Confluence of Lambda Calculus Modulo μ-Equivalence

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1 Unlabelled Syntax

Definition 1 (Unlabelled Terms).

$t ::= x \mid \lambda x.t \mid tt' \mid \mu t$

I use Λ to denote the set of all unlabelled terms.

Definition 2 (Substitution).

$[t'/x]x := t'$

$[t'/x]y := y$

$[t'/x]\lambda y.t := \lambda y.[t'/x]t$

$[t'/x](t_1 t_2) := [t'/x]t_1[t'/x]t_2$

$[t'/x]\mu t := \mu([t'/x]t)$

Definition 3 (Beta-Reductions).

$$\frac{}{(\lambda x.t)t' \to_\beta [t'/x]t}$$

$$\frac{t \to_\beta t'}{\lambda x.t \to_\beta \lambda x.t'}$$

$$\frac{t' \to_\beta t''}{tt' \to_\beta tt''}$$

$$\frac{t \to_\beta t''}{tt' \to_\beta t''t'}$$

$$\frac{t \to_\beta t'}{\mu t \to_\beta \mu t'}$$

Definition 4 (Mu-Reductions).

$$\frac{}{\mu(\lambda x.t) \to_\mu \lambda x.\mu t}$$

$$\frac{}{\mu(t_1 t_2) \to_\mu (\mu t_1)(\mu t_2)}$$

$$\frac{}{\mu(\mu t) \to_\mu \mu t}$$

$$\frac{t \to_\mu t'}{\lambda x.t \to_\mu \lambda x.t'}$$

$$\frac{t' \to_\mu t''}{tt' \to_\mu tt''}$$

$$\frac{t \to_\mu t''}{tt' \to_\mu t''t'}$$

$$\frac{t \to_\mu t'}{\mu t \to_\mu \mu t'}$$

2 Labelled Terms

Definition 5 (Labelled Terms).

$$t \ ::= \ x \mid \lambda x.t \mid tt' \mid \mu t \mid (\underline{\lambda} x.t)t' \mid \underline{\mu}\mu t \mid \underline{\mu}(tt') \mid \underline{\mu}(\lambda x.t)$$

Definition 6 (Substitution).

$$[t'/x]x \ := \ t'$$

$$[t'/x]y \ := \ y$$

$$[t'/x]\lambda y.t \ := \ \lambda y.[t'/x]t$$

$$[t'/x](t_1 t_2) \ := \ [t'/x]t_1[t'/x]t_2$$

$$[t'/x]\mu t \ := \mu([t'/x]t)$$

$$[t'/x](\underline{\lambda} y.t_1)t_2 \ := (\underline{\lambda} y.[t'/x]t_1)[t'/x]t_2$$

$$[t'/x]\underline{\mu}\mu t \ := \underline{\mu}\mu([t'/x]t)$$

$$[t'/x]\underline{\mu}(t_1 t_2) \ := \underline{\mu}([t'/x]t_1[t'/x]t_2)$$

$$[t'/x]\underline{\mu}(\lambda y.t) \ := \underline{\mu}(\lambda y.[t'/x]t)$$

Notice:1. $\underline{\lambda} x.t$ is not a well-formed labelled term, but $(\underline{\lambda} x.t)t'$ is a well-formed labelled term. For the same reason, $\underline{\mu} t$ is not a well-formed term.

2. It turns out it is not that trivial to extend the notion of reductions to the labelled terms. Some efforts have to be taken to make sure the well-formness of the labelled terms after the reductions. For example, we cannot simply allow this rule:

$$\frac{t \to_{\underline{\beta}} t'}{\underline{\mu} t \to_{\underline{\beta}} \underline{\mu} t'}$$

Definition 7 (Labelled Beta-Reductions).

$$\frac{}{(\underline{\lambda} x.t)t' \to_{\underline{\beta}} [t'/x]t}$$

$$\frac{}{(\underline{\lambda} x.t)t' \to_{\underline{\beta}} [t'/x]t}$$

$$\frac{t \to_{\underline{\beta}} t'}{\lambda x.t \to_{\underline{\beta}} \lambda x.t'}$$

$$\frac{t \to_{\underline{\beta}} t''}{tt' \to_{\underline{\beta}} t''t'}$$

$$\frac{t' \to_{\underline{\beta}} t''}{tt' \to_{\underline{\beta}} tt''}$$

$$\frac{t \to_{\underline{\beta}} t'}{\mu t \to_{\underline{\beta}} \mu t'}$$

$$\frac{u \to_{\underline{\beta}} u'}{(\underline{\lambda} x.u)t' \to_{\underline{\beta}} (\underline{\lambda} x.u')t'}$$

$$\frac{t' \to_{\underline{\beta}} t''}{(\underline{\lambda} x.u)t' \to_{\underline{\beta}} (\underline{\lambda} x.u)t''}$$

$$\frac{t \to_{\underline{\beta}} t'}{\underline{\mu}\mu t \to_{\underline{\beta}} \underline{\mu}\mu t'}$$

$$\frac{t \to_{\underline{\beta}} t'}{\underline{\mu}(\lambda x.t) \to_{\underline{\beta}} \underline{\mu}(\lambda x.t')}$$

$$\frac{t' \to_{\underline{\beta}} t''}{\underline{\mu}(tt') \to_{\underline{\beta}} \underline{\mu}(tt'')}$$

$$\frac{t \to_{\underline{\beta}} t''}{\underline{\mu}(tt') \to_{\underline{\beta}} \underline{\mu}(t''t')}$$

Definition 8 (Labelled Mu-Reductions).

$$\frac{}{\mu(\lambda x.t) \to_{\underline{\mu}} \lambda x.\mu t}$$

$$\frac{}{\mu(t_1 t_2) \to_{\underline{\mu}} (\mu t_1)(\mu t_2)}$$

$$\frac{}{\mu(\mu t) \to_{\underline{\mu}} \mu t}$$

$$\frac{}{\underline{\mu}(\lambda x.t) \to_{\underline{\mu}} \lambda x.\mu t}$$

$$\frac{}{\underline{\mu}(t_1 t_2) \to_{\underline{\mu}} (\mu t_1)(\mu t_2)}$$

$$\frac{}{\underline{\mu}(\mu t) \to_{\underline{\mu}} \mu t}$$

$$\frac{t \to_{\underline{\mu}} t'}{\lambda x.t \to_{\underline{\mu}} \lambda x.t'}$$

$$\frac{t' \to_{\underline{\mu}} t''}{tt' \to_{\underline{\mu}} tt''}$$

$$\frac{t \to_{\underline{\mu}} t''}{tt' \to_{\underline{\mu}} t''t'}$$

$$\frac{t \to_{\underline{\mu}} t'}{\mu t \to_{\underline{\mu}} \mu t'}$$

$$\frac{t \to_{\underline{\mu}} t'}{\underline{\mu}\mu t \to_{\underline{\mu}} \underline{\mu}\mu t'}$$

$$\frac{t' \to_{\underline{\mu}} t''}{\underline{\mu}(tt') \to_{\underline{\mu}} \underline{\mu}(tt'')}$$

$$\frac{t \to_{\underline{\mu}} t''}{\underline{\mu}(tt') \to_{\underline{\mu}} \underline{\mu}(t''t')}$$

$$\frac{t \to_{\underline{\mu}} t'}{\underline{\mu}(\lambda x.t) \to_{\underline{\mu}} \underline{\mu}(\lambda x.t')}$$

Remarks:

-1. It is natural to make sure that: if t is a well-formed labelled term and $t \to_{\underline{m}} t'$, then t' is also a well-formed labelled term. We can do this by induction on the derivation of $t \to_{\underline{m}} t'$.

0. I use $\underline{\Lambda}$ to denote the set of all labelled terms.

1. $\Lambda \subset \underline{\Lambda}$

3. $\to_{\underline{m}} := \to_{\underline{\mu}} \cup \to_{\underline{\beta}}$ and $\to_m := \to_\mu \cup \to_\beta$.

4. $\twoheadrightarrow_m$ denotes the reflexive and transitive closure of $\to_m$, $\twoheadrightarrow_{\underline{m}}$ denotes the reflexive and transitive closure of $\to_{\underline{m}}$

5. $\twoheadrightarrow_{\underline{m}}, \to_{\underline{m}}, \to_{\underline{\mu}}, \to_{\underline{\beta}}$ are all defined on terms in $\underline{\Lambda}$

Definition 9 (Erasure). *We define $e(t)$ to be the unlabelled term obtains by ignoring all the underlines in t:*

$e(x) = x$

$e(tt') = e(t)e(t')$

$e(\lambda x.t') = \lambda x.e(t')$

$e(\mu t') = \mu e(t')$

$e(\underline{\mu}\mu t') = \mu\mu e(t')$

$e(\underline{\mu}(\lambda x.t')) = \mu(\lambda x.e(t'))$

$e(\underline{\mu}(tt')) = \mu(e(t)e(t'))$

$e((\underline{\lambda}x.t)t') = (\lambda x.e(t))e(t')$

graphically denoted by $\rightarrow_e$

Definition 10 (Contraction).
We define a contraction function $\phi : \underline{\Lambda} \rightarrow \Lambda$ *as below:*

$\phi(x) = x$

$\phi(tt') = \phi(t)\phi(t')$

$\phi(\lambda x.t') = \lambda x.\phi(t')$

$\phi(\mu t') = \mu\phi(t')$

$\phi(\underline{\mu}\mu t') = \mu\phi(t')$

$\phi(\underline{\mu}(\lambda x.t')) = \lambda x.\mu\phi(t')$

$\phi(\underline{\mu}(tt')) = (\mu\phi(t))(\mu\phi(t'))$

$\phi((\underline{\lambda}x.t)t') = [\phi(t')/x]\phi(t)$

graphically denoted by $\rightarrow_\phi$

3 Lemmas

Lemma 1. *If* $t_1 \rightarrow_m t_2$ *and* $t_1' \rightarrow_e t_1$*, then there exist* t_2' *such that* $t_1' \rightarrow_{\underline{m}} t_2'$*, and* $t_2' \rightarrow_e t_2$*.*

$$\begin{tikzcd} t_1' \arrow[r, dashed, "\underline{m}"'] \arrow[d, "e"] & t_2' \arrow[d, dashed, "e"] \\ t_1 \arrow[r, "m"'] & t_2 \end{tikzcd}$$

Note: From now on, I will use diagram instead of English to express the lemmas or theorems.

Proof. If $t_1 \rightarrow_m t_2$, then t_2 is obtained by contracting one redex Δ in t_1. We simply reduce Δ in t_1' we get t_2', which is in fact has $e(t_2') = t_2$.

Lemma 2.

$$\begin{tikzcd} t_1' \arrow[r, dashed, two heads, "\underline{m}"'] \arrow[d, "e"] & t_2' \arrow[d, dashed, "e"] \\ t_1 \arrow[r, two heads, "m"'] & t_2 \end{tikzcd}$$

Proof. Using lemma 1. By transitivity.

Lemma 3.

$$\begin{tikzcd} t_1 \arrow[r, "\underline{\mu}"'] \arrow[d, "\phi"] & t_2 \arrow[d, "\phi"] \\ \phi(t_1) \arrow[r, dashed, two heads, "\mu"'] & \phi(t_2) \end{tikzcd}$$

Proof. By induction on the derivation of $t_1 \rightarrow_{\underline{\mu}} t_2$.

Lemma 4. $\phi([t'/x]t) = [\phi(t')/x]\phi(t)$.

Proof. By induction on the structure of t.

Base Case: $t = x$. Obvious.

Step Case: $t = \lambda y.t_1$.

$\phi(\lambda y.[t'/x]t_1) = \lambda y.\phi([t'/x]t_1) =_{IH} \lambda y.[\phi(t')/x]\phi(t_1) = [\phi(t')/x]\phi(\lambda y.t_1)$.

Step Case: $t = t_1 t_2$.

$\phi([t'/x]t_1[t'/x]t_2) = \phi([t'/x]t_1)\phi([t'/x]t_2) =_{IH} [\phi(t')/x]\phi(t_1)[\phi(t')/x]\phi(t_2) = [\phi(t')/x]\phi(t_1 t_2)$.

Step Case: $t = \mu t_1$.

$\phi([t'/x]\mu t_1) = \phi(\mu([t'/x]t_1)) = \mu\phi([t'/x]t_1) =_{IH} \mu[\phi(t')/x]\phi(t_1) = [\phi(t')/x]\phi(\mu t_1)$.

Step Case: $t = (\underline{\lambda} y.t_1)t_2$.

$\phi((\underline{\lambda} y.[t'/x]t_1)[t'/x]t_2) = [\phi([t'/x]t_1)/y]\phi([t'/x]t_2) =_{IH} [[\phi(t')/x]\phi(t_1)/y][\phi(t')/x]\phi(t_2) = [\phi(t')/x]([\phi(t_1)/y]\phi(t_2)) = [\phi(t')/x]\phi((\underline{\lambda} y.t_1)t_2)$.

Step Case: $t = \underline{\mu}\mu t_1$.

$\phi(\underline{\mu}\mu([t'/x]t_1)) = \mu\phi([t'/x]t_1) =_{IH} \mu[\phi(t')/x]\phi(t_1) = [\phi(t')/x]\phi(\underline{\mu}\mu t_1)$.

Step Case: $t = \underline{\mu}(t_1 t_2)$.

$\phi(\underline{\mu}([t'/x]t_1[t'/x]t_2)) = \mu\phi([t'/x]t_1)\mu\phi([t'/x]t_1) =_{IH} \mu[\phi(t')/x]\phi(t_1)\mu[\phi(t')/x]\phi(t_2) = [\phi(t')/x]\phi(\underline{\mu}(t_1 t_2))$.

Step Case: $t = \underline{\mu}(\lambda y.t_1)$.

$\phi(\underline{\mu}(\lambda y.[t'/x]t_1)) = \lambda y.\mu\phi([t'/x]t_1) =_{IH} \lambda y.\mu[\phi(t')/x]\phi(t_1) = [\phi(t')/x]\phi(\underline{\mu}(\lambda y.t_1))$.

Lemma 5.

$$
\begin{tikzcd}
t_1 \arrow[r, "\underline{\beta}"'] \arrow[d, "\phi"] & t_2 \arrow[d, "\phi"] \\
\phi(t_1) \arrow[r, dashed, ->>, "\beta"'] & \phi(t_2)
\end{tikzcd}
$$

Proof. By induction on the derivation of $t_1 \to_{\underline{\beta}} t_2$. Using lemma 4.

Lemma 6.

$$
\begin{tikzcd}
t_1 \arrow[r, ->>, "\underline{m}"'] \arrow[d, "\phi"] & t_2 \arrow[d, "\phi"] \\
\phi(t_1) \arrow[r, dashed, ->>, "m"'] & \phi(t_2)
\end{tikzcd}
$$

Proof. By the two lemmas above.

Lemma 7.

$$
\begin{tikzcd}
& t \arrow[dl, "e"'] \arrow[dr, "\phi"] & \\
e(t) \arrow[rr, dashed, ->>, "m"'] & & \phi(t)
\end{tikzcd}
$$

Proof. By induction on the structure of t.

Lemma 8 (Strip Lemma).

$$
\begin{tikzcd}
& t \arrow[dl, ->>, "m"'] \arrow[dr, ->>, "m"] & \\
t_1 \arrow[dr, dashed, ->>, "m"'] & & t_2 \arrow[dl, dashed, ->>, "m"] \\
& t' &
\end{tikzcd}
$$

Proof. By the following diagram:

$$
\begin{tikzcd}
& t \arrow[dl, "m"'] \arrow[dr, "m"] & \\
t_1 \arrow[dr, dashed, "m"'] & t_3 \arrow[l, "\phi"] \arrow[u, "e"] \arrow[dr, dashed, "m"] & t_2 \arrow[dl, dashed, "m"'] \\
& t' & t_4 \arrow[l, "\phi"] \arrow[u, dashed, "e"']
\end{tikzcd}
$$