تهران
دانشگاه

Hello!

Advanced Engineering Mathematics

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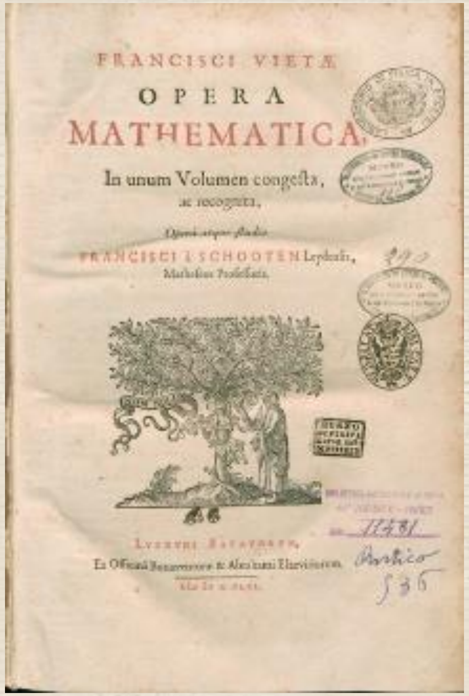
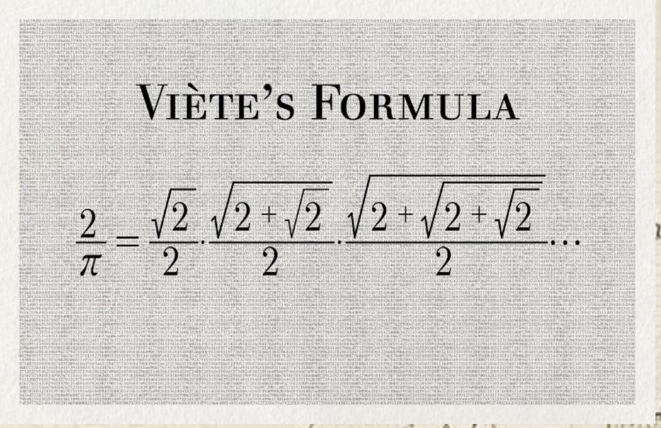
Infinite Products

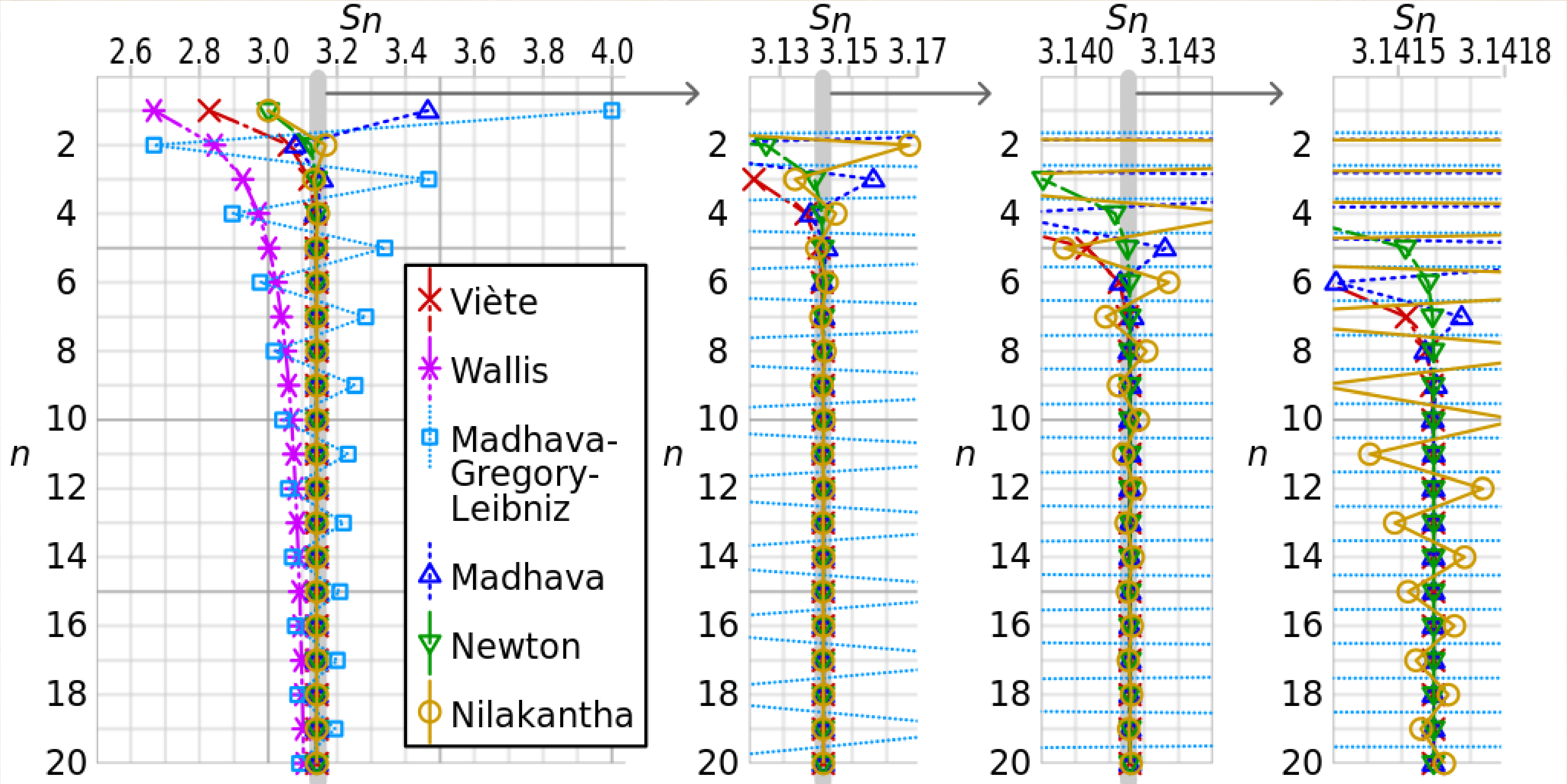
History & Introductions





Viète

In 1831 , Joseph Fourier , in his analysis of equations [225] , mentions Viète as one of the most illustrious mathematicians, the second inventor of algebra, after Al Kwarizmi .



Comparison of the convergence of the Wallis product (purple asterisks) and several historical infinite series for π. S_n is the approximation after taking n terms. Each subsequent subplot magnifies the shaded area horizontally by 10 times.(Wikipedia)

$$\prod_{k=1}^{5} k^2 = 1 \cdot 4 \cdot 9 \cdot a(4)$$

- Laufvariable: k
- Startwert: 1
- Endwert: 5
- Funktion: $a(k) = k^2$
- k kann folgende Werte annehmen:

  $k = 1$

  $k = 2$

  $k = 3$

  Setze $k = 4$ es ist $a(4) = 4^2 = 16$

  $k = 5$

Main Sections of Presentation

Infinite Products

The Gamma Function

Prime Numbers

1.

Infinite Products

- Convergence of Products and Series
- Conditionally Convergent Products
- Uniform Convergence of Products of Functions
- Infinite Products of Real Functions
- Infinite Product Expansions for sin x and cos x
- Abel’s Limit Theorem for Infinite Products
- Weierstrass Products
- The Weierstrass Factorization Theorem
- Blaschke Products
- Double Infinite Products

Example 1

Cosider the product

$$\prod_{j=1}^{\infty} \frac{1}{2^{1/2^j}}$$

The partial products are given by

$$P_n = \frac{1}{2^{1/2}} \cdot \frac{1}{2^{1/4}} \cdots \frac{1}{2^{1/2^n}}$$

$$= 2^{-\sum_{j=1}^{n} (1/2)^j}$$

$$= 2^{-1+1/2^n}$$

Evidently $P_n \to 1/2$ as $n \to \infty$ and therefore the product converges.

Cosider the product

$$\prod_{j=1}^{\infty} \frac{1}{2^j}$$

Here,

$$P_n = \frac{1}{2^1} \cdot \frac{1}{2^2} \cdots \frac{1}{2^n}$$

$$= 2^{-(1+2+\cdots+n)}$$

$$= 2^{-\frac{n(n+1)}{2}}$$

Consequently $P_n \to 0$ as $n \to \infty$ thus the product diverge to zero.

2.

The Gamma Function

- Representations of the Gamma Function
- Some Identities Involving the Gamma Function
- Analytic Functions Related to Gamma
- Stirling’s Formula
- Applications to Products and Series
- The Beta Function

Example 2

A function f is said to be asymptotic to a function g as x $\to \infty$ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$$

This relationship is denoted by

$$f \sim g$$

Ex: $f(x)=\log x+x^2+e^x \quad and \quad g(x)=e^x$

Stirling's Formula

$$\Gamma(x+1)\sim\sqrt{2\pi}\,x^{x+\frac{1}{2}}e^{-x}$$

As x $\to \infty$

3.

Prime Numbers & Partition

- Prime Numbers and Euler’s Identity
- Partition Functions
- The Jacobi Triple Product Identity

Example 3

Corollary 4.2.2 *For any $A \subseteq \mathbb{N}$ there exist sequences $\{a_j\}$ and $\{b_j\}$ such that*

$$\prod_{j \in A} \frac{1}{1 - x^j} = \sum_{k=0}^{\infty} a_k x^k$$

and

$$\prod_{j \in A} (1 + x^j) = \sum_{k=0}^{\infty} b_k x^k$$

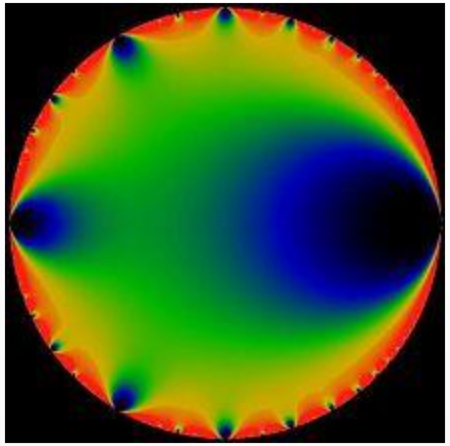
for all $x \in (-1, 1)$.

Theorem 4.2.3 *For all x such that $|x| < 1$,*

$$\prod_{j=0}^{\infty} \left(1 + x^{2^j}\right) = \sum_{j=0}^{\infty} x^j. \tag{4.2.3}$$

Euler function

$$\phi(q) = \prod_{k=1}^{\infty} (1 - q^k) \quad |q| < 1$$



Modulus of ϕ on the complex plane, colored so that black = 0, red = 4

Named after Leonhard Euler, it is a model example of a q-series, a modular form (as a function tau), and provides the prototypical example of a relation between combinatorics and complex analysis.

q-Pochhammer symbol

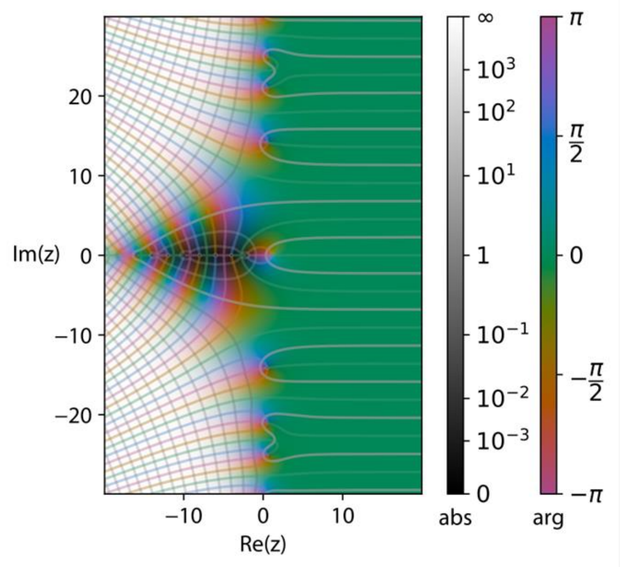
$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n) \xrightarrow{z=q} (q;q)_\infty = \phi(q)$$

In mathematics, in the area of combinatorics, a ***q*-Pochhammer symbol**, also called a ***q*-shifted factorial**, is a *q*-analog of the Pochhammer symbol

*The Euler function is a special case.

Euler Product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \ \text{prime}}^{\infty} \frac{1}{1 - p^{-s}}$$



The Riemann zeta function $\zeta(z)$ plotted with domain coloring.[1]

In number theory, an **Euler product** is an expansion of a Dirichlet series into an infinite product indexed by prime numbers. The original such product was given for the sum of all positive integers raised to a certain power as proven by Leonhard Euler. This series and its continuation to the entire complex plane would later become known as the Riemann zeta function.

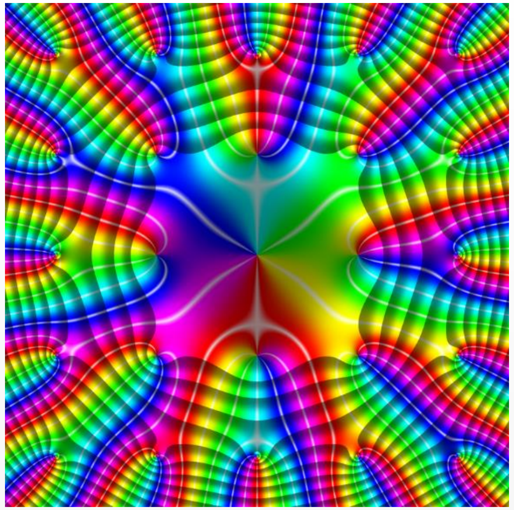
Wallis Product

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right) \longleftarrow \frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

In mathematics, the **Wallis product** for π, published in 1656 by John Wallis,[1]

Weierstrass Sigma Function

$$\sigma(z) = z \prod_{\omega \in \Lambda_*}^{\infty} \left(1 - \frac{z}{\omega}\right) e^{\frac{z^2}{2\omega} + \frac{z}{\omega}}$$

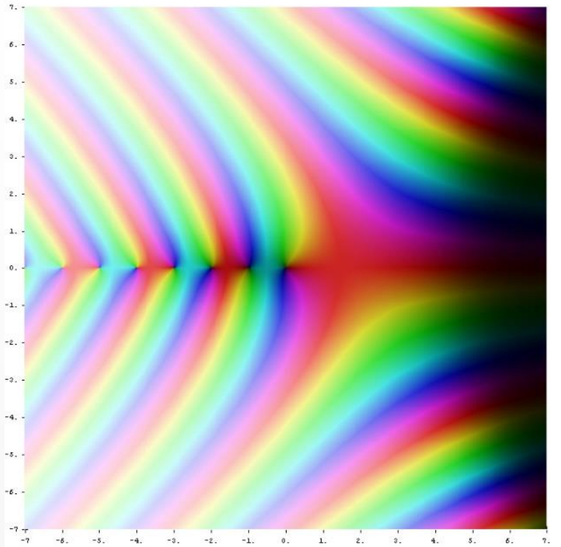


Plot of the sigma function using Domain coloring.

In mathematics, the Weierstrass functions are special functions of a complex variable that are auxiliary to the Weierstrass elliptic function. They are named for Karl Weierstrass. The relation between the sigma, zeta, and $\wp$ functions is analogous to that between the sine, cotangent, and squared cosecant functions: the logarithmic derivative of the sine is the cotangent, whose derivative is negative the squared cosecant.

Reciprocal Gamma Function

$$\frac{1}{\Gamma(z)} = z\prod_{n=1}^{\infty}\frac{1+\frac{z}{n}}{(1+\frac{z}{n})^{z}} = \frac{i}{2\pi}\oint_{H}(-t)^{-z}e^{-t}dt$$



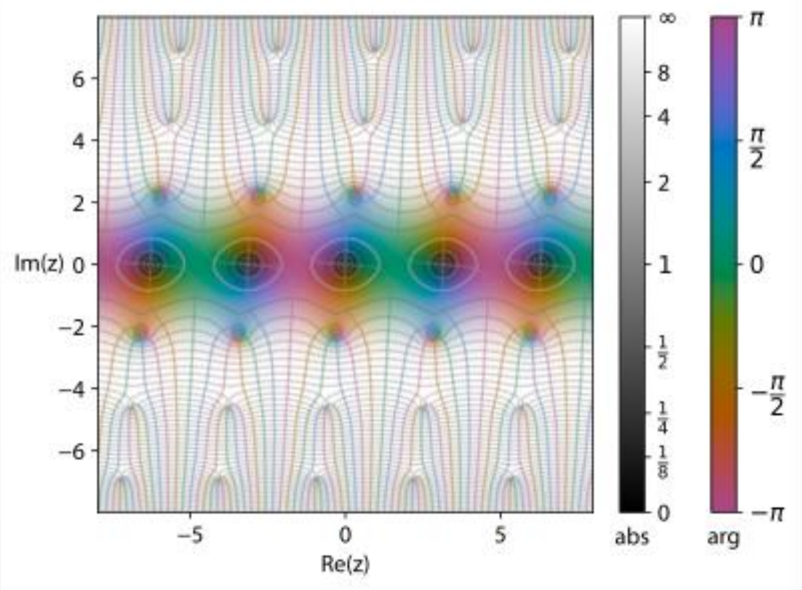
Reciprocal gamma function $1/\Gamma(z)$ in the complex plane. The color of a point z encodes the value of $1/\Gamma(z)$. Strong colors denote values close to zero and hue encodes the value's argument.

The reciprocal is sometimes used as a starting point for numerical computation of the gamma function, and a few software libraries provide it separately from the regular gamma function.

Karl Weierstrass called the reciprocal gamma function the "factorielle" and used it in his development of the Weierstrass factorization theorem.

Ramanujan Theta Function

$$f(a,b)=\prod_{n=0}^{\infty}(1+a^{n+1}b^{n})(1+a^{n}b^{n+1})(1-a^{n+1}b^{n+1})$$



Jacobi's theta function θ_1 with nome $q = e^{i\pi\tau} = 0.1e^{0.1i\pi}$:

An expression of the Jacobi triple product, also used in the expression of the Jacobi theta function

References and Usefull Links:

- Charles H. C. Little, Kee L. Teo, Bruce van Brunt, **An Introduction to Infinite Products**, Springer Nature, Switzerland, 2022.

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- https://ckrao.wordpress.com/2011/08/09/a-collection-of-infinite-products-ii/

Any questions?

Thanks!

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