On Stability of the Newton Schulz Iteration in an Approximate Algebra

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I. INTRODUCTION

Decay principles, often very sparse but very ill-conditioned problems.

A. Incomplete Schemes

Incompleteness -¿ sparse approximations dense problems, uses conventional sparse infrastructure, second order errors in matrix multiplication. Often adhoc.

$\tilde{\boldsymbol{a}} = \boldsymbol{a} + \delta\boldsymbol{a}$

$\tilde{\boldsymbol{a}} \cdot \tilde{\boldsymbol{b}} = \boldsymbol{a}\cdot\boldsymbol{b} + \delta\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\delta\boldsymbol{b} + \delta\boldsymbol{a}\cdot\delta\boldsymbol{b}$

The variations do not express in the overal context of the product. Because the error in the incomplete case is additive, the

For example, $\boldsymbol{a}$ may be small, but $\delta\boldsymbol{a}\cdot\boldsymbol{b}$ large, leading extra work. Also, once a truncation error is commited, it is encounted in all subsiquent steps; it becomes difficult to impossilbe to manage error flows of differing magnitude in complex maps.

B. On "Variational" and "non-Variational" Iteration

Gradients lack convergence properties Iteration without orig drives away from basis NS has both. Difference between scalar iteration, Higham page 92.

C. Methods that accumulate incompleteness (purification, and Krylov subspace):

D. "Variational" Methods Stay Close to the Basis (LNV Gradients, RQI and Newton Shulz)

E. Approximation with Incompleteness and Iteration: Corruption of the Eigen Basis

F. Approximate Algebras as N-Body Problem

SpAMM is the recursive Cauchy-Schwarz occlusion product $\otimes_\tau$ on matrix quadtrees

$$\boldsymbol{a}^i = \begin{bmatrix} \boldsymbol{a}^{i+1}_{00} & \boldsymbol{a}^{i+1}_{01} \\ \boldsymbol{a}^{i+1}_{10} & \boldsymbol{a}^{i+1}_{11} \end{bmatrix} \tag{1}$$

$$\boldsymbol{a}^i \otimes_\tau \boldsymbol{b}^i = \begin{cases} \emptyset \quad \texttt{if} \quad \|\boldsymbol{a}^{\mathrm{i}}\|\|\boldsymbol{b}^{\mathrm{i}}\| < \tau \\ \boldsymbol{a}^i \cdot \boldsymbol{b}^i \quad \texttt{if}(\mathtt{i} = \mathtt{leaf}) \\ \begin{bmatrix} \boldsymbol{a}^{i+1}_{00} \otimes_\tau \boldsymbol{b}^{i+1}_{00} + \boldsymbol{a}^{i+1}_{01} \otimes_\tau \boldsymbol{b}^{i+1}_{10} \;, & \boldsymbol{a}^{i+1}_{00} \otimes_\tau \boldsymbol{b}^{i+1}_{01} + \boldsymbol{a}^{i+1}_{01} \otimes_\tau \boldsymbol{b}^{i+1}_{11} \\ \boldsymbol{a}^{i+1}_{00} \otimes_\tau \boldsymbol{b}^{i+1}_{01} + \boldsymbol{a}^{i+1}_{01} \otimes_\tau \boldsymbol{b}^{i+1}_{11} \;, & \boldsymbol{a}^{i+1}_{00} \otimes_\tau \boldsymbol{b}^{i+1}_{01} + \boldsymbol{a}^{i+1}_{01} \otimes_\tau \boldsymbol{b}^{i+1}_{11} \end{bmatrix} \quad \texttt{else} \end{cases} \tag{2}$$

database orientation, Cauchy sch Approximate Algebra, SpAMM Cauchy Schwarz occlusion, n-body approach to numerical linear algebra, first order errors in matrix multiplication. Based on Cauchy Schwarz inequality.

error accumulation may be better than row-col single programing model. generacity. communication optimality and strong scaling.

G. Incomplete Algorithms and Aproximate Algebras

Incompleteness -¿ sparse approximations dense problems, uses conventional sparse infrastructure, second order errors in matrix multiplication. Often adhoc.

Philosophy: Don't know which elements to drop, because only make sense in context. It is very difficult to

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incorporate this context in a dropping strategies.

Approximate Algebra, SpAMM Cauchy Schwarz occlusion, n-body approach to numerical linear algebra, first order errors in matrix multiplication. Based on Cauchy Schwarz inequality.

We show that this first order error in algebraic context follows the functional analysis, and enables the consideration *and seperate treatment* of first order error flows in the analysis of complex maps.

$$\boldsymbol{a} \otimes_\tau \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{\Delta}_\tau^{a \cdot b} \tag{3}$$

where $\boldsymbol{\Delta}_\tau^{a \cdot b}$ is a deterministic first order variation cooresponding to the branch pattern set by Cauchy-Schwarz culling, with length $\|\boldsymbol{\Delta}_\tau^{a \cdot b}\| \leq \tau \|\boldsymbol{a}\| \|\boldsymbol{b}\|$.

For a vector space S with $\boldsymbol{a}, \boldsymbol{b} \in S$, the opperator $\otimes_\tau$ leads to a non-associative algebra with Lie bracket

$$[\boldsymbol{a}, \boldsymbol{b}]_\tau = \boldsymbol{a} \otimes_\tau \boldsymbol{b} - \boldsymbol{b} \otimes_\tau \boldsymbol{a} = \boldsymbol{\Delta}_\tau^{a \cdot b} - \boldsymbol{\Delta}_\tau^{b \cdot a}, \tag{4}$$

determined by the occlusion field. Our challenge is to master the error flows of these occlusion fields under iteration, for ill-conditioned problems and with permisive values of τ.

II. NEWTON SHULZ ITERATION

A. Idempotence

B. The Scaled Map

C. Alternative Formulations

III. MAPPING ERROR FLOWS

$$\delta \boldsymbol{x}_k^{\text{naiv}} = \delta \widetilde{\boldsymbol{z}}_k \cdot \boldsymbol{s} \cdot \widetilde{\boldsymbol{z}}_k + \widetilde{\boldsymbol{z}}_k \cdot \boldsymbol{s} \cdot \delta \widetilde{\boldsymbol{z}}_k \tag{5}$$

$$\delta \boldsymbol{x}_k^{\text{dual}} = \delta \widetilde{\boldsymbol{y}}_k \cdot \widetilde{\boldsymbol{z}}_k + \widetilde{\boldsymbol{y}}_k \cdot \delta \widetilde{\boldsymbol{z}}_k \tag{6}$$

$$\begin{aligned} \widetilde{\boldsymbol{x}}_k &= f\left[\widetilde{\boldsymbol{z}}_{k-1}, \widetilde{\boldsymbol{x}}_{k-1}\right] \\ &= \mathtt{m}\left[\widetilde{\boldsymbol{x}}_{\mathtt{k-1}}\right] \cdot \widetilde{\boldsymbol{z}}_{\mathtt{k-1}}^\dagger \cdot \boldsymbol{s} \cdot \widetilde{\boldsymbol{z}}_{\mathtt{k-1}} \cdot \mathtt{m}\left[\widetilde{\boldsymbol{x}}_{\mathtt{k-1}}\right] \end{aligned} \tag{7}$$

$$\delta \boldsymbol{x}_k = f_{\delta \boldsymbol{z}_{k-1}} \|\delta \boldsymbol{z}_{k-1}\| + f_{\delta \boldsymbol{x}_{k-1}} \|\delta \boldsymbol{x}_{k-1}\| + \mathcal{O}\left(\tau^2\right) \tag{8}$$

generalized Gateaux differential

$$\begin{aligned} f_{\delta \boldsymbol{z}_{k-1}} &= \lim_{\tau \to 0} \frac{f[\boldsymbol{z}_{k-1} + \tau \delta \widehat{\boldsymbol{z}}_{k-1}, \widetilde{\boldsymbol{x}}_{k-1}] - f[\,\boldsymbol{z}_{k-1}, \widetilde{\boldsymbol{x}}_{k-1}]}{\tau} \\ &= L_{\widetilde{\boldsymbol{x}}_k}\left(\widetilde{\boldsymbol{z}}_k, \delta \widehat{\boldsymbol{z}}_{k-1}\right) \end{aligned} \tag{9}$$

$$\begin{aligned} f_{\delta \boldsymbol{x}_{k-1}} &= \lim_{\tau \to 0} \frac{f[\widetilde{\boldsymbol{z}}_{k-1}, \boldsymbol{x}_{k-1} + \tau \delta \widehat{\boldsymbol{x}}_{k-1}] - f[\widetilde{\boldsymbol{z}}_{k-1}, \boldsymbol{x}_{k-1}]}{\tau} \\ &= L_{\widetilde{\boldsymbol{x}}_k}\left(\widetilde{\boldsymbol{z}}_k, \delta \widehat{\boldsymbol{x}}_{k-1}\right) \end{aligned} \tag{10}$$

$$\begin{aligned} L_{\widetilde{\boldsymbol{x}}_k}\left(\widetilde{\boldsymbol{z}}_k, \delta \widehat{\boldsymbol{x}}_{k-1}\right) = {} & \delta \widehat{\boldsymbol{x}}_{k-1}^\dagger \cdot \mathtt{m}'\left[\boldsymbol{x}_{\mathtt{k-1}}\right] \cdot \{\widetilde{\boldsymbol{z}}_{\mathtt{k-1}}^\dagger \cdot \boldsymbol{s} \cdot \widetilde{\boldsymbol{z}}_{\mathtt{k}}\} \\ & + \{\widetilde{\boldsymbol{z}}_k^\dagger \cdot \boldsymbol{s} \cdot \widetilde{\boldsymbol{z}}_{k-1}\} \cdot \mathtt{m}'\left[\boldsymbol{x}_{\mathtt{k-1}}\right] \cdot \delta \widehat{\boldsymbol{x}}_{\mathtt{k-1}} \end{aligned} \tag{11}$$

$$\{\widetilde{\boldsymbol{z}}_k^\dagger \cdot \boldsymbol{s} \cdot \widetilde{\boldsymbol{z}}_{k-1}\} \to \boldsymbol{p}_+\left[\boldsymbol{s}\right] \tag{12}$$

$$\begin{aligned} L_{\widetilde{\boldsymbol{x}}_k}\left(\widetilde{\boldsymbol{z}}_k, \delta \widehat{\boldsymbol{z}}_{k-1}\right) = {} & \{\mathtt{m}\left[\boldsymbol{x}_{\mathtt{k-1}}\right] \cdot \delta \widehat{\boldsymbol{z}}_{\mathtt{k-1}}^\dagger \cdot \boldsymbol{s}\} \cdot \widetilde{\boldsymbol{z}}_{\mathtt{k}} \\ & + \widetilde{\boldsymbol{z}}_k^\dagger \cdot \{\boldsymbol{s} \cdot \delta \widehat{\boldsymbol{z}}_{k-1} \cdot \mathtt{m}\left[\boldsymbol{x}_{\mathtt{k-1}}\right]\} \end{aligned} \tag{13}$$

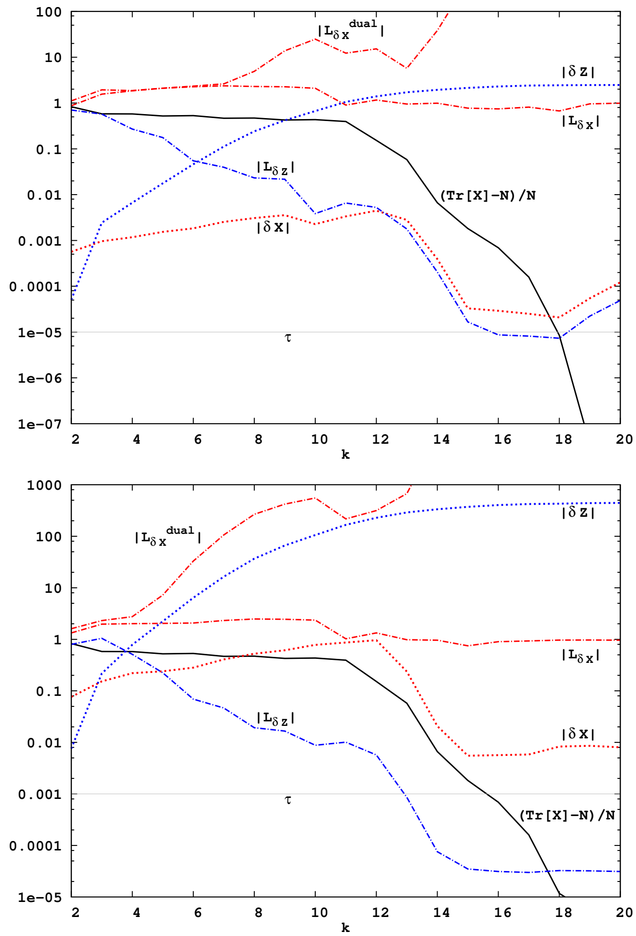
$$\{\boldsymbol{s} \cdot \delta \widehat{\boldsymbol{z}}_{k-1} \cdot \mathtt{m}\left[\boldsymbol{x}_{\mathtt{k-1}}\right]\} \to \boldsymbol{n}\left[\boldsymbol{s}\right] \tag{14}$$

$$\begin{aligned} \delta \boldsymbol{z}_{k-1} \approx {} & \Delta_\tau^{\widetilde{\boldsymbol{z}}_{k-2} \cdot \mathtt{m}[\widetilde{\boldsymbol{x}}_{\mathtt{k-2}}]} + \boldsymbol{z}_{k-2} \cdot \mathtt{m}'\left[\widetilde{\boldsymbol{x}}_{\mathtt{k-2}}\right] \cdot \delta \boldsymbol{x}_{\mathtt{k-2}} \\ & + \delta \boldsymbol{z}_{k-2} \cdot \mathtt{m}\left[\widetilde{\boldsymbol{x}}_{\mathtt{k-2}}\right] \end{aligned} \tag{15}$$

$$\begin{aligned} \|\delta \boldsymbol{z}_{k-1}\| \lesssim {} & \|\boldsymbol{z}_{k-2}\| \left(\ \tau \left\|\mathtt{m}\left[\widetilde{\boldsymbol{x}}_{\mathtt{k-2}}\right]\right\| \right. \\ & \left. + \|\delta \boldsymbol{x}_{k-2}\| \left\|\mathtt{m}'\left[\widetilde{\boldsymbol{x}}_{\mathtt{k-2}}\right]\right\|\ \right) \end{aligned} \tag{16}$$

$$\|\boldsymbol{z}_k\| \to \sqrt{\kappa\left(\boldsymbol{s}\right)} \tag{17}$$

FIG. 1: equation...



A. An Experiment

$$\boldsymbol{\epsilon}_{\perp}\left(\boldsymbol{x}\right) = \frac{\|[\boldsymbol{s},\boldsymbol{x}]\|}{\|\boldsymbol{s}\|\|\boldsymbol{x}\|} \tag{18}$$

$$\boldsymbol{z}_{k+1} \leftarrow \boldsymbol{z}_k \;\cdot\; m(\boldsymbol{x}_k) \tag{19}$$

$$\boldsymbol{x}_{k+1} \leftarrow \boldsymbol{z}_{k+1} \;\cdot\; m(\boldsymbol{x}) \;\cdot\; \boldsymbol{z}_{k+1} \tag{20}$$

$$\widetilde{\boldsymbol{z}}_{k+1} \leftarrow \boldsymbol{z}_k \otimes_\tau m(\boldsymbol{x}_k) \tag{21}$$

$$\widetilde{\boldsymbol{z}}_k = \boldsymbol{z}_k + \boldsymbol{\delta}^\tau_k \tag{22}$$

$$\|\boldsymbol{\delta}^\tau_k\| \sim \tau \|\boldsymbol{z}_k\| \|m\left(\boldsymbol{x}_k\right)\| \tag{23}$$

$$\widetilde{\boldsymbol{x}}_{k+1} \leftarrow \widetilde{\boldsymbol{z}}_{k+1} \otimes_\tau m(\boldsymbol{x}_k) \;\otimes_\tau \widetilde{\boldsymbol{z}}_{k+1} \tag{24}$$

$$\widetilde{\boldsymbol{x}}_k = \boldsymbol{x}_k + \boldsymbol{\epsilon}^\tau_k \tag{25}$$

$$\widetilde{\boldsymbol{x}}_{k+1} \leftarrow \left(\boldsymbol{z}_{k+1} + \boldsymbol{\delta}_k\right) \otimes_\tau m(\boldsymbol{x}_k) \;\otimes_\tau \left(\boldsymbol{z}_{k+1} + \boldsymbol{\delta}_k\right) \tag{26}$$

$$\widetilde{\boldsymbol{x}}^{\rm stab}_{k+1} \leftarrow \left(\boldsymbol{z}^\dagger_{k+1} + \boldsymbol{\delta}\dagger_k\right) \otimes_\tau m(\boldsymbol{x}_k) \;\otimes_\tau \left(\boldsymbol{z}_{k+1} + \boldsymbol{\delta}_k\right) \tag{27}$$

$$\widetilde{\boldsymbol{x}}_{\rm n} = \boldsymbol{z} \otimes_\tau [\boldsymbol{s} \otimes_\tau \boldsymbol{z}] \tag{28}$$
$$= \boldsymbol{z} \otimes_\tau [\boldsymbol{s}\cdot\boldsymbol{z} + \boldsymbol{\delta}^{s,z}_\tau] \tag{29}$$
$$= \boldsymbol{x}_{\rm n} + \boldsymbol{z}\cdot\boldsymbol{\delta}^{s,z}_\tau + \boldsymbol{\delta}^{z,\delta^{s,z}}_\tau \tag{30}$$

$$\widetilde{\boldsymbol{x}}^\dagger_{\rm n} = \boldsymbol{z}^\dagger \otimes_\tau \left[\boldsymbol{s} \otimes_\tau \boldsymbol{z}^\dagger\right] \tag{31}$$
$$= \boldsymbol{z}^\dagger \otimes_\tau \left[\boldsymbol{s}\cdot\boldsymbol{z}^\dagger + \boldsymbol{\delta}^{[sz^\dagger]}\right] \tag{32}$$
$$= \boldsymbol{x}^\dagger_{\rm n} + \boldsymbol{z}^\dagger\cdot\boldsymbol{\delta}^{s,z^\dagger}_\tau + \boldsymbol{\delta}^{z^\dagger,\delta^{s,z^\dagger}}_\tau \tag{33}$$

$$\boldsymbol{\epsilon}^{\rm n}_\perp(\boldsymbol{z},\boldsymbol{s},\tau) = \left[\tilde{\boldsymbol{x}}_{\rm n}, \tilde{\boldsymbol{x}}^\dagger_{\rm n}\right] \tag{34}$$
$$= \left[\boldsymbol{x} + \boldsymbol{z}\cdot\boldsymbol{\delta}^{s,z}_\tau + \mathcal{O}(\tau^2)\,, \right. \tag{35}$$
$$\left. \boldsymbol{x}^\dagger + \boldsymbol{z}^\dagger\cdot\boldsymbol{\delta}^{s,z^\dagger}_\tau + \mathcal{O}(\tau^2)\right]$$
$$= \left[\boldsymbol{x}_{\rm n}, \boldsymbol{x}^\dagger_{\rm n}\right] + \left[\boldsymbol{x}_{\rm n}, \boldsymbol{z}^\dagger\cdot\boldsymbol{\delta}^{s,z^\dagger}_\tau\right] \tag{36}$$
$$+ \left[\boldsymbol{z}\cdot\boldsymbol{\delta}^{s,z}_\tau, \boldsymbol{x}^\dagger_{\rm n}\right] + \mathcal{O}(\tau^2) \tag{37}$$

$$\widetilde{\boldsymbol{x}}_{\rm stab} = \boldsymbol{z}^\dagger \otimes_\tau [\boldsymbol{s} \otimes_\tau \boldsymbol{z}] \tag{38}$$
$$= \boldsymbol{z}^\dagger \otimes_\tau \left[\boldsymbol{s}\cdot\boldsymbol{z} + \boldsymbol{\delta}^{[sz]}\right] \tag{39}$$
$$= \boldsymbol{x} + \left(\boldsymbol{z}^\dagger\cdot\boldsymbol{\delta}^{[sz]}\right) + \boldsymbol{\delta}^{z^\dagger[sz]} \tag{40}$$

$$\widetilde{\boldsymbol{x}}_{\rm s} = \boldsymbol{z}^\dagger \otimes_\tau [\boldsymbol{s} \otimes_\tau \boldsymbol{z}] \tag{41}$$
$$= \boldsymbol{z}^\dagger \otimes_\tau \left[\boldsymbol{s}\cdot\boldsymbol{z} + \boldsymbol{\delta}^{[sz]}\right] \tag{42}$$
$$= \boldsymbol{x} + \left(\boldsymbol{z}^\dagger\cdot\boldsymbol{\delta}^{[sz]}\right) + \boldsymbol{\delta}^{z^\dagger[sz]} \tag{43}$$

a. potential for data oriented mathematics

$$\widetilde{\boldsymbol{x}}^\dagger_{\rm s} = \left\{\boldsymbol{z}^\dagger \otimes_\tau [\boldsymbol{s} \otimes_\tau \boldsymbol{z}]\right\}^\dagger \tag{44}$$
$$= \left[\boldsymbol{s}\cdot\boldsymbol{z} + \boldsymbol{\delta}^{[sz]}\right]^\dagger \otimes_\tau \boldsymbol{z} \tag{45}$$
$$= \left[\boldsymbol{z}^\dagger\cdot\boldsymbol{s} + \boldsymbol{\delta}^{[z^\dagger s]}\right] \otimes_\tau \boldsymbol{z} \tag{46}$$
$$= \boldsymbol{x}_{\rm s} + \boldsymbol{\delta}^{[z^\dagger s]}\cdot\boldsymbol{z} + \boldsymbol{\delta}^{[z^\dagger s]z} \tag{47}$$
$$= \boldsymbol{x}_{\rm s} + \left(\boldsymbol{z}^\dagger\cdot\boldsymbol{\delta}^{[sz]}\right)^\dagger + \boldsymbol{\delta}^{[z^\dagger s]z} \tag{48}$$

$$\boldsymbol{\epsilon}^{\rm s}_\perp(\boldsymbol{z},\boldsymbol{s},\tau) = \left[\boldsymbol{x}_{\rm s} + \boldsymbol{z}\cdot\boldsymbol{\delta}^{s,z}_\tau + \mathcal{O}(\tau^2)\,, \right. \tag{49}$$
$$\left. \boldsymbol{x}^\dagger_s + \boldsymbol{z}^\dagger \cdot^{s,z^\dagger}_\tau + \mathcal{O}(\tau^2)\right] \tag{50}$$

$$\boldsymbol{\epsilon}^{\rm s}_\perp(\boldsymbol{z},\boldsymbol{s},\tau) = \left[\tilde{\boldsymbol{x}}_{\rm s}, \tilde{\boldsymbol{x}}^\dagger_{\rm s}\right] \tag{51}$$
$$= \left[\boldsymbol{x} + \boldsymbol{z}^\dagger\cdot\boldsymbol{\delta}^{s,z}_\tau + \mathcal{O}(\tau^2)\,, \right. \tag{52}$$
$$\left. \boldsymbol{x}^\dagger + \left(\boldsymbol{z}^\dagger\cdot\boldsymbol{\delta}^{s,z}_\tau\right)^\dagger + \mathcal{O}(\tau^2)\right]$$
$$= \left[\boldsymbol{x}_{\rm s}, \boldsymbol{x}^\dagger_{\rm s}\right] + \left[\boldsymbol{x}_{\rm s}, \left(\boldsymbol{z}^\dagger\cdot\boldsymbol{\delta}^{s,z}_\tau\right)^\dagger\right] \tag{53}$$
$$+ \left[\boldsymbol{z}^\dagger\cdot\boldsymbol{\delta}^{s,z}_\tau, \boldsymbol{x}^\dagger_{\rm s}\right] + \mathcal{O}(\tau^2) \tag{54}$$

IV. STABILIZATION OF THE ACCELERATED MAP

V. CONCLUSION