GARCH(1,1) model for stressed markets

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September 2020

1 Introduction

During the recent Covid crisis the markets showed an anomalous up-down rapid move of the volatility measures. Figure 1 shows the plots of VIX and of the realized volatility measure from the Oxford Man realized volatility library (more precisely the *rk_th2* Tukey-Hanning kernel measure) over the time period Jan-Mar 20. This is especially clear in the realized variance measure which has large up-down moves, as seen in the right plot of Figure 1.

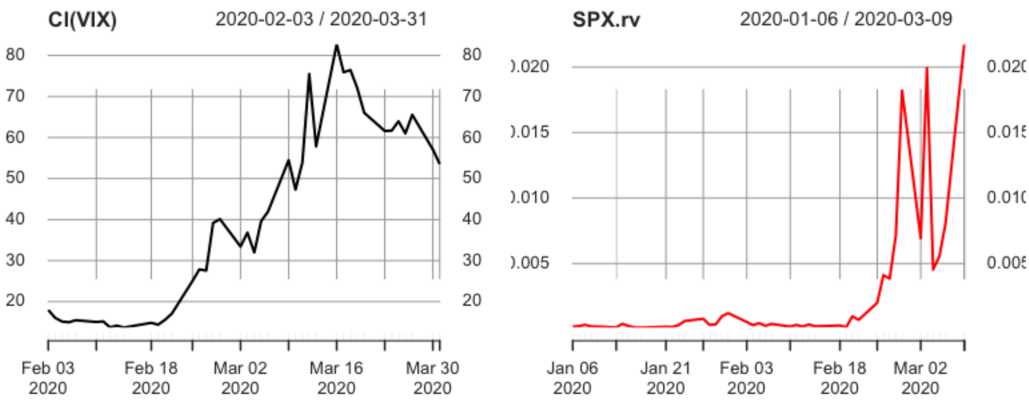

Figure 1: The daily closing VIX index (left) and the rk_th2 realized volatility measure of the SP500 index (right) over the first few months of 2020.

These rapid see-saw moves impact the autocorrelation structure of the realized volatility measures. This is observed in Figure **??** which shows the autocorrelation (ACF) of the daily realized variance on a data set covering the period Jan-19 to Mar-20. We note here an anomalously small autocorrelation at lag-1, superimposed on a regular structure at larger lags.

This effect is similar to the bid-ask bounce in market microstructure, and we propose here a modification of the GARCH(1,1) model which can accommodate such large moves. The model is somewhat similar to the Roll model, as it uses an iid binomial random variable to simulate this effect.

2 GARCH(1,1)

Recall the GARCH(1,1) model for the conditional variance v_t.

$$v_t = w + \alpha Z_{t-1}^2 v_{t-1} + \beta v_{t-1} \tag{1}$$

This model has the following well-known properties. Assume that the process has reached stationary state, by taking $t \to \infty$. This corresponds to the unconditional variance.

1) The mean of the unconditional variance is

$$V_0 = \mathbb{E}[v_t] = \frac{w}{1-\alpha-\beta} \tag{2}$$

and its second moment is

$$\gamma_0 = \mathbb{E}[v_t^2] = w^2 \frac{1+\alpha+\beta}{1-\alpha-\beta} \frac{1}{1-(k_4\alpha^2 + 2\alpha\beta + \beta^2)} \tag{3}$$

where $k_4 = \mathbb{E}[Z^4]$.

Denote the autocovariance as

$$\gamma_k := \mathbb{E}[v_t v_{t+k}] \tag{4}$$

This can be computed in closed form for any $k \geq 1$. For small values of k this can be evaluated by direct computation

$$\gamma_1 = \mathbb{E}[v_t v_{t+1}] = wV_0 + (\alpha+\beta)\gamma_0 \tag{5}$$

$$\gamma_2 = wV_0 + (\alpha+\beta)\gamma_1 \tag{6}$$

This is a general pattern and we note the recursion

$$\gamma_k = wV_0 + (\alpha+\beta)\gamma_{k-1} \tag{7}$$

which can be solved in closed form

$$\gamma_k = \frac{wV_0}{1-\alpha-\beta} + (\alpha+\beta)^k \left(\gamma_0 - \frac{wV_0}{1-\alpha-\beta}\right) . \tag{8}$$

In GARCH(1,1) autocovariances decrease as a power of lag k. Empirically the decrease is slower, like $\sim 1/k^n$ with n a small power.

3 Proposed model

We propose the following modification of the GARCH(1,1) model for the conditional variance $v_t = \sigma_t^2$, which allows for a rapid up-down change of the variance.

The model:

$$v_t = w + \alpha Z_{t-1}^2 v_{t-1} + \beta v_{t-1} + cY_t \tag{9}$$

where $Y_t = \{+1, -1\}$ is a Bernoulli random variable taking the values shown with equal probabilities 50%, and is independent of the innovations Z_t.

Denote γ_n is the covariance between σ_t^2 and σ_{t-n}^2

$$\begin{aligned}\gamma_0 &= Var[\sigma_t^2] \\ &= \mathbb{E}[\sigma_t^4] - (\mathbb{E}[\sigma_t^2])^2\end{aligned}$$

To get the second term $\mathbb{E}[\sigma_t^2]$:

$$\mathbb{E}[\sigma_t^2] = w + a\mathbb{E}[Z_{t-1}^2\sigma_{t-1}^2] + b\mathbb{E}[\sigma_{t-1}^2] + c\mathbb{E}[Y_t]$$

since

$$\begin{aligned}[Z_{t-1}^2\sigma_{t-1}^2] &= Cov(Z_{t-1}^2, \sigma_{t-1}^2) + \mathbb{E}[Z_{t-1}^2]\mathbb{E}[\sigma_{t-1}^2] \\ &= 0 + \mathbb{E}[\sigma_{t-1}^2]\end{aligned}$$

Thus we have:

$$\mathbb{E}[\sigma_t^2] = w + a\mathbb{E}[\sigma_{t-1}^2] + b\mathbb{E}[\sigma_{t-1}^2] + c * 0$$

Finaly we get

$$\mathbb{E}[\sigma_t^2] = \frac{w}{1-a-b} \tag{10}$$

To get the $\mathbb{E}[\sigma_t^4]$:

$$\begin{aligned}\mathbb{E}[\sigma_t^4] &= \mathbb{E}[(w + aZ_{t-1}^2\sigma_{t-1}^2 + b\sigma_{t-1}^2 + cY_t)^2] \\ &= \mathbb{E}[a^2\sigma_{t-1}^4 Z_{t-1}^4 + 2ab\sigma_{t-1}^4 Z_{t-1}^2 + 2ac\sigma_{t-1}^2 Y_t Z_{t-1}^2 + 2a\sigma_{t-1}^2 w Z_{t-1}^2 + \\ &\quad b^2\sigma_{t-1}^4 + 2bc\sigma_{t-1}^2 Y_t + 2b\sigma_{t-1}^2 w + c^2 Y_t^2 + 2cwY_t + w^2] \\ &= 3a^2\mathbb{E}[\sigma_{t-1}^4] + 2ab\mathbb{E}[\sigma_{t-1}^4] + 0 + \frac{2aw^2}{1-a-b} \\ &\quad b^2\mathbb{E}[\sigma_{t-1}^4] + 0 + \frac{2bw^2}{1-a-b} + c^2 + 0 + w^2\end{aligned}$$

Thus we have

$$\mathbb{E}[\sigma_t^4] = \frac{w^2(\frac{1+a+b}{1-a-b}) + c^2}{1 - 2a^2 - (a+b)^2} \tag{11}$$

In the steps above, we need:

- $\mathbb{E}[\sigma_{t-1}^4 Z_{t-1}^4] = Cov(\sigma_{t-1}^4, Z_{t-1}^4) + \mathbb{E}[\sigma_{t-1}^4]\mathbb{E}[Z_{t-1}^4] = 3\mathbb{E}[\sigma_{t-1}^4]$
- $\mathbb{E}[\sigma_{t-1}^4 Z_{t-1}^2] = Cov(\sigma_{t-1}^4, Z_{t-1}^2) + \mathbb{E}[\sigma_{t-1}^4]\mathbb{E}[Z_{t-1}^2] = \mathbb{E}[\sigma_{t-1}^4]$
- $\mathbb{E}[\sigma_{t-1}^2 Y_t Z_{t-1}^2] = Cov(Y_t\sigma_{t-1}^2, Z_{t-1}^2) + \mathbb{E}[Y_t\sigma_{t-1}^2]\mathbb{E}[Z_{t-1}^2] = 0$
- $\mathbb{E}[\sigma_{t-1}^2 Y_t] = Cov(\sigma_{t-1}^2, Y_t) + \mathbb{E}[\sigma_{t-1}^2]\mathbb{E}[Y_t] = 0$

- $\mathbb{E}[\sigma_{t-1}^2 Z_{t-1}^2] = Cov(\sigma_{t-1}^2, Z_{t-1}^2) + \mathbb{E}[\sigma_{t-1}^2]\mathbb{E}[Z_{t-1}^2] = \mathbb{E}[\sigma_{t-1}^2]$

  By using 10 and 11 we have:

$$\gamma_0 = \frac{w^2(\frac{1+a+b}{1-a-b}) + c^2}{1 - 2a^2 - (a+b)^2} - \frac{w^2}{(1-a-b)^2} \tag{12}$$

For lag-1 covariance we have:

$$\begin{aligned}
\gamma_1 &= Cov(\sigma_t^2, \sigma_{t-1}^2) = \mathbb{E}[\sigma_t^2 \sigma_{t-1}^2] - \mathbb{E}[\sigma_t^2]\mathbb{E}[\sigma_{t-1}^2] \\
&= \mathbb{E}[(w + aZ_{t-1}^2\sigma_{t-1}^2 + b\sigma_{t-1}^2 + cY_t)\sigma_{t-1}^2] - \frac{w^2}{(1-a-b)^2} \\
&= \mathbb{E}[w\sigma_{t-1}^2 + a\sigma_{t-1}^4 Z_{t-1}^2 + b\sigma_{t-1}^4 + c\sigma_{t-1}^2 Y_t] - \frac{w^2}{(1-a-b)^2} \\
&= \frac{w^2}{1-a-b} + (a+b)\frac{w^2(\frac{1+a+b}{1-a-b}) + c^2}{1 - 2a^2 - (a+b)^2} - \frac{w^2}{(1-a-b)^2}
\end{aligned} \tag{13}$$

The similiar recursion pattern:

$$\gamma_k = \frac{wV_0}{1-\alpha-\beta} + (\alpha+\beta)^k \left(\gamma_0 - \frac{wV_0}{1-\alpha-\beta}\right) + (\alpha+\beta)^k \left(\frac{c^2}{1 - (k_4\alpha^2 + 2\alpha\beta + \beta^2)}\right) \tag{14}$$

Compared to autocovariance closed form 8, there is an additional term

4 Parameters(w, a, b, c) Estimation

To be continue...(MLE) or ....