A Brief Introduction of Natural Transformation

Fungus

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Definition 1. A category C consists of:

- A collection Ob(C) of objects $X, Y, Z \cdots$
- A collection

$$\operatorname{Mor}(\mathcal{C}) = \bigcup_{X \in \operatorname{Ob}(\mathcal{C})} \operatorname{dom}_{\mathcal{C}}(X) = \bigcup_{Y \in \operatorname{Ob}(\mathcal{C})} \operatorname{cod}_{\mathcal{C}}(Y) = \bigcup_{X,Y \in \operatorname{Ob}(\mathcal{C})} \operatorname{Hom}_{\mathcal{C}}(X,Y)$$

of morphisms $f, g, h \cdots$ with a binary operation " \circ " which is defined on the subclass of $\operatorname{Mor}(\mathcal{C}) \times \operatorname{Mor}(\mathcal{C})$, where $\operatorname{dom}_{\mathcal{C}}(X)$ is the domain of its elements as well as $\operatorname{cod}_{\mathcal{C}}(Y)$ is the codomain of its elements, and $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is the intersection of both. For $W, X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ and $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \wedge g \in \operatorname{Hom}_{\mathcal{C}}(Y,Z)$, the binary operation that defines composite morphism satisfies:

- 1. $g \circ f \in \text{Hom}_{\mathcal{C}}(X, Z)$;
- 2. $\forall h \in \text{Hom}_{\mathcal{C}}(Z, W)[h \circ (g \circ f) = (h \circ g) \circ f];$
- 3. $\forall h \in \operatorname{Hom}_{\mathcal{C}}(Y, X) \exists \mathbf{1}_X \in \operatorname{Hom}_{\mathcal{C}}(X, X) [f \circ \mathbf{1}_X = f \wedge \mathbf{1}_X \circ h = h].$

It's easy to verify that the *identity morphism* $\mathbf{1}_X$ is unique for all $X \in \mathrm{Ob}(\mathcal{C})$. In commutative diagram,

$$X \xrightarrow{f} Y \quad \text{means} \quad f \in \operatorname{Hom}_{\mathcal{C}}(X,Y), \qquad \bigvee_{f} \xrightarrow{h} \qquad \text{means} \quad g \circ f = h.$$

Definition 2. A functor $F: \mathcal{C} \to \mathcal{D}$, between category \mathcal{C} and \mathcal{D} , consists the following data:

- Mapping $F: \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
- Mapping $F: \operatorname{Mor}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{D})$, which satisfies:
 - 1. $F[\operatorname{Hom}_{\mathcal{C}}(X,Y)] \subseteq \operatorname{Hom}_{\mathcal{D}}(F(X),G(Y))$ for all $X,Y \in \operatorname{Ob}(\mathcal{C})$;
 - 2. For all $f, g \in \text{Mor}(\mathcal{C})$, if $g \circ f$ is defined in $\text{Mor}(\mathcal{C})$, then $F(g \circ f) = F(g) \circ F(f)$;
 - 3. $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$ for all $X \in \text{Ob}(\mathcal{C})$.

In commutative diagram,

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

means that F is the functor between \mathcal{C} and \mathcal{D} .

For functors $F: \mathcal{C}_1 \to \mathcal{C}_2, G: \mathcal{C}_2 \to \mathcal{C}_3$, the composition $GF: \mathcal{C}_1 \to \mathcal{C}_3$ between both satisfies:

$$GF(X) = G(F(X))$$
 and $GF(f) = G(F(f))$ for all $X \in Ob(\mathcal{C}_1), f \in Mor(\mathcal{C}_1)$.

It's trivial to verify that the composition is also a functor and it satisfy associative law.

For any category \mathcal{C} , there exists a identity functor $id_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ that satisfies

$$id_{\mathcal{C}}(X) = X$$
 and $id_{\mathcal{C}}(f) = f$ for all $X \in Ob(\mathcal{C}_1), f \in Mor(\mathcal{C}_1)$.

It's easy to verify that $F \operatorname{id}_{\mathcal{C}} = F$ and $\operatorname{id}_{\mathcal{C}} G = G$ for all functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.

Definition 3. The natural transformation θ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a mapping from $Ob(\mathcal{C})$ to $Mor(\mathcal{D})$ whose each value satisfies

$$\theta_X := \theta(X) \in \operatorname{Hom}_{\mathcal{D}}(F(X), G(X))$$

and the commutative diagram below:

$$F(X) \xrightarrow{\theta_X} G(X)$$

$$F(f) \downarrow \qquad \qquad \downarrow G(f)$$

$$F(Y) \xrightarrow{\theta_Y} G(Y),$$

$$(1)$$

where $X, Y \in \text{Ob}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(X, Y)$. In other words, we can record the above natural transformation as $\theta : F \Rightarrow G$, or in such a commutative diagram:

$$\mathcal{C}$$
 θ \mathcal{D} .

We may use symbol " $F(\theta)_X$ " instead of " $F((\theta)_X)$ " in some particular case (such as there are more than one symbols of natural transformations in the brackets).

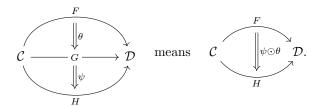
For any functor $F: \mathcal{C} \to \mathcal{D}$, there exists a identity transformation $\mathrm{id}^F: F \to F$ that satisfies

$$\forall X \in \mathrm{Ob}(\mathcal{C})[\mathrm{id}_X^F = \mathbf{1}_{F(X)}].$$

For functors $F: \mathcal{C}_1 \to \mathcal{C}_2, G: \mathcal{C}_2 \to \mathcal{C}_3$, it's trivial to prove the following formula using the definition of identity morphism and transformation:

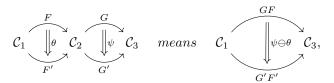
$$\forall X \in \mathrm{Ob}(\mathcal{C}_1)[G(\mathbf{id}_X^F) = \mathbf{id}_{F(X)}^G]. \tag{2}$$

Definition 4. For functors $F, G, H : \mathcal{C} \to \mathcal{D}$, natural transformations $\theta : F \Rightarrow G$ and $\psi : G \Rightarrow H$, the element of vertical composition of the natural transformations is defined as $(\psi \odot \theta)_X = \psi_X \circ \theta_X$. In commutative diagrams forms,



Actually, we need to prove that the definition is well-defined, i.e., to verify that $(\psi \odot \theta)_X \in \text{Hom}_{\mathcal{D}}(F(X), H(X))$ for all $X \in \text{Ob}(\mathcal{C})$, it's easy to do so.

Definition 5. For functors $F, F' : \mathcal{C}_1 \to \mathcal{C}_2$ and $G, G' : \mathcal{C}_2 \to \mathcal{C}_3$, natural transformations $\theta : F \Rightarrow F'$ and $\psi : G \Rightarrow G'$, the element of *horizontal composition* of natural transformations $(\psi \ominus \theta)_X$ is defined as $G'(\theta_X) \circ \psi_{F(X)} = \psi_{F'(X)} \circ G(\theta_X)$. In commutative diagrams forms,



which satisfy

$$GF(X) \xrightarrow{G(\theta_X)} GF'(X)$$

$$\psi_{F(X)} \downarrow \qquad \qquad \downarrow \psi_{F'(X)}$$

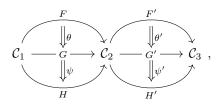
$$G'F(Y) \xrightarrow{G'(\theta_X)} G'F'(Y).$$

$$(3)$$

Actually, we need to prove that the definition is well-defined, i.e., to verify the commutative diagram and that $(\psi \ominus \theta)_X \in \operatorname{Hom}_{\mathcal{C}_3}(GF(X), G'F'(X))$ for all $X \in \operatorname{Ob}(\mathcal{C}_1)$, it's easy to do so observing commutative diagram (1).

Theorem 1. The vertical and horizontal compositions of natural transformations are natural transformations.

Proof. For



we need to verify the following commutative diagrams:

$$F(X) \xrightarrow{(\psi \odot \theta)_X} H(X) \qquad GF(X) \xrightarrow{(\theta' \ominus \theta)_X} G'F'(X)$$

$$(a)_{F(f)} \downarrow \qquad \downarrow_{H(f)} \text{ and (b)} \qquad \downarrow_{GF(f)} \qquad \downarrow_{G'F'(f)}.$$

$$F(y) \xrightarrow{(\psi \odot \theta)_X} H(y) \qquad GF(Y) \xrightarrow{(\theta' \ominus \theta)_Y} G'F'(Y)$$

From (a), we have

$$\begin{array}{ll} H(f)\circ (\psi\odot\theta)_X & \text{(Assumption)} \\ = H(f)\circ \psi_X\circ\theta_X & \text{(Def: vertical composition)} \\ = (\psi_Y\circ G(f))\circ\theta_X & \text{(Property of natural transformation }\psi) \\ = \psi_Y\circ\theta_Y\circ F(f) & \text{(Property of natural transformation }\theta) \\ = (\psi\odot\theta)_X\circ F(f), & \text{(Def: vertical composition)} \end{array}$$

thus $(\psi \odot \theta)$ is natural transformation.

From (b), we have

thus $(\psi \ominus \theta)$ is natural transformation.

What's more, we can prove that $(\psi \odot \theta) \ominus (\psi' \odot \theta') = (\psi' \ominus \psi) \odot (\theta' \ominus \theta)$:

$$\begin{array}{ll} ((\psi' \odot \theta') \ominus (\psi \odot \theta))_X & \text{(Assumption)} \\ = (\psi' \odot \theta')_{H(X)} \circ F'(\psi \odot \theta)_X & \text{(Def: horizontal composition)} \\ = \psi'_{H(X)} \circ \theta_{H(X)} \circ F'(\theta_X) \circ F'(\theta_X) & \text{(Def: vertical composition, Property of functor } F') \\ = \psi'_{H(X)} \circ (G'(\psi_X) \circ \theta'_{G(X)}) \circ F'(\theta_X) & \text{(Commutative diagram (3))} \\ = (\psi' \ominus \psi)_X \circ (\theta' \ominus \theta)_X & \text{(Def: horizontal composition)} \\ = ((\psi' \ominus \psi) \odot (\theta' \ominus \theta))_X, & \text{(Def: vertical composition)} \end{array}$$

where $X \in \mathrm{Ob}(\mathcal{C})$.

Theorem 2. Both vertical and horizontal compositions of natural transformations satisfy associative law.

Proof. For vertical composition, observe the following commutative diagram and natural transformations:

$$\begin{array}{ccc}
F \\
C & \mathcal{D} \\
H & A
\end{array}$$
 as well as
$$\begin{cases}
\theta : F \Rightarrow G, \\
\psi : G \Rightarrow H, \\
\phi : H \Rightarrow K,
\end{cases}$$

it's trivial to prove that $((\phi \odot \psi) \odot \theta)_X = (\phi \odot (\psi \odot \theta))_X$ for all $X \in \text{Ob}(\mathcal{C})$, thus the vertical composition satisfies associative law.

For

we have

$$\begin{array}{ll} (\phi\ominus(\psi\ominus\theta))_X & \text{(Assumption)} \\ =\phi_{G'F'(X)}\circ H(\psi\ominus\theta)_X & \text{(Def: horizontal composition)} \\ =\phi_{G'F'(X)}\circ H(\psi_{F'(X)}\circ G(\theta_X)) & \text{(Ditto)} \\ =\phi_{G'F'(X)}\circ H(\psi_{F'(X)})\circ HG(\theta_X) & \text{(Property of functor H)} \\ =(\phi\ominus\psi)_{F'(X)}\circ HG(\theta_X) & \text{(Def: horizontal composition)} \\ =((\phi\ominus\psi)\ominus\theta)_X, & \text{(Ditto)} \end{array}$$

thus the horizontal composition satisfies associative law.

Theorem 3. Observe the following natural commutative:

$$C_1 \xrightarrow{F} C_2 \underbrace{\downarrow \theta}_H C_3 \xrightarrow{K} C_4,$$

we have

$$\theta \odot \mathbf{id}^G = \theta = \mathbf{id}^H \odot \theta, \tag{4}$$

$$(\theta \ominus \mathbf{id}^F)_X = \theta_{F(X)},\tag{5}$$

$$(\mathbf{id}^K \ominus \theta)_Y = K(\theta_Y), \tag{6}$$

for all $X \in \text{Ob}(\mathcal{C}_1), Y \in \text{Ob}(\mathcal{C}_2)$.

Proof. It's trivial to prove (4) and using definition of identity transformation and vertical composition. For (5), we have

$$(\theta \ominus \mathbf{id}^F)_X$$
 (Assumption)
$$=\theta_{F(X)} \circ G(\mathbf{id}_X^F)$$
 (Def: horizontal composition)
$$=\theta_{F(X)} \circ \mathbf{id}_{F(X)}^G$$
 (formula 2)
$$=\theta_{F(X)}$$
 (formula 4)

for all $X \in \text{Ob}(\mathcal{C}_1)$. For (6), we have

$$\begin{aligned} (\mathbf{id}^K \ominus \theta)_Y & \text{(Assumption)} \\ = & K(\theta_Y) \circ \mathbf{id}_{G(Y)}^K & \text{(Def: horizontal composition)} \\ = & K(\theta_Y) \circ K(\mathbf{id}_Y^G) & \text{(formula 2)} \\ = & K(\theta_Y \circ \mathbf{id}_Y^G) & \text{(Property of functor } K) \\ = & K(\theta_Y) & \text{(formula 4)} \end{aligned}$$

for all $Y \in \mathrm{Ob}(\mathcal{C}_2)$.

Lemma 1 (ZF). For any sets A, B and mapping $f: A \to B$, we have

$$\exists g: B \to A[g \circ f = \mathrm{id}_A] \iff f \text{ is a injection } \iff \forall C \forall h, h': C \to A[f \circ h = f \circ h' \Longrightarrow h = h'],$$
$$\exists g: B \to A[f \circ g = \mathrm{id}_B] \Longrightarrow f \text{ is a surjection } \iff \forall C \forall h, h': B \to C[h \circ f = h' \circ f \Longrightarrow h = h'],$$

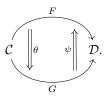
and f is a surjection $\Longrightarrow \exists g: B \to A[f \circ g = \mathrm{id}_B]$ can be proved in ZFC. It's easy to see that f is bijection if and only if it has both left and right inversal mappings, and obviously the two inversal mappings are the same one, which is unique.

Definition 6. Consider $X, Y \in \text{Ob}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(X, Y)$. If there exists a morphism $g \in \text{Hom}_{\mathcal{C}}(Y, X)$ that $f \circ g = \mathbf{1}_Y$, then we say g is the *right inverse* of it; if $g \circ f = \mathbf{1}_X$, then we say g is the *left inverse* of it. If f has both inverses, then we say f is *isomorphic*, it's easy to verify that the two inverses are the same one, which is unique as well, so we say $f^{-1} := g$ is the *inverse* (*inversal morphism*) of it.

What's more, it's trivial to verify that the composition of isomorphic morphisms are also isomorphic, so we can find that the collection of automorphisms $\operatorname{Aut}_{\mathcal{C}}(X) := \{f \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \mid f \text{ is isomorphic}\}\$ is a group $\langle \operatorname{Aut}_{\mathcal{C}}(X), \circ, \mathbf{1}_X \rangle$. Therefore if f has a left inverse, then f is a $injective\ morphism$, which satisfies the $left\ cancellation\ law$: $\forall Z \in \operatorname{Ob}(C) \forall g, h \in \operatorname{Hom}_{\mathcal{C}}(Z,X)[f \circ g = f \circ h \iff g = h]$; if f has a right inverse, then f is a $surjective\ morphism$, which satisfies the $right\ cancellation\ law$: $\forall Z \in \operatorname{Ob}(C) \forall g, h \in \operatorname{Hom}_{\mathcal{C}}(Y,Z)[g \circ f = h \circ f \iff g = h]$.

Consider a functor $F: \mathcal{C} \to \mathcal{D}$, we can define the inverse of functor in the same way: If there exists a functor $G: \mathcal{D} \to \mathcal{C}$ that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$, then we say F is isomorphic (F is a isomorphism between \mathcal{C} and \mathcal{D}), and $F^{-1} := G$ is the unique inverse (inversal functor) of F. From Lemma 1 we know that F is isomorphic if and only if $F \upharpoonright_{\mathrm{Ob}(\mathcal{C})}$ and $F \upharpoonright_{\mathrm{Mor}(\mathcal{C})}$ are both bijection.

Consider the following diagram:



If $\psi \odot \theta = \mathbf{id}^F$ and $\theta \odot \psi = \mathbf{id}^G$, then we say θ is isomorphic between F and G, and $\theta^{-1} := \psi$ is the unique inverse (inversal transformation) of θ , we record it as $\theta : F \overset{\sim}{\Rightarrow} G$. It's trivial to prove that θ is isomorphic if and only if θ_X is isomorphic for each $X \in \mathrm{Ob}(\mathcal{C})$, thus we have $(\theta^{-1})_X = (\theta_X)^{-1}$, we abbreviate it as " θ_X^{-1} ".

For functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$, if there exist isomorphic $\theta: FG \stackrel{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$ and $\psi: GF \stackrel{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{C}}$, then we say G is the *quasi-inverse* of F, and F is a *equivalence* between \mathcal{C} and \mathcal{D} . If there exist an equivalence between two categories \mathcal{C} and \mathcal{D} , we say they are *equivalent* and record it as $\mathcal{C} \simeq \mathcal{D}$.

It's tirvial to verify that the composition of any isomorphic morphisms/functors/transformations is isomorphic, and the composition of any equivalences are equivalence.