Entanglement: From EPR to the CHSH Game

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Single qubit gates

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \end{cases}$$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} Z|0\rangle = |0\rangle \\ Z|1\rangle = -|1\rangle \end{cases}$$

Single qubit gates

$$\frac{\mathrm{Z}+\mathrm{X}}{\sqrt{2}}=\frac{|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|-|1\rangle\langle 1|}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=\mathrm{H}$$

$$\frac{\mathrm{Z}-\mathrm{X}}{\sqrt{2}}=\frac{|0\rangle\langle 0|-|0\rangle\langle 1|-|1\rangle\langle 0|-|1\rangle\langle 1|}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}$$

Bell states

$$\left|\beta_{xy}\right\rangle = \frac{\left|0, y\right\rangle + \left(-1\right)^{x}\left|1, 1-y\right\rangle}{\sqrt{2}}, \qquad x, y \in \mathrm{Z}_2 = \{0, 1\}$$

Bell states

In	Out
$\|00\rangle$	$\dfrac{\|00\rangle+\|11\rangle}{\sqrt{2}}=\|\beta_{00}\rangle$
$\|01\rangle$	$\dfrac{\|01\rangle+\|10\rangle}{\sqrt{2}}=\|\beta_{01}\rangle$
$\|10\rangle$	$\dfrac{\|00\rangle-\|11\rangle}{\sqrt{2}}=\|\beta_{10}\rangle$
$\|11\rangle$	$\dfrac{\|01\rangle-\|10\rangle}{\sqrt{2}}=\|\beta_{11}\rangle$

Bell states

$$|\beta_{00}\rangle \quad \mapsto \quad |\Phi^{+}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\beta_{01}\rangle \quad \mapsto \quad |\Psi^{+}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\beta_{10}\rangle \quad \mapsto \quad |\Phi^{-}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\beta_{11}\rangle \quad \mapsto \quad |\Psi^{-}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

<u>Part 1</u>

From EPR to Bell’s Theorem

Einstein, in his **EPR** paper (co-authored with Nathan **Rosen** and Boris **Podolsky**, in 1935), proposed a «Gedanken-Experiment» (thought experiment) which, he believed, demonstrated that **QM** is not a complete theory of Nature.

In a paper published in 1964, John Stewart **Bell** proposed an experimental test that could be used to check whether or not the **EPR** scenario was valid. This experimental test is now known as **Bell's theorem**.

<u>Part 2</u>
The CHSH Game

Alice		Bob
$\mathsf{Q} = \pm 1$	$\xleftrightarrow{\textbf{ebit}}$	$\mathsf{S} = \pm 1$
$\mathsf{R} = \pm 1$		$\mathsf{T} = \pm 1$

[redacted]

Outcomes $\rightarrow$ $\boxed{m = \pm 1}$

$\mathsf{N} =$ **nonlocal correlation**

CHSH Game $\rightarrow$ $\boxed{\begin{aligned} \mathsf{N} &= \mathsf{QS} + \mathsf{RS} + \mathsf{QT} - \mathsf{RT} \\ &= (\mathsf{Q} + \mathsf{R})\mathsf{S} + (\mathsf{Q} - \mathsf{R})\mathsf{T} \end{aligned}}$

Shared ebit $\rightarrow$ $\left|\Phi^{+}\right\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

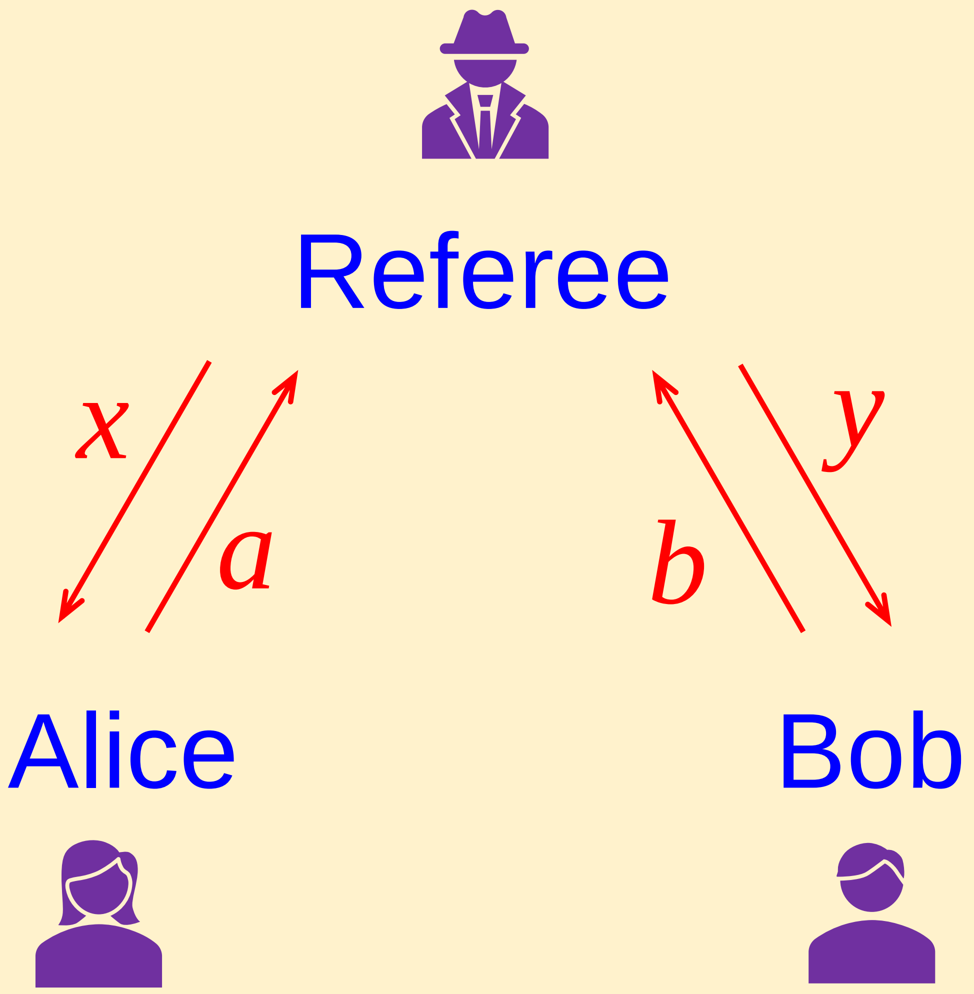
$$\boxed{\mathrm{B} = \mathrm{E}(\mathsf{N}) = \mathrm{E}(\mathsf{QS}) + \mathrm{E}(\mathsf{RS}) + \mathrm{E}(\mathsf{QT}) - \mathrm{E}(\mathsf{RT})}$$

Winning Probability $\rightarrow$ $\boxed{\mathrm{P} = \dfrac{2 + \mathrm{B}}{4}}$

Convention

$$\left\{ \begin{array}{lll} \text{Alice} & \mapsto & \left\{ \begin{array}{lll} m=+1 & \Rightarrow & a=0 \\ m=-1 & \Rightarrow & a=1 \end{array} \right. \\ \text{Bob} & \mapsto & \left\{ \begin{array}{lll} m=+1 & \Rightarrow & b=0 \\ m=-1 & \Rightarrow & b=1 \end{array} \right. \end{array} \right.$$

The EPR thought experiment uncovered a very strange aspect of QM – **entanglement**. One of the simplest means for demonstrating the power of entanglement is with a two-player game known as the **CHSH game**, which is a particular variation of the original setup in Bell’s theorem.



Referee
x
a
b
y
Alice
Bob

CHSH Game

Winning condition $\rightarrow$ $\boxed{x \wedge y = a \oplus b}$

x	y	$x \wedge y$	$= a_x \oplus b_y$
0	0	0	$= a_0 \oplus b_0$
0	1	0	$= a_0 \oplus b_1$
1	0	0	$= a_1 \oplus b_0$
1	1	1	$= a_1 \oplus b_1$

$a \oplus b$	$b = 0$	$b = 1$
$a = 0$	$a \oplus b = 0$	$a \oplus b = 1$
$a = 1$	$a \oplus b = 1$	$a \oplus b = 0$

$$\boxed{\begin{array}{ccc} a \oplus b = 0 & \Leftrightarrow & a = b \\ a \oplus b = 1 & \Leftrightarrow & a \neq b \end{array}}$$

Part 3

The Classical Strategy to Win the CHSH Game

$$\boxed{P \leq \frac{3}{4} = 75\%}$$

Part 4

The Quantum Strategy to Win the CHSH Game

Quantum strategy $\rightarrow$ Alice and Bob share a maximally entangled state $\left|\Phi^{+}\right\rangle$

$$\begin{cases} x=0 & \Rightarrow \quad \text{Alice performs a measurement of } \mathsf{Q}=\mathsf{Z}_1 \text{ on her system} \\ x=1 & \Rightarrow \quad \text{Alice performs a measurement of } \mathsf{R}=\mathsf{X}_1 \text{ on her system} \end{cases}$$

$$\begin{cases} y=0 & \Rightarrow \quad \text{Bob performs a measurement of } \mathsf{S}=\dfrac{\mathsf{Z}_2+\mathsf{X}_2}{\sqrt{2}} \text{ on his system} \\ y=1 & \Rightarrow \quad \text{Bob performs a measurement of } \mathsf{T}=\dfrac{\mathsf{Z}_2-\mathsf{X}_2}{\sqrt{2}} \text{ on his system} \end{cases}$$

$$QS = \frac{Z_1 Z_2 + Z_1 X_2}{\sqrt{2}}$$

$$E(QS) = \frac{\langle \Phi^+ | Z_1 Z_2 | \Phi^+ \rangle}{\sqrt{2}} + \frac{\langle \Phi^+ | Z_1 X_2 | \Phi^+ \rangle}{\sqrt{2}}$$

$$\left\{ \begin{array}{l} \langle \Phi^+ | Z_1 Z_2 | \Phi^+ \rangle = 1 \\ \langle \Phi^+ | Z_1 X_2 | \Phi^+ \rangle = 0 \end{array} \right.$$

Conclusion $\rightarrow \quad E(QS) = \frac{1}{\sqrt{2}}$

$$\mathrm{RS} = \frac{X_1 Z_2 + X_1 X_2}{\sqrt{2}}$$

$$E(\mathrm{RS}) = \frac{\langle \Phi^+ | X_1 Z_2 | \Phi^+ \rangle}{\sqrt{2}} + \frac{\langle \Phi^+ | X_1 X_2 | \Phi^+ \rangle}{\sqrt{2}}$$

$$\begin{cases} \langle \Phi^+ | X_1 Z_2 | \Phi^+ \rangle = 0 \\ \langle \Phi^+ | X_1 X_2 | \Phi^+ \rangle = 1 \end{cases}$$

Conclusion $\rightarrow$ $E(\mathrm{RS}) = \frac{1}{\sqrt{2}}$

$$QT = \frac{Z_1 Z_2 - Z_1 X_2}{\sqrt{2}}$$

$$E(QT) = \frac{\langle \Phi^+ | Z_1 Z_2 | \Phi^+ \rangle}{\sqrt{2}} - \frac{\langle \Phi^+ | Z_1 X_2 | \Phi^+ \rangle}{\sqrt{2}}$$

$$\begin{cases} \langle \Phi^+ | Z_1 Z_2 | \Phi^+ \rangle = 1 \\ \langle \Phi^+ | Z_1 X_2 | \Phi^+ \rangle = 0 \end{cases}$$

Conclusion $\rightarrow$ $E(QT) = \frac{1}{\sqrt{2}}$

$$RT = \frac{X_1 Z_2 - X_1 X_2}{\sqrt{2}}$$

$$E(RT) = \frac{\langle \Phi^+ | X_1 Z_2 | \Phi^+ \rangle}{\sqrt{2}} - \frac{\langle \Phi^+ | X_1 X_2 | \Phi^+ \rangle}{\sqrt{2}}$$

$$\begin{cases} \langle \Phi^+ | X_1 Z_2 | \Phi^+ \rangle = 0 \\ \langle \Phi^+ | X_1 X_2 | \Phi^+ \rangle = 1 \end{cases}$$

Conclusion $\rightarrow$ $E(RT) = -\dfrac{1}{\sqrt{2}}$

$$E(QS)=\frac{1}{\sqrt{2}},\quad E(RS)=\frac{1}{\sqrt{2}},\quad E(QT)=\frac{1}{\sqrt{2}},\quad E(RT)=-\frac{1}{\sqrt{2}}$$

Conclusion (QM) $\rightarrow$ $\boxed{B=E(QS)+E(RS)+E(QT)-E(RT)=2\sqrt{2}}$

Maximal Winning Probability (QM) $\rightarrow$ $\boxed{P=\frac{4+B}{8}=\frac{2+\sqrt{2}}{4}=\cos^2\left(\frac{\pi}{8}\right)\approx 85.36\%}$

Part 5

Conclusion

Appendix

Acronyms

CHSH	Clause-Horne-Shimony-Holt (1969)
EPR	Einstein-Podolsky-Rosen (1935)
QM	Quantum Mechanics

Questions

The main goal of the 3rd Project is to develop a clear account of the problem outlined in this file (this is only a sketch). That is: Each project should develop its coherent account starting from this overall structure.

<u>Problem</u>: From EPR to the CHSH Game.

Two main parts are:
1) To explain why local realism (EPR) predicts a winning probability less or equal to 75%;
2) To explain why QM predicts a winning probability of 85%.

The QM treatment herein presented should be translated into an explanation in terms of photons and their polarization.

Neither EPR nor QM allow a winning probability of 100%. Comment.

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