A Study Note on Quantified Modal Logic

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Abstract. This is a study note, I do not claim any originality on any of the results presented in this note. In this note, I try to clarify my confusions on two styles of quantified modal logics, namely, one with Carnap-Barcan formula and the other without it.

1 Motivation

The formalization of quantified modal logics in Hughes and Cresswell's book [1] is enough to enable philosophical discssions upon these systems. Sometimes we might want more than that, for example, one might want to see their proof theoretical formalization. There should be a hope to achieve this once we understand the axiomatic formalization better.

According to what I hear in the class, in order to embrace Kripke's semantics, both the notions of validity and underlying first order logic need to be revised. I would like to see clearly what kinds of changes that we really need and why we need these changes. I also like to see how exactly free logic come in to play in this process.

Terminology: I will use *system* $\mathfrak{B}$ to mean the quantified modal logic with Carnap-Barcan formula as axiom; I will use *system* $\mathfrak{K}$ to mean the quantified modal logic in which Carnap-Barcan formula is not valid. LPC is short for lower predicate logic. PC is short for proposition logic. S is short for modal propositional logic.

2 System $\mathfrak{B}$

In this section, we first characterize LPC. We will extend it to $\mathfrak{B}$. Then we will investigate the semantics of it.

2.1 LPC

Syntax For simplicity, I will use Backus Naur form(BNF) to characterize the well-formed formula of LPC:

$\alpha, \beta, \gamma ::= p \mid \phi(x_1, ..., x_n) \mid \neg\alpha \mid \alpha \vee \beta \mid \forall x\alpha$

The notation above means: α, β denote well-formed formula. $::=$ means syntactic characterization. $\mid$ is used just for the separation of different syntactical elements. p is called proposition variable, which is a wff. $\phi(x_1, ..., x_n)$ is a wff, where ϕ is a predicate with arity $n > 0$ and $x_1, ..., x_n$ are variable. If α is a wff, then $\neg\alpha$ is a wff. If α, β is wff, then $\alpha \vee \beta$ is a wff. If α is wff, then $\forall x\alpha$ is wff, where x is a bounded variable. We also have following short hand:

$\alpha \to \beta =_{def} \neg\alpha \vee \beta$

$\alpha \wedge \beta =_{def} \neg\alpha \vee \neg\beta$

$\alpha \equiv \beta =_{def} (\alpha \to \beta) \wedge (\beta \to \alpha)$

$\exists x\alpha =_{def} \neg\forall x\neg\alpha$

Axiomazation $\vdash_{LPC} \alpha$ denotes α is derivable from the axioms and the inference rules of LPC.

$$\frac{}{\vdash_{LPC} \alpha \to (\alpha \wedge \alpha)}\ PC1$$

$$\frac{}{\vdash_{LPC} (\alpha \wedge \beta) \to \alpha}\ PC2$$

$$\frac{}{\vdash_{LPC} (\alpha \to \beta) \to (\neg(\beta \wedge \gamma) \to \neg(\gamma \wedge \alpha))}\ PC3$$

$$\frac{}{\vdash_{LPC} \forall x\alpha \to [y/x]\alpha}\ UI$$

$$\frac{\vdash_{LPC} \alpha \to \beta \quad x\ is\ not\ free\ in\ \alpha}{\vdash_{LPC} \alpha \to \forall x\beta}\ UG$$

$$\frac{\vdash_{LPC} \alpha \to \beta \quad \vdash_{LPC} \alpha}{\vdash_{LPC} \beta}\ MP$$

2.2 System $\mathfrak{B}$

Syntax The wff of $\mathfrak{B}$ is an extension of LPC:

$\alpha, \beta ::= p \mid \phi(x_1, ..., x_n) \mid \neg\alpha \mid \alpha \vee \beta \mid \forall x\alpha \mid L\alpha$

Axiomazation $\vdash_{\mathfrak{B}} \alpha$ denotes α is derivable from the axioms and the inference rules of $\mathfrak{B}$.

$$\frac{}{\vdash_{\mathfrak{B}} \alpha \to (\alpha \wedge \alpha)}\ PC1$$

$$\frac{}{\vdash_{\mathfrak{B}} (\alpha \wedge \beta) \to \alpha}\ PC2$$

$$\frac{}{\vdash_{\mathfrak{B}} (\alpha \to \beta) \to (\neg(\beta \wedge \gamma) \to \neg(\gamma \wedge \alpha))}\ PC3$$

$$\frac{}{\vdash_{\mathfrak{B}} L(\alpha \to \beta) \to (L\alpha \to L\beta)}\ K$$

$$\frac{}{\vdash_{\mathfrak{B}} \forall xL\alpha \to L\forall x\alpha}\ BF$$

$$\frac{}{\vdash_{\mathfrak{B}} \forall x\alpha \to [y/x]\alpha}\ UI$$

$$\frac{\vdash_{\mathfrak{B}} \alpha \to \beta \quad x\ is\ not\ free\ in\ \alpha}{\vdash_{\mathfrak{B}} \alpha \to \forall x\beta}\ UG$$

$$\frac{\vdash_{\mathfrak{B}} \alpha \to \beta \quad \vdash_{\mathfrak{B}} \alpha}{\vdash_{\mathfrak{B}} \beta}\ MP$$

$$\frac{\vdash_{\mathfrak{B}} \alpha}{\vdash_{\mathfrak{B}} L\alpha}\ N$$

2.3 Semantics of $\mathfrak{B}$ and Soundness

Definition 1. *A $\mathfrak{B}$-Model is a quadruple $\langle W, R, D, V\rangle$, where W is a set of worlds, R is a relation on W, D is a set of domain and V is a function such that, where ϕ is an n-place predicate, $V(\phi) \subseteq D \times D \times ... \times D \times W$. Specially, $V(p) \subseteq W$.*

Definition 2 (Barcan Model). *Let μ be an assignment that map each individual variable to an element in D, namely $\mu(x) \in D$. Let $\mu[a/x]$ be an assignment just like μ, except it assign a to x, where $a \in D$. We define the interpretation of a well-formed formula α in a given world w under the assignment μ–$[\![\alpha]\!]^w_\mu \in \{0,1\}$ as follows:*

$[\![p]\!]^w_\mu = 1$ *iff* $w \in V(p)$.

$[\![\phi(x_1, ..., x_n)]\!]^w_\mu = 1$ *iff* $(\mu(x_1), ..., \mu(x_n), w) \in V(\phi)$.

$[\![\neg\alpha]\!]^w_\mu = 1$ *iff* $[\![\alpha]\!]^w_\mu = 0$.

$[\![\alpha \vee \beta]\!]^w_\mu = 1$ *iff* $[\![\alpha]\!]^w_\mu = 1$ *or* $[\![\beta]\!]^w_\mu = 1$.

$[\![L\alpha]\!]^w_\mu = 1$ *iff* $[\![\alpha]\!]^{w'}_\mu = 1$ *for any* w' *such that* wRw'.

$[\![\forall x\alpha]\!]^w_\mu = 1$ *iff* $[\![\alpha]\!]^w_{\mu[a/x]} = 1$ *for any* $a \in D$.

Definition 3. *We say a wff α is valid in a Barcan model $\langle W, R, D, V\rangle$ iff $[\![\alpha]\!]^w_\mu = 1$ for any $w \in W$ and any assignment μ.*

We say a wff α is valid on a frame $\langle W, R\rangle$ iff α is valid in every Barcan model based on $\langle W, R\rangle$.

Theorem 1. *$\forall xL\phi(x) \to L\forall x\phi(x)$ is valid on a frame $\langle W, R\rangle$.*

Proof. If $\forall xL\phi(x) \to L\forall x\phi(x)$ is not valid on the frame, then there exist a Barcan model $\langle W, R, D, V\rangle$, in which there is a world w and an assignment μ, such that $[\![\forall xL\phi(x) \to L\forall x\phi(x)]\!]^w_\mu = 0$. By definition of $[\![\forall xL\phi(x) \to L\forall x\phi(x)]\!]^w_\mu$, we have $[\![\forall xL\phi(x)]\!]^w_\mu = 1$ and $[\![L\forall x\phi(x)]\!]^w_\mu = 0$. $[\![L\forall x\phi(x)]\!]^w_\mu = 0$ implies $\exists w', wRw', [\![\forall x\phi(x)]\!]^{w'}_\mu = 0$, thus we have $\exists a \in D, [\![\phi(x)]\!]^{w'}_{\mu[a/x]} = 0$. Now we consider $[\![\forall xL\phi(x)]\!]^w_\mu = 1$. This implies for the exact a in $[\![\phi(x)]\!]^{w'}_{\mu[a/x]} = 0$, we have $[\![L\phi(x)]\!]^w_{\mu[a/x]} = 1$. Thus we have $[\![\phi(x)]\!]^{w'}_{\mu[a/x]} = 1$. So we reach a contradiction.

Notice this theorem holds due the the way we define $[\![\forall x\alpha]\!]^w_\mu$ in definition 2.

Theorem 2 (Soundness). *If $\vdash_{\mathfrak{B}} \alpha$, then α is valid on a frame $\langle W, R\rangle$.*

We can show this theorem by the same method in the proof of last theorem.

3 System $\mathfrak{K}$

Both Carnap-Barcan and converse Carnap-Barcan formula assume there is some connection of objects in different worlds. We want to abandon this kind of assumption. We will first develop a semantics to capture the idea of abandoning Carnap-Barcan or its converse, then we try to formalize a axiomatic system to capture this semantics.

3.1 Toward Semantics of $\mathfrak{K}$

Definition 4. *A $\mathfrak{K}$-Model is a quintuple $\langle W, R, D, Q, V\rangle$, where W is a set of worlds, R is a relation on W, D is a set of domain, Q is a function from members of W to subsets of D, namely: $W \to \mathcal{P}(D)$, and V is a function such that, where ϕ is an n-place predicate, $V(\phi) \subseteq D \times D \times ... \times D \times W$. Specially, $V(p) \subseteq W$.*

Definition 5. *We define the interpretation of a well-formed formula α in a given world w under the assignment μ–$[\![\alpha]\!]_\mu^w \in \{0,1\}$ and $\mu : FV(\alpha) \to Q(w)$. We have:*

$[\![p]\!]_\mu^w = 1$ *iff* $w \in V(p)$.

$[\![\phi(x_1, ..., x_n)]\!]_\mu^w = 1$ *iff* $(\mu(x_1), ..., \mu(x_n), w) \in V(\phi)$.

$[\![\neg\alpha]\!]_\mu^w = 1$ *iff* $[\![\alpha]\!]_\mu^w = 0$.

$[\![\alpha \vee \beta]\!]_\mu^w = 1$ *iff* $[\![\alpha]\!]_\mu^w = 1$ *or* $[\![\beta]\!]_\mu^w = 1$.

$[\![L\alpha]\!]_\mu^w = 1$ *iff* $[\![\alpha]\!]_\mu^{w'} = 1$ *for any* w' *such that* wRw'.

$[\![\forall x\alpha]\!]_\mu^w = 1$ *iff* $[\![\alpha]\!]_{\mu[a/x]}^w = 1$ *for any* $a \in Q(w)$.

We notice that for given $[\![\alpha]\!]_\mu^w$, the assignment μ is closely related to w, i.e. the range of μ-$ran(\mu)$ is a subset of $Q(w)$. A problem could arise when we define: $[\![L\alpha]\!]_\mu^w = 1$ iff $[\![\alpha]\!]_\mu^{w'} = 1$ for any w' such that wRw'. It may happen that $ran(\mu) \cap Q(w') = \emptyset$. In this sense, $[\![L\alpha]\!]_\mu^w$ is not well-defined. Thus definition 5 is not giving us a well-defined semantics.

One way to deal with this problem is to again change the restriction of μ, namely, we don't require the range of μ is a subset of $Q(w)$, but let $ran(\mu)$ to be a subset of the union of all the objects of the ancestry worlds of w, denoted by $Q^+(w)$. So if $a \in Q^+(w)$, then $a \in Q^+(w_1)$ and w_1Rw. Thus we have following new definition:

Definition 6 (Closure Model). *We define the interpretation of a well-formed formula α in a given world w under the assignment μ–$[\![\alpha]\!]_\mu^w \in \{0,1\}$ and $\mu : FV(\alpha) \to Q^+(w)$. We have:*

$[\![p]\!]_\mu^w = 1$ *iff* $w \in V(p)$.

$[\![\phi(x_1, ..., x_n)]\!]_\mu^w = 1$ *iff* $(\mu(x_1), ..., \mu(x_n), w) \in V(\phi)$.

$[\![\neg\alpha]\!]_\mu^w = 1$ *iff* $[\![\alpha]\!]_\mu^w = 0$.

$[\![\alpha \vee \beta]\!]_\mu^w = 1$ *iff* $[\![\alpha]\!]_\mu^w = 1$ *or* $[\![\beta]\!]_\mu^w = 1$.

$[\![L\alpha]\!]_\mu^w = 1$ *iff* $[\![\alpha]\!]_\mu^{w'} = 1$ *for any* w' *such that* wRw'.

$[\![\forall x\alpha]\!]_\mu^w = 1$ *iff* $[\![\alpha]\!]_{\mu[a/x]}^w = 1$ *for any* $a \in Q^+(w)$.

Now again we look at the definition of $[\![L\alpha]\!]_\mu^w$. We know that $ran(\mu) \subseteq Q^+(w)$. For $[\![\alpha]\!]_\mu^{w'}$, we require that $ran(\mu) \subseteq Q^+(w')$. Since $Q^+(w) \subseteq Q^+(w')$, we know $[\![\alpha]\!]_\mu^{w'}$ is well-defined.

Definition 7. *We say a wff α is valid in a closure model $\langle W, R, D, Q, V\rangle$ iff $[\![\alpha]\!]_\mu^w = 1$ for any $w \in W$ and any assignment μ under the definition 6.*

Theorem 3. *Converse Carnap-Barcan formula $L\forall x\phi(x) \to \forall xL\phi(x)$ is valid in the closure model $\langle W, R, D, Q, V\rangle$.*

Proof. Assume $\exists w \in W, \mu, [\![L\forall x\phi(x) \to \forall xL\phi(x)]\!]_\mu^w = 0$. Thus we have $[\![L\forall x\phi(x)]\!]_\mu^w = 1$ and $[\![\forall xL\phi(x)]\!]_\mu^w = 0$. $[\![L\forall x\phi(x)]\!]_\mu^w = 1$ implies $[\![\forall x\phi(x)]\!]_\mu^{w'} = 1$ for any w', wRw'. $[\![\forall x\phi(x)]\!]_\mu^{w'} = 1$ implies $[\![\phi(x)]\!]_{\mu[a/x]}^{w'} = 1$ for any $a \in Q^+(w')$. $[\![\forall xL\phi(x)]\!]_\mu^w = 0$ implies $\exists b \in Q^+(w) \subseteq Q^+(w'), [\![L\phi(x)]\!]_{\mu[b/x]}^w = 0$. $[\![L\phi(x)]\!]_{\mu[b/x]}^w = 0$ implies $[\![\phi(x)]\!]_{\mu[b/x]}^{w'} = 0$. This contradicts the fact that $[\![\phi(x)]\!]_{\mu[a/x]}^{w'} = 1$ for any $a \in Q^+(w')$.

We can see that the closure model doesn't give us a semantics that invalids converse Carnap-Barcan formula. The reason is that the closure model doesn't really abandon the notion of inclusion requirement between two related worlds.

3.2 System $\mathfrak{K}$

We want to reject converse Carnap-Barcan formula as well. One way to approach this it is change the range of μ to D in definition 5 instead of $Q^+(w)$. But $\forall x\phi(x) \to \phi(y)$ could be invalid: Let $[\![\forall x\phi(x) \to \phi(y)]\!]^w_{[d/y]} = 0$, where $d \notin Q(w)$. Thus $[\![\forall x\phi(x)]\!]^w_{[d/y]} = 1$ and $[\![\phi(y)]\!]^w_{[d/y]} = 0$. $[\![\forall x\phi(x)]\!]^{w'}_{[d/y]} = 1$ implies $[\![\phi(x)]\!]^w_{[a/x,d/y]} = 1$ for any $a \in Q(w)$. Since $d \notin Q(w)$, we could not reach any contradiction here.

Invalidating $\forall x\phi(x) \to \phi(y)$ seems reasonable in modal context, we admit that it is indeed possible for y to refer to something that is not exist in the current world, while quantifiers are restricted to the objects exist in current world. Thus we introduce the primitive predicate $E(x)$ to mean x must refer to the objects in the current world. So instead of trying to validate $\forall x\phi(x) \to \phi(y)$, we can validate $\forall x\phi(x) \wedge E(y) \to \phi(y)$.

So that is how free logic come into play here. Instead of adapting classic lower predicate logic's axioms, we will now use free logic's axioms. First we characterize system $\mathfrak{K}$, which reflects the idea we just discussed. Then we will present the semantics of $\mathfrak{K}$.

3.3 Axiomatization

$\alpha, \beta ::= p \mid \phi(x_1, ..., x_n) \mid E(x) \mid \neg\alpha \mid \alpha \vee \beta \mid \forall x\alpha \mid L\alpha$

axiomatization $\vdash_{\mathfrak{K}} \alpha$ denotes α is derivable from the axioms and the inference rules of $\mathfrak{B}$.

$$\frac{}{\vdash_{\mathfrak{K}} \alpha \to (\alpha \wedge \alpha)}\ PC1$$

$$\frac{}{\vdash_{\mathfrak{K}} (\alpha \wedge \beta) \to \alpha}\ PC2$$

$$\frac{}{\vdash_{\mathfrak{K}} (\alpha \to \beta) \to (\neg(\beta \wedge \gamma) \to \neg(\gamma \wedge \alpha))}\ PC3$$

$$\frac{}{\vdash_{\mathfrak{K}} \forall x E(x)}\ UE$$

$$\frac{}{\vdash_{\mathfrak{K}} (\forall x\alpha \wedge E(y)) \to [y/x]\alpha}\ UIE$$

$$\frac{}{\vdash_{\mathfrak{K}} \forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)}\ DB$$

$$\frac{x \notin FV(\alpha)}{\vdash_{\mathfrak{K}} \alpha \equiv \forall x\alpha}\ VQ$$

$$\frac{}{\vdash_{\mathfrak{K}} L(\alpha \to \beta) \to (L\alpha \to L\beta)}\ K$$

$$\frac{\vdash_{\mathfrak{K}} \alpha}{\vdash_{\mathfrak{K}} \alpha \to \forall x\alpha}\ UG$$

$$\frac{\vdash_{\mathfrak{K}} \alpha \to \beta \quad \vdash_{\mathfrak{K}} \alpha}{\vdash_{\mathfrak{K}} \beta}\ MP$$

$$\frac{\vdash_{\mathfrak{K}} \alpha}{\vdash_{\mathfrak{K}} L\alpha}\ N$$

Semantics of $\mathfrak{K}$

Definition 8 ($\mathfrak{K}$ Model). *We define the interpretation of a well-formed formula α in a given world w under the assignment μ–$[\![\alpha]\!]^w_\mu \in \{0,1\}$ and $\mu : FV(\alpha) \to D$. We have:*

$[\![p]\!]^w_\mu = 1$ *iff* $w \in V(p)$.

$[\![E(x)]\!]^w_\mu = 1$ *iff* $(\mu(x), w) \in V(E)$ *iff* $\mu(x) \in Q(w)$.

$[\![\phi(x_1, ..., x_n)]\!]^w_\mu = 1$ *iff* $(\mu(x_1), ..., \mu(x_n), w) \in V(\phi)$.

$[\![\neg\alpha]\!]^w_\mu = 1$ *iff* $[\![\alpha]\!]^w_\mu = 0$.

$[\![\alpha \vee \beta]\!]^w_\mu = 1$ *iff* $[\![\alpha]\!]^w_\mu = 1$ *or* $[\![\beta]\!]^w_\mu = 1$.

$[\![L\alpha]\!]^w_\mu = 1$ *iff* $[\![\alpha]\!]^{w'}_\mu = 1$ *for any* w' *such that* wRw'.

$[\![\forall x\alpha]\!]^w_\mu = 1$ *iff* $[\![\alpha]\!]^w_{\mu[a/x]} = 1$ *for any* $a \in Q(w)$.

Definition 9. *We say a wff* α *is valid in a* $\mathfrak{K}$ *model* $\langle W, R, D, Q, V\rangle$ *iff* $[\![\alpha]\!]^w_\mu = 1$ *for any* $w \in W$ *and any assignment* μ *under the definition 8.*

Theorem 4 (Soundness). *If* $\vdash_{\mathfrak{K}} \alpha$*, then* α *is valid in any* $\mathfrak{K}$ *model.*

It would be interesting to give a proof of this soundness theorem. The method is somewhat similar to tableau method.

4 Conclusion

In this note, I use the interpretational style to formalize the semantics of two different quantified modal logic, namely, $\mathfrak{B}$ and $\mathfrak{K}$. I use Hilbert style of axiomatization [2]. It would be interesting to develop Gentzen style of deduction system for quantified modal logic.

In the process of writing this note, I think I clarify some of the confusions that I had before. For instance, what kind of situation will arise when we eliminate Carnap-Barcan formula, how to abandon its converse verion and how the free logic come in this process.

References

1. G. E. Hughes and M. J. Cresswell. *A New Introduction to Modal Logic.* Routledge, September 1996.
2. A. S. Troelstra and H. Schwichtenberg. *Basic proof theory.* Cambridge University Press, New York, NY, USA, 1996.