A Novel Rewriting Approach to System F a la Curry

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1 System F

1.1 Syntax

Types T ::= $X \mid T_1 \to T_2 \mid \forall X.T$

Terms t ::= $x \mid (t_1\ t_2) \mid \lambda x.t$

Reduction rules Contexts

$$C ::= * \mid C\ t \mid \lambda x.C \mid t\ C$$

Values v ::= $\lambda x.t$

Reductions

Full Beta Reduction.

$$C[(\lambda x.t)\ t'] \rightsquigarrow C[[t'/x]t]$$

1.2 Typing

Context $\Gamma ::= \cdot \mid \Gamma, x : T$

$$\frac{(x : T) \in \Gamma}{\Gamma \vdash x : T}\ T\_Var$$

$$\frac{\Gamma, x : T_1\ \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2}\ \to\_intro$$

$$\frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1\ t_2 : T_2}\ \to\_elim$$

$$\frac{\Gamma \vdash t : T \quad X \notin FV(\Gamma)}{\Gamma \vdash t : \forall X.T}\ \forall\_intro$$

Notice: $FV(\Gamma)$ is the set of all free type variables in Γ.

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t : [T'/X]T}\ \forall\_elim$$

2 Rewriting Simulation

2.1 Syntax

Types $T ::= X \mid T_1 \to T_2 \mid \forall X.T$

Terms $t ::= x \mid (t_1\ t_2) \mid \lambda x.t$

TypContext $\Gamma ::= \cdot \mid \Gamma, x : T.$

PreTypes $P ::= X \mid \Gamma x \mid \Gamma X \mid P \to P' \mid P_1 P_2 \mid \lambda x.P \mid \forall X.P$

RedContext $E ::= [] \mid E\ P \mid P\ E \mid \lambda x.E \mid \forall X.E \mid P \to E \mid E \to P$

Definition 1. *We define $FV(\Gamma)$ inductively:*

$FV(\cdot) = \emptyset.$

$FV(\Gamma, x : T) = FV(\Gamma) \cup FV(T)$, *where $FV(T)$ means the set of free type variables of T.*

Definition 2. *We define $\Gamma V(P)$ inductively:*

$\Gamma V(X) = \emptyset.$

$\Gamma V(\Gamma x) = FV(\Gamma).$

$\Gamma V(\Gamma X) = FV(\Gamma).$

$\Gamma V(P_1 P_2) = \Gamma V(P_2) \cup \Gamma V(P_1)$

$\Gamma V(P' \to P) = \Gamma V(P') \cup \Gamma V(P)$

$\Gamma V(\lambda x.P) = \Gamma V(P)$

$\Gamma V(\forall X.P) = \Gamma V(P) - \{X\}$

Notice:

1. We want to distict X from $[\cdot]X$.

2. X and ΓX belong to different syntactic category.

2.2 Syntactical Definitions

From now on, we will assume modulo renaming up to alpha equivalence. And we will assume capture avoiding for substitution.

Definition 3. *We can extend the constructions of Γx and ΓX to arbitrary terms and types as following.*

$\Gamma(t_1 t_2) \equiv (\Gamma t_1)(\Gamma t_2).$

$\Gamma(\lambda x.t) \equiv \lambda x.(\Gamma t)$. *This will invoke term variable renaming as nessesary.*

$\Gamma(T_1 \to T_2) \equiv (\Gamma T_1) \to (\Gamma T_2)$.

$\Gamma(\forall X.T) \equiv \forall X.(\Gamma T)$. *This will invoke type variable renaming as nessesary.*

Notice: $\Gamma t \in$ **PreTypes** and $\Gamma T \in$ **PreTypes** by the definition above.

Definition 4.
We define substitution $[T/X]P$ on pretypes P:

$[T/X](\Gamma x) \equiv ([T/X]\Gamma)x$, *where $[T/X]\Gamma$ means apply the substitution $[T/X]$ on each type in Γ.*

$[T/X]X \equiv T$.

$[T/X]\Gamma X' \equiv ([T/X]\Gamma)([T/X]X')$.

$[T/X](P' \to P) \equiv [T/X]P' \to [T/X]P$

$[T/X]P_1P_2 \equiv ([T/X]P_1)([T/X]P_2)$

$[T/X]\lambda x.P \equiv \lambda x.[T/X]P$

$[T/X]\forall Y.P \equiv \forall Y.([T/X]P)$. *This will invoke renaming and capture avoiding if nessesary.*

Definition 5. *Define a new operation $[x : T] \cdot P$ inductively on the structure of P:*

$[x : T] \cdot X \equiv X$.

$[x : T] \cdot \Gamma y \equiv [\Gamma, x : T]y$ *if $x \notin dom(\Gamma)$. Else if $x \in dom(\Gamma)$, then $[x : T] \cdot \Gamma y \equiv \Gamma y$.*

$[x : T] \cdot \Gamma Y \equiv [\Gamma, x : T]Y$ *if $x \notin dom(\Gamma)$. Else if $x \in dom(\Gamma)$, then $[x : T] \cdot \Gamma Y \equiv \Gamma Y$.*

$[x : T] \cdot (P_1P_2) \equiv ([x : T] \cdot P_1)([x : T] \cdot P_2)$

$[x : T] \cdot (P' \to P) \equiv [x : T] \cdot P' \to [x : T] \cdot P$

$[x : T] \cdot \lambda y.P \equiv \lambda y.[x : T] \cdot P$. *This will invoke renaming if nessesary.*

$[x : T] \cdot \forall X.P \equiv \forall X.[x : T] \cdot P$. *This will invoke renaming if nessesary.*

2.3 Reductions

$E[(\lambda x.P)] \rightsquigarrow_\lambda E[T \to [x : T] \cdot P]$.

$E[(T \to P)T] \rightsquigarrow_\epsilon E[P]$.

$E[P] \rightsquigarrow_\pi E[\forall X.P]$, where $X \notin \Gamma V(P)$ and $P \notin$ **Types**.

$E[\forall X.P] \rightsquigarrow_\iota E[[T/X]P]$.

$E[\Gamma x] \rightsquigarrow_s E[\Gamma T]$, where $(x : T) \in \Gamma$.

$E[\mathit{\Gamma} T] \rightsquigarrow_r E[T]$.

Notice: we use $\rightsquigarrow$ as a shorthand for $\rightsquigarrow_\lambda \cup \rightsquigarrow_\epsilon \cup \rightsquigarrow_\pi \cup \rightsquigarrow_\iota \cup \rightsquigarrow_s \cup \rightsquigarrow_r$.

We can go ahead and define $\overset{*}{\rightsquigarrow}$:

$$\frac{}{P \overset{*}{\rightsquigarrow} P}\ \mathit{ref}$$

$$\frac{P \overset{*}{\rightsquigarrow} P'' \quad P'' \overset{*}{\rightsquigarrow} P'}{P \overset{*}{\rightsquigarrow} P'}\ \mathit{trans}$$

Notations: we use $\forall X^n.P$ as a shorthand for $\forall X_n.\forall X_{n-1}...\forall X_1.P$ and $\forall X^0.P \equiv P$.

3 Important Lemmas I

Lemma 1 (Congruence). *If $P \overset{*}{\rightsquigarrow} P'$, then $E[P] \overset{*}{\rightsquigarrow} E[P']$.*

This lemma is straightforward by the definition of reductions.

Lemma 2 (Closed Under Substitution). *If $P \rightsquigarrow P'$, then for any type level substitution δ, we have $\delta P \rightsquigarrow \delta P'$.*

Proof. By induction on the derivation of $P \rightsquigarrow P'$.

Case:

$$\frac{}{\lambda x.P \rightsquigarrow_\lambda T \to [x : T] \cdot P'}$$

We have $\lambda x.\delta P \rightsquigarrow_\lambda \delta T \to [x : \delta T] \cdot (\delta P) \equiv \delta(T \to [x : T] \cdot P)$. So it is the case.

Case:

$$\frac{}{(T \to P)T \rightsquigarrow_\epsilon P}$$

We have $\delta((T \to P)T) \equiv (\delta T \to \delta P)\delta T \rightsquigarrow_\epsilon \delta P$. So it is the case.

Case:

$$\frac{}{\forall X.P \rightsquigarrow_\iota [T/X]P}$$

We have $\forall X.\delta P \rightsquigarrow_\iota [T/X](\delta P) \equiv \delta([T/X]P)$. Assuming modulo renaming and capture avoiding.

Case:

$$\frac{X \notin \mathit{\Gamma} V(P) \quad P \notin \textbf{Types}}{P \rightsquigarrow_\pi \forall X.P}$$

$\delta P \rightsquigarrow_\pi \forall X.(\delta P) \equiv \delta(\forall X.P)$, since $\delta P \notin \textbf{Types}$ and $X \notin \mathit{\Gamma} V(\delta P)$.

Case:

$$\frac{(x:T)\in\Gamma}{\Gamma x \leadsto_s \Gamma T}$$

$\delta\Gamma x \leadsto_s \delta\Gamma\delta T$, since $(x:\delta T)\in\delta\Gamma$.

Case:

$$\frac{}{\Gamma T \leadsto_r T}$$

$\delta\Gamma\delta T \leadsto_r \delta T$. So it is the case.

Case:

$$\frac{P \leadsto P'}{\lambda x.P \leadsto \lambda x.P'}$$

By IH, we have $\delta P \leadsto \delta P'$. Thus we have $\lambda x.\delta P \equiv \delta(\lambda x.P) \leadsto \lambda x.\delta P' \equiv \delta(\lambda x.P')$.

Case:

$$\frac{P_2 \leadsto P_2'}{P_1 P_2 \leadsto P_1 P_2'}$$

By IH,we have $\delta P_2 \leadsto \delta P_2'$. So $\delta P_1\delta P_2 \leadsto \delta P_1 \delta P_2'$. So it is the case.

The other cases are similar.

Lemma 3 (Compatible with TypContext Action). *If $P \leadsto P'$, then $[x:T]\cdot P \leadsto [x:T]\cdot P'$.*

Proof. By induction on the derivation of $P \leadsto P'$.

Base Case 1: $\lambda x.P \leadsto_\lambda T_1 \to [x:T_1]\cdot P$. We also have $[y:T]\cdot(\lambda x.P) \equiv \lambda x.([y:T]\cdot P) \leadsto_\lambda T_1 \to [x:T_1]\cdot([y:T]\cdot P) \equiv [y:T]\cdot(T_1 \to [x:T_1]\cdot P)$. So it is the case.

Base Case 2: $(T \to P)T \leadsto_\epsilon P$. We also have $[x:T']\cdot((T\to P)T) \equiv (T \to [x:T']\cdot P)T \leadsto_\epsilon [x:T']\cdot P$.

Base Case 3: $P \leadsto_\pi \forall X.P$. Then $[x:T]\cdot P \leadsto_\pi \forall X.[x:T]\cdot P \equiv [x:T]\cdot(\forall X.P)$.

Base Case 4: $\forall X.P \leadsto_\iota [U/X]P$. Then $[x:T]\cdot(\forall X.P) \equiv \forall X.([x:T]\cdot P) \leadsto_\iota [x:T]\cdot([U/X]P)$.

Base Case 5: $\Gamma x \leadsto_s \Gamma T$, where $(x:T)\in\Gamma$. It is the case.

Base Case 6: $\Gamma T \leadsto_r T$. Obvious it is the case.

Step Case: $P \equiv E[P_1] \leadsto P' \equiv E[P_2]$, where $P_1 \leadsto P_2$. We need to show $[x:T]\cdot E[P_1] \leadsto [x:T]\cdot E[P_2]$. By case split on the form of E:

If $E \equiv E \to P''$. Then $[x:T]\cdot(P_1 \to P'') \equiv [x:T]\cdot P_1 \to [x:T]\cdot P''$. By IH, we have $[x:T]\cdot P_1 \leadsto [x:T]\cdot P_2$. Thus $[x:T]\cdot P_1 \to [x:T]\cdot P'' \leadsto [x:T]\cdot P_2 \to [x:T]\cdot P''$.

If $E \equiv EP''$. Then $[x:T]\cdot(P_1P'') \equiv [x:T]\cdot P_1([x:T]\cdot P'')$. By IH, we have $[x:T]\cdot P_1 \leadsto [x:T]\cdot P_2$. Thus $[x:T]\cdot P_1([x:T]\cdot P'') \leadsto [x:T]\cdot P_2([x:T]\cdot P'')$.

If $E \equiv \forall X.E$. Then $[x : T] \cdot (\forall X.P_1) \equiv \forall X.([x : T] \cdot P_1)$. By IH, we have $[x : T] \cdot P_1 \leadsto [x : T] \cdot P_2$. Thus $\forall X.([x : T] \cdot P_1) \leadsto \forall X.([x : T] \cdot P_2)$.

If $E \equiv \lambda y.E$. Then $[x : T] \cdot (\lambda y.P_1) \equiv \lambda y.([x : T] \cdot P_1)$. By IH, we have $[x : T] \cdot P_1 \leadsto [x : T] \cdot P_2$. Thus $\lambda y.([x : T] \cdot P_1) \leadsto \lambda y.([x : T] \cdot P_2)$.

Lemma 4. *If $\Gamma t \stackrel{*}{\leadsto} \forall Y^m.P$, then $\{Y_1, ..., Y_m\} \cap FV(\Gamma) = \emptyset$. Assuming modulo alpha equivalence.*

Proof. By induction on the length of $\Gamma t \stackrel{*}{\leadsto} \forall Y^m.P$.

Base Case: $m = 0$ and $P \equiv \Gamma t$. So it is the case.

Step Case: $\Gamma t \stackrel{*}{\leadsto} P' \leadsto \forall Y^m.P$. Now case split on the last step $\leadsto$:

$P' \equiv \forall Y^{m-1}.P \leadsto_\pi \forall Y^m.P$. By IH, we have $\{Y_1, ..., Y_{m-1}\} \cap FV(\Gamma) = \emptyset$. And by the restriction on the $\leadsto_\pi$, we know that $Y_m \notin \Gamma V(P)$. We can use renaming to make sure $Y_m \notin FV(\Gamma)$. Thus we have $\{Y_1, ..., Y_m\} \cap FV(\Gamma) = \emptyset$.

$P' \equiv \forall Y^m.P \leadsto_\iota \forall Y^{m-1}.[U/Y_m]P$. By IH, we have $\{Y_1, ..., Y_m\} \cap FV(\Gamma) = \emptyset$. Thus $\{Y_1, ..., Y_{m-1}\} \cap FV(\Gamma) = \emptyset$.

$P' \equiv \forall Y^m.P_1 \leadsto \forall Y^m.P$, where $P_1 \leadsto P$. By IH, we have $\{Y_1, ..., Y_m\} \cap FV(\Gamma) = \emptyset$. So it is the case.

Lemma 5 (Abstraction Inversion). *If $\forall X^n.(\lambda x.P) \stackrel{*}{\leadsto} T$, then there are T_1, P', m such that $\forall X^n.\lambda x.P \stackrel{*}{\leadsto} \forall Y^m.\lambda x.P' \leadsto_\lambda \forall Y^m.(T_1 \to [x : T_1] \cdot P') \stackrel{*}{\leadsto} T$ and $\forall X^n.P \stackrel{*}{\leadsto} P'$.*

Proof. By induction on the length of $\forall X^n.(\lambda x.P) \stackrel{*}{\leadsto} T$.

Base Case: It is impossible to arise.

Step Case: $\forall X^n.(\lambda x.P) \leadsto P' \stackrel{*}{\leadsto} T$. Case split on the first step $\leadsto$.

If $\forall X^n.(\lambda x.P) \leadsto_\lambda \forall X^n.(T_1 \to [x : T_1]P) \stackrel{*}{\leadsto} T$. So it is the case. In this case, $\forall X^n.P \stackrel{*}{\leadsto} P$.

If $\forall X^n.\lambda x.P \leadsto_\pi \forall X^{n+1}.\lambda x.P \stackrel{*}{\leadsto} T$. By IH, we have $\forall X^{n+1}.P \stackrel{*}{\leadsto} P'$. Thus we have $\forall X^n.P \leadsto_\pi \forall X^{n+1}.P \stackrel{*}{\leadsto} P'$.

If $\forall X^n.\lambda x.P \leadsto_\iota \forall X^{n-1}.\lambda x.[U/X_n]P$. By IH, $\forall X^{n-1}.[U/X_n]P \stackrel{*}{\leadsto} P'$. Thus we have $\forall X^n.P \leadsto_\iota \forall X^{n-1}.[U/X_n]P \stackrel{*}{\leadsto} P'$.

If $\forall X^n.\lambda x.P \leadsto \forall X^n.\lambda x.P''$, where $P \leadsto P''$. Thus $\forall X^n.P \leadsto \forall X^n.P'' \stackrel{*}{\leadsto} P'$.

Lemma 6 (Arrow Inference). *If $\forall X^n.(T \to P) \stackrel{*}{\leadsto} T'$, then $T' \equiv \forall Y^m.(T_1 \to T_2)$, $\delta T \equiv T_1$, $\delta P \stackrel{*}{\leadsto} T_2$ for some type level substitution δ.*

Proof. By induction on the length of $\forall X^n.(T \to P) \stackrel{*}{\leadsto} T'$.

Base Case: $\forall X^n.(T \to P) \equiv T'$. It is the case.

Step Case: $\forall X^n.(T \to P) \leadsto P' \stackrel{*}{\leadsto} T'$. Case split on the first step.

If $\forall X^n.(T \to P) \leadsto_\pi \forall X^{n+1}.(T \to P) \stackrel{*}{\leadsto} T'$. By IH, it is the case.

If $\forall X^n.(T \to P) \leadsto_\iota \forall X^{n-1}.([U/X_n]T \to [U/Xn]P) \overset{*}{\leadsto} T'$. By IH, we have $\delta([U/X_n]T) \equiv [\delta U/X_n]\delta T \equiv T_1$ and $\delta([U/Xn]P) \overset{*}{\leadsto} T_2$. So it is the case.

If $\forall X^n.(T \to P) \leadsto \forall X^n.(T \to P'') \overset{*}{\leadsto} T'$, where $P \leadsto P''$. By IH, we have $\delta T \equiv T_1$ and $\delta P'' \overset{*}{\leadsto} T_2$. So $\delta P \overset{*}{\leadsto} \delta P'' \overset{*}{\leadsto} T_2$ by compatible with substitution. Thus it is the case.

Notice: We can acutally construct a δ from the proof above. And moreover, we can construct a δ in which $dom(\delta) \cap V = \emptyset$ for any given set of type variables V.

Lemma 7 (Application Inversion). *If $\forall X^n.P_1 P_2 \overset{*}{\leadsto} T$, then there exists P', m, T_1 such that $\forall X^n.P_1 P_2 \overset{*}{\leadsto} \forall Y^m.(T_1 \to P')T_1 \leadsto_\epsilon \forall Y^m.P' \overset{*}{\leadsto} T$. Also we have $\forall X^n.P_1 \overset{*}{\leadsto} T_1 \to P'$ and $\forall X^n.P_2 \overset{*}{\leadsto} T_1$.*

Proof. By induction on the length of $\forall X^n.P_1 P_2 \overset{*}{\leadsto} T$.

Base Case: Impossible to arise.

Step Case: $\forall X^n.P_1 P_2 \leadsto P_3 \overset{*}{\leadsto} T$. Case Split on $\leadsto$:

If $P_1 \equiv T_1 \to P'$ and $P_2 \equiv T_1$. Then $\forall X^n.P_1 P_2 \leadsto_\epsilon P_3 \equiv \forall X^n.P' \overset{*}{\leadsto} T$. Thus it is the case.

If $\forall X^n.P_1 P_2 \leadsto_\pi \forall X^{n+1}.P_1 P_2 \overset{*}{\leadsto} T$. By IH, it is the case.

If $\forall X^n.P_1 P_2 \leadsto_\iota \forall X^{n-1}.[U/X_n](P_1 P_2) \overset{*}{\leadsto} T$. By IH, we have $\forall X^{n-1}.[U/X_n]P_1 \overset{*}{\leadsto} T_1 \to P', \forall X^{n-1}.[U/X_n]P_2 \overset{*}{\leadsto} T_1$. Thus we have $\forall X^n.P_1 \leadsto_\iota \forall X^{n-1}.[U/X_n]P_1 \overset{*}{\leadsto} T_1 \to P'$ and $\forall X^n.P_2 \leadsto_\iota \forall X^{n-1}.[U/X_n]P_2 \overset{*}{\leadsto} T_1$.

If $\forall X^n.P_1 P_2 \leadsto \forall X^n P_1' P_2 \overset{*}{\leadsto} T$, where $P_1 \leadsto P_1'$. By IH, $\forall X^n.P_1' \overset{*}{\leadsto} T_1 \to P'$, $\forall X^n.P_2 \overset{*}{\leadsto} T_1$. Thus $\forall X^n.P_1 \leadsto \forall X^n.P_1' \overset{*}{\leadsto} T_1 \to P$. So it is the case.

Lemma 8. *If $\forall X^n.P \overset{*}{\leadsto} T$, then there exists δ such that $\delta P \overset{*}{\leadsto} T$.*

Proof. By induction on the length of $\forall X^n.P \overset{*}{\leadsto} T$.

Base Case: Obvious.

Step Case: $\forall X^n.P \leadsto P' \overset{*}{\leadsto} T$. Case split on the first step.

$\forall X^n.P \leadsto_\pi \forall X^{n+1}.P \equiv P' \overset{*}{\leadsto} T$. By IH, it is the case.

$\forall X^n.P \leadsto_\iota \forall X^{n-1}.[U/X_n]P \equiv P' \overset{*}{\leadsto} T$. By IH, $\delta([U/X_n]P) \overset{*}{\leadsto} T$. So it is the case.

$\forall X^n.P \leadsto \forall X^n.P' \overset{*}{\leadsto} T$, where $P \leadsto P'$. By IH, we have $\delta P' \overset{*}{\leadsto} T$. Thus $\delta P \leadsto \delta P' \overset{*}{\leadsto} T$. So it is the case.

Notice: We can acutally construct a δ from the proof above. And moreover, we can construct a δ in which $dom(\delta) \cap V = \emptyset$ for any given set of type variables V.

4 Correctness of the Simulation

Theorem 1. *If $\Gamma \vdash t : T$, then $\Gamma t \overset{*}{\leadsto} T$.*

Proof. By induction on the derivation of $\Gamma \vdash t : T$.

Base Case:

$$\frac{(x : T) \in \Gamma}{\Gamma \vdash x : T} \ T\_Var$$

We know that $\Gamma x \rightsquigarrow_s \Gamma T \rightsquigarrow_r T$, where $(x : T) \in \Gamma$. So it is the case.

Step Case:

$$\frac{\Gamma, x : T_1 \ \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2} \ \to\_intro$$

We have $\Gamma \lambda x.t \rightsquigarrow_\lambda T_1 \to [\Gamma, x : T_1]t$. By IH, we know that $[\Gamma, x : T_1]t \overset{*}{\rightsquigarrow} T_2$. By congruence lemma, we know that $\Gamma \lambda x.t \equiv \lambda x.\Gamma t \overset{*}{\rightsquigarrow} T_1 \to [\Gamma, x : T_1]t \overset{*}{\rightsquigarrow} T_1 \to T_2$.

Step Case:

$$\frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 \ t_2 : T_2} \ \to\_elim$$

By IH, we know that $\Gamma t_1 \overset{*}{\rightsquigarrow} T_1 \to T_2$ and $\Gamma t_2 \overset{*}{\rightsquigarrow} T_1$. Thus we have $\Gamma t_1 t_2 \equiv (\Gamma t_1)(\Gamma t_2) \overset{*}{\rightsquigarrow} (T_1 \to T_2)T_1 \rightsquigarrow_\epsilon T_2$. Thus it is the case.

Step Case:

$$\frac{\Gamma \vdash t : T \quad X \notin FV(\Gamma)}{\Gamma \vdash t : \forall X.T} \ \forall\_intro$$

By IH, we know that $\Gamma t \rightsquigarrow T$. And $\Gamma t \rightsquigarrow_\pi \forall X.(\Gamma t) \overset{*}{\rightsquigarrow} \forall X.T$, where $X \notin FV(\Gamma)$.

Step Case:

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t : [T'/X]T} \ \forall\_elim$$

By IH, we have $\Gamma t \overset{*}{\rightsquigarrow} \forall X.T \rightsquigarrow_\iota [T'/X]T$. Thus it is the case.

Theorem 2. *If $\Gamma t \overset{*}{\rightsquigarrow} T$, then $\Gamma \vdash t : T$.*

Proof. By induction on the structure of t.

Base Case:$t \equiv x$.

So we have $\Gamma x \overset{*}{\rightsquigarrow} T$. So $\overset{*}{\rightsquigarrow}$ can only be a combination of $\rightsquigarrow_r, \rightsquigarrow_s, \rightsquigarrow_\pi, \rightsquigarrow_\iota$. Due to the restriction on the $\rightsquigarrow_\pi$. $\rightsquigarrow_\iota$ can not change the type in Γ. So we have $(x : T') \in \Gamma$, where $\forall Y^n.T' \equiv T$. So $\Gamma \vdash x : T$.

Step Case:$t \equiv t_1 t_2$.

We have $(\Gamma t_1)(\Gamma t_2) \overset{*}{\rightsquigarrow} T$. By lemma 7, we have $(\Gamma t_1)(\Gamma t_2) \overset{*}{\rightsquigarrow} \forall Y^n.(T_1 \to P')T_1 \rightsquigarrow_\epsilon \forall Y^n.P' \overset{*}{\rightsquigarrow} T$, $\Gamma t_1 \overset{*}{\rightsquigarrow} T_1 \to P$, $\Gamma t_2 \overset{*}{\rightsquigarrow} T_1$ and $\forall Y^n.P' \overset{*}{\rightsquigarrow} T$. By lemma 8, we have $\delta P \overset{*}{\rightsquigarrow} T$ and $dom(\delta) \cap FV(\Gamma) = \emptyset$. So we have $\delta(\Gamma t_1) \equiv \Gamma t_1 \stackrel{*}{\leadsto} \delta T_1 \to \delta P \stackrel{*}{\leadsto} \delta T_1 \to T$. And we also have $\delta(\Gamma t_2) \equiv \Gamma t_2 \stackrel{*}{\leadsto} \delta T_1$. By IH, we have $\Gamma \vdash t_1 : \delta T_1 \to T$ and $\Gamma \vdash t_2 : \delta T_1$. So we have $\Gamma \vdash t_1 t_2 : T$.

Step Case: $t \equiv \lambda x.t'$.

We have $\lambda x.(\Gamma t') \stackrel{*}{\leadsto} T$. By lemma 5, we have $\lambda x.\Gamma t' \stackrel{*}{\leadsto} \forall Y^n.(\lambda x.P) \leadsto_\lambda \forall Y^n.(T_1 \to [x : T_1]P) \stackrel{*}{\leadsto} T$ and $\Gamma t' \stackrel{*}{\leadsto} P$. Thus $[\Gamma, x : T_1]t' \stackrel{*}{\leadsto} [x : T_1] \cdot P$ by compatible with typing context. By lemma 6, we have $T \equiv \forall Z^n.(T_3 \to T_4)$, $\delta T_1 \equiv T_3$ and $[x : \delta T_1]\delta P \stackrel{*}{\leadsto} T_4$ and $dom(\delta) \cap FV(\Gamma) = \emptyset$. So we have $\delta([\Gamma, x : T_1]t') \equiv [\Gamma, x : \delta T_1]t' \stackrel{*}{\leadsto} [x : \delta T_1]\delta P \stackrel{*}{\leadsto} T_4$. Thus we have $[\Gamma, x : T_3]t' \stackrel{*}{\leadsto} T_4$. By IH, we have $\Gamma, x : T_3 \vdash t' : T_4$. Thus by lemma 4 we have $\Gamma \vdash \lambda x.t' : \forall Z^n.(T_3 \to T_4)$.

5 Type Preservation

5.1 Important Lemma II

Lemma 9. *If $[\Gamma, x : T_1]t_1 \stackrel{*}{\leadsto} T$ and $\Gamma t_2 \stackrel{*}{\leadsto} T_1$, then $\Gamma([t_2/x]t_1) \stackrel{*}{\leadsto} T$.*

Proof. By induction on the structure of t_1.

Base Case: $t_1 \equiv x$. We have $[\Gamma, x : T_1]x \stackrel{*}{\leadsto} T$ and $\Gamma t_2 \stackrel{*}{\leadsto} T_1$. We know that $T \equiv \forall X^n.T_1$, where $\{X_1, ..., X_n\} \cap FV(\Gamma) = \emptyset$. Thus $\Gamma([t_2/x]x) \equiv \Gamma t_2 \stackrel{*}{\leadsto} \forall X^n.(\Gamma t_2) \stackrel{*}{\leadsto} \forall X^n.T_1$. So it is the case.

Step Case: $t_1 \equiv \lambda y.t'$. We have $[\Gamma, x : T_1](\lambda y.t') \stackrel{*}{\leadsto} T$ and $\Gamma t_2 \stackrel{*}{\leadsto} T_1$. By lemma 5, we have $[\Gamma, x : T_1](\lambda y.t') \equiv \lambda y.[\Gamma, x : T_1]t' \stackrel{*}{\leadsto} \forall X^n.\lambda y.P \leadsto_\lambda \forall X^n.(T_x \to [y : T_x] \cdot P) \stackrel{*}{\leadsto} T$ and $[\Gamma, x : T_1]t' \stackrel{*}{\leadsto} P$. By lemma 6, we have $T \equiv \forall Z^m.(T_a \to T_b)$. And we have a type substitution δ, where $dom(\delta) \cap (FV(\Gamma) \cup FV(T_1)) = \emptyset$, such that $\delta([y : T_x]P) \stackrel{*}{\leadsto} T_b$ and $\delta T_x \equiv T_a$. So by compatible with typcontext action, we have $[\Gamma, x : T_1, y : T_x]t' \stackrel{*}{\leadsto} [y : T_x] \cdot P$. By closed under substitution, we have $[\Gamma, x : T_1, y : \delta T_x]t' \stackrel{*}{\leadsto} [y : \delta T_x] \cdot \delta P \stackrel{*}{\leadsto} T_b$. So we have $\Gamma(\lambda y.[t_2/x]t') \equiv \lambda y.\Gamma([t_2/x]t') \leadsto_\lambda \delta T_x \to [\Gamma, y : \delta T_x]([t_2/x]t')$. By IH, we have $[\Gamma, y : \delta T_x]([t_2/x]t') \stackrel{*}{\leadsto} T_b$. So we have $\Gamma(\lambda y.[t_2/x]t') \equiv \lambda y.\Gamma([t_2/x]t') \leadsto_\lambda \delta T_x \to [\Gamma, y : \delta T_x]([t_2/x]t') \stackrel{*}{\leadsto} \forall Z^m.(\delta T_x \to [\Gamma, y : \delta T_x]([t_2/x]t')) \stackrel{*}{\leadsto} \forall Z^m.(\delta T_x \to T_b) \equiv \forall Z^m.(T_a \to T_b)$. So it is the case.

Step Case: $t_1 \equiv t_a t_b$. We have $[\Gamma, x : T_1](t_a t_b) \stackrel{*}{\leadsto} T_1$ and $\Gamma t_2 \stackrel{*}{\leadsto} T_1$. By lemma 7, we have $[\Gamma, x : T_1](t_a t_b) \stackrel{*}{\leadsto} \forall Y^n.(T_x \to P)T_x \leadsto_\epsilon \forall Y^n.P \stackrel{*}{\leadsto} T_1$, $[\Gamma, x : T_1]t_a \stackrel{*}{\leadsto} T_x \to P$ and $[\Gamma, x : T_1]t_b \stackrel{*}{\leadsto} T_x$. By lemma 8, we know there is a type substitution δ, where $dom(\delta) \cap (FV(\Gamma) \cup FV(T_1)) = \emptyset$, such that $\delta P \stackrel{*}{\leadsto} T_1$. Thus $[\Gamma, x : T_1]t_a \stackrel{*}{\leadsto} \delta T_x \to \delta P \stackrel{*}{\leadsto} \delta T_x \to T_1$ and $[\Gamma, x : T_1]t_b \stackrel{*}{\leadsto} \delta T_x$. By IH, we have $\Gamma([t_2/x]t_a) \stackrel{*}{\leadsto} \delta T_x \to T_1$ and $\Gamma([t_2/x]t_b) \stackrel{*}{\leadsto} \delta T_x$. Thus $\Gamma([t_2/x]t_a)[t_2/x]t_b \stackrel{*}{\leadsto} (\delta T_x \to T_1)\delta T_x \leadsto_\epsilon T_1$. So it is the case.

5.2 Type Preservation

Theorem 3. *If $\Gamma(\lambda x.t_1)t_2 \stackrel{*}{\leadsto} T$ and $(\lambda x.t_1)t_2 \leadsto_\beta [t_2/x]t_1$, then $\Gamma[t_2/x]t_1 \stackrel{*}{\leadsto} T$.*

Proof. Since we know that $\Gamma(\lambda x.t_1)t_2 \stackrel{*}{\leadsto} T$. By lemma 7, we know that $\Gamma \lambda x.t_1 \stackrel{*}{\leadsto} T_x \to P$, $\Gamma t_2 \stackrel{*}{\leadsto} T_x$ and $\forall Y^m.P \stackrel{*}{\leadsto} T$. By lemma 8, there is a type level subsitution δ, where $dom(\delta) \cap FV(\Gamma) = \emptyset$, such that $\delta P \stackrel{*}{\leadsto} T$. Thus we have $\Gamma \lambda x.t_1 \stackrel{*}{\leadsto} \delta T_x \to T$ and $\Gamma t_2 \stackrel{*}{\leadsto} \delta T_x$. By lemma 5, $\lambda x.\Gamma t_1 \stackrel{*}{\leadsto} \forall Z^q.\lambda x.P_1 \leadsto_\lambda \forall Z^q.(T_a \to [x : T_a] \cdot P_1) \stackrel{*}{\leadsto} \delta T_x \to T$ and $\Gamma t_1 \stackrel{*}{\leadsto} P_1$. By lemma 6, we have a type level substitution δ', where $dom(\delta') \cap FV(\Gamma) = \emptyset$, such that $\delta' T_a \equiv \delta T_x$, and $[x : \delta' T_a] \cdot \delta' P_1 \stackrel{*}{\leadsto} T$. Thus $[\Gamma, x : \delta T_x]t_1 \stackrel{*}{\leadsto} [x : \delta' T_a] \cdot \delta' P_1 \stackrel{*}{\leadsto} T$. By lemma 9, we have $\Gamma([t_2/x]t_1) \stackrel{*}{\leadsto} T$. So it is the case.

Theorem 4. *If $\Gamma t \stackrel{*}{\leadsto} T$ and $t \leadsto_\beta t'$, then $\Gamma t' \stackrel{*}{\leadsto} T$.*

This theorem can be obtain by induction on the derivation of $t \leadsto_\beta t'$. I don't expect any difficulty in this situation since we have the help of lemma 5, lemma 6, lemma 7, lemma 8. In another word, proved by confident.