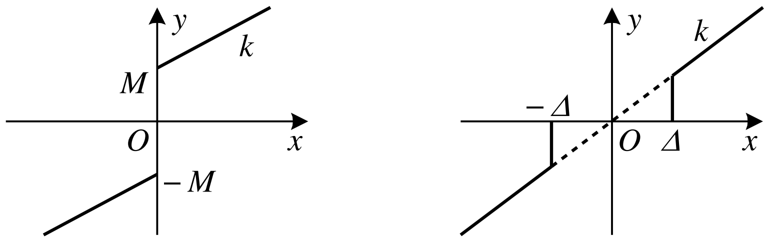
Problems of
Analysis of Nonlinear Control Systems

Part I: Describing Function method

7.3 Calculate the describing functions $N(X)$ of nonlinearities as shown in Fig. 7.E.2, and sketch the plots of $N(X)$ and $-1/N(X)$.



(a) (b)

Fig. 7.E.2 Nolinearity in Problem 7.3

7.5 Given a nonlinear system as shown in Fig. 7.E.3, where $K > 0$. Solve the following problems with describing function method:

(1) To discuss the motion of the system when $K = 5$;

(2) To analyze the frequency and amplitude of the sustained oscillation in the output $c(t)$ when $K = 5$.

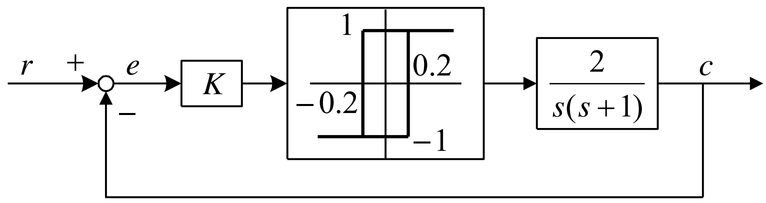


Fig. 7.E.3 The system of Problem 7.5

7.6 Given a system as shown in Fig. 7.E.4, where $K > 0$, $k = 1$. Solve the following problems with the describing function method:

(1) To discuss the motion of the system when $K = 5$;

(2) To analyze the frequency and amplitude of the sustained oscillation in the output $c(t)$ when $K = 5$.

(3) Determine the stability boundary of gain K.

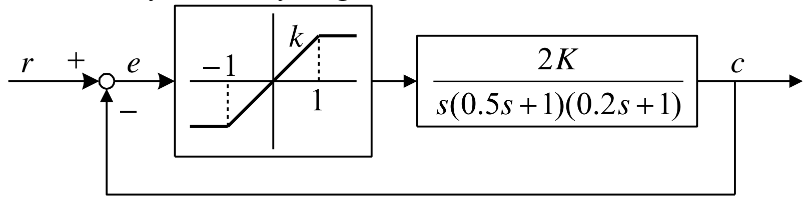
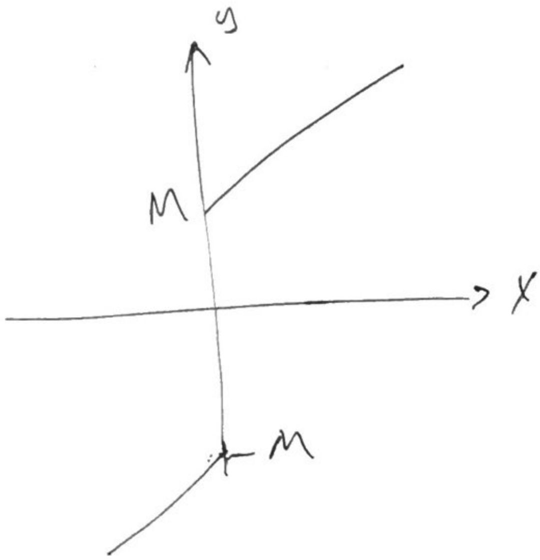
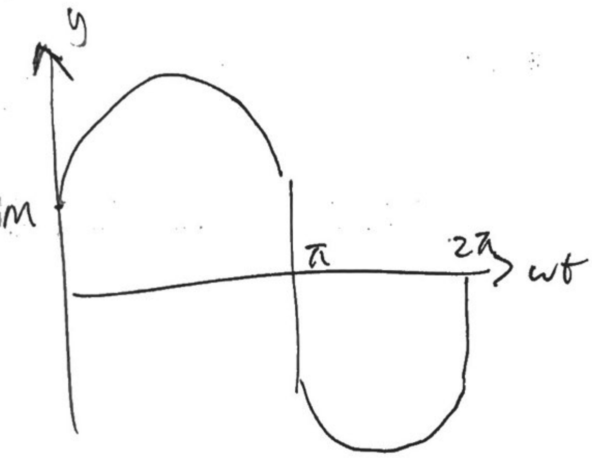
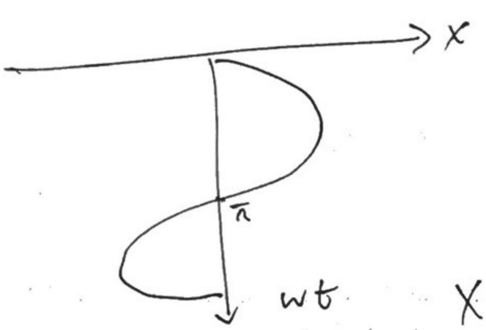


Fig. 7.E.4 The system of Problem 7.6

Q7.3 (a)







$$y = \begin{cases} M + kX & x \geq 0 \\ -M - kX & x < 0 \end{cases}$$

$$X(t) = X \sin \omega t$$

$$y(t) = M + kX \sin \omega t$$

Odd function

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t) \sin \omega t \, d\omega t$$

$$= \frac{4}{\pi}\int_0^{\pi/2} (M + kX \sin \omega t) \sin \omega t \, d\omega t$$

$$\cancel{= \frac{4(M+kX)}{\pi}\left[-\cos \omega t\right]_0^{\frac{\pi}{2}}}$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \left(M \sin \omega t + KX \sin^2 \omega t\right) dt$$

$$= \frac{4}{\pi}\left\{\left[-M \cos \omega t\right]_0^{\frac{\pi}{2}} + KX \left[\frac{1}{2}\omega t - \frac{1}{4}\sin 2\omega t\right]_0^{\frac{\pi}{2}}\right\}$$

$$= \frac{4}{\pi}\left[-M \cos \omega t\right]_0^{\frac{\pi}{2}} + \frac{2kX}{\pi}\left[\omega t - \sin \omega t \cos \omega t\right]_0^{\frac{\pi}{2}}$$

$$B_1 = \frac{4}{\pi} M + \frac{\cancel{2}kX}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{\cancel{2}}\right)$$

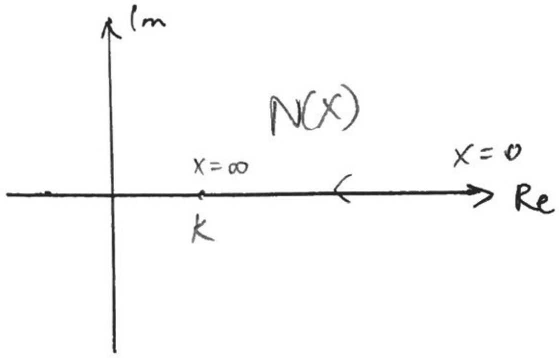
$$= \frac{4M}{\pi} + kX$$

$$N = \frac{B_1}{X} = \frac{4M}{\pi X} + k \quad //$$

$N(X)$ only has real parts

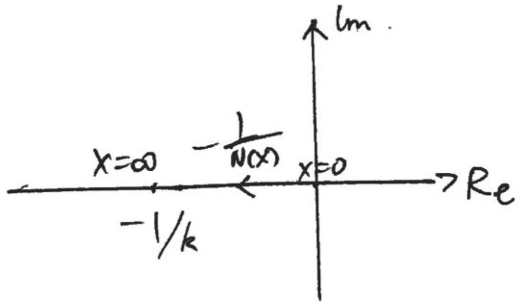
$$N(X) = \frac{4M}{\pi X} + k$$

$$\lim_{X \to 0} N(X) = \infty \qquad \lim_{X \to \infty} N(X) = k \qquad \cancel{N(X)\big|_{X=}}$$

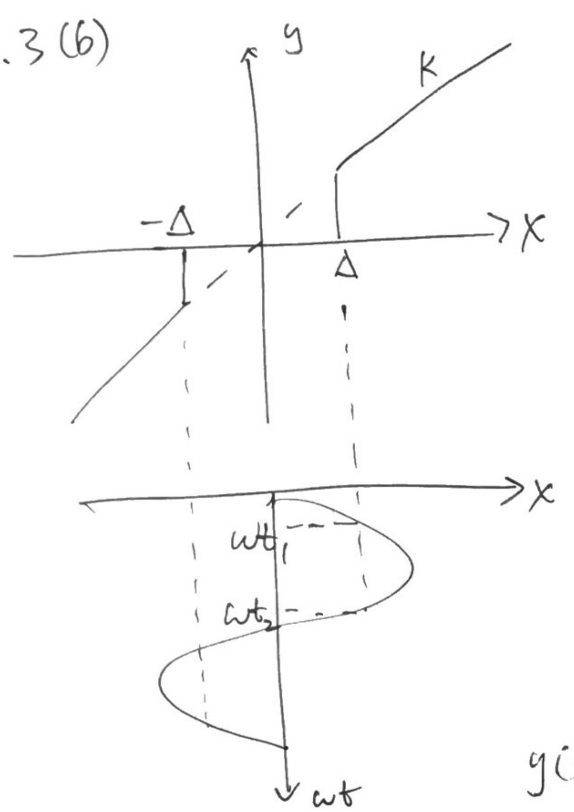


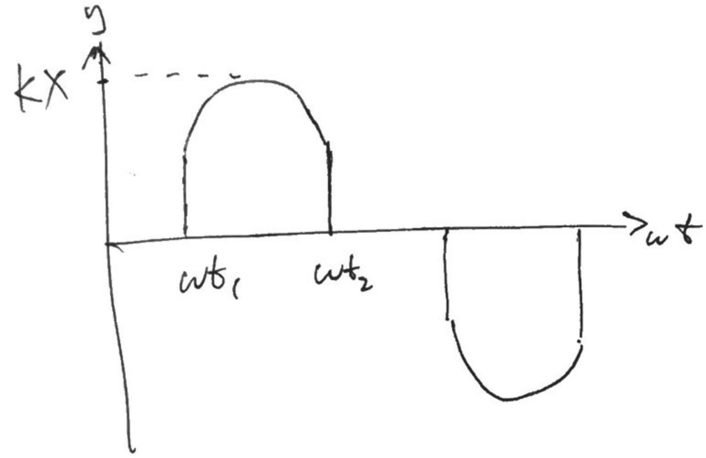
$$-\frac{1}{N(X)} = -\frac{\pi X}{4M + k\pi X}$$

$$\lim_{X \to 0} -\frac{1}{N(X)} = 0 \qquad \lim_{X \to \infty} -\frac{1}{N(X)} = -\frac{1}{1/k}$$



Q7.3 (6)





$$y = \begin{cases} kX & X \geq \Delta \\ 0 & -\Delta \leq X < \Delta \\ kX & X < -\Delta \end{cases}$$

$$y(t) = \begin{cases} k\,X \sin \omega t \\ 0 \\ k\,X \sin \omega t \end{cases}$$

$x(t) = X \sin \omega t$

$\Delta = X \sin \omega t_1$

$\omega t_1 = \sin^{-1} \dfrac{\Delta}{X}$ Odd function

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} y(t) \sin \omega t \, d\omega t$$

$$= \frac{4}{\pi} \int_{\omega t_1}^{\pi/2} KX \sin \omega t \sin \omega t \, d\omega t$$

$$= \frac{4KX}{\pi} \left[\frac{1}{2} \omega t - \cancel{\tfrac{1}{4}} \frac{1}{2} \sin \omega t \cos \omega t \right]_{\omega t_1}^{\frac{\pi}{2}}$$

$$= \frac{2KX}{\pi} \left[\omega t - \sin \omega t \cos \omega t \right]_{\omega t_1}^{\frac{\pi}{2}}$$

$$= \frac{2KX}{\pi} \left(\frac{\pi}{2} - \omega t_1 + \sin \omega t_1 \cos \omega t_1 \right)$$

$$= \frac{\cancel{2}KX}{\cancel{\pi}} \frac{\cancel{\pi}}{\cancel{2}} + \frac{2KX}{\pi} \left(-\sin^{-1} \frac{\Delta}{X} + \sin\left(\sin^{-1} \frac{\Delta}{X}\right) \cos\left(\sin^{-1} \frac{\Delta}{X}\right) \right)$$

$$= kX + \frac{2KX}{\pi} \left(-\sin^{-1} \frac{\Delta}{X} + \frac{\Delta}{X} \sqrt{1 - \left(\frac{\Delta}{X}\right)^2} \right)$$

$$N = \frac{B_1}{X} = k + \frac{2k}{\pi} \left(-\sin^{-1} \frac{\Delta}{X} + \frac{\Delta}{X} \sqrt{1 - \left(\frac{\Delta}{X}\right)^2} \right)$$

Q 7.3 (b)

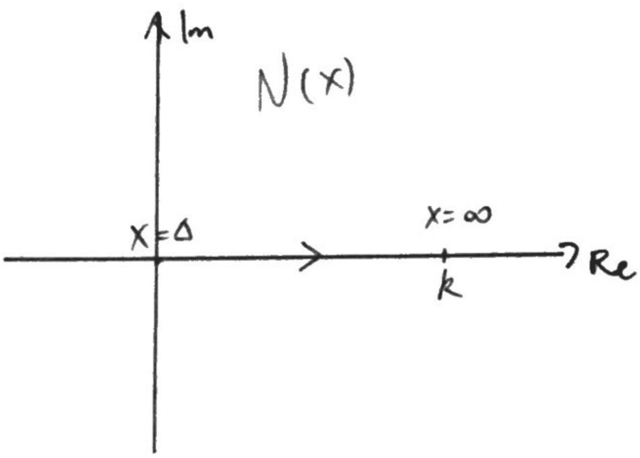
$$N(x) = k + \frac{2k}{\pi}\left(-\sin^{-1}\frac{\Delta}{x} + \frac{\Delta}{x}\sqrt{1-\left(\frac{\Delta}{x}\right)^2}\right)$$

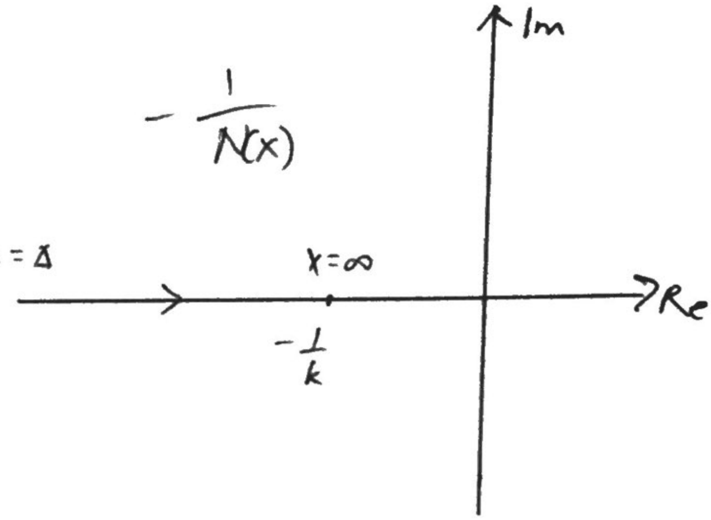
$$\lim_{x\to\Delta} N(x) = k + \frac{\cancel{2}k}{\cancel{\pi}}\left(-\frac{\cancel{\pi}}{\cancel{2}}\right) = 0$$

$$\lim_{x\to\infty} N(x) = k + \frac{2k}{\pi}(0) = k$$

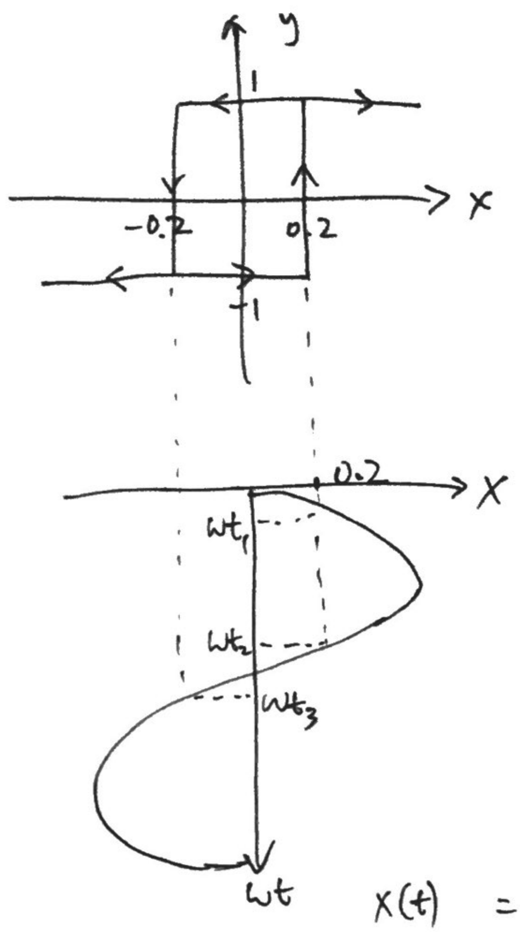
$$\lim_{x\to\Delta} -\frac{1}{N(x)} = -\infty$$

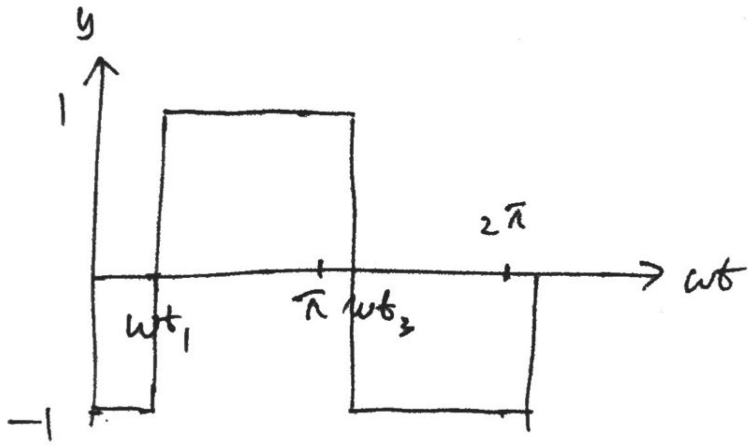
$$\lim_{x\to\infty} -\frac{1}{N(x)} = -\frac{1}{k}$$





Q7.5





$$y = \begin{cases} 1 & X \geq 0.2 \\ -1 & X < 0.2 \end{cases} \qquad \dot{x} > 0$$

$$y = \begin{cases} 1 & X \geq -0.2 \\ -1 & X < -0.2 \end{cases} \qquad \dot{X} < 0$$

$$x(t) = X \sin \omega t$$

$$\omega t_1 = \sin^{-1} \frac{0.2}{X}$$

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y(t) \cos(\omega t)\, d\omega t$$

$$= \frac{2}{\pi}\int_0^{\pi} y(t) \cos(\omega t)\, d\omega t$$

$$= \frac{2}{\pi}\int_{\omega t_1}^{\pi} \cos(\omega t)\, d\omega t - \frac{2}{\pi}\int_0^{\omega t_1} \cos(\omega t)\, d\omega t$$

$$= \frac{2}{\pi}\left[\sin \omega t\right]_{\omega t_1}^{\pi} - \frac{2}{\pi}\left[\sin \omega t\right]_0^{\omega t_1}$$

$$= \frac{4}{\pi}\left(-\sin \sin^{-1}\frac{0.2}{X}\right)$$

$$= -\frac{4}{\pi}\frac{0.2}{X}$$

$$= -\frac{0.8}{X\pi}$$

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\,\sin(\omega t)\,\cancel{dt}\,d\omega t$$

$$= \frac{2}{\pi}\int_{\omega t_1}^{\pi} \sin(\omega t)\,d\omega t - \frac{2}{\pi}\int_0^{\omega t_1} \sin(\omega t)\,d\omega t$$

$$= \frac{2}{\pi}\Big[-\cos\omega t\Big]_{\omega t_1}^{\pi} + \frac{2}{\pi}\Big[\cos\omega t\Big]_0^{\omega t_1}$$

$$= \frac{2}{\pi}\left(\cancel{1}+\cos\omega t_1\right) + \frac{2}{\pi}\left(\cos\omega t_1 - \cancel{1}\right)$$

$$\cancel{= \frac{2}{\pi}\left[1+\cos\left(\sin^{-1}\frac{0.2}{x}\right)\right]}$$

$$= \frac{4}{\pi}\cos\left(\sin^{-1}\frac{0.2}{x}\right)$$

$$= \frac{4}{\pi}\sqrt{1-\left(\frac{0.2}{x}\right)^2}$$

$$N(x) = B_1 + A_1 j$$

$$|N(x)| = \sqrt{B_1^2 + A_1^2}$$

$$= \sqrt{\left(\frac{4}{\pi}\right)^2\left(1-\left(\frac{0.2}{x}\right)^2\right) + \left(\frac{4}{\pi}\right)^2\left(\frac{0.2}{x}\right)^2}$$

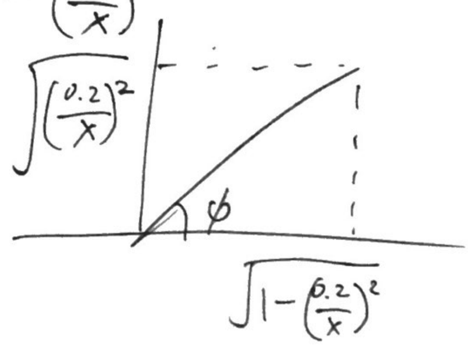
$$= \frac{4}{\pi}$$

$$\phi = \tan^{-1}\frac{A_1}{B_1}$$

$$= \tan^{-1}\frac{-\frac{\cancel{4}}{\cancel{\pi}}\frac{0.2}{x}}{\frac{\cancel{4}}{\cancel{\pi}}\sqrt{1-\left(\frac{0.2}{x}\right)^2}}$$

$$= -\sin^{-1}\frac{0.2}{x}$$

$$\frac{\left(\frac{0.2}{x}\right)^2}{1-\left(\frac{0.2}{x}\right)^2}$$



$$N(x) = \frac{4}{\pi x}\,e^{-j\sin^{-1}\frac{0.2}{x}} \quad //$$

$x > 0.2$

Q7.5 (1) $N(X)\ G_p(j\omega) = \frac{4}{\pi X} e^{-j\left(\sin^{-1}\frac{0.2}{X}\right)} \frac{2K}{(j\omega)(j\omega+1)}$

$\left|N(X)\ G_p(j\omega)\right| = 1 \qquad \angle N(X)\ G_p(j\omega) = -\pi$

$$\left|N(X)\ G_p(j\omega)\right| = \frac{8K}{\pi X} \left|\frac{1}{-\omega^2 + j\omega}\right|$$

$$1 = \frac{8K}{\pi X} \frac{1}{\omega\sqrt{1+\omega^2}}$$

$$K = \frac{\pi X}{8} \omega\sqrt{1+\omega^2}$$

$$\angle N(X)\ G_p(j\omega) = -\sin^{-1}\frac{0.2}{X} - \frac{\pi}{2} - \tan^{-1}\omega$$

$$\cancel{-\pi} = -\sin^{-1}\frac{0.2}{X} \cancel{-\frac{\pi}{2}} \cancel{-\frac{\pi}{2}} + \tan^{-1}\frac{1}{\omega}$$

$$\sin^{-1}\frac{0.2}{X} = \tan^{-1}\frac{1}{\omega}$$

$$\sqrt{1+\omega^2} = 5X$$

$$X = \frac{\sqrt{1+\omega^2}}{5}$$

Q7.5 (2) $K = \frac{\pi}{8} \frac{\sqrt{1+\omega^2}}{5} \omega\sqrt{1+\omega^2}$

$K=5$

$$5 = \frac{\pi\omega(1+\omega^2)}{40}$$

$$\frac{200}{\pi} = \cancel{\pi}\ \omega(1+\omega^2)$$

$$0 = \omega^3 + \omega - \frac{200}{\pi}$$

$$\omega = 3.909$$

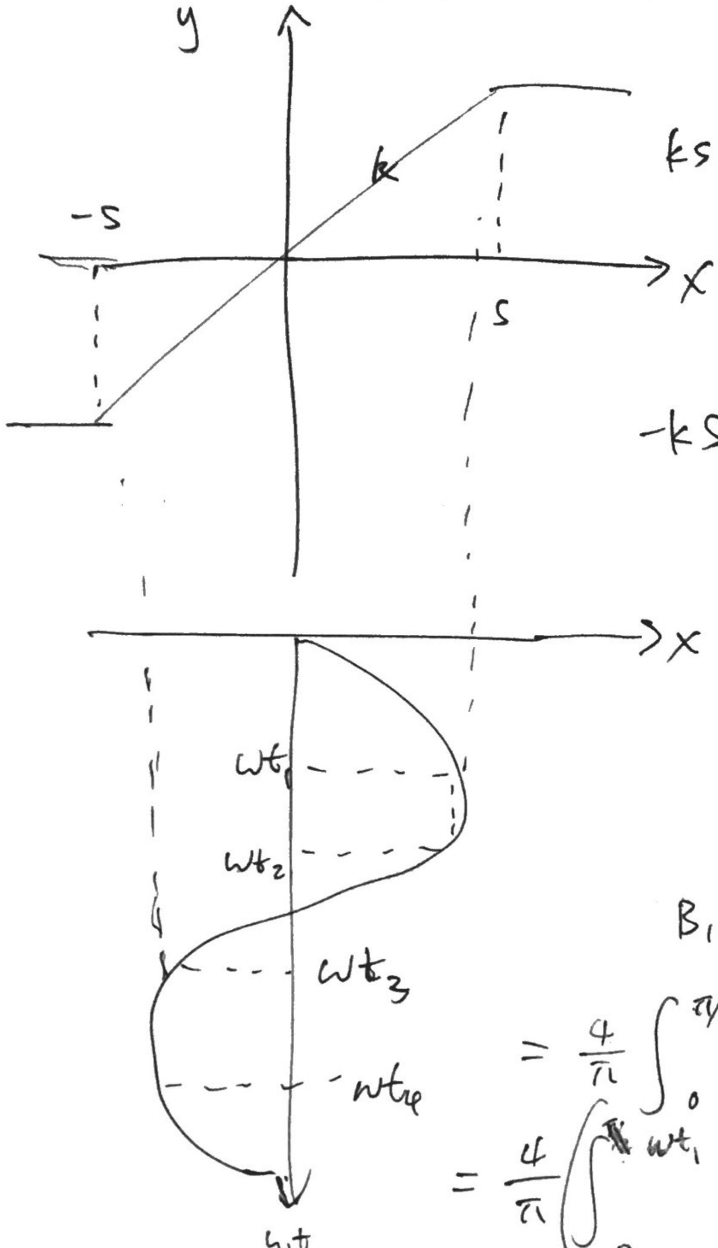
$$X = \frac{\sqrt{1+\omega^2}}{5} = 0.807$$

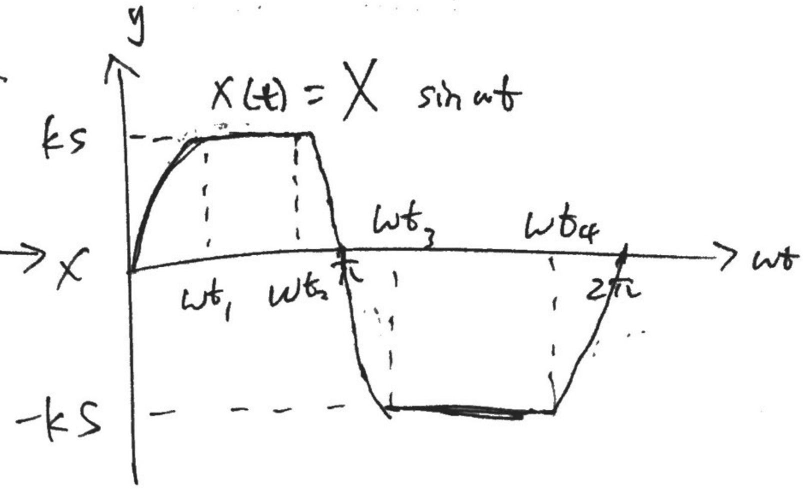
sustained oscillation

$$C = \frac{X}{K} = 0.16$$

Q7.6

$$y = \begin{cases} kS & x \geq S \\ kX & -S \leq X \leq S \\ -kS & -S > X \end{cases}$$





$$x(t) = X \sin \omega t$$

$$X \sin \omega t_1 = S$$

$$\omega t_1 = \sin^{-1} \frac{S}{X}$$

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} y(t) \sin(\omega t)\, d(\omega t)$$

$$= \frac{4}{\pi} \int_0^{\pi/2} y(t) \sin \omega t \, d\omega t$$

$$= \frac{4}{\pi} \left(\int_0^{\omega t_1} \left(kX \sin^2 \omega t\right) d\omega t + \int_{\omega t_1}^{\pi/2} \left(kS \sin \omega t\right) d\omega t \right)$$

$$= \frac{4k}{\pi} \left\{ \left[X \left(\frac{1}{2}\omega t - \frac{1}{4} \sin 2\omega t \right) \right]_0^{\omega t_1} - \left[S \cos \omega t \right]_{\omega t_1}^{\frac{\pi}{2}} \right\}$$

$$= \frac{4k}{\pi} \left\{ \frac{X\omega t_1}{2} - \frac{X}{4} \sin 2\omega t_1 + S \cos \omega t_1 \right\}$$

$$= \frac{k}{\pi} \left(2X\omega t_1 - X \sin 2\omega t_1 + 4S \cos \omega t_1 \right)$$

$$\cancel{= \frac{k}{\pi} \left(2X \sin^{-1} \frac{S}{X} - X \sin 2 \sin^{-1} \frac{S}{X} + 4S \cos \sin^{-1} \frac{S}{X} \right)}$$

$$= \frac{k}{\pi} \left(2X\omega t_1 - 2X \sin \omega t_1 \cos \omega t_1 + 4S \cos \omega t_1 \right)$$

$$= \frac{2k}{\pi}\left(X\omega t_1 - X\sin\omega t_1\cos\omega t_1 + 2s\cos\omega t_1\right)$$

$$= \frac{2k}{\pi}\left(X\sin^{-1}\frac{s}{X} - \cancel{X}\frac{s}{\cancel{X}}\cos\sin^{-1}\frac{s}{X} + 2s\cos\sin^{-1}\frac{s}{X}\right)$$

$$= \frac{2kX}{\pi}\left(\sin^{-1}\frac{s}{X} + \frac{s}{X}\sqrt{1-\left(\frac{s}{X}\right)^2}\right)$$

$$N = \frac{B_1}{X} = \frac{2k}{\pi}\left(\sin^{-1}\frac{s}{X} + \frac{s}{X}\sqrt{1-\left(\frac{s}{X}\right)^2}\right)$$

$$N(X) = \frac{2}{\pi}\left(\sin^{-1}\frac{1}{X} + \frac{1}{X}\sqrt{1-\left(\frac{1}{X}\right)^2}\right) //$$

Q7.6(1)

$$N(X)\,G_p(j\omega) = \frac{4K}{\pi}\left(\sin^{-1}\frac{1}{X} + \frac{1}{X}\sqrt{1-\left(\frac{1}{X}\right)^2}\right)\frac{1}{(j\omega)(0.5j\omega+1)(0.2j\omega+1)}$$

$$\left|N(X)\,G_p(j\omega)\right| = \frac{4K}{\pi}\left(\sin^{-1}\frac{1}{X} + \frac{1}{X}\sqrt{1-\left(\frac{1}{X}\right)^2}\right)\frac{1}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+0.04\omega^2}}$$

....

$$\angle N(X)G_p(j\omega) = -\frac{\pi}{2} - \tan^{-1}0.5\omega - \tan^{-1}0.2\omega$$

$$\frac{\pi}{2} = \tan^{-1}0.5\omega + \tan^{-1}0.2\omega$$

$$\omega = 3.16 //$$

Q7.6(2)

$K = 5$

$$1 = \frac{20}{\pi}\left(\sin^{-1}\frac{1}{X} + \frac{1}{X}\sqrt{1-\left(\frac{1}{X}\right)^2}\right)(0.1429)$$

$$\frac{\pi}{20\times0.1429} = \sin^{-1}\frac{1}{X} + \frac{1}{X}\sqrt{1-\left(\frac{1}{X}\right)^2}$$

$$X = 1.71$$

//

$$C = 1.71$$

//

Q7.6 (3)

$$\cancel{\lim_{x\to 1} \sin^{-1}\frac{1}{x} + \frac{1}{x}\sqrt{1-\left(\frac{1}{x}\right)^2}} =$$

$$\lim_{x\to 1} N(x) = \frac{2}{\pi}\left(\sin\frac{1}{x} + \frac{1}{x}\sqrt{1-\left(\frac{1}{x}\right)^2}\right)$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{2}$$

$$= 1$$

$$\lim_{x\to 1} -\frac{1}{N(x)} = -1$$

$$\sin^{-1}\frac{1}{x} + \frac{1}{x}\sqrt{1-\left(\frac{1}{x}\right)^2} = \frac{5.496}{K}$$

$$\frac{5.496}{K} < \frac{\pi}{2}$$

$$K > 3.50$$