Modern Portfolio Theory (MPT) Primer

At least 2 measures are required to capture relevant information about a frequency function

① measure of average value / mean:

(a) Returns:

$$\boxed{R_{ij} = \frac{P_j - P_{j-1}}{P_{j-1}}}$$

i'th stock (portfolio)   j day

(b) mean:

$$\boxed{\bar{R}_i = E[R_i] = \frac{1}{M}\sum_{j=1}^{M} R_{ij}}$$

② measure of dispersion around the average value

↳ variance / standard deviation

variance:

$$\boxed{\sigma_i^2 = E[(R_i - \bar{R}_i)^2] = \frac{1}{M}\sum_{j=1}^{M} (R_{ij} - \bar{R}_i)^2}$$

standard deviation:

$$\boxed{\sigma_i = \sqrt{\sigma_i^2}}$$

MEASURE of RISK

(3) covariance:

measure of how different returns on assets move together

$i \Rightarrow$ Asset #1
$k \Rightarrow$ " #2
} Portfolio

if $R_i \uparrow$ when $R_k \uparrow$ (+) covariance

$\downarrow$	$\downarrow$	(+)
$\uparrow$	$\downarrow$	(−)
$\downarrow$	$\uparrow$	(−)

$$\boxed{\sigma_{ik} = E\left[(R_i - \bar{R}_i)(R_k - \bar{R}_k)\right] = \frac{1}{m}\sum_{j=1}^{m}(R_{ij} - \bar{R}_i)(R_{kj} - \bar{R}_k)}$$

correlation coefficient: ρ_{ik}

→ standardizes the covariance
same properties of covariance
but with range of −1 to +1

$$\boxed{\rho_{ik} = \frac{\sigma_{ik}}{\sigma_i \sigma_k}}$$

$$-1 \leq \rho_{ik} \leq +1$$

Investors choice:

- if σ_i constant, choose investment with $\uparrow R_i$
- if R_i constant, choose investment with $\downarrow \sigma_i$

Prefer more return to less

exhibit risk avoidance

Delineating Efficient Portfolios

Efficient Set (Efficient Frontier)

Find a subset of portfolio that will be preferred by all investors who exhibit risk avoidance AND who prefer more to less.

X_i = fraction (or weight) of how you will invest in the i'th asset in a portfolio.

$$\sum_{i=1}^{N} X_i = 1 \quad (100\%)$$

E.g. $i = 2$ assets

N: # of assets

$x_1 = .80 \quad x_2 = .20$
$x_1 = .60 \quad x_2 = .40$
etc.

Avg / mean of portfolio

$$\boxed{\bar{R}_p = \sum_{i=1}^{N} x_i \bar{R}_i}$$

Standard deviation of portfolio:

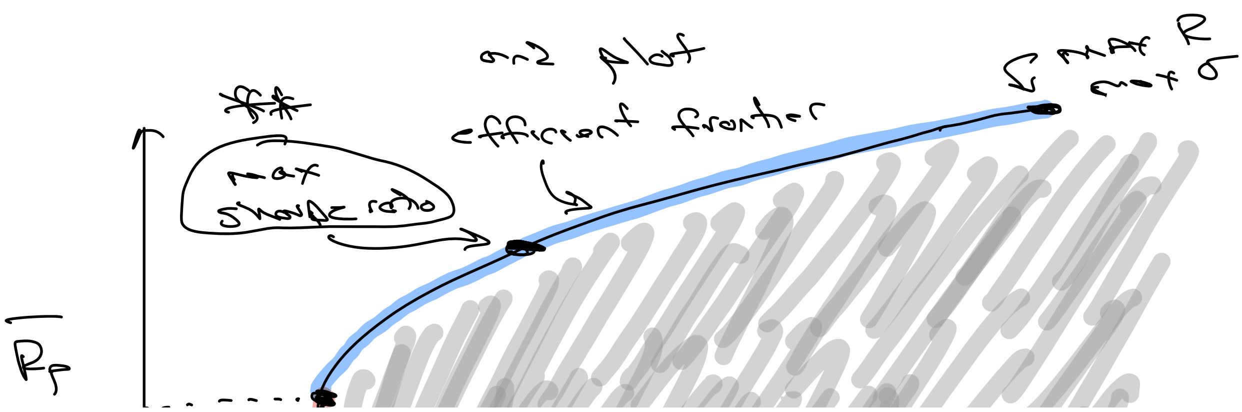
$$\boxed{\sigma_p^2 = \sum_{j=1}^{N} \left(x_j^2 \sigma_j^2 \right) + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ j \neq k}}^{N} \left(x_j x_k \sigma_{jk} \right)}$$

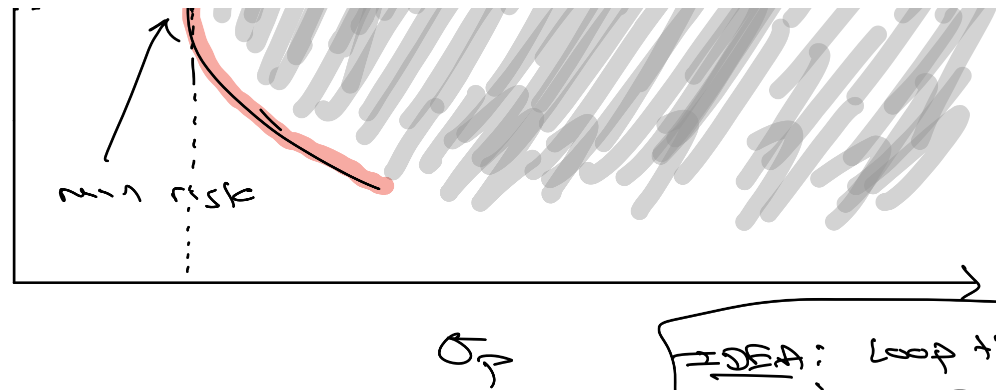
$$\boxed{\sum_{i=1}^{N} x_i = 1}$$

$\Rightarrow$ sweep through x_i's

and find $\bar{R}_p$ & σ_p

and plot





IDEA: Loop through 1000's of randomized weights and plot

Efficient Frontier

- Does not indicate which composition of assets in the portfolio to invest in.

- Based on the investors risk preference (σ), the efficient frontier gives the best portfolio composition (x) to invest in to maximize the profit (R)

- No one point on the efficient frontier is better or worse than any other point

optimum portfolio selection!

$\Rightarrow$ NEVER invest in portfolio selection on the "red" line.... taking a higher risk (σ) for lower return.

SHORT SELLING !!

- if allowed to short sell, this can be referenced by <u>negative</u> weights...

range of $x \Rightarrow -1 : +1$ → or (+2)?

⇑

tells you how to allocate the weights of assets in a portfolio....

<u>BUT</u> how do we select the assets in the first place

⇓

The Single Index Model

↳ assumes the co-movement between stocks is due to a single common influence or index → "market" (e.g. S&P 500)

- casual observance:

market ↑↓ most stocks ↑↓

Common response to market changes

∴ useful to relate the return of the stock to the return of the stock market index.

$$R_i = \alpha_i + \beta_i R_m + \boxed{e_i}$$

Return of the stock

Return of market

ERROR

Alpha: component of return of the stock that is insensitive (independent) to the return of the market.

Beta: measure of expected change in R_i given a change in R_m

Step A: Return on the market

$$\boxed{\bar{R}_m = E[R_m] = \frac{1}{M}\sum_{j=1}^{M} R_{mj}}$$

Step B: Variance of market.

$$\boxed{\sigma_m^2 = E\left[(R_m - \bar{R}_m)^2\right] = \frac{1}{M}\sum_{j=1}^{M}(R_{mj} - \bar{R}_m)^2}$$

Step C: Return on asset/stock.

$$\boxed{\bar{R}_i = E[R_i] = \frac{1}{n}\sum_{j=1}^{n} R_{ij}}$$

Step D : covariance of each security and the market.

$$\boxed{\sigma_i^2 = E[(R_i - \bar{R}_i)^2] = \frac{1}{n}\sum_{j=1}^{n}(R_{ij} - \bar{R}_i)^2}$$

$$\boxed{\sigma_m^2 = E[(R_m - \bar{R}_m)^2] = \frac{1}{n}\sum_{j=1}^{n}(R_{mj} - \bar{R}_m)^2}$$

Step E : correlation coefficient between each security and the market.

$$\sigma_i = \sqrt{\sigma_i^2} \qquad \sigma_m = \sqrt{\sigma_m^2}$$

$$\rho_{im} = \frac{\sigma_{im}}{\sigma_i \sigma_m}$$

Step F : Beta for each stock.

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$$

→ volatility of an asset relative to market.

Step 6: Alpha for each stock.

$$\boxed{\alpha_i = \bar{R}_i - \beta_i \bar{R}_M}$$

→ excess return after adjusting for market volatility

Step 7: standard deviation of the residuals (error) from each regression

$$e_i = R_i - \alpha_i - \beta_i R_M$$

⇒ Evaluate performance

mean of residual error:

$$\bar{R}_{ei} = E[e_i] = \frac{1}{m} \sum_{j=1}^{m} e_j$$

standard deviation of error:

$$\boxed{\sigma_{ei} = \sqrt{\sigma_{ei}^2} = \sqrt{E[(e_i - \bar{R}_{ei})^2]} = \sqrt{\frac{1}{m} \sum_{j=1}^{m} (e_{ij} - \bar{R}_{ei})^2}}$$

Determining the Efficient Frontier

Step #1 : Compute the "excess Return over Beta"

$$\boxed{\frac{\bar{R}_i - R_F}{\beta}} \Rightarrow \text{Sharpe Ratio}$$

Expected Return of asset i

expected change of rate of Return on asset i associated with change in market Return.

Step #2 : Rank the securities in ascending order based on "excess Return over Beta"

(highest → lowest)

Step #3 : Compute Security Ratio C_i

cutoff point →

excess Return over Beta > C_i ⇒ include in portfolio

< C_i ⇒ exclude from portfolio

$$C_i^* = \frac{\sigma_m^2 \sum_{j=1}^{i} \frac{(\bar{R}_j - R_f)\beta_j}{\sigma_{ej}^2}}{1 + \sigma_m^2 \sum_{j=1}^{i} \left(\frac{\beta_j^2}{\sigma_{ej}^2} \right)}$$

Step # 5: construct the portfolio

selected the stocks in portfolio, now to calculate the percent invested in each security.

$$X_i = \frac{Z_i}{\sum_{j=1}^{N} Z_j}$$

$$Z_i = \frac{\beta_i}{\sigma_{ei}^2} \left(\frac{\bar{R}_i - R_f}{\beta_i} - C^* \right)$$