Numerical characterization of the receiving dipole antenna

A. Introduction

Consider a dipole antenna with the antenna terminals connected to a lumped load $\underline{Z}_L$. The antenna is illuminated by a plane wave that arrives along the direction θ, as illustrated in the figure. The electric field lies in the "incidence plane", i.e., the plane that contains the axis of the dipole (z-direction) and the direction of propagation of the incoming wave.

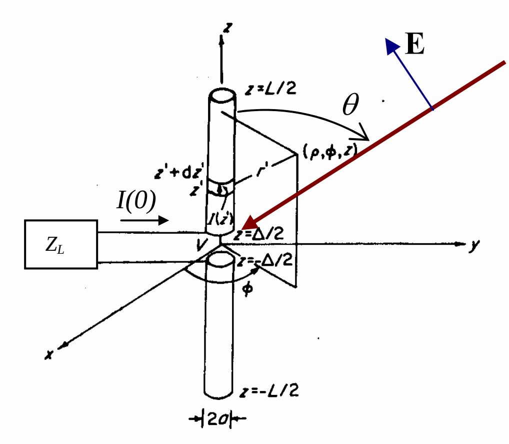

Fig. 1 A dipole antenna is illuminated by a plane wave that arrives along the direction θ. The distance between the antenna terminals is infinitesimal.

It is supposed that the metal is a perfect conductor. The incident wave induces a current distribution $\underline{I}(z')$. The induced current must guarantee that the electric field tangential to the antenna surface vanishes.

The field radiated by the induced current (i.e., the field scattered by the antenna) is denoted by $\mathbf{E}^s$. The scattered field is determined by a vector potential $\mathbf{A} = A_z\hat{\mathbf{z}}$ oriented along z such that [1, 2]:

$$A_z(\rho=0,z)=\frac{\mu_0}{4\pi}\int_{-L/2}^{L/2} I(z')\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}dz', \tag{1}$$

where $|\mathbf{r}-\mathbf{r}'|=\sqrt{a^2+(z-z')^2}$ and a is the dipole radius. The formula assumes that the vector potential is evaluated along the antenna axis (z-axis). For the considered geometry, the z-component of $\mathbf{E}^s$ and the vector potential are related as:

$$E_z^s=\frac{1}{j\omega\mu_0\varepsilon_0}\left(\frac{\partial^2 A_z}{\partial z^2}+k_0^2 A_z\right). \tag{2}$$

The total electric field is the superposition of the incident wave and of the scattered wave. Thereby, one has:

$$E_z=E_z^{inc}+E_z^s, \tag{3}$$

where E_z^{inc} is the z-component of the incident plane wave, given by (when evaluated over the antenna axis),

$$E_z^{inc}=E_0\sin\theta\, e^{jk_0 z\cos\theta}, \qquad (\rho=0). \tag{4}$$

B. Integral equation

To find the unknown current distribution $I(z')$, we use the fact that the z-component E_z of the total electric field must vanish at the antenna surface. Assuming that the spacing between the antenna terminals is infinitesimal (Δ), it follows that:

$$E_z(\rho=0,z)\approx -V\delta(z), \qquad -L/2<z<L/2 \tag{5}$$

where $V=-I(0)Z_L$ is the voltage at the antenna terminals. Using Eqs. (3)-(4), one sees that the scattered field evaluated over the antenna axis is:

$$E_z^s(\rho=0,z)=I(0)\underline{Z}_L\delta(z)-E_0\sin\theta\, e^{jk_0\cos\theta z}, \qquad -L/2<z<L/2 \tag{6}$$

Solving the differential equation (2) (with the observation point in the antenna axis) with E_z^s given by (6), it can be shown that:

$$A_z(\rho=0,z) = -\frac{jE_0}{ck_0\sin\theta}e^{jk_0\cos\theta z} + j\frac{I(0)\underline{Z}_L}{2c}\sin(k_0|z|) + \tilde{C}_1\cos(k_0 z) + \tilde{C}_2\sin(k_0 z) \tag{7}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are unknown constants and c is the speed of light. Substituting (1) into (7), one finally finds that the unknown current satisfies the integral equation:

$$\frac{1}{4\pi}\int_{-L/2}^{L/2} I(z')\frac{e^{-jk_0\sqrt{a^2+(z-z')^2}}}{\sqrt{a^2+(z-z')^2}}dz' + C_1\cos(k_0 z) + C_2\sin(k_0 z) - j\frac{I(0)\underline{Z}_L}{2\eta_0}\sin(k_0|z|) = -\frac{jE_0}{\eta_0 k_0\sin\theta}e^{jk_0\cos\theta z} \tag{8}$$

for $-L/2 < z < L/2$, where $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the free-space impedance and C_1 and C_2 are unknown parameters.

C. Solution of the integral equation

The objective of the computational work is to numerically solve the integral equation (8). To this end, one needs to find the current distribution $I(z')$ and the constants C_1 e C_2 such that Eq. (8) is satisfied for any z in the range $-L/2 < z < L/2$. The input parameters are a, L, λ_0 , θ and E_0.

The key idea is to write the current in a suitable basis of functions [2, 3]. The simplest approach is to approximate $I(z')$ by a "staircase" type function, as illustrated in Figure 2. Such approximation is assumes that the current is constant in intervals spaced by $\Delta = L/N$ [1], with N the number[2] of intervals in the segment $-L/2 \le z' \le L/2$. Let us define

$$z_n = -\frac{L}{2} + \left(n+\frac{1}{2}\right)\Delta, \qquad n=0,1,\ldots N-1. \tag{9}$$

[1] This Δ gives the amplitude of the discretization intervals, and should not be confused with the Δ shown in Figure 1, which represents the spacing between the antenna terminals.
[2] N must be an odd number: 1, 3, 5, etc.

The current is a constant in the interval centered about z_n with amplitude Δ, that is:

$$I(z') = I_n\,, \qquad z_n - \Delta/2 < z < z_n + \Delta/2\,. \tag{10}$$

Since the current must vanish at the antenna ends $(z = \pm L/2)$, it is assumed *a priori* that $I_0 = I_{N-1} = 0$. Thus, the unknowns are I_1, I_2, ... I_{N-2}.

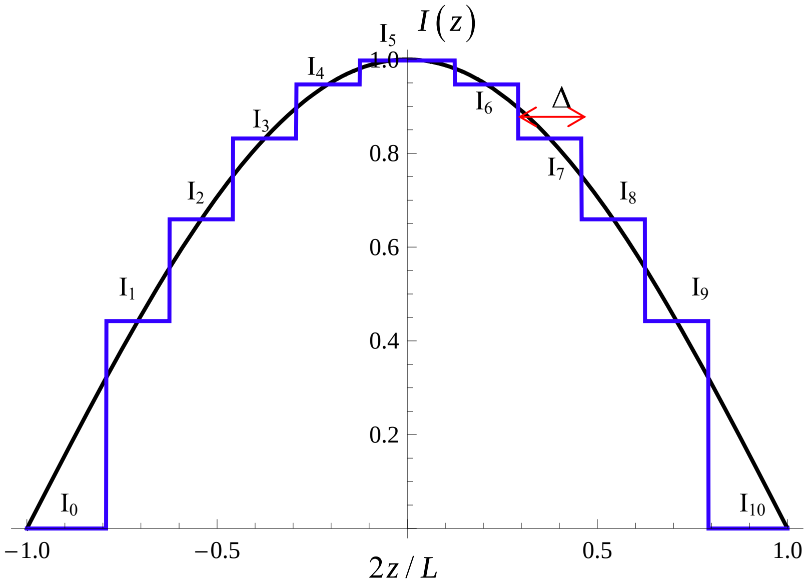

Fig. 2 Approximation of a sinusoidal-type current distribution (black line) by a staircase function with *N*=11 "steps" (blue line). The first and the last "steps" have zero amplitude to guarantee that the current vanishes at the dipole ends.

Using the approximation (10) one can write:

$$\frac{1}{4\pi}\int_{-L/2}^{L/2} I(z')\frac{e^{-jk_0\sqrt{a^2+(z-z')^2}}}{\sqrt{a^2+(z-z')^2}}dz' \approx \sum_{n=1}^{N-2} Z_n(z)\, I_n \tag{11}$$

where

$$Z_n(z) = \int_{z_n-\Delta/2}^{z_n+\Delta/2} K(z\,|\,z')\,dz'\,, \qquad K(z\,|\,z') = \frac{1}{4\pi}\frac{e^{-jk_0\sqrt{a^2+(z-z')^2}}}{\sqrt{a^2+(z-z')^2}}\,. \tag{12}$$

Therefore, substituting this result in (8), the integral equation becomes:

$$\begin{aligned}&\sum_{n=1}^{N-2} Z_n(z)\,I_n + C_1\cos(k_0 z) + C_2\sin(k_0 z) - j\,I_{(N-1)/2}\frac{Z_L}{2\eta_0}\sin(k_0|z|)\\ &= -\frac{j\,E_0}{\eta_0\,k_0\sin\theta}e^{j\,k_0\cos\theta\, z}\end{aligned} \tag{13}$$

It was taken into account that the current at the feeding point is $I(0) = I_{(N-1)/2}$. For a given fixed *N*, the problem unknowns are I_1, I_2, ... I_{N-2} and the constants C_1 e C_2 (a total of N unknowns). The equation (10) should be satisfied for every z in the range $-L/2 < z < L/2$. To find an approximate solution it is enough to take N samples of this interval: $z = z_m \equiv -\frac{L}{2} + \left(m + \frac{1}{2}\right)\Delta$, *m*=0,…*N-1*. In this way, the problem is reduced to the following set of linear equations,

$$\sum_{n=1}^{N-2} Z_{mn} I_n + C_1 \cos(k_0 z_m) + C_2 \sin(k_0 z_m) - j\, I_{(N-1)/2} \frac{Z_L}{2\eta_0} \sin(k_0 |z_m|) = -\frac{j\, E_0}{\eta_0\, k_0 \sin\theta} e^{j k_0 \cos\theta\, z_m}, \qquad m = 0, \ldots N-1 \tag{14}$$

where,

$$Z_{mn} = Z_n(z_m) = \int_{z_n - \Delta/2}^{z_n + \Delta/2} K(z_m \mid z')\, dz'. \tag{15}$$

The coefficients Z_{mn} can be numerically calculated using known quadrature rules or with suitable numerical software [4]. By numerically solving the linear system one finds the unknown current distribution.

D. Report and delivery of the work

Using the procedure delineated in the previous sections, implement a numerical code that solves numerically the integral equation of the receiving dipole antenna (you can use your favorite numerical platform: Matlab, Mathematica, C,...). The work can be done in groups of two or individually.

A report with the results and answers to the questions must be sent by email (mario.silveirinha@tecnico.ulisboa.pt) until the end of 8th May. I will not accept reports delivered after the deadline.

The report should contain the answers to the following questions (in all the questions suppose that $E_0 = 1V/m$ and $\lambda = 1m$):

- **Q1-** Make a graphical representation of the current (real and imaginary parts) for a dipole antenna with length $L = 0.47\lambda$ and radius $a = 0.005\lambda$, calculated for the number of steps *N*= 21, 51, 71 (the curves associated with different *N* should be represented in the same plot). Suppose that the incidence angle is $\theta = 90^{\mathrm{o}}$ and that the antenna terminals are in short-circuit.
- **Q2-** Repeat Q1 for $L = 1.0\lambda$ and $L = 1.5\lambda$.
- **Q3-** Rewrite Eq. (13) in a way that it is more suitable (free of singularities) to find V_{oc}, i.e. the voltage induced at the antenna terminals when they are in open-circuit. Recall that in open-circuit one has $I(0) = 0$ and $\underline{Z}_L = \infty$. Explain clearly what are the required modifications and what are the unknowns of the new problem. Implement a code to solve this new problem.
- **Q4-** Make a polar plot of the voltage $|V_{oc}|$ as a function of incidence angle θ for the cases $L/\lambda = 0.47$, $L/\lambda = 0.75$, e $L/\lambda = 1.5$ supposing that $a = 0.005\lambda$. Use *N*=51 in the calculations. Compare with the result predicted by the sinusoidal current approximation (superimpose the sinusoidal current approximation and the integral equation results in the same polar plot).

  Note: if you did not solve *Q3, find* V_{oc} *using the solution of eq. (14) with the load impedance* $\underline{Z}_L = 10^5\,\Omega$.
- **Q5-** Using the procedure of Q1 one can find the current at the feeding point of the antenna, when its terminals are short-circuited I_{sc}. On the other hand, using the procedure of Q4 one can find the voltage V_{oc} when the antenna terminals are in open-circuit. Taking into account such properties and using

the equivalent circuit of the receiving antenna, find the antenna impedance for a dipole with length $L = 0.47\lambda$ and radius $a = 0.005\lambda$. Specifically, calculate the impedance for N= 21, 51, 71. Suppose that the incidence angle is $\theta = 90^\circ$.

- **Q6-** Using the same procedure as in the previous question, make a plot of the antenna impedance $\underline{Z}_L = R_L + jX_L$ (resistance and reactance) of the dipole antenna as a function of L/λ in the range $0 < L/\lambda < 1.3$ for the case $a = 0.005\lambda$. The results should be obtained using N=51. Indicate for which values of L the antenna is resonant.
- **Q7-** Make a polar plot showing the voltage $|V|$ induced at the antenna terminals (calculated numerically with N=51) as a function of the angle θ for the cases $L/\lambda = 0.47$, and $L/\lambda = 0.75$ with $a = 0.005\lambda$. Suppose that the load impedance is $\underline{Z}_L = 73\Omega$. Represent in the same polar plot the result predicted by the equivalent circuit of the antenna with the parameters V_{oc} and $\underline{Z}_L$ obtained as in the previous questions.
- **Q8-** Annex with the numerical implementation (matlab or mathematica code).

References:

[1] K. F. Lee, *Principles of Antenna Theory*, John Wiley, 1984.

[2] C. Balanis, *Antenna Theory*, Wiley, 2005, Chapter 8.

[3] R. Harrington, *Field Computation by Moment Methods*, IEEE Press, 1993.

[4] *Numerical Recipes in C*, Chapter 4.