The Early History of the Brick Factory Problem

LOWELL BEINEKE AND ROBIN WILSON

Few mathematical concepts can trace their origins to the labor camps of World War II and to the mathematical jottings of an abstract artist. In this article we trace the origins of the crossing number of a graph, the minimum number of crossings that arise when the graph is drawn in the plane, with particular reference to the war-time experiences of the Hungarian number-theorist Paul Turán and to the geometrical explorations of the British artist Anthony Hill.

Origins

> *In July, 1944 the danger of deportation was real in Budapest and a reality outside Budapest. We worked near Budapest, in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yards. All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yards, and unload them there. We had a reasonable piece rate for the trucks, and the work itself was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short, this caused a lot of trouble and loss of time which was rather precious to all of us (for reasons not to be discussed here). We were all sweating and cursing at such occasions, I too; but* nolens-volens *the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings? I realized after several days that the actual situation could have been improved, but the exact solution of the general problem with m kilns and n storage yards seemed to be very difficult…*

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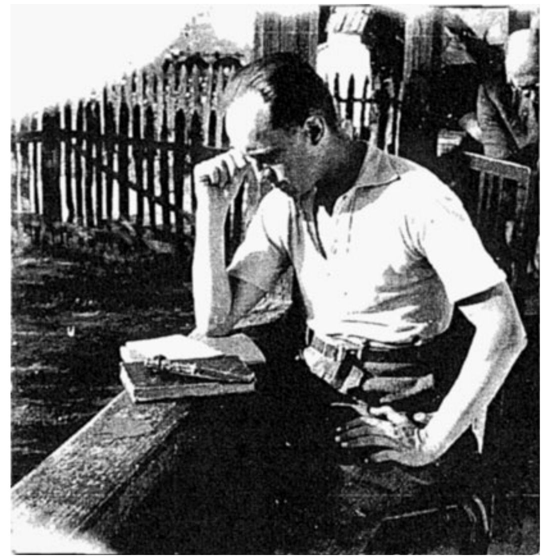

Figure 1. Paul Turán in a war-time labor camp.

This description of the brick factory problem was written by Paul Turán for the first issue of the *Journal of Graph Theory* [25]. Earlier, a slightly different version had appeared in a letter to Richard Guy in February 1968 and was recounted in [12] (see below). Figure 1 shows Turán in the labor camp during the war-time years.

Paul Turán's letter to Richard Guy

In 1944 our labor combattation had the extreme luck to work —thanks to some very rich comrades—in a brick factory near Budapest. Our work was to bring out bricks from the ovens where they were made and carry them on small vehicles which run on rails in some of several open stores which happened to be empty. Since one could never be sure which store will be available, each oven was connected by rail with each store. Since we had to settle a fixed amount of loaded cars daily it was our interest to finish it as soon as possible. After being loaded in the (rather warm) ovens the vehicles run smoothly with not much effort; the only trouble arose at the crossing of two rails. Here the cars jumped out, the bricks fell down; a lot of extra work and loss of time arose. Having this experience a number of times it occurred to me why on earth did they build the rail system so uneconomically; minimizing the number of crossings the production could be made much more economical.

A related problem, the houses-and-utilities problem, is of unknown origin. It was described by the mathematical puzzler Henry Dudeney [4, 5] as 'as old as the hills', and as

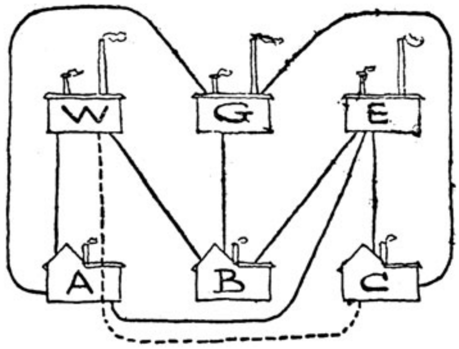

Figure 2. A 'solution' of the utilities problem.

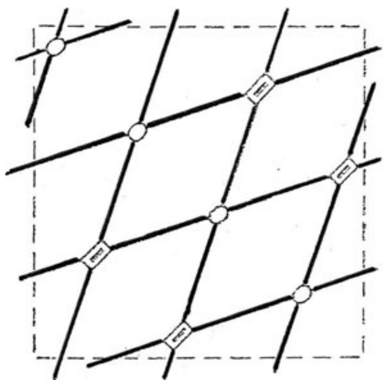

Figure 3. A utilities problem on a torus.

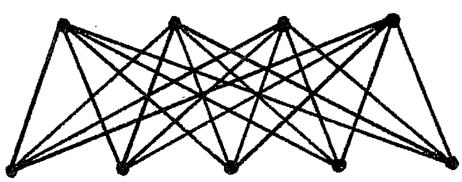

Figure 4. The complete bipartite graph $K_{4,5}$.

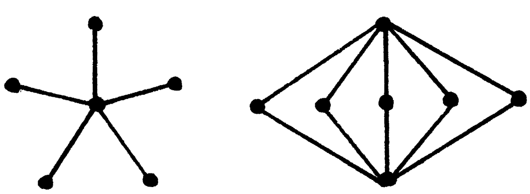

Figure 5. Plane drawings of $K_{1,5}$ and $K_{2,5}$.

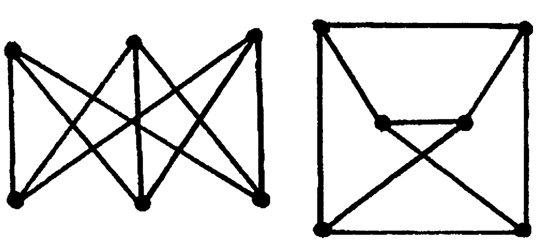

Figure 6. Two drawings of $K_{3,3}$.

'an extinct volcano [that] bursts into eruption in a surprising manner' which is 'much older than electric lighting, or even gas'.

The puzzle concerns three neighbours A, B, and C who wish their homes to be connected to the utilities of water, gas and electricity (W, G, and E) in such a way that no connections cross. According to Sam Loyd [20], his father, the American puzzler Sam Loyd, described it as a 'hoary old chestnut … which I first brought out in 1900'. A number of entertaining variations on the problem are described by David Kullman [18].

It turns out that no solution exists unless we flout the rules (as Dudeney does in [4] and in Problem 251 of [5]) and allow one of the connections to pass through a house, as shown in Figure 2.

In 1961, the Scottish mathematician T. H. O'Beirne [22] described a utilities problem involving four utilities and four houses situated on a torus; this version does have a solution (see Figure 3).

We can describe these problems in mathematical terms. The complete bipartite graph $K_{m,n}$ is obtained by joining each of a set of m vertices to each of a set of n other vertices (see Figure 4, which shows $K_{4,5}$); for Turán's brick factory problem, the m vertices correspond to the kilns and the n vertices to the storage yards. The problem asks how many crossings are required if the graph $K_{m,n}$ is drawn in the plane.

It is easily seen that if m or n is 1 or 2, then $K_{m,n}$ can be drawn in the plane without any edges crossing; Figure 5 shows plane drawings of $K_{1,5}$ and $K_{2,5}$.

However, no such drawing is possible if both m and n are 3 or more. In particular, the 'utilities graph' $K_{3,3}$ has no plane drawing, although a drawing exists with just one edge-crossing (see Figure 6).

In general, we define the crossing number $\mathrm{cr}(G)$ of a graph G to be the minimum number of edge-crossings needed when G is drawn in the plane (assuming only two edges appear at each crossing), and we see that $\mathrm{cr}(K_{1,5}) = \mathrm{cr}(K_{2,5}) = 0$ and $\mathrm{cr}(K_{3,3}) = 1$. Turán's brick factory problem asks for the crossing number $\mathrm{cr}(K_{m,n})$, for any natural numbers m and n. The following table lists some values of $\mathrm{cr}(K_{m,n})$; for example, $\mathrm{cr}(K_{4,5}) = 8$.

m/n	3	4	5	6	7
3	1	2	4	6	9
4	2	4	8	12	18
5	4	8	16	24	36
6	6	12	24	36	54

Some time after the end of the war, according to Guy [11], Turán communicated the brick factory problem to other mathematicians. In October 1952, during his first visit to Poland, he posed it in lectures in Wrocław and Warsaw. Solutions were proposed almost simultaneously by the probabilist Kazimierz Urbanik [26], who had attended the Wrocław lecture and who described his solution at a topological seminar there on November 12 1952, and by the topologist Kazimierz Zarankiewicz [28], who was present at the Warsaw lecture.

Zarankiewicz subsequently submitted his solution [29] to *Fundamenta Mathematicae* on December 15 1952.[1] His statement of the brick factory problem follows; note that he makes explicit the restriction that no three edges may intersect at an internal point.

Theorem i. *If*

(α) *in the Euclidean plane two sets of points, A and B, are given, A consisting of p points* $a_1, a_2, a_3, \ldots, a_p$, *and B consisting of q points* $b_1, b_2, b_3, \ldots, b_q$ *(p and q are natural numbers)*;
(β) *for each pair of points* a_i, b_j, *where* $i = 1, 2, 3, \ldots, p$, $j = 1, 2, 3, \ldots, q$, *there exists a simple arc lying in the plane and having the points* a_i, b_j *as its end points*;
(γ) *the arcs lie in such a way that no three arcs have an interior point (i.e., a point that is not an end point) in common*;
(δ) $K(p, q)$ *denotes the smallest number of intersection points of arcs*;

then the following formulas hold:

$$K(2k, 2n) = \left(k^2 - k\right)\left(n^2 - n\right), \tag{1}$$

$$K(2k, 2n+1) = \left(k^2 - k\right)n^2, \tag{2}$$

$$K(2k+1, 2n+1) = k^2 n^2. \tag{3}$$

In an endnote, Zarankiewicz mentions Urbanik's interest in the problem:

As has been found by K. Urbanik and noticed by A. Rényi and P. Turán, independently of one another, formulas (1), (2) and (3) can be written in the form of a single formula,

$$K(p, q) = (p - 1 - E(p/2))E(p/2)(q - 1 - E(q/2))E(q/2),$$

where $E(x)$ *denotes the greatest integer* $\leq x$.

So Zarankiewicz's claim is the following:

Zarankiewicz's conjecture: *The minimum number of crossings in any drawing of the complete bipartite graph* $K_{p,q}$ *is*

$$[p/2](p - [p/2] - 1) \cdot [q/2](q - [q/2] - 1),$$

where $[\cdot]$ *is the 'integer part'.*

Note that his formula can be rewritten more conveniently as

$$[p/2][(p-1)/2] \cdot [q/2][(q-1)/2] \text{ or as } \left[(p-1)^2/4\right] \cdot \left[(q-1)^2/4\right];$$

for example, the minimum number of crossings in any drawing of $K_{4,5}$ is

$$[4/2][3/2] \cdot [5/2][4/2] = \left[3^2/4\right] \cdot \left[4^2/4\right] = 8.$$

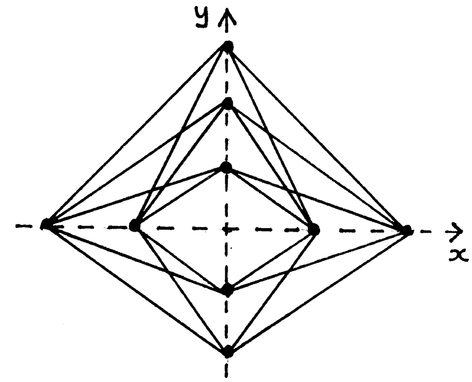

Figure 7. The crossing number of $K_{4,5}$.

The standard approach to crossing-number problems is to find a drawing with a certain number of crossings, and then to show that no drawing can have fewer. As Zarankiewicz observed, the above number of crossings can be attained by the following construction:

> *Divide the p vertices into two sets of equal (or nearly equal) sizes and place the two sets equally spaced on the x-axis on either side of the origin. Do the same for the q vertices, placing them on the y-axis, and then join appropriate pairs of vertices by straight-line segments.*

Figure 7 illustrates the construction for $K_{4,5}$, showing how it can be drawn with 8 crossings.

Zarankiewicz's proof of the case $p = 3$ was correct, but, as observed independently by the graph-theorists Paul Kainen (1965) and Gerhard Ringel (1966) (see Guy [12]), his inductive argument was deficient; it is a 'one-legged induction' that works easily when going from odd to even values of p or q, but not from even to odd. Thus, his formula yields only an upper bound for the minimum number of crossings.

Since then, Daniel Kleitman [16], who learned of the problem from Richard Guy, has shown that the formula yields the correct minimum number of crossings of $K_{p,q}$ when p or q is at most 6. Douglas Woodall [27] has extended these results to include the crossing numbers of $K_{7,q}$ and $K_{8,q}$, for $q = 7$, 8, 9, and 10. The problem remains unsolved in general.

Developments

Without any formal training in higher mathematics, the British artist Anthony Hill (Figure 8) conducted his own explorations into a wide range of geometrical and combinatorial objects (see box). In particular, unaware of the brick factory problem, he drew a number of points in the plane, joined them all in pairs by curves, and investigated how many times these curves must cross one another.

In mathematical terms, the problem concerns the complete graph K_n, obtained by taking n vertices and joining each pair by an edge; Figure 9 shows drawings of K_4, K_5,

[1] A contemporary paper, *On a Problem of K. Zarankiewicz* [17] by Tomás Kövari, Vera Sós, and Paul Turán, refers to a different problem.

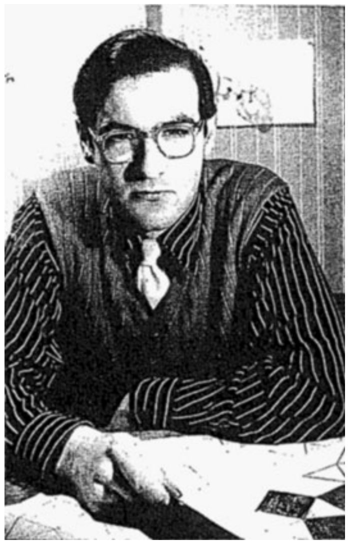

Figure 8. Anthony Hill with geometrical objects.

Anthony Hill
Anthony Hill (b. 1930) describes himself as a 'constructivist working as a geometric formalist' and has been described by the architect Yona Friedman (see [15, p. 84]) as follows:

Anthony Hill is a 'discoverer-artist': Each work of his is an act of discovery, either of an abstract mathematical structure (which he succeeds to translate into an artwork) or of an aesthetic structure (which he transposes into graph theory).

In 1952 Hill attended lectures by the philosopher Imre Lakatos in London, and in 1958 embarked on a collaboration with fellow-artist John Ernest on the crossing-number problem. He was later awarded a Leverhulme Fellowship to research on symmetry as an Honorary Research Fellow in the Department of Mathematics at University College, London, and was an invited speaker at the International Conference on Combinatorial Mathematics at the New York Academy of Sciences in 1970. He has written a dozen papers in graph theory.

and K_6. It can be shown that if $n \leq 4$, then K_n can be drawn in the plane without any crossings, whereas K_5 needs at least one crossing and K_6 needs at least three; these drawings confirm that $\mathrm{cr}(K_4) = 0$, $\mathrm{cr}(K_5) = 1$ and $\mathrm{cr}(K_6) = 3$ (Figure 9).

The following table lists the known values of $\mathrm{cr}(K_n)$ for $n \geq 5$ (the last two values were determined by Pan and Richter [23]):

n	5	6	7	8	9	10	11	12
$\mathrm{cr}(K_n)$	1	3	9	18	36	60	100	150

As Friedman [6] has remarked: "These crossovers are like rabbits . . . they have a tendency to multiply at a terrifying rate".

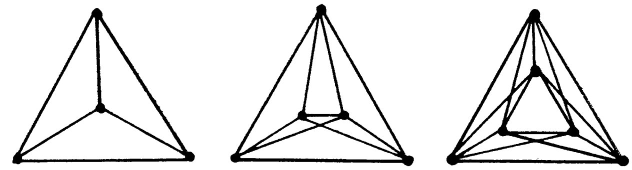

Figure 9. The complete graphs K_4, K_5, and K_6.

Finding the crossing numbers of the complete graphs has a more confused history than that of the complete bipartite graphs. For one thing, the complete graph problem seems to be a more natural place to start and various people may have considered it until its difficulty discouraged them from pursuing it. Those who heard Turán describe the brick factory problem may also have thought about this problem; certainly Paul Erdős claimed in 1960 to have been looking at the problem for at least 20 years, but uncharacteristically seems to have told no one else about it. However, it does seem as though serious investigations into the complete graph problem originated with Anthony Hill around 1958, and that the earliest (albeit unpublished) records exist in the form of his notes and correspondence dating back to that time.

After consulting some colleagues as to whether the problem was known, and following a great deal of experimentation, Hill found drawings of K_6 with 3 crossings, K_7 with 9 crossings, K_8 with 18 crossings, and K_9 with 36 crossings. Figure 10 shows some of his geometrical jottings.

After a great deal of experimentation, Hill also produced a construction that can be described as follows:

Label the vertices 1, 2, ..., n, and arrange the odd numbered ones equally around the inner of two concentric circles and the even ones around the outer circle. Then join all pairs of odd vertices inside the inner circle, join all pairs of even vertices outside the outer circle, and join even vertices to odd ones in the region between the circles.

As he recalled: "Looking at diagonals of polygons—some inside and some outside—that's how I came to it". Figure 11 illustrates his construction for K_7.

This construction led Hill to make the following conjecture in the late 1950s, probably for the first time:

Hill's conjecture: *The minimum number of crossings in any drawing of the complete graph K_n is*

$$(1/64)(n-1)^2(n-3)^2 \quad \textit{for } n \textit{ odd},$$
$$(1/64)n(n-2)^2(n-4) \quad \textit{for } n \textit{ even}.$$

He also noted that the formula in the odd case $n = 2r + 1$ is the square of the triangular number $1/2\ r(r-1)$, while that in the even case $n = 2r$ is the product of the consecutive triangular numbers $1/2\ (r-1)(r-2)$ and $1/2\ r(r-1)$.

Using a variation of Hill's construction, with the vertices placed on the two ends of a tin can, J. Blažek and M. Koman [2] confirmed Hill's conjectured results as upper bounds that can be combined into a single formula as

$$\mathrm{cr}(K_n) \leq (1/4)[n/2][(n-1)/2][(n-2)/2][(n-3)/2],$$

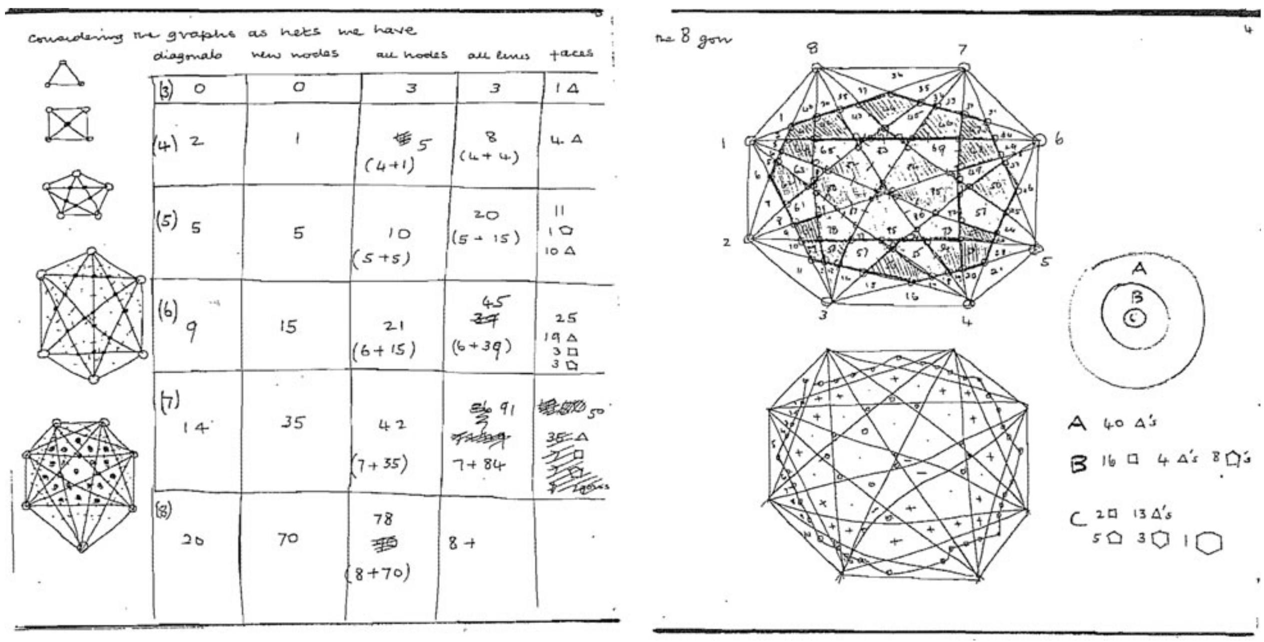

Figure 10. Some jottings from Anthony Hill's notebooks.

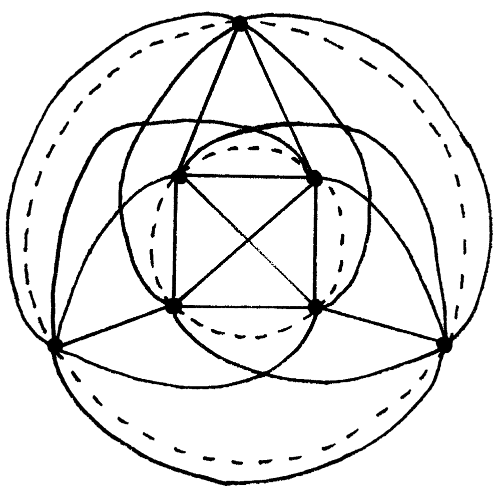

Figure 11. The crossing number of K_7.

or equivalently,

$$\operatorname{cr}(K_n) \le (1/4)\left[(n-1)^2/4\right]\left[(n-3)^2/4\right];$$

these are now known to yield the correct values for all $n \le 12$ (see [23]).

It follows from these bounds that

$$\operatorname{cr}(K_n) \le (1/64)n(n-1)(n-2)(n-3).$$

On the other hand, as Guy [12] observed, each copy of K_5 in K_n gives rise to at least one crossing, and such a crossing occurs in $n - 4$ copies of K_5 (since the crossing involves only four vertices and there are $n - 4$ possibilities for the fifth vertex); thus,

$$\operatorname{cr}(K_n) \ge (1/120)n(n-1)(n-2)(n-3).$$

It follows that Hill's construction yields the right order of magnitude for $\operatorname{cr}(K_n)$.

By the spring of 1959, Hill and his friend the American artist John Ernest had arrived at the above formulas and approached some professional mathematicians about the problem. Among these was the geometer Bernard Scott of King's College, London, who offered to try to learn whether the problem was known and whether it had been solved. Scott suggested contacting Andrew Booth at Birkbeck College, University of London; Booth, a computer pioneer in the 1940s, had been programming a computer to work on the classification of knots, and a student of his was sent to Hill to investigate the possibility of finding a suitable program for the complete graph problem.

Hill also paid the first of a series of visits to the Dutch mathematician and philosopher L. E. J. Brouwer in April of that year. Brouwer was of the opinion that the crossing-number problem might be like the four-colour problem and present great difficulties, in spite of its simple sounding nature.

In May 1959, Hill communicated the problem to Professor Ambrose Rogers of University College, the geometer John Todd in Cambridge, and the combinatorialist Richard Rado at the University of Reading. At Rado's suggestion, he wrote also to the French graph-theorist Claude Berge. Rado believed the problem to be difficult, but no one could shed any light on it.

In November 1959, Richard Guy gave a seminar at University College on unsolved elementary problems, attended by Hill. Hearing about the complete graph problem, probably via Rogers, Guy wrote the first paper [11] on it in *Nabla*, the Bulletin of the Malayan Mathematical Society—a natural place for him to publish since for 10 years he had taught in Singapore (then part of Malaya). In the same paper, Guy also investigated the corresponding problem for drawings of complete graphs on a torus.

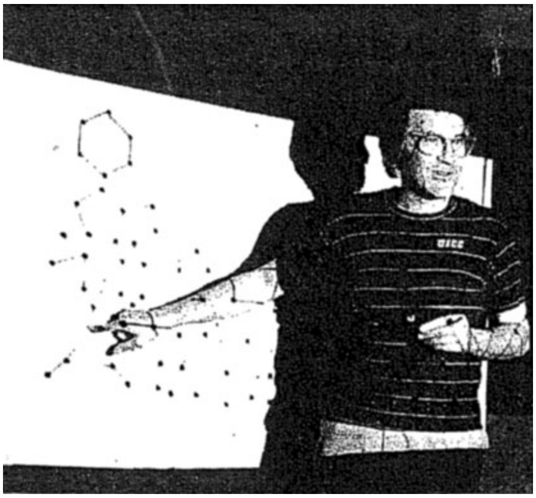

Figure 12. Anthony Hill explaining some graph drawings[2].

In May 1961, H. P. Goodman, a student of Booth's, wrote a letter to *Nature* [9] from the Department of Numerical Automation at Birkbeck College, describing their attempts to solve this seemingly intractable problem using a computer:

> *This problem does not appear tractable analytically, so it was programmed for the University of London* Mercury *computer. The programme was written on the assumption that a minimum for n + 1 points can be obtained by adding an extra point in a suitable place on a minimum solution [with m crossings] for n points. However, the computations have proved that this apparently natural assumption is false: Two different minimum configurations for n = 7, m = 9 were taken, and one led to the true minimum n = 8, m = 18, while the other led to n = 8, m = 19.*

By December 1960, their computer had yielded the values of 60 for $n = 10$ and 100 for $n = 11$, results that have since been proved correct.

After reading Goodman's letter in *Nature*, two chemists, Joseph P. Manfreda and Martin B. Sheratte of North Haven, Connecticut, wrote to Hill offering a proof. They submitted their paper to *Nature*, whose editors sent it to Paul Erdős and the graph-theorist Frank Harary. Hill was witness to a consultation between Erdős and Harary in which, after a great deal of uncertainty, they arrived at the conclusion that Manfreda and Sheratte had been unsuccessful in obtaining a correct argument.

Around this time, Hill communicated his results to Harary who (according to Hill) 'took no interest in my algorithm, and so it has not been properly exposed'. Nevertheless, Harary and Hill produced a joint paper [13], summarizing the progress on the two crossing-number problems that had been made up to that point.

Variations

Although the main problems in the area remain unsolved, there have been a number of other directions for research. We conclude by briefly summarizing three of these.

Straight-Line Drawings

In the 1930s and 1940s, Klaus Wagner and István Fáry proved independently that every graph that can be drawn in the plane without crossings can be so drawn in such a way that all the edges are straight lines. Following from this, in 1958 Anthony Hill defined the straight-line crossing number $\overline{\text{cr}}(K_n)$ (later called the linear or rectilinear crossing number) to be the smallest possible number of crossings needed when the complete graph is drawn with straight lines in the plane.

It is tempting to believe that the values of $\text{cr}(K_n)$ and $\overline{\text{cr}}(K_n)$ are equal for all values of n, but surprisingly this is not the case. These values are indeed equal for $n \leq 7$ and for $n = 9$, but for $n = 8$ we have $\text{cr}(K_8) = 18$ and $\overline{\text{cr}}(K_8) = 19$; a drawing of K_8 with 18 crossings (due to Hill's colleague John Henderson) and a straight-line drawing with $\overline{\text{cr}}(K_8) = 19$ crossings appear in Figure 13 (see Harary and Hill [13]). For $n = 10$, $\text{cr}(K_{10}) = 60$ and $\overline{\text{cr}}(K_{10}) = 62$ (see Brodsky, Durocher, and Gethner [3]).

Dan Bienstock and Nate Dean [1] proved some interesting results on the straight-line crossing numbers of graphs with low crossing numbers. Extending the result of Wagner and Fáry mentioned above, they showed that if $\text{cr}(G) \leq 3$, then $\overline{\text{cr}}(G) = \text{cr}(G)$. They also showed that, counter-intuitively, there are graphs with $\text{cr}(G) = 4$ but arbitrarily high straight-line crossing numbers.

Products of Cycles

In addition to the complete bipartite graphs and the complete graphs, much attention has been given to the products of cycles. If C_p and C_q are cycles, then their Cartesian product $C_p \times C_q$ is the result of taking q copies of C_p and joining corresponding vertices in a cyclic manner; the graph $C_4 \times C_5$ is shown in Figure 14.

The original motivation for studying these graphs was that they can all be drawn on a torus without any crossings, but have arbitrarily large crossing numbers in the plane, as shown by Harary, Kainen, and Schwenk [14]. They noted that the general version of Figure 14 yields the inequality

$$\text{cr}\left(C_p \times C_q\right) \leq (p-2)q, \quad \text{for } 3 \leq p \leq q,$$

and conjectured that equality always holds. As with the complete graph and complete bipartite graph conjectures, a proof has been elusive. Attempts to prove it have led to quite a rich theory beyond this particular problem—see the survey of crossing numbers by Richter and Salazar [24]. Through the successive efforts of a number of mathematicians (see Myers [21] for a survey of early results), equality has been established for all q when $p \leq 7$; it has also been confirmed by Glebskiĭ and Salazar [8] for all values of p and q when $q \geq p(p + 1)$.

Finding Crossing Numbers Efficiently

As M. R. Garey and D. S. Johnson [7] observed in 1983, crossing number problems have practical applications—for example, for providing lower bounds on the amount of chip area required by a graph in a VSLI (very large scale integration) circuit layout (see also Leighton [19]). It is

[2]This photograph was taken by Jérome Ducrot and appears in [15, p. 85].

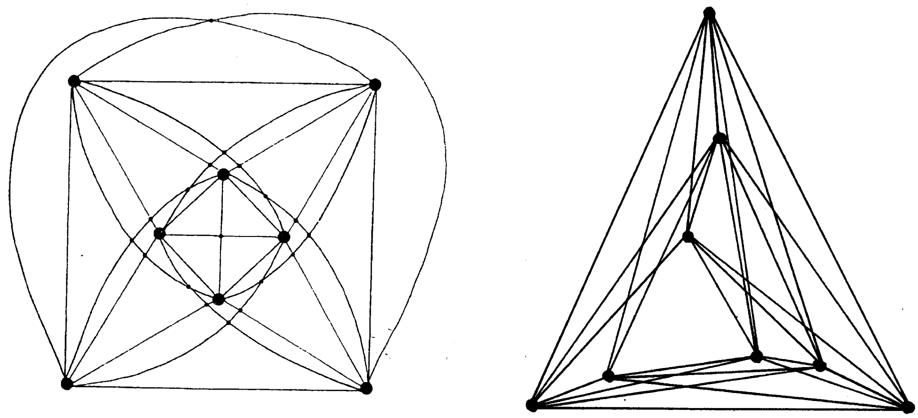

Figure 13. The usual and straight-line crossing numbers of K_8.

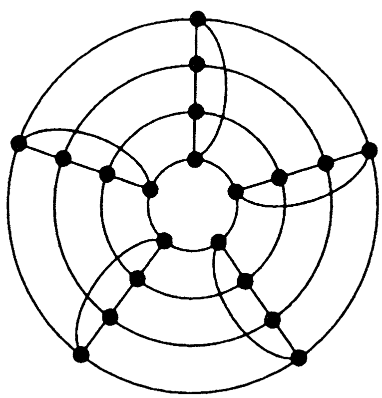

Figure 14. The product $C_4 \times C_5$.

therefore important to know whether the crossing number of a given graph can be found efficiently.

It is well known that there are efficient linear-time algorithms for testing whether a given graph is planar—that is, for testing whether its crossing number is 0. Furthermore, Martin Grohe [10] showed in 2004 that, for any fixed value of k, there is a quadratic-time algorithm for determining whether a given graph has crossing number k. However, Garey and Johnson [7] have shown that determining the crossing numbers of graphs in general is an NP-complete problem, so that no polynomial-time algorithms are likely.

Aftermath

As we have seen, there has recently been a great deal of progress in our knowledge and understanding of crossing numbers. In spite of this, the two basic challenges of proving Zarankiewicz's conjecture and Hill's conjecture remain. In spite of massive efforts by many people, these conjectures have withstood all attempts, remaining unproved for over 50 years.

ACKNOWLEDGEMENTS

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