Interim Report and Risk Assessment

Project Title: Interactive 3D visualisation of the Hopf fibration

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1.1 Background literature

Introduction

The fourth dimension is a geometric space with four dimensions. Algebraically the rules of vectors and coordinate geometry is applied to a vector with four components which can be used to represent a position in four dimensional space. But we cannot draw this position, nor can we draw an object. However, we can represent a mapping of an object in four dimensional space in a similar way a three dimensional object can be mapped to two dimensional space.

The Hopf fibration (HF) is a representation of the sphere in 4D, in 3D space. The sphere in 4D will be referred to as S3 or 3-Sphere. By mapping the 3D surface of the S3 in the ordinary 3D space (R3) we will be able to develop a digital 3 dimensional mapping of S3 (the HF) which we can use to help us understand the special properties of the S3. The HF is a fibration of spheres; the base is the 2-sphere; the fibre is the 1-sphere. With this information about the fibration we can gain some insight about the 3-sphere.

Topology is a field in mathematics where the HF describes the four dimensional sphere S3. Specifically, it is a map from S3 to S2 where S2 is a sphere in R3.

HF is a mapping of the four dimensional sphere (S3). It shows that the S3 can be produced by a set of circles that are arranged like points on the 3 dimensional sphere (S2). The circles that appear in the HF are known as fibres. The project aims to show the user the symmetry of the HF on a 3D display.

The problem is that this object has been drawn and projected to a 2D display like a monitor. But to experience the best of the beauty of this object, a 3D display would help the user admire and study the properties of the fibration to its fullest in its native dimension.

The S2 can be described as a shape that is formed of all the points which are of constant distance from a central point. S3 is a four dimensional sphere that is made up of all the points which are constant distance from a centre. This means that the boundary of the S3 can collapse into one single point in four dimensions, without collapsing the interior.

Imagining the S3 can be difficult for us, which is what the HF aims to solve. One way can be a two dimensional plane of complex dimension, a space with real dimension 4. Where each point in this plane is determined by two coordinates, each coordinate is a complex number. Each axis is a complex line and intersects the S3 in a circle. For each complex line that intersects the axis at the origin and S3, produces a decomposition of the sphere into circles, known as the Hopf fibration. There are other generalisations of the HF as well. This can be achieved by rotations of the S2 in R3 using quaternions which are also regarded as being R4.

<u>Motivation</u>

The HF is named after Heinz Hopf who studied it in his paper from 1931. Heinz Hopf was a German mathematician who worked in the fields of topology and geometry.

He discovered the Hopf invariant of maps $\mathbf{S^3} \rightarrow \mathbf{S^2}$ and proved that the HF has invariant 1. HF is a map from a higher dimensional sphere to a lower dimensional sphere which is not null-homotopic. Which meant that the $\mathbf{S^3}$ could be mapped in the form of the Hopf map described as; for any point *p* on a sphere is mapped to a circle S1 in S3, however we do not know what this looks like because $\mathbf{S^3}$ is embedded in $\mathbf{R^4}$. But it is possible to map $\mathbf{S^3}$ to $\mathbf{R^3}$.

Since this mapping belongs in 3 dimensional space, the motivation is to represent this projection in 3D space. This will mean we must be able to write any point $\mathbf{P}$ = (X1, X2, X3, X4) $\in \mathbf{S^3}$ in terms of $\mathbf{p}$ = (x, y, z) $\in \mathbf{R^3}$.

The program intends to help us to understand the structure of the HF and demonstrate the 3-sphere's symmetry. Unity3D is a cross-platform game engine that can distribute the HF's stereographic visualisation to many different 3D viewing platforms.

The HF visualised in a 3 dimensional space can display a wide variety of different parametrisations. The point *p* on $\mathbf{S^2}$, will determine the mapping; these points will be classed as a path on the two-sphere. This can entitle a path to contain points that are not bound by a set. Such a set can have a path around the S2 which does mean researching different patterns extracted using equations that take points on the surface of a unit sphere.

There are different methods in which you can achieve to draw this shape. But the shape itself has a wide variety of physical applications including magnetic monopoles, rigid body mechanics and quantum information theory. The approach that is popularly taken is the algebra of quaternions but it can also be achieved by complex projective line.

Since we know the properties of lower dimensional spheres, the principals must be the same. Taking a 3 dimensional sphere and projecting it down to R2 from R3 will result in the projection being a circle. This shows that it is feasible to map a higher dimensional object to a lower dimension. Such that the 4 dimensional sphere can be projected down to R3. Which draws the Hopf map.

Hopf Construction

There are different ways in which the HF can be constructed. In this research summary I go through 2 different methods. I am going to use the implementation using the algebra of quaternions in this project.

Paths on S2

First we must collect points from S2. These paths have a big impact on the fibration, because this is the primary input that results to produce various features of the HF. If we took all the points on a sphere, then this will fill R3 space. Using different patterns on S2 will result in a different parametrisation of the HF. The more different subsets of parametrisations we can find the more we can find out about the HF symmetry.

The points should be collected using the spherical coordinate system. The coordinates used are rho, theta, phi.

Rho is the distance from the origin to the point.

Theta is the angle in radians between the line from the x-axis and the origin to the point.

Phi is the angle between the z-axis and the line from the origin to the point.

These coordinates can be translated to Cartesian coordinates as follows:

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

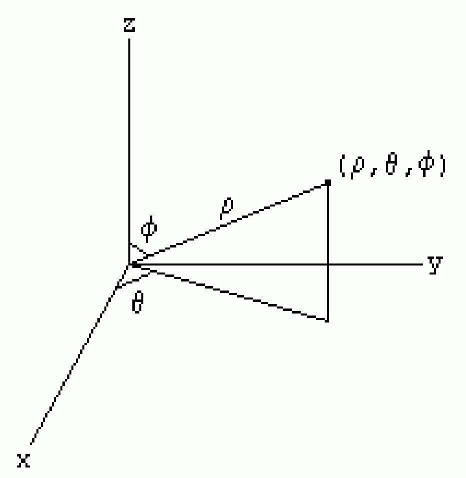


Figure 1

The points collected as a path on S2 are base coordinates. An example of a path on S2 can be the fibres over a circle of latitude, that forms a torus in S3. Projecting multiple circles over different latitudes can produce multiple toruses in R3 can fill space. The torus itself is not constructed by horizontal slices layering the torus, nor is it constructed by arranging vertical circles arranged around in a ring shape (the torus of revolution). The torus in the HF made from the circles over a latitude on S2, where each individual fibre map to a linking villarceau circles.

Figure 2 is an example of the above description. Each torus represents a circular path where 0 ≤ θ ≤ 2π, sweeping out a circle that lies parallel to the x z plane.

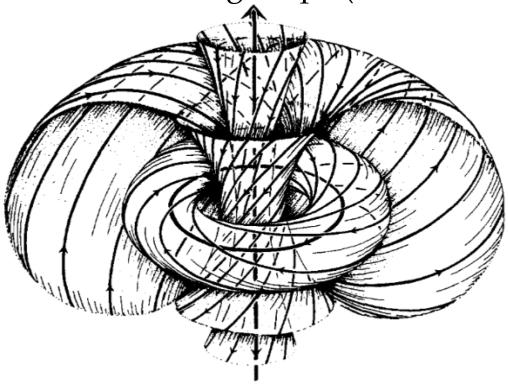


Figure 2

Parametrisation of fibres

The approaches taken to parameterise the fibres will deal primarily with the S2 and S3, the unit spheres from R3 and R4.

They are defined as follows:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

$$S^3 = \{(x, y, z) \in \mathbb{R}^4 : a^2 + b^2 + c^2 + d^2 = 1\}$$

The unit sphere is defined as all points 1 unit from the central point. The Base points are also unit points, if not they must be normalised.

There are a couple of different approaches that can be taken to achieve the HF mapping. The algebra of quaternions, relates the S3 to the S2 which is embedded in R3. Another approach is by using C2, which is defined as being in R4, drawing a circle in C2 equates the sphere S3 in R4.

Interpretation using Quaternions

By using S3 of unit quaternions to rotate S2 we can create the HF. This means we must define a map S3 to S2. We start by taking a unit quaternion which means $||q|| = 1$ such that the quaternion is an element of S3. Quaternion is equivalent to a rotation matrix which the HF is extracted.

Quaternions are an extension of complex numbers, where *w, x, y, z* are real numbers and *i, j* and *k* are imaginary.

A fibre over the point (*a, b, c*) in S2 will be determined a rotation qθ where $0 \leq \theta \leq 2\pi$ this will draw a circle for each rotation of θ so long as (a, b, c) is not (1, 0, 0) or (-1, 0, 0).

To plot the 3-sphere as a quaternion with the base (*a, b, c*) and rotation qθ, you can use the function:

$$\frac{1}{\sqrt{2(1+\mathbf{c})}}.\left((1+\mathbf{c})cos(\theta),\ \mathbf{a}sin(\theta) - \mathbf{b}cos(\theta), \mathbf{a}cos(\theta) + \mathbf{b}sin(\theta), (1+\mathbf{c}).sin(\theta)\right)$$

This will produce a quaternions in the form:

$$q = (w, x, y, z) = w + xi + yj + zk$$

Interpretation of 3-Sphere in C2

The real coordinates of S3 in R4 will be:

$$x_1{}^2 + y_1{}^2 + x_2{}^2 + y_2{}^2 = 1$$

But it can also be thought of as $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$ being complex numbers. So we can say that the S3 is the pair of complex numbers(z_2, z_1). This means that S3 is regarded as the unit sphere in a complex dimension 2, which can be analysed as a circle in a complex plane.

For the equation $z_2 = a.z_1$ *a* defines a complex number such that a line of the form of the equation meets the S3 on a circle and so there is a circle for each complex line from the complex number *a*, the circle will be the fibre.

Stereographic projection

After parameterising the points to a collection of quaternions, the 4 dimensional sphere must be projected to R3. This follows the same principle as projection S2 to R2 space.

Projection of S2 to R2 can be described with a ball in R3 and a light source. By placing the light source above the ball you will get a shadow on the floor. Similarly you can project a line from the north of the S2 to the projection plane which is n-1 dimension. This can mean that the projection will preserve the circle but the size will be deformed to an extent. Though it is still recognisable as a point in this case.

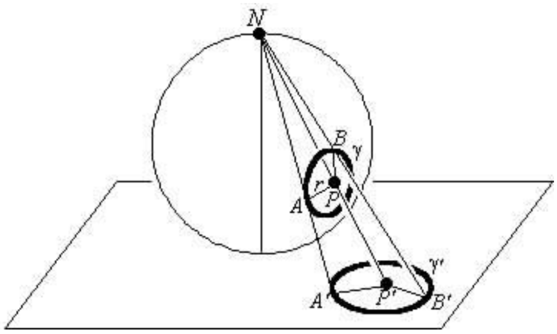


Figure 3

This is applied to the S3 to project it onto a R3 plane, the point will be projected as a circle but will not preserve the size, though it is recognisable as a circle. This can be done using the projection:

$$(w, x, y, z) \mapsto x^2 \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w}\right)$$

Displays

Since the HF is a fibration that belongs to R3, stereographic projection will help to 'get inside' the fibration to an extent. Stereographic projection will help achieve the HF is displayed in 3D. This also means exploring different applications that may support this and how I must build my project so that it can be integrated to work on different 3D displays.

A 3D TV is a good example of a platform that could host the visualisation of the HF. During this semester I have investigated into how to project to a 3D display and compatibility with Unity projects. I have also been involved in an experiment which used a 3D TV to test stereo comfort, by showing me (the subject) different stereographic 3D images. Doing this experiment made me understand what types of problems can occur to cause discomfort to a user and also what is very comfortable to look at.

The development in Unity3D mainly aims, but not only at the Oculus Rift. Virtual reality has become a fast and growing trend in the software area. This project hopes to use this trend in mathematical areas such as topology where virtual shapes can be made in order to study and admire them and this project aims to achieve this by rendering the HF, a fundamental object in topology.

Colour

The colour of the HF can vary from fibre to fibre. But the colour can represent different features of the HF. The colour must have a reference to something related to the HF, one example of many can be the latitude of the base points that were collected on the two-sphere.

Colour is an interactive feature that the user should be able to change. The meaning of what the colour represents should also be explicit to the user. This feature should allow the user to change the colour reference to their preference.

Various colourisation of the HF will increase the interactivity of the program. It will also help users understand the links between the S1, S2, S3 and HF. So far the colours represent either the latitude or the longitude of the base points.

1.2 Aims and objectives

The problem of visualising a higher dimensional object is hard to achieve as we live in a space with 3 spacial dimensions. Four special dimensions is the next step. The program should help the user understand more about the four dimensional sphere and hopefully give them an idea of what it may look like.

To develop software that is able to project the HF and help visualise one of the most important fibrations in topology in order to solve the problem of visualising and understand better, the four dimensional sphere in a three dimensional space.

This can be used as an introductory to topology for a mathematician. The HF is one of the fundamental objects in topology. Circles, spheres and tori are among the simplest objects that are studied. A topologist will try to understand the connection between these object, which is what the program will also try to help create.

Objectives

- To undertake research on the four dimensional sphere so that we can draw a part of it in a three dimensional space.
- Build a system that is capable to render the HF to a 3D display.
- Implement the program so that it can interact with the user and use the capabilities and functionality of a 3D display. The user must be able to interact with the shape and be able to look at the fibration in detail.
- Incorporate any other hardware that may seem compatible to improve the comfort and experience of the end user.
- Design and implement a system that can render the HF efficiently by using any design patterns necessary.
- Investigate the user experience of the 3D display from any user experiments and assess the data so that it can be used to improve their experience.
- Design a feature that will let the user explore many different paths on S2 and render what it would look like on S3.

1.3 Risk Assessment

Description of Risk	Description of impact	Likelihood rating	Impact rating	Preventative actions
The project requirements are too ambiguous	Can result in producing software that doesn't align with the specifications	Low	High	Review specifications with supervisor to always check if the project is going on the right tracks.
Requirements are incomplete	Can lead to confusion when implementing software, cause software to be implemented incorrectly	Low	Medium	Requirements for my project are straightforward, but it is always good practise to evaluate requirements.
System outages	Critical systems go down or corrupt data. Can be caused by power cuts, fire, flooding or any other damage corrupting data.	Low	High	Keep backup of program on an external drive in case of any physical or software corruption to original program.
Components lack of stability	Display or laptop crashing	Medium	High	Make sure the software is compatible with the Oculus rift or 3D display. Check versions and stay updated on display firmware
Failure to integrate external components	Not being able to use any hardware, hinders the interactivity the program will have.	Low	High	Using software such as Unity3D will ensure compatibility with hardware components such as Oculus Rift and Controllers
Design lacks flexibility	If the software is designed poorly it can be difficult to make changes	Medium	Medium	Design using UML diagrams to ensure flexibility and feasibility of a task.

Table continued

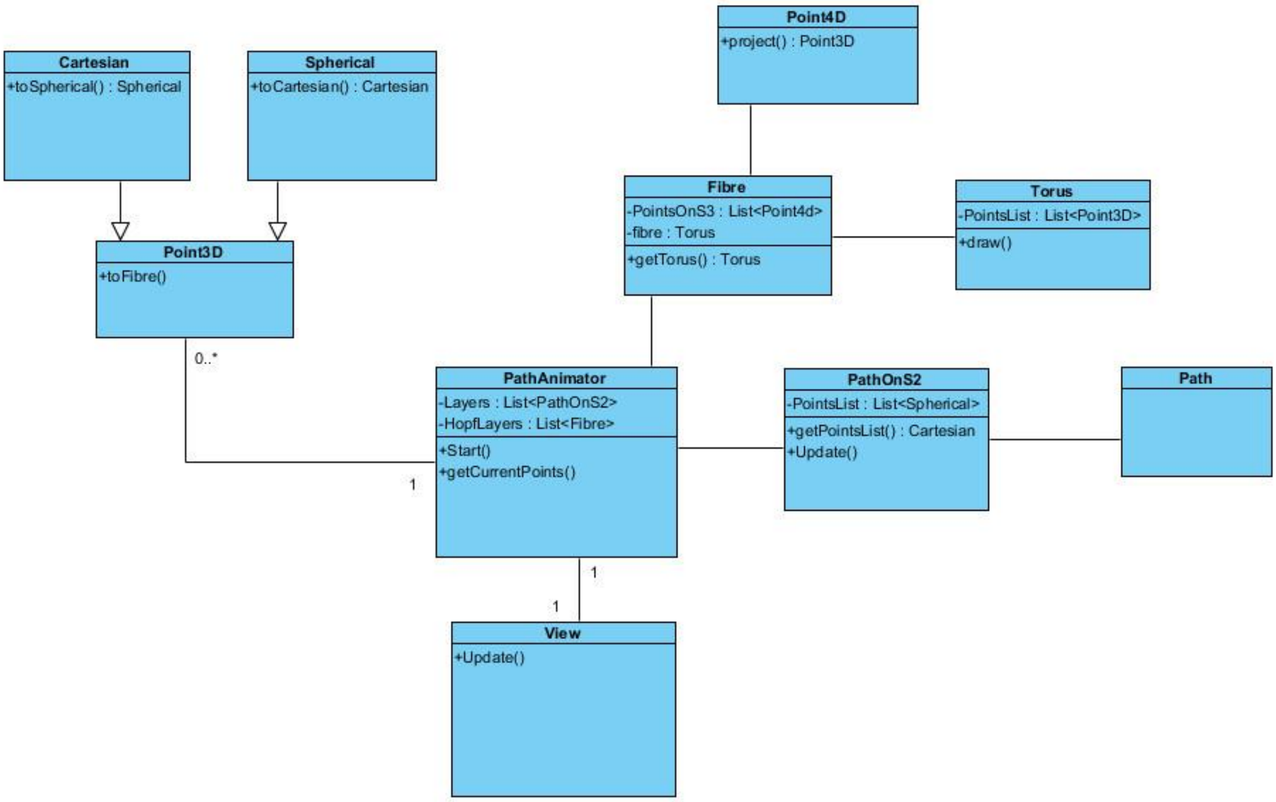
Description of Risk	Description of impact	Likelihood rating	Impact rating	Preventative actions
Trailing wires likely to cause trips	Can cause anyone, not only user to trip, that is in the room with trailing wires	High	High	Wires to be covered where possible, appropriately by cable management equipment. Where trailing wires are unavoidable, they must be identified to any personnel entering the room.
Occlusion of vision when using Oculus Rift	The user will not be able to see any obstructions and obstacles in their surroundings. Can cause injury.	Medium	High	Assess the room and space before using the Oculus Rift, making sure the environment is suitable. The Oculus Rift is tethered to a computer therefore it must be used seated.
Prolonged use may cause nausea and dizziness	Use of the Oculus rift can potentially cause the user to get dizzy, blurred vision or any other adverse effects caused by the Oculus Rift or other 3D displays	High	High	All of the users must be advised of the potential adverse effects that occur, before use. If any of the problems do occur; eye strain, dizziness, blurred vision or nausea then stop use of system immediately. User is advised to use the 3D device for no longer than 15 minutes without a break.

2.1 Design and Implementation

Class Diagram

I have produced a class diagram which shows the structure of my design to develop the previously described in chapter 1. The HF is composed of Base points which the class `PathsOnS2` will compute over a collection of `Point3D` objects which have the actual type `Spherical`. This collection will hold as a path on the two-sphere. The `PathAnimator` will hold a collection of objects that are of type `PathOnS2`. A collection will help to distinguish which collection of points belongs to which set of paths. The animator class also holds a collection called `HopfLayers`; this holds a collection of `Fibre` objects, which hold a collection four dimensional points which is also referred to as quaternions in the HF construction.

This class diagram will design the HF so that the path on the 2-sphere will be able to update and display an animated visualisation of the HF. The next step is to enhance the features of this design by inhabiting the features of interactivity with the object. To be able to control the viewing aspect (navigation, zoom and other user controls) and to control the HF's colour scheme, the rotation of the HF (animation) and the paths being collected on S2



Implementation

Over the first semester I have developed a prototype of what the system skeleton may look like. The picture on the right shows the output of the prototype, designed in Unity3D. This is the HF from the path of 5 sets of circles around the latitude of the S2. This creates what looks like figure 2. The colour scheme in that is used in this is the colour of fibres from the same points of latitude. This was achieved by giving each layer of points 1/numberOfLayers * LayerNumber equating to the hue value. Where the colour is represented as HSV. The program will need to improve the time taken to render this drawing, and it should incorporate interactive features.

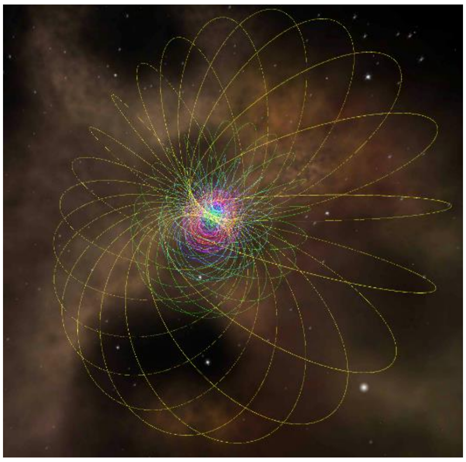


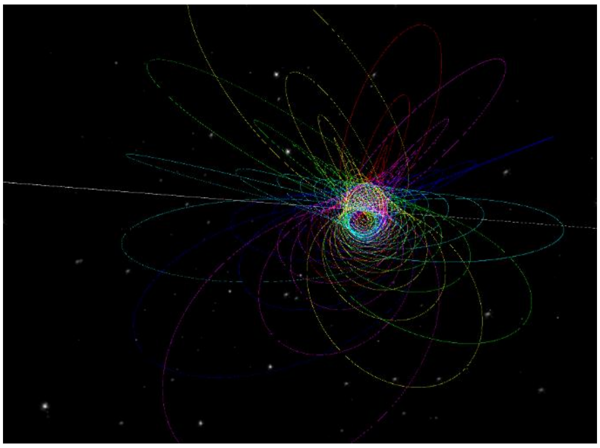
Figure 4

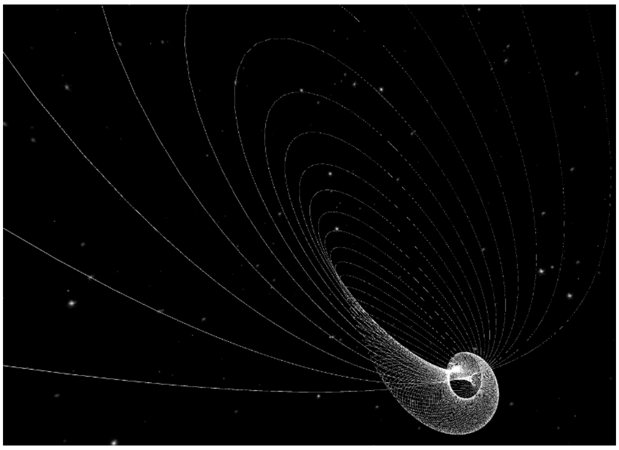
The program is capable of drawing any points from the 2-sphere, this is the result of drawing a set of circles along the longitude on S2 and the spiral points. In this picture you can also see the circle that looks like a straight line. The stereographic projection projects this point to what looks an infinite line.

5 sets of points on the S2, such that each set contains the set of points of a circle drawn with the longitude of S2 positioned equally apart from each other.

The second image represents a spiral from the top to the bottom of S2

For the next stage of my program I will develop the interactivity of the object with the user on a 3D display. This will be achieved by evolving the animation of the object relevant to the user.





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