Linear Algebra

Last Updated January 21, 2023

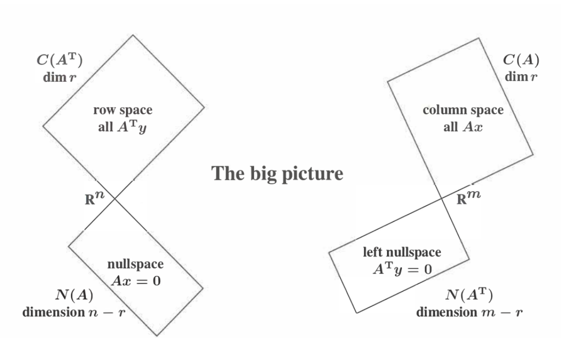
Concepts

Determinant - $det(A)$
Singular matrix - $det(A) = 0$

Vector Spaces and Subspaces

Four Subspaces

Let A a m by n matrix



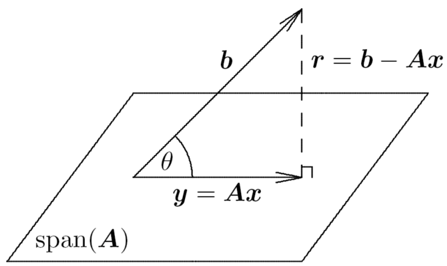
- $r = rank(A) = dim(C(A)) =$ Number of pivots
- $N(A) =$ Null space of $A = Ker(A) =$(FR) Noyau de A
- $C(A) =$Column Space of $A = Im(A) =$(FR) Image de A

Fundamental Theorem of Linear Algebra

- The column space and row space both have dimension r.
- The nullspaces have dimension $n - r$ and $m - r$
- $N(A)$ is the orthogonal complement of the row space $C(A^T)$ (in) $\mathbb{R}^n$
- $N(A^T)$ is the orthogonal complement of the column space $C(A)$ (in) $\mathbb{R}^m$

Projection & Least Squares

Projection



- $span(A) \perp r$
- $span(A) = C(A) = Im(A)$
- $y \in span(A)$

$$\Rightarrow span(A) \perp r$$
because each column of A is perpendicular to r
$$\Leftrightarrow A^T r = \vec{0}$$
$$\Leftrightarrow A^T(b - Ax) = \vec{0}$$
$$\Leftrightarrow x = (A^TA)^{-1}A^Tb$$
$$\Leftrightarrow y = A(A^TA)^{-1}A^Tb$$
$$\Leftrightarrow \rho = A(A^TA)^{-1}A^T$$
ρ being the projection matrix

Least Squares

Let A be a matrix of $m \times n$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, where m is the number of samples and n is the number of dimension/features. The dimension of Ax, which is the $span(A)$, is at maximum n, and b have dimension m.
Usually $m > n$ because there are more samples than features, so $dim(span(A)) < dim(b)$, in that case it is not guaranteed that it can be found a x that satified the equation:
$$Ax = b$$
That is why the make a projection of b over $span(A)$ to have the closest Ax to b. Taking the equations from projection section:
$$x = (A^TA)^{-1}A^Tb$$

Matrix factorizations

1. $A = LU$
2. $A = LDU$
3. $PA = LU$
4. $EA = R$
5. $S = C^TC$
6. $A = QR$ - orthonormal columns in Q, upper triangular R

   Requirements
7. $A = X\Lambda X^{-1}$ - Eigenvectors in X, eigenvalues in Λ
8. $A = Q\Lambda Q^T$
9. $A = BJB^{-1}$
10. $A = U\Sigma V^T$ - SVD: (orthogonal U is $m \times m$)($m \times n$ singular value matrix $\sigma_1, ..., \sigma_r$ on its diagonal)(orthogonal V is $n \times n$)
11. $A^+ = V\Sigma^+U^T$ - Pseudoinverse
12. $A = QS$
13. $A = U\Sigma U^{-1}$
14. $A = QTQ^{-1}$

Frank Facundo