Rewriting Approach to Type Assignment a la Curry

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Simply Typed Lambda Calculus a la Curry

- Terms: $t ::= x \mid \lambda x.t \mid tt$
- Types: $T ::= B \mid T_1 \Rightarrow T_2$
- Typing Context: $\Gamma ::= \cdot \mid \Gamma, x : T$
- Reduction: $(\lambda x.t)t' \to_\beta [t'/x]t$
- Type assignment:

$$\frac{(x : T) \in \Gamma}{\Gamma \vdash x : T} \qquad \frac{\Gamma, x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \Rightarrow T_2}$$

$$\frac{\Gamma \vdash t_1 : T_1 \Rightarrow T_2 \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1\ t_2 : T_2}$$

Rewrite rules for Curry-Style STLC

- *mixed terms*: $m \ ::= \ x \mid \lambda x.\, m \mid m\ m' \mid B \mid T \Rightarrow m$
- *standard terms*: $t \ ::= \ x \mid \lambda x.\, t \mid t\ t'$
- Reductions:
  $\hat{m}[(T \Rightarrow m)\ T] \ \to_\epsilon \ \hat{m}[m]$
  $\hat{m}[\lambda x.\, m] \ \to_\lambda \ \hat{m}[T \Rightarrow [T/x]m]$
- Soundness and Completeness: $\Gamma \vdash t : T$ iff $\Gamma t \stackrel{*}{\to} T$. Here the Γ in Γt acts a a substitution.
- Example: $[y : T](\lambda x.x)y \equiv (\lambda x.x)T \to_\lambda (T \Rightarrow T)T \to_\epsilon T$, which corresponds to $y : T \vdash (\lambda x.x)y : T$.
- Notice: $\to_\lambda$ is infinite branching.

Curry System F

We can extend curry-STLC to curry-sytem F by adding new types and type assignments.

- Types: $T ::= ... \mid X \mid \forall X.T$
- New typing rules: $\dfrac{\Gamma \vdash t : T \quad X \notin FV(\Gamma)}{\Gamma \vdash t : \forall X.T}$ $\dfrac{\Gamma \vdash t : \forall X.T'}{\Gamma \vdash t : [T/X]T'}$
- Two temptative new rewriting rules:
  $\hat{m}[\forall X.m] \to_\iota \hat{m}[[T/X]m]$
  $\hat{m}[m] \to_\pi \hat{m}[\forall X.m]$
  Assuming $m ::= x \mid X \mid \lambda x.m \mid m \Rightarrow m' \mid mm' \mid \forall X.m$.
- We want to show: $\Gamma \vdash t : T$ iff $\Gamma t \xrightarrow{*} T$.
- But: $\lambda x.x \to_\lambda X \Rightarrow X \to_\pi X \Rightarrow \forall X.X$. We know that $\nvdash \lambda x.x : X \Rightarrow \forall X.X$.

How to resolve the problem?

- Attempt: $\hat{m}[m] \to_\pi \hat{m}[\forall X.m]$, where $X \notin FV(\hat{m})$.
- We lose compatible with reduction. i.e. it is no longer the case that: $m_1 \to m_2$ implies $\hat{m}[m_1] \to \hat{m}[m_2]$. Since $X \Rightarrow X \not\to_\pi X \Rightarrow \forall X.X$.
- A more fundamental problem: $[x : X]x \equiv X \to_\pi \forall X.X$. But $x : X \not\vdash x : \forall X.X$.
- We lose all the typing context information when we use Γ in Γt as a substitution.
- A new point of view: what if we do not use Γ in Γt as a substitution?
- What if Γt is a new syntactical category?

A New formulation

- Pretypes: $P ::= T \mid \Gamma t \mid \Gamma T \mid P_1 P_2 \mid P_1 \Rightarrow P_2 \mid \lambda x.P \mid \forall X.P$
- A new meaning of: $\Gamma t \stackrel{*}{\to} T$.
- If we allow Γ distributed over the structure of t and T. Meaning $\Gamma t_1 t_2 \equiv \Gamma t_1 \Gamma t_2$ etc..
- $P ::= X \mid \Gamma x \mid \Gamma X \mid P_1 P_2 \mid P_1 \Rightarrow P_2 \mid \lambda x.P \mid \forall X.P$.
- What about the reduction rules?

New Reduction Rules

- $\hat{P}[\Gamma x] \to_s \hat{P}[\Gamma T]$ if $(x : T) \in \Gamma$.

$$\frac{(x : T) \in \Gamma}{\Gamma \vdash x : T}$$

- $\hat{P}[\lambda x.P'] \to_\lambda \hat{P}[T \Rightarrow [x : T] \cdot P']$. Notice that $[x : T] \cdot P'$ will be defined later.

$$\frac{\Gamma, x : T_1 \ \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \Rightarrow T_2}$$

- $\hat{P}[(T \Rightarrow P')T] \to_\epsilon \hat{P}[P']$.

$$\frac{\Gamma \vdash t_1 : T_1 \Rightarrow T_2 \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 \ t_2 : T_2}$$

New Reduction Rules

- $\hat{P}[\forall X.P'] \to_\iota \hat{P}[[T/X]P']$. We will define $[T/X]P'$ later.

$$\frac{\Gamma \vdash t : \forall X.T'}{\Gamma \vdash t : [T/X]T'}$$

- $\hat{P}[P'] \to_\pi \hat{P}[\forall X.P']$, where $X \notin \Gamma V(P')$ and P' is not a type. We will define $\Gamma V(P')$ later.

$$\frac{\Gamma \vdash t : T \quad X \notin FV(\Gamma)}{\Gamma \vdash t : \forall X.T}$$

- $\hat{P}[\Gamma T] \to_r \hat{P}[T]$.

Definitions: I

We define substitution $[T/X]P$ on pretypes P:

$[T/X](\Gamma x) \equiv ([T/X]\Gamma)x$, where $[T/X]\Gamma$ means apply the substitution $[T/X]$ on each type in Γ.
$[T/X]X \equiv T$.
$[T/X]\Gamma X' \equiv ([T/X]\Gamma)([T/X]X')$.
$[T/X](P' \to P) \equiv [T/X]P' \to [T/X]P$
$[T/X]P_1 P_2 \equiv ([T/X]P_1)([T/X]P_2)$
$[T/X]\lambda x.P \equiv \lambda x.[T/X]P$
$[T/X]\forall Y.P \equiv \forall Y.([T/X]P)$. This will invoke renaming and capture avoiding if nessesary.

Definitions: II

Define a new operation $[x : T] \cdot P$ inductively on the structure of P:

$[x : T] \cdot X \equiv X$.
$[x : T] \cdot \Gamma y \equiv [\Gamma, x : T] y$ if $x \notin dom(\Gamma)$. Else if $x \in dom(\Gamma)$, then $[x : T] \cdot \Gamma y \equiv \Gamma y$.
$[x : T] \cdot \Gamma Y \equiv [\Gamma, x : T] Y$ if $x \notin dom(\Gamma)$. Else if $x \in dom(\Gamma)$, then $[x : T] \cdot \Gamma Y \equiv \Gamma Y$.
$[x : T] \cdot (P_1 P_2) \equiv ([x : T] \cdot P_1)([x : T] \cdot P_2)$
$[x : T] \cdot (P' \to P) \equiv [x : T] \cdot P' \to [x : T] \cdot P$
$[x : T] \cdot \lambda y.P \equiv \lambda y.[x : T] \cdot P$. This will invoke renaming if nessesary.
$[x : T] \cdot \forall X.P \equiv \forall X.[x : T] \cdot P$. This will invoke renaming if nessesary.

Definitions: III

We define $\Gamma V(P)$ inductively:

$\Gamma V(X) = \emptyset$.
$\Gamma V(\Gamma x) = FV(\Gamma)$.
$\Gamma V(\Gamma X) = FV(\Gamma)$.
$\Gamma V(P_1 P_2) = \Gamma V(P_2) \cup \Gamma V(P_1)$
$\Gamma V(P' \to P) = \Gamma V(P') \cup \Gamma V(P)$
$\Gamma V(\lambda x.P) = \Gamma V(P)$
$\Gamma V(\forall X.P) = \Gamma V(P) - \{X\}$

Properties of New Rewrite Rules

- **Compatible with Reduction:** If $P \to P'$, then $\hat{P}_1[P] \to \hat{P}_1[P']$.
- **Compatible with Type-Context Action:** If $P \to P'$, then $[x : T] \cdot P \to [x : T] \cdot P'$.
- **Closed under Type Substitution:** For any type substitution δ, if $P \to P'$, then $\delta P \to \delta P'$.
- **Soundness:** If $\Gamma \vdash t : T$, then $\Gamma t \stackrel{*}{\to} T$.

Completeness

Goal: If $\Gamma t \xrightarrow{*} T$, then $\Gamma \vdash t : T$.

We need more lemmas!

- **Abstraction Inversion**:

  If $\forall X^n.(\lambda x.P) \xrightarrow{*} T$, then there are T_1, P', m such that $\forall X^n.\lambda x.P \xrightarrow{*} \forall Y^m.\lambda x.P' \to_\lambda \forall Y^m.(T_1 \Rightarrow [x : T_1] \cdot P') \xrightarrow{*} T$ and $\forall X^n.P \xrightarrow{*} P'$.

- **Arrow Inference**:

  If $\forall X^n.(T \Rightarrow P) \xrightarrow{*} T'$, then $T' \equiv \forall Y^m.(T_1 \Rightarrow T_2)$, $\delta T \equiv T_1$, $\delta P \xrightarrow{*} T_2$ for some type level substitution δ.

Completeness

Goal: If $\Gamma t \xrightarrow{*} T$, then $\Gamma \vdash t : T$.

The dual lemmas.

- **Application Inversion**:

  If $\forall X^n.P_1 P_2 \xrightarrow{*} T$, then there exists P', m, T_1 such that $\forall X^n.P_1 P_2 \xrightarrow{*} \forall Y^m.(T_1 \Rightarrow P')T_1 \to_\epsilon \forall Y^m.P' \xrightarrow{*} T$. Also we have $\forall X^n.P_1 \xrightarrow{*} T_1 \Rightarrow P'$ and $\forall X^n.P_2 \xrightarrow{*} T_1$.

- **Reduction Inference**:

  If $\forall X^n.P \xrightarrow{*} T$, then there exists δ such that $\delta P \xrightarrow{*} T$.

- Concluding the completeness by induction on the structure of t.

Type Preservation and Conclusion

- **Type Preservation:** If $\Gamma \vdash (\lambda x.t_1)t_2 : T$, then $\Gamma \vdash [t_2/x]t_1 : T$.
- **Rewriting Version:** If $\Gamma((\lambda x.t_1)t_2) \stackrel{*}{\to} T$, then $\Gamma([t_2/x]t_1) \stackrel{*}{\to} T$.
- **Conclusion:** The new rewriting formulation for curry system F could be useful.

Maybe go through some proofs?