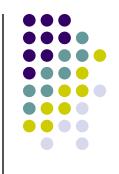
二、高阶偏导数

函数z = f(x,y)的二阶偏导数为



$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y), \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y)$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y), \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y)$$

后者称为混合偏导数

定义: 二阶及二阶以上的偏导数统称为高阶偏导数.

类似可以定义更高阶的偏导数.

例如,z = f(x, y) 关于 x 的三阶偏导数为

$$\frac{\partial}{\partial x}(\frac{\partial^2 z}{\partial x^2}) = \frac{\partial^3 z}{\partial x^3}$$

z = f(x, y) 关于 x 的 n-1 阶偏导数,再关于

y的一阶偏导数为

$$\frac{\partial}{\partial y}(\frac{\partial^{n-1}z}{\partial x^{n-1}}) = \frac{\partial^n z}{\partial x^{n-1}\partial y}$$



例4: 求函数 $z = e^{x+2y}$ 的二阶偏导数及 $\frac{\partial^3 z}{\partial y \partial x^2}$. 解: $\frac{\partial z}{\partial x} = e^{x+2y}$ $\frac{\partial z}{\partial y} = 2e^{x+2y}$

解:
$$\frac{\partial z}{\partial x} = e^{x+2y}$$

$$\frac{\partial z}{\partial y} = 2e^{x+2y}$$

$$\frac{\partial^2 z}{\partial x^2} = e^{x+2y}$$

$$\frac{\partial y}{\partial x \partial y} = 2e^{x+2y}$$

$$\frac{\partial^2 z}{\partial x^2} = e^{x+2y}$$

$$\frac{\partial^2 z}{\partial x^2} = 2e^{x+2y}$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2e^{x+2y}$$

$$\frac{\partial^2 z}{\partial v^2} = 4e^{x+2y}$$

$$\frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial y \partial x} \right) = 2e^{x+2y}$$



注意:上例中
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$
,

与求偏导数的次序无关

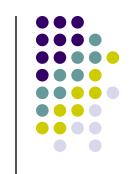
一般而言: 这一结论并不总成立(如: 例5).

为此,我们可证明以下结论:

定理:如果二元函数两个混合偏导数在区域D上

连续,则它们必然在D上相等

例5、
$$f(x,y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$$



$$f_{x}(x,y) = \begin{cases} y \frac{x^{4} + 4x^{2}y^{2} - y^{4}}{(x^{2} + y^{2})^{2}}, & x^{2} + y^{2} \neq 0 \\ 0, & x^{2} + y^{2} = 0 \end{cases}$$

$$\begin{cases} x^{4} - 4x^{2}y^{2} - y^{4}, & x^{2} + y^{2} \neq 0 \\ 0, & x^{2} + y^{2} = 0 \end{cases}$$

$$f_{y}(x,y) = \begin{cases} x \frac{x^{4} - 4x^{2}y^{2} - y^{4}}{(x^{2} + y^{2})^{2}}, & x^{2} + y^{2} \neq 0 \\ 0, & x^{2} + y^{2} = 0 \end{cases}$$

$$f_{xy}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-\Delta y}{\Delta y} = -1$$

$$f_{yx}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x,0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$f_{yx}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$