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Crossing Numbers and Combinatorial Characterization of Monotone Drawings of K_n

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Abstract In 1958, Hill conjectured that the minimum number of crossings in a drawing of K_n is exactly $Z(n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$. Generalizing the result by Ábrego et al. for 2-page book drawings, we prove this conjecture for plane drawings in which edges are represented by x-monotone curves. In fact, our proof shows that the conjecture remains true for x-monotone drawings of K_n in which adjacent edges may cross an even number of times, and instead of the crossing number we count the pairs of edges which cross an odd number of times. We further discuss a generalization of this result to shellable drawings, a notion introduced by Ábrego et al. We also give a combinatorial characterization of several classes of x-monotone drawings of complete graphs using a small set of forbidden configurations. For a similar local characterization of shellable drawings, we generalize Carathéodory's theorem to simple drawings of complete graphs.

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1 Introduction

Let G be a graph with no loops or multiple edges. In a *drawing D of a graph G* in the plane, the vertices are represented by distinct points and each edge is represented by a simple continuous arc connecting the images of its endpoints. As usual, we identify the vertices and their images, as well as the edges and the arcs representing them. We require that the edges pass through no vertices other than their endpoints. We also assume for simplicity that any two edges have only finitely many points in common, no two edges *touch* at an interior point and no three edges meet at a common interior point.

A *crossing in D* is a common interior point of two edges where they properly cross. The *crossing number* $\mathrm{cr}(D)$ *of a drawing D* is the number of crossings in D. The *crossing number* $\mathrm{cr}(G)$ *of a graph G* is the minimum of $\mathrm{cr}(D)$, taken over all drawings D of G. A drawing D is called *simple* if no two adjacent edges cross and no two edges have more than one common crossing. It is well known and easy to see that every drawing of G which minimizes the crossing number is simple.

According to the famous conjecture of Hill [23,25] (also known as Guy's conjecture), the crossing number of the complete graph K_n on n vertices satisfies $\mathrm{cr}(K_n) = Z(n)$, where

$$Z(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

This conjecture has been verified for $n \leq 10$ by Guy [24] and recently for $n \leq 12$ by Pan and Richter [32]. Moreover for each n, there are drawings of K_n with exactly $Z(n)$ crossings [12,23,25,26]. Current best asymptotic lower bound, $\mathrm{cr}(K_n) \geq 0.8594Z(n)$, follows from the lower bound on the crossing number of the complete bipartite graph [15] by an elementary double-counting argument [36].

A curve α in the plane is *x-monotone* if every vertical line intersects α in at most one point. A drawing of a graph G in which every edge is represented by an x-monotone curve and no two vertices share the same x-coordinate is called *x-monotone* (or *monotone*, for short). The *monotone crossing number* $\text{mon-cr}(G)$ *of a graph G* is the minimum of $\mathrm{cr}(D)$, taken over all monotone drawings D of G.

The *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ *of a graph G* is the smallest number of crossings in a drawing of G where every edge is represented by a straight-line segment. Since every rectilinear drawing of G in which no two vertices share the same x-coordinate is x-monotone, we have $\mathrm{cr}(G) \leq \text{mon-cr}(G) \leq \overline{\mathrm{cr}}(G)$ for every graph G.

The *odd crossing number* $\mathrm{ocr}(G)$ of a graph G is the minimum number of pairs of edges crossing an odd number of times in a drawing of G in the plane. The *monotone odd crossing number*, $\text{mon-ocr}(G)$, is the minimum number of pairs of edges crossing an odd number of times in a monotone drawing of G. For these two notions of the

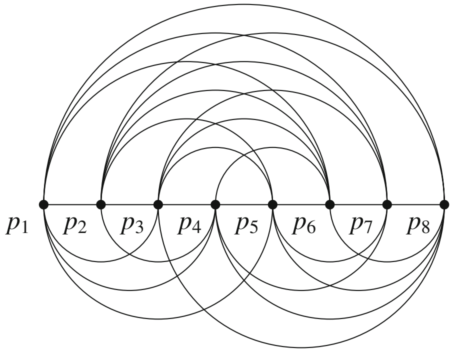


Fig. 1 An example of a 2-page book drawing of K_8 with $Z(8) = 18$ crossings obtained by Blažek and Koman [12]

crossing number, optimal drawings do not have to be simple. Moreover, there are graphs G with $\text{ocr}(G) < \text{cr}(G)$ [33,42], and for every n, there is a graph G with $\text{mon-ocr}(G) = 1$ and $\text{mon-cr}(G) \geq n$ [19].

We call a drawing of a graph *semisimple* if adjacent edges do not cross but independent edges may cross more than once. The *monotone semisimple odd crossing number* of G (called *monotone odd* + by Schaefer [38]), denoted by $\text{mon-ocr}_+(G)$, is the smallest number of pairs of edges that cross an odd number of times in a monotone semisimple drawing of G. We call a drawing of a graph *weakly semisimple* if every pair of adjacent edges cross an even number of times; independent edges may cross arbitrarily. The *monotone weakly semisimple odd crossing number* of G, denoted by $\text{mon-ocr}_\pm(G)$, is the smallest number of pairs of edges that cross an odd number of times in a monotone weakly semisimple drawing of G. Clearly, $\text{mon-ocr}(G) \leq \text{mon-ocr}_\pm(G) \leq \text{mon-ocr}_+(G) \leq \text{mon-cr}(G)$.

The monotone crossing number has been introduced by Valtr [43] and recently further investigated by Pach and Tóth [31], who showed that $\text{mon-cr}(G) < 2\text{cr}(G)^2$ holds for every graph G. On the other hand, they showed that the monotone crossing number and the crossing number are not always the same: there are graphs G with arbitrarily large crossing numbers such that $\text{mon-cr}(G) \geq \frac{7}{6}\text{cr}(G) - 6$.

We study the monotone crossing numbers of complete graphs. The drawings of complete graphs with $Z(n)$ crossings obtained by Blažek and Koman [12] (see also [26]) are 2-*page book* drawings. In such drawings the vertices are placed on a line l and each edge is fully contained in one of the half-planes determined by l. Since 2-page drawings may be considered as a strict subset of x-monotone drawings, we have $\text{mon-cr}(K_n) \leq Z(n)$ (Fig. 1).

Ábrego et al. [1] recently proved that Hill's conjecture holds for 2-page book drawings of complete graphs. We generalize their techniques and show that Hill's conjecture holds for all x-monotone drawings of complete graphs, and even for the monotone weakly semisimple odd crossing number.

Theorem 1.1 *For every $n \in \mathbb{N}$, we have*

$$\text{mon-ocr}_\pm(K_n) = \text{mon-ocr}_+(K_n) = \text{mon-cr}(K_n) = Z(n).$$

The rectilinear crossing number of K_n is known to be asymptotically larger than $Z(n)$: this follows from the best current lower bound $\overline{\mathrm{cr}}(K_n) \geq (277/729)\binom{n}{4} - O(n^3)$ [5,7] and from the simple upper bound $Z(n) \leq \frac{3}{8}\binom{n}{4} + O(n^3)$.

See a recent survey by Schaefer [38] for an encyclopedic treatment of all known variants of crossing numbers.

During the preparation of this paper, we were informed that the authors of [1] achieved the result mon-cr$(K_n) = Z(n)$ already during discussions after their presentation at SoCG 2012 and that Silvia Fernandez-Merchant was going to present it in her keynote talk at LAGOS 2013. The proceedings of the conference were recently published [2]. Pedro Ramos [35] then presented the results and some further developments at the XV Spanish Meeting on Computational Geometry (ECG 2013) in his invited talk. Very recently, Ábrego et al. [3] made their paper containing a more general result publicly available.

In Sect. 2, we first prove Theorem 1.1 for semisimple monotone drawings. Then we extend the result to weakly semisimple monotone drawings, by showing that even crossings of adjacent edges can be easily eliminated in such drawings.

In Sect. 3 we introduce a combinatorial characterization of x-monotone drawings of K_n. We show that there is a one-to-one correspondence between semisimple, simple or pseudolinear x-monotone drawings of K_n and mappings $\binom{[n]}{3} \to \{+,-\}$, called *signature functions*, avoiding a finite number of certain sub-configurations. The signature functions were introduced by Peters and Szekeres [41] as a generalization of order types of planar points sets.

In Sect. 4 we show a further generalization of Theorem 1.1 to shellable drawings and weakly shellable drawings; we define these notions in the beginning of Sect. 4. We show a local characterization of shellable drawings, for which we generalize Caratheodory's theorem to simple drawings of complete graphs. We also show that shellable drawings form a more general class than monotone drawings. Finally, we further generalize a key lemma from [1], which implies a generalization of the main result of [3] to weakly semisimple drawings.

In the last section we state our stronger version of Hill's conjecture.

2 Monotone Crossing Number of the Complete Graph

Let P denote a set of n points in the plane in general position and let k be an integer satisfying $0 \leq k \leq n$. The line segment joining a pair of points p and q in P is a *k-edge* (*$\leq k$-edge*) if there are exactly (at most, respectively) k points of P in one of the open half-planes defined by the line pq.

Ábrego and Fernández-Merchant [6] and Lovász et al. [29] discovered a relation between the numbers of *k-edges* (or *$\leq k$-edges*) in P and the number of convex 4-tuples of points in P, which is equal to the number of crossings of the complete geometric graph with vertex set P. This relation transforms every lower bound on the number of $\leq k$-edges to a lower bound on the number of crossings. Using this method, many incremental improvements on the rectilinear and pseudolinear crossing number of K_n have been achieved [4–6,8,11,29].

To prove the lower bound on the 2-page crossing number of K_n, Ábrego et al. [1] generalized the notion of k-edges to arbitrary simple drawings of complete graphs. They also introduced the notion of $\leq\leq k$-edges, which capture the essential properties of 2-page book drawings better than $\leq k$-edges. We show that the approach using $\leq\leq k$-edges can be generalized to arbitrary semisimple x-monotone drawings.

For a semisimple drawing D of K_n and distinct vertices u and v of K_n, let γ be the oriented arc representing the edge $\{u, v\}$. If w is a vertex of K_n different from u and v, then we say that w is *on the left (right) side of* γ if the topological triangle uvw with vertices u, v and w traced in this order is oriented counter-clockwise (clockwise, respectively). This generalizes the definition introduced by Ábrego et al. [1] for simple drawings. Further generalization is possible for weakly semisimple drawings, where every two edges of the triangle uvw cross an even number of times; see Sect. 4. However, we were not able to find a meaningful generalization of this notion to arbitrary drawings, where the edges of the triangle uvw can cross an odd number of times.

A *k-edge* in D is an edge $\{u, v\}$ of D that has exactly k vertices on the same side (left or right). Since every k-edge has $n-2-k$ vertices on the other side, every k-edge is also an $(n-2-k)$-edge and so every edge of D is a k-edge for some integer k where $0 \leq k \leq \lfloor n/2 \rfloor - 1$.

Analogously to the case of point sets, an i-edge in D with $i \leq k$ is called a *$\leq k$-edge*. Let $E_i(D)$ be the number of i-edges and $E_{\leq k}(D)$ the number of $\leq k$-edges of D. Clearly, $E_{\leq k}(D) = \sum_{i=0}^{k} E_i(D)$. Similarly, the *number $E_{\leq\leq k}(D)$ of $\leq\leq k$-edges* of D is defined by the following identity.

$$E_{\leq\leq k}(D) = \sum_{j=0}^{k} E_{\leq j}(D) = \sum_{i=0}^{k} (k+1-i) E_i(D). \tag{1}$$

Considering the only three different simple drawings of K_4 up to a homeomorphism of the plane, Ábrego *et al.* [1] showed that the number of crossings in a simple drawing D of K_n can be expressed in terms of the number of k-edges in the following way.

Lemma 2.1 ([1]) *For every simple drawing D of K_n we have*

$$\mathrm{cr}(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n-2-k) E_k(D), \tag{2}$$

which can be equivalently rewritten as

$$\mathrm{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor - \frac{1}{2}\left(1+(-1)^n\right) E_{\leq\leq \lfloor n/2 \rfloor - 2}(D).$$

Lemma 2.1 generalizes the relation found by Ábrego and Fernández-Merchant [6]. We further generalize it to semisimple drawings of K_n where $\mathrm{cr}(D)$ is replaced by

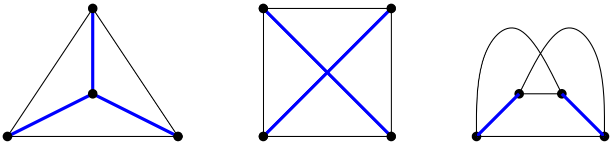


Fig. 2 The three homeomorphism classes of simple drawings of K_4. The fat edges are 1-edges

$\operatorname{ocr}(D)$, which counts the number of pairs of edges that cross an odd number of times in D.

Lemma 2.2 *For every semisimple drawing D of K_n we have*

$$\operatorname{ocr}(D) = 2\sum_{k=0}^{\lfloor n/2\rfloor-2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor - \frac{1}{2}\big(1+(-1)^n\big)E_{\leq\leq\lfloor n/2\rfloor-2}(D).$$

We recall that a *face* of a drawing D in the plane is a connected component of the complement of all the edges and vertices of D in $\mathbb{R}^2$. The *outer face* of D is the unbounded face of D.

Proof (sketch) We just sketch the main idea, which is common with the proof of Lemma 2.1, and then explain the generalization to semisimple drawings. For the details, we refer the reader to [1, Theorem 1 and Proposition 1].

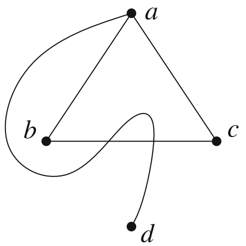
Let D be a semisimple drawing of K_n. A *separation* in D is an unordered triple $\{ab, c, d\}$, where ab is an edge of D, c, d are vertices of D distinct from a, b, and the orientations of the two triangles abc and abd are opposite. Observe that $\{ab, c, d\}$ is a separation in D if and only if ab is a 1-edge (and also a *halving edge*) in the complete subgraph of D induced by the vertices a, b, c, d. The total number of separations in D relates to both the crossing number and the numbers of k-edges in the following way.

(i) Every k-edge belongs to exactly $k(n-k-2)$ separations.
(ii) Every 4-tuple of vertices inducing a crossing contributes two separations, and every 4-tuple of vertices inducing a planar drawing of K_4 contributes three separations. In particular, for every complete subgraph D with 4 vertices we have the equality $\operatorname{cr}(D) + E_1(D) = 3$.

Fact (i) is a direct consequence of the definitions. Fact (ii) is easily seen by inspecting all three homeomorphism classes of simple drawings of K_4 in the plane: there is one class with no crossing, and two classes with one crossing, which would form just one class on the sphere; see Fig. 2. Lemma 2.1 follows from the facts (i) and (ii) by elementary computations.

To generalize Lemma 2.1 to semisimple drawings, we observe that semisimple drawings of K_4 can be classified analogously as the simple drawings of K_4. In particular, the following claim implies that the equality $\operatorname{ocr}(D)+E_1(D) = 3$ is still satisfied for every semisimple drawing D of K_4.

Claim *A semisimple drawing D of K_4 has at most one pair of edges crossing an odd number of times. Moreover, D has three separations if* $\operatorname{ocr}(D) = 0$ *and two separations if* $\operatorname{ocr}(D) = 1$.



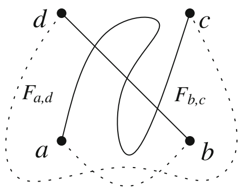


Fig. 3 Illustration to the proof of Lemma 2.2

In the rest of the proof we prove the claim. Let D be a semisimple drawing of K_4. Suppose that $\mathrm{ocr}(D) = 0$. Let abc be a triangle in D and let d be the fourth vertex of D. See Fig. 3, left. If the edge da crosses bc, then either d and b share no face in the drawing of the subgraph with edges ab, bc, ad, or d and c share no face in the drawing of the subgraph with edges ac, bc, ad. This means that one of the edges bd or cd either crosses an adjacent edge or crosses another edge an odd number of times. Therefore, the edge da has no crossing with the triangle abc. Analogous argument for the edges db and dc shows that D has no crossings at all. In particular, D has three separations; see Fig. 2, left.

Now suppose that $\mathrm{ocr}(D) \geq 1$ and let ac and bd be two edges that cross an odd number of times. Since all the other edges are adjacent to both ac and bd, the vertices a, b, c, d share a common face F in the drawing of the subgraph with edges ac, bd. Moreover, the cyclic order of the vertices along the boundary of F is a, b, c, d, either clockwise or counter-clockwise. See Fig. 3, right.

We show that at most one more pair of edges can cross, either ab and cd, or ad and bc, but only an even number of times. For example, in the drawing of the subgraph with edges ac, bd, ab, the vertices c and d belong to the same face, and the edge cd is allowed to cross only the edge ab, each time switching faces. If ab and cd cross, then a and d share a unique face $F_{a,d}$ in the drawing of the graph K with edges ac, bd, ab, cd, and c and b share a unique face $F_{b,c}$ different from $F_{a,d}$. Since the edges ad and bc are adjacent to all edges of K, the edge ad lies completely in $F_{a,d}$, the edge bc lies completely in $F_{b,c}$ and thus ad and bc cannot cross. A symmetric argument shows that if ab and cd are disjoint, then ad and bc are either disjoint or cross an even number of times. In any case, we have $\mathrm{ocr}(D) \leq 1$ (and the pair crossing number of D is at most 2).

It remains to show that every semisimple drawing D of K_4 with $\mathrm{ocr}(D) = 1$ has exactly two 1-edges. More precisely, we show that the two 1-edges always form a perfect matching.

Let e be an edge in D incident with the outer face. An *edge flip* is an operation where the portion of e incident with the outer face is redrawn along the other side of the drawing; see Fig. 4. For drawings on the sphere, the edge flip is just a homeomorphism of the sphere. For every bounded face F of D, there is a sequence of edge flips that makes F the outer face.

If D is a semisimple drawing of K_4, then every edge flip of an edge e changes the orientation of the two triangles adjacent to e. Consequently, exactly the four edges

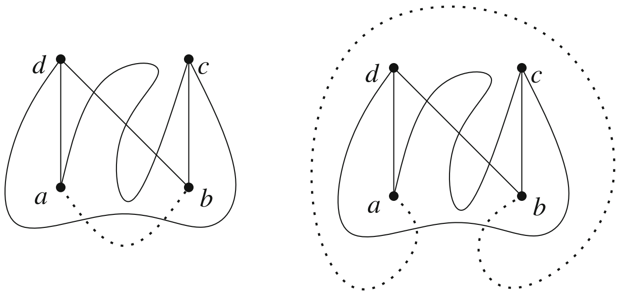


Fig. 4 An edge flip of ab

adjacent to e, forming a 4-cycle, change from 1-edges to 0-edges or vice versa. Also observe that the edge flip of e can be performed only if e is a 0-edge. It follows that 1-edges form a perfect matching in D if and only if they form a perfect matching in the drawing obtained by the edge flip.

Let D be a semisimple drawing of K_4 with $\mathrm{ocr}(D) = 1$. Let ac and bd be the two edges that cross an odd number of times. By performing edge flips, we may assume that all the vertices are adjacent to the outer face of the drawing of the subgraph H with edges ac and bd. Each edge e of the remaining four edges can be drawn in two essentially different ways with respect to H, which differ just by an edge flip of e in $H + e$; see Fig. 4. In total, there are 16 possible combinations. We cannot, however, assume any particular combination, since not all edge flips are always available. Observe that the orientations of all triangles are determined by the four binary choices for the edges ab, bc, cd, ad. Also, changing the choice for one edge e has the same effect on the orientations of the triangles as the edge flip of e. For one particular choice, for example the one yielding the middle drawing in Fig. 2, the 1-edges form a perfect matching. Changing the choice for a subset of edges yields either a perfect matching of 1-edges or a complete graph of 1-edges. However, the latter option is excluded by the fact that in every semisimple drawing the edges incident with the outer face are 0-edges. This finishes the proof of the claim and the lemma. □

Considering $\leq k$-edges, Ábrego and Fernández-Merchant [6] and Lovász et al. [29] proved that for rectilinear drawings of K_n, the inequality $E_{\leq k} \geq 3\binom{k+2}{2}$ together with (2) gives $\overline{\mathrm{cr}}(G) \geq Z(n)$. However, there are simple x-monotone (even 2-page) drawings of K_n where $E_{\leq k} < 3\binom{k+2}{2}$ for $k = 1$ [1]. Ábrego et al. [1] showed that the inequality $E_{\leq\leq k} \geq 3\binom{k+3}{3}$, which is implied by inequalities $E_{\leq j} \geq 3\binom{j+2}{2}$ for $j \leq k$, is satisfied by all 2-page book drawings. We show that the same inequality is satisfied by all x-monotone semisimple drawings of K_n.

Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n. Note that we can assume that all vertices in an x-monotone drawing lie on the x-axis. We also assume that the x-coordinates of the vertices satisfy $x(v_1) < x(v_2) < \cdots < x(v_n)$.

The following observation describes the structure of k-edges incident to vertices on the outer face in semisimple drawings of complete graphs. See Fig. 5, left.

Observation 2.3 *Let D be a semisimple drawing of K_n, not necessarily x-monotone. Let v be a vertex incident to the outer face of D and let γ_i be the ith edge incident*

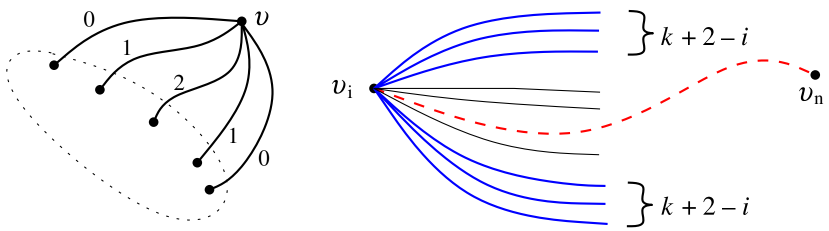


Fig. 5 *Left* k-edges incident with a vertex on the outer face. *Right* after removing v_n, at least $k+2-i$ right edges at v_i are invariant $\leq k$-edges

to v in the counter-clockwise order so that γ_1 and γ_{n-1} are incident to the outer face in a small neighborhood of v. Let v_{k_i} be the other endpoint of γ_i. Then for every i, j, $1 \leq i < j \leq n-1$, the triangle $v_{k_i} v v_{k_j}$ is oriented clockwise. Consequently, for every k with $1 \leq k \leq (n-1)/2$, the edges γ_k and γ_{n-k} are $(k-1)$-edges.

For an x-monotone drawing D of K_n, we use Observation 2.3 for the vertex v_n and the drawing D and then for each i, for the vertex v_i and the drawing of the subgraph induced by $v_i, v_{i+1}, \ldots, v_n$.

The following definitions were introduced by Ábrego *et al.* [1] for 2-page book drawings. Let D be a semisimple x-monotone drawing of K_n and let D' be the drawing obtained from D by deleting the vertex v_n together with its adjacent edges. A k-edge in D is a (D, D')-*invariant k-edge* if it is also a k-edge in D'. It is easy to see that every $\leq k$-edge in D' is also a $\leq (k+1)$-edge in D. If $0 \leq j \leq k \leq \lfloor n/2 \rfloor - 1$, then a (D, D')-invariant j-edge is called a (D, D')-*invariant $\leq k$-edge*. Let $E_{\leq k}(D, D')$ denote the number of (D, D')-invariant $\leq k$-edges.

For $i < j$, the edge $v_i v_j$ is called a *right edge at v_i*. The right edges at v_i have a natural vertical order, which coincides with the order of their crossings with an arbitrary vertical line separating v_i and v_{i+1}. The set of j *topmost* (*bottommost*) right edges at v_i is the set of j right edges at v_i that are above (below, respectively) all other right edges at v_i in their vertical order.

Lemma 2.4 *Let D be a semisimple x-monotone drawing of K_n and let k be a fixed integer such that $0 \leq k \leq (n-3)/2$. For every $i \in \{1, 2, \ldots, k+1\}$, the $k+2-i$ bottommost and the $k+2-i$ topmost right edges at v_i are $\leq k$-edges in D. Moreover, at least $k+2-i$ of these $\leq k$-edges are (D, D')-invariant $\leq k$-edges.*

Proof See Fig. 5, right. The first part of the lemma follows directly from Observation 2.3. If the edge $v_i v_n$ is one of the $k+2-i$ topmost right edges at v_i, then the $k+2-i$ bottommost right edges at v_i are (D, D')-invariant $\leq k$-edges. Otherwise the $k+2-i$ topmost right edges at v_i are (D, D')-invariant $\leq k$-edges. $\square$

Corollary 2.5 *We have*

$$E_{\leq k}(D, D') \geq \sum_{i=1}^{k+1} (k+2-i) = \binom{k+2}{2}.$$

The following theorem gives a lower bound on the number of $\leq\leq k$-edges. The proof is essentially the same as in [1], we only extracted Lemma 2.4, which needed to be generalized. Together with Lemma 2.2, Theorem 2.6 yields the second and the third equality in Theorem 1.1, by the same computation as in [1].

Theorem 2.6 *Let $n \geq 3$ and let D be a semisimple x-monotone drawing of K_n. Then for every k satisfying $0 \leq k < n/2 - 1$, we have $E_{\leq\leq k}(D) \geq 3\binom{k+3}{3}$.*

Proof The proof proceeds by induction on n and k starting at $n = 3$ and $k = -1$. The case $n = 3$ is trivially true, and the case $k = -1$ is taken care of by setting $E_{\leq\leq -1}(D) = 0$ for every drawing D. Let $n \geq 4$ and let D be a semisimple x-monotone drawing of K_n. For the induction step we remove the point v_n together with its adjacent edges to obtain a drawing D' of K_{n-1}, which is also semisimple and x-monotone.

Using Observation 2.3 we see that, for $0 \leq i \leq k < n/2 - 1$, there are two i-edges adjacent to v_n in D and together they contribute with $2\sum_{i=0}^{k}(k+1-i) = 2\binom{k+2}{2}$ to $E_{\leq\leq k}(D)$ by (1).

Let γ be an i-edge in D'. If $i \leq k$, then γ contributes with $(k - i)$ to the sum

$$E_{\leq\leq k-1}(D') = \sum_{i=0}^{k-1}(k-i)E_i(D').$$

We already observed that γ is either an i-edge or an $(i + 1)$-edge in D. If γ is also an i-edge in D (that is, γ is a (D, D')-invariant i-edge), then it contributes with $(k+1-i)$ to $E_{\leq\leq k}(D)$. This is a gain of +1 towards $E_{\leq\leq k-1}(D')$. If γ is an $(i + 1)$-edge in D, then it contributes only with $(k - i)$ to $E_{\leq\leq k}(D)$. Therefore we have

$$E_{\leq\leq k}(D) = 2\binom{k+2}{2} + E_{\leq\leq k-1}(D') + E_{\leq k}(D, D').$$

By the induction hypothesis we know that $E_{\leq\leq k-1}(D') \geq 3\binom{k+2}{3}$ and thus we obtain

$$E_{\leq\leq k}(D) \geq 3\binom{k+3}{3} - \binom{k+2}{2} + E_{\leq k}(D, D').$$

The theorem follows by plugging the lower bound from Corollary 2.5. □

2.1 Removing Even Adjacent Crossings

Here we finish the proof of Theorem 1.1 by showing that allowing adjacent edges to cross evenly yields no substantially new monotone drawings of K_n.

The *rotation* at a vertex v in a drawing is the clockwise cyclic order of the neighbors of v in which the corresponding edges appear around v. The *rotation system* of a drawing is the set of rotations of all its vertices.

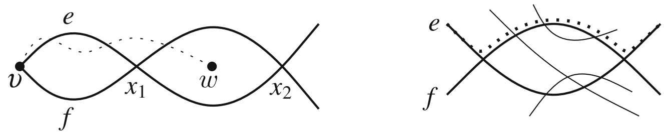


Fig. 6 *Left* the edge vw is forced to cross e or f an odd number of times. *Right* decreasing the total number of crossings

Proposition 2.7 *Let D be a weakly semisimple monotone drawing of K_n. Then there is a semisimple monotone drawing D' of K_n such that for every two edges e, f of K_n, the parity of the number of crossings between e and f in D' is the same as in D. Moreover, D' and D have the same rotation system and the same above/below relations of vertices and edges.*

Proof Let $O(D)$ be the set of pairs of edges of K_n that cross an odd number of times in D. Let D' be a weakly semisimple monotone drawing of K_n with minimum total number of crossings such that D' is *strongly equivalent* to D', that is, D' and D have the same rotation system, the same above/below relations of vertices and edges and $O(D') = O(D)$. We show that D' is semisimple.

Suppose for contrary that D' has two adjacent edges e, f that cross. Since D' is weakly semisimple, e and f cross at least twice. Let v be the common vertex of e and f and suppose that e is above f in the neighborhood of v. Let x_1 and x_2 be the two crossings of e and f closest to v. See Fig. 6, left. Let B be the closed topological disc bounded by the two portions of e and f between x_1 and x_2. Clearly, B has no vertex on its boundary. Moreover, we claim that B has no vertex in its interior. For if B contains a vertex w in its interior, then w is below f and above e. This implies that the edge vw is below f and above e in the neighborhood of v, which is absurd.

Since B contains no vertices, every edge other than e and f crosses the boundary of B an even number of times. Therefore, by redrawing an open segment of e or f containing x_1 and x_2 along the other side of B, we obtain a drawing strongly equivalent to D' with at most $\mathrm{cr}(D') - 2$ crossings. See Fig. 6, right. □

We note that using slightly more careful redrawing operations (such as those in the proof of Theorem 3.2 [10]), we may obtain a semisimple monotone drawing D'' strongly equivalent to D such that for every two edges, the number of their common crossings in D'' is not larger than in D.

By Proposition 2.7, the odd crossing number of a weakly semisimple monotone drawing of K_n is equal to the odd crossing number of some semisimple monotone drawing of K_n. This proves the first equality in Theorem 1.1.

3 Combinatorial Description of Monotone Drawings

In this section we develop a combinatorial characterization of x-monotone drawings based on the signature functions introduced by Peters and Szekeres [41] as generalizations of order types of planar point sets. Let T_n be the set of ordered triples (i, j, k) with $i < j < k$, of the set $[n] = \{1, 2, \ldots, n\}$ and let Σ_n be the set of *signature functions*

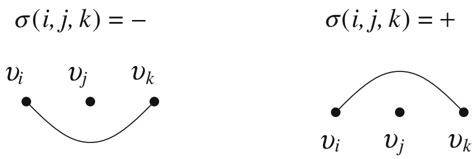


Fig. 7 The negative and the positive signature $\sigma(i, j, k)$

$\sigma\colon T_n \to \{-, +\}$. The set T_n may be also regarded as the set $\binom{[n]}{3}$ of all unordered triples, since we write all the triples in the increasing order of their elements.

Let D be an x-monotone drawing of the complete graph $K_n = (V, E)$ with vertices $v_1, v_2, \dots, v_n$ such that their x-coordinates satisfy $x(v_1) < x(v_2) < \cdots < x(v_n)$. We assign a signature function $\sigma \in \Sigma_n$ to the drawing D according to the following rule. For every edge $e = v_i v_k \in E$ and every integer $j \in (i, k)$, let $\sigma(i, j, k) = -$ if the point v_j lies above the arc representing the edge e and $\sigma(i, j, k) = +$ otherwise. See Fig. 7. Note that if the drawing D is also semisimple, then a triangle $v_i v_j v_k$, with $j \in (i, k)$, is oriented clockwise (counter-clockwise) if and only if $\sigma(i, j, k) = -$ ($\sigma(i, j, k) = +$, respectively).

It is easy to see that, for every signature function $\sigma \in \Sigma_n$, there exists an x-monotone drawing D which induces σ. However, some signature functions are induced only by drawings that are not semisimple. We show a characterization of simple and semisimple x-monotone drawings by small forbidden configurations in the signature functions.

For integers $a, b, c, d \in [n]$ with $a < b < c < d$, signs $\xi_1, \xi_2, \xi_3, \xi_4 \in \{-, +\}$ and a signature function $\sigma \in \Sigma_n$, we say that the 4-tuple (a, b, c, d) *is of the form* $\xi_1\xi_2\xi_3\xi_4$ *in* σ if

$$\sigma(a, b, c) = \xi_1,\ \sigma(a, b, d) = \xi_2,\ \sigma(a, c, d) = \xi_3,\ \text{and}\ \sigma(b, c, d) = \xi_4.$$

Alternatively, we write $\sigma(\{\pi(a), \pi(b), \pi(c), \pi(d)\}) = \xi_1\xi_2\xi_3\xi_4$ for any permutation π of the set $\{a, b, c, d\}$.

For a sign $\xi \in \{-, +\}$ we use $\overline{\xi}$ to denote the opposite sign, that is, if $\xi = +$ then $\overline{\xi} = -$ and conversely, if $\xi = -$ then $\overline{\xi} = +$.

3.1 Simple and Semisimple x-Monotone Drawings

Theorem 3.1 *A signature function $\sigma \in \Sigma_n$ can be realized by a semisimple x-monotone drawing if and only if every 4-tuple of indices from $[n]$ is of one of the forms*

$$++++,\ ----,\ ++--,\ --++,\ -++-,\ +--+,$$
$$---+,\ +++-,\ +---,\ -+++$$

in σ. The signature function σ can be realized by a simple x-monotone drawing if, in addition, there is no 5-tuple (a, b, c, d, e) with $a < b < c < d < e$ such that

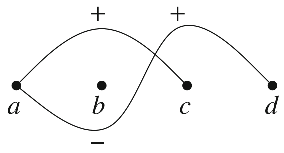


Fig. 8 A 4-tuple (a, b, c, d) of the form $+-+\xi$ forces two adjacent edges to cross

$$\sigma(a,b,e) = \sigma(a,d,e) = \sigma(b,c,d) = \overline{\sigma(a,c,e)}.$$

See Figs. 13 and 10 for an illustration of the first and the second part of the theorem.

Proof Let σ be a signature function with a *forbidden* 4*-tuple*, that is, an ordered 4-tuple (a, b, c, d) whose form is not listed in the statement of the theorem. Such a 4-tuple (a, b, c, d) is one of the forms $\xi_1\overline{\xi_1}\xi_1\xi_2$ or $\xi_2\xi_1\overline{\xi_1}\xi_1$ where $\xi_1, \xi_2 \in \{-,+\}$. If (a, b, c, d) is of the form $+-+\xi$ where $\xi \in \{-,+\}$ is an arbitrary sign, then the edges $v_a v_c$ and $v_a v_d$ are forced to cross between the vertical lines going through v_b and v_c; see Fig. 8. But this is not allowed in a semisimple drawing and we have a contradiction. The other cases are symmetric.

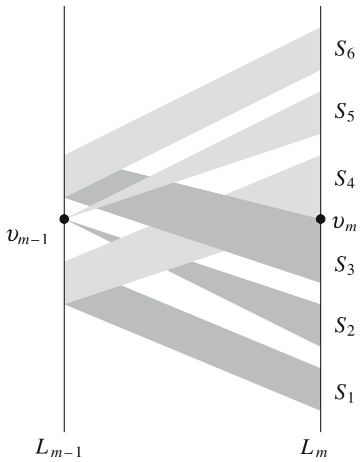
On the other hand, let σ be a signature function such that every 4-tuple is of one of the ten allowed forms in σ. We will construct a semisimple x-monotone drawing D of K_n which induces σ. We use the points $v_i = (i, 0)$, $i \in [n]$, as vertices and connect consecutive pairs of vertices by straight-line segments.

For $m \in [n]$, let L_m be the vertical line containing v_m. In every x-monotone drawing, the line L_m intersects every edge $\{v_i, v_j\}$ with $1 \le i < m \le j \le n$ exactly once. To draw the edges of K_n, it suffices to specify the positions of their intersections with the lines L_m and to draw the edges as polygonal lines with bends at these intersections. Instead of the absolute position of these intersections on L_m, we only need to determine their vertical total ordering, which we represent by a total ordering $\prec_m$ of the corresponding edges. The edges whose right endpoint is v_m will be ordered by $\prec_m$ according to their vertical order in the left neighborhood of v_m. The edges with left endpoint v_m are not considered in $\prec_m$.

The idea of the construction is to interpret the signature function as the set of above/below relations for vertices and edges and take a set of orderings $\prec_m$ that obey these relations and minimize the total number of crossings. In the rest of the proof we show a detailed, explicit construction of the orderings $\prec_m$ which induce an x-monotone semisimple drawing.

For $i \in [n]$, we define an ordering $\lessdot_i$ of the edges with a common left endpoint v_i (that is, the right edges at v_i) in the following way. If $e = \{v_i, v_j\}$ and $f = \{v_i, v_k\}$, $i < j, k$, are two such edges, then we set $e \lessdot_i f$ if either $j < k$ and $\sigma(i, j, k) = +$, or $k < j$ and $\sigma(i, k, j) = -$. Clearly, the relation $\lessdot_i$ is irreflexive, antisymmetric and for every two right edges e, f at v_i either $e \lessdot_i f$ or $f \lessdot_i e$. To show that $\lessdot_i$ is a total ordering, it remains to prove that it is transitive. Suppose for contrary that there are three edges $e = \{v_i, v_j\}$, $f = \{v_i, v_k\}$ and $g = \{v_i, v_l\}$ with $i < j < k < l$ such that $e \lessdot_i f$, $f \lessdot_i g$ and $g \lessdot_i e$. Then $\sigma(i, j, k) = +$, $\sigma(i, k, l) = +$ and $\sigma(i, j, l) = -$, so the 4-tuple i, j, k, l is of the form $+-+\xi$, which is forbidden. Similarly, if $f \lessdot_i e$, $e \lessdot_i g$ and $g \lessdot_i f$, then the 4-tuple i, j, k, l is of the form $-+-\xi$, which is forbidden as well.

Fig. 9 Placing edges and minimizing the number of crossings



We proceed by induction on m. In the case $m = 1$ the ordering $\prec_1$ is empty. For $m = 2$ the ordering $\prec_2$ compares only edges with the common endpoint v_1, so we can set $\prec_2 = \lessdot_1$. Since all the edges are drawn by line segments starting in a common endpoint, no crossings appear between L_1 and L_2.

Let $m > 2$. For the inductive step we consider the following sets $S_1, \ldots, S_6$ of edges which intersect L_{m-1} and L_m (see Fig. 9):

$$
\begin{aligned}
S_1 &= \{\{v_i, v_j\} \mid \sigma(i, m-1, j) = -, \sigma(i, m, j) = -\},\\
S_2 &= \{\{v_{m-1}, v_j\} \mid \sigma(m-1, m, j) = -\},\\
S_3 &= \{\{v_i, v_j\} \mid \sigma(i, m-1, j) = +, \sigma(i, m, j) = - \text{ or } j = m\},\\
S_4 &= \{\{v_i, v_j\} \mid \sigma(i, m-1, j) = -, \sigma(i, m, j) = + \text{ or } j = m\},\\
S_5 &= \{\{v_{m-1}, v_j\} \mid \sigma(m-1, m, j) = +\},\\
S_6 &= \{\{v_i, v_j\} \mid \sigma(i, m-1, j) = +, \sigma(i, m, j) = +\}.
\end{aligned}
$$

The edges within sets S_2 and S_5 are ordered according to $\lessdot_{m-1}$ and the edges in each of the remaining sets S_k according to $\prec_{m-1}$. For $e \in S_k$ and $f \in S_l$ where $k < l$, we set $e \prec_m f$. Observe that $\prec_m$ is a total ordering.

We show that the drawing D determined by the orders $\prec_m$ is semisimple. Suppose for contradiction that two adjacent edges $e = \{v_i, v_j\}$ and $f = \{v_i, v_k\}$, with $i < j, k$ and $e \lessdot_i f$, cross. Their leftmost crossing occurs between lines L_{m-1} and L_m, where $i < m-1$ and $m \le j, k$. There are three cases:

(i) $e \in S_6$ and $f \in S_3$,
(ii) $e \in S_4$ and $f \in S_1$, or
(iii) $e \in S_4$ and $f \in S_3$.

We analyze the cases (i) and (iii) together, case (i) and case (ii) are symmetric. If $j < k$ then $\sigma(i, m, k) = -$ and by the definition of the relation $\lessdot_i$, we have $\sigma(i, j, k) = +$. This further implies that $m < j$ and $\sigma(i, m, j) = +$. Thus (i, m, j, k) forms a forbid-

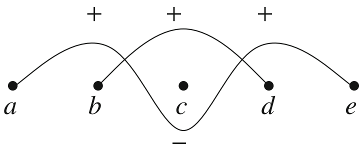


Fig. 10 A forbidden 5-tuple (a, b, c, d, e) forces at least two crossings between $v_a v_e$ and $v_b v_d$

den 4-tuple. If $k < j$, then $\sigma(i, m, j) = +, \sigma(i, k, j) = -$, which implies that $m < k$ and $\sigma(i, m, k) = -$, and so we obtain a forbidden 4-tuple (i, m, k, j).

Now suppose that two adjacent edges $e = \{v_i, v_k\}$ and $f = \{v_j, v_k\}$, with $i, j < k$, cross. Their leftmost crossing occurs between lines L_{m-1} and L_m, where $i, j \le m-1$ and $m < k$. We may assume that $f \prec_m e$ and $e \prec_{m-1} f$. There are five cases:

(i) $e \in S_6$ and $f \in S_3$,
(ii) $e \in S_4$ and $f \in S_1$,
(iii) $e \in S_4$ and $f \in S_3$,
(iv) $e \in S_4$ and $f \in S_2$, or
(v) $e \in S_5$ and $f \in S_3$.

Case (i) and case (ii) are symmetric, as well as case (iv) and case (v). Therefore it is sufficient to consider cases (i), (iii) and (v). In all these three cases $\sigma(j, m, k) = -$ and $\sigma(i, m, k) = +$. If $j < i$, then $\sigma(j, i, k) = +$ since $e \prec_{m-1} f$ and the edges e and f do not cross to the left of L_{m-1}. Hence (j, i, m, k) forms a forbidden 4-tuple. If $i < j$, then analogously $\sigma(i, j, k) = -$ and (i, j, m, k) forms a forbidden 4-tuple. This finishes the proof that D is semisimple.

It remains to show the second part of the theorem. If D is a drawing with a signature function σ with a *forbidden 5-tuple* (a, b, c, d, e), then D is not simple as the edges $v_a v_e$ and $v_b v_d$ are forced to cross at least twice; see Fig. 10.

In the rest of the proof we show the second part of the theorem.

Given a signature function σ with no forbidden 4-tuples and 5-tuples we apply the same construction as before to obtain a semisimple x-monotone drawing D. We show that D is, in addition, simple. Since D is semisimple, no two crossing edges have an endpoint in common. By the construction of D, every crossing c of two edges e and f occurs between lines L_m and L_{m+1} for some $m \in [n-1]$ and we say that v_{m+1} is the *right neighbor* of c. The right neighbor is either an endpoint of e or f or it separates the crossings of L_{m+1} with e and f. Suppose that there are edges $e = v_i v_j$ and $f = v_k v_l$ with $i < k < j, l$ that cross at least twice. We show that then there is always a forbidden 4-tuple or a forbidden 5-tuple in σ.

Let v_m be the right neighbor of the leftmost crossing and $v_{m'}$ the right neighbor of the second leftmost crossing of e and f. Observe that $i, k < m < m' \le j, l$.

First assume that $l < j$. Refer to Fig. 11. If $\sigma(i, k, j) = \sigma(i, l, j) = \xi$ for some $\xi \in \{-, +\}$, then $\xi = \sigma(k, m, l) = \overline{\sigma(i, m, j)}$ and so (i, k, m, l, j) forms a forbidden 5-tuple. If $\sigma(i, k, j) = \overline{\sigma(i, l, j)} = \xi$ for some $\xi \in \{-, +\}$, then e and f cross at least three times and so $m' < l, j$. We have $\xi = \sigma(k, m, l) = \overline{\sigma(i, m, j)} = \overline{\sigma(k, m', l)} = \sigma(i, m', j)$. If $\sigma(k, m, m') = \overline{\xi}$, then (k, m, m', l) forms a forbidden 4-tuple. If $\sigma(k, m, m') = \xi$, then (i, k, m, m', j) forms a forbidden 5-tuple.

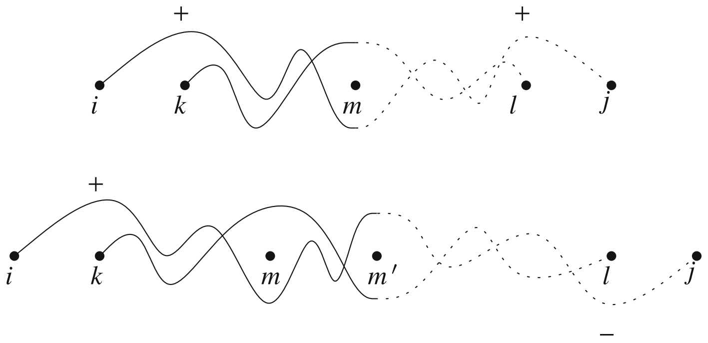


Fig. 11 Edges $v_i v_j$ and $v_k v_l$ crossing twice imply a forbidden 5-tuple or 4-tuple; case $l < j$

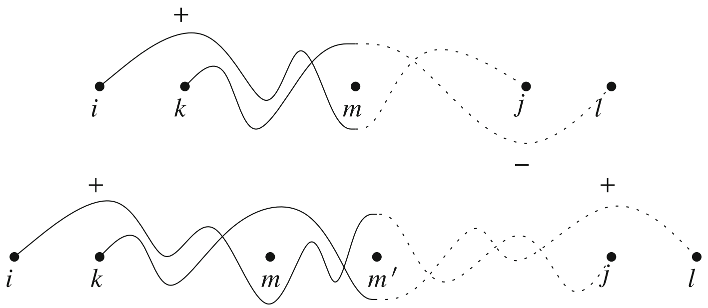


Fig. 12 Edges $v_i v_j$ and $v_k v_l$ crossing twice imply a forbidden 5-tuple or 4-tuple; case $j < l$

Conversely let $j < l$. Refer to Fig. 12. Assume that $\sigma(i, k, j) = \overline{\sigma(k, j, l)} = \xi$ for some $\xi \in \{-, +\}$. Then $\xi = \sigma(k, m, l) = \overline{\sigma(i, m, j)}$. If $\sigma(k, m, j) = \xi$, we get a forbidden 4-tuple (i, k, m, j), otherwise $\sigma(k, m, j) = \overline{\xi}$ and we get a forbidden 4-tuple (k, m, j, l). Finally, assume that $\sigma(i, k, j) = \sigma(k, j, l) = \xi$ for some $\xi \in \{-, +\}$. The proof in this case is identical to the proof of the case $l < j$ and $\sigma(i, k, j) = \overline{\sigma(k, j, l)} = \xi$ in the previous paragraph. □

3.2 Pseudolinear x-Monotone Drawings

A drawing D of a complete graph K_n is *pseudolinear* (also *pseudogeometric* or *extendable*) if the edges of D can be extended to unbounded simple curves that cross each other exactly once, thus forming an *arrangement of pseudolines*. The vertices of D together with the $\binom{n}{2}$ pseudolines extending the edges are said to form a *pseudoarrangement of points* (also *generalized configuration of points*). Note that the pseudoarrangement of points extending D is usually not unique as there is a certain freedom in choosing where the pseudolines extending disjoint noncrossing edges of D cross.

It is well known that every arrangement of pseudolines can be made x-monotone by a suitable isotopy of the plane (this follows, for example, by the duality transform established by Goodman [20,22]). Therefore, every pseudolinear drawing of K_n is isotopic to an x-monotone pseudolinear drawing. Every rectilinear drawing of K_n is

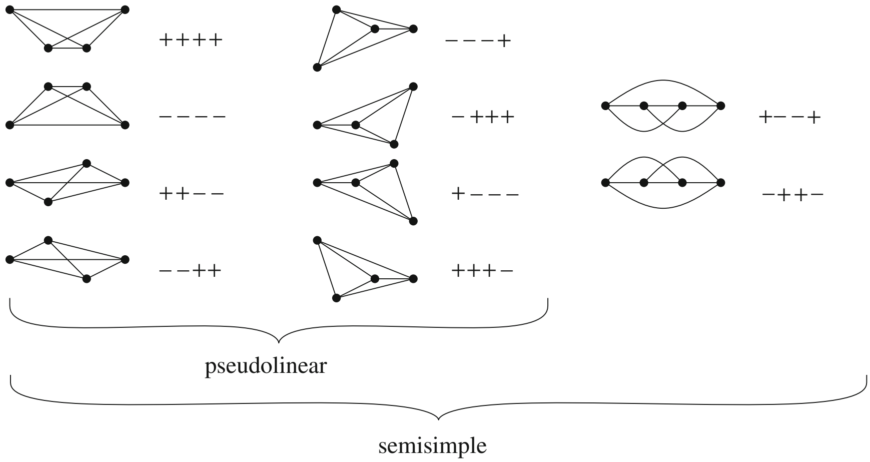


Fig. 13 The 4-tuples in pseudolinear and semisimple drawings

x-monotone and pseudolinear, but there are pseudolinear drawings of K_n that cannot be "stretched" to rectilinear drawings.

We show that x-monotone pseudolinear drawings of K_n can be characterized in a combinatorial way by forbidden 4-tuples in the corresponding signature function, by further restricting the conditions on the signatures in Theorem 3.1. In fact, the conditions in Theorem 3.2 are precisely the *geometric constraints* that Peters and Szekeres [41] used to restrict the set of signature functions in their investigation of the Erdős–Szekeres problem. Fig. 13 illustrates the classification of 4-tuples from Theorem 3.1 and Theorem 3.2.

Theorem 3.2 *A signature function* $\sigma \in \Sigma_n$ *can be realized by a pseudolinear* x*-monotone drawing if and only if every ordered* 4*-tuple of indices from* $[n]$ *is of one of the forms*

$$++++,\ +++-,\ ++--,\ +---,$$
$$----,\ ---+,\ --++,\ -+++$$

in σ.

Pseudolinear drawings of complete graphs are equivalent to *CC systems* introduced by Knuth [27], although this equivalence is not easily seen. The CC systems are ternary *counter-clockwise relations* of finite sets satisfying a certain set of five axioms involving triples, 4-tuples or 5-tuples of elements. CC systems generalize the *order types* of planar point sets in general position: an ordered triple in the counter-clockwise relation is interpreted as a triple of points in the plane placed in the counter-clockwise order, like a triple with signature $+$ in the signature function. Unlike the signature functions, the CC systems have no fixed ordering of the elements. Therefore, some of the axioms for CC systems involve 5-tuples of elements, whereas 4-tuples are sufficient in the case of signature functions. In fact, the axioms of CC systems specify exactly that every 5-tuple of elements can be realized as a point set in the plane.

Knuth [27] established a correspondence between CC systems and *reflection networks* (also called *wiring diagrams*), which are simple arrangements of pseudolines dual to the pseudoarrangements of points extending the pseudolinear drawings of complete graphs. Knuth [27] also showed a two-to-one correspondence between CC systems and *uniform acyclic oriented matroids of rank 3* on the same underlying set. Here the CC system is, in fact, the *chirotope* of the corresponding oriented matroid.

Streinu [40] characterized sets of signed circular permutations (*directed clusters of stars*) that arise from generalized configurations of n points as circular sequences of pseudolines at each of the n points, and provided an $O(n^2)$ drawing algorithm. It is easy to show that the set of signed circular permutations determines the orientation of all triangles (and thus the corresponding CC system) and vice versa. However, many details are omitted in the extended abstract [40].

Felsner and Weil [16,17] proved that *triangle-sign functions* of simple arrangements of n pseudolines are precisely those functions $f : \binom{[n]}{3} \to \{+, -\}$ that are monotone on all 4-tuples. This is the same condition as the condition on signature functions in Theorem 3.2. That is, Theorem 3.2 is a dual analogue of Felsner's and Weil's result. Felsner and Weil [16,17] also introduced *r-signotopes*, a notion unifying permutations, allowable sequences and monotone triangle-sign functions of simple arrangements. In this notation, the signature functions satisfying the conditions of Theorem 3.2 are 3-signotopes.

Theorem 3.2 can be deduced from each of these previous results. However, we did not find any of these ways particularly easy or straightforward. We provide a direct, self-contained proof of Theorem 3.2 in the extended version of this paper [10].

3.3 A Remark on Rectilinear Drawings

A similar characterization of *rectilinear* drawings of K_n (equivalently, order types of planar point sets in general position) in terms of signature functions or CC systems with a finite number of forbidden configurations is impossible: for example, Bokowski and Sturmfels [13] constructed infinitely many minimal CC systems (simplicial affine 3-chirotopes) that are not realizable as sets of points in the plane. This and related results were also referred to by the phrase "missing axiom for chirotopes is lost forever".

Moreover, recognizing signature functions of rectilinear drawings of K_n (or, order types of planar point sets in general position), is polynomially equivalent to rectilinear realizability of complete abstract topological graphs and to stretchability of pseudoline arrangements [28], which is polynomially equivalent to the existential theory of the reals [30]. In the terminology introduced by Schaefer [37], these problems are $\exists\mathbb{R}$-complete. It is known that $\exists\mathbb{R}$-complete problems are in PSPACE [14] and NP-hard, but they are not known to be in NP.

3.4 Crossing Minimal x-Monotone Drawings

Note that in a simple x-monotone drawing of K_n, the crossings appear only between edges whose endpoints induce a 4-tuple of one of the forms $++++$, $----$, $++--$, $--++$, $-++-$, $+--+$. Analogously as for the rectilinear drawings of K_n, we may call these 4-tuples *convex*. Then, for a simple x-monotone drawing D of K_n the crossing number of D equals the number of convex 4-tuples. A similar notion of convexity for general k-tuples was used by Peters and Szekeres [41].

This description of crossings is convenient for computer calculations. Using it, we have obtained a complete list of optimal x-monotone drawings of K_n for $n \leq 10$. To enumerate "essentially different" drawings we used the following approach.

Let D be an x-monotone drawing of K_n which induces a signature function σ. We can assume that the vertices are points placed on the same horizontal line (the x-axis). The following operations on D and σ produce a signature function σ' of a simple monotone drawing D' that is homeomorphic to D on the sphere, by a homeomorphism that does not necessarily preserve the labels of vertices. In some cases we just describe the transformation of the drawing; the new signature function σ' can be then computed in a straightforward way.

(a) *Vertical reflection*: setting $\sigma'(i, j, k) = \overline{\sigma(i, j, k)}$ for every $(i, j, k) \in T_n$.
(b) *Horizontal reflection*: setting $\sigma'(i, j, k) = \sigma(n+1-k, n+1-j, n+1-i)$ for every $(i, j, k) \in T_n$.
(c) *Shifting* v_1: if every edge incident to v_1 lies completely above or completely below the x-axis, that is, $\sigma(1, i, k) = \sigma(1, j, k)$ for every $k \in \{3, \ldots, n\}$ and $1 < i, j < k$, then we can move v_1 to the position of v_n and move every v_{i+1} to the position of v_i, for every $1 \leq i \leq n-1$.
(d) *Switching consecutive points*: let $j \in [n-1]$. If there is a $\xi \in \{-, +\}$ such that $\sigma(j, j+1, k) = \xi$ for every $j+1 < k \leq n$ and $\sigma(i, j, j+1) = \overline{\xi}$ for every $1 \leq i < j$, then we can switch the positions of v_j and v_{j+1}. After the switch, we have $\sigma'(j, j+1, k) = \overline{\xi}$ for every $j+1 < k \leq n$ and $\sigma'(i, j, j+1) = \xi$ for every $1 \leq i < j$.
(e) *Redrawing the edge* $v_1 v_n$: in every crossing minimal x-monotone drawing, the edge $v_1 v_n$ crosses no other edge, since we can always redraw this edge along the top or the bottom part of the boundary of the outer face. The signature function σ thus satisfies $\sigma(1, i, n) = \xi$ for some $\xi \in \{+, -\}$ and for every i, $1 < i < n$. We may thus simultaneously change all the signatures $\sigma(1, i, n)$.

We say that two x-monotone drawings D and D' are *switching equivalent* if there is a sequence of operations (a)–(e) such that, when applied to D, we obtain a drawing which has the same signature function as D'. We have found representatives of all switching equivalence classes of crossing minimal x-monotone drawings of K_n, for $n \leq 10$. Their numbers are given in Table 1.

Ábrego et al. [1] proved that for every even n, there is a unique crossing minimal 2-page book drawing of K_n, up to a homeomorphism of the sphere. We have found crossing minimal x-monotone drawings of K_8 and K_{10} that are not homeomorphic to 2-page book drawings. There are exactly two such drawings of K_8; see Fig. 14. We do not have a construction of such drawings of K_n for arbitrarily large n.

4 Weakly Semisimple and Shellable Drawings

Our proof of Theorem 1.1 for semisimple monotone drawings, as well as the earlier proof by Ábrego *et al.* [2, Theorem 1.1], do not use all properties of monotone draw-

Table 1 Numbers of switching equivalence classes of crossing minimal x-monotone drawings of K_n for $n \leq 10$

Number of vertices	5	6	7	8	9	10
Number of drawings	1	1	5	3	510	38

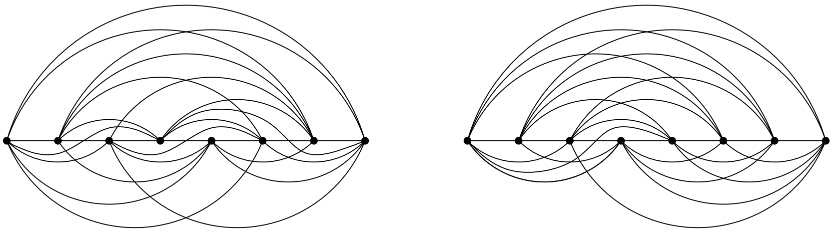


Fig. 14 *Left* a crossing minimal x-monotone drawing of K_8 homeomorphic to the cylindrical drawing. *Right* a crossing minimal x-monotone drawing of K_8 that is not homeomorphic to a 2-page book drawing and neither to the cylindrical drawing

ings. Both rely only on the fact that the vertices of the drawing can be ordered as $v_1, v_2, \ldots, v_n$ so that for every pair i, j with $1 \leq i < j \leq n$, the vertices v_i and v_j are on the outer face of the drawing induced by the interval of vertices $v_i, v_{i+1}, \ldots, v_j$. Pedro Ramos [35] introduced the term *shellable drawings* for these drawings of K_n. Ábrego et al. [3] later observed that a still more general condition, s-shellability for some $s \geq n/2$, is sufficient, since the depth of the recursion in the proof is only $n/2$. A drawing of a complete graph with a vertex set V is called *s-shellable* if there is a subset of vertices $v_1, v_2, \ldots, v_s \in V$ such that for every pair i, j with $1 \leq i < j \leq s$, the vertices v_i and v_j are on the outer face of the drawing induced by $V \setminus \{v_1, v_2, \ldots, v_{i-1}, v_{j+1}, v_{j+2}, \ldots, v_s\}$. In our version of this definition, we require v_1 and v_s to be incident with the outer face; this is slightly more restrictive compared to the original definition in [3]. Informally speaking, s-shellable drawings consist of two parts: the first part is a shellable drawing of K_s, the second part is an arbitrary drawing of the remaining vertices and edges that does not block the shelling of the first part. If $s \geq 3$, this means, in particular, that all vertices from the second part "see" the vertices in the first part in the same cyclic order. The class of s-shellable drawings includes, for example, all drawings with a crossing-free cycle of length s, with at least one edge of the cycle incident with the outer face [3]. Note that the notions *shellable* and *n-shellable* coindide for drawings of K_n.

Following this notation, we call the sequence $v_1, v_2, \ldots, v_n$ from the definition of a shellable drawing of K_n a *shelling sequence* of the drawing, which is similar to the term *s-shelling* introduced by Ábrego *et al.* [3].

Ábrego et al. [3] also considered the class of x-bounded drawings, which form a subclass of shellable drawings and generalize x-monotone drawings. A drawing of a graph is *x-bounded* if no two vertices share the same x-coordinate and every interior point of every edge uv lies in the interior of the strip bounded by two vertical lines passing through the vertices u and v. Fulek *et al.* [19] showed that every x-bounded drawing D can be transformed into an x-monotone drawing D', while keeping the

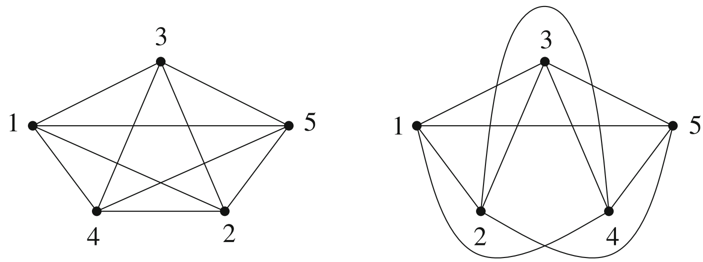


Fig. 15 A simple shellable drawing (*left*) and the corresponding semisimple monotone drawing (*right*). Note that the left drawing is both shellable and monotone; however, its shelling sequence 1, 2, 3, 4, 5 is not its monotone sequence

rotation system and the parity of the number of crossings of every pair of edges fixed. This implies, in particular, that $\mathrm{ocr}(D) = \mathrm{ocr}(D')$. Also D' is weakly semisimple if and only if D is weakly semisimple. Therefore, the lower bound from Theorem 1.1 extends to all weakly semisimple x-bounded drawings of K_n.

It is not a priori clear that shellable drawings are essentially different from monotone or x-bounded drawings, since the conditions for shellability and x-boundedness are very similar at first sight. In Sect. 4.2 we show that simple shellable drawings are indeed more general than simple monotone drawings, but the difference is rather subtle. By a somewhat detailed analysis, which we do not include here, it can be shown that every simple shellable drawing of K_n can be decomposed into three monotone drawings, in a very specific way.

Apart from following the proof of Theorem 1.1, we may obtain a lower bound on the crossing number of shellable drawings of K_n by the following straightforward reduction to the monotone crossing number of K_n, using the combinatorial characterization of x-monotone drawings.

Proposition 4.1 *Let D be a semisimple shellable drawing of K_n. There is a semisimple x-monotone drawing D' of K_n with $ocr(D') = ocr(D)$.*

We note that the drawing D' obtained in Proposition 4.1 does not necessarily preserve the parity of the number of crossings between a given pair of edges. Moreover, it is also possible that for a simple shellable drawing D, we obtain a monotone drawing D' where some pair of edges cross more than once; see Fig. 15.

Let $v_1, v_2, \ldots, v_n$ be the vertices of a semisimple drawing D of K_n. The *order type* of D is the function $\sigma : \binom{[n]}{3} \to \{+, -\}$ defined in the following way: for $1 \le i < j < k \le n$, $\sigma(i, j, k) = +$ if the triangle $v_i v_j v_k$ is drawn counter-clockwise and $\sigma(i, j, k) = -$ if the triangle $v_i v_j v_k$ is drawn clockwise. This generalizes the definition of the signature function for semisimple monotone drawings. As in the previous section, we use the shortcut $\sigma(i, j, k, l)$ for the string of four signs $\sigma(i, j, k)$, $\sigma(i, j, l)$, $\sigma(i, k, l)$, $\sigma(j, k, l)$.

Proof Let $v_1, v_2, \ldots, v_n$ be a shelling sequence of D. Let σ be the order type of D. We show that σ satisfies the assumptions of Theorem 3.1, and therefore can be realized by a semisimple monotone drawing. Let v_i, v_j, v_k, v_l be a 4-tuple of vertices with

$1 \le i < j < k < l \le n$. Then the drawing of K_4 induced by v_i, v_j, v_k, v_l has v_i and v_l on its outer face. To verify the assumptions of Theorem 3.1, it is sufficient to show that none of the cases $\sigma(i, j, k, l) = +-+\xi$, $\sigma(i, j, k, l) = -+-\xi$, $\sigma(i, j, k, l) = \xi+-+$ or $\sigma(i, j, k, l) = \xi-+-$, with $\xi \in \{+, -\}$, occurs. Suppose the contrary. Due to symmetry, we may suppose that $\sigma(i, j, k, l) = +-+\xi$. This means that reading the linear counter-clockwise order of the edges incident with v_i starting from the outer face, we encounter the edge v_iv_j before the edge v_iv_k, v_iv_k before v_iv_l, and v_iv_l before v_iv_j; a contradiction.

Let D' be a semisimple monotone drawing realizing σ. Every 4-tuple of vertices in D induces a drawing of K_4 with at most one pair of edges crossing oddly. This is clear if D is simple; for semisimple drawings this is proved in the claim in the proof of Lemma 2.2. Call a 4-tuple of vertices in D or D' *odd* if it induces exactly one pair of edges crossing oddly and *even* otherwise. To finish the proof, it remains to show that odd (even) 4-tuples of vertices in D correspond to odd (even, respectively) 4-tuples in D'.

Odd (also convex) 4-tuples in D' are of one of the forms $++++$, $----$, $++--$, $--++$, $-++-$, $+--+$. Even 4-tuples in D' are of one of the forms $+++-$, $-+++$, $---+$, $+---$.

Let v_i, v_j, v_k, v_l, with $i < j < k < l$, be a 4-tuple of vertices in D, inducing a drawing H of K_4. By deforming the plane, we may assume that $v_i = (0, 0)$, $v_l = (1, 0)$, and that the vertices v_j, v_k and the interiors of all six edges of H lie in the interior of the strip between the vertical lines passing through v_i and v_l. Note, however, that H is not necessarily deformable to an x-bounded drawing with v_j to the left of v_k: see Fig. 17, left.

Due to symmetry, we may assume that $\sigma(i, j, l) = +$. That is, the vertex v_j and the interiors of the edges v_iv_j and v_jv_l lie below the edge v_iv_l. Now if $\sigma(i, k, l) = -$, then the vertex v_k and the interiors of the edges v_iv_k and v_kv_l lie above the edge v_iv_l. See Fig. 16a. Thus, the edges v_iv_l and v_jv_k are forced to cross an odd number of times, and no other pair of edges in H cross. Also, the triangle $v_iv_jv_k$ is drawn counter-clockwise and the triangle $v_jv_kv_l$ clockwise, so we have $\sigma(i, j, k, l) = ++--$. Therefore, the 4-tuple v_i, v_j, v_k, v_l is odd in both drawings D and D'.

If $\sigma(i, k, l) = +$, then the vertex v_k and the interiors of the edges v_iv_k and v_kv_l lie below the edge v_iv_l. We have four cases according to the signs $\sigma(i, j, k)$ and $\sigma(j, k, l)$, which determine the vertical order of the edges near v_i and v_l, respectively, but do not determine completely which edges cross oddly. This is true even when the drawing H is simple; see Fig. 17. If $\sigma(i, j, k, l) = ++++$ or $\sigma(i, j, k, l) = -++-$, then either the edges v_iv_k and v_jv_l cross oddly, or the edges v_iv_j and v_kv_l cross oddly, and some other pair of edges may cross evenly; see Fig. 16 b, c. In both cases, the 4-tuple v_i, v_j, v_k, v_l is odd in both drawings D and D'. If $\sigma(i, j, k, l) = -+++$ or $\sigma(i, j, k, l) = +++-$, then no two edges cross; see Fig. 16 d, e. In these last two cases, the 4-tuple v_i, v_j, v_k, v_l is even in both drawings D and D'. □

Proposition 4.1 can be generalized to weakly semisimple shellable drawings, but the equality of the odd crossing numbers has to be replaced by inequality, since there are weakly semisimple shellable drawings of K_4 with odd crossing number 2; see Fig. 18, left.

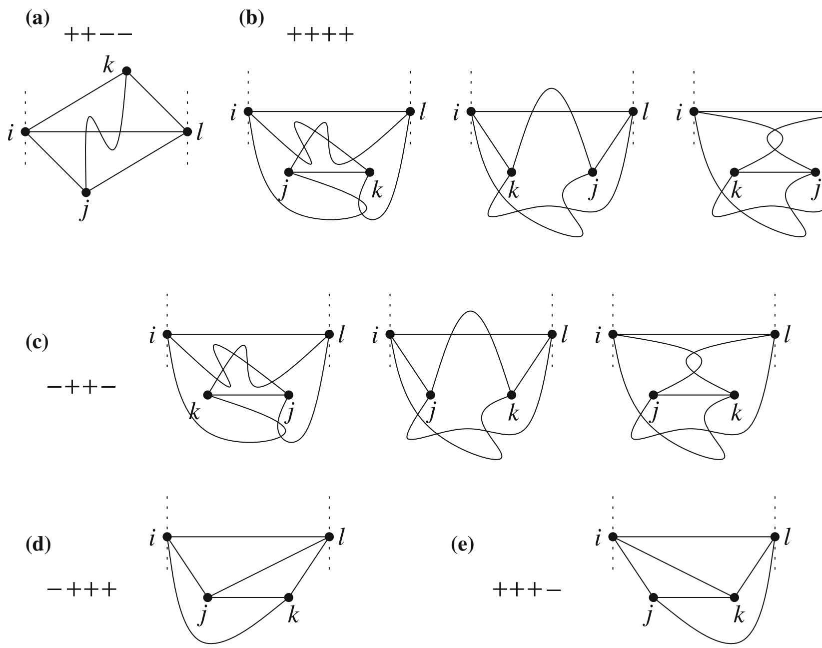


Fig. 16 Examples of semisimple shellable drawings of K_4

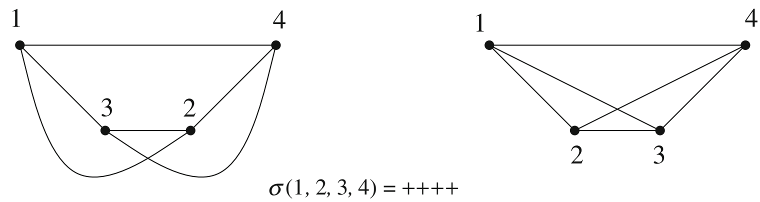


Fig. 17 Two drawings of K_4 with the same order type

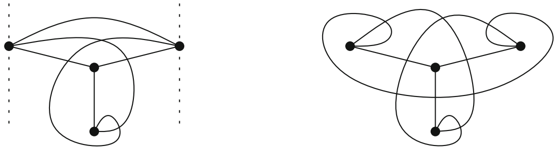


Fig. 18 *Left* a weakly semisimple shellable drawing of K_4 with two pairs of edges crossing oddly. *Right* a weakly semisimple drawing of K_4 with three pairs of edges crossing oddly

For general weakly semisimple drawings, the triangles are not necessarily simple closed curves. Nevertheless, we may still define the orientation of a triangle when every two of its edges cross evenly. Let uvw be a triangle in a weakly semisimple drawing D of K_n. Orient the closed curve γ representing the triangle uvw so that it passes through the vertices u, v, w in this cyclic order. Then for each point p on γ that is not a crossing, a sufficiently small neighborhood of p is divided by γ into the *right neighborhood* and the *left neighborhood* of p, consistently with the chosen orientation of γ.

Let x be a point in the complement of γ in the plane. The *winding number of* γ *around* x is, informally speaking, the number of counter-clockwise turns of γ around x. More formally, if γ is parametrized by continuous polar coordinates $(r(t), \varphi(t)) : [0, 1] \to (0, \infty) \times \mathbb{R}$, with center at x, then the winding number of γ around x is $\frac{\varphi(1)-\varphi(0)}{2\pi}$. We use only the parity of the winding number, which is independent of the chosen orientation of γ.

We say that the triangle uvw, represented by the curve γ, is oriented *counter-clockwise* if for some point x in the right neighborhood of u, the winding number of γ around x is even. Similarly, the triangle uvw is oriented *clockwise* if the winding number of γ around x is odd. Due to the fact that every two edges of uvw cross an even number of times, the definition does not change if we choose x in the right neighborhood of v or w. We may thus generalize the notion of the *order type* to every weakly semisimple drawing of K_n with vertices labeled $v_1, v_2, \ldots, v_n$.

Proposition 4.2 *Let D be a weakly semisimple shellable drawing of K_n. There is a semisimple x-monotone drawing D' of K_n with $ocr(D') \leq ocr(D)$.*

Proof We proceed in the same way as in the proof of Proposition 4.1. Let $v_1, v_2, \ldots, v_n$ be a shelling sequence of D and let σ be the order type of D. The fact that σ satisfies the assumptions of Theorem 3.1 can be proved exactly in the same way as in the proof of Proposition 4.1. Let D' be a semisimple monotone drawing with signature function σ.

To prove the inequality, it is sufficient to show that every 4-tuple of vertices in D that induces a K_4 subgraph with odd crossing number 0, corresponds to a 4-tuple with no crossing in D'. For that, we only need to show that the 4-tuple in D is of the type $+++-$, $-+++$, $---+$ or $+---$. All other 4-tuples in D induce subgraphs with odd crossing number 1 or 2, which is at least as large as the odd crossing number of any K_4 subgraph in D'.

Let v_i, v_j, v_k, v_l, with $i < j < k < l$, be vertices in D inducing a subgraph H with all pairs of edges crossing evenly. We will show that there is a planar drawing H'' of the complete graph with vertices v_i, v_j, v_k, v_l, with v_i and v_l on its outer face, such that the orientation of each triangle in H'' is the same as in H. This will finish the proof, since such a drawing H'' is homeomorphic to one of the drawings in Fig. 16 d, e.

The drawing H satisfies the assumptions of the weak Hanani–Tutte theorem [39]. The weak Hanani–Tutte theorem says that for every drawing D of a graph G in the plane where every two edges cross an even number of times, there is a planar drawing D' of G which has the same rotation system as D (that is, the cyclic orders of the edges around each vertex are preserved). The shortest proof of the weak Hanani–Tutte theorem, based on a more general version for arbitrary surfaces [34], was given by Fulek et al. [18, Lemma 3].

We may assume that v_i is the unique point in H with smallest x-coordinate and that v_l is the unique point in H with largest x-coordinate. We extend the drawing H to a drawing K by adding a vertex y placed below H, a vertex z placed above H, and adding four edges $v_i y$, $y v_l$, $v_i z$, $z v_l$, drawn as monotone curves and forming a simple cycle $v_i y v_l z$. The cycle $v_i y v_l z$ forms the boundary of the outer face of K. By the weak Hanani–Tutte theorem, there is a planar drawing K' having the same rotation system as K. In particular, the cycle $v_i y v_l z$ bounds a face F in K'. Without loss of generality, we may assume that F is the outer face of K'. Let H' be the drawing obtained from K' by removing the vertices y, z and their adjacent edges. Clearly, the drawings H' and H have the same rotation system, H' has no crossings, and v_i and v_l are on the boundary of the outer face of H'. The orientation of triangles $v_i v_j v_k$, $v_i v_j v_l$ and $v_i v_k v_l$ is determined by the rotation at v_i, and the orientation of the triangle $v_j v_k v_l$ is determined by the rotation at v_l. It follows that H and H' have the same order type, and the proof is finished. □

An attempt to generalize the approach in Proposition 4.1 to general non-shellable drawings fails, for the following reason. If $v_1, v_2, \ldots, v_n$ is a chosen ordering of the vertices which is not a shelling sequence, we can have a 4-tuple v_i, v_j, v_k, v_l, with $i < j < k < l$, inducing a planar drawing of K_4 such that v_i or v_l is the only vertex not incident with the outer face. These 4-tuples are of type $+-++$, $++-+$, $-+--$, or $--+-$. In monotone drawings, such 4-tuples are not semi-simple and, moreover, have monotone odd crossing number 2. On the other hand, this is the only obstacle in generalizing Proposition 4.1 to all simple drawings. Indeed, it is easy to see that all simple drawings of K_4 with one crossing and arbitrary ordering of the vertices are of type $++++$, $++--$, $--++$, $+--+$, $-++-$, or $----$, and thus correspond to a simple monotone drawing of K_4 with one crossing. In fact, this is still true also for semisimple drawings, by the claim in the proof of Lemma 2.2.

We may thus generalize Proposition 4.1 and consequently Theorem 1.1 to every drawing of K_n such that there is an ordering $v_1, v_2, \ldots, v_n$ of its vertices such that for every 4-tuple v_i, v_j, v_k, v_l, with $i < j < k < l$, inducing a planar drawing H of K_4, the vertices v_i and v_l are on the outer face of H. We call such a drawing *weakly shellable*. Trivially, every drawing of K_n with $\binom{n}{4}$ crossings is weakly shellable, with arbitrary ordering of its vertices.

Corollary 4.3 *Let D be a semisimple weakly shellable drawing of K_n. There is a semisimple x-monotone drawing D' of K_n with $ocr(D') = ocr(D)$.*

Corollary 4.4 *Let D be a semisimple weakly shellable drawing of K_n. Then $ocr(D) \geq Z(n)$.*

We note that there are simple drawings of complete graphs that are not weakly shellable. For example, the drawing F_6 of K_6 in Fig. 23, left, has the property that every vertex is the central vertex of a planar drawing of K_4 induced by some 4-tuple of vertices. Moreover, by taking two disjoint copies F_6 and adding all remaining 36 edges, we obtain a simple drawing of K_{12} which will not become weakly shellable even if we change its outer face by an arbitrary sequence of edge flips.

By removing the central vertex in F_6 we obtain a weakly shellable simple drawing of K_5 that is not shellable. This shows that weakly shellable drawings are more general than shellable drawings.

4.1 Local Characterization of Shellable Drawings

The definition of a shellable drawing of a complete graph involves testing a quadratic number of subgraphs. It is easy to see that only linearly many of the subgraphs are sufficient.

Observation 4.5 *A sequence of vertices $v_1, v_2, \dots, v_n$ is a shelling sequence of a drawing of a complete graph if and only if for every $i \in [n]$, the vertex v_i is on the outer face of the two subgraphs induced by the subsets of vertices $\{v_1, v_2, \dots, v_i\}$ and $\{v_i, v_{i+1}, \dots, v_n\}$.*

In a similar spirit as in Theorem 3.1, we may obtain a local characterization of shellable drawings, by testing only the subgraphs with four vertices. Like in Theorem 3.1, we need to assume a fixed ordering of the vertices, as there are arbitrarily large minimal non-shellable (and non-monotone) drawings of complete graphs—for example, "flowers" generalizing the drawing F_6 in Fig. 23, left. Unlike in the case of monotone drawings, the order type does not necessarily determine a unique shellable drawing; see Fig. 17.

Theorem 4.6 *Let D be a simple drawing of K_n. A sequence $v_1, v_2, \dots, v_n$ of the vertices is a shelling sequence of D if and only if every 4-tuple v_i, v_j, v_k, v_l, with $i < j < k < l$, induces a drawing of K_4 having v_i and v_l on its outer face.*

To show Theorem 4.6, we use the following generalization of Carathéodory's theorem.

Lemma 4.7 (Carathéodory's theorem for simple complete topological graphs). *Let D be a simple drawing of K_n and let x be a point in the interior of a bounded face of D. Then there is a triangle uvw in D containing x in its interior. Moreover, there is a set of at most $n-2$ triangles covering all bounded faces of D and such that every edge of D is in at most two of these triangles.*

We use only the first part of the lemma. The stronger conclusions are included since they follow easily from the proof and might be interesting on their own.

Proof We proceed by induction on the number of vertices. For $n \leq 2$ the assumptions are vacuous and for $n = 3$ the statement is obvious. Now let $n \geq 4$ and suppose that the lemma has been proved for drawings with at most $n-1$ vertices. Let $v_1, v_2, \dots, v_n$ be the vertices of D. Let D_{n-1} be the drawing of the complete subgraph induced by $v_1, v_2, \dots, v_{n-1}$. Let C be the simple curve forming the boundary of the outer face of D_{n-1}. By induction, all bounded faces of D_{n-1} are covered by a set $\mathscr{T}_{n-1}$ of at most $n-3$ triangles so that no edge is contained in more than two triangles from $\mathscr{T}_{n-1}$. We assume (and prove) an even stronger induction statement: if two triangles from $\mathscr{T}_{n-1}$ share an edge e, then they do not cover the same face incident with e. That is, the two triangles are "attached" to e from the opposite sides of e.

By adding v_n with its incident edges to D_{n-1}, the outer face of D_{n-1} is partitioned into the outer face of D_n and several bounded faces. We show that all these new bounded faces can be covered by a single triangle. We distinguish two cases.

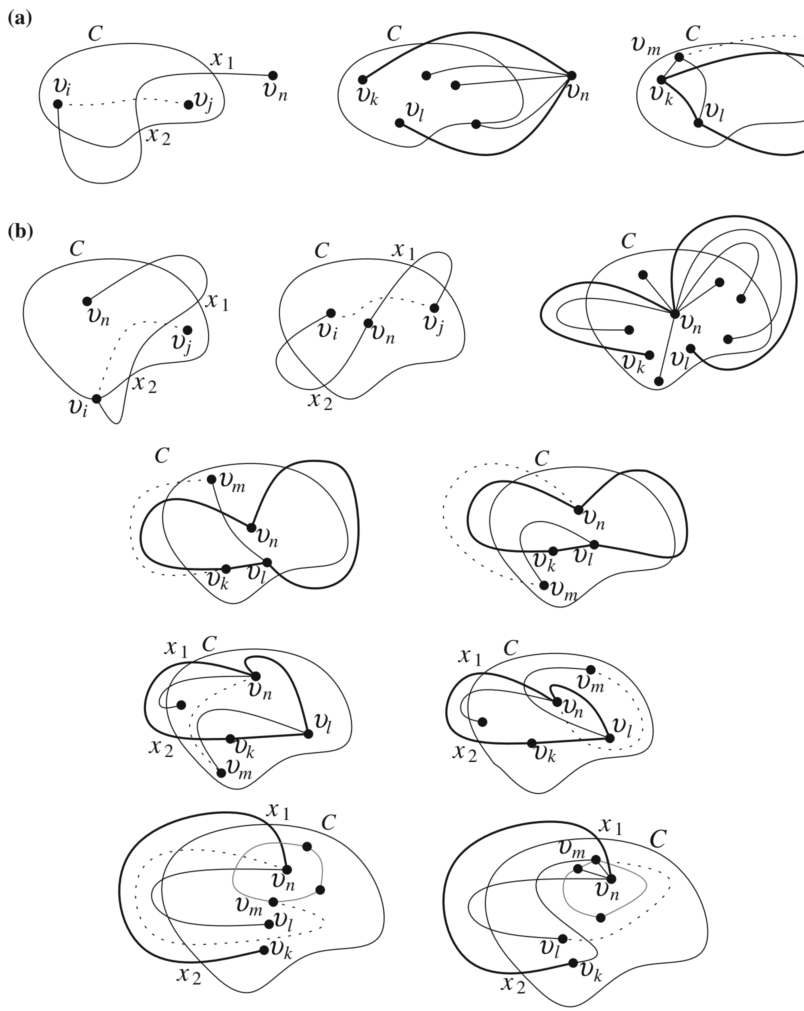


Fig. 19 **a** Adding a vertex v_n to the outer face. **b** adding a vertex v_n to a bounded face. *Dotted curves* represent forbidden edges

(a) The vertex v_n is in the outer face of D_{n-1}. First we observe that no edge $v_i v_n$ has more than one crossing with C. See Fig. 19a. Suppose the contrary and let x_1 and x_2 be two crossings of $v_n v_i$ with C closest to v_n. Then the portion of $v_n v_i$ between x_1 and x_2 separates the drawing D_{n-1} into two parts, each of them containing at least one vertex. In particular, the part that does not contain v_i contains some other vertex v_j. The edge $v_i v_j$ has to lie in the closed region bounded by C, thus it is forced to cross the edge $v_i v_n$; a contradiction.

It follows that for every edge $v_n v_i$, either the relative interior of $v_n v_i$ lies outside C and v_i lies on C, or $v_n v_i$ crosses C in exactly one point, x_i, and the portion of $v_n v_i$ between x_i and v_i lies in the closed region bounded by C. In all cases, only the initial portion of the edge $v_n v_i$ lies in the outer face of D_{n-1}. Consequently, only two edges incident with v_n are incident with the outer face of D_n.

Let $v_n v_k$ and $v_n v_l$ be the two edges incident with v_n and with the outer face of D_n. Since the relative interior of the edge $v_k v_l$ lies inside C, the triangle $v_n v_k v_l$ covers all bounded faces of D_n lying outside C. If no triangle from $\mathscr{T}_{n-1}$ has the edge $v_k v_l$, or if exactly one such triangle, $v_m v_k v_l$, exists but has the opposite orientation from $v_n v_k v_l$, we let $\mathscr{T}_n = \mathscr{T}_{n-1} \cup \{v_n v_k v_l\}$. If some triangle $v_m v_k v_l$ from $\mathscr{T}_{n-1}$ has the edge $v_k v_l$ and has the same orientation as $v_n v_k v_l$, then v_m cannot lie outside $v_n v_k v_l$, as then the edge $v_n v_m$ would be incident with the outer face. Hence v_m is inside $v_n v_k v_l$. The orientation of the triangle $v_m v_k v_l$ then implies that the whole triangle $v_m v_k v_l$ is covered by $v_n v_k v_l$, and so we let $\mathscr{T}_n = (\mathscr{T}_{n-1} \setminus \{v_m v_k v_l\}) \cup \{v_n v_k v_l\}$.
(b) The vertex v_n is in the interior of some bounded face of D_{n-1}. By a similar argument as in part (a), every edge $v_n v_i$ has at most two crossings with C. See Fig. 19 b. If no edge incident with v_n is incident with the outer face of D_n, then C is the boundary of the outer face of D_n and thus we let $\mathscr{T}_n = \mathscr{T}_{n-1}$. If two edges $v_n v_i$ and $v_n v_j$ cross C, they separate the closed region bounded by C into two parts. The vertices v_i and v_j must be in the same part, otherwise the edge $v_i v_j$ would cross $v_n v_i$ or $v_n v_j$, which is forbidden.

It follows that at most two edges incident with v_n, $v_n v_k$ and, possibly, $v_n v_l$, are incident with the outer face of D_n. All other edges $v_n v_i$ that cross C do so in a “nested fashion” in the interval bounded by the crossings of $v_n v_k$ with C, or in the interval bounded by the crossings of $v_n v_l$ with C; see Fig. 19b. Hence, if $v_n v_k$ and $v_n v_l$ are incident with the outer face, then the triangle $v_n v_k v_l$ covers all bounded faces of D_n that lie outside C.

If there is no triangle $v_m v_k v_l$ in $\mathscr{T}_{n-1}$ with the same orientation as $v_n v_k v_l$, we let $\mathscr{T}_n = \mathscr{T}_{n-1} \cup \{v_n v_k v_l\}$. If there is a triangle $v_m v_k v_l$ in $\mathscr{T}_{n-1}$ with the same orientation as $v_n v_k v_l$, then v_m has to be inside $v_n v_k v_l$. For if v_m was outside $v_n v_k v_l$ in the region bounded by $v_n v_k$, $v_n v_l$ and C, then one of the edges $v_m v_k$ or $v_m v_l$ would be forced to cross an adjacent edge or C. Similarly, if v_m was in the other region outside $v_n v_k v_l$ and inside (or on) C, then the edge $v_m v_n$ would be forced to cross an adjacent edge or it would separate $v_n v_k$ or $v_n v_l$ from the outer face. Like in case (a), if v_m is inside $v_n v_k v_l$, then the orientation of $v_m v_k v_l$ implies that $v_m v_k v_l$ is covered by $v_n v_k v_l$. We let $\mathscr{T}_n = (\mathscr{T}_{n-1} \setminus \{v_m v_k v_l\}) \cup \{v_n v_k v_l\}$.

We are left with the case when $v_n v_k$ is the only edge incident with v_n and with the outer face. Let x_1 and x_2 be the crossings of $v_n v_k$ with C, so that x_1 is between v_n and x_2. Without loss of generality, assume that the portion of the edge $v_n v_k$ starting at x_1 and ending at x_2 is oriented counter-clockwise on the boundary of the outer face. Let $v_n v_l$ be the edge following $v_n v_k$ clockwise in the rotation at v_n.

If $v_n v_l$ does not cross C, then the triangle $v_n v_k v_l$ covers all bounded faces of D_n outside C. Similarly as in the previous case, we argue that if there is a triangle $v_m v_k v_l$ in $\mathscr{T}_{n-1}$ with the same orientation as $v_n v_k v_l$, then v_m is inside $v_n v_k v_l$ and so $v_m v_k v_l$ is covered by $v_n v_k v_l$, otherwise the edge $v_n v_m$ would have have to cross some adjacent edge. Here we use the fact that no edge leaves v_n outside the triangle $v_n v_k v_l$. Again, we let $\mathscr{T}_n = \mathscr{T}_{n-1} \setminus \{v_m v_k v_l\} \cup \{v_n v_k v_l\}$ or $\mathscr{T}_n = \mathscr{T}_{n-1} \cup \{v_n v_k v_l\}$, according to the existence of the triangle $v_m v_k v_l$ covered by $v_n v_k v_l$.

Finally, suppose that $v_n v_l$ crosses C. By induction, there is a triangle $v_m v_i v_j \in \mathscr{T}_{n-1}$ containing v_n in its interior. Hence, each of the edges $v_n v_k$ and $v_n v_l$ crosses at least one edge of $v_m v_i v_j$. If $v_n v_k$ and $v_n v_l$ cross the same edge, say, $v_m v_i$, then the edge

$v_n v_m$ also crosses $v_m v_i$, a contradiction. Otherwise, the region bounded by $v_n v_k$, $v_n v_l$ and C that does not contain x_2, contains at least one vertex of the triangle $v_m v_i v_j$, say, v_m. Then it is impossible to draw the edges $v_m v_k$ and $v_m v_l$ so that the resulting drawing is simple. Therefore $v_n v_l$ cannot cross C and we are finished. □

Proof of Theorem 4.6 The condition on 4-tuples is clearly necessary. We show that it is also sufficient. Suppose that D is a simple drawing of K_n and that for some $i \in [n]$, the vertex v_i is not incident with the outer face of the subgraph induced by the subset $\{v_1, v_2, \ldots, v_i\}$ (the case with the subset $\{v_i, v_{i+1}, \ldots, v_n\}$ is symmetric). By Lemma 4.7, there is a triangle $v_j v_k v_l$ with $1 \le j < k < l < i$ containing v_i in its interior. In particular, v_i is not incident with the outer face of the drawing of K_4 induced by the 4-tuple v_j, v_k, v_l, v_i. □

4.2 Shellable Drawings and Monotone Drawings

Here we show that shellable drawings form a more general class than monotone drawings. We also show how monotone drawings may be characterized as a special case of shellable drawings.

Two drawings D_1, D_2 of a graph $G = (V, E)$ are *weakly isomorphic* if for every two edges $e, f \in E$, e and f cross in D_1 if and only if they cross in D_2. Let D be a simple drawing of K_n with vertex set $\{v_1, v_2, \ldots, v_n\}$. We say that a sequence of vertices $v_1, v_2, \ldots, v_n$ is an *x-monotone sequence* of D if v_1 and v_n are incident with the outer face of D and D is weakly isomorphic to a simple monotone drawing where $v_i = (i, 0)$ for every $i \in [n]$.

We have the following characterization of x-monotone sequences in terms of shelling sequences.

Lemma 4.8 *Let D be a simple drawing of K_n. A sequence of vertices $v_1, v_2, \ldots, v_n$ is an x-monotone sequence of D if and only if it is a shelling sequence of D and the path $v_1 v_2 \ldots v_n$ does not cross itself.*

Proof The "only if" part is obvious. Let $v_1, v_2, \ldots, v_n$ be a shelling sequence such that the path $v_1 v_2 \ldots v_n$ does not cross itself. We claim that for every v_i, v_j, v_k, v_l with $1 \le i < j < k < l \le n$, the path $v_i v_j v_k v_l$ does not cross itself. Let H be the drawing of K_4 induced by the vertices v_i, v_j, v_k, v_l.

Suppose for contrary that the path $v_i v_j v_k v_l$ in H crosses itself. That is, the edges $v_i v_j$ and $v_k v_l$ cross. Let i, j, k, l be such a 4-tuple with the pair $(l - i, j - i)$ lexicographically smallest. The drawing H is homeomorphic to one of the drawings in Fig. 20. Since v_l is on the outer face of the complete subgraph $D_{i,l}$ with vertices $v_i, v_{i+1}, \ldots, v_l$, there is an unbounded curve γ_l starting at v_l going to infinity and avoiding all edges of $D_{i,l}$. Similarly, there is an unbounded curve γ_k starting at v_k going to infinity and avoiding all edges of the complete graph induced by $v_i, v_{i+1}, \ldots, v_k$. In particular, γ_l and γ_k do not cross the path $P_{i,j} = v_i v_{i+1} \ldots v_j$, the curve γ_l lies completely in the outer face of H, and γ_k lies completely outside the triangle $v_i v_j v_k$.

By the minimality of $l - i$, the edge $v_k v_l$ crosses no edge $v_{i+a} v_{i+a+1}$ with $1 \le a \le j - i - 1$. By the minimality of $j - i$, the edge $v_i v_{i+1}$ does not cross $v_k v_l$, unless

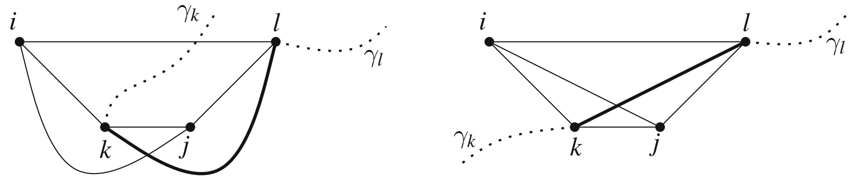


Fig. 20 Drawings of K_4 with shelling sequence v_i, v_j, v_k, v_l where the edges $v_i v_j$ and $v_k v_l$ cross. *Dotted curves* cannot cross the path $P_{i,j}$. *Thick curves* cannot cross the path $P_{i,j}$ either except for the case $j = i+1$

$j = i + 1$. Since the double-infinite curve formed by γ_k, $v_k v_l$ and γ_l separates v_i from v_j, it must cross the path $P_{i,j}$. This implies that $j = i + 1$.

Similarly, there are unbounded curves γ_i and γ_j starting at v_i and v_j, respectively, that do not cross the path $P_{k,l} = v_k v_{k+1} \dots v_l$, the curve γ_i lies completely in the outer face of H, and γ_j lies completely outside the triangle $v_j v_k v_l$. Since $j = i + 1$ and by the assumption, the edge $v_i v_j$ does not cross the path $P_{k,l}$ either. The double-infinite curve formed by γ_i, $v_i v_j$ and γ_j thus separates v_k from v_l but does not cross $P_{k,l}$; this is a contradiction.

Since the path $v_i v_j v_k v_l$ does not cross itself, the order type of H determines the drawing up to an isotopy. Indeed, the drawings in Fig. 17 represent, up to relabeling, the only two isotopy classes of simple shellable drawings of K_4 that have the same order type. There are two possible shelling sequences common for both drawings. For the shelling sequence 1, 2, 3, 4, the corresponding path is noncrossing only in the right drawing. For the shelling sequence 1, 3, 2, 4, the corresponding path is noncrossing only in the left drawing.

Let σ be the order type of D. By the proof of Proposition 4.1, there is a semisimple monotone drawing D' with signature function σ such that two edges cross oddly in D if and only if they cross oddly in D'.

It remains to show that D' is simple. By Theorem 3.1, it is sufficient to show that there is no 5-tuple (a, b, c, d, e) with $a < b < c < d < e$ such that $\sigma(a, b, e) = \sigma(a, d, e) = \sigma(b, c, d) = \overline{\sigma(a, c, e)} = \xi$, where $\xi \in \{+, -\}$. Suppose for contrary that there is such a 5-tuple. By symmetry, we may assume that $\xi = +$. The vertices v_a, v_b, v_c, v_d, v_e induce a shellable drawing K of K_5 in D. We may deform the plane by an isotopy so that $v_a = (0, 0)$, $v_e = (1, 0)$, and so that all edges of K are drawn between the vertical lines going through v_a and v_e. From $\sigma(a, b, e) = +$ and $\sigma(a, c, e) = -$ we have $\sigma(a, b, c, e) = {+}{+}{-}{-}$. Similarly, from $\sigma(a, c, e) = -$ and $\sigma(a, d, e) = +$ we have $\sigma(a, c, d, e) = {-}{-}{+}{+}$. This further implies that $\sigma(a, b, c, d) = {+}{-}{-}{+}$. In particular, the edges $v_a v_c$ and $v_b v_d$ cross. The signatures also imply that v_b and v_d are below the edge $v_a v_e$ and v_c is above the edge $v_a v_e$. For a simple drawing this means that the edge $v_b v_d$ is below $v_a v_e$ and the relative interior of the edge $v_a v_c$ is above $v_a v_e$, therefore the edges $v_a v_c$ and $v_b v_d$ cannot cross; a contradiction. □

It is easy to see that some drawings of K_n have more shellable sequences than x-monotone sequences. For example, for the convex geometric drawing of K_n, all $n!$ permutations of vertices are shelling sequences, whereas at most $n \cdot 2^{n-2}$ permutations of vertices, inducing a noncrossing Hamiltonian path, are x-monotone sequences.

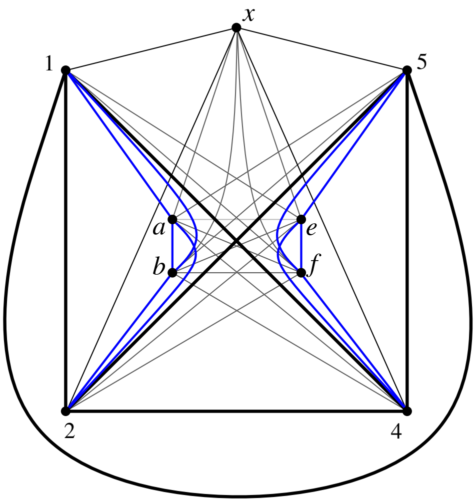


Fig. 21 A simple drawing S_9 of K_9 with shelling sequence $1, 4, f, e, x, a, b, 2, 5$ which has no x-monotone sequence

To show that shellable drawings are indeed more general than monotone drawings, we provide an example of a shellable drawing that has no x-monotone sequence.

Theorem 4.9 *The drawing in Fig. 21 is a simple shellable drawing of K_9 which is not weakly isomorphic to a simple monotone drawing.*

Proof Clearly, the sequence $1, 4, f, e, x, a, b, 2, 5$ is a shelling sequence of the drawing S_9 in Fig. 21. Suppose that μ is an x-monotone sequence of S_9. We write $v \prec w$ for vertices v, w if v precedes w in μ. By symmetry, we may assume that $1 \prec 5$. The subgraphs induced by 4-tuples $\{1, 2, 4, 5\}$, $\{1, 2, a, b\}$ and $\{4, 5, e, f\}$ have unique x-monotone sequences, up to reversal. In particular, we have $1 \prec 2 \prec 4 \prec 5$, which in turn implies that $1 \prec a \prec b \prec 2 \prec 4 \prec f \prec e \prec 5$. To uncover the vertex a, it is not sufficient to remove the vertex 1, we have to remove at least one more vertex. Since all vertices except for x are preceded by a in μ, we have $x \prec a$. Similarly, to uncover the vertex e, it is not sufficient to remove the vertex 5, and the only available vertex is x. Therefore, $e \prec x$. These conditions cannot be fulfilled, thus S_9 has no x-monotone sequence. □

4.3 Crossing Number and k-Edges in Weakly Semisimple Drawings

Here we show a generalization of Lemma 2.2 to weakly semisimple drawings, which may be used to generalize Theorem 1.1 and the result of Ábrego *et al.* [3] to weakly semisimple s-shellable drawings with $s \geq n/2$. As in Proposition 4.2, the equality has to be replaced by an inequality. Since the orientation of triangles and hence the order type can be still defined in weakly semisimple drawings (see the definition before Proposition 4.2), the notions of k-edges, $\leq k$-edges, $\leq\leq k$-edges and separations generalize to weakly semisimple drawings as well.

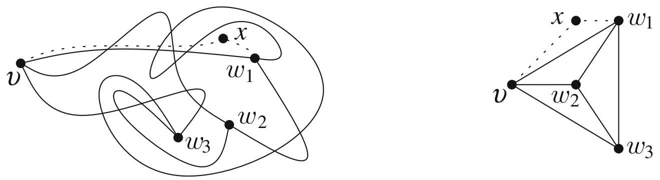


Fig. 22 *Left* adding an auxiliary vertex and two edges to a drawing of K_4 before applying the weak Hanani–Tutte theorem. *Right* a planar drawing of the extended graph with the same rotation system

Lemma 4.10 *For every weakly semisimple drawing D of K_n we have*

$$\operatorname{ocr}(D) \geq 2\sum_{k=0}^{\lfloor n/2\rfloor-2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor - \frac{1}{2}\big(1+(-1)^n\big)E_{\leq\leq\lfloor n/2\rfloor-2}(D).$$

Proof The lemma follows in the same way as Lemma 2.2 or Lemma 2.1, after proving that every weakly semisimple drawing D of K_4 satisfies the inequality $\operatorname{ocr}(D) + E_1(D) \geq 3$. The equality is not always attained as there are weakly semisimple drawings of K_4 with odd crossing number 3 and with six separations; see Fig. 18, right.

Let D be a weakly semisimple drawing of K_4. The *separation graph* of D is the subgraph of D formed by the 1-edges in D. The separation graph depends only on the order type of D. Every order type can be obtained from each other by changing the orientation of some triangles. By changing the orientation of a triangle uvw, the edges uv, uw, vw change from 0-edges to 1-edges and vice versa. It follows that the degree of each vertex in the separation graph either remains the same or changes by 2. Since in the planar drawing of K_4 the separation graph is isomorphic to $K_{1,3}$, it follows that the separation graph of D has all vertices of odd degree. That is, it is isomorphic to $K_2 + K_2$, $K_{1,3}$, or K_4. In particular, $E_1(D) \geq 2$.

Therefore, the inequality is proved for drawings with $\operatorname{ocr}(D) \geq 1$. Now suppose that $\operatorname{ocr}(D) = 0$. We show that the separation graph of D is isomorphic to $K_{1,3}$. We achieve this by transforming D into a drawing D'' by a sequence of edge flips and then to a planar drawing D' which has the same order type as D''. Performing the steps in reverse order will imply that the separation graph of each of D', D'' and D is isomorphic to $K_{1,3}$.

Every edge flip (see the definition in the proof of Lemma 2.2) in a drawing of K_4 changes the orientation of two adjacent triangles. The separation graph is thus transformed by taking the symmetric difference with a cycle C_4. Clearly, if the separation graph is isomorphic to $K_{1,3}$, then its symmetric difference with arbitrarily positioned C_4 is isomorphic to $K_{1,3}$ as well. We may transform D by a sequence of edge flips into a drawing D'' which has at least one vertex v on the outer face. Let w_1, w_2, w_3 be the other three vertices of D'', so that the initial portions of the edges vw_1 and vw_3 are incident with the outer face of D'' and the rotation at v is w_1, w_2, w_3. See Fig. 22, left.

We extend the drawing D'' by adding one auxiliary vertex x close to w_1 and edges vx and xw_1, so that x follows immediately after v in the rotation at w_1, the rotation at v is x, w_1, w_2, w_3, the triangle vxw_1 is oriented clockwise and the path vxw_1 is drawn close to the edge vw_1. We denote this new drawing as K.

Since every two edges cross evenly in D'', the same is true for the drawing K and thus we may apply the weak Hanani–Tutte theorem to K. We obtain a planar drawing K' with the same rotation system as K. We may assume that vxw_1w_3 forms a boundary of the outer face of K'. See Fig. 22, right. Let D' be the subgraph of K' obtained after removing x and its adjacent edges. The orientations of all three triangles incident to v are the same in D'' and in D', since v is on the outer face in both drawings and the rotation at v is the same in K and in K'.

It remains to compare the orientation of the triangle $w_1w_2w_3$ in D'' and D'. Let γ (γ') be the closed curve formed by the edges of the triangle $w_1w_2w_3$ in K (K', respectively). Since the curve vx crosses every edge of D'' an even number of times, the winding number of γ around x has the same parity as the winding number of γ around v. Since v is in the outer face of D'', both winding numbers are even. Since x is outside γ' in K', the winding number of γ' around x is even as well. Together with the fact that in both drawings K and K', the rotation at w_1 is the same, this implies that the triangle $w_1w_2w_3$ is oriented counter-clockwise in both drawings. Therefore, D'' and D' have the same order type. □

Combining Lemma 4.10 with the proof by Ábrego *et al.* [3], we obtain the following generalization.

Corollary 4.11 *Let $s \geq n/2$ and let D be a weakly semisimple s-shellable drawing of K_n. Then $ocr(D) \geq Z(n)$.*

5 Concluding Remarks

It would be interesting to see if techniques similar to those used in the proof of Theorem 1.1 can be used to prove Hill's conjecture for general drawings of complete graphs. We note that the same approach does not generalize to all drawings. For example, a particular planar realization of the so-called *cylindrical drawing* [23,25] of K_{10}, with crossing number $Z(10)$, does not satisfy the lower bound on $\leq\leq 1$-edges from Theorem 2.6. See Fig. 23, right. Fig. 23, left, shows an even smaller example, but this drawing of K_6 is not crossing optimal. Analogous cylindrical drawings of K_{4k+6}, for $k \geq 2$, violate the lower bound on $\leq\leq k$-edges from Theorem 2.6.

Extrapolating the definitions of $\leq k$-edges and $\leq\leq k$-edges, we define the number of *$\leq\leq\leq k$-edges*, $E_{\leq\leq\leq k}(D)$, by the following identity.

$$E_{\leq\leq\leq k}(D) = \sum_{j=0}^{k} E_{\leq\leq j}(D) = \sum_{i=0}^{k} \binom{k+2-i}{2} E_i(D).$$

In our context, using $\leq\leq\leq k$-edges seems to be even more natural than using $\leq\leq k$-edges, since the formula from Lemma 2.1 can be rewritten in the following compact form:

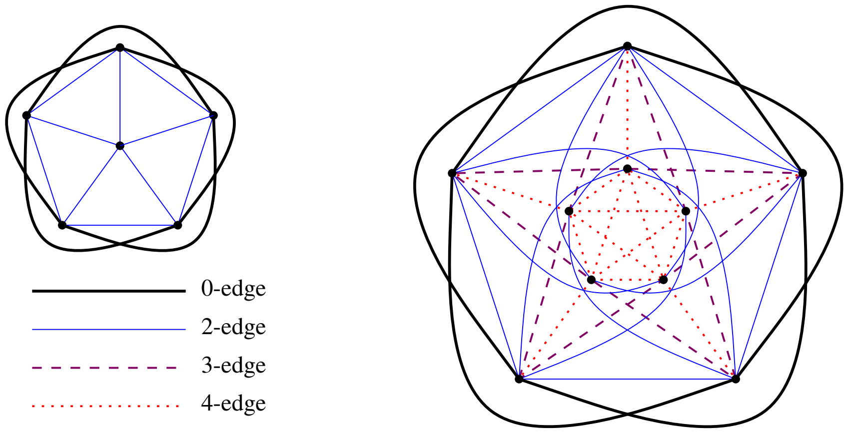


Fig. 23 A general simple drawing of K_6 (*left*) and a cylindrical drawing of K_{10} (*right*) where $E_0 = 5$ and $E_1 = 0$, hence $E_{\leq\leq 1} = 10 < 12 = 3\binom{1+3}{3}$

$$\mathrm{cr}(D) = 2E_{\leq\leq\leq \lfloor n/2 \rfloor -2}(D) - \frac{1}{8}n(n-1)(n-3) \text{ for } n \text{ odd, and}$$

$$\mathrm{cr}(D) = E_{\leq\leq\leq \lfloor n/2 \rfloor -3}(D) + E_{\leq\leq\leq \lfloor n/2 \rfloor -2}(D) - \frac{1}{8}n(n-1)(n-2) \text{ for } n \text{ even.}$$

We conjecture that the following lower bound on $\leq\leq\leq k$-edges is satisfied by all simple drawings of complete graphs.

Conjecture 1 *Let $n \geq 3$ and let D be a simple drawing of K_n. Then for every k satisfying $0 \leq k < n/2 - 1$, we have*

$$E_{\leq\leq\leq k}(D) \geq 3\binom{k+4}{4}.$$

Conjecture 1 is stronger than Hill's conjecture. Theorem 2.6 implies Conjecture 1 for all simple x-monotone drawings. All our examples of simple drawings of complete graphs, including the cylindrical drawings, also satisfy Conjecture 1. We note that Conjecture 1 is trivially satisfied for $k = 0$, since every simple drawing of a complete graph with at least three vertices has at least three 0-edges—those incident with the outer face.

We have no counterexample even to the following conjecture, which further generalizes Conjecture 1 to arbitrary graphs.

Conjecture 2 *Let $k \geq 0$ and let D be a simple drawing of a graph with at least $\binom{2k+3}{2}$ edges. Then*

$$E_{\leq\leq\leq k}(D) \geq 3\binom{k+4}{4}.$$

Note that in a drawing of a general graph with n vertices, a k-edge contained in t triangles is also a $(t-k)$-edge, but not necessarily an $(n-2-k)$-edge. Thus, for example, in every drawing of a triangle-free graph, every edge is a 0-edge. This suggests that it might be easier to prove Conjecture 2 for non-complete graphs. Also, Conjecture 2 or some still stronger variant might be susceptible to a proof by induction on the number of edges.

Further, it would be interesting to generalize Theorem 1.1 to arbitrary monotone drawings, where adjacent edges are also allowed to cross oddly. For such drawings, two notions of the crossing number are of interest. The *monotone odd crossing number*, mon-ocr(G), counting the minimum number of pairs of edges crossing an odd number of times, and the *monotone independent odd crossing number*, mon-iocr(G), or, mon-ocr$_{-}(K_n)$, counting the number of pairs of nonadjacent edges crossing an odd number of times. By definition, for every graph G we have mon-ocr$_{-}(G) \leq$ mon-ocr$(G) \leq$ mon-ocr$_{\pm}(G)$.

5.1 Order Types and λ-Matrices

By Lemma 2.2, the crossing number of a semisimple drawing of K_n is determined by the number of k-edges for all k. For a set of points $p_1, p_2, \dots, p_n$ in the plane, Goodman and Pollack [21] introduced the λ*-matrix* $(\lambda(i, j))$, where for every $i \neq j$, $\lambda(i, j)$ is the number of points to the left of the directed line $p_i p_j$, and $\lambda(i, i) = 0$. They showed that the λ-matrix determines the order type of the point set. Aichholzer et al. [9] used λ-matrices to represent point sets for computing lower bounds on the rectilinear crossing number of complete graphs.

The λ-matrix may be defined for semisimple drawings of K_n with vertices $v_1, v_2, \dots, v_n$ in a similar way: for every $i \neq j$, $\lambda(i, j)$ is the number of triangles $v_i v_j v_l$ oriented counter-clockwise. Clearly, $v_i v_j$ is a k-edge if and only if $\lambda(i, j) \in \{k, n-2-k\}$. The order type of a drawing determines its λ-matrix, but not the drawing itself (see Fig. 17 or Fig. 15). Therefore, the λ-matrix does not determine the drawing either. However, a generalization of Goodman's and Pollack's result to semisimple drawings is true.

Observation 5.1 *The λ-matrix of a semisimple drawing of K_n determines its order type.*

This is easily seen by induction over all subgraphs of K_n: in every semisimple drawing of a graph with at least one edge, all edges incident with the outer face are 0-edges. In particular, there is an edge $v_i v_j$ such that $\lambda(i, j) = 0$. Every 0-edge determines the orientation of all incident triangles. Therefore, we may remove such an edge, update the λ-matrix and use induction for the smaller graph.

The same observation is no longer true for weakly semisimple drawings: in the drawing in Fig. 18, right, every edge is a 1-edge. Therefore, its λ-matrix is identical with the λ-matrix of a mirror-symmetric drawing, but these two drawings have mutually inverse order types.

Since the crossing number of a semisimple drawing of a complete graph is determined by its λ-matrix, it might be interesting to investigate the properties of λ-matrices that can be realized by semisimple drawings of complete graphs.

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