FVA on Vanilla Option

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1 Summary

This document is to calculate the strike implied vol for forward volatility agreement (FVA) on vanilla option with replication method.

To replicate the forward contract, we long 1 call with expiry at Dec 17, 2021 and short $\frac{C(0,T_2)}{C(0,T_1)}$ call with expiry at Jun 18, 2021. $C(0,T_2), C(0,T_1)$ stands for the call value with maturity T_2 and T_1, respectively. The NPV when enter (Sep 29, 2020) is zero. See *Section 2* for details.

Stirke volatility is $X = 16.10671\%$. The calculation is to get expected forward start implied vol by using implied total variance additive. The expected total implied variance is calculated by a strip of OTM options, which is similar to VIX/VSTOXX calculation. See *Section 2* for details.

2 Methodology

2.1 Replication

To replicate a forward contract, which is zero payment when enter into, we use a self-financing portfolio as below. $C(0,T_2), C(0,T_1)$ stands for the call value with maturity T_2 and T_1, respectively.

- long 1 call with expiry at T_2 (Dec 17, 2021), $cash = -C(0,T_2)$
- To finance the first trade, short selling $\frac{C(0,T_2)}{C(0,T_1)}$ unit call with expiry at T_1 (Jun 18, 2021), thus get the $cash = +C(0,T_2)$

The *Table 1* shows that asset and cash change of replicated portfolio is the same the forward.

The NPV(Net Present Value) at Sep 29, 2020 will be equal to $V_0 = C(0,T_2) - \frac{C(0,T_2)}{C(0,T_1)} * C(0,T_1) = 0$, which is consistent with entering a forward contact.

	t_0	T_1	T_2
Forward	0	$C(S_{T_1}, \sigma(T_1, T2)) - C(S_{T_1}, X)$	$(S_{T_2} - K)^+$
long	$-C(0, T_2)$	$C(S_{T_1}, \sigma(T_1, T2))$	$(S_{T_2} - K)^+$
short	$+C(0, T_2)$	$-\frac{C(0,T_2)}{C(0,T_1)}(S_{T_1} - K)^+$	0

Table 1: Replicated Portfolio v.s Forward Contract

2.2 Forward-start Implied Variance

Define total implied variance from t_1 to t_2:

$$W(t_1, t_2) = \int_{t_1}^{t_2} \sigma_{imp}^2(t)dt$$

Thus the total implied variance can additive. For our trade, we have:

$$W(0, T_2) = W(0, T_1) + W(T_1, T_2)$$

$$\mathbb{E}\left[W(0, T_2)\right] = \mathbb{E}\left[W(0, T_1)\right] + \mathbb{E}\left[W(T_1, T_2)\right]$$

Thus we have:

$$\mathbb{E}\left[\sigma_{imp}^2(T_1, T_2)\right] = \frac{\mathbb{E}\left[W(0, T_2)\right] - \mathbb{E}\left[W(0, T_1)\right]}{T_2 - T_1}$$

for $\mathbb{E}\left[W(0, T_2)\right]$ and $\mathbb{E}\left[W(0, T_1)\right]$, it can be replicated by a strip of weighted average of OTM options per Derman, Kamal, Kani, and Zou (1996)[1]. The approximate is due to the accurate replication portfolio consists of infinite OTM option in continuous strike, but in practice this is not true.

$$\mathbb{E}\left[W(0, T)\right] \approx 2\sum_i \frac{\Delta K_i}{K_i^2} Q_i(K_i) - \left[\frac{F}{K_0} - 1\right]^2$$

K_0 is the highest strike below underlying forward F, in our trade, $F = 3350$, and $K_0 = 3300$. $Q_i(K_i)$ is the OTM put and call option with strike $K_i < K_0$

2.3 Convexity Adjustment

Per Jensen's inequality, we have[2]:

$$X = \mathbb{E}\left[\sqrt{\sigma_{imp}^2(T_1, T_2)}\right] \leq \sqrt{\mathbb{E}\left[\sigma_{imp}^2(T_1, T_2)\right]}$$

[1] Similar to VIX/VSTOXX calculation (see *Gatheral, J. Ref. [1]* for derivation details).
[2] $f(x) = \sqrt{x}$ is s concave function

We will use Heston model with Heston-Nandi parameters ($v_0 = 0.04, \bar{v} = 0.04, \lambda = 10.0, \eta = 1.0, \rho = -1$) to calculate convexity adjustment ([1]).

$$\mathbb{E}\left[\sqrt{W(0,T)}\right] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - \mathbb{E}\left[e^{-\psi W(0,T)}\right]}{\psi^{3/2}} \, d\psi$$

$$\mathbb{E}\left[W(0,T)\right] = \frac{1 - e^{-\lambda T}}{\lambda} (v_0 - \bar{v}) + \bar{v}T$$

$$\textit{Convexity Adj.} = \sqrt{\mathbb{E}\left[W(0,T)\right]} - \mathbb{E}\left[\sqrt{W(0,T)}\right]$$

$$X = \mathbb{E}\left[\sqrt{\sigma^2_{imp}(T_1,T_2)}\right] = \sqrt{\mathbb{E}\left[\sigma^2_{imp}(T_1,T_2)\right]} - \frac{\textit{Convexity Adj.}}{\sqrt{T_2 - T_1}}$$

NOTE that the convexity adjustment is model dependent (based on Heston model and Heston-Nandi parameters).

In this case, the convexity adjustment is 0.04511077 (from R code) and final $X = 0.161067053 (\approx 16.11\%)$**.**

3 Discussion of Pricing Risk

The methodology used in this document is a model-free way (except for convexity adjustment), however, here are some concerns:

- The expected total implied variance is an infinite strip of weighted average of OTM options with all continuous strikes. In practice, we only have a finite number of strike prices listed per expiration. This may lead to pricing error. One way to solve is to use model-dependent way.
- One way of model-dependent solution is to fit the SVI model by using variance swap market quotes. But pricing will be also problematic. (e.g. extrapolation method for long-dated expiry)

4 Appendix: Convexity Adjustment

This section we report the convexity adjustment of total implied variance and volatility for different maturity. See Figure 1. The methodology refers to the [1] and R code is in submission folder.

References

[1] Gatheral, J. (2006). *The Volatility Surface: A Practitioner's Guide, New York: Wiley*

[2] Mohney, D., Bassman, H.(2013). *The FVA: Forward Volatility Agreement*

[3] Derman, E., I Kani, and J. Z. Zou (1996). *The Local Volatility Surface: Unlocking the Information in Index Options Prices.* Financial Analysts Journal, (July-August, 1996), pp. 25-36

Figure 1: Convexity Adjustment with Maturity

Adjustment.PNG