Problems of
Analysis of Nonlinear Control Systems

Part II: Phase Plane Method

7.16 Discuss the singular points for the given nonlinear system $\ddot{x}+\dot{x}+|x|=0$, and sketch the phase plane portrait with the isocline method.

7.17 Given the system as shown in Fig 7.E.10. Assume that the input $r=0$, the system is subject only to the initial conditions. Sketch the phase plane portraits in the e - $\dot{e}$ plane for cases $K=0$ and $K=1$, respectively.

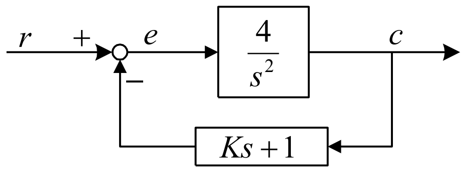


Fig 7.E.10 The ystem of Problem 7.17

7.18 Fig 7.E.11 illustrates a second-order system with nonlinear feedback gain, where $K=5$, $J=1$ and $a=1$

(1) Assume $r=0$, sketch the typical phase trajectories in the e - $\dot{e}$ plane for different initial conditions;

(2) Let a ramp input $r=Vt$ be applied to the system when the system is in the static condition, sketch the phase plane portrait of the system in the e - $\dot{e}$ plane.

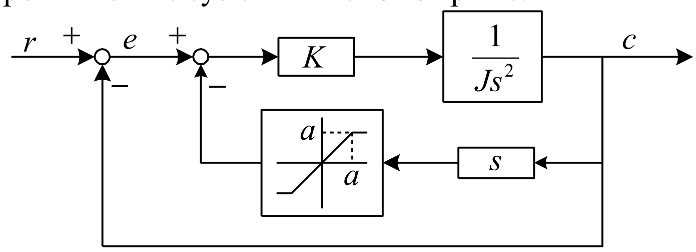


Fig. 7.E.11 The nonlinear system of Problem 7.18

Q7.16 $\ddot{x} + \dot{x} + |x| = 0$

$$\dot{x}\frac{d\dot{x}}{dx} + \dot{x} + |x| = 0$$

set $\frac{d\dot{x}}{dx} = a$ $\quad a\dot{x} + \dot{x} + |x| = 0$

$$\dot{x} = -\frac{|x|}{1+a}$$

singular point:

~~$x_1 = \dot{x}$~~

$x_1 = x, \quad x_2 = \dot{x}$

$$\frac{dx_1}{dt} = \dot{x} = 0$$

$$\frac{dx_2}{dt} = \ddot{x} = 0$$

$$0 = -\dot{x} - |x|$$

$$= \frac{|x|}{1+a} - |x|$$

$$0 = |x| \frac{-a}{1+a}$$

$$|x| = 0$$

$$x = 0$$

singular point $(0, 0)$ //

$f_1 = \dot{x}_1$ $\quad f_2 = \dot{x}_2$

$f_1 = x_2$ $\quad f_2 = -\dot{x} - |x| = -x_2 - |x_1|$

$$\frac{\partial f_1}{\partial x_1} = 0 \qquad \frac{\partial f_2}{\partial x_1} = \begin{cases} -1 & x_1 \geq 0 \\ 1 & x_1 < 0 \end{cases}$$

$$\frac{\partial f_1}{\partial x_2} = 1 \qquad \frac{\partial f_2}{\partial x_2} = -1$$

linearized system of differential equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 \end{cases} \text{ for } x_1 \geq 0 \qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases} \text{ for } x_1 < 0$$

for $x_1 > 0$: $\dot{x}_1 = x_2$, $\ddot{x}_1 = \dot{x}_2$

for $x_1 < 0$: $\dot{x}_1 = x_2$, $\ddot{x}_1 = \dot{x}_2$

$$\dot{x}_2 = -x_1 - x_2 \qquad\qquad \dot{x}_2 = x_1 - x_2$$

$$\ddot{x} = -x - \dot{x} \qquad\qquad \ddot{x} = x - \dot{x}$$

$$\ddot{x} + \dot{x} + x = 0 \qquad\qquad \ddot{x} + \dot{x} - x = 0$$

$$\lambda^2 + \dot{\lambda} + 1 = 0 \qquad\qquad \lambda^2 + \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} \qquad\qquad \lambda_{1,2} = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2} \qquad\qquad = \frac{-1 \pm \sqrt{5}}{2}$$

stable focal point

Saddle point

Isoclines:

$$\dot{x} = -\frac{|x|}{1+a}$$

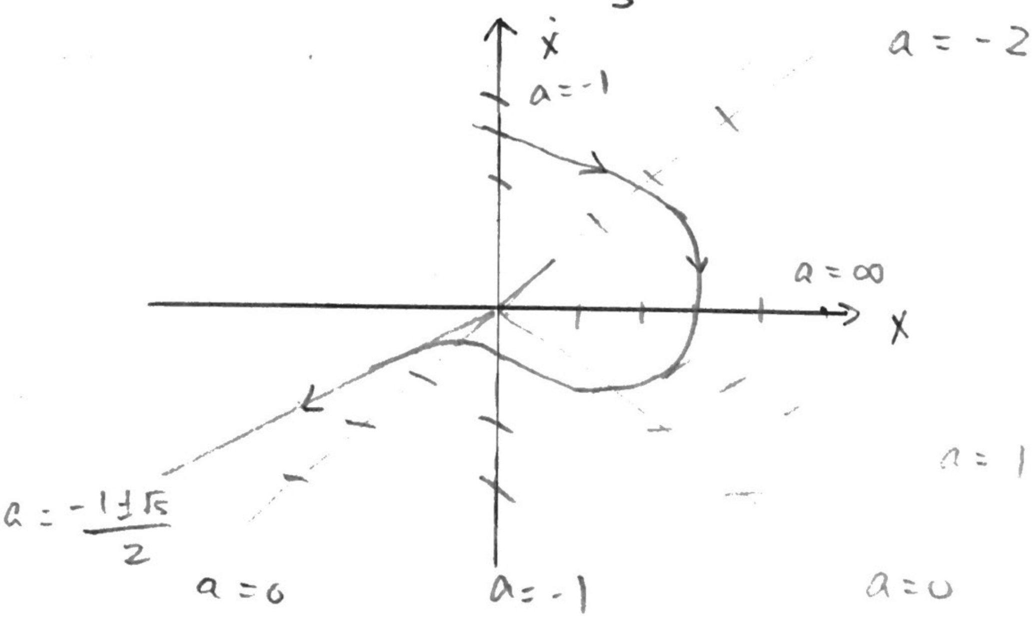
$a \;\; -1 \Rightarrow \dot{x} = -\infty$

$a = \infty \Rightarrow \dot{x} = 0$

$a = 0 \Rightarrow \dot{x} = -|x|$

$a = 1 \Rightarrow \dot{x} = -\frac{|x|}{2}$

$a = 2 \Rightarrow \dot{x} = -\frac{|x|}{3}$



Q 7.17 $$\frac{C(s)}{R(s)} = \frac{\frac{4}{s^2}}{1+\frac{4}{s^2}(ks+1)}$$

$$C(s)\left[1+\frac{4}{s^2}(ks+1)\right] = \frac{4}{s^2}R(s)$$

$$C(s)\left[s^2+4(ks+1)\right] = 4R(s)$$

$$\ddot{c}+4k\dot{c}+4c = 4r \qquad r=0$$

$$\ddot{c}+4k\dot{c}+4c=0$$

$$E(s) = R(s) - C(s) \qquad R(s) = \left(\frac{s^2}{4} \cancel{k} + ks+1\right)C(s)$$

$$\cancel{= \left(\frac{s^2}{4}+ks\right)C(s)}$$

$$\cancel{\frac{E(s)}{C(s)} = \frac{s^2}{4}+ks}$$

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)}$$

$$= \frac{1+\frac{4}{s^2}(ks+1)-\frac{4}{s^2}}{1+\frac{4}{s^2}(ks+1)}$$

$$E(s) = \frac{1+\frac{4}{s^2}ks}{1+\frac{4}{s^2}(ks+1)}R(s)$$

$$e(\infty) = \lim_{s\to\infty} sE(s) = \frac{s+4k}{1+\frac{4k}{s}+\frac{4}{s^2}}r(s) = 0 \qquad ???$$

$$e = r - c \qquad \ddot{c}+4k\dot{c}+4c=0$$

$$\dot{e} = \dot{r}-\dot{c} \qquad (\ddot{r}-\ddot{e})+4k(\dot{r}-\dot{e})+4(r-e)=0$$

$$\ddot{e} = \ddot{r}-\ddot{c} \qquad \ddot{e}+4k\dot{e}+4e=0$$

$$\dot{r}=\ddot{r}=0 \qquad k=0 \qquad k=1$$

$$① \; \ddot{e}+4e=0 \qquad ② \; \ddot{e}+4\dot{e}+4e=0$$

$x_1 = e$

$x_2 = \dot{e}$

① $\ddot{e} + 4e = 0$

$\cancel{\dot{x}_1 \frac{d\dot{x}_1}{dx_1} + 4x_1 = 0}$

$\dot{e}\,\frac{d\dot{e}}{de} + 4e = 0$

$x_2 \,\frac{dx_2}{dx_1} + 4x_1 = 0$

$\int x_2 \, dx_2 = -4 \int x_1 \, dx_1$

$\frac{1}{2} x_2^2 + \frac{4}{2} x_1^2 = C$

$x_2^2 + 4x_1^2 = C'$

$\dot{e}^2 + 4e^2 = C'$

singular point $(0, 0)$

$\ddot{e} = 0$ & $\dot{e} = 0$

$4e = 0$

$e = 0$

$\cancel{f_2 = -4x_1}$

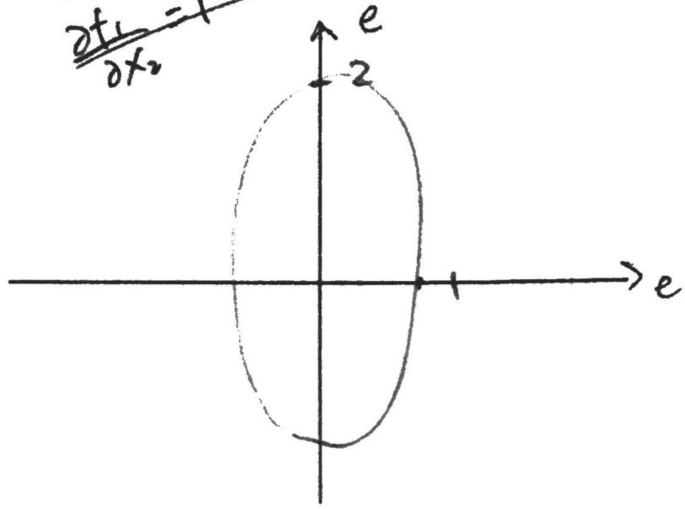
$\cancel{f_1 = x_2}$

$\cancel{\frac{\partial f_2}{\partial x_1} = -4}$

$\cancel{\frac{\partial f_1}{\partial x_1} = 0}$

$\cancel{\frac{\partial f_2}{\partial x_2} = 0}$

$\cancel{\frac{\partial f_1}{\partial x_2} = 1}$



② $\ddot{e} + 4\dot{e} + 4e = 0$

singular point : $(0, 0)$

$\dot{x}_1 = 0$ $\quad \dot{x}_2 = 0$

$\ddot{e} = 0$

$4\dot{e} + 4e = 0$

$e = 0$

~~Linearize system of differential equations~~

$\cancel{f_1 = \dot{x}_1 = \dot{e} = x_2}$

$\cancel{f_2 = \dot{x}_2 = \ddot{x}_1 = \ddot{e} = -4\dot{e} - 4e = -4(x_2 + x_1)}$

$\cancel{\frac{\partial f_1}{\partial x_1} = 0}$ $\quad \cancel{\frac{\partial f_2}{\partial x_1} = -4}$

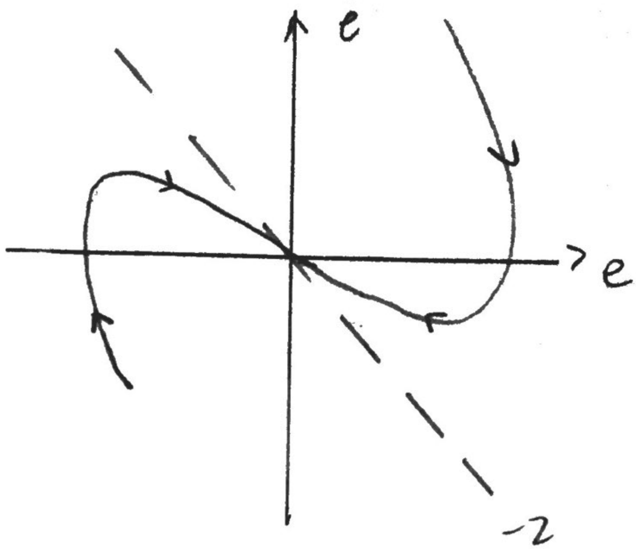
$\cancel{\frac{\partial f_1}{\partial x_2} = 1}$ $\quad \cancel{\frac{\partial f_2}{\partial x_2} = -4}$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-16}}{2}$$

$$= -2$$

稳定结点



Q 7.18

① $|m| \le 1$ $\qquad \dfrac{\frac{5}{s^2}}{1+\frac{5}{s^2}s} = \dfrac{5}{s^2+5s}$

$$\frac{C(s)}{R(s)} = \frac{5}{s^2+5s+5}$$

$$\frac{E(s)}{R(s)} = \frac{s^2+5s}{s^2+5s+5}$$

$\ddot{c}+5\dot{c}+5c = 5r \qquad \ddot{e}+5\dot{e}+5e = \ddot{r}+5\dot{r}$

② $m>1$: $\cancel{\frac{E(s)}{R}}$ $\quad E(s) = \dfrac{s^2R(s)+5}{5+s^2}$

$$C(s) = \frac{5(R(s)+1)}{5+s^2}$$

$\ddot{e}+5e = \ddot{r}+5 \qquad \ddot{c}+5c = 5r+5$

③ $m<-1$: $\quad E(s) = \dfrac{s^2R(s)-5}{5+s^2}$

$$C(s) = \frac{5(R(s)-1)}{5+s^2}$$

$\ddot{e}+5e = \ddot{r}-5 \qquad \ddot{c}+5c = 5r-5$

Q 7.18 (1) $r=\dot{r}=\ddot{r}=0$

$x_1 = e$

$x_2 = \dot{e}$

$\dot{x}_1 = \dot{e} = 0$

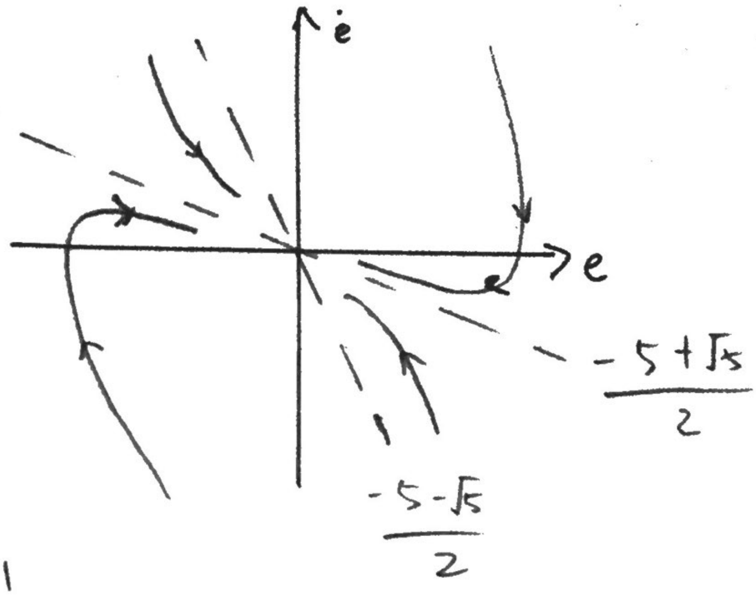
① $\dot{x}_2 = \ddot{e} = -5\dot{e}-5e = 0 \qquad e=0$

Singular point $(0,0)$

$\lambda^2+5\lambda+5=0 \qquad \lambda_{1,2} = \dfrac{-5\pm\sqrt{5}}{2}$

Q7.18 (1)

①



$m > 1$

② $\ddot{e} + 5e - 5 = 0$ $\qquad$ $\ddot{c} + 5c - 5 = 0$

$$c(0) = \lim_{s\to\infty} s\,C(s) = \lim_{s\to\infty} \frac{5s\left(R(s)-1\right)}{5+s^2} = 0$$

$$\dot{c}(0) = \lim_{s\to\infty} s^2 C(s) = \lim_{s\to\infty} \frac{5s^2\left(R(s)-1\right)}{5+s^2} = 5\left(R(s)-1\right) = -5$$

$$e = r - c = 0$$

$$\dot{e} = \dot{r} - \dot{c} = 5$$

令 $\dot{e} = 0$

$$\ddot{e} = 5 - 5e = 0$$

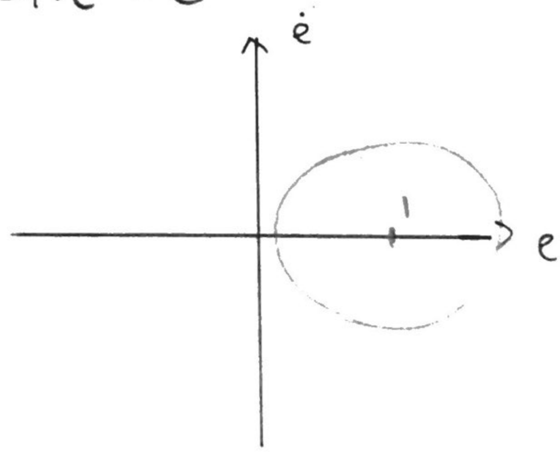
$$e = 1$$

singular point $(1, 0)$

$$\ddot{e} + 5e = 5$$

$$\int \dot{e}\, d\dot{e} = \int (5 - 5e)\, de$$

$$\dot{e}^2 + 5e^2 - 10e = C$$



Q 7.18 (1)

③ $m < -1$

$\ddot{e} + 5e + 5 = 0 \qquad \ddot{c} + 5c + 5 = 0$

$c(0) = \lim_{s\to\infty} sC(s) = \lim_{s\to\infty} \frac{5s(R(s)+1)}{5+s^2} = 0$

$\dot{c}(0) = \lim_{s\to\infty} s^2 C(s) = \lim_{s\to\infty} \frac{5s^2(R(s)+1)}{5+s^2} = 5$

$e = r - c = 0$

$\dot{e} = \dot{r} - \dot{c} = -5$

$\dot{e} = 0$

$\ddot{e} = -5(e+1) = 0$

$e = -1$

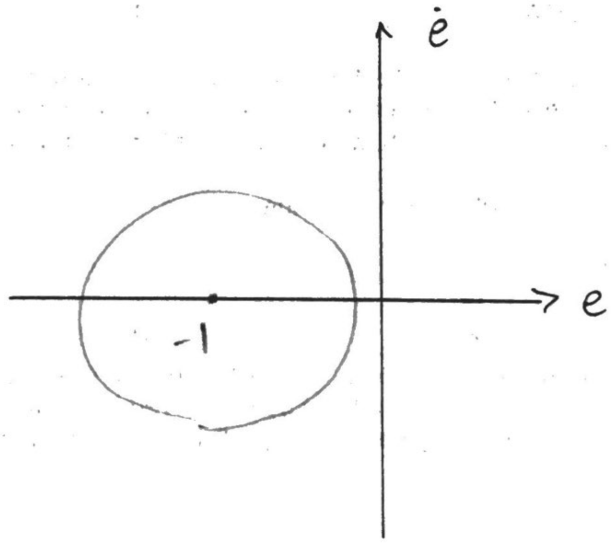
singular point ~~(0, =~~ $(-1, 0)$

$\ddot{e} + 5e = -5$

$\dot{e}\frac{d\dot{e}}{de} = -5 - 5e$

$\int \dot{e}\, d\dot{e} = -5\int (e+1)\, de$

$\dot{e}^2 + 5e^2 + 10e = C$



Q 7.18 (2) $r = Vt$ $\dot{r} = V$ $\ddot{r} = 0$

① $|m| \leq 1$

$\ddot{c} + 5\dot{c} + 5c = 5Vt$ $\ddot{e} + 5\dot{e} + 5e = 5V$

$\dot{e} = 0$

$\ddot{e} = -5(\dot{e} + e - V) = 0$

$e = V$

singular point $(V, 0)$

$m > 1$

② $\ddot{e} + 5e = \ddot{r} + 5$

$\ddot{e} + 5e = 5$

$\dot{e} = 0$

$\ddot{e} = 5 - 5e = 0$

$e = 1$

singular point $(1, 0)$

③ $m < -1$

$\ddot{e} = -5e - 5 + \ddot{r}$

$\ddot{e} + 5e = -5$

$\dot{e} = 0$

$\ddot{e} = -5 - 5e = 0$

$e = -1$

singular point $(-1, 0)$