FrontISTR Theory Manual

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October 5, 2020

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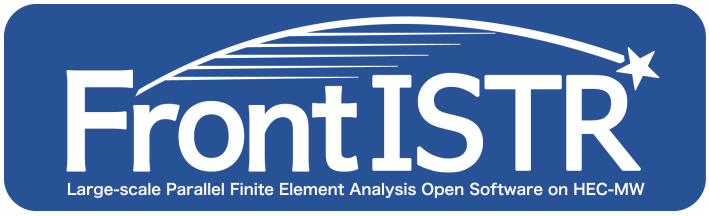

Item	Content
Name of Software	FrontISTR
Version	5.1
License	MIT License
Correnponding Clerks	FrontISTR Commons2-11-16 Yayoi, Bunkyo-ku, Tokyoc/o Institute of Engineering Innovation, School of EngineeringE-mail support@frontistr.com

1.1 Manuals

- Introduction
- How to install
- Theory
- User's manual
- Tutorial
- FAQ

This manual describes the analysis method by the finite element method (FEM) used in FrontISTR.

Regarding the stress analysis method of solids, the infinitesimal deformation linear elasticity static analyais method is described first, and geometric nonlinear analysis method and elastoplasticity analysis method which are required when handling finite deformation problems are described next. Furthermore, a summarized evaluation method of the facture mechanics parameters which can be acquired using the results of the stress analysis by FEM is described. Finally, th eigenvalue analysis and heat conduction analysis method is described.

1.2 List of description on this manual

- PDF
- Static Analysis
  - Infinitesimal Deformation Linear Elasticity Static Analysis
  - Nonlinear Static Analysis Method
- Dynamic Analysis
  - Dynamic Analysis Method
  - Eigenvalue Analysis
  - Frequency Response Analysis
- Heat Conduction Analysis
- References

1.3 Infinitesimal Deformation Linear Elastic Static Analysis

In this section, the elastic static analysis is formulated on the basis of the infinitesimal deformation theory, which assumes linear elasticity as a stress-strain relationship.

1.3.1 Basic equations

The equilibrium equation, mechanical boundary conditions, and geometric boundary conditions (basic boundary conditions) of solid mechanics are given by the following equations (see Fig. 2.1.1):

$$\nabla \cdot \sigma + \overline{b} = 0 \quad in\ V \tag{1}$$

$$\sigma \cdot n = \bar{t} \quad on\ S_t \tag{2}$$

$$u = \bar{u} \quad on\ S_u \tag{3}$$

where σ, $\bar{t}$ and S_t denote stress, surface force, and body force, respectively. S_t and S_u represent the geometric and mechanical boundaries, respectively.

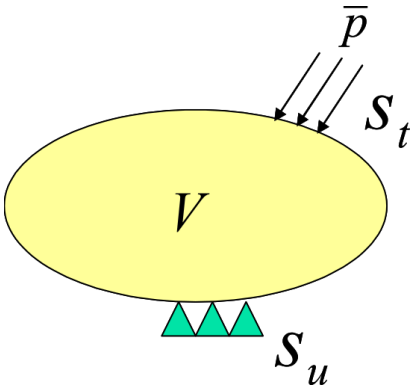

Figure 1: Boundary value problem in solid mechanics (infinitesimal deformation problem)

Fig. 2.1.1 Boundary value problem in solid mechanics (infinitesimal deformation problem)

The strain-displacement relation in infinitesimal deformation problems is given by the following equation:

$$\varepsilon = \nabla_S u \tag{4}$$

Furthermore, the stress-strain relationship (constitutive equation) in linear elastic bodies is given by the following equation:

$$\sigma = C : \varepsilon \tag{5}$$

where, C is a fourth-order elasticity tensor.

1.3.2 Principle of Virtual Work

The principle of the virtual work related to the infinitesimal deformation linear elasticity problem, which is equivalent to the basic equation Eq.(1), Eq.(2) and Eq.(3), is expressed as:

$$\int_V \sigma : \delta\varepsilon \, dV = \int_{S_t} \bar{t} \cdot \delta u \, dS + \int_V \bar{b} \cdot \delta u \, dV \tag{6}$$

$$\delta u = 0 \quad on\ S_u \tag{7}$$

Moreover, considering the constitutive equation Eq.(5), Eq.i(6), is expressed as follows:

$$\int_V (C : \varepsilon) : \delta\varepsilon \, dV = \int_{S_t} \bar{t} \cdot \delta u \, dS + \int_V \bar{b} \cdot \delta u \, dV \tag{8}$$

In Eq.(8), ε is the strain tensor and C is the forth-order enasticity tensor. In this case, if the strain tensor σ and ε are represented by vector formats $\hat{\sigma}$ and $\hat{\varepsilon}$, respectively, the consitutive equation Eq.(5) is expressed as follows

$$\hat{\sigma} = D\hat{\varepsilon} \tag{9}$$

where D is an elastic matrix.

Considering that the $\hat{\sigma}$, $\hat{\varepsilon}$ and Eq.(9) are expressed in vector format, Eq.(8) is expressed as follows:

$$\int_V \hat{\varepsilon}^T D \delta\hat{\varepsilon}\, dV = \int_{S_t} \delta u^T \overline{t}\, dS + \int_V \delta u^T \overline{b}\, dV \tag{10}$$

Eq.(10) and Eq.(7) are the principles of the virtual work discretized in this development code.

1.3.3 Formulation

If the principle of virtual work, Eq.(10), is discretized for each finite element, the following equation is obtained:

$$\sum_e = \int_{V^e} \hat{\varepsilon}^T D \delta\hat{\varepsilon}\, dV = \sum_e \int_{S_t^e} \delta u^T \overline{t}\, dS + \sum_e \int_{V^e} \delta u^T \overline{b}\, dV \tag{11}$$

Using the displacement of the nodes that compose each element, the displacement field is interpolated as follows:

$$u = \sum_{i=1}^{m} N_i u_i = NU \tag{12}$$

The strain at this moment, using Eq.(4), is given as follows:

$$\hat{\varepsilon} = BU \tag{13}$$

When Eq.(12) and Eq.(13) are substituted into Eq.(11), the following equation is obtained:

$$\sum_e \delta U^T \left(\int_{V^e} B^T D B\, dV \right) U = \sum_e \delta U^T \cdot \int_{S_t^e} N^T \overline{t}\, dS + \sum_e \delta U^T \int_{V^e} N^T \overline{b}\, dV \tag{14}$$

Eq.(14) can be summarized as

$$\delta U^T K U = \delta U^T F \tag{15}$$

In this case, the components of the matrix and vector defined by Eq.(16) and Eq.(17) can be calculated for each finite and overlapped element:

$$K = \sum_e \int_{V^e} B^T D B dV \tag{16}$$

$$F = \sum_e (\int_{S_t^e} N^T \overline{t}\, dS + \int_{V^e} N^T \overline{b}\, dV) \tag{17}$$

if Eq.(15) is true for an arbitary virtual displacement δU, tha following equation is obtained:

$$KU = F \tag{18}$$

Meanwhile, the displacement boundary conditioni Eq.(3) is expressed as follows:

$$U = \overline{U} \tag{19}$$

By solving Eq.(18) based on the constraint condition Eq.(19), it is possible to define the node displacement U.

1.4 Non-linear Static Analysis Method

As mentioned previously, in the analysis of infinitesimal deformation problems, it is possible to use the principle of virtual work, which is equivalent to the basic equations (the equilibrium equation), to perform finite element analysis by discretizing this equation with finite elements. Therefore, analyses of finite deformation problems that deal with large deformations of structures are typically conducted with the principle of virtual work.

However, in finite deformation problems, the equation of the principle of virtual work is non-linear in relation to displacement even if it the material is assumed to be linear.

Generally, non-linear equations are solved by iteration methods.

These type of iterative calculations are performed through incremental analysis, which is performed on small load increments that are accumulated until the final deformation state is reached. If an infinitesimal deformation problem is assumed, there will be no distinction in the arrangement to define the strain and stress before and after the deformation. Thus, the basic equation can be described before or after the deformation with no problem.

However, in the case of increment analysis in finite deformation problems, it is possible to choose between the initial state or the starting point of the increment. The former is called the total Lagrange method and the latter is called the updated Lagrange method. For more details, refer to the references at the end of this chapter.

In this development code, both the total and updated Lagrange methods were adopted.

1.4.1 Geometric non-linear analysis method

1.4.1.1 Incremental decomposition of virtual work equation

In this section, an increment analysis is performed wherein the status is known until time t and unknown until $t' = t + \Delta t$ (See to Fig. 2.2.1). The equilibrium equation, of the static boundary value problem, mechanical boundary conditions, and geometric boundary conditions (basic boundary condidions) are as follows:

$$\nabla_{t'_x} \cdot {}^{t'}\sigma + {}^{t'}\bar{b} = 0 \qquad in\ V \tag{20}$$

$${}^{t'}\sigma \cdot {}^{t'}n = {}^{t'}\bar{t} \qquad on\ {}^{t'}S_t \tag{21}$$

$${}^{t'}u = {}^{t'}\bar{u} \qquad on\ {}^{t'}S_u \tag{22}$$

${}^{t'}\sigma\ {}^{t'}\bar{b}\ {}^{t'}n\ {}^{t'}\bar{t}$ and ${}^{t'}\bar{u}$ are the Cauchy stress (true stress), body force, outward until normal vector on body surface, predetermined surfacxe force, and predetermined displacement at time t', respectively. These equations are described for the arrangements of ${}^{t'}v$, ${}^{t'}s_t$, ${}^{t'}s_u$ at time $'t$.

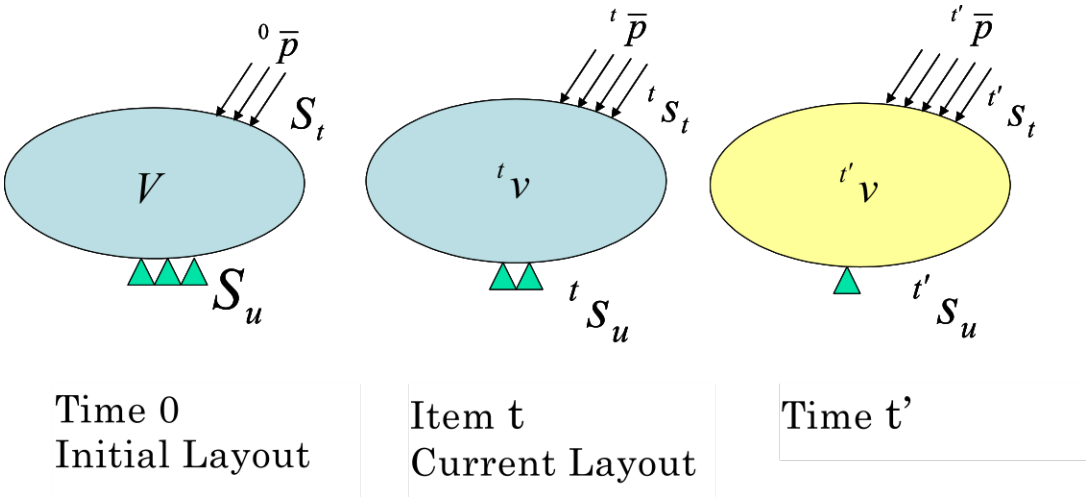

Fig. 2.2.1: Concept of incremental analysis

1.4.2 Principle of Virtual Work

The principle of virtual work equivalent to the equilibrium equation of Eq.(20) and mechanical boundary conditions of Eq.(21) is given by the following equation:

$$\int_{{}^{t'}v} {}^{t'}\sigma : \delta {}^{t'}A_{(L)}\, d^{t'}v = \int_{{}^{t'}s_t}^{t'} \bar{t} \cdot \delta u\, d^{t'}s + \int_V^{t'} \bar{b} \cdot \delta u\, d^{t'}v \tag{23}$$

where ${}^{t'}A_{(L)}$ is the linear part of the Almansi strain tensor, which is expressed by the following equation:

$${}^{t'}A_{(L)} = \frac{1}{2}\left\{ \frac{\partial^{t'}u}{\partial^{t'}x} + \left(\frac{\partial^{t'}u}{\partial^{t'}x}\right)^{\mathrm{T}} \right\} \tag{24}$$

Eq.(23) should be solved along with the geometric boundary conditions, strain displacement relation, and stress-strain relationship equation; however Eq.(23) is described with the arrangement at time t', which is still unknown at this stage. Therefor, a formulation with reference to arrangement V at time 0 or arrangement ${}^{t'}v$ at time t has to be performed.

1.4.2.1 Formulation of total Lagrange method

In this section, a formulation based on the total Lagrange method used in the development code is described.

The principle of virtual work equation at time t' with reference to the initial arrangement at time 0 is given by the following equation:

$$\int_V {}^{t'}_{0}S : \delta {}^{t'}_{0}E\, dV = {}^{t'}\, \delta R \tag{25}$$

$${}^{t'}\delta R = \int_{S_t} {}^{t'}_{0}\bar{t} \cdot \delta u\, dS + \int_V {}^{t'}_{0}\bar{b} \cdot \delta u\, dV \tag{26}$$

where ${}^{t'}_{0}S$ and ${}^{t'}_{0}E$ represent the second Piola–Kirchhoff strain tensor and Green–Lagrange strain tensor, respectively, at time t' with reference to the initial arrengement at time 0. Furthermore, ${}^{t'}_{0}\bar{t}$ and ${}^{t'}_{0}\bar{b}$ are the surface force vector and body force coverted per unit volume of the initial arrangement, respectively, and are expressed as follows when associated with Eq.(20), Eq.(21) and Eq.(22):

$${}^{t'}_{0}\bar{t} = \frac{d^{t'}s_{t'}}{dS}\bar{t} \tag{27}$$

$${}^{t'}_{0}\bar{b} = \frac{d^{t'}v_{t'}}{dV}\bar{b} \tag{28}$$

The Green-Langrange strain tensor at time t is defined by the following equation:

$${}^{t'}_{0}E = \frac{1}{2}\left\{ \frac{\partial^{t}u}{\partial X} + \left(\frac{\partial^{t}u}{\partial X}\right)^{T} + \left(\frac{\partial^{t}u}{\partial X}\right)^{T} \cdot \frac{\partial^{t}u}{\partial X} \right\} \tag{29}$$

The displacement at time t' and the second Piola-Kirchhoff stress ${}^{t'}u$, ${}^{t'}_{0}S$ can be represented with incremental decomposition as follows:

$${}^{t'}u = {}^{t}\, u + \Delta u \tag{30}$$

$${}^{t'}_{0}S = {}^{t}_{0}\, S + \Delta S \tag{31}$$

The increment of Green-Lagrrange strain, in connection with the displacement increment, is defined by the following equation:

$$ {}_0^{t'}E = {}_0^{t}E + \Delta E \tag{32} $$

$$ \Delta E = \Delta E_L + \Delta E_{NL} \tag{33} $$

$$ \Delta E_L = \frac{1}{2}\left\{ \frac{\partial \Delta u}{\partial X} + \left(\frac{\partial \Delta u}{\partial X}\right)^T + \left(\frac{\partial \Delta u}{\partial X}\right)^T \cdot \frac{\partial {}^t u}{\partial X} + \left(\frac{\partial {}^t u}{\partial X}\right)^T \cdot \frac{\partial \Delta u}{\partial X} \right\} \tag{34} $$

$$ \Delta E_{NL} = \frac{1}{2}\left(\frac{\partial \Delta u}{\partial X}\right)^T \cdot \frac{\partial \Delta u}{\partial X} \tag{35} $$

If Eq.(30), Eq.(31), Eq.(32), Eq.(33), Eq.(34) and Eq.(35) are substituted into Eq.(25) and Eq.(26), the following equation is obtained:

$$ \int_V \Delta S : (\delta \Delta E_L + \delta \Delta E_{NL}) dV + \int_V {}_0^t S : \delta \Delta E_{NL}\, dV = {}^{t'}\, \delta R - \int_V {}_0^t S : \delta \Delta E_L\, dV \tag{36} $$

In this case, it is assumed that ΔS is associated with ΔE_L and the forth-order tensor ${}_0^t C$, and is expressed as follows:

$$ \Delta S = {}_0^t\, C : \Delta_t E_L \tag{37} $$

By substituting Eq.(37) into Eq.(36), and omitting $\Delta S : \delta \Delta E_{NL}$ with Δu of second or higher order, the following equation is obtained:

$$ \int_V ({}_0^t C \Delta E_L) : \delta \Delta E_L\, dV + \int_V {}_0^t S : \delta \Delta E_{NL}\, dV = {}_0^{t'}\, \delta R - \int_V {}_0^t S : \delta \Delta E_L\, dV \tag{38} $$

Further, if Eq.(38) is discretized by the finite element, following equation is obtained:

$$ \delta U^T ({}_0^t K_L + {}_0^t K_{NL}) \Delta U = \delta U^T\, {}_0^{t'} F - \partial U^T\, {}_0^t Q \tag{39} $$

where ${}_0^t K$, ${}_0^t K_{NL}$, ${}_0^{t'} F$, ${}_0^t Q$ denote the initial displacement matrix, initial stress matrix, external force vector, and internal stress vector, respectively.

Therefore, the recurrence formula to determine the status from time t to time t' is given by the following equation:

$i = 0$

Step1 : ${}_0^{t'} K^{(0)} = {}_0^t\, K_L + {}_0^t\, K_{NL}$; ${}_0^{t'} Q^{(0)} = {}_0^t\, Q$; $U^{(0)} = {}^t\, U$

Step2 : ${}_0^{t'} K^{(i)} \Delta U^{(i)} = {}_0^{t'}\, F - {}_0^{t'}\, Q^{(i-1)}$

Step3 : ${}^{t'} U^{(i)} = {}^{t'}\, U^{(i-1)} + \Delta U^{(i)}$

$i = i + 1$

1.4.2.2 Formulation of the Updated Lagrange Method

The principle of the virtual work equation at time t' with reference to the arrangement at time t given by the following equation:

$$\int_V {}_t^{t'}S : \delta {}_t^{t'}E dV = {}^{t'}\delta R \tag{40}$$

$${}^{t'}\delta R = \int_{S_t} {}_t^{t'}\overline{t} \cdot \delta u \, dS + \int_V {}_t^{t'}\overline{b} \cdot \delta u \, dV \tag{41}$$

However,

$${}_t^{t'}\overline{t} = \frac{d^{t'}s_{t'}}{d^t s}\overline{t} \tag{42}$$

$${}_t^{t'}\overline{b} = \frac{d^{t'}v_{t'}}{d^t v}\overline{b} \tag{43}$$

While tensor ${}_t^{t'}S$ and ${}_t^{t'}E$ and vector ${}_t^{t'}\overline{t}$ and ${}_t^{t'}\overline{b}$ are are based on arrangement at time t, the Green-Lagrange strain does not include the initial displacement (displacement until time t ${}^t u$:

$${}_t^{t'}E = \Delta_t E_L + \Delta_t E_{NL} \tag{44}$$

Further, this becomes

$$\Delta_t E_L = \frac{1}{2}\left\{\frac{\partial \Delta u}{\partial^t x} + \left(\frac{\partial \Delta u}{\partial^t x}\right)^T\right\} \tag{45}$$

$$\Delta_t E_{NL} = \frac{1}{2}\left(\frac{\partial \Delta u}{\partial^t x}\right)^T \cdot \frac{\partial \Delta u}{\partial^t x} \tag{46}$$

However,

$${}_{t'}^{t}S = {}_t^t S + \Delta_t S \tag{47}$$

Thus, if this is substituted into Eq.(40), Eq.(41), and Eq.(44), the eqation to be solved is as follows:

$$\int_{t_v} \Delta_t S : (\delta \Delta_t E_L + \delta \Delta_t E_{NL}) d^t v + \int_{t_v} {}_t^{t'}S : \delta \Delta_t E_{NL} \, d^t v = {}^{t'}\ \delta R - \int_{t_v} {}_t^t S : \delta \Delta_t E_L \, d^t v \tag{48}$$

In this case, it is assumed that $\Delta_t S$ with $\Delta_t E_t$ and forth-order tensor ${}_t^t C$, and is expressed as follows:

$$\Delta_t S = {}_t^t C : \Delta_t E_L \tag{49}$$

If this is substituted Eq.(48), the following equation is obtained:

$$\int_V ({}_t^t C \Delta_t E_L) : \delta \Delta_t E_L \, dV + \int_V {}_t^t S : \delta \Delta_t E_{NL} \, dV = {}^{t'}\ \delta R - \int_V {}_t^t S : \delta \Delta_t E_L \, dV \tag{50}$$

By discretizing Eq.(50) with finite elements, as following equation is acquired:

$$\delta U^T ({}_t^t K_L + {}_t^t K_{NL}) \Delta U = \delta U^T {}_t^{t'} F - \partial U^T {}_t^t Q \tag{51}$$

where ${}^t_tK_L$, ${}^t_tK_{NL}$, ${}^{t'}_tF$ and ${}^t_tQ$ denote the initial displacement matrix, initial stress matrix, external force vector, and internal stress vector, respectively.

Therefore, the recurrence formula to determine the status from time t to t' is given by the following equation:

$i = 0$

Step1 : ${}^{t'}_tK^{(i)} = {}^t_tK_L + {}^t_tK_{NL}$; ${}^{t'}_tQ^{(i)} = {}^t_tQ$; $U^{(i)} = {}^tU$

Step2 : ${}^{t'}_tK^{(i)}\Delta U^{(i)} = {}^{t'}_tF - {}^{t'}_tQ^{(i-1)}$

Step3 : ${}^{t'}U^{(i)} = {}^{t'}U^{(i-1)} + \Delta U^{(i)}$

$i = i + 1$

1.4.3 Material Non-linear Analysis Method

With this development code, it is possible to analyze two types of non-linear materials; materials with isotropic hyperelasticity and elastoplasticity.

If the material to be analyzed is elastoplastic, the updated Lagrange method is applied. If it is hyperelastic, the total Lagrange method. Furthermore, the Newton–Raphson method is applied to the iterative analysis method.

These material constitutive equations are discussed ahead.

1.4.3.1 Hyperelastic Material

The elastic potential energy in isotropic hyperelastic materials is obtained from an isotropic response from an unstressed initial state. It can be represented as a function of the principal invariants of the Cauchy–Green deformation tensor $C(I_1, I_2, I_3)$ or the principal invariants of deformation tensor $(\overline{I_1}, \overline{I_2}, \overline{I_3})$ excluding volume change; that is, as $W = W(I_1, I_2, I_3)$ or $W = W(\overline{I_1}, \overline{I_2}, \overline{I_3})$.

The constitutive equation of a hyperelastic material is defined by the relationship between the second Piola–Kirchhoff stress and Green–Lagrange strain, and the total Lagrange method is applicable for its deformation analysis.

The elastic potential energy W of the hyperelastic models included in this development code is listed below. If the elastic potential energy W is known, the second Piola–Kirchhoff stress and the stress-strain relationship can be calculated as follows:

$$S = 2\frac{\partial W}{\partial C} \tag{52}$$

$$C = 4\frac{\partial^2 W}{\partial C \partial C} \tag{53}$$

1.4.3.1.1 (1) Neo-Hookean hyperelasticity model

The Neo-Hookean hyperelasticity model is an expansion of the isotropic linear law (Hooke's law); thus, it is compatible with large deformation problems. Its elastic potential is as follows:

$$W = C_{10}(\overline{I}_1 - 3) + \frac{1}{D_1}(J-1)^2 \tag{54}$$

where C_{10} and D_1 are the material constants.

1.4.3.1.2 (2) Mooney-Rivlin hyperelasticity model

$$W = C_{10}(\overline{I_1} - 3) + C_{01}(\overline{I_2} - 3) + \frac{1}{D_1}(J-1)^2 \tag{55}$$

where, C_{10}, C_{01} and D_1 are the material constants.

1.4.3.1.3 (3) Arruda Boyce hyperelasticity model

$$W = \mu \left[\frac{1}{2}(\overline{I_1} - 3) + \frac{1}{20\lambda_m^2}(\overline{I_1}^2 - 9) + \frac{11}{1050\lambda_m^2}(\overline{I_1}^3 - 27) + \frac{19}{7000\lambda_m^2}(\overline{I_1}^4 - 81) + \frac{519}{673750\lambda_m^2}(\overline{I_1}^5 - 243) \right] + \frac{1}{D}\left(\frac{J^2 - 1}{2} - \ln J \right) \tag{56}$$

$$\mu = \frac{\mu_0}{1 + \frac{3}{5\lambda_m^2} + \frac{99}{175\lambda_m^4} + \frac{513}{875\lambda_m^6} + \frac{42039}{67375\lambda_m^8}} \tag{57}$$

where μ, λ_m and D are the material constants.

1.4.3.2 Elastoplastic materials

In this development code, an elastoplastic constitutive equation that follows the associated flow rule is applied. Furthermore, its constitutive equation represents the relationship between the Jaumman speed of Kirchhoff stress and deformation speed tensor, and the updated Lagrange method is applicable for its deformation analysis.

1.4.3.2.1 (1) Elastoplastic constitutive Equation

The yield criteria of an elasto-plastic solid is assumed to be given as follows:

Initla yield conditions:

$$F(\sigma, \sigma_{y_0}) \tag{58}$$

Subsequent yield conditions:

$$F(\sigma, \sigma_y(\overline{e}^p)) \tag{59}$$

where

- F : Yield function
- σ_{y_0} : Initial yield stress
- σ_y : Consecutive yield stress
- σ : Stress tensor
- e : Infinitesimal strain tensor
- e^p : Plastic strain tensor
- $\overline{e}^p$: Equivalent plastic strain

It is assumed that the relationship between yield stress and equivalent plastic strain corresponds to that between stress in uniaxial state and plastic strain.

The relationship between stress in uniaxial state and plastic strain

$$\sigma = H(e^p) \tag{60}$$

$$\frac{d\sigma}{de^p} = H' \tag{61}$$

where H' is the modulus of strain hardening

The relationship between equivalent stress and equivalent plastic strain

$$\overline{\sigma} = H(\overline{e}^p) \tag{62}$$

$$\dot{\bar{\sigma}} = H' \dot{\bar{e}}^p \tag{63}$$

The subsequent yield function is normally a function of temperature and plastic strain work; however, to simplify, it is a function of only equivalent plastic strain e^{-p} in this case. Moreover, $F = 0$ continues to be satisfied during the plastic deformation; thus, the following equation must hold:

$$\dot{F} = \frac{\partial F}{\partial \sigma} : \dot{\sigma} + \frac{\partial F}{\partial e^p} : \dot{e}^p = 0 \tag{64}$$

where $\dot{F}$ represents the time derivative of F, and the time derivative of a certain amount A is represented by $\dot{A}$.

In this case, assuming the existence of plastic potential Θ, the plastic strain speed is speed represented by the following equation:

$$\dot{e}^p = \dot{\lambda} \frac{\partial \Theta}{\partial \sigma} \tag{65}$$

where $\dot{\lambda}$ is a coefficient.

Moreover, considering that the plastic potential Θ is equivalent to the yield function F, the associated flow rule of the following equation is assumed:

$$\dot{e}^p = \dot{\lambda} \frac{\partial F}{\partial \sigma} \tag{66}$$

If this is substituted into Eq.(64), the following equation is obtained:

$$\dot{\lambda} = \frac{a^T : d_D}{A + a^T : D : a} \dot{e} \tag{67}$$

Where D is an elasticity matrix,

$$a^T = \frac{\partial F}{\partial \sigma} \qquad d_D = D a^T \qquad A = -\frac{a}{\dot{\lambda}} \frac{\partial F}{\partial e^p} : \dot{e}^p \tag{68}$$

The stress-strain relationship equation of elastoplasticity can be expressed as follows

$$\sigma = \left\{ D - \frac{d_D \otimes d_D^T}{A + d_D^T a} \right\} : \dot{e} \tag{69}$$

When the yield function Eq.(69) of an elastoplastic material is known, the constitutive equation can be acquired from this equation.

1.4.3.2.2 (1) Yield Function

The elastoplastic yield functions included in this development code are as follows:

- Von Mises yield function:

$$F = \sqrt{3 J_2} - \sigma_y = 0 \tag{70}$$

- Mohr-Coulomb yield function:

$$F = \sigma_1 - \sigma_3 + (\ \sigma_1 + \sigma_3\ ) \sin\phi - 2\ c \cos\phi = 0 \tag{71}$$

- Drucker-Prager yield function:

$$F = \sqrt{J_2} - \ \alpha \ \sigma \ : I - \sigma_y = 0 \tag{72}$$

where the material constants α and σ_y are calculated from viscosity and friction angle of the material.

$$\alpha = \frac{2\sin\phi}{3+\sin\phi} \qquad\qquad \sigma_y = \frac{6\ c\cos\phi}{3+\sin\phi} \tag{73}$$

1.4.3.3 Viscoelastic material

In this development code, the generalized Maxwell model is applied for viscoelastic materials. The constitutive equation is a function of deviatoric strain e and deviatoric viscous strain q.

$$\sigma\ (t) = Ktr\varepsilon I + 2G(\mu_0 e + \mu q) \tag{74}$$

where

$$\mu q = \sum_{m=1}^{M} \mu_m q^{(m)} \qquad\qquad \sum_{m=0}^{M} \mu_m = 1 \tag{75}$$

Furthermore, q can be determined from

$$\dot{q}^{(m)} + \frac{1}{\lambda_m} q^{(m)} = \dot{e} \tag{76}$$

where λ_m is relaxation, and the relaxation coefficient G is represented by the following Prony series:

$$G(t) = G\left[\mu_0 + \sum_{i=1}^{M} \mu_m \exp\left(\frac{-t}{\lambda_m}\right)\right] \tag{77}$$

1.4.3.4 Creep material

A displacement under constant stress with time dependence is a phenomenon called "creep".

The previously mentioned viscoelastic behavior can also be considered as a type of linear creep phenomenon. In this section, a few types of non-linear creep are explained. For this phenomenon, a method that creates a constitutive equation by adding it to an instantaneous strain is typically used, and the strain when a constant load is applied is defined as creep strain ε^c. The most commonly used constitutive equation that considers creep is creep strain speed $\dot{\varepsilon^c}$, which is defined as a function of stress and total creep strain:

$$\dot{\varepsilon}^c \equiv \frac{\partial \varepsilon^c}{\partial t} = \beta(\ \sigma,\ \varepsilon^c\ ) \tag{78}$$

In this case, if the instantaneous strain is assumed as the elasticity strain ε^e, the total strain is expressed as an addition of creep strain to it.

$$\varepsilon = \varepsilon^e + \varepsilon^c \tag{79}$$

where

$$\varepsilon^e = c^{e-1}\ : \sigma \tag{80}$$

As previously mentioned in plastic materials, it is necessary to show the method of time integration on numerical analysis for the constitutive equation that indicates creep. The constitutive equation when creep is considered is

$$\sigma_{n+1} = c \ : \ (\varepsilon_{n+1} - \varepsilon^c_{n+1}) \tag{81}$$

$$\varepsilon^c_{n+1} = \varepsilon^c_n + \ \Delta t \ \beta_{n+\theta} \tag{82}$$

where $\beta_{n+\theta}$ is

$$\beta_{n+\theta} = (1-\theta)\beta_n + \theta\beta_{n+1} \tag{83}$$

Furthermore, the creep strain increment $\Delta\varepsilon^c$ is a simplified non-linear equation,

$$R_{n+1} = \varepsilon_{n+1} - \ c^{-1} \ : \sigma_{n+1} - \ \varepsilon^c_n - \ \Delta t \ \beta_{n+\theta} = \mathbf{0} \tag{84}$$

In the iterative calculation of the Newton-Raphson method, using the initial value as a strain increment determined from $\sigma_{n+1} = \sigma_n$ and the finite element method, the iterative and incremental solution are as follows:

$$R^{(k+1)}_{n+1} = \mathbf{0} = \ R^{(k)}_{n+1} - (\ c^{-1} + \Delta t \ c^c_{n+1} \ ) d\sigma^{(k)}_{n+1} \tag{85}$$

where

$$c^c_{n+1} = \left.\frac{\partial\beta}{\partial\sigma}\right|_{n+\theta} = \theta\left.\frac{\partial\beta}{\partial\sigma}\right|_{n+1} \tag{86}$$

When the iterative solution is performed using the solutions of Eq.(84) and Eq.(85) until the residual R becomes 0, stress σ_{n+1} and the tangent coefficient are used as follows:

$$c^*_{n+1} = [c^{-1} + \Delta t c^c_{n+1}]^{-1} \tag{87}$$

As a specific equation of Eq.(77), this development code applies the Norton model below. Its constitutive equation is represented as follows, where the equivalent creep strain $\dot{\varepsilon}^{cr}$ is a function of Mises stress q and time t:

$$\dot{\varepsilon}^{cr} = Aq^n t^m \tag{88}$$

where A, m and n are the material constants.

1.4.4 Contact Analysis Method

When two objects contact each other, the contact force t_c is conducted through the contact surface. The principle of virtual work Eq.(23) can be expressed as follows:

$$\int_{t'v}^{t'} {}^{t'}\sigma : \delta^{t'} A_{(L)} d^{t'} v = \int_{t'S_t}^{t'} {}^{t'}\overline{t} \cdot \delta u d^{t'} s + \int_V^{t'} \overline{b} \cdot \delta u d^{t'} v + \int_{t'S_c}^{t'} t_c[\delta u^{(1)} - u^{(2)}] \tag{89}$$

where S_c is contact area, and $u^{(1)}$ and $u^{(2)}$ represent the displacement of contact objects 1 and 2, respectively.

In the contact analysis, the surfaces with possible contact are designated as pairs; one of the surfaces is the master surface, and the other is the slave surface. In this master-slave analysis method, the following contact constraint conditions are assumed:

1. The slave nodes do not perforate the master surface.
2. When the contact occurs, the slave node is defined as the contact position through which the master and slave surface transfer the contact and frictional forces to each other.

If the last term of Eq.(74) is discretized by finite elements, the following equation is obtained:

$$\int_{t'}^{t'} {}_{S_c} t_c[\delta u^{(1)} - \delta u^{(2)}] \approx \delta U K_c \Delta U + \delta U F_c \tag{90}$$

where K_c and F_c represent the contact stiffness matrix and contact force, respectively. By substituting this equation into Eq.(39) or Eq.(51), the finite element equation of the total Lagrange method (which considers the contact constraint) and the updated Lagrange method become:

$$\delta U^T({}_0^t K_L + {}_0^t K_{NL} + K_c)\Delta U = \delta U {}_0^{Tt'} F - \partial U {}_0^{Tt} Q + \delta U F_c \tag{91}$$

$$\delta U^T({}_t^t K_L + {}_t^t K_{NL} + K_c)\Delta U = \delta U {}_t^{Tt'} F - \partial U {}_t^{Tt} Q + \delta U F_c \tag{92}$$

With this development software, it is possible to analyze the contact deformation between two deformable bodies, and the user can choose from the following analysis functions:

- Infinitesimal sliding contact problem: This analysis assumes that the position of the contact point does not change.
- Finite sliding contact problem: This analysis supports cases where the contact position changes because of deformation.
- Contact problem without friction
- Contact problem with friction: This analysis supports the Coulomb friction law.

However, when the infinitesimal deformation linear elastic analysis is chosen, it becomes a problem without infinitesimal sliding friction.

Furthermore, at this point, it only supports contact analysis of primary solid elements (element numbers 341, 351, and 361).

1.5 Dynamic Analysis Method

In this section, the dynamic problem analysis method with a direct time integration method applied is described. As presented below, with this development code, it is possible to perform time history response analysis by an implicit or explicit method.

1.5.1 Formulation of the implicit method

Focusing on dynamic problems, the direct time integration method was applied to solve the following equation motion indicated:

$$M(t+\Delta t)\ddot{U}(t+\Delta t) + C(t+\Delta t)\dot{U}(t+\Delta t) + Q(t+\Delta t) = F(t+\Delta t) \tag{93}$$

where M and the mass matrix, C is the damping matrix, and Q is the internal stress vector, and F is the external force vector. This software does not consider the changes in mass; thus, the mass matrix is non-linear and constant regardless of deformation.

The displacement within time increment Δt, and the change in speed and acceleration are approximated with the Newmark-β method, as expressed in Eq.(94) and Eq.(95):

$$\dot{U}(t+\Delta t) = \frac{\gamma}{\beta \Delta t}\Delta U(t+\Delta t) - \frac{\gamma-\beta}{\beta}\dot{U}(t) - \Delta t \frac{\gamma-2\beta}{2\beta}\ddot{U}(t) \tag{94}$$

$$\ddot{U}(t+\Delta t) = \frac{1}{\beta \Delta t^2}\Delta U(t+\Delta t) - \frac{1}{\beta \Delta t}\dot{U}(t) - \frac{1-2\beta}{2\beta}\ddot{U}(t) \tag{95}$$

where γ and β are the parameters of the Newmark-β method.

if γ and β have the following values, it coincides with the linear acceleration method or the trapezoidal rule.

As it is already known, when γ and β are substituted into the following values, it will match the linear acceleration method, or trapezoid rule.

- $\gamma = \frac{1}{2}$, $\beta = \frac{1}{6}$ (Linear acceleration method)
- $\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$ (Trapezoid rule)

If Eq.(94) and Eq.(95) are substituted into Eq.(93), the following equation is obtaind:

$$\begin{aligned}\left(\frac{1}{\beta\Delta t^2}\mathbf{M} + \frac{\gamma}{\beta\Delta t}C + K\right)\Delta U(t+\Delta t) = F(t+\Delta t) - Q(t+\Delta t) \\ + \frac{1}{\beta\Delta t}M\dot{U}(t) + \frac{1-2\beta}{2\beta}M\ddot{U}(t) \\ + \frac{\gamma-\beta}{\beta}C\dot{U}(t) + \Delta t\frac{\gamma-2\beta}{2\beta}C\ddot{U}(t)\end{aligned} \tag{96}$$

K_L is linear stiffness matrix for linear problem; thus, $Q(t+\Delta t) = K_L U(t+\Delta t)$. If this equation is substituted into the equation above, the following equation is obtained:

$$\left\{M\left(-\frac{1}{(\Delta t)^2\beta}U(t) - \frac{1}{(\Delta t)\beta}\dot{U}(t) - \frac{1-2\beta}{2\beta}\ddot{U}(t)\right) + C\left(-\frac{\gamma}{(\Delta t)\beta}U(t) + \left(1-\frac{\gamma}{\beta}\right)\dot{U}(t) + \Delta t\frac{2\beta-\gamma}{2\beta}\ddot{U}(t)\right)\right\} + \tag{97}$$

In the portion, where the acceleration is specified as a geometric boundary condition, the displacement of the following equation is obtained from Eq.(94).

$$u_{is}(t+\Delta t) = u_{is}(t) + \Delta t\,\dot{u}_{is}(t) + (\Delta t)^2\left(\frac{1}{2}-\beta\right)\ddot{u}_{is}(t) + (\Delta t)^2\beta\ddot{u}_{is}(t+\Delta t) \tag{98}$$

Similarly, if the speed is specified, the displacement of the following equation is obtained from Eq.(98):

$$u_{is}(t+\Delta t) = u_{is}(t) + \Delta t\frac{\gamma-\beta}{\gamma}\dot{u}_{is} + (\Delta t)^2\frac{\gamma-2\beta}{2\gamma}\ddot{u}_{is} + \Delta t\frac{\beta}{\gamma}\dot{u}_{is}(t+\Delta t) \tag{99}$$

Where

- $u_{is}(t+\Delta t)$ is the nodal displacement at $t+\Delta t$,
- $\dot{u}_{is}(t+\Delta t)$ is the nodal speed at time $t+\Delta t$
- $\ddot{u}_{is}(t+\Delta t)$ is the nodal acceleration at time $t+\Delta t$
- i is the nodal degree-of-freedom (DOF)
- s is the Node number.

Furthermore, the mass and damping terms were handled as follows:

1.5.1.1 (1) Handling of mass term

The mass matrices are handled as concentrated mass matrices.

1.5.1.2 (2) Handling of damping terms

The damping terms are handled as Rayleigh damping expressed in Eq.(100).

$$C = R_m M + R_k K_L \tag{100}$$

where R_m and R_k are parameters of Rayleigh damping.

1.5.2 Formulation of Explicit Method

The explicit method is based on the motion equation at time t expressed in the following equation:

$$M\ddot{U}(t) + C(t)\dot{U}(t) + Q(t) = F(t) \tag{101}$$

where the displacement at time $t + \Delta t$ and that at time $t - \Delta t$ are expressed by the Taylor expansion at time t. If it is expanded up to the secondary term with Δt, it becomes

$$U(t + \Delta t) = U(t) + \dot{U}(t)(\Delta t) + \frac{1}{2!}\ddot{U}(t)(\Delta t)^2 \tag{102}$$

$$U(t - \Delta t) = U(t) - \dot{U}(t)(\Delta t) + \frac{1}{2!}\ddot{U}(t)(\Delta t)^2 \tag{103}$$

From the difference and sum of Eq.(95) and Eq.(96), the following equation is obtained:

$$\dot{U}(t) = \frac{1}{2\Delta t}(U(t + \Delta t) - U(t - \Delta t)) \tag{104}$$

$$\ddot{U} = \frac{1}{(2\Delta t)^2}(U(t + \Delta t) - 2U(t) + U(t - \Delta t)) \tag{105}$$

If Eq.(104) and Eq.(105) are substituted into Eq.(101), the following equation is obtained:

$$\begin{aligned}\left(\frac{1}{\Delta t^2}M + \frac{1}{2\Delta t}C\right)U(t + \Delta t) &= F(t) - Q(t) \\ &- \frac{1}{\Delta t^2}M[2U(t) - U(t - \Delta t)] - \frac{1}{2\Delta t}CU(t - \Delta t)\end{aligned} \tag{106}$$

For linear problems, specifically, $Q(t) = K_L U(t)$, the above equation becomes

$$\begin{aligned}\left(\frac{1}{\Delta t^2}M + \frac{1}{2\Delta t}C\right)U(t + \Delta t) &= F(t) - K_L U(t) \\ &- \frac{1}{\Delta t^2}M[2U(t) - U(t - \Delta t)] - \frac{1}{2\Delta t}CU(t - \Delta t)\end{aligned} \tag{107}$$

In this case, if mass matrix M is set as a concentrated mass matrix, and the damping matrix as the proportional damping matrix $C = R_m M$, Eq.(107) eliminates the requirement of solving operations for simultaneous equations.

Therefore, from Eq.(107), $U(t + \Delta t)$ can be determined by the following equation:

$$U(t + \Delta t) = \frac{1}{\left(\frac{1}{\Delta t^2}M + \frac{1}{2\Delta t}C\right)}\left\{F(t) - Q(t) - \frac{1}{\Delta t^2}M[2U(t) - U(t - \Delta t)] - \frac{1}{2\Delta t}C(t - \Delta t)U\right\} \tag{108}$$

1.6 Heat Conduction Analysis

In this section, the method of heat conduction analysis for solid bodies with the finite element methods used in this development code is described.

1.6.1 Basic Equation

The heat conduction equation in a continuous body is as follows:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(kxx \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(kyy \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(kzz \frac{\partial T}{\partial z}\right) + Q \tag{109}$$

where,

$\rho = \rho(x)$	mass (density)
$c = c(x, T)$	specific heat
$T = T(x, t)$	temperature
$K = k(x, T)$	thermal conductivity
$Q = Q(x, T, t)$	calorific value

x represents the position, T represents the temperature and t represents the time.

The area being considered is defined as S and its surroundings as Γ. Assuming that Dirichet or Neumann-type boundary conditions are given throughout Γ, the boundary conditions become as follows:

$$T = T_1(x, 5) \qquad X \in \Gamma_1 \tag{110}$$

$$k\frac{\partial T}{\partial n} = q(x, T, t) \qquad X \in \Gamma_2 \tag{111}$$

The function form of T_1 and q is known. q is the outflow heat flux from the boundaries.

With this program, it is possible to consider three types of heat flux.

$$q = -q_s + q_c + q_r \tag{112}$$

$$q_s = q_s(x, t) \tag{113}$$

$$q_c = hc(T - T_c) \tag{114}$$

$$q_r = hr(T^4 - {T_r}^4) \tag{115}$$

where q_s is the distributed heat flux, q_c is the heat flux by the convective heat transfer, and q_r is the heat flux by the radiant heat transfer.

Furthermore,

$Tc = Tc(x, t)$	Convective heat transfer coefficient atomospheric temperature
$hc = hc(x, t)$	Convective heat transfer coeffcient
$Tr = Tr(x, t)$	Radiation heat transfer coefficient atmospheric temperature
$hr = \varepsilon\sigma F = hr(x, t)$	Radiation heat transfer coefficient

ε: radiation rate, σ: Stefan-Boltzmann constant, F: shape factor

1.6.2 Discretization

If Eq.(109) is discretized with the Galerkin method,

$$[K]\{T\} + [M]\frac{\partial T}{\partial t} = \{F\} \tag{116}$$

However,

$$K = \int \left(k_x \frac{\partial N^T}{\partial x}\frac{\partial N}{\partial x} + k_y \frac{\partial N^T}{\partial y}\frac{\partial N}{\partial y} + k_z \frac{\partial N^T}{\partial z}\frac{\partial N}{\partial z} \right) dV + \int hcN^T N ds + \int hrN^T N ds \tag{117}$$

$$M = \int \rho c N^T N dV \tag{118}$$

$$F = \int Q N^T dV - \int q_s N^T dS + \int hcTcN^T dS + \int hcTr(T + Tr)(T^2 + Tr^2)N^T dS \tag{119}$$

$$\{N\} = (N^1, N^2, ..., Ni) \tag{120}$$

Eq.(116) is a formula of non-linear and non-steady-state. The objective now is to discretize it in time by the backward Euler method and calculate the temperature at time $t = t_0$ (when the temperature at $t = t_0 + \Delta t$ is known) with the following equation:

$$[K]_{t=t_0+\Delta t}\{T\}_{t=t_0+\Delta t} + [M]_{t=t_0+\Delta t}\frac{\{T\}_{t=t_0+\Delta t} - \{T\}_{t=t_0}}{\Delta t} = \{F\}_{t=t_0+\Delta t} \tag{121}$$

The next step is to improve the temperature vector $\{T\}^{(i)}_{t=t_0+\Delta t}$ which approximately satisfies Eq.(121) to determine the solution $\{T\}^{(i)+1}_{t=t_0+\Delta t}$ with a good precision.

Therefore, the temperature vector must be expressed as follows:

$$\{T\}_{t=t_0+\Delta t} = \{T\}^{(i)}_{t=t_0+\Delta t} + \Delta\{T\}^{(i)}_{t=t_0+\Delta t} \tag{122}$$

The product of the heat transfer matrix and temperature vector, as well as the mass matrix, are expressed approximately by the following equations:

$$\begin{aligned}[K]_{t=t_{0+\Delta t}}\{T\}_{t=t_0+\Delta t} \cong\ & [K]^{(i)}_{t=t_0+\Delta t}\{T\}^{(i)}_{t=t_0+\Delta t} \\ & + \frac{\partial[K]^{(i)}_{t=t_0+\Delta t}\{T\}^{(i)}_{t=t_0+\Delta t}}{\partial\{T\}^{(i)}_{t=t_0+\Delta t}}\{\Delta T\}^{(i)}_{t=t_0+\Delta t}\end{aligned} \tag{123}$$

$$[M]_{t=t_0+\Delta t} \cong [M]^{(i)}_{t=t_0+\Delta t} + \frac{\partial[M]^{(i)}_{t=t_0+\Delta t}}{\partial\{T\}^{(i)}_{t=t_0+\Delta t}}\Delta\{T\}^{(i)}_{t=t_0+\Delta t} \tag{124}$$

By substituting Eq.(122), Eq.(123) and Eq.(124) into Eq.(121), and omitting the terms of second or higher order, the following equation is obtained:

$$\left(\frac{[M]^{(i)}_{t=t_0+\Delta t}}{\Delta t} + \frac{\partial[M]^{(i)}_{t=t_0+\Delta t}\{T\}^{(i)}_{t=t_0+\Delta t}}{\partial\{T\}^{(i)}_{t=t_0+\Delta t}}\frac{\{T\}^{(i)}_{t=t_0+\Delta t} - \{T\}_{t=t_0}}{\Delta t} + \frac{\partial[K]^{(i)}_{t=t_0+\Delta t}\{T\}^{(i)}_{t=t_0+\Delta t}}{\partial\{T\}^{(i)}_{t=t_0+\Delta t}}\right)\{\Delta T\}^{(i)}_{t=t_0+\Delta t} = \{F\}_{t=t_0+\Delta t} - [M]^{(i)}_{t=t} \tag{125}$$

Moreover, the coefficient matrix of the left side is approximately evaluated with the following equation:

$$\begin{aligned}[K^*]^{(i)} &= \frac{[M]^{(i)}_{t=t_0+\Delta t}}{\Delta t} + \frac{\partial[K]^{(i)}_{t=t_0+\Delta t}}{\partial\{T\}^{(i)}_{t=t_0+\Delta t}}\{T\}^{(i)}_{t=t_0+\Delta t} \\ &= \frac{[M]^{(i)}_{t=t_0+\Delta t}}{\Delta t} + [K_T]^{(i)}_{t=t_0+\Delta t}\end{aligned} \tag{126}$$

where $[K_T]^{(i)}_{t=t_0+\Delta t}$ is a tangent stiffness matrix.

Finally, it is possible to calculate the temperature at time $t = t_0 + \Delta t$ through iterative calculation using the following equation:

$$
\begin{aligned}
[K^*]^{(i)}\{T\}^{(i)}_{t=t_0+\Delta t} &= \{F\}_{t=t_0+\Delta t} \\
&\quad - [M]^{(i)}_{t=t_0+\Delta t}\frac{\{T\}^{(i)}_{t=t_0+\Delta t} - \{T\}_{t=t_0}}{\Delta t} \\
&\quad - [K]^{(i)}_{t=t_0+\Delta t}\{T\}^{(i)}_{t=t_0+\Delta t} \\
\{T\}^{(i+1)}_{t=t_0+\Delta t} &= \{T\}^{(i)}_{t=t_0+\Delta t} + \{\Delta T\}^{(i)}_{t=t_0+\Delta t}
\end{aligned} \tag{127}
$$

In steady-state analysis, the iterative calculation is performed with the following equation:

$$
\begin{aligned}
[K_T]^{(i)}\{\Delta T\}^{(i)}_{t=\infty} &= \{F\}_{t=\infty} - [K_T]^{(i)}\{\Delta T\}^{(i)}_{t=\infty} \\
\{T\}^{(i+1)}_{t=\infty} &= \{T\}^{(i)}_{t=\infty} + \{\Delta T\}^{(i)}_{t=\infty}
\end{aligned} \tag{128}
$$

In non-steady-state analysis, the discretization in time is done through the implicit method; thus, the analysis is normally not affected by the restriction of the size of the time increment Δt. However, if the time increment Δt is to large, the number of convergences in the iterative calculation increases. Therefore, this program is equipped with an automatic increment function, which constantly monitors the dimension of the residual vector in the iterative calculation process. If the convergence of the iterative calculation is too slow, it decreases the time increment Δt. Moreover, when the number of iterative calculations is too small, it increases the time increment Δt.

1.7 Eigenvalue Analysis

1.7.1 Generalized Eigenvalue Problems

In free oscillation analysis of continuous bodies, a spatial discretization is performed, and it is modeled with a multi-DOF system with concentrated mass points as shown in Fig. 2.3.1. In the case of free oscillation problems without damping, the governing equation (motion equation) is as follows:

$$
M\ddot{u} + Ku = 0 \tag{129}
$$

where u is the generalized displacement vector, M is the mass matrix and K is the stiffness matrix. Further, the function is defined with ω as as the inherent angular frequency; a, b and c as arbitary constants; and x as the vector:

$$
u(t) = (a\sin\omega t + b\cos\omega t)x \tag{130}
$$

In this case, this equation and its second derivative, that is,

$$
\ddot{u}(t) = -\omega^2(a\sin\omega t + b\cos\omega t)x \tag{131}
$$

is substituted into Eq.(129), which becomes

$$
M\ddot{u} + Ku = (a\sin\omega t + b\cos\omega t)(-\omega^2 M + Kx) = (-\lambda M + Kx) = 0 \tag{132}
$$

That is, the following equation is obtained:

$$
Kx = \lambda M x \tag{133}
$$

Therefore, if coefficient $\lambda(=\omega^2)$ and vector x that satisfy Eq.(133) can be determined, function $u(t)$ becomes the solution of formula.

The coefficient λ and vector x are called eigenvalue and eigenvector, respectively, and the problem that determines these from Eq.(129) is known as a generalized eigenvalue problem.

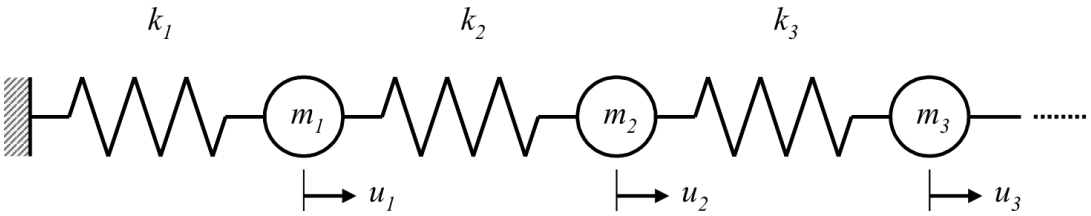

Fig. 2.3.1: Example of a multi-DOF system of free oscillation without damping

1.7.2 Problem Settings

Eq.(133), which can be expanded to any order, appears in many situations. When dealing with physical problems, the matrix is often Hermitian (symmetric.) In a complex matrix, the transpose is a conjugate complex number, and the real matrix is a symmetric matrix. Therefore, when the ij components of matrix K are defined as k_{ij}, if the conjugate complex number k is set as $\bar{k}$, the relationship becomes

$$k_{ij} = \bar{k}_{ji} \tag{134}$$

In this study, it is assumed that the matrices are symmetric and positive definite. A positive definite matrix is a symmetric matrix with all positive eigenvalues; that is, it always satisfies Eq.(135):

$$x^t Ax > 0 \tag{135}$$

1.7.3 Shifted Inverse Iteration Method

Structural analyses with the finite element method do not require all eigenvalues. In many cases, just a few low-order eigenvalues are sufficient. As for HEC-MW, it was designed to deal with large-scale problems thus, the matrices are large and very sparse (with many zeros). Therefore, it is important to consider this and determine eigenvalues of low-order mode efficiently.

When the lower limit of eigenvalues is set to σ, Eq.(133) is modified according to the following equation (which is mathematically equivalent):

$$(K - \sigma M)^{-1} Mx = \frac{1}{(\lambda - \sigma)} x \tag{136}$$

This equation has the following convenient properties for calculation:

1. The mode is inverted.
2. The eigenvalue around ρ are maximized.

In actual calculations, the maximum eigenvalue is often determined at the beginning. Therefore, the main convergence calculation is applied to Eq.(136), rather than Eq.(133) to determine from the eigenvalues around ρ. This method is called shifted inverse iteration.

1.7.4 Algorithm for Eigenvalue Solution

The Jacobi method is another such orthodox and popular method.

It is an effective method for small and dense matrices; however, the matrices dealt with by HEC-MW are large and sparse; thus, the Lanczos iterative is preferred.

1.7.5 Lanczos Method

The Lancos method was proposed by C. Lanczos in the 1950s and is a calculation algorithm for triply diagonalizing a matrix. The following are some of its characteristics:

1. It is an iterative convergence method that allows calculation of a matrix even if it is sparse.

2. The algorithm is focused on matrices and vector product, and suitable for parallelization.
3. It is suitable for the geometric segmentation associated with finite element mesh.

4. It is possible to limit the number of eigenvalues to be determined and mode range to make the calculation more efficient.

The Lanczos method creates sequential orthogonal vectors, starting from the initial vector, to calculate the basis of subspaces. It is faster than the other subspace methods and is widely used in finite element method programs. However, this method is easily influenced by computer errors, which may impair the orthogonality of the vectors and interrupt it in the middle of the process. Therefore, it is essential to apply measures against errors.

1.7.6 Geometric Significance of the Lanczos Method

By converting Eq.(136) into a variable

$$a^T = \frac{\partial F}{\partial \sigma} , \qquad \left[\frac{1}{(\lambda - \sigma)}\right] = \zeta \tag{137}$$

and rewriting the problem, the following equation is obtained:

$$Ax = \zeta x \tag{138}$$

An appropriate vector q_0 linearly transformed with matrix A (see Fig. 2.3.2).

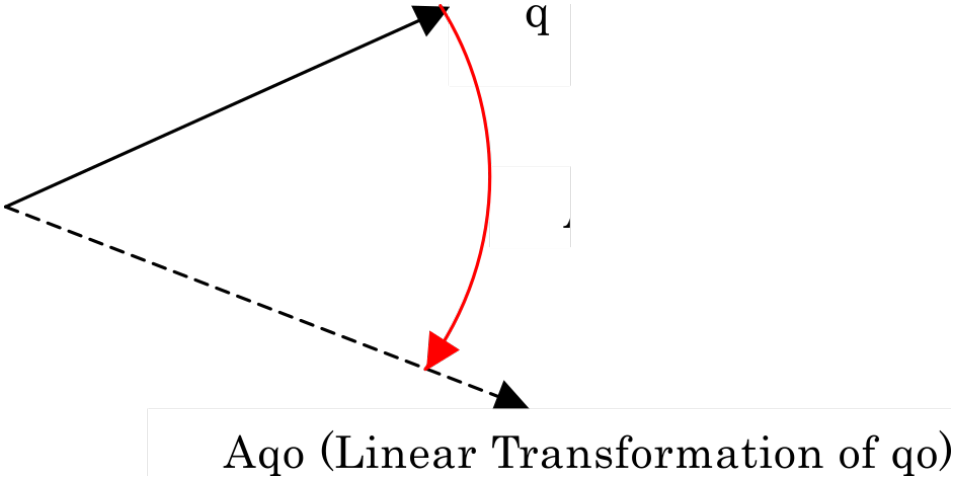

Fig. 2.3.2: Linear Transformation of q_0 with Matrix A

The transformed vector is orthogonalized within the space created by the original vector. That is, it is subjected to a so-called Gram–Schmidt orthogonalization shown in Fig. 2.3.2. Thus, if the vector obtained is defined as r_1 and normalized (to length 1), it generates q_1 (Fig. 2.3.3). With a similar calculation, q_2 is obtained from q_1 (Fig. 2.3.4), which is orthogonal to both q_1 and q_0. If the same calculation is repeated, mutually orthogonal vectors are determined up to the order of the maximum matrix.

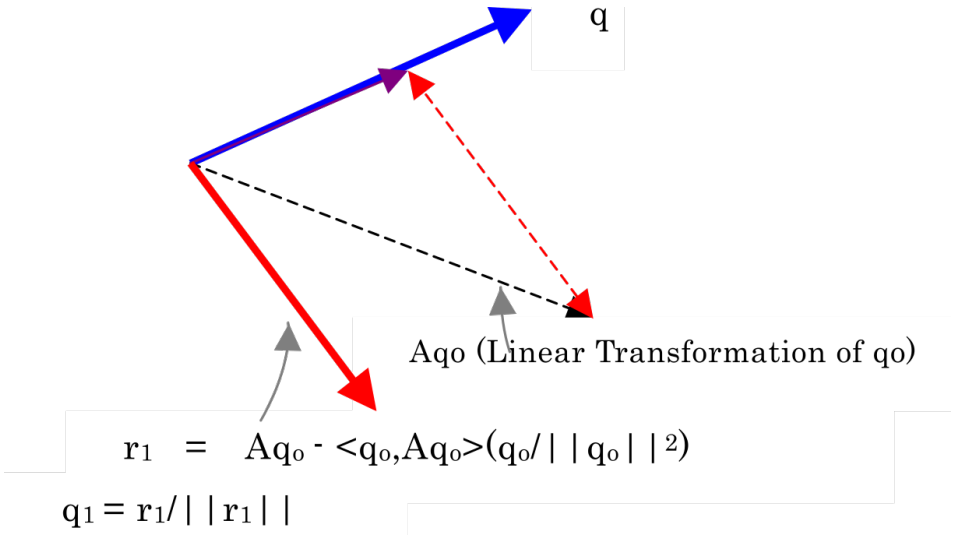

Fig. 2.3.3: Vector q_1 orthogonal to q_0

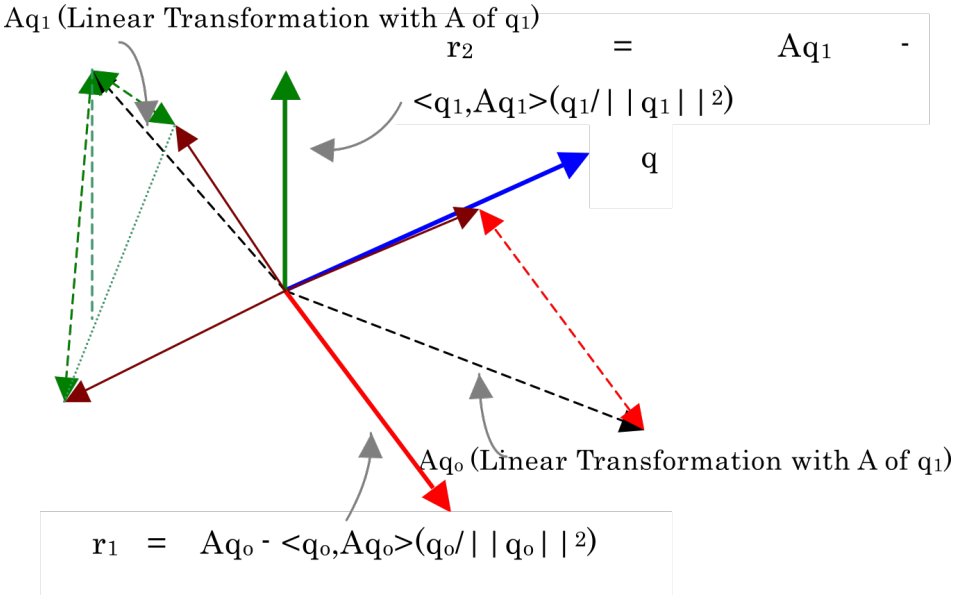

Fig. 2.3.4: Vector q_2 Orthogonal to q_1 and q_0

The algorithm of the Lanczos method is a Gram–Schmidt orthogonalization on vector sequence $A_{q_0}, A^2_{q_0}, A^3_{q_0}, \dots, A^n_{q_0}$ or, in other words, $A_{q_0}, A_{q_1}, A_{q_2}, \dots$. This vector sequence is called Krylov sequence, and the space it creates is called Krylov subspace. If Gram–Schmidt orthogonalization is performed in this space, two adjacent vectors determine another vector. This is called the principle of Lanczos.

1.7.7 Triple Diagonalization

The $i+1$th calculation in the iteration above can be expressed as

$$\beta_{i+1} q_{i+1} + \alpha_{i+1} q_i + \gamma_{i+1} q_{i-1} = A q_i \tag{139}$$

In this case,

$$\beta_{i+1} = \frac{1}{|r_{i+1}|} , \qquad \alpha_{i+1} = \frac{(q_i, Aq_i)}{q_i, q_i} , \qquad \gamma_{i+1} = \frac{(q_{i-1}, Aq_i)}{(q_{i-1}, q_{i-1})} \tag{140}$$

In matrix notation, this becomes

$$AQ_m = Q_m T_m \tag{141}$$

In this case,

$$Q_m = [q_1, q_2, q_3, ..., q_m] \ , \qquad T = \begin{pmatrix} \alpha_1 & \gamma_1 & & & \\ \beta_2 & \alpha_2 & \gamma_2 & & \\ & \cdots & & & \\ & & & \beta_m & \alpha_m \end{pmatrix} \tag{142}$$

That is, the eigenvalues are obtained through eigenvalue calculation on the triply diagonalized matrix obtained with Eq.(141).

1.8 Frequency Response Analysis

1.8.1 Formulation

The motion equation of frequency response analysis when damping is not considered becomes as follows:

$$M\ddot{U} + KU = 0 \tag{143}$$

If this is expanded for each eigenmode, it becomes

$$U = U_j e^{i\omega_j t} \tag{144}$$

If this is substituted into Eq.(143), the following equation is obtained:

$$KU_j = \omega_j^2 M U_j \tag{145}$$

The following is the proof that this eigenfrequency is real. By defining $\omega_j^2 = \lambda_j$ removing the complex conjugate of Eq.(145), Eq.(146), the following equation is obtained:

$$\begin{aligned} KU_j &= \lambda_j M U_j \\ K\overline{U_J} &= \overline{\lambda_J} M \overline{U_J} \end{aligned} \tag{146}$$

If this multiplied by $\overline{U_J}^T$, the following equation is obtained:

$$\begin{aligned} {U_j}^T K \overline{U_J} &= \overline{\lambda_J} {U_j}^T M \overline{U_J} \\ \overline{U_J}^T K U_j &= \lambda_j \overline{U_J}^T M U_j \end{aligned} \tag{147}$$

From Eq.(147), it becomes

$$0 = (\lambda_j - \overline{\lambda_J}) \overline{U_J}^T M U_j \tag{148}$$

In this case, the mass matrix is a positive-definite symmetric matrix; thus,

$$\overline{U_J} M U_j > 0 \tag{149}$$

holds for eigenvectors that are not zero vectors. Therefore,

$$\lambda_j = \overline{\lambda_J} \tag{150}$$

and ${\omega_j}^2 = \lambda_j$ becomes a real number. In this case, two different modes are analyzed.

$$\begin{aligned} KU_i &= \lambda_i M U_i \\ KU_j &= \lambda_j M U_j \end{aligned} \tag{151}$$

From this, the following is obtained:

$$(\lambda_i - \lambda_j) {U_j}^T M U_j = 0 \tag{152}$$

If the eigenvalue is different, it becomes

$${U_j}^T M U_i = 0 \tag{153}$$

That is, different eigenmodes are orthogonal to the mass matrix. The advantage of same modes is that if they are normalized for the mass matrix Eq.(154), the handling becomes easier.

$${U_i}^T M U_i = 1 \tag{154}$$

Further, the frequency response analysis is formulated when damping is considered. The motion equation to be analyzed is expressed in Eq. Eq.(155).

$$M\ddot{U} + C\dot{U} + KU = F \tag{155}$$

The damping term, assuming a Rayleigh-type damping, can be expressed as Eq.(155).

$$C = \alpha M + \beta K \tag{156}$$

With the eigenvector obtained in eigenvalue analysis, the displacement vector can be expanded at time t as in Eq.(157):

$$U(t) = \sum_i b_i(t) U_i \tag{157}$$

where the external force term,

$$F(t) = \{F_R + iF_I\} e^{i\Omega t} \tag{158}$$

defines $b_j(t)$ in the harmonic oscillator equation. The following motion equation Eq.(155) acquires the form of forced vibration holds:

$$b_j(t) = (b_{jR} + b_{jI}) e^{j\Omega t} \tag{159}$$

If the real and imaginary parts of the expansion coefficient of $b_i(t)$ are determined, it becomes Eq.(160) and Eq.(161):

$$b_{jR} = \frac{{U_j}^T F_R({\omega_j}^2 - \Omega^2) + {U_j}^T F_I(\alpha + \beta {\omega_j}^2)\Omega}{({\omega_j}^2 - \Omega^2)^2 + (\alpha + \beta {\omega_j}^2)^2 \Omega^2} \tag{160}$$

$$b_{jI} = \frac{{U_j}^T F_I({\omega_j}^2 - \Omega^2) + {U_j}^T F_R(\alpha + \beta {\omega_j}^2)\Omega}{({\omega_j}^2 - \Omega^2)^2 + (\alpha + \beta {\omega_j}^2)^2 \Omega^2} \tag{161}$$

1.9 References

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