第五节、 隐函数求导法则

一. 一个方程的情形

一般地说，能用$y=f(x)$, $z=f(x, y)$等形式将因变量解出为自变量的函数,称之为**显函数**;

如：方程形式:$F(x, y)=0$

能确定出函数$y=f(x)$,

如：方程形式:$F(x, y, z)=0$,

能确定出函数$z=f(x, y)$,

未解出因变量,而由方程形式确定的函数称为**隐函数**

定理1

设 函数$F(x,y)$在点$P_0(x_0,y_0)$的某个邻域内有一阶连续偏导数，且 $F(x_0,y_0)=0$, $F_y(x_0,y_0)\neq 0$,

则 方程$F(x,y)=0$在点P_0的某个邻域内唯一确定函数$y=f(x)$,

$y=f(x)$具有一阶连续导数, 满足$y_0=f(x_0)$, $F(x,f(x))=0$,

且有求导公式 $$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

注: 具体求由隐函数方程所确定的导数或偏导数有两个方法①按上述公式②按证明公式的方法

定理2 设函数$F(x, y, z)$在点$P_0(x_0, y_0, z_0)$的某个邻域 内有连续偏导数,

$$F(x_0, y_0, z_0)=0, \quad F_z(x_0, y_0, z_0))\neq 0,$$

则 方程$F(x, y, z)=0$在点P_0的某个邻域内唯一确定 函数$z=f(x, y)$,

该函数具有一阶连续偏导数, 满足$z_0=f(x_0, y_0)$,

且有求导公式

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

说明:

因$y=f(x)$是由$F(x, y)=0$确定的隐函数，故成立恒等式$F(x, f(x))\equiv 0$.

等式两边同时对x求导，利用链式法则,有:

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

得:
$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

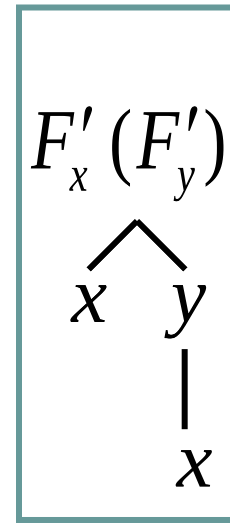
二阶导数 计算:

若$F(x,y)$的二阶偏导数也都连续, 则有

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{F_x'}{F_y'}\right)=-\frac{\dfrac{\partial F_x'}{\partial x}F_y'-\dfrac{\partial F_y'}{\partial x}F_x'}{(F_y')^2}$$

$$=-\frac{F_y'\left(F_{xx}''+F_{xy}''\dfrac{dy}{dx}\right)-F_x'\left(F_{yx}''+F_{yy}''\dfrac{dy}{dx}\right)}{F_y'^2}$$

$$=-\frac{F_{xx}''F_y'^2-2F_{xy}''F_x'F_y'+F_{yy}''F_x'^2}{F_y'^3}$$



例1 设函数$z=f(x,y)$由方程$\sin z=xyz$确定，

求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$

解法1 设$F(x,y,z)=\sin z-xyz$,

则 $\dfrac{\partial F}{\partial x}=-yz$ $\dfrac{\partial F}{\partial y}=-xz$ $\dfrac{\partial F}{\partial z}=\cos z-xy$

故 $\dfrac{\partial z}{\partial x}=\dfrac{yz}{\cos z-xy}$ $\dfrac{\partial z}{\partial y}=\dfrac{xz}{\cos z-xy}$

解法2

对方程sin$z$=xyz两边分别对x求偏导，得

$$\cos z \cdot \frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x}$$

故 $$\frac{\partial z}{\partial x} = \frac{yz}{\cos z - xy}$$

同理可得 $$\frac{\partial z}{\partial y} = \frac{xz}{\cos z - xy}$$

例2 设 $x^2+2y^2+3z^2+xy-z-9=0$

求 $\left.\dfrac{\partial^2 z}{\partial x^2}\right|_{(1,-2,1)}$

解：欲求 $\left.\dfrac{\partial^2 z}{\partial x^2}\right|_{(1,-2,1)}$ **，应先求出** $\dfrac{\partial z}{\partial x}$ **，再求** $\dfrac{\partial^2 z}{\partial x^2}$ **，**

最后以*x*=1, *y*=－2, *z*=1代入即可.

由*z*=*f* (*x*, *y*)是由方程确定的隐函数，

所以，设*F*= $x^2+2y^2+3z^2+xy-z-9$

故 $\dfrac{\partial F}{\partial x}=2x+y,\qquad \dfrac{\partial F}{\partial z}=6z-1$

所以 $\dfrac{\partial z}{\partial x}=\dfrac{2x+y}{1-6z}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x+y}{1-6z}\right)$$

$$= \frac{2(1-6z)-(2x+y)(-6)\dfrac{\partial z}{\partial x}}{(1-6z)^2}$$

$$= \frac{2(1-6z)^2+6(2x+y)^2}{(1-6z)^3}$$

故 $$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(1,-2,1)} = -\frac{2}{5}$$

作业

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二. 由方程组确定的一元或多元隐函数的导数或偏导数

方程组 $\begin{cases} F(x,y,u,v)=0, \\ G(x,y,u,v)=0, \end{cases}$ (*) ，在满足一定条件下

确定了隐函数 $\begin{cases} u=u(x,y), \\ v=v(x,y), \end{cases}$ 求 $\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$

求解思路:

1、利用复合函数求导法则, 在方程*F*(*x*, *y*, *u*,*v*)=0及*G*(*x*, *y*, *u*, *v*)=0两端同时对*x*求偏导数,

(注意：*u*, *v*是　自变量*x*, *y*的函数)

$$\begin{cases} F_u \dfrac{\partial u}{\partial x} + F_v \dfrac{\partial v}{\partial x} = -F_x \\ G_u \dfrac{\partial u}{\partial x} + G_v \dfrac{\partial v}{\partial x} = -G_x \end{cases}$$

2、视 $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$ 为未知量，即可求得 $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$

同理可求得 $\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$

记号：用 $\dfrac{\partial(F,G)}{\partial(u,v)}$ 表示行列式 $\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$.

即 $\dfrac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$

称为函数F, G关于 u, v的雅可比行列式.

定理3:

方程组 $\begin{cases} F(x,y,u,v)=0, \\ G(x,y,u,v)=0, \end{cases}$ (*) $\begin{cases} F_u \dfrac{\partial u}{\partial x} + F_v \dfrac{\partial v}{\partial x} = -F_x \\ G_u \dfrac{\partial u}{\partial x} + G_v \dfrac{\partial v}{\partial x} = -G_x \end{cases}$

设$P_0=(x_0, y_0, u_0, v_0)\in R^4$, 若

1) $F(x, y, u, v)$, $G(x, y, u, v)$在$U(P_0)$有一阶连续偏导数

2) $F(x_0, y_0, u_0, v_0)=G(x_0, y_0, u_0, v_0)=0$

3) 雅可比行列式 $\left.\dfrac{\partial(F,G)}{\partial(u,v)}\right|_{P_0} \neq 0.$

则方程组(1)唯一确定两个二元函数 $u = u(x, y)$,

$v = v(x, y)$, 满足 $u_0 = u(x_0, y_0)$, $v_0= v(x_0, y_0)$. 且

$$\begin{cases} F_u \dfrac{\partial u}{\partial x} + F_v \dfrac{\partial v}{\partial x} = -F_x \\ G_u \dfrac{\partial u}{\partial x} + G_v \dfrac{\partial v}{\partial x} = -G_x \end{cases}$$

且 $$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}, \qquad \frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}},$$

$$\frac{\partial v}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}, \qquad \frac{\partial v}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}},$$

***说明：** 设F, G满足条件，从而存在隐函数

$$u=u(x, y),\ v= v(x, y),$$

代入方程组(*)
$$\begin{cases} F(x, y, u(x, y), v(x, y))\equiv 0 \\ G(x, y, u(x, y), v(x, y))\equiv 0 \end{cases}$$

方程两边对 x 求编导，得

$$\begin{cases} F_u \dfrac{\partial u}{\partial x} + F_v \dfrac{\partial v}{\partial x} = -F_x \\ G_u \dfrac{\partial u}{\partial x} + G_v \dfrac{\partial v}{\partial x} = -G_x \end{cases}$$

将 $\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$ 看作未知量，解二元线性方程组.

由克莱姆法则. $\begin{cases} F_x+F_u\dfrac{\partial u}{\partial x}+F_v\dfrac{\partial v}{\partial x}=0 \\ G_x+G_u\dfrac{\partial u}{\partial x}+G_v\dfrac{\partial v}{\partial x}=0 \end{cases}$

当 $\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}=\dfrac{\partial(F,G)}{\partial(u,v)}\neq 0$，时

有 $$\frac{\partial u}{\partial x}=\frac{\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}}{\dfrac{\partial(F,G)}{\partial(u,v)}}=-\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}.$$

有

$$\frac{\partial v}{\partial x}=\frac{\begin{vmatrix} F_u & -F_x \\ G_u & -G_x \end{vmatrix}}{\dfrac{\partial(F,G)}{\partial(u,v)}}=-\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}.$$

同理可得 $\dfrac{\partial u}{\partial y},\ \dfrac{\partial v}{\partial y}.$

$$\begin{cases} F(x, y, u(x, y), v(x, y)) \equiv 0, \\ G(x, y, u(x, y), v(x, y)) \equiv 0, \end{cases}$$

两边对 y 求偏导，得

$$\begin{cases} F_y + F_u \dfrac{\partial u}{\partial y} + F_v \dfrac{\partial v}{\partial y} = 0 \\ G_y + G_u \dfrac{\partial u}{\partial y} + G_v \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

再解出 $\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}.$

例3 由 $\begin{cases} u+v=x+y, \\ y\sin u = x\sin v, \end{cases}$ 求 $\dfrac{\partial u}{\partial x}$ $\dfrac{\partial u}{\partial y}$ $\dfrac{\partial v}{\partial x}$

解法1 方程组两端分别对x求偏导数

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial x}=1, \\ y\cos u\cdot\dfrac{\partial u}{\partial x}=\sin v+x\cos v\cdot\dfrac{\partial v}{\partial x}, \end{cases}$$

用消元法解此方程组得

$$\frac{\partial u}{\partial x}=\frac{\sin v+x\cos v}{x\cos v+y\cos u} \qquad \frac{\partial v}{\partial x}=-\frac{\sin v-y\cos u}{x\cos v+y\cos u}$$

同理，方程组两端分别对y求偏导数，解相应的未知量为$\frac{\partial u}{\partial y}$，$\frac{\partial v}{\partial y}$ 的线性方程组，可求得

$$\frac{\partial u}{dy}=\frac{x\cos v-\sin u}{x\cos v+y\cos u}$$

解法2 利用一阶全微分形式不变性，**方程组两端分别微分**，有

$$\begin{cases} du + dv = dx + dy, \\ \sin u dy + y\cos u du = \sin v dx + x\cos v dv, \end{cases}$$

以**du, dv**为未知量，解此方程组得

$$du = \frac{1}{x\cos v + y\cos u}\left[(\sin v + x\cos v)dx - (\sin u - x\cos v)dy\right]$$

$$dv = \frac{1}{x\cos v + y\cos u}\left[-(\sin v - y\cos u)dx + (\sin u + y\cos u)dy\right]$$

由全微分定义，可求得

$$\frac{\partial u}{\partial x} = \frac{\sin v + x\cos v}{x\cos v + y\cos u} \qquad \frac{\partial u}{\partial y} = \frac{x\cos v - \sin u}{x\cos v + y\cos u}$$

$$\frac{\partial v}{\partial x} = \frac{y\cos u - \sin v}{x\cos v + y\cos u} \qquad \frac{\partial v}{\partial y} = \frac{\sin u + y\cos u}{x\cos v + y\cos u}$$

例4. 设$\begin{cases} x^2u+y^2v=2 \\ yu^2+xv^2=0 \end{cases}$ 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$.

(其中$x^3v-y^3u\neq 0$)

解: 先求$\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$. **方程两边对 x 求偏导,**

注意u, v 都是 x 的函数，y 看作常数.

得 $$\begin{cases} 2xu+x^2u_x+y^2v_x=0 \\ y\cdot 2u\cdot u_x+v^2+x\cdot 2v\cdot v_x=0 \end{cases}$$

即 $\begin{cases} x^2u_x + y^2v_x = -2xu & \cdots (1) \\ 2yuu_x + 2xvv_x = -v^2 & \cdots (2) \end{cases}$

由于系数行列式

$$D = \begin{vmatrix} x^2 & y^2 \\ 2yu & 2xv \end{vmatrix} = 2x^3v - 3y^2u \neq 0$$

该方程组有唯一解. 解得

$$u'_x = \begin{vmatrix} -2xu & y^2 \\ -v^2 & 2xv \end{vmatrix} \Big/ D = \frac{y^2v^2 - 4x^2uv}{2x^3v - 2y^3u}$$

$$v_x = \begin{vmatrix} x^2 & -2xu \\ 2yu & -v^2 \end{vmatrix} \Big/ D$$

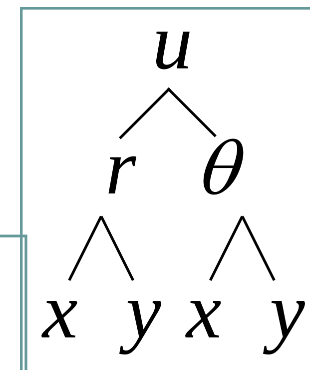
$$= \frac{4xyu^2 - x^2v^2}{2x^3v - 2y^3u}$$

$$\left(自求 \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} \right)$$

例5 设 $u=f(x,y)$ 二阶偏导数连续, 求下列表达式在 $x=r\cos\theta$, $y=r\sin\theta$ 下关于r,θ的形式

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$

解: 已知 $r=\sqrt{x^2+y^2}$, $\theta=\arctan\frac{y}{x}$



(1) $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\boxed{\frac{\partial r}{\partial x}=\frac{x}{r},\quad \frac{\partial \theta}{\partial x}=\frac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2}=\frac{-y}{x^2+y^2}}$$

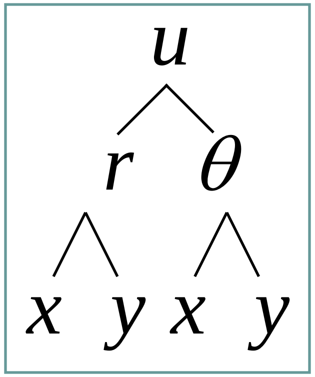
$$=\frac{\partial u}{\partial r}\frac{x}{r}-\frac{\partial u}{\partial \theta}\frac{y}{r^2}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\boxed{\frac{\partial r}{\partial y}=\frac{y}{r},\quad \frac{\partial \theta}{\partial y}=\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}}$$

$$=\frac{\partial u}{\partial r}\frac{y}{r}+\frac{\partial u}{\partial \theta}\frac{x}{r^2}$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$



$$\therefore \quad (\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

作业

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