二、高阶偏导数

函数$z=f(x,y)$的二阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f_{xx}(x,y),\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f_{xy}(x,y),\ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=f_{yx}(x,y)$$

后者称为混合偏导数

定义：二阶及二阶以上的偏导数统称为高阶偏导数.

类似可以定义更高阶的偏导数.

例如， $z=f(x,y)$ 关于 x 的三阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right)=\frac{\partial^3 z}{\partial x^3}$$

$z=f(x,y)$ 关于 x 的 $n-1$ 阶偏导数，再关于 y 的一阶偏导数为

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} z}{\partial x^{n-1}}\right)=\frac{\partial^n z}{\partial x^{n-1}\partial y}$$

例4： 求函数 $z=e^{x+2y}$ 的二阶偏导数及 $\frac{\partial^3 z}{\partial y\partial x^2}$.

解： $\frac{\partial z}{\partial x}=e^{x+2y}$ $\qquad$ $\frac{\partial z}{\partial y}=2e^{x+2y}$

（2）

$$\frac{\partial^2 z}{\partial x^2}=e^{x+2y} \qquad \frac{\partial^2 z}{\partial x\partial y}=2e^{x+2y}$$

（3）

$$\frac{\partial^2 z}{\partial y\partial x}=2e^{x+2y} \qquad \frac{\partial^2 z}{\partial y^2}=4e^{x+2y}$$

$$\frac{\partial^3 z}{\partial y\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y\partial x}\right)=2e^{x+2y}$$

注意:上例中 $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$,

与求偏导数的次序无关

一般而言：这一结论并不总成立(如：例5).

为此，我们可证明以下结论:

定理：如果二元函数两个混合偏导数在区域D上连续，则它们必然在D上相等

例5、 $$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_x(x,y)=\begin{cases} y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_{xy}(0,0)=\lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y}{\Delta y}=-1$$

$$f_{yx}(0,0)=\lim_{\Delta x\to 0}\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1$$

二者不等