Introduction to Machine Learning

Fall 2019

University of Science and Technology of China

Lecturer: Jie Wang — Homework 1

Posted: Seq. 20, 2019 — Due: Seq. 30, 2019

Name: Bowen Zhang — ID: PB17000215

Exercise 1: Rank of matrices 20pts

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$.

1. Please show that

   (a) $\mathbf{rank}(\mathbf{A}) = \mathbf{rank}(\mathbf{A}^\top)$;

   (b) $\mathbf{rank}(\mathbf{AB}) \leq \mathbf{rank}(\mathbf{A})$;

   (c) $\mathbf{rank}(\mathbf{AB}) \leq \mathbf{rank}(\mathbf{B})$;

   (d) $\mathbf{rank}(\mathbf{A}) = \mathbf{rank}(\mathbf{A}^\top \mathbf{A})$.

2. The *column space* of $\mathbf{A}$ is defined by

$$\mathcal{C}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \mathbf{A}\mathbf{x}, \, \mathbf{x} \in \mathbb{R}^n\}.$$

   The *null space* of $\mathbf{A}$ is defined by

$$\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = 0\}.$$

   Notice that, the rank of $\mathbf{A}$ is the dimension of the column space of $\mathbf{A}$.

   Please show that:

   (a) $\mathbf{rank}(\mathbf{A}) + \dim(\mathcal{N}(\mathbf{A})) = n$;

   (b) let $\mathbf{y} \in \mathbb{R}^m$, show that $\mathbf{y} = 0$ if and only if $\mathbf{a}_i^\top \mathbf{y} = 0$ for $i = 1, \ldots, m$, where $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is a basis of $\mathbb{R}^m$.

Solution:

1. 设$rank(A) = r$

■