We use Petri ' s theorem in order to conclude if the curve X_0 (112) / < ω_{16} > corresponding to pair (112; < ω_{16} >) is bielliptic or not.

For N = 112, the jacobian of X_0 (112) / < ω_{16} > denoted by $J_0(112)^{<\omega_{16}>}$ is isogenous over Q to $\prod_{i=1}^{5} A^{ni}{}_{f_i}$, for some $f_i \in New_{M_i}$ with $M_i \mid N$ and the abelian varieties A_{f_i} are pairwise non isogenous over Q. Any f_i determines n_i normalized eigenforms g_j in $S_2(112)^{<\omega_{16}>}$ such that $J_0(112)^{<\omega_{16}>}$ isogenous over Q to $\prod_{j=1}^{6} A_{g_j}$, where g_1, ..., g_6 are all of these eigenforms. We can see this in the following

According to the Magma algorithm the Jacobian of the modular curve X_0 (112) / < ω_{16} > is isogeny over Q to the product $E^2{}_{14\,a} \times E_{56\,b} \times E_{56\,a} \times E_{112\,a} \times E_{112\,b}$ of elliptic curves. Hence there are normalized newforms f_1, f_2, f_3, f_4, and f_5 such that the product $A^2{}_{f_1} \times A_{f_2} \times A_{f_3} \times A_{f_4} \times A_{f_5}$ of elliptic curves is isogeny over Q to $E^2{}_{14\,a} \times E_{56\,b} \times E_{56\,a} \times E_{112\,a} \times E_{112\,b}$, where the q expansion of f_i, i = 1, ..., 5 are

```
In[1]:= f1 = q - q^2 - 2 * q^3 + q^4 + 2 * q^6 + q^7 - q^8 + q^9 - 2 * q^12 - 4 * q^13 - q^14 + q^16 +
      6 * q^17 - q^18 + 2 * q^19 - 2 * q^21 + 2 * q^24 - 5 * q^25 + 4 * q^26 + 4 * q^27 +
      q^28 - 6 * q^29 - 4 * q^31 - q^32 - 6 * q^34 + q^36 + 2 * q^37 - 2 * q^38 + 8 * q^39;
f2 = q + 2 * q^3 - 4 * q^5 + q^7 + q^9 - 8 * q^15 - 2 * q^17 - 2 * q^19 + 2 * q^21 +
    8 * q^23 + 11 * q^25 - 4 * q^27 + 2 * q^29 + 4 * q^31 - 4 * q^35 - 6 * q^37;
f3 = q + 2 * q^5 - q^7 - 3 * q^9 - 4 * q^11 + 2 * q^13 - 6 * q^17 +
    8 * q^19 - q^25 + 6 * q^29 + 8 * q^31 - 2 * q^35 - 2 * q^37;
f4 = q - 2 * q^3 - 4 * q^5 - q^7 + q^9 + 8 * q^15 - 2 * q^17 + 2 * q^19 + 2 * q^21 -
      8 * q^23 + 11 * q^25 + 4 * q^27 + 2 * q^29 - 4 * q^31 + 4 * q^35 - 6 * q^37;
f5 = q + 2 * q^5 + q^7 - 3 * q^9 + 4 * q^11 + 2 * q^13 - 6 * q^17 -
      8 * q^19 - q^25 + 6 * q^29 - 8 * q^31 + 2 * q^35 - 2 * q^37;
```

In order to obtain Petri ' s model, we have to lift these modular forms to level 112.
For f_1, it has level 14. To lift it to level 112 = 14 * 8,
we need to lift it 3 times by 2. We can see,
in Magma code or we can use also Caremona table,
that the action of ω_2 is 1. Hence the action of $\omega_8 = \omega_2 * \omega_2 * \omega_2$ is 1 due to the action of ω_{16} is 1 when we lift to level 112 . Thus we have 2 cases of the action of ω_8
Case1 : the action is 1,
-1,
and - 1 (the other cases as - 1, 1, and - 1 or - 1, -1, and 1 etc) are equivalent.
Case 2 : the action is 1, 1,
and 1. For this reason f_1 repeated twoic in the Jacobian decomposion $J_0\ (112)^{<\omega_{16}>}$ and we obtain two independent newforms g1111 and g1211 of level 112.

For f_2, it has level 56. To lift it to level 112 = 56 * 2,
we need to lift it one time by 2. From the q expansion of eigenform correspondence to the elliptic curve $E_{56\,b}$ in Caremona table we see that the action of ω_8 is 1. Since the action of ω_{16} must be 1 when we lift f_2 to level 112,
then the action of ω_2 is 1. For this reason we get one newform g_{21} of level 112.

For f_3, it has level 56. To lift it to level 112 = 56 * 2,
we need to lift it one time by 2. From the q expansion of eigenform correspondence to the elliptic curve $E_{56\,a}$ in Caremona table we see that the action of ω_8 is - 1. Since the action of ω_{16} must be 1 when we lift f_2 to level 112,
then the action of ω_2 is - 1. For this reason we get one newform g_{31} of level 112.

For f_4 and f_5, they are of levels 112 and we did not need to lift them.

```
In[5]:= g11 = f1 + 2 (f1 /. q → q^2) // Expand;
g111 = g11 - 2 (g11 /. q → q^2) // Expand;
g1111 = g111 - 2 (g111 /. q → q^2) // Expand;

g12 = f1 + 2 (f1 /. q → q^2) // Expand;
g121 = g12 + 2 (g12 /. q → q^2) // Expand;
g1211 = g121 + 2 (g121 /. q → q^2) // Expand;

g21 = f2 + 2 (f2 /. q → q^2) // Expand;

g31 = f3 - 2 (f3 /. q → q^2) // Expand;

g41 = f4;

g51 = f5;

m = 39;
h1 = Series[g1111, {q, 0, m}];
h2 = Series[g1211, {q, 0, m}];
h3 = Series[g21, {q, 0, m}];
h4 = Series[g31, {q, 0, m}];
h5 = Series[g41, {q, 0, m}];
h6 = Series[g51, {q, 0, m}];
```

During the next steps we apply the following proposition
Let X be a non-hyperelliptic curve of genus g ≥ 3 defined over a subfield K of the complex field C.

For a fixed basis $\omega_1, \ldots, \omega_9$ of $\Omega^1{}_{X/K}$ and an iteger i ≥ 2, we denote by $\mathcal{L}_i$ the K-vector space formed by the homogenous polynomials $Q \in K[x_1, \ldots, x_g]$ of degree i such that $Q(\omega_1, \ldots, \omega_g) = 0$. Assume that Jac(X) is isogeny over K to $E^m \times A$, where E is an elliptic curve and A an abelian variety that does not have E as a quotient defined over K. Denote by $I_{g-m} \in M_{g-m}(\mathbb{Q})$ the identity matrix. Take the basis $\{\omega_i\}$ such that $\omega_1, \ldots, \omega_m$ and $\omega_{m+1}, \ldots, \omega_g$ are base of the pullbacks of $\Omega^1{}_{E^m/K}$ and $\Omega^1{}_{A/K}$ respectively. Then, E is K-isogenous to the jacobian of a bielliptic quotient of X over K if, and only if, there exists a matrix $\gamma \in GL_m(K)$ that satisfies $Q((-x_1, x_2, \ldots, x_g) . B) \in \mathcal{L}'_i$ for all $Q \in \mathcal{L}_i$ and for all i ⩾ 2, where B is the matrix $\begin{pmatrix} \gamma & 0 \\ 0 & I_{g-m} \end{pmatrix} \in GL_g(K)$ and $\mathcal{L}'_i=\{Q((x_1, x_2, \ldots, x_g) . B) \mid Q \in \mathcal{L}_i\}$.

Now h_1, h_2, h_3, h_4, h_5 and h_6 gives us the q expansion of 6 modular forms each one of level 112. Since the genus of the modular curve $X_0(112) / < \omega_{16} >$ is 6, then we consider the quadratic form $Q \in \mathbb{Q}[x, y, z, s, t, u]$

```
In[21]:= Q = {x^2, y^2, z^2, s^2, t^2, u^2, x y , x z, x s, x t , x u, y z,
    y s, y t, y u, z s, z t, z u, s t, s u, t u}.{a1, a2, a3, a4, a5, a6,
    a7, a8, a9, a10, a11, a12, a13, a14, a15, a16, a17, a18, a19, a20, a21}
```

Out[21]= $a4\, s^2 + a19\, s\, t + a5\, t^2 + a20\, s\, u + a21\, t\, u + a6\, u^2 + a9\, s\, x + a10\, t\, x + a11\, u\, x + a1\, x^2 + a13\, s\, y + a14\, t\, y + a15\, u\, y + a7\, x\, y + a2\, y^2 + a16\, s\, z + a17\, t\, z + a18\, u\, z + a8\, x\, z + a12\, y\, z + a3\, z^2$

Substitution by the q expansion of h_1, h_2, h_3, h_4, h_5 and h_6 in the quadratic form Q and then equating the coefficients of q^i, i = 2, 3, ..., 30 by 0 yields

```
In[22]:= L = Q /. {x → h1, y → h2, z → h3, s → h4, t → h5, u → h6} // Expand;
l = Table[Coefficient[L, q, i], {i, 2, 30}];
T = Solve[{l == {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
        0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}, {a1, a2, a3, a4, a5, a6, a7, a8,
      a9, a10, a11, a12, a13, a14, a15, a16, a17, a18, a19, a20, a21}][[1]]
```

Solve::svars : Equations may not give solutions for all "solve" variables. >>

Out[23]= $\left\{a4 \to -2\, a3,\ a6 \to -\frac{48\, a1}{19} - \frac{80\, a2}{19} + 2\, a3 - \frac{11\, a5}{19},\ a7 \to \frac{5\, a1}{19} + \frac{21\, a2}{19} - \frac{4\, a5}{19},\right.$
$a8 \to \frac{15\, a1}{19} + \frac{25\, a2}{19} - \frac{5\, a3}{8} - \frac{5\, a5}{38},\ a11 \to \frac{77\, a1}{38} - \frac{125\, a2}{38} + \frac{5\, a3}{4} + \frac{3\, a5}{38},$
$a12 \to \frac{9\, a1}{19} + \frac{15\, a2}{19} - \frac{3\, a3}{8} - \frac{3\, a5}{38},\ a13 \to -\frac{16\, a10}{29} + \frac{11\, a9}{29},\ a14 \to \frac{11\, a10}{29} + \frac{16\, a9}{29},$
$a15 \to \frac{27\, a1}{38} - \frac{107\, a2}{38} - \frac{5\, a3}{4} + \frac{5\, a5}{38},\ a16 \to -\frac{16\, a10}{29} + \frac{40\, a9}{29},\ a17 \to 0,\ a18 \to 0,$
$\left. a19 \to -\frac{52\, a1}{19} + \frac{116\, a2}{19} - \frac{4\, a5}{19},\ a20 \to -\frac{40\, a10}{29} - \frac{16\, a9}{29},\ a21 \to \frac{32\, a10}{29} - \frac{80\, a9}{29}\right\}$

Here we plug in Q by the previous values of a4, a6, ... in terms of a1, a2, a3, a5, a9, a10.

In[25]:= **QQ = Q /. T // Expand // Factor // Numerator**

Out[25]= $-8816\ a3\ s^2 - 12064\ a1\ s\ t + 26912\ a2\ s\ t - 928\ a5\ s\ t + 4408\ a5\ t^2 - 6080\ a10\ s\ u - 2432\ a9\ s\ u + 4864\ a10\ t\ u - 12160\ a9\ t\ u - 11136\ a1\ u^2 - 18560\ a2\ u^2 + 8816\ a3\ u^2 - 2552\ a5\ u^2 + 4408\ a9\ s\ x + 4408\ a10\ t\ x + 8932\ a1\ u\ x - 14500\ a2\ u\ x + 5510\ a3\ u\ x + 348\ a5\ u\ x + 4408\ a1\ x^2 - 2432\ a10\ s\ y + 1672\ a9\ s\ y + 1672\ a10\ t\ y + 2432\ a9\ t\ y + 3132\ a1\ u\ y - 12412\ a2\ u\ y - 5510\ a3\ u\ y + 580\ a5\ u\ y + 1160\ a1\ x\ y + 4872\ a2\ x\ y - 928\ a5\ x\ y + 4408\ a2\ y^2 - 2432\ a10\ s\ z + 6080\ a9\ s\ z + 3480\ a1\ x\ z + 5800\ a2\ x\ z - 2755\ a3\ x\ z - 580\ a5\ x\ z + 2088\ a1\ y\ z + 3480\ a2\ y\ z - 1653\ a3\ y\ z - 348\ a5\ y\ z + 4408\ a3\ z^2$

The dimension of the space $\mathcal{L}_2$ is 6 and it is generated by the following quadratic forms

In[27]:=
```
Coefficient[QQ, a1]
Coefficient[QQ, a2]
Coefficient[QQ, a3]
Coefficient[QQ, a5]
Coefficient[QQ, a9]
Coefficient[QQ, a10]
```

Out[27]= $-12064\ s\ t - 11136\ u^2 + 8932\ u\ x + 4408\ x^2 + 3132\ u\ y + 1160\ x\ y + 3480\ x\ z + 2088\ y\ z$

Out[28]= $26912\ s\ t - 18560\ u^2 - 14500\ u\ x - 12412\ u\ y + 4872\ x\ y + 4408\ y^2 + 5800\ x\ z + 3480\ y\ z$

Out[29]= $-8816\ s^2 + 8816\ u^2 + 5510\ u\ x - 5510\ u\ y - 2755\ x\ z - 1653\ y\ z + 4408\ z^2$

Out[30]= $-928\ s\ t + 4408\ t^2 - 2552\ u^2 + 348\ u\ x + 580\ u\ y - 928\ x\ y - 580\ x\ z - 348\ y\ z$

Out[31]= $-2432\ s\ u - 12160\ t\ u + 4408\ s\ x + 1672\ s\ y + 2432\ t\ y + 6080\ s\ z$

Out[32]= $-6080\ s\ u + 4864\ t\ u + 4408\ t\ x - 2432\ s\ y + 1672\ t\ y - 2432\ s\ z$

QQ /. {x → h1, y → h2, z → h3, s → h4, t → h5, u → h6} // Expand

$O[q]^{41}$

Here we check if QQ (x, y, z, s, t, u) - QQ (x, y, -z, s, t, u) equal zero or not. Similarly for s, t and u.

QQ2 = (QQ - (QQ /. z → -z)) / (2 z) // Factor

$-2432\ a10\ s + 6080\ a9\ s + 3480\ a1\ x + 5800\ a2\ x - 2755\ a3\ x - 580\ a5\ x + 2088\ a1\ y + 3480\ a2\ y - 1653\ a3\ y - 348\ a5\ y$

QQ3 = (QQ - (QQ /. s → -s)) / (2 s) // Factor

$-8\ (1508\ a1\ t - 3364\ a2\ t + 116\ a5\ t + 760\ a10\ u + 304\ a9\ u - 551\ a9\ x + 304\ a10\ y - 209\ a9\ y + 304\ a10\ z - 760\ a9\ z)$

QQ4 = (QQ - (QQ /. t → -t)) / (2 t) // Factor

$-8\ (1508\ a1\ s - 3364\ a2\ s + 116\ a5\ s - 608\ a10\ u + 1520\ a9\ u - 551\ a10\ x - 209\ a10\ y - 304\ a9\ y)$

QQ5 = (QQ - (QQ /. u → -u)) / (2 u) // Factor

$-2\ (3040\ a10\ s + 1216\ a9\ s - 2432\ a10\ t + 6080\ a9\ t - 4466\ a1\ x + 7250\ a2\ x - 2755\ a3\ x - 174\ a5\ x - 1566\ a1\ y + 6206\ a2\ y + 2755\ a3\ y - 290\ a5\ y)$

Since f_1 repeated twice in $J_0\ (112)^{\langle\omega_{16}\rangle}$, we replace x, y by (aa1 x + aa2 y) and (bb1 x + bb2 y) respectively . Then we check if QQ (aa1 x + aa2 y, bb1 x + bb2 y, z, s, t, u) - QQ (-aa1 x + aa2 y, -bb1 x + bb2 y, z, s, t, u) equal zero or not.

```
R = QQ /. {x → aa1 x + aa2 y, y → bb1 x + bb2 y};

QQ1 = (R - (R /. x → -x)) / (2 x) // Factor

4408 a9 aa1 s - 2432 a10 bb1 s + 1672 a9 bb1 s + 4408 a10 aa1 t + 1672 a10 bb1 t +
 2432 a9 bb1 t + 8932 a1 aa1 u - 14 500 a2 aa1 u + 5510 a3 aa1 u + 348 a5 aa1 u +
 3132 a1 bb1 u - 12 412 a2 bb1 u - 5510 a3 bb1 u + 580 a5 bb1 u + 8816 a1 aa1 aa2 y +
 1160 a1 aa2 bb1 y + 4872 a2 aa2 bb1 y - 928 a5 aa2 bb1 y + 1160 a1 aa1 bb2 y +
 4872 a2 aa1 bb2 y - 928 a5 aa1 bb2 y + 8816 a2 bb1 bb2 y + 3480 a1 aa1 z + 5800 a2 aa1 z -
 2755 a3 aa1 z - 580 a5 aa1 z + 2088 a1 bb1 z + 3480 a2 bb1 z - 1653 a3 bb1 z - 348 a5 bb1 z

l1 = {Coefficient[QQ1, x, 1], Coefficient[QQ1, y, 1],
   Coefficient[QQ1, z, 1], Coefficient[QQ1, t, 1],
   Coefficient[QQ1, s, 1], Coefficient[QQ1, u, 1]} // Factor

{0, 232 (38 a1 aa1 aa2 + 5 a1 aa2 bb1 + 21 a2 aa2 bb1 -
    4 a5 aa2 bb1 + 5 a1 aa1 bb2 + 21 a2 aa1 bb2 - 4 a5 aa1 bb2 + 38 a2 bb1 bb2),
 29 (24 a1 + 40 a2 - 19 a3 - 4 a5) (5 aa1 + 3 bb1), 152 (29 a10 aa1 + 11 a10 bb1 + 16 a9 bb1),
 152 (29 a9 aa1 - 16 a10 bb1 + 11 a9 bb1),
 58 (154 a1 aa1 - 250 a2 aa1 + 95 a3 aa1 + 6 a5 aa1 +
    54 a1 bb1 - 214 a2 bb1 - 95 a3 bb1 + 10 a5 bb1)}

ll1 = (l1 /. {aa1 → 0}) / bb1 // Factor

{0, 232 (5 a1 aa2 + 21 a2 aa2 - 4 a5 aa2 + 38 a2 bb2), 87 (24 a1 + 40 a2 - 19 a3 - 4 a5),
 152 (11 a10 + 16 a9), -152 (16 a10 - 11 a9), 58 (54 a1 - 214 a2 - 95 a3 + 10 a5)}

{50 736 a1 - 5268 a2 - 3760 a3 + 3315 a4,
 10 336 (2 a7 - a9), -28 080 a1 - 7836 a2 + 11 312 a3 - 1479 a4,
 -16 (5168 a7 aa2 - 2584 a9 aa2 - 2112 a1 bb2 + 696 a2 bb2 - 704 a3 bb2 - 1173 a4 bb2),
 0, -32 (5634 a1 - 249 a2 - 706 a3 + 357 a4)}
ll1[[5]];

{50 736 a1 - 5268 a2 - 3760 a3 + 3315 a4,
 10 336 (2 a7 - a9), -28 080 a1 - 7836 a2 + 11 312 a3 - 1479 a4,
 -16 (5168 a7 aa2 - 2584 a9 aa2 - 2112 a1 bb2 + 696 a2 bb2 - 704 a3 bb2 - 1173 a4 bb2),
 0, -32 (5634 a1 - 249 a2 - 706 a3 + 357 a4)}

ll1 = (l1 /. {aa1 → 1}) // Factor

{0, 232 (38 a1 aa2 + 5 a1 aa2 bb1 + 21 a2 aa2 bb1 -
    4 a5 aa2 bb1 + 5 a1 bb2 + 21 a2 bb2 - 4 a5 bb2 + 38 a2 bb1 bb2),
 29 (24 a1 + 40 a2 - 19 a3 - 4 a5) (5 + 3 bb1), 152 (29 a10 + 11 a10 bb1 + 16 a9 bb1),
 152 (29 a9 - 16 a10 bb1 + 11 a9 bb1),
 58 (154 a1 - 250 a2 + 95 a3 + 6 a5 + 54 a1 bb1 - 214 a2 bb1 - 95 a3 bb1 + 10 a5 bb1)}

Solve[ll1[[3]] == 0, bb1]
```

$$\left\{\left\{bb1 \to -\frac{5}{3}\right\}\right\}$$

$$ll1 = \left(l1 \,/.\, \left\{aa1 \to 1,\ bb1 \to -\frac{5}{3}\right\}\right) \,//\, Factor$$

$$\left\{0,\ \frac{232}{3}\,(89\,a1\,aa2 - 105\,a2\,aa2 + 20\,a5\,aa2 + 15\,a1\,bb2 - 127\,a2\,bb2 - 12\,a5\,bb2),\ 0,\ \frac{2432}{3}\,(2\,a10 - 5\,a9),\ \frac{2432}{3}\,(5\,a10 + 2\,a9),\ \frac{464}{3}\,(24\,a1 + 40\,a2 + 95\,a3 - 4\,a5)\right\}$$

```
Solve[ll1[[2]] == 0, aa2]
Solve[ll1[[2]] == 0, bb2]
```

$$\left\{\left\{\text{aa2} \to \frac{-15\ \text{a1 bb2} + 127\ \text{a2 bb2} + 12\ \text{a5 bb2}}{89\ \text{a1} - 105\ \text{a2} + 20\ \text{a5}}\right\}\right\}$$

$$\left\{\left\{\text{bb2} \to \frac{-89\ \text{a1 aa2} + 105\ \text{a2 aa2} - 20\ \text{a5 aa2}}{15\ \text{a1} - 127\ \text{a2} - 12\ \text{a5}}\right\}\right\}$$

Since QQi ≠ 0 for every i=1, 2, 3, 4, 5, then we conclude that $X_0(112)/\langle \omega_{16} \rangle$ is not bielliptic in any case.