Therefore we obtain (22.5.53) by dividing this inequality by n. □

22.5.3 Entov–Polterovich quasimorphism on $\widetilde{\mathrm{Ham}}(M, \omega)$

Now we explain the procedure of obtaining a quasimorphism out of spectral numbers $\rho(\cdot; e)$ for any idempotent $e \in QH^*(M)$, i.e., those e satisfying $e^2 = e$. (For example, $e = 1$, the unit of the quantum cohomology, is a natural candidate for this construction.) This is the procedure invented by Entov and Polterovich (EnP03), which was further refined by McDuff and Ostrover (Os03).

Consider the subring $QH^{(0)}(M) \subset QH(M)$ of degree zero. This is a commutative Frobenius algebra with unit $1 = PD(M)$ over the field $K = \Lambda_\omega^{(0)}$. Let e be any non-zero idempotent of $QH^{(0)}(M)$. We consider the spectral map

$$\rho_e : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$$

defined by $\rho_e(f) = \rho(f; e)$ for $f \in \widetilde{\mathrm{Ham}}(M, \omega)$. The triangle inequality together with the idempotency $e^2 = e$ implies that

$$\rho_e(fg) \le \rho_e(f) + \rho_e(g).$$

We now ask under which condition of (M, ω) ρ_e satisfies the inequality

$$\rho_e(fg) \ge \rho_e(f) + \rho_e(g) - R \tag{22.5.54}$$

for some constant R independent of f, g, which would ensure that ρ_e is a quasimorphism on the *infinite-dimensional* group $\widetilde{\mathrm{Ham}}(M, \omega)$. The following theorem is a slight variation of that of Entov and Polterovich (EnP03) refined by McDuff and Ostrover (Os03). We closely follow the proof given in (EnP03).

Theorem 22.5.11 ((EnP03)) *Assume that $Q = QH^{(0)}(M)$ has a decomposition $Q = Q_1 \oplus Q'$ as a K-vector space such that Q_1 and Q' are subrings of Q and Q_1 is a field. Denote by e the unit of Q_1. Then there exists some $R > 0$ such that* (22.5.54) *holds, and hence ρ_e defines a quasimorphism on* $\widetilde{\mathrm{Ham}}(M, \omega)$.

Proof The valuations ν on Λ_ω and on $QH^*(M)$ are restricted to those defined on $K = \Lambda_\omega^{(0)}$, and on $Q = QH^{(0)}(M)$ and on Q_1, respectively.

Lemma 22.5.12 *The function $\zeta \in K \mapsto e^{-\nu(\zeta)} := |\zeta|$ is multiplicative, i.e., satisfies $|\zeta_1\zeta_2| = |\zeta_1||\zeta_2|$ for all ζ_1, $\zeta_2 \in K$.*

Proof We need only show that $\nu(\zeta_1\zeta_2) = \nu(\zeta_1) + \nu(\zeta_2)$. When $\zeta_1 = 0$, $\zeta_1\zeta_2 = 0$ and hence the identity trivially holds as both sides become $-\infty$. Therefore we assume that ζ_1, $\zeta_2 \neq 0$. Since K is a field it does not have a zero-divisor and so $\zeta_1\zeta_2 \neq 0$. By definition, we have

$$\zeta_i = a_{i,0} T^{\nu(\zeta_i)} + (\zeta_i - a_{i,0} T^{\nu(\zeta_i)})$$

with $0 \neq a_{i,0} \in \mathbb{Q}$ and $\nu(\zeta_i - a_{i,0}T^{\nu(\zeta_i)}) > \nu(\zeta_i)$ for $i = 1, 2$. Therefore we have

$$\zeta_1\zeta_2 = a_{1,0}a_{2,0}T^{\nu(\zeta_1)+\nu(\zeta_2)} + \text{‘higher-order terms’}.$$

Since $a_{1,0}a_{2,0} \neq 0$, we obtain $\nu(\zeta_1\zeta_2) = \nu(\zeta_1) + \nu(\zeta_2)$. □

Proposition 22.5.13 *There exists R' such that $\nu(b) + \nu(b^{-1}) \geq R'$ for every $b \in Q_1 \setminus \{0\}$.*

Proof We note that the field Q_1 is a finite-dimensional vector space over the field K and hence a finite field extension of the field K. Therefore the above multiplicative norm $|\cdot|$ on K extends to a multiplicative norm on Q_1. We denote this extension on Q_1 again by $|\cdot|$.

On the other hand the function $b \in Q_1 \mapsto e^{-\nu_{Q_1}(b)}$ defines a norm $\|\cdot\|$ on the vector space Q_1 over K. Recall that all norms on a finite-dimensional vector space over K are equivalent and hence there exists $r > 0$ such that

$$\|b\| \leq r|b|$$

for all $b \in Q_1$. Therefore for $b \neq 0$, we have

$$\|b\| \cdot \|b^{-1}\| \leq r^2|b| \cdot |b^{-1}| = r^2.$$

But we also have

$$\|b\| \cdot \|b^{-1}\| = e^{-\nu_{Q_1}(b)}e^{-\nu_{Q_1}(b^{-1})}.$$

Combining the two, we obtain

$$\nu_{Q_1}(b) + \nu_{Q_1}(b^{-1}) \geq -2\log r \tag{22.5.55}$$

for all $0 \neq b \in Q_1$. On setting $R' = -2\log r$, we are done. □

Now we are ready to give the proof of Theorem 22.5.11.

Let $f, g \in \widetilde{\mathrm{Ham}}(M, \omega)$. By the triangle inequality, we have

$$\rho(fg; e) \geq \rho(f; e) - \rho(g^{-1}; e).$$

By applying Lemma 22.5.5 to $\rho(g^{-1}; e)$, we get

$$\rho(fg; e) \geq \rho(f; e) + \inf_{b;\Pi(b,e)\neq 0} \rho(g; b). \tag{22.5.56}$$

We now estimate $\inf_{b;\Pi(b,e)\neq 0} \rho(g; b)$. By writing $b = b_1 + b'$, where $b_1 \in Q_1$, $b' \in Q'$, we obtain

$$0 \neq \Pi(b, e) = \pi_0(\langle b \cdot e, 1\rangle) = \pi_0(\langle b_1 \cdot e, 1\rangle) = \pi_0(\langle b_1, 1\rangle)$$

from (20.4.43). Therefore $b_1 \neq 0$ and $\nu_{Q_1}(b_1) \leq 0$. Since Q_1 is a field, b_1 is invertible and $\nu_{Q_1}(b_1^{-1}) \geq -\nu_{Q_1}(b_1) - 2\log r \geq R'$ by (22.5.55).

On the other hand, upon applying the triangle inequality twice, we obtain

$$\rho(g;b) \geq \rho(g;b\cdot e) - \rho(\mathbf{1};e) = \rho(g;b_1\cdot e) - \rho(\mathbf{1};e) \geq \rho(g;e) \\ - \rho(\mathbf{1};b_1^{-1}) - \rho(\mathbf{1};e)$$

where $\mathbf{1} \in \widetilde{\mathrm{Ham}}(M,\omega)$ is the identity element corresponding to the constant Hamiltonian path associated with the zero Hamiltonian function $\underline{0}$. By the normalization axiom of the spectral invariants, we have $\rho(\mathbf{1};b_1^{-1}) = -\nu(b_1^{-1})$ and $\rho(\mathbf{1};e) = -\nu(e)$. Therefore, on combining these, we obtain

$$\rho(g;b) \geq \rho(g;e) + \nu(b_1^{-1}) + \nu(e) \geq \rho(g;e) + R' + \nu(e)$$

for all b with $\Pi(b,e) \neq 0$. Therefore we have

$$\inf_{b;\Pi(b,e)\neq 0} \rho(g;b) \geq \rho(g;e) + R' + \nu(e).$$

Substituting this into (22.5.56), we obtain

$$\rho(fg;e) \geq \rho(f;e) + \rho(g;e) + R' + \nu(e).$$

This proves that $\rho(\cdot;e)$ defines a quasimorphism with defect smaller than $R = |R' + \nu(e)|$. This finishes the proof of Theorem 22.5.11. □

Homogenizing the above quasimorphism ρ_e, we obtain a homogeneous quasimorphism on $\widetilde{\mathrm{Ham}}(M,\omega)$:

$$\widetilde{\mu}_e(f) = \lim_{m\to\infty} \frac{\rho_e(f^m)}{m}. \tag{22.5.57}$$

Remark 22.5.14 We would like to point out that, due to the different conventions used in (EnP03), the negative sign in Equation (17) of (EnP03) does not appear in our definition.

We now prove the following important property of the quasimorphism $\widetilde{\mu}$.

Proposition 22.5.15 *Suppose that $U \subset M$ that is displaceable, i.e., there exists $\phi \in \mathrm{Ham}(M,\omega)$ such that $\phi(\overline{U}) \cap \overline{U} = \emptyset$. Then we have the identity*

$$\widetilde{\mu}(f) = \mathrm{Cal}_U(F)$$

for all $f = [\phi_F] \in \widetilde{\mathrm{Ham}}(M,\omega)$ with $\mathrm{supp}\, F \subset U$.

Proof Denote $\lambda = \phi_F$. By definition of $\rho(\lambda;1)$ for general Hamiltonian path λ,

$$\rho(\lambda^m;1) = \rho(\mathrm{Dev}(\lambda^m);1).$$

(Recall Definition 2.3.15 for the definition of $\mathrm{Dev}(\mu)$ for a general Hamiltonian path.) But

$$\begin{aligned}\mathrm{Dev}(\lambda^m) &= \underline{F^{\#m}} \\ &= F^{\#m} - \frac{1}{\mathrm{vol}_\omega(M)} \int F_t^{\#m}\, d\mu_\omega \\ &= F^{\#m} - \frac{1}{\mathrm{vol}_\omega(M)} \int \sum_{k=1}^m F\left(t, ((\phi_F^t)^{-1})^{k-1}(x)\right) d\mu_\omega \\ &= F^{\#m} - \frac{m}{\mathrm{vol}_\omega(M)} \int F(t,x) d\mu_\omega,\end{aligned}$$

where the last equality arises since $((\phi_F^t)^{-1})^{k-1}$ preserves the measure μ_ω for all $k = 1, \ldots, m$. Therefore we derive

$$\rho(\mathrm{Dev}(\lambda^m); 1) = \rho(F^{\#m}; 1) + m\,\mathrm{Cal}(F). \tag{22.5.58}$$

We note that $\mathrm{supp}\, F^{\#m} \subset U$ since $\mathrm{supp}\, F \subset U$. Choose any Hamiltonian H such that $\phi = \phi_H^1$. Then, since $\phi(U) \cap \overline{U} = \emptyset$ by the hypothesis, Proposition 22.3.7 implies

$$\rho(F^{\#m}; 1) \leq \gamma(H).$$

By substituting this into (22.5.58), dividing the resulting equality by m and then taking the limit as $m \to \infty$, we obtain

$$\widetilde{\mu}(f) \leq \mathrm{Cal}(F).$$

Similarly, by replacing λ by λ^{-1}, we obtain

$$\widetilde{\mu}(f^{-1}) \leq \mathrm{Cal}(\overline{F}) = -\mathrm{Cal}(F).$$

On the other hand, by the homogeneity of μ, we also have $\widetilde{\mu}(f^{-1}) = -\widetilde{\mu}(f)$. On combining the two, we have obtained $-\widetilde{\mu}(f) \leq -\mathrm{Cal}(F)$, which is equivalent to $\widetilde{\mu}(f) \geq \mathrm{Cal}(F)$. This proves $\widetilde{\mu}(f) = \mathrm{Cal}(F)$ and hence the proposition for $e = 1$.

The same proof also proves the second statement for general idempotent e. □

Remark 22.5.16 We established in Section 2.5.1 that on any exact (open) symplectic manifold, the value $\mathrm{Cal}^{\mathrm{path}}(\lambda)$ can be expressed as

$$\mathrm{Cal}^{\mathrm{path}}(\lambda) = \frac{1}{n+1} \int_M (\phi^*\alpha - \alpha) \wedge \alpha \wedge \omega^{n-1}. \tag{22.5.59}$$

In particular $\mathrm{Cal}^{\mathrm{path}}(\lambda)$ depends only on the final point $\lambda(1) = \phi$ and hence defines a non-trivial homomorphism

$$\mathrm{Cal} : \mathrm{Ham}^c(M, d\alpha) \to \mathbb{R}.$$

Applying this construction to $\mathcal{P}_U^{\mathrm{ham}}$ for contractible U, we can define Cal_U : $\mathrm{Ham}^c(U, \omega) \to \mathbb{R}$ for each contractible U. Then the above proof shows that, *whenever* $\widetilde{\mu}_e$ descends to a quasimorphism on $\mathrm{Ham}(M, \omega)$, we have

$$\mu_e(\phi) = \mathrm{Cal}_U(\phi)$$

for all $\phi \in \mathrm{Ham}^c(U) \subset \mathrm{Ham}(M, \omega)$.

Now we state a simple sufficient condition for $\widetilde{\mu}_e$ to descend to $\mathrm{Ham}(M, \omega)$.

Proposition 22.5.17 (Proposition 3.4 (EnP03)) *Suppose that* $\pi_1(\mathrm{Ham}(M, \omega))$ *is finite. Then the quasimorphism* $\widetilde{\mu}_e$ *descends to a Calabi quasimorphism on* $\mathrm{Ham}(M, \omega)$.

Proof Note that $\pi_1(\mathrm{Ham}(M, \omega))$ is nothing but the kernel of the covering projection $\widetilde{\mathrm{Ham}}(M, \omega) \to \mathrm{Ham}(M, \omega)$.

First we claim that $\pi_1(\mathrm{Ham}(M, \omega)) \subset \widetilde{\mathrm{Ham}}(M, \omega)$ lies in the center of $\widetilde{\mathrm{Ham}}(M, \omega)$. It suffices to prove $ghg^{-1}h^{-1} = id$ for any $g \in \pi_1(\mathrm{Ham}(M, \omega))$ and $h \in \widetilde{\mathrm{Ham}}(M, \omega)$. Represent $g = [\lambda]$ and $h = [\delta]$, where λ, $\delta : [0, 1] \to \mathrm{Ham}(M, \omega)$ are smooth paths, $\lambda(1) = \lambda(0) = id$ and $\delta(0) = id$. Note that for each $s \in [0, 1]$ we have

$$\lambda(1)\delta(s)\lambda(1)^{-1}\delta(s)^{-1} = id$$

and hence the paths $t \in [0, 1] \mapsto \lambda(t)\delta(st)\lambda(1)^{-1}\delta(st)^{-1}$ for $s \in [0, 1]$ define a homotopy of loops based at id. By definition, the path for $s = 0$ is the trivial loop and the path for $s = 1$ is the path $t \mapsto \lambda(t)\delta(t)\lambda(1)^{-1}\delta(t)^{-1}$ which represents $ghg^{-1}h^{-1}$ in $\widetilde{\mathrm{Ham}}(M, \omega)$. This proves $ghg^{-1}h^{-1} = id$, i.e., $gh = hg$ and hence the claim.

Therefore, on combining this with the quasimorphism property, we obtain

$$|\widetilde{\mu}_e(g^m) + \widetilde{\mu}_e(h^m) - \widetilde{\mu}_e((gh)^m)| = |\widetilde{\mu}_e(g^m) + \widetilde{\mu}_e(h^m) - \widetilde{\mu}_e(g^m h^m)| \le R$$

for all m. On the other hand, from the homogeneity of $\widetilde{\mu}_e$ we have that $\widetilde{\mu}_e((gh)^m) = m\widetilde{\mu}_e(gh)$ and $\widetilde{\mu}_e(g^m) + \widetilde{\mu}_e(h^m) = m(\widetilde{\mu}_e(g) + \widetilde{\mu}_e(h))$. On substituting this into the above, we obtain

$$m(|\widetilde{\mu}_e(g) + \widetilde{\mu}_e(h) - \widetilde{\mu}_e(gh)|) \le R$$

for all m. This proves

$$\widetilde{\mu}_e(g) + \widetilde{\mu}_e(h) = \widetilde{\mu}_e(gh). \tag{22.5.60}$$

Upon applying this identity repeatedly for $h = g$, we obtain

$$k\widetilde{\mu}_e(g) = \widetilde{\mu}_e(g^k)$$

for all $k \in \mathbb{N}$. But, since $\pi_1(\mathrm{Ham}(M,\omega))$ is assumed to be finite, $g^{k_0} = id$ for some choice of $k_0 \neq 0$ and hence $\widetilde{\mu}_e(g^{k_0}) = \widetilde{\mu}_e(id) = 0$. This implies that $\widetilde{\mu}_e(g) = 0$ for any $g \in \pi_1(\mathrm{Ham}(M,\omega))$.

On substituting this back into (22.5.60), we obtain $\widetilde{\mu}_e(gh) = \widetilde{\mu}_e(h)$ for all $g \in \pi_1(\mathrm{Ham}(M,\omega))$. This proves that $\widetilde{\mu}_e$ descends to a map $\mu_e : \mathrm{Ham}(M,\omega) \to \mathbb{R}$, which we define to be $\mu_e(\phi) = \widetilde{\mu}_e(h)$ for $\phi \in \mathrm{Ham}(M,\omega)$, where $h \in \widetilde{\mathrm{Ham}}(M,\omega)$ is a particular (and hence any) lift of ϕ.

Exercise 22.5.18 Finish the proof by proving the Calabi quasimorphism property of $\mu_e : \mathrm{Ham}(M,\omega) \to \mathbb{R}$ defined as above.

This finishes the proof. □

For the later purpose of extending the quasimorphisms to the topological Hamiltonian context in Section 22.6, we now lift the quasimorphism $\widetilde{\mu}_e$ to one on the path space $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega), id)$ instead of the universal covering space $\widetilde{\mathrm{Ham}}(M,\omega)$. In the smooth context, this lifting is automatic thanks to the homotopy invariance of the $\rho(\cdot; e)$ with the following definition.

Definition 22.5.19 We define a homogeneous quasimorphism

$$\mu^{\mathrm{path}} = \mu_1^{\mathrm{path}} : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega), id) \to \mathbb{R}$$

by the same formula

$$\mu^{\mathrm{path}}(\lambda) = \lim_{i\to\infty} \frac{\rho(\lambda^m; 1)}{m}. \tag{22.5.61}$$

For any given open subset $U \subset M$, we denote by

$$\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}_U(M,\omega), id)$$

the set of Hamiltonian paths supported in U as before. We recall that $\mathrm{Cal}_U(F)$ is defined by the integral

$$\mathrm{Cal}_U(F) = \frac{1}{\mathrm{vol}_\omega(M)} \int_0^1 \int_M F(t,x)\,\mu_\omega. \tag{22.5.62}$$

Now we repeat the same construction as above verbatim, replacing $\widetilde{\mathrm{Ham}}_U(M,\omega)$ by $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}_U(M,\omega), id)$. Then all the above properties of $\widetilde{\mu}_e$ hold also for μ_e^{path}.

We end this section with the discussion of the spectral diameter of the 2-sphere S^2.

Example 22.5.20 Consider the spectral function

$$\rho_1 : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$$

defined by $\rho_1(f) = \rho(f;1)$. Obviously it satisfies

$$\rho_1(fg) \le \rho_1(f) + \rho_1(g).$$

We note that S^2 and the quantum cohomology class $1 \in QH^0(S^2)$ satisfy the hypothesis required in Theorem 22.5.11. Therefore

$$\rho_1(fg) \ge \rho_1(f) + \rho_1(g) - R \tag{22.5.63}$$

for some constant R independent of f, g. We now compute this bound R explicitly.

We first remark that the right-hand side of (22.5.46) becomes

$$-\inf_{a \in QH^0(S^2)\setminus\{0\}} \rho(g;a).$$

Furthermore, $QH^0(S^2)$ is factorized into

$$QH^0(S^2) = \mathbb{Q}\mathbf{1} \oplus \mathbb{Q}\{[\omega]q^{-1}\},$$

where $\mathbf{1} = PD[S^2]$ and q is the formal parameter of degree 2 with its valuation $\nu(q) = 4\pi$, and any non-zero element in $QH^0(S^2)$ is invertible.

Now, for any $a \in QH^0(S^2) \setminus \{0\}$ that is invertible,

$$\rho(g;a) \ge \rho(g; a\cdot a^{-1}) - \rho(1;a^{-1}) = \rho(g;1) + \nu(a^{-1})$$

by the normalization axiom of the spectral invariants, which gives rise to $\rho(1;a^{-1}) := -\nu(a^{-1})$. But we have

$$\inf_{b \in QH^0(S^2)\setminus\{0\}} \nu(b^{-1}) = -4\pi. \tag{22.5.64}$$

This implies that $\rho(g;a) \ge \rho(g;1) - 4\pi$ for all $a \in QH^0(S^2) \setminus \{0\}$.

Let f, $g \in \widetilde{\mathrm{Ham}}(M,\omega)$. By the triangle inequality, we have

$$\rho(fg;1) \ge \rho(f;1) - \rho(g^{-1};1).$$

Upon applying Lemma 22.5.5 and (22.5.64), we get

$$\rho(fg;1) \ge \rho(f;1) + \rho(g;1) - 4\pi,$$

which finishes the proof with $R = 4\pi$ by substituting $f = g^{-1}$. Summarizing the above discussion, we obtain the following proposition.

Proposition 22.5.21 *For any Hamiltonian H on S^2 of area 4π,*

$$\rho((\phi_H^1)^{-1}; 1) \leq -\rho(\phi_H; 1) + 4\pi. \tag{22.5.65}$$

One interesting consequence of the above discussion gives rise to the following bound for the spectral diameter of $\mathrm{Ham}(S^2)$.

Theorem 22.5.22 *Denote by* Diam_ρ *the spectral diameter defined by*

$$\mathrm{Diam}_\rho(\mathrm{Ham}(S^2)) := \sup_H\{\gamma(\phi_H) \mid H \in C^\infty([0,1] \times M, \mathbb{R})\}.$$

Then we have $\mathrm{Diam}_\rho(\mathrm{Ham}(S^2)) \leq 4\pi$*, or, equivalently,*

$$\gamma(\phi_H) \leq 4\pi \ (= \mathrm{Area}(S^2)) \tag{22.5.66}$$

for all H.

The above upper bound of the spectral diameter of S^2 is quite a contrast to Polterovich's theorem, Theorem 17.6.3, on the Hofer diameter of S^2 in (Po98b), which states that the diameter of $\mathrm{Ham}(S^2)$ with respect to the Hofer distance is infinite. In fact, a similar bound holds whenever the quantum cohomology $QH^*(M)$ becomes a field, e.g., such as $\mathbb{C}P^n$.

The following is an interesting question to ask.

Problem 22.5.23 Find out the precise criterion for (M, ω) to have a finite (or infinite) Hofer diameter. How about the spectral diameter? What is its implication for Hamiltonian dynamics?

22.6 Back to topological Hamiltonian dynamics

In Section 6, we introduced the notion of topological Hamiltonian flow and its associated Hamiltonian. In this section, we explain how we can extend the various spectral quantities to the topological Hamiltonian paths.

22.6.1 Spectral invariants of topological Hamiltonians

In this section, we extend the definition and basic properties of the spectral invariants of *Hamiltonian paths* to the topological Hamiltonian category.

For this extension, it is crucial to have the definition on the level of Hamiltonian paths, i.e., on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), id)$ as formulated in (Oh05c), not just on the covering space of $\mathrm{Ham}(M, \omega)$. Furthermore, a uniqueness theorem of topological Hamiltonians from Section 6.3 will be crucial for the extension to this C^0 category.

We start by rewriting the spectral number $\rho(H; a)$ in terms of its Hamiltonian path ϕ_H:

$$\rho(\lambda; a) := \rho(\underline{H}; a) = \inf_{\alpha \in \ker \partial_H} \{\lambda_{\underline{H}}(\alpha) \mid [\alpha] = a^\flat\}.$$

For a smooth Hamiltonian H, this rewriting is a trivial matter. To extend this definition to the topological Hamiltonian λ by taking the limit of $\rho(\underline{H}_i; a)$ for a sequence H_i whose Hamiltonian flow ϕ_{H_i} uniformly converges to the given λ, one needs to prove the independence of the limit $\lim_{i\to\infty} \rho(\underline{H}_i; a)$ of the choice of such a sequence H_i.

Theorem 22.6.1 *For a smooth Hamiltonian path ϕ_H with normalized Hamiltonian H, we define*

$$\rho(\phi_H; a) = \rho(\underline{H}; a).$$

Then the map $\rho_a : \phi_H \mapsto \rho(\phi_H; a)$ extends to a continuous function

$$\rho_a = \rho(\cdot; a) : \mathcal{P}^{\text{ham}}(\text{Sympeo}(M, \omega), id) \to \mathbb{R}$$

(in the Hamiltonian topology) and satisfies the triangle inequality

$$\rho(\lambda\mu; a \cdot b) \le \rho(\lambda; a) + \rho(\mu; b). \tag{22.6.67}$$

Proof The first statement is an immediate consequence of the Hamiltonian continuity of

$$\rho_a : \mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), id) \to \mathbb{R}$$

and Theorem 6.3.3: for any topological Hamiltonian path λ, we define

$$\rho(\lambda; a) = \lim_{i\to\infty} \rho(\underline{H}_i; a) \tag{22.6.68}$$

for any sequence H_i with $\phi_{H_i} \to \lambda$ and the H_i converge in $L^{(1,\infty)}$. The uniqueness theorem implies that this definition is well defined. Furthermore, Proposition 21.3.6 implies continuity of the extension to $\mathcal{P}^{\text{ham}}(\text{Sympeo}(M, \omega), id)$.

The triangle inequality immediately follows from the continuity of ρ by taking the limit of the triangle inequality for the smooth case. □

Now we focus on the invariant $\rho(\lambda; 1)$ for $1 \in QH^*(M)$. Recall that the function

$$\gamma(H) = \rho(H; 1) + \rho(\overline{H}; 1)$$

was introduced for a smooth Hamiltonian path $\lambda = \phi_H$ before. In order not to confuse the reader, we denote this by norm_γ when we regard γ as a function on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega), id)$ instead.

Definition 22.6.2 (Spectral pseudonorm) Let $\lambda \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega), id)$ and H be a Hamiltonian such that $\lambda = \phi_H$. Then we define the function

$$\mathrm{norm}_\gamma : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega), id) \to \mathbb{R}_+$$

by setting $\mathrm{norm}_\gamma(\lambda) = \gamma(H)$. We call $\mathrm{norm}_\gamma(\lambda)$ the *spectral pseudonorm* of λ.

This was proven to be non-negative and to depend only on the path-homotopy class of $\lambda = \phi_H$ in $\mathrm{Ham}(M,\omega)$.

Again the uniqueness of topological Hamiltonians enables us to extend the definition to the topological Hamiltonian paths.

Proposition 22.6.3 *The spectral pseudonorm γ extends to a continuous function*

$$\mathrm{norm}_{\overline{\gamma}} : \mathcal{P}^{\mathrm{ham}}(\mathrm{Sympeo}(M,\omega), id) \to \mathbb{R}$$

with the definition

$$\mathrm{norm}_{\overline{\gamma}}(\lambda) = \lim_{i\to\infty} \mathrm{norm}_\gamma(\phi_{H_i}) \tag{22.6.69}$$

for a particular (and hence any) sequence $\phi_{H_i} \to \lambda$ in $L^{(1,\infty)}$-Hamiltonian topology.

Recall that for a smooth Hamiltonian H each $\rho(\phi_H; a) = \rho(\underline{H}; a)$ is associated with a periodic orbit of Hamilton's equation $\dot{x} = X_H(t,x)$ and corresponds to the action of the periodic orbit, at least for *arbitrary* Hamiltonians on the rational symplectic manifold (see (Oh05c, Oh09a) for the proof) and for the *nondegenerate* Hamiltonians on general irrational symplectic manifolds (see (Oh09a), (Ush08)). In this regard, the following question seems to be of fundamental importance.

Question 22.6.4 What is the meaning of the extended spectral pseudonorm $\mathrm{norm}_{\overline{\gamma}}(\lambda)$ in regard to the dynamics of topological Hamiltonian flows?

The following is the analog of the definition from (Oh05d) of the spectral displacement energy in the topological Hamiltonian category.

Definition 22.6.5 (Spectral displacement energy) Let $A \subset M$ be a compact subset. We define the spectral displacement energy, denoted by $e_{\overline{\gamma}}(A)$, of A by

$$e_{\overline{\gamma}}(A) = \inf_{\lambda}\{\mathrm{norm}_{\overline{\gamma}}(\lambda) \mid A \cap \lambda(1)(A) = \emptyset,\ \lambda \in \mathcal{P}^{\mathrm{ham}}_{\infty}(\mathrm{Sympeo}(M,\omega), id)\}.$$

By unravelling the definitions of Hamiltonian homeomorphisms and of the spectral displacement energy, we also have the following theorem.

Theorem 22.6.6 *We have $e_{\overline{\gamma}}(A) = e_{\gamma}(A)$ for any compact subset $A \subset M$.*

Again on the basis of this theorem, we just denote the spectral displacement energy of A even in the topological Hamiltonian category by $e_{\gamma}(A)$. Then we have the following theorem.

Theorem 22.6.7 *For every $\psi \in \mathrm{Sympeo}(M,\omega)$ we have*

$$e_{\gamma}(A) = e_{\gamma}(\psi(A)).$$

Proof We note that $h(A) \cap A = \emptyset$ if and only if $\psi h \psi^{-1}(\psi(A)) \cap \psi(A) = \emptyset$. Furthermore, $h \in \mathrm{Hameo}(M,\omega)$ if and only if $\psi h \psi^{-1} \in \mathrm{Hameo}(M,\omega)$. This finishes the proof. □

Next we consider the spectral norm

$$\gamma(\phi) := \inf_{ev_1(\lambda)=\phi} \mathrm{norm}_{\gamma}(\lambda) = \inf_{H \mapsto \phi}\{\rho(H;1) + \rho(\overline{H};1)\}.$$

The following definition extends the spectral norm to $\mathrm{Hameo}(M,\omega)$.

Definition 22.6.8 (Spectral norm) Let $h \in \mathrm{Hameo}(M,\omega)$ and consider topological Hamiltonian paths $\lambda \in \mathcal{P}^{\mathrm{ham}}_{\infty}(\mathrm{Sympeo}(M,\omega), id)$ with $\overline{\mathrm{ev}}_1(0) = h$. We denote by $\lambda \mapsto h$ if $\overline{\mathrm{ev}}_1(\lambda) = h$. We define $\overline{\gamma}$ by

$$\overline{\gamma}(h) = \inf_{\lambda}\{\mathrm{norm}_{\overline{\gamma}}(\lambda) \mid \lambda \in \mathcal{P}^{\mathrm{ham}}_{\infty}(\mathrm{Sympeo}(M,\omega), id),\ \overline{\mathrm{ev}}_1(\lambda) = h\}. \quad (22.6.70)$$

The following establishes the analogs to all the properties of an invariant norm in this topological Hamiltonian context.

Theorem 22.6.9 *The generalized spectral function $\overline{\gamma} : \mathrm{Hameo}(M,\omega) \to \mathbb{R}_+$ satisfies all the properties of an invariant norm.*

The proof will be essentially the same as that of the Hofer norm once the following continuity lemma for the smooth case has been proved.

Lemma 22.6.10 *The function $\gamma : \mathrm{Ham}(M,\omega) \to \mathbb{R}_+$ is continuous in the Hofer topology and so also in Hamiltonian topology of* $\mathrm{Ham}(M,\omega)$.

Proof Let $H \mapsto \phi$ and $K \mapsto \psi$. Then the triangle inequality of γ and the inequality $\gamma(\phi) \leq \|\phi\|$ imply that

$$|\gamma(\phi) - \gamma(\psi)| \leq \gamma(\phi^{-1}\psi) \leq \|\phi^{-1}\psi\| = d_{\mathrm{Hofer}}(\phi, \psi)$$

This finishes the proof. □

The following foundational questions are interesting questions to ask.

Question 22.6.11

(1) Is γ (or $\overline{\gamma}$) continuous in the C^0-topology?
(2) Does the identity

$$\overline{\gamma}|_{\mathrm{Ham}(M,\omega)} = \gamma \tag{22.6.71}$$

hold?

Recently, Seyfaddini (Sey13a) answered the first question above affirmatively on two-dimensional surfaces by combining various elements in C^0 symplectic topology and Hamiltonian dynamics. This in turn gives rise to an interesting corollary in two-dimensional area-preserving dynamical systems (Sey13b).

22.6.2 Calabi quasimorphism on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Sympeo}(S^2), id)$

In the rest of this section, we will restrict the discussion to the case of the sphere S^2 with the standard symplectic form ω_{S^2} on it. Omitting the symplectic form ω_{S^2} from their notation, we just denote by $\mathcal{P}^{\mathrm{ham}}(\mathrm{Sympeo}(S^2), id)$ and $\mathrm{Hameo}(S^2)$ the groups of topological Hamiltonian paths and of Hamiltonian homeomorphisms on S^2, respectively, and so on.

From the definition above and the Hamiltonian-continuity of $\rho(\cdot; 1)$, it follows that μ^{path} is also Hamiltonian-continuous. An immediate consequence of the Calabi property of μ^{path} is the following homomorphism property of μ restricted to $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}_U(S^2), id)$.

Corollary 22.6.12 *Suppose that U is an open subset of S^2 such that $\overline{U}$ is displaceable on S^2 and let*

$$\lambda_1,\ \lambda_2 \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}_U(S^2), id).$$

Then we have

$$\mu^{\mathrm{path}}(\lambda_1\lambda_2) = \mu^{\mathrm{path}}(\lambda_1) + \mu^{\mathrm{path}}(\lambda_2).$$

We can extend all of the above discussions to the level of *topological* Hamiltonian paths. These generalizations immediately follow once we know the facts that

(1) $\rho(\cdot; 1)$ has been extended to $\mathcal{P}^{\mathrm{ham}}(\mathrm{Sympeo}(M, \omega), id)$ for an arbitrary closed symplectic manifold, i.e., in particular for (S^2, ω_{S^2}) in Section 12.6; and
(2) in addition, this extension is Hamiltonian-continuous, i.e., continuous in the Hamiltonian topology.

We summarize the above discussion in the following theorem.

Theorem 22.6.13 *We have an extension of the Calabi quasimorphism* $\mu^{\mathrm{path}} : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(S^2), id) \to \mathbb{R}$ *to a quasimorphism*

$$\overline{\mu}^{\mathrm{path}} : \mathcal{P}^{\mathrm{ham}}(\mathrm{Sympeo}(S^2), id) \to \mathbb{R}$$

that satisfies all the analogs to Proposition 22.5.17 and the Calabi property.

Now we state the following conjecture, which we believe will play an important role in the study of the simpleness question of the area-preserving group of S^2. Recall from (EnP03) that the corresponding fact was proved by Entov and Polterovich for the group $\mathrm{Ham}(S^2)$ of *smooth* Hamiltonian diffeomorphisms on S^2.

Conjecture 22.6.14 *Let* $\overline{\mu}^{\mathrm{path}} : \mathcal{P}^{\mathrm{ham}}(\mathrm{Sympeo}(S^2), id) \to \mathbb{R}$ *be the above extension of the homogeneous Calabi quasimorphism given in (22.5.61). This pushes down to a homogeneous quasimorphism* $\overline{\mu} : \mathrm{Hameo}(S^2) \to \mathbb{R}$ *that satisfies*

$$\overline{\mu}^{\mathrm{path}} = \overline{\mu} \circ \overline{\mathrm{ev}}_1. \tag{22.6.72}$$

In particular, $\overline{\mu}^{\mathrm{path}}(\lambda)$ *depends only on the time-one map* $\lambda(1)$ *of* λ *as long as* λ *lies in* $\mathcal{P}^{\mathrm{ham}}(\mathrm{Sympeo}(S^2), id)$*. Furthermore, it satisfies*

$$\overline{\mu}(\phi) = \overline{\mathrm{Cal}}_{D^+}(\phi) \tag{22.6.73}$$

for any $\phi \in \mathrm{Hameo}(S^2)$ *as long as* $\lambda(1) = \phi$ *for a topological Hamiltonian path* λ *supported in* $\mathrm{Int}(D^+)$*.*

An immediate corollary of Conjecture 22.6.14 and of the Calabi property of $\overline{\mu}^{\mathrm{path}}$ would be the solution to the following conjecture.

Conjecture 22.6.15 *The Calabi homomorphism* $\mathrm{Cal} : \mathrm{Ham}(D^2, \partial D^2) \to \mathbb{R}$ *is extended to a homomorphism*

$$\overline{\mathrm{Cal}} : \mathrm{Hameo}(D^2, \partial D^2) \to \mathbb{R}$$

that is continuous in Hamiltonian topology.

22.7 Wild area-preserving homeomorphisms on D^2

In this section, we will describe an example of a compactly supported area-preserving homeomorphism in $\mathrm{Sympeo}(D^2, \partial D^2)$ that would not be contained in $\mathrm{Hameo}(D^2, \partial D^2)$, *if Conjecture 22.6.15 were to hold*. Then this would imply that $\mathrm{Hameo}(D^2, \partial D^2)$ is a proper normal subgroup of $\mathrm{Sympeo}(D^2, \partial D^2)$, by Theorem 6.4.1. Combining the above chain of statements would give rise to the non-simpleness of $\mathrm{Homeo}^{\Omega}(D^2, \partial D^2)$, via the following smoothing theorem from (Oh06b), (Sik07).

Theorem 22.7.1 *We have*

$$\mathrm{Sympeo}(D^2, \partial D^2) = \mathrm{Homeo}^{\Omega}(D^2, \partial D^2)$$

for the standard area form Ω on D^2, regarding it also as the symplectic form $\omega = \Omega$.

This being said, we will focus on the construction of an example of a wild area-preserving homeomorphism on D^2. For this description, we will need to consider the conjugate action of rescaling maps of D^2

$$R_a : D^2(1) \to D^2(a) \subset D^2(1)$$

for $0 < a < 1$ on $\mathrm{Hameo}(D^2, \partial D^2)$, where $D^2(a)$ is the disc of radius a with its center at the origin. We note that R_a is a conformally symplectic map and hence its conjugate action maps a symplectic map to a symplectic map whenever it is defined.

Furthermore, the right composition by R_a defines a map

$$\phi \mapsto \phi \circ R_a\,;\ \mathrm{Hameo}(D^2, \partial D^2) \to \mathrm{Hameo}(D^2(a), \partial D^2(a)) \subset \mathrm{Hameo}(D^2, \partial D^2)$$

and the composition by R_a^{-1} defines a map

$$\mathrm{Hameo}(D^2(a), \partial D^2(a)) \to \mathrm{Hameo}(D^2, \partial D^2).$$

We have the following important formula for the behavior of Calabi invariants under the Alexander isotopy.

Lemma 22.7.2 *Let λ be a given topological Hamiltonian path on D^2 and λ_a be the map defined by*

$$\lambda_a(t,x) = \begin{cases} a\lambda(t,x/a) & \textit{for } |x| \le a(1-\eta), \\ x & \textit{otherwise} \end{cases}$$

for $0 < a \le 1$. Then λ_a is also a topological Hamiltonian path on D^2 and satisfies

$$\overline{\mathrm{Cal}}^{\mathrm{path}}(\lambda_a) = a^4 \overline{\mathrm{Cal}}^{\mathrm{path}}(\lambda). \tag{22.7.74}$$

Proof A straightforward calculation proves that λ_a is generated by the (unique) topological Hamiltonian defined by

$$\mathrm{Dev}(\lambda_a)(t,x) = \begin{cases} a^2 H\,(t,x/a) & \text{for } |x| \le a(1-\eta), \\ 0 & \text{otherwise,} \end{cases}$$

where $H = \mathrm{Dev}(\lambda)$. Obviously the right-hand-side function is the Hamiltonian limit of $\mathrm{Dev}(\lambda_{i,a})$ for a sequence λ_i of smooth Hamiltonian approximations of λ, where $\lambda_{i,a}$ is defined by the same formula as that for λ_i.

From this, we derive the formula

$$\begin{aligned} \overline{\mathrm{Cal}}^{\mathrm{path}}(\lambda_a) &= \int_0^1 \int_{D^2(a(1-\eta))} a^2 H\left(t, \frac{x}{a}\right) \Omega \wedge dt \\ &= a^4 \int_0^1 \int_{D^2} H(t,y)\Omega \wedge\ dt = a^4 \overline{\mathrm{Cal}}^{\mathrm{path}}(\lambda). \end{aligned}$$

This proves (22.7.74). □

Here there follows the construction of an example of a wild area-preserving homeomorphism (Oh10).

Example 22.7.3 With the above preparations, we consider the set of dyadic numbers $1/2^k$ for $k = 0, \ldots$. Let (r,θ) be polar coordinates on D^2. Then the standard area form is given by

$$\omega = r\,dr \wedge d\theta.$$

Consider maps $\phi_k : D^2 \to D^2$ of the form given by

$$\phi_k = \phi_{\rho_k} : (r,\theta) \to (r, \theta + \rho_k(r)),$$

where $\rho_k : (0,1] \to [0,\infty)$ is a smooth function supported in $(0,1)$. It follows that ϕ_{ρ_k} is an area-preserving map generated by an autonomous Hamiltonian given by

$$F_{\phi_k}(r,\theta) = -\int_1^r s\rho_k(s)ds.$$

Therefore its Calabi invariant becomes

$$\mathrm{Cal}(\phi_k) = -\int_{D^2}\left(\int_1^r s\rho_k(s)\,ds\right) r\,dr\,d\theta = \pi\int_0^1 r^3\rho_k(r)dt. \qquad (22.7.75)$$

We now choose ρ_k in the following way.

(1) ρ_k has support in $1/2^k < r < 1/2^{k-1}$.
(2) For each $k = 1,\ldots,$ we have

$$\rho_k(r) = 2^4\rho_{k-1}(2r) \qquad (22.7.76)$$

for $r \in (1/2^k, 1/2^{k-1})$.
(3) $\mathrm{Cal}(\phi_1) = 1$.

Since the ϕ_k have disjoint supports by construction, we can freely compose without concerning ourselves about the order of composition. It follows that the infinite product

$$\Pi_{k=0}^{\infty}\phi_k$$

is well defined and defines a continuous map that is smooth except at the origin, at which ϕ_ρ is continuous but not differentiable. This infinite product can also be written as the homeomorphism, having its values given by $\phi_\rho(0) = 0$ and

$$\phi_\rho(r,\theta) = (r, \theta + \rho(r)),$$

where the smooth function $\rho : (0,1] \to \mathbb{R}$ is defined by

$$\rho(r) = \rho_k(r) \quad \text{for } [1/2^k, 1/2^{k-1}],\ k = 1,\ 2, \ldots.$$

It is easy to check that ϕ_ρ is smooth on $D^2 \setminus \{0\}$ and is a continuous map, even at 0, which coincides with the above infinite product. Obviously the map $\phi_{-\rho}$ is the inverse of ϕ_ρ, which shows that it is a homeomorphism. Furthermore, we have

$$\phi_\rho^*(r\,dr \wedge d\theta) = r\,dr \wedge d\theta \quad \text{on } D^2 \setminus \{0\},$$

which implies that ϕ_ρ is indeed area-preserving.

The following lemma will play an important role in our proof of Theorem 22.7.6.

Lemma 22.7.4 *Let ϕ_k be the diffeomorphisms given in Example 22.7.3. We have the identity*

$$R_{\frac{1}{2}} \circ \phi_{k-1}^{2^4} \circ R_{\frac{1}{2}}^{-1} = \phi_k. \tag{22.7.77}$$

In particular, we have

$$\mathrm{Cal}(\phi_k) = \mathrm{Cal}(\phi_{k-1}). \tag{22.7.78}$$

Proof Using (22.7.76), we compute

$$R_{\frac{1}{2}} \circ \phi_{k-1} \circ R_{\frac{1}{2}}^{-1}(r,\theta) = (r, \theta + \rho_{k-1}(2r)) = \left(r, \theta + \frac{1}{2^4}\rho_k(r)\right),$$

where the second identity follows from (22.7.76). By iterating this identity 2^4 times, we obtain (22.7.77) from (22.7.76). The equality (22.7.78) follows from this and (22.7.74). □

An immediate corollary of this lemma and (22.7.76) is the following.

Corollary 22.7.5 *We have*

$$\mathrm{Cal}(\phi_k) = 1$$

for all $k = 1, \ldots$.

Now we are ready to give the proof of the following theorem.

Theorem 22.7.6 *Validity of Conjecture 22.6.15 implies that ϕ_ρ cannot be contained in* $\mathrm{Hameo}(D^2, \partial D^2)$.

Proof Suppose to the contrary that $\phi_\rho \in \mathrm{Hameo}(D^2, \partial D^2)$, i.e., there exists a path $\lambda \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Sympeo}(D^2, \partial D^2), id)$ with $\lambda(1) = \phi_\rho$.

Then its Calabi invariant has a finite value which we denote

$$\overline{\mathrm{Cal}}(\phi_\rho) = \overline{\mathrm{Cal}}^{\mathrm{path}}(\lambda) = C_1 \tag{22.7.79}$$

for some $C_1 \in \mathbb{R}$.

Writing $\phi_\rho = \psi_N \widetilde{\psi}_N$, where

$$\begin{aligned} \psi_N &= \Pi_{i=1}^N \phi_i, \\ \widetilde{\psi}_N &= \Pi_{i=N+1}^\infty \phi_i, \end{aligned}$$

for $N = 1$, we derive

$$C_1 = \overline{\mathrm{Cal}}(\psi_1) + \overline{\mathrm{Cal}}(\widetilde{\psi}_1) \tag{22.7.80}$$

from the homomorphism property of $\overline{\mathrm{Cal}}$. Here we note that ψ_N is smooth and thus obviously lies in $\mathrm{Hameo}(D^2, \partial D^2)$. Therefore it follows from the group property of $\mathrm{Hameo}(D^2, \partial D^2)$ that $\widetilde{\psi}_N$ lies in $\mathrm{Hameo}(D^2, \partial D^2)$ if ϕ_ρ does so. We also derive

$$\overline{\mathrm{Cal}}(\psi_1) = \overline{\mathrm{Cal}}(\phi_1) = 1$$

from Corollary 22.7.5, and hence

$$\overline{\mathrm{Cal}}(\widetilde{\psi}_1) = C_1 - 1. \tag{22.7.81}$$

On the other hand, applying (22.7.77) iteratively to the infinite product

$$\widetilde{\psi}_1 = \prod_{i=2}^{\infty} \phi_i,$$

we show that $\widetilde{\psi}_1$ satisfies the identity

$$\widetilde{\psi}_1(r,\theta) = \begin{cases} R_{\frac{1}{2}} \circ \phi_\rho^{2^4} \circ R_{\frac{1}{2}}^{-1}(r,\theta) & \text{for } 0 < r \le \frac{1}{2}, \\ (r,\theta) & \text{for } \frac{1}{2} \le r \le 1. \end{cases} \tag{22.7.82}$$

In particular, we have

$$\widetilde{\psi}_1 = \mathrm{ev}_1((\lambda^{2^4})_{1/2}). \tag{22.7.83}$$

This property, Lemma 22.7.2 applied for $a = 1/2$ and the homomorphism property of $\overline{\mathrm{Cal}}^{\mathrm{path}}$ give rise to

$$\begin{aligned} \overline{\mathrm{Cal}}(\widetilde{\psi}_1) &= \overline{\mathrm{Cal}}^{\mathrm{path}}((\lambda^{2^4})_{1/2}) = \left(\frac{1}{2}\right)^4 \overline{\mathrm{Cal}}^{\mathrm{path}}(\lambda^{2^4}) \\ &= \left(\frac{1}{2}\right)^4 2^4\, \overline{\mathrm{Cal}}^{\mathrm{path}}(\lambda) = \overline{\mathrm{Cal}}^{\mathrm{path}}(\lambda) = C_1. \end{aligned} \tag{22.7.84}$$

It is manifest that (22.7.81) and (22.7.84) contradict each other. This finishes the proof.

□

Appendix A

The Weitzenböck formula for vector-valued forms

In this appendix, we provide a derivation of the standard Weitzenböck formulae for the general E-valued forms on a Riemannian manifold (P, h), and applied to our current circumstance where

$$(P, h) = \left(\sum, h\right), \quad E = u^*TM$$

for a map $u : \sum \to M$. An elegant exposition on the general Weitzenböck formula directly working on the bundle of frames is given in the appendix of (FU84). The exposition here is borrowed from that of (OhW12).

Assume that (P, h) is a Riemannian manifold of dimension n with metric h, and D is the Levi-Civita connection. Let $E \to P$ be any vector bundle with inner product $\langle \cdot, \cdot \rangle$, and assume that ∇ is a connection of E that is compatible with $\langle \cdot, \cdot \rangle$.

For any vector bundle E-valued form s, we calculate the (Hodge) Laplacian of the energy density of s and get

$$-\frac{1}{2}\Delta|s|^2 = |\nabla s|^2 + \langle \mathrm{Tr}\ \nabla^2 s, s \rangle,$$

where by $|\nabla s|$ we mean the induced norm in the vector bundle $T^*P \otimes E$, i.e., $|\nabla s|^2 = \sum_i |\nabla_{e_i} s|^2$ with $\{e_i\}$ an orthonormal frame of TP. $\mathrm{Tr}\,\nabla^2$ denotes the connection Laplacian, which is defined as $\mathrm{Tr}\,\nabla^2 = \sum_i \nabla^2_{e_i,e_i} s$, where $\nabla^2_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$.

Denote the space of E-valued k-forms on P by $\Omega^k(M; E) = \Gamma(\lambda^k(M) \otimes E)$. The connection ∇ induces an exterior derivative by $d^\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$, which is characterized by the equations

$$d^\nabla \zeta = \nabla \zeta,$$
$$d^\nabla(\alpha \otimes \zeta) = d\alpha \otimes \zeta + (-1)^k \alpha \wedge \nabla \zeta$$

for $\alpha \in \Omega^*(M)$, $\zeta \in \Gamma(E)$.

For example, for a one-form β, we have

$$d^\nabla \beta(v_1, v_2) = (\nabla_{v_1}\beta)(v_2) - (\nabla_{v_2}\beta)(v_1),$$

with $v_1, v_2 \in TP$.

We extend the Hodge star operator to E-valued forms by

$$\begin{aligned} * \, : \, \Omega^k(E) &\to \Omega^{n-k}(E), \\ *\beta = *(\alpha \otimes \zeta) &= (*\alpha) \otimes \zeta \end{aligned}$$

for $\beta = \alpha \otimes \zeta \in \Omega^k(E)$.

Define the Hodge Laplacian of the connection ∇ by

$$\Delta^\nabla := d^\nabla \delta^\nabla + \delta^\nabla d^\nabla,$$

where δ^∇ is defined by

$$\delta^\nabla := (-1)^{nk+n+1} * d^\nabla * .$$

The following lemma is important for the derivation of the Weitzenböck formula.

Lemma A.1 *Assume that $\{e_i\}$ is an orthonormal frame of P and $\{\alpha^i\}$ is the dual frame. Then we have*

$$\begin{aligned} d^\nabla &= \sum_i \alpha^i \wedge \nabla_{e_i}, \\ \delta^\nabla &= -\sum_i e_i \rfloor \nabla_{e_i}. \end{aligned}$$

Proof Assume that $\beta = \alpha \otimes \zeta \in \Omega^k(E)$, then

$$\begin{aligned} d^\nabla(\alpha \otimes \zeta) &= (d\alpha) \otimes \zeta + (-1)^k \alpha \wedge \nabla\zeta, \\ &= \sum_i \alpha^i \wedge \nabla_{e_i}\alpha \otimes \zeta + (-1)^k \alpha \wedge \nabla\zeta. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_i \alpha^i \wedge \nabla_{e_i}(\alpha \otimes \zeta) &= \sum_i \alpha^i \wedge \nabla_{e_i}\alpha \otimes \zeta + \alpha^i \wedge \alpha \otimes \nabla_{e_i}\zeta \\ &= \sum_i \alpha^i \wedge \nabla_{e_i}\alpha \otimes \zeta + (-1)^k \alpha \wedge \nabla\zeta, \end{aligned}$$

so we have proved the first equality.

For the second equality, we compute

$$\begin{aligned}
\delta^\nabla(\alpha\otimes\zeta) &= (-1)^{nk+n+1} * d^\nabla * (\alpha\otimes\zeta)\\
&= (\delta\alpha)\otimes\zeta + (-1)^{nk+n+1} * (-1)^{n-k}(*\alpha)\wedge\nabla\zeta\\
&= -\sum_i e_i\rfloor\nabla_{e_i}\alpha\otimes\zeta + \Sigma_i(-1)^{nk-k+1} * ((*\alpha)\wedge\alpha^i)\otimes\nabla_{e_i}\zeta\\
&= -\sum_i e_i\rfloor\nabla_{e_i}\alpha\otimes\zeta - \Sigma_i e_i\rfloor\alpha\otimes\nabla_{e_i}\zeta\\
&= -\sum_i e_i\rfloor\nabla_{e_i}(\alpha\otimes\zeta)
\end{aligned}$$

and then we are done with this lemma. □

Theorem A.2 (Weitzenböck formula) *Assume that $\{e_i\}$ is an orthonormal frame of P and $\{\alpha^i\}$ is the dual frame. Then for any vector bundle E-valued forms*

$$\Delta^\nabla = -\mathrm{Tr}\ \nabla^2 + \sum_{i,j}\alpha^j\wedge(e_i\rfloor R(e_i,e_j)),$$

where R is the curvature tensor of the bundle E with respect to the connection ∇.

Proof Since the right-hand side of the equality is independent of the choice of orthonormal basis, and it is a pointwise formula, we can take the normal coordinates $\{e_i\}$ at a point $p\in P$ (and $\{\alpha^i\}$ the dual basis), i.e., $h_{ij} := h(e_i,e_j)(p) = \delta_{ij}$ and $dh_{i,j}(p) = 0$, and prove that such formula holds at p for such coordinates. For the Levi-Civita connection, the condition $dh_{i,j}(p) = 0$ of the normal coordinate is equivalent to letting $\Gamma^k_{i,j}(p) := \alpha^k(D_{e_i}e_j)(p) = 0$.

For $\beta\in\Omega^k(E)$, using Lemma A.1 we calculate

$$\begin{aligned}
\delta^\nabla d^\nabla\beta &= -\sum_{i,j} e_i\rfloor\nabla_{e_i}(\alpha^j\wedge\nabla_{e_j}\beta)\\
&= -\sum_{i,j} e_i\rfloor(D_{e_i}\alpha^j\wedge\nabla_{e_j}\beta + \alpha^j\wedge\nabla_{e_i}\nabla_{e_j}\beta).
\end{aligned}$$

At the point p, the first term vanishes, and we get

$$\begin{aligned}
\delta^\nabla d^\nabla\beta(p) &= -\sum_{i,j} e_i\rfloor(\alpha^j\wedge\nabla_{e_i}\nabla_{e_j}\beta)(p)\\
&= -\sum_i \nabla_{e_i}\nabla_{e_i}\beta(p) + \Sigma_{i,j}\alpha^j\wedge(e_i\rfloor\nabla_{e_i}\nabla_{e_j}\beta)(p)\\
&= -\sum_i \nabla^2_{e_i,e_i}\beta(p) + \Sigma_{i,j}\alpha^j\wedge(e_i\rfloor\nabla_{e_i}\nabla_{e_j}\beta)(p).
\end{aligned}$$

Also,

$$
\begin{aligned}
d^\nabla \delta^\nabla \beta &= -\sum_{i,j} \alpha^i \wedge \nabla_{e_i}(e_j \rfloor \nabla_{e_j}\beta) \\
&= -\sum_{i,j} \alpha^i \wedge (e_j \rfloor \nabla_{e_i}\nabla_{e_j}\beta) - \sum_{i,j} \alpha^i \wedge ((D_{e_i}e_j)\rfloor \nabla_{e_j}\beta),
\end{aligned}
$$

for which, at the point p, the second term vanishes.

Now we sum the two parts $d^\nabla\delta^\nabla$ and $\delta^\nabla d^\nabla$ and get

$$
\Delta^\nabla \beta(p) = -\sum_i \nabla^2_{e_i,e_i}\beta(p) + \sum_{i,j} \alpha^j \wedge (e_i \rfloor R(e_i, e_j)\beta)(p). \qquad \square
$$

In particular, when acting on zero forms, i.e., sections of E, the second term on the right-hand side vanishes, and so the formula becomes

$$
\Delta^\nabla = -\mathrm{Tr}\ \nabla^2.
$$

When acting on full-rank forms, the above formula can be easily checked.

When $\beta \in \Omega^1(E)$, which is the case we use here, the following corollary applies.

Corollary A.3 *For $\beta = \alpha \otimes \zeta \in \Omega^1(E)$, the Weitzenböck formula can be written as*

$$
\Delta^\nabla \beta = -\sum_i \nabla^2_{e_i,e_i}\beta + \mathrm{Ric}^{D*}(\alpha)\otimes\zeta + \mathrm{Ric}^\nabla \beta,
$$

where Ric^{D*} *denotes the adjoint of* Ric^D*, which acts on* 1*-forms.*

In particular, when $P = \Sigma$ is a surface, we have

$$
\begin{aligned}
\Delta^\nabla \beta &= -\sum_i \nabla^2_{e_i,e_i}\beta + K\cdot\beta + \mathrm{Ric}^\nabla(\beta), \\
-\frac{1}{2}\,\Delta|\beta|^2 &= |\nabla\beta|^2 - \langle \Delta^\nabla\beta, \beta\rangle + K\cdot|\beta|^2 + \langle \mathrm{Ric}^\nabla(\beta), \beta\rangle,
\end{aligned}
\tag{A.1}
$$

where K is the Gaussian curvature of the surface $\sum$ and $\mathrm{Ric}^\nabla(\beta) := \alpha \otimes \sum_{i,j} R(e_i, e_j)\zeta$.

Appendix B

The three-interval method of exponential estimates

The main purpose of this appendix is to give a proof of Lemma 9.5.29. This is the prototype of the general exponential behavior that often appears in geometric PDE problems of the minimal-surface type. An elegant approach to obtain such an estimate was used by Mundet i Riera and Tian (MT09) for the pseudoholomorphic maps, which is called the three-interval method. We partially follow the exposition given in (OhZ11a) in a different context. The same kind of argument can be applied to many other contexts that appear in other geometric problems. Here we reproduce their argument in the current context where the diameter of pseudoholomorphic maps on a long cylinder is small.

We first restate Lemma 9.5.29 here.

Lemma B.1 *There exists $L_0 > 0$ and $\epsilon_1 > 0$ such that, for any $L \geq L_0$, if $u : [-L-1, L+1] \times S^1 \to (M, J)$ is J-holomorphic and* $\mathrm{diam}(u) \leq \epsilon_1$, *then*

$$\left|\frac{\partial u}{\partial \tau}(\tau, t)\right| + \left|\frac{\partial u}{\partial t}(\tau, t)\right| \leq C e^{-\lambda\, \mathrm{dist}(\tau, \partial[-L-1, L+1])}$$

for $\tau \in [-L, L] \times S^1$, where $\lambda > 0, C > 0$ is independent of $L \geq L_0$. In fact, we can choose $\lambda = 2\pi\upsilon > 0$ with $\upsilon = \upsilon(\epsilon_1)$ as close to 1 as possible by letting ϵ_1 become sufficiently small.

Let $\iota_g > 0$ be the injectivity radius of the given compatible metric g on M. Let $\epsilon > 0$, assume that $u([-L-1, L+1] \times S^1) \subset B_p(\epsilon)$ at some point $p \in M$ and consider the equation

$$\frac{\partial u}{\partial \tau} + J\frac{\partial u}{\partial t} = 0.$$

If we identify the neighborhood $B_p(2\epsilon)$ with the ball of radius 2ϵ centered at the origin in (T_pM, J_p) by the exponential map, we may assume that $B_p(2\epsilon)$ is a subset of the linear space T_pM. We identify (T_pM, J_p) with $\mathbb{C}^n$ equipped with the standard metric on $\mathbb{C}^n$. The J-holomorphic map u in M will be transformed into an $\exp_p^* J$-holomorphic map, which we again denote by $J = J(x)$ for $x \in \mathbb{C}^n \cong T_pM$.

In this setting the observation made in (MT09) is that an exponential estimate can be reduced to a local L^2 estimate

$$\|du\|_{L^2(Z_{II})} \leq \frac{1}{2}\left(\|du\|_{L^2(Z_I)} + \|du\|_{L^2(Z_{III})}\right) \tag{B.1}$$

on three sequential cylinders $Z_I, Z_{II}, Z_{III} \subset [-L-1, L+1] \times S^1$ of unit length, namely the cylinders

$$[i-1, i] \times S^1, [i, i+1] \times S^1, [i+1, i+2] \times S^1$$

for some integer i.

To get the best σ in the exponential decay (we need σ to be very close to 2π), we recall that the authors of (MT09) defined the constant

$$\gamma(c) = \frac{1}{e^c + e^{-c}}.$$

The importance of $\gamma(c)$ lies in the identity

$$\int_0^1 e^{c\tau}\, d\tau = \gamma(c)\left[\int_{-1}^0 e^{c\tau}\, d\tau + \int_1^2 e^{c\tau}\, d\tau\right], \tag{B.2}$$

which will appear in the L^2-energy of du on three sequential unit-length cylinders later. Notice that, when $c > 0$, $\gamma(c)$ is a strictly decreasing function of c.

We recall the following elementary lemma.

Lemma B.2 ((MT09) Lemma 9.4) *For non-negative numbers x_k ($k = 0, 1, \ldots, N$), if*

$$x_k \leq \gamma(x_{k-1} + x_{k+1})$$

holds for some fixed constant $\gamma \in (0, 1/2)$ for all $1 \leq k \leq N-1$, then

$$x_k \leq x_0 \xi^{-k} + x_N \xi^{-(N-k)},$$

for all $1 \leq k \leq N-1$, where $\xi = (1 + \sqrt{1 - 4\gamma^2})/(2\gamma)$.

Remark B.3 If $\gamma = \gamma(c) = (e^c + e^{-c})^{-1}$, then we can check $\xi = e^c$ and the above inequality becomes the exponential decay estimate

$$x_k \leq x_0 e^{-ck} + x_N e^{-c(N-k)} \tag{B.3}$$

for $1 \leq k \leq N-1$.

Proposition B.4 *For any* $0 < \upsilon < 1$, *there exists* $L_0 > 0$, *and* $\epsilon_1 = \epsilon_1(\upsilon) > 0$ *depending on* υ *but independent of* $L \geq L_0$, *such that, for all cases in which* $u : [-L-1, L+1] \times S^1 \to (M, J)$ *is* J*-holomorphic and* $\mathrm{diam}(u) \leq \epsilon$ *for* $0 < \epsilon < \epsilon_1$ *and* $p \in M$, *we have*

$$\|du\|_{L^2(Z_{II})} \leq \gamma(4\pi\upsilon) \cdot \left(\|du\|_{L^2(Z_I)} + \|du\|_{L^2(Z_{III})}\right) \tag{B.4}$$

on any three sequential cylinders Z_I, Z_{II} *and* Z_{III} *in* $[-L, L] \times S^1$.

Proof We prove (B.4) by contradiction. Suppose that there exist some sequences $L_j \to \infty$ and u_j with $\mathrm{diam}\left(u_j([-L_j, L_j] \times S^1)\right) \to 0$ but (B.4) is violated on three sequential cylinders Z_I^j, Z_{II}^j and Z_{III}^j in $\left[-L_j, L_j\right] \times S^1$:

$$\left\|du_j\right\|_{L^2\left(Z_{II}^j\right)} > \gamma(4\pi\upsilon)\left(\left\|du_j\right\|_{L^2\left(Z_I^j\right)} + \left\|du_j\right\|_{L^2\left(Z_{III}^j\right)}\right).$$

On $Z_I^j \cup Z_{II}^j \cup Z_{III}^j$ consider the rescaled sequence

$$\widehat{u}_j = u_j / \left\|u_j\right\|_{L^\infty\left(Z_I^j \cup Z_{II}^j \cup Z_{III}^j\right)},$$

where the denominator $\|u\|_{L^\infty\left(Z_I^j \cup Z_{II}^j \cup Z_{III}^j\right)}$ is never 0, otherwise $u_j \equiv 0$ on $Z_I^j \cup Z_{II}^j \cup Z_{III}^j$, contradicting the assumption of u_j being non-constant. Then

$$\begin{aligned}
&\left\|\widehat{u}_j\right\|_{L^\infty\left(Z_I^j \cup Z_{II}^j \cup Z_{III}^j\right)} = 1,\\
&\qquad\left\|d\widehat{u}\right\|_{L^2\left(Z_{II}^j\right)} > \gamma(4\pi\upsilon) \cdot \left(\left\|d\widehat{u}_j\right\|_{L^2\left(Z_I^j\right)} + \left\|d\widehat{u}_j\right\|_{L^2\left(Z_{III}^j\right)}\right),\\
&\left(\frac{\partial}{\partial\tau} + J(u_j)\frac{\partial}{\partial t}\right)\widehat{u}_j = 0.
\end{aligned}$$

After possibly shifting $Z_I^j \cup Z_{II}^j \cup Z_{III}^j$, and recalling that $\mathrm{diam}(u_j) \to 0$, we may choose a subsequence, still denoted by u_j, such that

(1) the rescaled map $\widehat{u}_j(\tau, t)$ converges to $\widehat{u}_\infty(\tau, t)$, and
(2) the original map $u_j(\tau, t)$ converges to a point p_∞ for some point $p_\infty \in M$

on $[0, 3] \times S^1$.

Next the following C^1-convergence to C^1-behavior of $J(u_j(\tau, t))$ as $j \to \infty$ will be needed.

Lemma B.5 *After choosing a subsequence if necessary, we have*

$$\lim_{j\to\infty} \|du\|_{C^0;K} = 0$$

for any given compact subset $K \subset \mathbb{R} \times S^1$. *In particular,* $J(u_j) \to J_0$ *in the compact* C^1 *topology on* $\mathbb{R} \times S^1$.

Proof Let $K \subset \mathbb{R} \times S^1$ be a given compact subset. Since $L_j \to \infty$, eventually $K \subset [-L_j, L_j] \times S^1$. Now suppose to the contrary that there exist some $\epsilon > 0$ and $(\tau_j, t_j) \in K$ such that $|du_j(\tau_j, t_j)| > \epsilon$ for all j, after extracting a subsequence if needed. Considering the translated and rotated maps $\widetilde{u}_j$ defined by $\widetilde{u}_j(\tau, t) = u_j(\tau - \tau_j, t - t_j)$ whose domain contains $[-L_j + \tau_j, L_j + \tau_j] \times S^1$ and $|d\widetilde{u}_j(0,0)| > \epsilon$, the standard bubbling argument either creates a non-constant bubble or produces a subsequence such that $\widetilde{u}_j \to u_\infty$ in compact C^1 topology, where $u_\infty : \mathbb{R} \times S^1 \to M$ is a J_0-holomorphic map satisfying

$$|du_\infty(0,0)| \geq \epsilon$$

and hence is not constant, which gives rise to a contradiction since $\operatorname{im} \widetilde{u}_j = \operatorname{im} u_j \subset B_{p_j}(2\epsilon_j)$ and thus $\operatorname{diam}(\operatorname{im} u_\infty) = 0$. This finishes the proof of the lemma. □

In particular, this lemma implies that $J(u_j(\tau, t)) \to J_0$ in C^1 topology on a *fixed* $[0,3] \times S^1$ $(= Z_{\mathrm{I}} \cup Z_{\mathrm{II}} \cup Z_{\mathrm{III}})$.

Therefore it follows from the above discussion that the limit map $\widehat{u}_\infty$ satisfies

$$\begin{aligned}
&\left\|\widehat{u}_\infty\right\|_{L^\infty(Z_{\mathrm{I}} \cup Z_{\mathrm{II}} \cup Z_{\mathrm{III}})} = 1, \\
&\qquad \left\|d\widehat{u}_\infty\right\|_{L^2(Z_{\mathrm{II}})} \geq \gamma\,(4\pi\upsilon) \cdot \left(\left\|d\widehat{u}_\infty\right\|_{L^2(Z_{\mathrm{I}})} + \left\|d\widehat{u}_\infty\right\|_{L^2(Z_{\mathrm{III}})}\right), \\
&\left(\frac{\partial}{\partial\tau} + J_0 \frac{\partial}{\partial t}\right)\widehat{u}_\infty = 0.
\end{aligned}$$

Then $\widehat{u}_\infty$ is a non-zero holomorphic map to $\mathbb{C}^n$ by the first and third identities. We write $\widehat{u}_\infty\,(\tau, t)$ in Fourier series as

$$\widehat{u}_\infty\,(\tau, t) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi k\tau} e^{2\pi kit} \text{ with } \left\|\widehat{u}_\infty\right\|_{L^2(Z_{\mathrm{I}} \cup Z_{\mathrm{II}} \cup Z_{\mathrm{III}})} \leq 3,$$

where the a_k are constant vectors in $\mathbb{C}^n$. We can explicitly compute

$$\left\|d\widehat{u}_\infty\right\|^2_{L^2\left([a,b] \times S^1\right)} = \sum_{k=-\infty}^{\infty} 4\pi^2 k^2 \left|a_k\right|^2 \cdot \int_a^b e^{4\pi\tau}\, d\tau. \tag{B.5}$$

On multiplying (B.2) by $e^{4\pi k}$ and letting $c = 4\pi$ there, we have

$$\int_k^{k+1} e^{4\pi\tau}\, d\tau = \gamma\,(4\pi)\left[\int_{k-1}^{k} e^{4\pi\tau}\, d\tau + \int_{k+1}^{k+2} e^{4\pi\tau}\, d\tau\right]. \tag{B.6}$$

From (B.5) and (B.6) we see that

$$\left\|d\widehat{u}_\infty\right\|_{L^2(Z_{\mathrm{II}})} = \gamma\,(4\pi) \cdot \left(\left\|d\widehat{u}_\infty\right\|_{L^2(Z_{\mathrm{I}})} + \left\|d\widehat{u}_\infty\right\|_{L^2(Z_{\mathrm{III}})}\right).$$

This contradicts

$$\left\|d\widehat{u}_\infty\right\|_{L^2(Z_{\mathrm{II}})} \geq \gamma\,(4\pi\upsilon) \cdot \left(\left\|d\widehat{u}_\infty\right\|_{L^2(Z_{\mathrm{I}})} + \left\|d\widehat{u}_\infty\right\|_{L^2(Z_{\mathrm{III}})}\right),$$

since $\gamma\,(4\pi) < \gamma\,(4\pi\upsilon)$. □

On combining the above Lemma and (B.3) we have the following corollary.

Corollary B.6 *For any* $0 < \upsilon < 1$, *there exists* $L_0 > 0$, *and* $\epsilon_0 = \epsilon_0(\upsilon)$ *depending on* υ *but independent of* $L \geq L_0$, *such that, for all* $0 < \epsilon \leq \epsilon_1$ *and* $[\tau, \tau + 1] \subset [-L, L]$, *we have*

$$\int_{[\tau,\tau+1]\times S^1} |du|^2 \leq e^{-4\pi\upsilon(L-|\tau|)} \left[\int_{[-L-1,-L]\times S^1} |du|^2 + \int_{[L,L+1]\times S^1} |du|^2 \right].$$

From these results and the standard elliptic estimate on the cylinder $\left[\tau - \frac{1}{2}, \tau + \frac{1}{2}\right] \times S^1$, we obtain the following pointwise exponential decay estimate of u.

Corollary B.7 *For any* $0 < \upsilon < 1$, *there exists* $L_0 > 0$, *and* $\epsilon_1 = \epsilon_0(\upsilon) > 0$ *depending on* υ *but independent of* $L \geq L_0$, *such that, for all* $0 < \epsilon \leq \epsilon_1$ *and* $\tau \in [-L, L]$, *we have*

$$|du| \leq Ce^{-2\pi\upsilon(L-|\tau|)} \left(\|du\|_{L^2([-L-1,-L]\times S^1)} + \|du\|_{L^2([L,L+1]\times S^1)} \right).$$

The υ *can be made arbitrarily close to* 1.

Remark B.8 The same scheme can be applied to the semi-strip $[0, \infty) \times [0, 1]$ for the Floer equation

$$\begin{cases} \partial u/\partial\tau + J\,\partial u/\partial t = 0, \\ u(\tau, 0) \in L_0,\ u(\tau, 1) \in L_1, \end{cases}$$

with its asymptotic limit being $\lim_{\tau\to\infty} u(\tau, \cdot) = p \in L_0 \cap L_1$ for a transversal intersection point; in this case the limiting equation will be

$$\begin{cases} \partial\widehat{u}_\infty/\partial\tau + J(p)\partial\widehat{u}_\infty/\partial t = 0, \\ \widehat{u}_\infty(\tau, 0) \in T_pL_0,\ \widehat{u}_\infty(\tau, 1) \in T_pL_1. \end{cases}$$

This will provide another proof of the exponential decay result proven in Section 14.1, which not only involves less computation but provides a precise value of δ appearing in the exponent of (8.6.51).

Exercise B.9 Complete the details of the above-mentioned proof.

Appendix C

The Maslov index, the Conley–Zehnder index and the index formula

The main purpose of the present appendix is to give the details of the proof of the index formulae appearing in Propositions 15.3.1 and 20.2.13. For this purpose, we need the Conley–Zehnder index for the symplectic path and the Maslov index for the pair of Lagrangian paths, especially their generalized version introduced by Robbin and Salamon (RS93). Following (RS93, RS95), we first summarize the basic properties of this generalized Maslov index of Lagrangian paths, and then prove the index formula appearing in Proposition 20.2.13.

C.1 The Maslov index for the Lagrangian paths

In this section, we summarize the basic properties of the Maslov index associated with each pair (Λ, Λ') of the Lagrangian paths by following verbatim Robbin and Salmon's approach given in Section 5 of (RS95) except for minor notational differences. This in turn follows their detailed exposition on the Maslov index provided in (RS93).

Let (S, Ω) be a symplectic vector space and denote by $\mathcal{L}(S)$ the set of Lagrangian subspaces of (S, Ω). The Maslov index as defined in (RS93) assigns to every pair of Lagrangian paths Λ, $\Lambda' : [a, b] \to \mathcal{L}(S)$ a half-integer $\mu(\Lambda, \Lambda')$. They are characterized by the following axioms.

(1) (Naturality) $\mu(\Psi\Lambda, \Psi\Lambda') = \mu(\Lambda, \Lambda')$ for any symplectomorphism $\Psi : (S, \Omega) \to (S', \Omega')$.
(2) (Homotopy) The Maslov index is invariant under fixed-end-point homotopies.
(3) (Zero) If $\Lambda(t) \cap \Lambda'(t)$ is of constant dimension, then $\mu(\Lambda, \Lambda') = 0$.

(4) (Direct sum) If $S = S_1 \oplus S_2$, then

$$\mu(\Lambda_1 \oplus \Lambda_2, \Lambda_1' \oplus \Lambda_2') = \mu(\Lambda_1, \Lambda_1') + \mu(\Lambda_2, \Lambda_2').$$

(5) (Catenation) For $a < c < b$,

$$\mu(\Lambda, \Lambda') = \mu(\Lambda|_{[a,c]}, \Lambda'|_{[a,c]}) + \mu(\Lambda|_{[c,b]}, \Lambda'|_{[c,b]}).$$

(6) (Localization) If $(S, \Omega) = (\mathbb{R}^{2n}, \omega_0)$, $\Lambda'(t) \equiv \mathbb{R}^n \times 0$ and $\Lambda(t) = \mathrm{Gr}(A(t))$ for a path $A : [a, b] \to \mathbb{R}^{n \times n}$ of symmetric matrices, then the Maslov index is given by

$$\mu(\Lambda, \Lambda') = \frac{1}{2}\,\mathrm{sign} A(b) - \frac{1}{2}\,\mathrm{sign} A(a).$$

Another important useful property is the following relationship with the Conley–Zehnder index given in Theorem 18.4.2. To explain, we also consider the product symplectic vector space

$$(S \times S, (-\Omega) \oplus \Omega)$$

and regard $\mathrm{Gr}(\Psi)$ and $V \times V$ as Lagrangian subspaces thereof.

Theorem C.1.1 *Let $S = \mathbb{R}^{2n} \simeq \mathbb{C}^n$. Let $\Psi : [a, b] \to Sp(2n)$ be a symplectic path and let $V = 0 \times \mathbb{R}^n$. Then the following statements hold.*

(1) *Consider the pair $(\Psi \cdot V, V)$, where ΨV is the Lagrangian path given by $t \mapsto \Psi(t)V$. Then*

$$\mu(\Psi V, V) = \mu(\mathrm{Gr}(\Psi), V \times V).$$

(2) *Supposing that Ψ is as in Theorem 18.4.2, i.e., it satisfies $\Psi(a) = 1$ and $\Psi(b)V \cap V = 0$, then $\mu(\Psi V, V) + n/2 \in \mathbb{Z}$.*

The following generalization of the Conley–Zehnder index appearing in Theorem 18.4.2 was given by Robbin and Salamon (RS93).

Definition C.1.2 Let $\Psi : [a, b] \to Sp(2n)$ be a symplectic path. The Conley–Zehnder index of Ψ is defined by

$$\mu_{CZ}(\Psi) = \mu(\mathrm{Gr}(\Psi), \Delta) \in \mathbb{Z} + \frac{1}{2}.$$

It is shown in (RS93) that the half-integer $\mu_{CZ}(\Psi)$ becomes an integer when Ψ satisfies $\Psi(a) = 1$ and $\Psi(b)V \cap V = 0$ and coincides with that of Theorem 18.4.2.

C.2 Reduction of the proof to the Lagrangian case

We now recall the set-up of Section 20.2.2.

Let (Σ, j) be a Riemann surface of genus 0 with three marked points. Denote the neighborhoods of the three by D_i, $i = 1, 2, 3$. We assume that the associated punctures for $i = 1, 2$ are positive with analytic coordinates given by

$$\varphi_i^+ : D_i \setminus \{z_i\} \to (-\infty, 0] \times S^1 \quad \text{for } i = 1,\ 2$$

and the puncture for $i = 0$ is negative with analytic coordinate given by

$$\varphi_3^- : D_3 \to [0, \infty) \times S^1.$$

Denote by (τ, t) the standard cylindrical coordinates on the cylinder

$$[0, \infty) \times S^1 = [0, \infty) \times \mathbb{R}/\mathbb{Z}$$

and fix a cut-off function $\rho^+ : (-\infty, 0] \to [0, 1]$ and $\rho^- : [0, \infty) \to [0, 1]$ by $\rho^-(\tau) = \rho^+(-\tau)$ as before.

Let $[z_i, w_i]$ be a critical point of $\mathcal{A}_{H_i}$ for $i = 1, 2$. We denote $\widehat{z}_i = [z_i, w_i]$ and $\widehat{z} = (\widehat{z}_1, \widehat{z}_2, \widehat{z}_3)$. Let $u \in \mathcal{M}(H, \widetilde{J}; \widehat{z})$ be an element, where $\mathcal{M}(H, \widetilde{J}; \widehat{z})$ is the moduli space given in Section 20.2.2 associated with the triple. We restate Proposition 20.2.13 here.

Proposition C.2.1 *The (virtual) dimension of $\mathcal{M}(H, \widetilde{J}; \widehat{z})$ is given by*

$$\begin{aligned} \dim \mathcal{M}(H, \widetilde{J}; \widehat{z}) &= 2n - (-\mu_{H_1}(\widehat{z}_1) + n) - (-\mu_{H_2}(\widehat{z}_2) + n) - (\mu_{H_3}(\widehat{z}_3) + n) \\ &= -n + (\mu_{H_1}(\widehat{z}_1) + \mu_{H_2}(\widehat{z}_2) - \mu_{H_3}(\widehat{z}_3)). \end{aligned}$$

It will be enough to compute the Fredholm index of the linearization operator at $u \in \mathcal{M}(H, \widetilde{J}; \widehat{z})$. The formula can then be obtained by the gluing formula (or excision formula) after capping u by the w_i.

We will then prove that each cap has the index given by $-\mu_*(z) + n$ when z is an *incoming* asymptotic limit in the cap. Note that the glued linearized operator is of the type of a perturbed Cauchy–Riemann operator acting on the set of sections of the *trivial bundle* $E = \Sigma \times \mathbb{R}^{2n}$ on the closed Riemann surface of genus 0. The triviality follows from the homotopy condition $[u] \coprod_{i=1}^3 [w_i] = 0$. The glued operator can be smoothly homotoped via a family of Fredholm operators to the standard Dolbeault operator acting on the space of sections of the trivial bundle. Since the trivial bundle has its first Chern number $c_1(E)$ zero, we derive that the Fredholm index of the latter operator becomes $2n$ by the index formula (10.4.14).

By the gluing formula for the Fredholm index, we have

$$\text{Index}\, D_u\overline{\partial}_{(H,\widehat{J})} + \sum_{i=1}^{3} \text{Index}\, D_{w_i}\overline{\partial}_{(H_i,J_i)} = 2n$$

and hence

$$\text{Index}\, D_u\overline{\partial}_{(H,\widehat{J})} = 2n - \sum_{i=1}^{3} \text{Index}\, D_{w_i}\overline{\partial}_{(H_i,J_i)}. \tag{C.1}$$

It remains to compute Index $D_{w_i}\overline{\partial}_{(H_i,J_i)}$.

Now we associate a canonical Cauchy–Riemann type operator with each cap acting on the space

$$W^{1,p}(C_+;\mathbb{R}^{2n}),$$

where C_+ is the surface

$$C_+ = (-\infty,0]\times S^1 \cup S^2_+/\!\sim.$$

Here S^2_+ is the right-half sphere and $\sim$ is the identification of $(0,t)\in(-\infty,0]\times S^1$ with $(r,\theta)=(1,t)\in S^2_+$, where (r,θ) is the polar coordinates of $S^2_+\cong D^2$.

With respect to the cylindrical coordinates (τ,t) on $(-\infty,0]\times S^1$ the operator $D_{w_i}\overline{\partial}_{(H_i,J_i)}$ is of the type

$$\overline{\partial}_{J,T} = \frac{\partial}{\partial\tau} + J\frac{\partial}{\partial t} + T,$$

where $T(\tau,\cdot)\equiv T(-\infty,\cdot)$ and $J(\tau,\cdot)\equiv J(-\infty,\cdot)$ for all $\tau\le -R$ for some $R>0$.

By deforming the operators on $(-\infty,0]\times S^1$ without changing the Fredholm index, we may further assume that, for sufficiently large $R>0$,

$$J(\tau,\cdot) = \begin{cases} J(\infty,\cdot) & \text{for} \quad \tau\ge R+1,\\ i & \text{for} \quad \tau\le R \end{cases}$$

and

$$T(\tau,\cdot) = \begin{cases} T(\infty,\cdot) & \text{for} \quad \tau\ge R+1,\\ 0 & \text{for} \quad \tau\le R. \end{cases}$$

Then the operator canonically extends to the region S^2_+ as an operator acting on $W^{1,p}(C_+;\mathbb{R}^{2n})$.

Theorem C.2.2 *Let T and J be as above, and let Ψ be the solution of the equation*

$$\begin{cases} \partial\Psi/\partial t - J(\infty,t)T(\infty,t)\Psi = 0,\\ \Psi(0) = id \end{cases} \tag{C.2}$$

on C_+. Then

$$\text{Index}\,\overline{\partial}_{J,T} = -\mu_{CZ}(\Psi) + n,$$

where Ψ is the fundamental solution of (C.4) *and $\mu_{CZ}(\Psi)$ is Robbin and Salamon's generalized Conley–Zehnder index of the symplectic path $t \mapsto \Psi(t) \in Sp(2n)$.*

C.3 The index formula for the Lagrangian case

In this section, we identity $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and $\mathbb{R}^n \oplus (\mathbb{R}^n)^*$ and often identify $\mathbb{R}^n \subset \mathbb{C}^n$ with $\mathbb{R}^n \times 0 \subset \mathbb{R}^n \oplus (\mathbb{R}^n)^*$.

We will derive the index of the operator given in Theorem C.3.1 from that of its Lagrangianization

$$\overline{\partial}_{J,T} = \frac{\partial}{\partial\tau} + J\frac{\partial}{\partial t} + T$$

acting on

$$W^{1,p}_{\Delta} = W^{1,p}_{\mathbb{R}^n}(\Theta; \mathbb{R}^{2n} \times \mathbb{R}^{2n}) := \{\zeta \in W^{1,p}(\Theta, \mathbb{R}^{4n}) \mid \zeta(\theta) \in \Delta \text{ for all } \theta \in \partial\Theta\},$$

where $\Theta = (-\infty, 0] \times [0,1] \# D^2_+$ is the 'unfolded' C_+ along the line $\{t = 0\}\#\{\theta = 0\} \subset C_+$. We note that Θ is conformally equivalent to $(-\infty, 0] \times [0,1]$ (relative to the boundary). For computational convenience, we will work on the latter and just denote $\Theta = (-\infty, 0] \times [0,1]$ henceforth.

We compute the index of this Fredholm operator in terms of Robbin and Salamon's generalized Maslov index of the Lagrangian path (RS93). (We refer readers to (Schw00) for a direct proof without going through the Lagrangianization.)

For this purpose, we recall the definition $\mu_{CZ}(\Psi) = \mu(\mathrm{Gr}(\Psi), \Delta)$ from Definition C.1.2, with respect to the symplectic vector space $(S \times S, (-\Omega) \oplus \Omega)$.

By a linear symplectic change of coordinates, we can symplectically transform the pair

$$(\mathrm{Gr}(\Psi), \Delta), \quad (S \times S, (-\Omega) \oplus \Omega)$$

to a pair

$$(\mathrm{Gr}(\widetilde{\Psi}), S \times 0), \quad (S \times S^*, \omega_0),$$

where $\widetilde{\Psi} : S \to S^*$ is the linear map associated with the $\mathrm{Gr}(\Psi)$ as a section of $T^*S \cong S \times S^*$. Then it follows that

$$\mu(\mathrm{Gr}(\Psi), \Delta; (-\Omega) \oplus \Omega) = \mu(\mathrm{Gr}(\widetilde{\Psi}), S \times 0; \omega_0) \tag{C.3}$$

Therefore this index can be derived from the following general theorem with the replacement of n by $2n$.

Theorem C.3.1 *Let T and J be as above, and let Ψ_∞ be the solution of the equation*

$$\begin{cases} \partial\Psi/\partial t - J(\infty, t)T(\infty, t)\Psi = 0, \\ \Psi(0) = id \end{cases} \tag{C.4}$$

on $(-\infty, 0] \times [0, 1]$. *Then*

$$\text{Index}\,\overline{\partial}_{J,T} = -\mu(\Psi\mathbb{R}^n, \mathbb{R}^n) + \frac{n}{2}, \tag{C.5}$$

where $\Psi\mathbb{R}^n$ is a Lagrangian path given by $t \mapsto \Psi(t)(\mathbb{R}^n \times 0) \subset \mathbb{C}^n$.

The remainder of this section will be occupied by the proof of Theorem C.3.1, which closely follows that of the appendix of (Oh99). Just like therein, we will consider the index problem on the semi-strip $[0, \infty) \times [0, 1]$ instead of on $(-\infty, 0] \times [0, 1]$ and prove

$$\text{Index}\,\overline{\partial}_{J,T} = \mu(\Psi\mathbb{R}^n, \mathbb{R}^n) + \frac{n}{2}. \tag{C.6}$$

Then (C.5) will immediately follow from this by the biholomorphism

$$(\tau, t) \in (-\tau, 1 - t)$$

which reverses the direction of the path Ψ. We denote $\Theta = [0, \infty) \times [0, 1]$ from now on.

We now transform the above operator $\overline{\partial}_{J,T} = \partial/\partial\tau + J\,\partial/\partial t + T$ acting upon $W^{1,p}_{\mathbb{R}^n}(\Theta; \mathbb{R}^{2n})$ to another operator defined on the space

$$W^{1,p}_{\widetilde{\Lambda}} = \{\zeta \in W^{1,p}(\Theta, \mathbb{R}^{2n}) \mid \zeta(\theta) \in \widetilde{\Lambda}(\theta)\,\},$$

where $\widetilde{\Lambda} : \partial\Theta \to \widetilde{\Lambda}(n)$ is the path defined as

$$\widetilde{\Lambda}(\theta) = \begin{cases} \mathbb{R}^n & \text{if} \quad \theta \in \partial_0\Theta, \\ \mathbb{R}^n & \text{if} \quad \theta = (\tau, 0) \quad \text{for } \tau \le R + 1, \\ \widetilde{\Lambda}_1(\theta) & \text{if} \quad \theta = (\tau, 1) \quad \text{for } R + 1 \le \tau \le R + 2, \\ \Psi(1) \cdot \mathbb{R}^n & \text{if} \quad \theta = (\tau, 1) \quad \text{for } \tau \ge R + 2. \end{cases}$$

Here $\partial_0\Theta$ is the portion of the boundary of the compact part Θ in $\partial\Theta$ as pictured in Figure C.1 and $\widetilde{\Lambda}_1 : [R + 1, R + 2] \to \widetilde{\Lambda}(n)$ is a path defined by

$$\widetilde{\Lambda}_1(\tau, 1) := \Psi(\rho(\tau - R - 1)) \cdot \mathbb{R}^n \quad \text{for} \quad \tau \ge R - 1$$

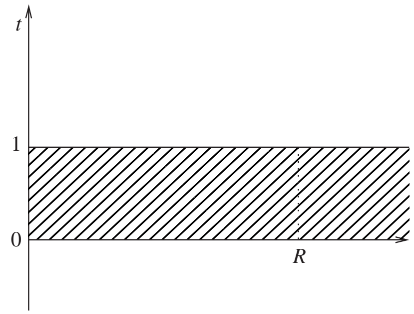

Figure C.1 Domain $\Theta_{0,1}$.

and by extending this definition to the whole of Θ by setting

$$\widetilde{\Lambda}(\theta) \equiv \mathbb{R}^n \quad \text{for} \quad \tau \leq R - 1.$$

Here the function $\rho : \mathbb{R} \to [0, 1]$ is the cut-off function we used before.

Note that, since we assume that $T \equiv 0$ on $\tau \leq R$, the map $\Psi : [R, \infty) \times [0, 1] \to Sp(2n)$ defined by

$$\Psi(\tau, t) := \Psi(\rho(\tau - R - 1)t) \quad \text{for} \quad \tau \geq R$$

can be smoothly extended to the whole of Θ by setting $\Psi \equiv id$ for $\tau \leq R$. Therefore we can now define the push-forward operator

$$\Psi_*(\overline{\partial}_{J,T}) = \Psi \circ \overline{\partial}_{J,T} \circ \Psi^{-1} : \quad W^{1,p}_{\widetilde{\Lambda}} \to L^p.$$

Then this push-forward will have the form

$$\overline{\partial}_{\widetilde{J},\widetilde{T},\widetilde{\Lambda}} = \frac{\partial}{\partial \tau} + \widetilde{J}\frac{\partial}{\partial t} + \widetilde{T} : \quad W^{1,p}_{\widetilde{\Lambda}} \to L^p$$

such that

$$\widetilde{T} \equiv 0 \quad \text{if } \tau \geq R + 1 \quad \text{or} \quad \tau \leq R,$$
$$\widetilde{J} \equiv i \quad \text{if } \tau \leq R.$$

It is obvious that the two operators $\overline{\partial}_{J,T}$ and $\overline{\partial}_{\widetilde{J},\widetilde{T},\widetilde{\Lambda}}$ will have the same Fredholm indices and hence it suffices to compute the index of $\overline{\partial}_{\widetilde{J},\widetilde{T},\widetilde{\Lambda}}$. We recall that we imposed the transversality $\Psi(1) \cdot \mathbb{R}^n \pitchfork \mathbb{R}^n$ and hence

$$\widetilde{\Lambda}_1(\theta) \pitchfork \mathbb{R}^n \quad \text{for} \quad \theta = (\tau, 1), \quad \tau \geq R + 2.$$

If we denote

$$\infty_+ = \lim_{\tau \to \infty} (\tau, 1), \qquad \infty_- = \lim_{\tau \to \infty} (\tau, 0)$$

the map $\widetilde{\Lambda} : \partial\Theta \to \widetilde{\Lambda}(n)$ satisfies

$$\begin{aligned}
\widetilde{\Lambda}(\infty_-) &= \widetilde{\Lambda}(\tau, R+1) = \mathbb{R}^n,\\
\widetilde{\Lambda}(\infty_+) &= \Psi(1)\cdot\mathbb{R}^n = \widetilde{\Lambda}_1(R+1),\\
\widetilde{\Lambda}(\tau, 1) &= \Psi(\rho(\tau - R - 1))\cdot\mathbb{R}^n \quad \text{for} \quad R+1 \le \tau.
\end{aligned}$$

By construction, the two paths $t \mapsto \Psi(t)\cdot\mathbb{R}^n$ and $\tau \mapsto \widetilde{\Lambda}_1$ are homotopic with the same fixed end points and hence have the same Maslov indices, i.e.,

$$\mu(\widetilde{\Lambda}_1, \mathbb{R}^n) = \mu(\Psi\cdot\mathbb{R}^n, \mathbb{R}^n). \tag{C.7}$$

Using the stratum-homotopy invariance of the Maslov index, Theorem 2.4 of [RS93], we can deform the operator $\overline{\partial}_{\widetilde{J},\widetilde{T},\widetilde{\Lambda}}$, without changing the Fredholm property and without changing the Maslov indices of $\widetilde{\Lambda}$, into

$$\overline{\partial}_{i,0,\Lambda} = \frac{\partial}{\partial\tau} + i\frac{\partial}{\partial t} = \overline{\partial} : \quad W^{1,p}_{\Lambda} \to L^p,$$

where $\Lambda : \partial\Theta_{0,1} \to \Lambda(n)$ is defined as

$$\Lambda(\theta) = \begin{cases} \mathbb{R}^n & \text{if} \quad \theta = (\tau, 0),\\ D(t)\cdot\mathbb{R}^n & \text{if} \quad \theta = (t, 0),\\ i\cdot\mathbb{R}^n & \text{if} \quad \theta = (\tau, 1) \end{cases}$$

with

$$D(t) = \begin{pmatrix} e^{-(\ell+\frac{1}{2})\pi i t} & 0 & 0 & \cdots & 0\\ 0 & e^{-\frac{1}{2}\pi i t} & 0 & \cdots & 0\\ 0 & 0 & . & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & e^{-\frac{1}{2}\pi i t} \end{pmatrix}$$

for some integer ℓ. Here we identify $\Theta_{0,1}$ with the semi-strip as drawn in Figure C.2.

C.4 Explicit calculation in one dimension

Noting that both the operator $\overline{\partial}$ and the boundary condition Λ are *separable*, we have reduced the computation of the index to the one-dimensional problem below. Recall that both the Fredholm index of $\overline{\partial}_{i,0,\Lambda}$ and the Maslov index of Λ are additive under the direct sum.

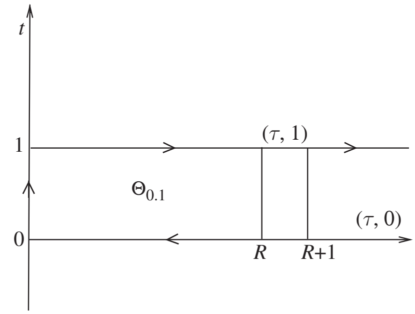

Figure C.2 Semi-strip $\Theta_{0,1}$.

We now study the following equation in one dimension:

$$\begin{cases} \overline{\partial}\zeta = 0, \\ \zeta(\tau, 0) \in \mathbb{R}, \quad \zeta(\tau, 1) \in i\mathbb{R}, \\ \zeta(0, t) \in e^{-(\ell+\frac{1}{2})\pi i t}\mathbb{R}. \end{cases} \tag{C.8}$$

Proposition C.4.1 *We denote by $\overline{\partial}_{i,0,\ell}$ the Cauchy–Riemann operator associated with this boundary-value problem. Then*

$$\operatorname{Index} \overline{\partial}_{i,0,\ell} = \dim \ker \overline{\partial}_{i,0,\ell} - \dim \operatorname{coker} \overline{\partial}_{i,0,\ell} = \ell + 1. \tag{C.9}$$

Proof Note that any ζ satisfying

$$\overline{\partial}\zeta = 0, \qquad \zeta(\tau, 0) \in \mathbb{R}, \quad \zeta(\tau, 1) \in i\mathbb{R}, \quad \int |D\zeta|^2 < \infty$$

must have the form

$$\zeta(z) = \sum_{k=1}^{\infty} a_k e^{-(k-\frac{1}{2})\pi z}, \quad a_k \in \mathbb{R} \quad \text{and } z \in \Theta.$$

This becomes

$$\zeta(0, t) = \sum_{k=1}^{\infty} a_k e^{-(k-\frac{1}{2})\pi i t}$$

on $\{0\} \times [0, 1]$. Since the third equation of (C.8) implies that

$$e^{(\ell+\frac{1}{2})\pi i t}\zeta(0, t) \in \mathbb{R},$$

we have

$$\sum_{k=1}^{\infty} a_k e^{-(k-\ell-1)\pi i t} \in \mathbb{R}.$$

Equivalently, we have

$$\sum_{k=-\ell}^{\infty} a_{k+\ell+1} e^{-k\pi i t} \in \mathbb{R}. \tag{C.10}$$

We consider the two cases in which $\ell \geq 0$ and $\ell \leq -1$ separately. First, let us assume that $\ell \geq 0$. Then, from (C.10), we derive

$$\begin{aligned} a_{k+\ell+1} &= 0 && \text{if } k \geq \ell + 2, \\ a_{\ell+1} & && \text{arbitrary}, \\ a_{k+\ell+1} &= a_{-k+\ell+1} && \text{if } -\ell \leq k \leq -1, \end{aligned}$$

i.e.,

$$\begin{aligned} a_k &= 0 && \text{if } k \geq \ell + 2, \\ a_{\ell+1} & && \text{arbitrary}, \\ a_k &= a_{-k+2(\ell+1)} && \text{if } 1 \leq k \leq \ell. \end{aligned}$$

On the other hand, if $\ell \leq -1$, it immediately follows from (C.10) that (C.8) has no non-trivial solution.

We denote by ker the set of solutions (C.8) and by coker (C.8) the set of solutions of its adjoint problem. Then

$$\dim \ker (\text{C.8}) = \begin{cases} \ell + 1 & \text{if} \quad \ell \geq 0, \\ 0 & \text{if} \quad \ell \leq -1. \end{cases} \tag{C.11}$$

Now let us study the L^2-adjoint problem of (C.8),

$$\begin{cases} \partial \eta = 0, \\ \eta(\tau, 0) \in \mathbb{R}, \quad \eta(\tau, 1) \in i\mathbb{R}, \\ i\eta(0, t) \in e^{-(\ell + \frac{1}{2})\pi i t} \cdot \mathbb{R} \end{cases} \tag{C.12}$$

and denote by coker (C.8) the solution space of (C.12). This equation can be derived by taking the L^2-inner product (C.8) with η and then integrating by parts. It follows from the equation $\partial \eta = 0$, $\eta(\tau, 0) \in \mathbb{R}$, $\eta(\tau, 1) \in i\mathbb{R}$ and $\int |D\eta|^2 < \infty$ that η must have the form

$$\eta(z) = \sum_{j=1}^{\infty} b_j e^{-(j-\frac{1}{2})\pi \bar{z}}, \quad b_j \in \mathbb{R}.$$

By substituting $z = (0, t)$ into this, we get

$$\eta(0, t) = \sum_{j=1}^{\infty} b_j e^{(j-\frac{1}{2})\pi i t}, \quad b_j \in \mathbb{R}.$$

The condition in the third equation of (C.12) is equivalent to

$$ie^{(\ell+\frac{1}{2})\pi it}\eta(0,t) \in \mathbb{R}, \qquad \text{i.e.,} \quad \sum_{j=1}^{\infty} ib_j e^{(j+\ell)\pi it} \in \mathbb{R},$$

and hence we have derived

$$\sum_{j=\ell+1}^{\infty} ib_{j-\ell} e^{j\pi it} \in \mathbb{R}. \tag{C.13}$$

From this, we immediately conclude that, if $\ell \geq 0$, then (C.12) has no non-trivial solution. When $\ell \leq -1$, we derive from (C.13)

$$\begin{aligned} b_{j-\ell} &= 0 \qquad \text{if} \quad j \geq -\ell, \\ b_{-\ell} &= 0, \\ b_{j-\ell} &= -b_{-j-\ell} \qquad \text{if} \quad 1 \leq j \leq -\ell - 1. \end{aligned}$$

From this, we conclude that

$$\dim \operatorname{coker}(\text{C.8}) = \begin{cases} 0 & \text{if} \quad \ell \geq 0, \\ -\ell - 1 & \text{if} \quad \ell \leq -1. \end{cases} \tag{C.14}$$

On combining (C.8) and (C.14), we have proven

$$\text{Index}\,(\text{C.8}) = \dim\ker(\text{C.12}) - \dim\ker(\text{C.12}) = \ell + 1 \tag{C.15}$$

for all ℓ. This finishes the proof. □

By applying (C.15) for $\ell = 0$ and adding the contributions from other components of $D(t)$, we get

$$\text{Index}\ \overline{\partial}_{i,0,\Lambda} = \ell + 1 + (n-1) = \ell + n. \tag{C.16}$$

We now compute the Maslov index $\mu(D(t)\cdot\mathbb{R}^n, \mathbb{R}^n)$ of the path

$$t \mapsto D(t)\cdot\mathbb{R}^n.$$

By virtue of the additivity of the Maslov index under the direct-sum operation, we have

$$\mu(D(t)\cdot\mathbb{R}^n, \mathbb{R}^n) = \mu(e^{-i(\ell+\frac{1}{2})\pi t}\cdot\mathbb{R}, \mathbb{R}) + (n-1)\mu(e^{-\frac{\pi}{2}it}\cdot\mathbb{R}, \mathbb{R}). \tag{C.17}$$

However, it is easy to check from the definition of the Maslov index from (RS93) that we have

$$\mu(e^{-i(\ell+\frac{1}{2})\pi t}\cdot\mathbb{R}, \mathbb{R}) = \ell + \frac{1}{2}.$$

By applying this to $\ell = 0$ for the second term in (C.17) as well, we conclude that

$$\mu(D \cdot \mathbb{R}^n, \mathbb{R}^n) = \ell + \frac{n}{2}.$$

However, from (C.7) and from the way in which we deform the operators afterwards, we have

$$\mu(\Psi \cdot \mathbb{R}^n, \mathbb{R}^n) = \mu(D \cdot \mathbb{R}^n, \mathbb{R}^n),$$

which finally finishes the proof of (C.6) and hence that of Theorem C.3.1. This in turn finishes the proof of Proposition 20.2.13.

One can apply the same argument to find the dimension formula for general $\mathcal{M}_{\Theta_{g,k}}(\vec{H}, J : \vec{z_-}, \vec{z_+})$, which is the space of solutions of the perturbed Cauchy–Riemann equation for arbitrary (g, k). We state the following theorem without proof. (We refer the reader to (Schw00) and others for its proof.)

Theorem C.4.2 *Let $\Theta_{g,k}$ be a surface of genus g with k punctures. Then*

$$\begin{aligned} \text{vir.}\dim \mathcal{M}_{\Theta_{g,k}}(\vec{H}, J : \vec{z_-}, \vec{z_+}) = n(1-g) &- \sum_{j=1}^{k_-}(-\mu_{CZ}(z_j^-) + n/2) \\ &- \sum_{j=1}^{k_+}(\mu_{CZ}(z_j^+) + n/2), \end{aligned}$$

where $k = k_- + k_+$, and k_- and k_+ are the numbers of outgoing and incoming edges, respectively.

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