A Brief Introduction of Natural Transformation

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Definition 1. *A* category $\mathcal{C}$ *consists of:*

- *A collection* $\mathrm{Ob}(\mathcal{C})$ *of* objects $X, Y, Z \cdots$
- *A collection*

$$\mathrm{Mor}(\mathcal{C}) = \bigsqcup_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{dom}_{\mathcal{C}}(X) = \bigsqcup_{Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{cod}_{\mathcal{C}}(Y) = \bigsqcup_{X,Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

  of morphisms $f, g, h \cdots$ *with a binary operation "$\circ$" which is defined on the subclass of* $\mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C})$*, where* $\mathrm{dom}_{\mathcal{C}}(X)$ *is the* domain *of its elements as well as* $\mathrm{cod}_{\mathcal{C}}(Y)$ *is the* codomain *of its elements, and* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ *is the intersection of both. For* $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \wedge g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$*, the binary operation satisfies:*

  1. $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X,Z)$;
  2. $\forall h \in \mathrm{Hom}_{\mathcal{C}}(Z,W)[h \circ (g \circ f) = (h \circ g) \circ f]$;
  3. $\forall h \in \mathrm{Hom}_{\mathcal{C}}(Y,X) \exists \mathbf{1}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)[f \circ \mathbf{1}_X = f \wedge \mathbf{1}_X \circ h = h]$.

It's easy to verify that the identity morphism $\mathbf{1}_X$ *is unique for all* $X \in \mathrm{Ob}(\mathcal{C})$.
In commutative diagram,

$$X \xrightarrow{f} Y \quad \textit{means} \quad f \in \mathrm{Hom}_{\mathcal{C}}(X,Y), \qquad \begin{tikzcd} X \arrow[d, "f"] \arrow[dr, "h"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd} \quad \textit{means} \quad g \circ f = h.$$

Definition 2. *A* functor $F : \mathcal{C} \to \mathcal{D}$*, between category* $\mathcal{C}$ *and* $\mathcal{D}$*, consists the following data:*

- *Mapping* $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
- *Mapping* $F : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$*, which satisfies:*

  1. $F[\mathrm{Hom}_{\mathcal{C}}(X,Y)] \subseteq \mathrm{Hom}_{\mathcal{D}}(F(X), G(Y))$ *for all* $X, Y \in \mathrm{Ob}(\mathcal{C})$;
  2. *For all* $f, g \in \mathrm{Mor}(\mathcal{C})$*, if* $g \circ f$ *is defined in* $\mathrm{Mor}(\mathcal{C})$*, then* $F(g \circ f) = F(g) \circ F(f)$;
  3. $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$ *for all* $X \in \mathrm{Ob}(\mathcal{C})$.

In commutative diagram,

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

means that F *is the functor between* $\mathcal{C}$ *and* $\mathcal{D}$.

For functors $F : \mathcal{C}_1 \to \mathcal{C}_2, G : \mathcal{C}_2 \to \mathcal{C}_3$*, the composition* $GF : \mathcal{C}_1 \to \mathcal{C}_3$ *between both satisfies:*

$$GF(X) = G(F(X)) \text{ and } GF(f) = G(F(f)) \text{ for all } X \in \mathrm{Ob}(\mathcal{C}_1), f \in \mathrm{Mor}(\mathcal{C}_1).$$

For any category $\mathcal{C}$*, there exists a* identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ *that satisfies*

$$\mathrm{id}_{\mathcal{C}}(X) = X \text{ and } \mathrm{id}_{\mathcal{C}}(f) = f \text{ for all } X \in \mathrm{Ob}(\mathcal{C}_1), f \in \mathrm{Mor}(\mathcal{C}_1).$$

It's easy to verify that $F\mathrm{id}_{\mathcal{C}} = F$ *and* $\mathrm{id}_{\mathcal{C}}G = G$ *for all functors* $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$.

Definition 3. *The* natural transformation θ *between functors* $F, G : \mathcal{C} \to \mathcal{D}$ *is a mapping from* $\mathrm{Ob}(\mathcal{C})$ *to* $\mathrm{Mor}(\mathcal{D})$ *whose each value satisfies*

$$\theta(X) = \theta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$$

and the commutative diagram below:

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"'] & G(Y), \end{tikzcd} \tag{1}$$

where $X, Y \in \mathrm{Ob}(\mathcal{C})$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. *In other words, we can record the above natural transformation as* $\theta : F \Rightarrow G$*, or in such a commutative diagram:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D}. \end{tikzcd}$$

We may use symbol "$F(\theta)_X$*" instead of "*$F((\theta)_X)$*" in some particular case (such as there are more than one symbols of natural transformations in the brackets).*

For any functor $F : \mathcal{C} \to \mathcal{D}$*, there exists a* identity transformation $\mathrm{id}^F : F \to F$ *that satisfies*

$$\forall X \in \mathrm{Ob}(\mathcal{C})[\mathrm{id}^F_X = \mathbf{1}_{F(X)}].$$

For functors $F : \mathcal{C}_1 \to \mathcal{C}_2, G : \mathcal{C}_2 \to \mathcal{C}_3$*, it's trivial to prove the following formula using the definition of identity morphism and transformation:*

$$\forall X \in \mathrm{Ob}(\mathcal{C}_1)[G(\mathrm{id}^F_X) = \mathrm{id}^G_{F(X)}]. \tag{2}$$

Definition 4. *For functors* $F, G, H : \mathcal{C} \to \mathcal{D}$*, natural transformations* $\theta : F \Rightarrow G$ *and* $\psi : G \Rightarrow H$*, the element of* longitudinal composition *of the natural transformations is defined as* $(\psi \odot \theta)_X = \psi_X \circ \theta_X$. *In commutative diagrams forms,*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\theta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\psi"] & \mathcal{D} \end{tikzcd} \quad \textit{means} \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\psi\odot\theta"] & \mathcal{D}. \end{tikzcd}$$

Actually, we need to prove that the definition is well-defined, i.e., to verify that $(\psi \odot \theta)_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), H(X))$ *for all* $X \in \mathrm{Ob}(\mathcal{C})$*, it's easy to do so.*

Definition 5. *For functors* $F, F' : \mathcal{C}_1 \to \mathcal{C}_2$ *and* $G, G' : \mathcal{C}_2 \to \mathcal{C}_3$*, natural transformations* $\theta : F \Rightarrow F'$ *and* $\psi : G \Rightarrow G'$*, the element of* horizontal composition *of natural transformations* $(\psi \ominus \theta)_X$ *is defined as* $G'(\theta_X) \circ \psi_{F(X)} = \psi_{F'(X)} \circ G(\theta_X)$. *In commutative diagrams forms,*

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}_2 \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{C}_3 \end{tikzcd} \quad \textit{means} \quad \begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "GF"] \arrow[r, bend right, "G'F'"'] \arrow[r, phantom, "\Downarrow\psi\ominus\theta"] & \mathcal{C}_3, \end{tikzcd}$$

which satisfy

$$\begin{tikzcd} GF(X) \arrow[r, "G(\theta_X)"] \arrow[d, "\psi_{F(X)}"'] & GF'(X) \arrow[d, "\psi_{F'(X)}"] \\ G'F(Y) \arrow[r, "G'(\theta_X)"'] & G'F'(Y). \end{tikzcd} \tag{3}$$

Actually, we need to prove that the definition is well-defined, i.e., to verify the commutative diagram and that $(\psi \ominus \theta)_X \in \mathrm{Hom}_{\mathcal{C}_3}(GF(X), G'F'(X))$ *for all* $X \in \mathrm{Ob}(\mathcal{C}_1)$*, it's easy to do so observing Commutative diagram (1).*

Theorem 1. *The longitudinal composition of natural transformations is natural transformation.*

Proof. For

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\theta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\psi"] & \mathcal{C}_2 \arrow[r, bend left=60, "F'"] \arrow[r, "G'" description] \arrow[r, bend right=60, "H'"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\theta'"] \arrow[r, phantom, shift right=3ex, "\Downarrow\psi'"] & \mathcal{C}_3 \end{tikzcd} ,$$

we need to verify the following commutative diagrams:

$$\text{(a)} \begin{tikzcd} F(X) \arrow[r, "(\psi\odot\theta)_X"] \arrow[d, "F(f)"'] & H(X) \arrow[d, "H(f)"] \\ F(y) \arrow[r, "(\psi\odot\theta)_X"'] & H(y) \end{tikzcd} \text{ and (b) } \begin{tikzcd} GF(X) \arrow[r, "(\theta'\ominus\theta)_X"] \arrow[d, "GF(f)"] & G'F'(X) \arrow[d, "G'F'(f)"] \\ GF(Y) \arrow[r, "(\theta'\ominus\theta)_Y"'] & G'F'(Y) \end{tikzcd} .$$

From (a), we have

$$
\begin{aligned}
& H(f)\circ(\psi\odot\theta)_X && \text{(Assumption)}\\
=& H(f)\circ\psi_X\circ\theta_X && \text{(Def: longitudinal composition)}\\
=& (\psi_Y\circ G(f))\circ\theta_X && \text{(Property of natural transformation } \psi\text{)}\\
=& \psi_Y\circ\theta_Y\circ F(f) && \text{(Property of natural transformation } \theta\text{)}\\
=& (\psi\odot\theta)_X\circ F(f), && \text{(Def: longitudinal composition)}
\end{aligned}
$$

thus $(\psi\odot\theta)$ is natural transformation.

From (b), we have

$$
\begin{aligned}
& G'F'(f)\circ(\theta'\ominus\theta)_X && \text{(Assumption)}\\
=& G'F'(f)\circ\theta'_{F'(X)}\circ G(\theta_X) && \text{(Def: horizontal composition)}\\
=& \theta'_{F'(Y)}\circ GF'(f)\circ G(\theta_X) && \text{(Property of natural transformation } \theta'\text{)}\\
=& \theta'_{F'(Y)}\circ G(F'(f)\circ G(\theta_X)) && \text{(Property of functor } G\text{)}\\
=& \theta'_{F'(Y)}\circ G(\theta_Y\circ F(f)) && \text{(Property of natural transformation } \theta\text{)}\\
=& \theta'_{F'(Y)}\circ G(\theta_Y)\circ GF(f) && \text{(Property of functor } G\text{)}\\
=& (\theta'\ominus\theta)_Y\circ GF(f), && \text{(Def: horizontal composition)}
\end{aligned}
$$

thus $(\psi\ominus\theta)$ is natural transformation.

What's more, we can prove that $(\psi\odot\theta)\ominus(\psi'\odot\theta') = (\psi'\ominus\psi)\odot(\theta'\ominus\theta)$:

$$
\begin{aligned}
& ((\psi'\odot\theta')\ominus(\psi\odot\theta))_X && \text{(Assumption)}\\
=& (\psi'\odot\theta')_{H(X)}\circ F'(\psi\odot\theta)_X && \text{(Def: horizontal composition)}\\
=& \psi'_{H(X)}\circ\theta_{H(X)}\circ F'(\theta_X)\circ F'(\theta_X) && \text{(Def: longitudinal composition, Property of functor } F'\text{)}\\
=& \psi'_{H(X)}\circ(G'(\psi_X)\circ\theta'_{G(X)})\circ F'(\theta_X) && \text{(Commutative diagram (3))}\\
=& (\psi'\ominus\psi)_X\circ(\theta'\ominus\theta)_X && \text{(Def: horizontal composition)}\\
=& ((\psi'\ominus\psi)\odot(\theta'\ominus\theta))_X, && \text{(Def: longitudinal composition)}
\end{aligned}
$$

where $X\in\mathrm{Ob}(\mathcal{C})$. □

Theorem 2. *Both longitudinal and horizontal compositions of natural transformations satisfy associative law.*

Proof. For longitudinal composition, observe the following commutative diagram and natural transformations:
For

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend left=20, "G"] \arrow[r, bend right=20, "H"'] \arrow[r, bend right=60, "K"'] & \mathcal{D}
\end{tikzcd}
\quad \text{as well as} \quad
\begin{cases}
\theta : F\Rightarrow G,\\
\psi : G\Rightarrow H,\\
\phi : H\Rightarrow K,
\end{cases}
$$

it's trivial to prove that $((\phi\odot\psi)\odot\theta)_X = (\phi\odot(\psi\odot\theta))_X$ for all $X\in\mathrm{Ob}(\mathcal{C})$, thus the longitudinal composition satisfies associative law.
For

$$
\begin{tikzcd}
\mathcal{C}_1 \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}_2 \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{C}_3 \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "H'"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{C}_4,
\end{tikzcd}
$$

we have

$$
\begin{aligned}
& (\phi\ominus(\psi\ominus\theta))_X && \text{(Assumption)}\\
=& \phi_{G'F'(X)}\circ H(\psi\ominus\theta)_X && \text{(Def: horizontal composition)}\\
=& \phi_{G'F'(X)}\circ H(\psi_{F'(X)}\circ G(\theta_X)) && \text{(Ditto)}\\
=& \phi_{G'F'(X)}\circ H(\psi_{F'(X)})\circ HG(\theta_X) && \text{(Property of functor } H\text{)}\\
=& (\phi\ominus\psi)_{F'(X)}\circ HG(\theta_X) && \text{(Def: horizontal composition)}\\
=& ((\phi\ominus\psi)\ominus\theta)_X, && \text{(Ditto)}
\end{aligned}
$$

thus the horizontal composition satisfies associative law. □

Theorem 3. *Observe the following natural commutative:*

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "F"] & \mathcal{C}_2 \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}_3 \arrow[r, "K"] & \mathcal{C}_4, \end{tikzcd}$$

we have

$$\theta \odot \mathrm{id}^G = \theta = \mathrm{id}^H \odot \theta, \tag{4}$$

$$(\theta \ominus \mathrm{id}^F)_X = \theta_{F(X)}, \tag{5}$$

$$(\mathrm{id}^K \ominus \theta)_Y = K(\theta_Y), \tag{6}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1), Y \in \mathrm{Ob}(\mathcal{C}_2)$.

Proof. It's trivial to prove (4) and using definition of identity transformation and longitudinal composition.

For (5), we have

$$\begin{aligned} &(\theta \ominus \mathrm{id}^F)_X && \text{(Assumption)} \\ =&\theta_{F(X)} \circ G(\mathrm{id}^F_X) && \text{(Def: horizontal composition)} \\ =&\theta_{F(X)} \circ \mathrm{id}^G_{F(X)} && \text{(formula(2))} \\ =&\theta_{F(X)} && \text{(formula(4))} \end{aligned}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

For (6), we have

$$\begin{aligned} &(\mathrm{id}^K \ominus \theta)_Y && \text{(Assumption)} \\ =&K(\theta_Y) \circ \mathrm{id}^K_{G(Y)} && \text{(Def: horizontal composition)} \\ =&K(\theta_Y) \circ K(\mathrm{id}^G_Y) && \text{(formula(2))} \\ =&K(\theta_Y \circ \mathrm{id}^G_Y) && \text{(Property of functor } K\text{)} \\ =&K(\theta_Y) && \text{(formula(4))} \end{aligned}$$

for all $Y \in \mathrm{Ob}(\mathcal{C}_2)$. □