

# Lambda Encoding, Types and Confluence

Peng Fu

Advisor: Prof. Aaron Stump  
Dept. of Computer Science

# Prelude: Programs and Data

- ▶ Program as data

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- ▶ Data as Program?
- ▶  $2f\ a = f\ (f\ a)$ .
- ▶ Lambda Calculus

# Outline

- ▶ Typed Lambda Calculus and Lambda Encoding
- ▶ Type Preservation and Confluence
- ▶ Limits of Dependent type
- ▶ Selfstar
- ▶ Summary

# Lambda Calculus

How to describe the *plus one* function.

- ▶  $x + 1$
- ▶  $\xi + 1$  (Frege, Grundgesetze der Arithmetik)
- ▶  $\hat{x} + 1$  (Principia Mathematica)
- ▶  $\hat{x}.x + 1$  (Church)<sup>1</sup>
- ▶  $\wedge x.x + 1$  ( Typewriter)
- ▶  $\lambda x.x + 1$  (Modern)
- ▶  $x + 1$  v.s.  $\lambda x.x + 1$

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<sup>1</sup>Come from Barendregt's Impact of Lambda Calculus

# Lambda Calculus

## Higher order expression reduction system

- ▶ Terms(expression)  $t ::= x \mid \lambda x.t \mid t t'$
- ▶ Reduction  $(\lambda x.t)t' \rightarrow_{\beta} [t'/x]t$
- ▶ In what sense it is higher order?
- ▶ Bind and free variable are primitives.  
free variable of  $\lambda x.x y$  is  $y$ ,  $x$  is called bind variable.
- ▶ Variable binding in  $\lambda x.\lambda y.x y \equiv \lambda z.\lambda x.z x$   
 $\lambda x.x \equiv \lambda y.y$
- ▶ Reduction examples:  
 $(\lambda x.x) y \rightarrow_{\beta} y$   
 $(\lambda x.x x) y \rightarrow_{\beta} y y.$



# Lambda Calculus

- ▶ To express higher order function
- ▶ Semantically, for an  $a \in A$ ,  $g : (A \rightarrow A) \rightarrow A$ , where  $f \mapsto f(a)$ .
- ▶ With Lambda Calculus:  
$$g := \lambda y. y \ a$$
$$g f \equiv (\lambda y. y \ a) f \rightarrow_{\beta} f \ a$$
- ▶ Function  $g$  is emulated at the syntactic level.

# Types

- ▶ Originated from Russell works, later used by Curry, Church and many others.
- ▶ Now common in programming language.
- ▶ E.g. types in C, java. Central in Ocaml, Haskell.
- ▶ Used to express certain assumptions, e.g.  $\text{String} \rightarrow \text{int}$
- ▶ The notion of types can be generalized to some sophisticated types, leads to theorem provers like Coq, Agda.

# Typed Lambda Calculus

Type is expression to govern the form of lambda term:  $\Gamma \vdash t : T$

- ▶ Simple type:  $T ::= X \mid T \rightarrow T'$
- ▶ Context(Environment)  $\Gamma ::= \cdot \mid \Gamma, x : T$
- ▶ Typing

$$\frac{(x : T) \in \Gamma}{\Gamma \vdash x : T} \textit{Var}$$

$$\frac{\Gamma, x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x. t : T_1 \rightarrow T_2} \textit{Abs}$$

$$\frac{\Gamma \vdash t : T_1 \rightarrow T_2 \quad \Gamma \vdash t' : T_1}{\Gamma \vdash t t' : T_2} \textit{App}$$

# Typed Lambda Calculus

Extend simple type:

- ▶ Polymorphic type:  $T ::= \dots \mid \forall X : \kappa. T$
- ▶ Kind:  $\kappa ::= *$
- ▶ Context:  $\Gamma ::= \dots \mid X : \kappa$
- ▶ Typing

$$\frac{\Gamma, X : \kappa \vdash t : T}{\Gamma \vdash t : \forall X : \kappa. T} \textit{Gen} \qquad \frac{\Gamma \vdash t : \forall X : \kappa. T \quad \Gamma \vdash T' : \kappa}{\Gamma \vdash t : [T'/X]T} \textit{Inst}$$

# Typed Lambda Calculus

Extends Polymorphic type(System F):

- ▶ Product type(dependent type):  $T ::= \dots \mid \Pi x : T.T' \mid T \ t$
- ▶ Kind:  $\kappa ::= * \mid \xi x : T.\kappa$ .
- ▶ Typing

$$\frac{\Gamma, x : T' \vdash t : T \quad x \in FV(T)}{\Gamma \vdash \lambda x.t : \Pi x : T'.T} \textit{Pi}$$

$$\frac{\Gamma \vdash t : \Pi x : T'.T \quad \Gamma \vdash t' : T'}{\Gamma \vdash t \ t' : [t'/x]T} \textit{Elim}$$

$$\frac{\Gamma \vdash t : [t_1/x]T \quad t_1 =_{\beta} t_2}{\Gamma \vdash t : [t_2/x]T} \textit{Conv}$$

# Typed Lambda Calculus

## Why Dependent Types?

- ▶ Curry-Howard correspondent for  $\Gamma \vdash t : T$   
     $\langle \text{Environment} \rangle \vdash \langle \text{program} \rangle : \langle \text{type} \rangle$   
     $\langle \text{Assumptions} \rangle \vdash \langle \text{proof} \rangle : \langle \text{formula} \rangle$
- ▶ Type Preservation:  
    If  $\Gamma \vdash t : T$  and  $t \rightarrow_{\beta} t'$ , then  $\Gamma \vdash t' : T$ .
- ▶ Strong Normalization:  
    If  $\Gamma \vdash t : T$ , then there is no infinite beta reduction sequence.

# Algebraic Data types

An example in Haskell:

```
data Nat = Zero
        | Succ Nat
```

```
add :: Nat -> Nat -> Nat
add n m = case n of
    Zero -> m
    | Succ p -> add p (Succ m)
```

```
data List a = Nil | Cons a (List a)
data Tree = Empty
        | Leaf Int
        | Node Tree Tree
```

# Core Language Design

To support (algebraic) data type, one way is to add data type and pattern matching(to core) as primitive.

- ▶ expression for data type declaration:  
data  $T (a_i : A_i)_{i \in I} : A$  where  $\{C_i : A_i\}_{i \in I}$
- ▶ expression for pattern matching :  
case  $a$  of  $\{C_i(x_1, \dots x_u) \Rightarrow a\}_{i \in I}$



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- ▶ expression for pattern matching :  
case  $a$  of  $\{C_i(x_1, \dots, x_u) \Rightarrow a\}_{i \in I}$
- ▶ Now typing rule become:

$$\begin{array}{l}
 r = n + m \\
 x_1 \dots x_n \notin \text{FV}(|t''|) \\
 \text{getArgs}(t') = [w_1, \dots, w_m] \\
 \text{buildCtx}(\Delta_2(\text{getHC}(t'))) = [y_1 : t'_1, \dots, y_n : t'_n] \\
 \text{cut}([y_1 : t'_1, \dots, y_n : t'_n], \text{buildCtx}(\text{getCType}(t', C, \Delta))) = [x_1 : t'_1, \dots, x_n : t'_n] \\
 \Delta, \Gamma \vdash^{TB} H \ t_1 \ t' \ y \ (l - \{C : \text{getCType}(t', C, \Delta)\}) : t'' \\
 \Delta, \Gamma, x_1 : [w_1/y_1]t'_1, \dots, x_n : [w_m/y_m]t'_n, y : t_1 = (C \ w'_{1 \in 1} \dots w'_{r \in n}) \vdash t_2 : t'' \\
 \hline
 \Delta, \Gamma \vdash^{TB} (C \ x_1 \varepsilon'_1 \dots x_n \varepsilon'_n \Rightarrow t_2 \mid H) \ t_1 \ t' \ y \ l : t''
 \end{array}
 \quad \text{TB\_BRANCH}$$

# Core Language Design

Complicate the analysis for core language.

- ▶ Type Preservation Proof for Standard ML
- ▶ *The machine-assisted proof of programming language properties*, 1996, Ph.D thesis by M. VanInwegen.
- ▶ Machine assisted proof by HOL.
- ▶ Quoted from abstract:  
“We were not able to complete the proof of type preservation because it is not true: we found counterexamples.”
- ▶ In Haskell Core Language <sup>2</sup>:  
“Case expressions are the most complicated bit of Core. ”

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<sup>2</sup><http://hackage.haskell.org/trac/ghc/wiki/Commentary/Compiler/CoreSynType>

# Scott Numerals

- ▶ Scott encoding with Recursive Definition:

```
Zero = lam s . lam z . z
Succ n = lam s . lam z . s n
add n m = case n of
            Zero -> m
            | Succ p -> add p (Succ m)
pred n    = case n of
            Zero -> Zero
            | Succ p -> p
```

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- Translate to lambda calculus with recursive definition

```
Zero :=  $\lambda s. \lambda z. z$ 
Succ :=  $\lambda n. \lambda s. \lambda z. s\ n$ 
add :=  $\lambda n. \lambda m. n\ (\lambda p. \text{add}\ p\ (\text{Succ}\ m))\ m$ 
pred :=  $\lambda n. n\ (\lambda p. p)\ \text{Zero}$ 
```

# Scott Numerals

- ▶ Isn't it great? No primitive data type and pattern matching needed!
- ▶ Beta reduction and definition unfolding are enough.
- ▶ the catch(or is it?): need recursive type and operation on Scott encoding data can not be typed in polymorphic type system.

$$\frac{\Gamma \vdash t : [\mu X.T/X]T}{\Gamma \vdash t : \mu X.T} \text{ Fold} \quad \frac{\Gamma \vdash t : \mu X.T}{\Gamma \vdash t : [\mu X.T/X]T} \text{ unFold}$$

$\text{Nat} := \mu X.(X \rightarrow U) \rightarrow U \rightarrow U$  for any type  $U$ .

- $\vdash \text{add} : \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat}$ .
- $\vdash \text{Zero} : \text{Nat}$ .
- $\vdash \text{Succ} : \text{Nat} \rightarrow \text{Nat}$ .

# Church Numerals

Idea,  $((2f) a) = f (f a)$ , so  $2 := \lambda f. \lambda a. f (f a)$ .

- Church encoding

```
Zero = lam s . lam z . z
Succ n = lam s . lam z . s (n s z)
iterator n f a = n f a
add n m = iterator n Succ m
```

- One can see

$1 = \text{Succ Zero} \rightarrow_{\beta}^* \lambda s. \lambda z. s z$

$2 = \text{Succ } 1 \rightarrow_{\beta}^* \lambda s. \lambda z. s (s z)$

- Easily translated to lambda calculus.

# Church Numerals

- ▶ This is also great.
- ▶ Even better, it can be typed in system F (Polymorphic type).  
$$\text{Nat} := \forall X : *. (X \rightarrow X) \rightarrow X \rightarrow X$$
  - $\vdash \text{Succ} : \text{Nat} \rightarrow \text{Nat}$
  - $\vdash \text{iterator} : \forall X : *. \text{Nat} \rightarrow (X \rightarrow X) \rightarrow X \rightarrow X$
  - $\vdash \text{add} : \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat}$
- ▶ The catch, inefficient predecessor.
- ▶ The predecessor function takes linear time (beta reductions) to compute, while with Scott encoding it only takes constant time.

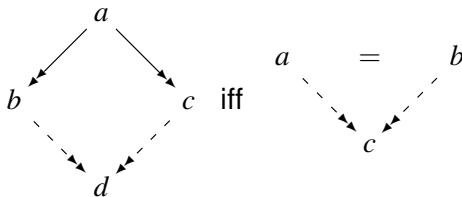
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# Abstract Reduction System

- ▶ Abstract Reduction System(ARS):  $(\mathcal{A}, \{\rightarrow_i\}_{i \in \mathcal{I}})$
- ▶ Basic concepts:  $a \xrightarrow{*}_i b$  (or  $a \twoheadrightarrow_i b$ ),  $a =_i b$ , *reducible*, *normal form*.
- ▶ Confluence and Church-Rosser property:  
 $\rightarrow := \bigcup_{i \in \mathcal{I}} \rightarrow_i$ .



# Confluence and Type Preservation

- Recall type preservation:

If  $\Gamma \vdash t : T$  and  $t \rightarrow t'$ , then  $\Gamma \vdash t' : T$ .

Usually prove by induction on derivation of  $\Gamma \vdash t : T$ .

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- ▶ Need to analyze the case:

$$\frac{\Gamma \vdash \lambda x.t : T' \rightarrow T \quad \Gamma \vdash t' : T'}{\Gamma \vdash (\lambda x.t)t' : T} \textit{App}$$

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- ▶ Present of non-syntax directed rule:

$$\frac{\Gamma \vdash t : T' \quad T = T'}{\Gamma \vdash t : T} \textit{Conv}$$

- ▶ Inversion on  $\Gamma \vdash \lambda x.t : T' \rightarrow T$  gives us only:

$\Gamma, x : T_1 \vdash t : T_2$  where  $T_1 \rightarrow T_2 = T' \rightarrow T$ .

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- ▶ Want *inverse structure congruence*:

If  $T_1 \rightarrow T_2 = T'_1 \rightarrow T'_2$ , then  $T_1 = T'_1$  and  $T_2 = T'_2$ .

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- ▶ By induction on derivation of  $T_1 \rightarrow T_2 = T'_1 \rightarrow T'_2$ , case:

$$\frac{T_1 \rightarrow T_2 = T_3 \quad T_3 = T'_1 \rightarrow T'_2}{T_1 \rightarrow T_2 = T'_1 \rightarrow T'_2} \textit{Trans}$$

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- ▶ Can't apply induction hypothesis!

# Confluence and Type Preservation

Inverse structure congruence:

If  $T_1 \rightarrow T_2 = T'_1 \rightarrow T'_2$ , then  $T_1 = T'_1$  and  $T_2 = T'_2$ .

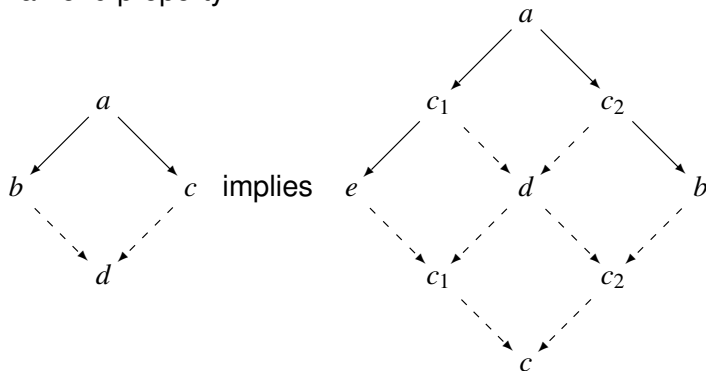
- ▶ We can define  $=$  to be the reflexive, symmetric, transitive closure of  $\succrightarrow$ .
- ▶ We show that  $\succrightarrow$  is confluent, thus Church-Rosser.
- ▶ So  $T_1 \rightarrow T_2 = T'_1 \rightarrow T'_2$  implies there is a  $T_3$  such that  $T_1 \rightarrow T_2 \xrightarrow{*} T_3$  and  $T'_1 \rightarrow T'_2 \xrightarrow{*} T_3$ .
- ▶ We design  $\succrightarrow$  in such a way that  $T_a \rightarrow T_b \succrightarrow T$  iff  $T \equiv T'_a \rightarrow T_b$  or  $T \equiv T_a \rightarrow T'_b$ , where  $T_a \succrightarrow T'_a$ ,  $T_b \succrightarrow T'_b$ .
- ▶ Thus we conclude inverse structure congruence, conquered one problem for proving type preservation.



# Confluence: Tait-Martin L f's Method

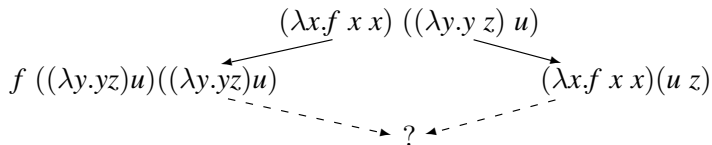
Lambda calculus  $(\Lambda, \rightarrow_\beta)$  as ARS is confluent.

- Diamond property:



# Confluence: Tait-Martin Löf's method

$\rightarrow_\beta$  reduction does not have diamond property.



Not joinable in one step, but joinable in “many” steps.

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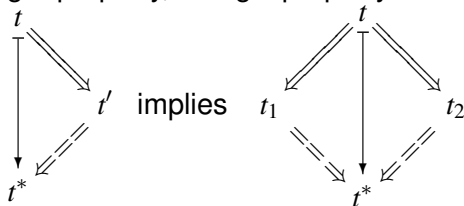
- ▶ Parallel Reduction(a notion of “many” steps).

$$\frac{t_1 \Rightarrow_{\beta} t'_1 \quad t_2 \Rightarrow_{\beta} t'_2}{(\lambda x.t_1)t_2 \Rightarrow_{\beta} [t'_2/x]t'_1}$$

- ▶ It has diamond property.
- ▶  $\rightarrow_{\beta} \subseteq \Rightarrow_{\beta} \subseteq \overset{*}{\rightarrow}_{\beta}$ , which implies  $(\overset{*}{\rightarrow}_{\beta}) = (\Rightarrow_{\beta}^*)$
- ▶ Confluence of  $\Rightarrow_{\beta}$  implies confluence of  $\rightarrow_{\beta}$

# Confluence: Takahashi's method

- ▶ A stronger property, triangle property:



- ▶ Parallel contractions:

$$x^* := x.$$

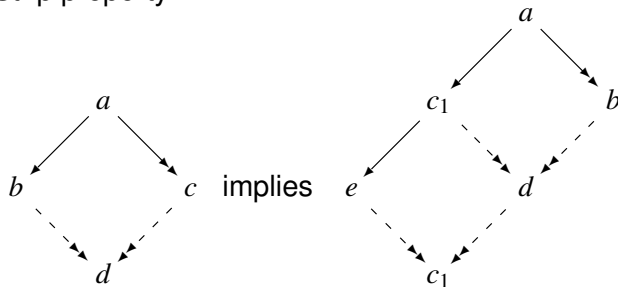
$$(\lambda x.t)^* := \lambda x.t^*.$$

$$(t_1 t_2)^* := t_1^* t_2^* \text{ if } t_1 t_2 \text{ is not a beta redex.}$$

$$((\lambda x.t_1) t_2)^* := [t_2^*/x]t_1^*.$$

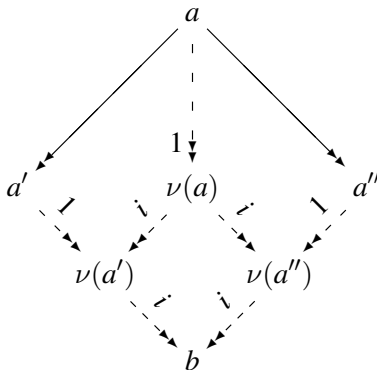
# Confluence: Barendregt's method

- Strip property:



# Confluence: Hardin's Interpretation method

- ▶ Assumption I:  $\rightarrow := \rightarrow_1 \cup \rightarrow_2$  and  $\rightarrow_1$  is strongly normalizing and confluent.
- ▶ Assumption II:  $\rightarrow_i \subseteq \rightarrow$  is defined on  $\rightarrow_1$  normal form  $\nu(t)$ .
- ▶ Assumption III: If  $a \rightarrow_2 b$ , then  $\nu(a) \rightarrow_i \nu(b)$ .
- ▶ Then: Confluence of  $\rightarrow_i$  implies confluence of  $\rightarrow$ .
- ▶



# Hardin's Interpretation method: Applications

- ▶ Confluence of  $(\Lambda_\mu, \rightarrow_\beta, \rightarrow_\mu)$ , originated from Selfstar.
- ▶  $\Lambda_\mu$  denotes terms  $t ::= x \mid \lambda x.t \mid tt' \mid \mu t$ .
- ▶ Closure:  $\mu ::= \{x_i \mapsto t_i\}_{i \in \mathcal{I}}$ . Locality.
- ▶ New reductions:

$$\frac{(x_i \mapsto t_i) \in \mu}{\mu x_i \rightarrow_\beta \mu t_i} \qquad \frac{\text{dom}(\mu) \# \text{FV}(t)}{\mu t \rightarrow_\mu t}$$
$$\frac{}{\mu(\lambda x.t) \rightarrow_\mu \lambda x.\mu t} \qquad \frac{}{\mu(t_1 t_2) \rightarrow_\mu (\mu t_1)(\mu t_2)}$$

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# Limits of Dependent Type

How to show some formula is unprovable.

- ▶ Formalized the notion of proof.
- ▶ If there is a proof of  $\perp$ , then draw a contradiction at meta-level.
- ▶ Under Curry Howard correspondent, showing some type is uninhabited.

# Limits of Dependent Type

Recall second order dependent type:

- Type as expression:

$$T := X \mid \forall X : \kappa. T \mid T_1 \rightarrow T_2 \mid \Pi x : T_1. T_2 \mid T \ t$$

Kind as expression  $\kappa := * \mid \xi x : T. \kappa$

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<sup>3</sup>Coquand's Metamathematical investigations of a calculus of constructions

<sup>4</sup>Werner's A Normalization Proof for an Impredicative Type System with Large Elimination over Integers

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- ▶ Induction principle:

$$\forall P : (\xi x : \text{Nat}.*) . P \ 0 \rightarrow (\Pi y : \text{Nat}. (P \ y) \rightarrow (P \ (\text{S } y))) \rightarrow \Pi x : \text{Nat}. P \ x$$

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- ▶  $P : (\xi x : \text{Nat}.*) , a : P \ 0 , b : \Pi y : \text{Nat}. (P \ y) \rightarrow (P \ (\text{S } y)) , x : \text{Nat} \vdash$   
 $(b \ 1 \ (b \ 0 \ a)) : P \ (\text{S } 1)$

Induction is not derivable(provable) within Dependent type system<sup>3</sup>

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- ▶  $P : (\xi x : \text{Nat}.*) , a : P \ 0 , b : \Pi y : \text{Nat}. (P \ y) \rightarrow (P \ (\text{S } y)) , x : \text{Nat} \vdash (b \ 1 \ (b \ 0 \ a)) : P \ (\text{S } 1)$

Induction is not derivable(provable) within Dependent type system<sup>3</sup>

- ▶ Since  $\perp$  is uninhabit, can not inhabit types like  $T \rightarrow \perp$ , where  $T$  is inhabited. Thus can not prove  $0 \neq 1$ .<sup>4</sup>

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<sup>3</sup>Coquand's Metamathematical investigations of a calculus of constructions

<sup>4</sup>Werner's A Normalization Proof for an Impredicative Type System with Large Elimination over Integers

# Limits of Dependent Type

Recall second order dependent type:

- ▶ Type as expression:

$$T := X \mid \forall X : \kappa. T \mid T_1 \rightarrow T_2 \mid \Pi x : T_1. T_2 \mid T \ t$$

Kind as expression  $\kappa := * \mid \xi x : T. \kappa$

- ▶ Induction principle:

$$\forall P : (\xi x : \text{Nat}.*) . P \ 0 \rightarrow (\Pi y : \text{Nat}. (P \ y) \rightarrow (P \ (\text{S } y))) \rightarrow \Pi x : \text{Nat}. P \ x$$

- ▶  $P : (\xi x : \text{Nat}.*) , a : P \ 0, b : \Pi y : \text{Nat}. (P \ y) \rightarrow (P \ (\text{S } y)), x : \text{Nat} \vdash (b \ 1 \ (b \ 0 \ a)) : P \ (\text{S } 1)$

Induction is not derivable(provable) within Dependent type system<sup>3</sup>

- ▶ Since  $\perp$  is uninhabit, can not inhabit types like  $T \rightarrow \perp$ , where  $T$  is inhabited. Thus can not prove  $0 \neq 1$ .<sup>4</sup>
- ▶ These compromise the usage of Church encoding in dependent type system.

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<sup>3</sup>Coquand's Metamathematical investigations of a calculus of constructions

<sup>4</sup>Werner's A Normalization Proof for an Impredicative Type System with Large Elimination over Integers

# Outline

- ▶ Typed Lambda Calculus and Lambda Encoding
- ▶ Type Preservation and Confluence
- ▶ Limits of Dependent type
- ▶ Selfstar
- ▶ Summary

# Selfstar: I

- ▶ Recall that:

$x : \text{Nat} \vdash \bar{n} : \forall P : (\xi x : \text{Nat}.*) . P\ 0 \rightarrow (\Pi y : \text{Nat}. (P\ y) \rightarrow (P\ (\text{S}\ y)))) \rightarrow P\ \bar{n}$ ,  
for any Church numerals  $\bar{n}$

- ▶ Dependent type system, namely,  $\Pi$  construct can not grasp this level of quantification.

- ▶ One way to try to capture (with the help of recursion) this is:

$x : \text{Nat} \vdash \bar{n} : \text{Nat} \rightarrow \text{Nat} . \forall P : (\xi x : \text{Nat}.*) . P\ 0 \rightarrow (\Pi y : \text{Nat}. (P\ y) \rightarrow (P\ (\text{S}\ y)))) \rightarrow P\ x$   
 $\text{Nat} := \text{Nat} \rightarrow \text{Nat} . \forall P : (\xi x : \text{Nat}.*) . P\ 0 \rightarrow (\Pi y : \text{Nat}. (P\ y) \rightarrow (P\ (\text{S}\ y)))) \rightarrow P\ x$

- ▶ add two typing rules:

$$\frac{\Gamma \vdash t : \text{Nat}.T}{\Gamma \vdash t : [t/x]T} \text{ SelfInst} \quad \frac{\Gamma \vdash t : [t/x]T}{\Gamma \vdash t : \text{Nat}.T} \text{ SelfGen}$$

- ▶ Introduce closure  $\mu := \{x_i \mapsto t_i\}_{i \in \mathcal{I}} \cup \{X_i \mapsto T_i\}_{i \in \mathcal{N}}$

- ▶ Wrap around term and type,  $\mu t, \mu T, \tilde{\mu} \in \Gamma$ .



# Selfstar: II

## Church encoding and Scott encoding in Selfstar

► Church encoding( $\tilde{\mu}_c$ ):

$(\text{Nat} : *) \mapsto$

$\iota x. \Pi C : \text{Nat} \rightarrow *. (\Pi n : \text{Nat}. (C\ n) \rightarrow (C\ (\text{S}\ n)))) \rightarrow (C\ 0) \rightarrow (C\ x)$

$(\text{S} : \text{Nat} \rightarrow \text{Nat}) \mapsto \lambda n. \lambda C. \lambda s. \lambda z. s\ n\ (n\ C\ s\ z)$

$(0 : \text{Nat}) \mapsto \lambda C. \lambda s. \lambda z. z.$

► Scott encoding( $\tilde{\mu}_s$ ):

$(\text{Nat} : *) \mapsto \iota x. \Pi C : \text{Nat} \rightarrow *. (\Pi n : \text{Nat}. C\ (\text{S}\ n)) \rightarrow (C\ 0) \rightarrow (C\ x)$

$(\text{S} : \text{Nat} \rightarrow \text{Nat}) \mapsto \lambda n. \lambda C. \lambda s. \lambda z. s\ n$

$(0 : \text{Nat}) \mapsto \lambda C. \lambda s. \lambda z. z$

# Selfstar: II

- ▶ Induction principle for Church encoding:

$$\tilde{\mu}_c \vdash (\text{Ind} := \lambda C. \lambda s. \lambda z. \lambda n. n \ C \ s \ z) : \\ \Pi C : \text{Nat} \rightarrow *. \Pi n : \text{Nat}. ((C \ n) \rightarrow (C \ (\text{S } n))) \rightarrow C \ 0 \rightarrow \Pi n : \text{Nat}. C \ n$$

- ▶ Addition function:

$$\tilde{\mu}_c \vdash (\text{add} := \lambda n. \lambda m. \text{Ind } (\lambda y. \text{Nat}) (\lambda x. \text{S}) \ m \ n) : \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat}.$$

- ▶ Case analysis principle for Scott encoding:

$$\tilde{\mu}_s \vdash (\text{Case} := \lambda C. \lambda s. \lambda z. \lambda n. n \ C \ s \ z) : \\ \Pi C : \text{Nat} \rightarrow *. \Pi n : \text{Nat}. (C \ (\text{S } n)) \rightarrow C \ 0 \rightarrow \Pi n : \text{Nat}. C \ n$$

- ▶ addition function:

$$(\text{add} : \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat}) \mapsto \\ \lambda n. \lambda m. \text{Case } (\lambda n. \text{Nat}) \ (\lambda p. (\text{S } (\text{add } p \ m))) \ m \ n$$

# Selfstar: III

- ▶ Due to  $* : *$  and recursive definition, term does not correspond to proof, type does not correspond to formula.
- ▶ Future work: show a fragment of Selfstar that can be erased to  $F^\omega$ .

# Outline

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# Summary

- ▶ We seen Church and Scott encoding data and as alternatives to implement algebraic data type.
- ▶ The use of confluence in preservation proof.
- ▶ Several methods to prove confluence.
- ▶ Limits of dependent type system give rise to Selfstar.
- ▶ Introduced Selfstar.

# Last But Not Least

- ▶ Thank my advisor Prof. Aaron Stump.
- ▶ Thank my exam committee:  
Prof. Cesare Tinelli, Prof. Kasturi Varadarajan, Prof. Ted Herman, Prof. Douglas Jones.
- ▶ For all the audiences.