A Brief Introduction of Basic Category Theory

Fungus

August 18, 2020

These are my notes of basic category theory. The category theory which talk about in this text is based on *Frege-Hilbert first-order logic axiomatic system* and ZF *axiomatic set theory* (sometimes including *the Axiom of Choice*), as for *metacategory*, please see Definition:Metacategory - ProofWiki. The main sources of this text come from [11, 第二章].

Definition 1. A *category* $\mathcal{C}$ consists of:

- A collection $\mathrm{Ob}(\mathcal{C})$ of *objects* $X, Y, Z \cdots$
- A collection
$$\mathrm{Mor}(\mathcal{C}) = \bigsqcup_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{sou}_{\mathcal{C}}(X) = \bigsqcup_{Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{tar}_{\mathcal{C}}(Y) = \bigsqcup_{X,Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$
of *morphisms* $f, g, h \cdots$ with a binary operation "$\circ$" which is defined on the subclass of $\mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C})$, where $\mathrm{sou}_{\mathcal{C}}(X)$ is the *domain* of its elements as well as $\mathrm{tar}_{\mathcal{C}}(Y)$ is the *codomain* of its elements, and *homology class* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the intersection of both. What's more, we define $\mathrm{dom}_{\mathcal{C}}(f) = X$ and $\mathrm{cod}_{\mathcal{C}}(f) = Y$ if $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. [*] For $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \wedge g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, the binary operation that defines *composite morphism* $g \circ^{\mathcal{C}} f$ (which is abbreviated as $g \circ f$) satisfies:
  1. $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$;
  2. $\forall h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)[h \circ (g \circ f) = (h \circ g) \circ f]$;
  3. $\forall h \in \mathrm{Hom}_{\mathcal{C}}(Y, X) \exists \mathit{1}_X^{\mathcal{C}} \in \mathrm{Hom}_{\mathcal{C}}(X, X)[f \circ \mathit{1}_X^{\mathcal{C}} = f \wedge \mathit{1}_X^{\mathcal{C}} \circ h = h]$.

It's easy to verify that the *identity morphism* $\mathit{1}_X^{\mathcal{C}}$ (which is abbreviated as $\mathit{1}_X$) is unique for all $X \in \mathrm{Ob}(\mathcal{C})$. In addition, we abbreviate $\mathrm{Hom}_{\mathcal{C}}(X, X)$ as $\mathrm{End}_{\mathcal{C}}(X)$, it's easy to see that $\langle \mathrm{End}_{\mathcal{C}}(X), \circ, \mathit{1}_X \rangle$ is a monoid group.

In commutative diagram,

$$X \xrightarrow{f} Y \quad \text{means} \quad f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), \qquad \begin{tikzcd} X \arrow[d, "f"] \arrow[dr, "h"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd} \quad \text{means} \quad g \circ f = h.$$

Example 1. Here are some special examples.

1. Consider a category Rel in ZF:
   - Objects are all sets.
   - Homomorphism between any sets X, Y is the power set of binary relations $\mathscr{P}(X \times Y)$.
   - The composition of morphisms is the composition of binary relations.
   - Identity morphism $\mathit{1}_X$ is the identity mapping $\mathrm{id}_X = \{\langle x, x \rangle \mid x \in X\}$.

   Obviously, it's indeed a category. This example shows that the morphisms are not only mappings, they may have looser structures. Compared to this, morphisms are more like binary relations.

2. Consider a set S, we can use it to construct a *discrete category* $\mathsf{Disc}(S)$, which $\mathrm{Ob}(\mathsf{Disc}(S)) = S$ and $\mathrm{Mor}(\mathsf{Disc}(S)) = \{\mathit{1}_x | x \in S\}$. A category without any objects and morphisms is called *zero* category $\mathbf{0}$. A discrete category which has exactly one object is written as $\mathbf{1}$.

3. Consider a category $\mathcal{C}$ and its object I, the *slice* category $\mathcal{C}/I$ satisfies:
   - Objects are all the morphisms $f \in \mathrm{Mor}(\mathcal{C})$ which satisfy $f \in \mathrm{Hom}_{\mathcal{C}}(X, I)$;
   - $\mathrm{Hom}_{\mathcal{C}/I}(f, g) = \{j \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{dom}_{\mathcal{C}}(f), \mathrm{dom}_{\mathcal{C}}(g)) | \, g \circ^{\mathcal{C}} j = f\}$;

[*] Strictly speaking, the morphism is composed by a 3-tuple $\langle X, f, Y \rangle$, otherwise, it will cause confusion. For instance, in set category Set (see Example 3.1), if we don't discriminate the same mapping in different homology class, i.e., which have different codomains (such as $f : \{0\} \to \{0\}$ and $g : \{0\} \to \{0, 1\}$ which satisfy $f(0) = g(0) = 0$), then they are the same morphism, it will contradict the disjoint of homology class. Of course, for convenience, we will omit the 3-tuples when describing morphisms.

- $1_f^{\mathcal{C}/I} = 1_{\mathrm{dom}_{\mathcal{C}}(f)}^{\mathcal{C}}$;
- if $\mathrm{cod}_{\mathcal{C}/I}(f) = \mathrm{dom}_{\mathcal{C}/I}(g)$ then $k \circ^{\mathcal{C}/I} j = k \circ^{\mathcal{C}} j$.

It's easy to verify that $\mathcal{C}/I$ is indeed a category.

Similarly, we can define *coslice* category $I/\mathcal{C}$, whose the objects are $f \in \mathrm{Mor}(\mathcal{C})$ which satisfy $f \in \mathrm{Hom}_{\mathcal{C}}(I, X)$, and $\mathrm{Hom}_{I/\mathcal{C}}(f, g) = \{j \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{cod}_{\mathcal{C}}(f), \mathrm{cod}_{\mathcal{C}}(g)) | \, j \circ^{\mathcal{C}} f = g\}$.

Definition 2. $\mathcal{C}'$ is a *subcategory* of category $\mathcal{C}$ if:

- $\mathcal{C}'$ is a category;
- $\mathrm{Ob}(\mathcal{C}') \subseteq \mathrm{Ob}(\mathcal{C})$;
- $\forall X, Y \in \mathrm{Ob}(\mathcal{C}')[\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)]$;
- $\forall f, g \in \mathrm{Mor}(\mathcal{C}')[f \circ^{\mathcal{C}'} g = f \circ^{\mathcal{C}} g]$ (if $f \circ^{\mathcal{C}} g$ is defined);
- for all $X \in \mathrm{Ob}(\mathcal{C}')$, the identity morphism 1_X in $\mathcal{C}'$ is also that in $\mathcal{C}$.

In particular, if $\forall X, Y \in \mathrm{Ob}(\mathcal{C}')[\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)]$, then we say $\mathcal{C}'$ is the *full subcategory* of $\mathcal{C}$.

A *opposite* category $\mathcal{C}^{\mathrm{op}}$ of category $\mathcal{C}$ satisfies:

- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$;
- $\forall X, Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})[\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)]$, i.e., $[f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \iff f^{\mathrm{op}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y)]$;
- $\forall f, g \in \mathrm{Mor}(\mathcal{C})[g^{\mathrm{op}} \circ^{\mathrm{op}} f^{\mathrm{op}} = (f \circ^{\mathcal{C}} g)^{\mathrm{op}}]$;
- for all $X \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, the identity morphism 1_X in $\mathcal{C}^{\mathrm{op}}$ is also that in $\mathcal{C}$.

It's easy to verify that $\mathcal{C}$ is also a category, and we have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$. $\mathcal{C}^{\mathrm{op}}$ has the symmetric algebraic properties as $\mathcal{C}$.

A category $\mathcal{C}$ is called *small* if both $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets in ZFC but not proper class, [*] and *large* otherwise. A *locally small* category is a category such that for all objects X and Y, $\mathrm{Hom}(X, Y)$ is a set in ZFC, called a *homset*.

Definition 3. A *functor* $F : \mathcal{C} \to \mathcal{D}$, between category $\mathcal{C}$ and $\mathcal{D}$, consists the following data:

- Mapping $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
- Mapping $F : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, which satisfies:
  1. $F[\mathrm{Hom}_{\mathcal{C}}(X, Y)] \subseteq \mathrm{Hom}_{\mathcal{D}}(F(X), G(Y))$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$;
  2. For all $f, g \in \mathrm{Mor}(\mathcal{C})$, if $\mathrm{cod}_{\mathcal{C}}(f) = \mathrm{dom}_{\mathcal{C}}(g)$, then $F(g \circ^{\mathcal{C}} f) = F(g) \circ^{\mathcal{D}} F(f)$;
  3. $F(1_X^{\mathcal{C}}) = 1_{F(X)}^{\mathcal{D}}$ for all $X \in \mathrm{Ob}(\mathcal{C})$.

In commutative diagram,

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

means that F is the functor between $\mathcal{C}$ and $\mathcal{D}$.

For functors $F : \mathcal{C}_1 \to \mathcal{C}_2, G : \mathcal{C}_2 \to \mathcal{C}_3$, the composition $GF : \mathcal{C}_1 \to \mathcal{C}_3$ between both satisfies:

$$GF(X) = G(F(X)) \text{ and } GF(f) = G(F(f)) \text{ for all } X \in \mathrm{Ob}(\mathcal{C}_1), f \in \mathrm{Mor}(\mathcal{C}_1).$$

It's trivial to verify that the composition is also a functor and it satisfy associative law.

For any category $\mathcal{C}$, there exists a *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that satisfies

$$\mathrm{id}_{\mathcal{C}}(X) = X \text{ and } \mathrm{id}_{\mathcal{C}}(f) = f \text{ for all } X \in \mathrm{Ob}(\mathcal{C}_1), f \in \mathrm{Mor}(\mathcal{C}_1).$$

It's easy to verify that for all functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and $C : \mathcal{C} \to \mathcal{C}$, $FC = F \wedge CG = G$ if and only if $C = \mathrm{id}_{\mathcal{C}}$.

Definition 4. The *natural transformation* θ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a mapping from $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Mor}(\mathcal{D})$ whose each value satisfies $\theta_X := \theta(X) \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ and the commutative diagram below:

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"'] & G(Y), \end{tikzcd} \tag{1}$$

[*] "X is a set in ZFC" has two meanings: we can prove X exists in ZFC, i.e., $\mathsf{ZFC} \vdash_{\mathbf{H}} \exists X$; or the existence of X in ZFC is consistent with ZFC, i.e., $\vdash_{\mathbf{H}} \mathsf{Con}(\mathsf{ZFC}) \to \mathsf{Con}(\mathsf{ZFC} + \exists X)$, where $\mathbf{H}$ means the Frege-Hilbert first-order logic axiomatic system. The meaning in the text is the former. Of course, to prove the consistency, we often need to add extra axioms. The provability of ZFC is limited, so we can only define the set in the model $(V_\kappa, \in)$, where κ is the least strongly inaccessible cardinal, but that's enough. See [7] for more details.

where $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. In other words, we can record the above natural transformation as $\theta : F \Rightarrow G$, [*] or in such a commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{D}. \arrow[from=U, to=D, Rightarrow, "\theta"] \end{tikzcd}$$

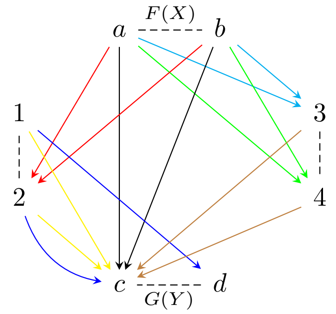
We may use symbol "$F(\theta)_X$" instead of "$F((\theta)_X)$" in some particular case (such as there are more than one symbols of natural transformations in the brackets).

For functor $F : \mathcal{C} \to \mathcal{D}$, there exists a *identity transformation* $\mathbf{id}^F : F \to F$ that satisfies $\forall X \in \mathrm{Ob}(\mathcal{C})[\mathbf{id}^F_X = \mathit{1}_{F(X)}]$.

Example 2. Consider two *finite* categories $\mathcal{C}, \mathcal{D}$ where $\mathrm{Ob}(\mathcal{C}) = \{X, Y\}$ and $\mathrm{Ob}(\mathcal{D}) = \{\{a, b\}, \{1, 2\}, \{3, 4\}, \{c, d\}\}$. There are three morphisms in $\mathcal{C}$: $\mathit{1}_X$, $\mathit{1}_Y$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Consider four functors $F, F', G, G' : \mathcal{C} \to \mathcal{D}$ such that

$$F(X) = \{a, b\} = F'(X), G(Y) = \{c, d\} = G'(Y).$$

And consider two natural transformations $\theta, \psi : F \Rightarrow G$, and all the morphisms (mappings) in $\mathcal{D}$ except identity morphisms are shown below:



Where the arrows with different colors mean the different mappings, balck arrows mean the morphism k, and the elements connected by one dashed line belong to the same set. There are 7 isomorphisms (except identity morphisms) in $\mathcal{D}$ in total, it's easy to see that $\mathcal{D}$ is indeed a category (we just need to verify that the compositions of any morphisms in $\mathcal{D}$ are also morphisms in it).

- Consider the following combination of objects and morphisms:

  $$G(X) = \{1, 2\} = G'(X), F(Y) = \{3, 4\} = F'(Y);$$

  $$\text{red:}\theta_X,\ \text{yellow:}G(f),\ \text{blue:}G'(f),\ \text{cyan:}F(f),\ \text{grenn:}F'(f),\ \text{brown:}\theta_Y.$$

  The four functors are indeed functors. What's more, it's trivial to verify that

  $$\theta_Y \circ F'(f) = \theta_Y \circ F(f) = k = G(f) \circ \theta_X = G'(f) \circ \theta_X,$$

  thus we know θ is indeed a natural transformation, and obviously θ have more than one "sources" and "targets".

- Consider the following combination of objects and morphisms:

  $$G(X) = \{3, 4\}, F(Y) = \{1, 2\};$$

  $$\text{red:}F(f),\ \text{yellow:}\theta_Y,\ \text{blue:}\psi_Y,\ \text{cyan:}\theta_X,\ \text{grenn:}\psi_X,\ \text{brown:}G(f).$$

  The two functors are indeed functors. What's more, it's trivial to verify that

  $$\theta_Y \circ F(f) = \psi_Y \circ F(f) = k = G(f) \circ \theta_X = G(f) \circ \psi_X,$$

  thus we know θ and ψ are indeed natural transformations between F and G.

These examples show us that one natural transformation can rely on different functors, and there may be different natural transformations "between" two functors. Hence it is necessary to label the natural transformations as 3-tuples.

Definition 5. For functors $F, G, H : \mathcal{C} \to \mathcal{D}$, natural transformations $\theta : F \Rightarrow G$ and $\psi : G \Rightarrow H$, the element of *vertical composition* of the natural transformations is defined as $(\psi \odot \theta)_X = \psi_X \circ \theta_X$. In commutative diagrams forms,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=70, "H"'{name=D}] & \mathcal{D} \arrow[from=U, to=M, Rightarrow, "\theta"] \arrow[from=M, to=D, Rightarrow, "\psi"] \end{tikzcd} \quad \text{means} \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "H"'{name=D}] & \mathcal{D}. \arrow[from=U, to=D, Rightarrow, "\psi\odot\theta"] \end{tikzcd}$$

Actually, we need to prove that the definition is well-defined, i.e., to verify that $(\psi \odot \theta)_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), H(X))$ for all $X \in \mathrm{Ob}(\mathcal{C})$, it's easy to do so.

[*] Like morphism, strictly speaking, functor $F : \mathcal{C} \to \mathcal{D}$ and natural transformation $\theta : F \to G$ are also composed by 3-tuples $\langle \mathcal{C}, F, \mathcal{D} \rangle$ and $\langle F, \theta, G \rangle$, rather than simple "mappings". For instance, consider a category $\mathcal{C}$ and its subcategory $\mathcal{C}'$, category $\mathcal{D}$, functor $F : \mathcal{D} \to \mathcal{C}'$, inclusion functor (see Example 4.1) $\iota : \mathcal{C}' \to \mathcal{C}$. Then the composition $\iota F : \mathcal{D} \to \mathcal{C}$ is different from single functor $F : \mathcal{D} \to \mathcal{C}'$, although they are the same if you regard them as mappings.

Definition 6. For functors $F, F' : \mathcal{C}_1 \to \mathcal{C}_2$ and $G, G' : \mathcal{C}_2 \to \mathcal{C}_3$, natural transformations $\theta : F \Rightarrow F'$ and $\psi : G \Rightarrow G'$, the element of *horizontal composition* of natural transformations $(\psi \ominus \theta)_X$ is defined as $G'(\theta_X) \circ \psi_{F(X)} = \psi_{F'(X)} \circ G(\theta_X)$. In commutative diagrams forms,

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}_2 \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{C}_3 \end{tikzcd} \quad \textit{means} \quad \begin{tikzcd} \mathcal{C}_1 \arrow[rr, bend left, "GF"] \arrow[rr, bend right, "G'F'"'] \arrow[rr, phantom, "\Downarrow\psi\ominus\theta"] & & \mathcal{C}_3, \end{tikzcd}$$

which satisfy

$$\begin{tikzcd} GF(X) \arrow[r, "G(\theta_X)"] \arrow[d, "\psi_{F(X)}"'] & GF'(X) \arrow[d, "\psi_{F'(X)}"] \\ G'F(Y) \arrow[r, "G'(\theta_X)"'] & G'F'(Y). \end{tikzcd} \tag{2}$$

Actually, we need to prove that the definition is well-defined, i.e., to verify the commutative diagram and that $(\psi \ominus \theta)_X \in \mathrm{Hom}_{\mathcal{C}_3}(GF(X), G'F'(X))$ for all $X \in \mathrm{Ob}(\mathcal{C}_1)$, it's easy to do so observing commutative diagram 1.

Theorem 1. The vertical and horizontal compositions of natural transformations are natural transformations, and the natural transformations satisfy the *interchange law* (formula 3).

Proof. For

$$\begin{tikzcd} \mathcal{C}_1 \arrow[rr, bend left=60, "F"] \arrow[rr, "G" description] \arrow[rr, bend right=60, "H"'] \arrow[rr, phantom, shift left=3, "\Downarrow\theta"] \arrow[rr, phantom, shift right=3, "\Downarrow\psi"] & & \mathcal{C}_2 \arrow[rr, bend left=60, "F'"] \arrow[rr, "G'" description] \arrow[rr, bend right=60, "H'"'] \arrow[rr, phantom, shift left=3, "\Downarrow\theta'"] \arrow[rr, phantom, shift right=3, "\Downarrow\psi'"] & & \mathcal{C}_3 \end{tikzcd},$$

we need to verify the following commutative diagrams:

$$\text{(a)} \quad \begin{tikzcd} F(X) \arrow[r, "(\psi\odot\theta)_X"] \arrow[d, "F(f)"'] & H(X) \arrow[d, "H(f)"] \\ F(y) \arrow[r, "(\psi\odot\theta)_X"'] & H(y) \end{tikzcd} \quad \text{and (b)} \quad \begin{tikzcd} GF(X) \arrow[r, "(\theta'\ominus\theta)_X"] \arrow[d, "GF(f)"] & G'F'(X) \arrow[d, "G'F'(f)"] \\ GF(Y) \arrow[r, "(\theta'\ominus\theta)_Y"'] & G'F'(Y) \end{tikzcd}.$$

From (a), we have

$$\begin{aligned} & H(f) \circ (\psi \odot \theta)_X \\ =& H(f) \circ \psi_X \circ \theta_X && \text{(Def: vertical composition)} \\ =& (\psi_Y \circ G(f)) \circ \theta_X && \text{(Property of natural transformation } \psi) \\ =& \psi_Y \circ \theta_Y \circ F(f) && \text{(Property of natural transformation } \theta) \\ =& (\psi \odot \theta)_X \circ F(f), && \text{(Def: vertical composition)} \end{aligned}$$

thus $(\psi \odot \theta)$ is natural transformation.

From (b), we have

$$\begin{aligned} & G'F'(f) \circ (\theta' \ominus \theta)_X \\ =& G'F'(f) \circ \theta'_{F'(X)} \circ G(\theta_X) && \text{(Def: horizontal composition)} \\ =& \theta'_{F'(Y)} \circ GF'(f) \circ G(\theta_X) && \text{(Property of natural transformation } \theta') \\ =& \theta'_{F'(Y)} \circ G(F'(f) \circ G(\theta_X)) && \text{(Property of functor } G) \\ =& \theta'_{F'(Y)} \circ G(\theta_Y \circ F(f)) && \text{(Property of natural transformation } \theta) \\ =& \theta'_{F'(Y)} \circ G(\theta_Y) \circ GF(f) && \text{(Property of functor } G) \\ =& (\theta' \ominus \theta)_Y \circ GF(f), && \text{(Def: horizontal composition)} \end{aligned}$$

thus $(\psi \ominus \theta)$ is natural transformation.

For interchange law

$$(\psi \odot \theta) \ominus (\psi' \odot \theta') = (\psi' \ominus \psi) \odot (\theta' \ominus \theta), \tag{3}$$

we can prove it in the below step:

$$
\begin{aligned}
&((\psi' \odot \theta') \ominus (\psi \odot \theta))_X \\
=&(\psi' \odot \theta')_{H(X)} \circ F'(\psi \odot \theta)_X && \text{(Def: horizontal composition)} \\
=&\psi'_{H(X)} \circ \theta_{H(X)} \circ F'(\theta_X) \circ F'(\theta_X) && \text{(Def: vertical composition, Property of functor } F'\text{)} \\
=&\psi'_{H(X)} \circ (G'(\psi_X) \circ \theta'_{G(X)}) \circ F'(\theta_X) && \text{(Commutative diagram 2)} \\
=&(\psi' \ominus \psi)_X \circ (\theta' \ominus \theta)_X && \text{(Def: horizontal composition)} \\
=&((\psi' \ominus \psi) \odot (\theta' \ominus \theta))_X, && \text{(Def: vertical composition)}
\end{aligned}
$$

where $X \in \mathrm{Ob}(\mathcal{C})$. □

Theorem 2. Both vertical and horizontal compositions of natural transformations satisfy associative law.

Proof. For vertical composition, observe the following commutative diagram and natural transformations:

For

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "F"{name=F}] \arrow[r, bend left=20, "G"{name=G, description}] \arrow[r, bend right=20, "H"{name=H, description}] \arrow[r, bend right=70, "K"'{name=K}] & \mathcal{D},
\arrow[from=F, to=G, Rightarrow, "\theta"]
\arrow[from=G, to=H, Rightarrow, "\psi"]
\arrow[from=H, to=K, Rightarrow, "\phi"]
\end{tikzcd}
$$

it's trivial to prove that $((\phi \odot \psi) \odot \theta)_X = (\phi \odot (\psi \odot \theta))_X$ for all $X \in \mathrm{Ob}(\mathcal{C})$, thus the vertical composition satisfies associative law.

For

$$
\begin{tikzcd}
\mathcal{C}_1 \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "F'"'{name=F1}] & \mathcal{C}_2 \arrow[r, bend left=50, "G"{name=G}] \arrow[r, bend right=50, "G'"'{name=G1}] & \mathcal{C}_3 \arrow[r, bend left=50, "H"{name=H}] \arrow[r, bend right=50, "H'"'{name=H1}] & \mathcal{C}_4,
\arrow[from=F, to=F1, Rightarrow, "\theta"]
\arrow[from=G, to=G1, Rightarrow, "\psi"]
\arrow[from=H, to=H1, Rightarrow, "\phi"]
\end{tikzcd}
\tag{4}
$$

we have

$$
\begin{aligned}
&(\phi \ominus (\psi \ominus \theta))_X \\
=&\phi_{G'F'(X)} \circ H(\psi \ominus \theta)_X && \text{(Def: horizontal composition)} \\
=&\phi_{G'F'(X)} \circ H(\psi_{F'(X)} \circ G(\theta_X)) && \text{(Ditto)} \\
=&\phi_{G'F'(X)} \circ H(\psi_{F'(X)}) \circ HG(\theta_X) && \text{(Property of functor } H\text{)} \\
=&(\phi \ominus \psi)_{F'(X)} \circ HG(\theta_X) && \text{(Def: horizontal composition)} \\
=&((\phi \ominus \psi) \ominus \theta)_X, && \text{(Ditto)}
\end{aligned}
$$

thus the horizontal composition satisfies associative law. □

Lemma 1 (ZF)**.** For any nonempty sets A, B and mapping $f : A \to B$, we have

$$\exists g : B \to A[g \circ f = \mathrm{id}_A] \iff f \text{ is a injection} \iff \forall C \neq \varnothing \forall h, h' : C \to A[f \circ h = f \circ h' \implies h = h'],$$

$$\exists g : B \to A[f \circ g = \mathrm{id}_B] \implies f \text{ is a surjection} \iff \forall C \neq \varnothing \forall h, h' : B \to C[h \circ f = h' \circ f \implies h = h'],$$

and f is a surjection $\implies \exists g : B \to A[f \circ g = \mathrm{id}_B]$ can be proved in using the Axiom of Choice. It's easy to see that f is a bijection if and only if it has both left and right inversal mappings, and obviously the two inversal mappings are the same one, which is unique.

Proof. The proofs are shown in [5, Theorem 3J] (left parts) and [4, (§2.1) Proposition 2.2 & Example 2.3] (right parts). □

Definition 7. Consider $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. If there exists a morphism $g \circ f = \mathit{1}_X$, then we say f is a *section*, and g is the *left inverse* of it; if $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ that $f \circ g = \mathit{1}_Y$, then we say f is a *retraction*, and g is the *right inverse* of it. If f has both inverses, then we say f is a *isomorphism*, it's easy to verify that the two inverses are the same one, which is unique as well, so we say $f^{-1} := g$ is the *inverse* (*inversal morphism*) of it. If there exists an isomorphism between two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, then we say they are *isomorphic* and record it as $X \overset{\mathrm{M}}{\simeq} Y$.

What's more, it's easy to verify that the composition of isomorphisms is also an isomorphism (see Theorem 3), so we can find that the collection of *automorphisms* $\mathrm{Aut}_{\mathcal{C}}(X) := \{f \in \mathrm{Hom}_{\mathcal{C}}(X, X) \mid f \text{ is an isomorphism}\}$ is a group $\langle \mathrm{Aut}_{\mathcal{C}}(X), \circ, \mathit{1}_X \rangle$. If f satisfies the *left cancellation law*: $\forall Z \in \mathrm{Ob}(\mathcal{C}) \forall g, h \in \mathrm{Hom}_{\mathcal{C}}(Z, X)[f \circ g = f \circ h \iff g = h]$, then we say f is *monic* (or f is a *monomorphism*); if f satisfies the *right cancellation law*: $\forall Z \in \mathrm{Ob}(\mathcal{C}) \forall g, h \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)[g \circ f = h \circ f \iff g = h]$, then we say f is *epic* (or f is a *epimorphism*). If f is monic as well as epic, then we say f is a *bimorphism*. We call a category *groupoid* if all the morphisms in it are isomorphisms, and call it is *balanced* if all the bimorphisms in $\mathrm{Mor}(\mathcal{C})$ are isomorphisms.

Consider a functor $F : \mathcal{C} \to \mathcal{D}$, we can define the inverse of functor in the same way: If there exists a functor $G : \mathcal{D} \to \mathcal{C}$ that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$, then we say F is an *isomorphism functor* between $\mathcal{C}$ and $\mathcal{D}$, and $F^{-1} := G$ is the unique inverse (*inversal functor*) of F. From Lemma 1 we know that F is isomorphic if and only if $F \restriction_{\mathrm{Ob}(\mathcal{C})} \to \mathrm{Ob}(\mathcal{D})$ and $F \restriction_{\mathrm{Mor}(\mathcal{C})} \to \mathrm{Mor}(\mathcal{D})$ are both bijection. If there exists an isomorphism functor between two categories $\mathcal{C}$ and $\mathcal{D}$, we say they are *isomorphic* and record it as $\mathcal{C} \overset{\mathrm{F}}{\simeq} \mathcal{D}$.

(Please see Corollary 1.1 first) Consider the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\theta \qquad \psi\Uparrow"] & \mathcal{D}. \end{tikzcd} \tag{5}$$

If $\psi \odot \theta = \mathbf{id}^F$ and $\theta \odot \psi = \mathbf{id}^G$, then we say θ is an *isomorphism transformation* between F and G, and $\theta^{-1} := \psi$ is the unique inverse (*inversal transformation*) of θ, we record it as $\theta : F \overset{\sim}{\Rightarrow} G$. If there exists an isomorphism transformation between two functors F and G, we say they are *isomorphic* and record it as $F \overset{\mathrm{T}}{\simeq} G$.

For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, if $GF \overset{\mathrm{T}}{\simeq} \mathrm{id}_{\mathcal{C}}$ and $FG \overset{\mathrm{T}}{\simeq} \mathrm{id}_{\mathcal{D}}$, then we say G is the *quasi-inverse* of F, and F is a *equivalence* between $\mathcal{C}$ and $\mathcal{D}$. If there exists an equivalence between two categories $\mathcal{C}$ and $\mathcal{D}$, we say they are *equivalent* and record it as $\mathcal{C} \sim \mathcal{D}$. We say categories $\mathcal{C}$ and $\mathcal{D}$ are *dual equivalent* if $\mathcal{C}^{\mathrm{op}} \sim \mathcal{D}$.

If there is no confusion in some context, we abbrviate $\overset{\mathrm{M}}{\simeq}, \overset{\mathrm{F}}{\simeq}, \overset{\mathrm{T}}{\simeq}$ as $\simeq$ and refer to "isomorphism", "isomorphism functor" and "isomorphism transformation" as "isomorphism" uniformly.

Corollary 1. Observe diagram 5, we have:

1. (For any θ, ψ and $\phi : F \Rightarrow F$) $\theta \odot \phi = \theta \wedge \phi \odot \psi = \psi$ if and only if $\phi = \mathbf{id}^F$.
2. θ is an isomorphism transformation if and only if θ_X is an isomorphism for each $X \in \mathrm{Ob}(\mathcal{C})$.
3. If θ is an isomorphism, then $(\theta^{-1})_X = (\theta_X)^{-1}$ for all $X \in \mathrm{Ob}(\mathcal{C})$, thus we abbreviate it as θ_X^{-1}.

Proof. It's easy to prove so using the definitions of identity transformation, vertical composition and identity morphism. □

Lemma 2. Observe commutative diagram 4, we have

$$\begin{aligned}
&G(\mathbf{id}_X^F) = \mathbf{id}_{F(X)}^G = \mathbf{id}_X^{GF}, && (6)\\
&(\psi \ominus \mathbf{id}^F)_X = \psi_{F(X)}, && (7)\\
&(\mathbf{id}^H \ominus \psi)_Y = H(\psi_Y), && (8)\\
&\mathbf{id}^G \ominus \mathbf{id}^F = \mathbf{id}^{GF}, && (9)\\
&\text{if } \psi \text{ and } \phi \text{ are isomorphisms, then } \phi \ominus \psi \text{ is also, and } (\phi \ominus \psi)^{-1} = \phi^{-1} \ominus \psi^{-1}, && (10)\\
&F \simeq F' \implies [G \simeq G' \implies GF \simeq G'F'] \wedge [H \simeq H' \implies \wedge HG \simeq H'G'], && (11)
\end{aligned}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1), Y \in \mathrm{Ob}(\mathcal{C}_2)$.

Proof. For (6), using the definition of identity morphism, properties of morphism and functor we have

$$\mathbf{id}_X^{GF} = 1_{GF(X)} = G(1_{F(X)}) = \mathbf{id}_{F(X)}^G = G(1_{F(X)}) = G(\mathbf{id}_X^F)$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

For (7), we have

$$\begin{aligned}
&(\psi \ominus \mathbf{id}^F)_X \\
=&\psi_{F(X)} \circ G(\mathbf{id}_X^F) && \text{(Def: horizontal composition)}\\
=&\psi_{F(X)} \circ \mathbf{id}_{F(X)}^G && \text{(Formula 6)}\\
=&\psi_{F(X)} && \text{(Corollary 1.1)}
\end{aligned}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

For (8), we have

$$\begin{aligned}
&(\mathbf{id}^H \ominus \psi)_Y \\
=&H(\psi_Y) \circ \mathbf{id}_{G(Y)}^H && \text{(Def: horizontal composition)}\\
=&H(\psi_Y) \circ H(\mathbf{id}_Y^G) && \text{(Formula 6)}\\
=&H(\psi_Y \circ \mathbf{id}_Y^G) && \text{(Property of functor } H\text{)}\\
=&H(\psi_Y) && \text{(Corollary 1.1)}
\end{aligned}$$

for all $Y \in \mathrm{Ob}(\mathcal{C}_2)$.

For (9), we have

$$\begin{aligned}
&(\mathbf{id}^G \ominus \mathbf{id}^F)_X \\
=&G(\mathbf{id}^F_X) \circ \mathbf{id}^G_{F(X)} && \text{(Def: horizontal composition)} \\
=&\mathbf{id}^{GF}_X \circ \mathbf{id}^{GF}_X && \text{(Corollary 1.1)} \\
=&\mathbf{id}^{GF}_X && \text{(Formula 6)}
\end{aligned}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

For (10), we have

$$\begin{aligned}
&(\phi \ominus \psi) \odot (\phi^{-1} \ominus \psi^{-1}) \\
=&(\phi^{-1} \odot \phi) \ominus (\psi^{-1} \odot \psi) && \text{(Interchange law)} \\
=&\mathbf{id}^H \ominus \mathbf{id}^G && \text{(Property of inverse)} \\
=&\mathbf{id}^{HG}. && \text{(Formula 9)}
\end{aligned}$$

Similarly, we can prove $(\phi^{-1} \ominus \psi^{-1}) \odot (\phi \ominus \psi) = \mathbf{id}^{H'G'}$.

For (11), we suppose that the three natural transformations are all isomorphisms, we claim that $\psi \ominus \theta : GF \overset{\sim}{\Rightarrow} G'F'$ and $\phi \ominus \psi : HG \overset{\sim}{\Rightarrow} H'G'$, it's trivial to prove using formula 10. $\square$

Theorem 3. If two particular morphisms/functors/transformations are isomorphic, than the isomorphism between them is unique. The composition of any morphisms/functors/transformations which are isomorphisms is an isomorphism, and the composition of any equivalences are equivalence. Therefore isomorphic objects, categories, functors and equivalent categories satisfy transitivity.

Proof. The uniqueness is trivial to prove.

Because the compositions of isomorphic morphisms, functors and transformations have some similar algebraic properties, we just need to prove the composition of isomorphic morphisms is an isomorphism:
Suppose isomorphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, we claim that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X,Z)$:

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ 1_Y \circ g^{-1} = g \circ g^{-1} = 1_Z,$$

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ 1_Y \circ f = f^{-1} \circ f = 1_X.$$

Observe the following diagram:

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{C}_1} \arrow[r, phantom, "\circlearrowright"] & \mathcal{C}_1 \arrow[r, bend left, "F"] & \mathcal{C}_2 \arrow[l, bend left, "F'"] \arrow[r, bend left, "G"] & \mathcal{C}_3 \arrow[l, bend left, "G'"] \arrow[r, phantom, "\circlearrowleft"] & \mathrm{id}_{\mathcal{C}_3}.
\end{tikzcd}$$

We need to prove that if F and G are equivalence then GF is also, we now assume that $F'F \simeq \mathrm{id}_{\mathcal{C}_1}$, $FF' \simeq \mathrm{id}_{\mathcal{C}_2}$, $G'G \simeq \mathrm{id}_{\mathcal{C}_2}$ and $GG' \simeq \mathrm{id}_{\mathcal{C}_3}$. Using formula 11 we have

$$\mathrm{id}_{\mathcal{C}_2} \simeq G'G \Longrightarrow \mathrm{id}_{\mathcal{C}_1} \simeq F'F = F'\mathrm{id}_{\mathcal{C}_2}F \simeq F'(G'G)F = (F'G')(GF),$$

$$\mathrm{id}_{\mathcal{C}_2} \simeq F'F \Longrightarrow \mathrm{id}_{\mathcal{C}_3} \simeq GG' = G\mathrm{id}_{\mathcal{C}_2}G' \simeq G(FF')G' = (GF)(F'G').$$

Using transitivity of isomorphic functors, we have $\mathrm{id}_{\mathcal{C}_1} \simeq (F'G')(GF)$ and $\mathrm{id}_{\mathcal{C}_3} \simeq (GF)(F'G')$. Thus GF is equivalence. $\square$

Corollary 2. If functors G, G' are quasi-inverses of equivalence $F : \mathcal{C} \to \mathcal{D}$, then $G \simeq G'$.

Proof. Using formula 11, we have

$$G'F \simeq \mathrm{id}_{\mathcal{C}} \wedge FG \simeq \mathrm{id}_{\mathcal{D}} \Longrightarrow G' = G'\mathrm{id}_{\mathcal{D}} \simeq G'(FG) = (G'F)G \simeq \mathrm{id}_{\mathcal{C}}G = G,$$

thus we have $G' \simeq G$ using transitivity of isomorphic functors. $\square$

Corollary 3. Consider two morphisms f and g in $\mathcal{C}$ which satisfy $\mathrm{cod}_{\mathcal{C}}(f) = \mathrm{dom}_{\mathcal{C}}(g)$.

1. Every section is monic, and every retraction is epic.

2. The composition of monomorphisms is monic, and the composition of epimorphisms is epic.

3. If $g \circ f$ is monic then f is monic, if $g \circ f$ is epic then g is epic.

4. The following propositions are equivalent:

   - f is an isomorphism.
   - f is a monomorphism as well as a retraction.

- f is an epimorphism as well as a section.

Proof. The proofs are trivial, the details are shown in [12, §1.4]. $\square$

Example 3. There are some examples of isomorphisms.

1. Consider two monomorphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, if f and g are *factor through* each other each other, i.e., $\exists i \in \mathrm{Hom}_{\mathcal{C}}(X, Y)\exists j \in \mathrm{Hom}_{\mathcal{C}}(Y, X)[f = g \circ i \wedge g = f \circ j]$, then the *factors* i and j are both isomorphisms and they are inverses to each other. In commutative diagram:

$$\begin{tikzcd} X \arrow[rr, shift right, "i"'] \arrow[dr, "f"'] & & Y \arrow[ll, shift right, "j"'] \arrow[dl, "g"] \\ & Z. & \end{tikzcd}$$

2. *Set category* Set (which the objects are all the sets, and morphisms are all the mappings), *group category* Grp (which the objects are all the groups, and morphisms are all the group homomorphisms), *vector space category* $\mathsf{Vect}(\Bbbk)$ (which the objects are all the vector spaces generated by field $\Bbbk$ and any abelian groups V, and the morphisms are all the linear mappings) are all balanced category.
However, consider a morphism, an inclusion ring homomorphism $f : \mathbb{Z} \to \mathbb{Q}$, in *ring category* Rng (which the objects are all the rings and morphisms are all the ring homomorphisms). We will find that it is a epimorphism but not a retraction, because there is no inverse of it in $\mathrm{Mor}(\mathsf{Rng})$. So Rng isn't a balanced category. Actually, the *topology category* Top (which the objects are all the topology spaces, and the morphisms are all the continuous mappings) is not neither.

Definition 8. For functor $F : \mathcal{C} \to \mathcal{D}$, we define:

- F is *essentially surjective* if $\forall Y \in \mathrm{Ob}(\mathcal{D})\exists X \in \mathrm{Ob}(\mathcal{C})[F(X) \overset{\mathrm{M}}{\simeq} F(Y)]$.
- F is *faithful* if for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, $F \upharpoonright_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective.
- F is *full* if for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, $F \upharpoonright_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective.

Example 4. Observe the following example.

1. For category $\mathcal{C}$ and its subcategory $\mathcal{C}'$, there exists an *inclusion functor* $\iota := \mathrm{id}_{\mathcal{C}}\upharpoonright_{\mathcal{C}'} : \mathcal{C}' \to \mathcal{C}$. It's obviously faithful, and ι is full if and only if $\mathcal{C}'$ is a full subcategory.

2. There are some *forgetful functors* such as $\mathsf{Set} \to \mathsf{Rel}$, $\mathsf{Grp} \to \mathsf{Set}$, $\mathsf{Top} \to \mathsf{Set}$, $\mathsf{Ab} \to \mathsf{Grp}$ (where Ab means *abelian group category*, which the objects are all the abelian groups, and the morphisms are all the abelian group homomorphisms), $\mathsf{Vect}(\Bbbk) \to \mathsf{Ab}$ (which forget field $\Bbbk$) that has a feature, that is, they forget some (order, algebraic, topological, etc.) structures. Except functor $\mathsf{Top} \to \mathsf{Set}$ (it is faithful as well as full because of discrete spaces), the other functors above are all faithful but not full.

Lemma 3. Consider a functor $F : \mathcal{C} \to \mathcal{D}$, for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ and morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we have the following propositions:

1. $F(f)$ is a section/retraction if f is a section/retraction; and the left/right inverse of $F(f)$ is $F(g)$, where g is the left/right inverse of f.

2. When F is faithful and full, we have f is a section/retraction if $F(f)$ is a section/retraction; and the left/right inverse of f is g, where $F(g)$ is the left/right inverse of $F(f)$.

3. $X \simeq Y \implies F(X) \simeq F(Y)$; if F is faithful and full, we have $F(X) \simeq F(Y) \implies X \simeq Y$.

4. The composition of faithful/full/essentially-surjective functors is also faithful/full/essentially-surjective.

Proof. Proof of **Proposition 1** is trivial.

For **Proposition 2**, we only suppose that $F(f)$ is a section, the proof that f is a retraction is similar: Because $F(f)$ has a left inverse in $\mathrm{Hom}_{\mathcal{C}}(F(Y), F(X))$ and F is full, we have $F \upharpoonright_{\mathrm{Hom}_{\mathcal{C}}(Y,X)} \to \mathrm{Hom}_{\mathcal{C}}(F(Y), F(X))$ is surjective, so there exists $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ that $F(g)$ is the left inverse of $F(f)$, that is,

$$F(1_X^{\mathcal{C}}) = 1_{F(X)}^{\mathcal{D}} = F(g) \circ^{\mathcal{D}} F(f) = F(g \circ^{\mathcal{C}} f).$$

Because F is faithful, that means, $F \upharpoonright_{\mathrm{Hom}_{\mathcal{C}}(X,X)}$ is injective, then we have $1_X^{\mathcal{C}} = g \circ^{\mathcal{C}} f$, thus f is a section.

Proof of **Proposition 3** is trivial using Propositions 1 and 2.

For **Proposition 4**, it's trivial to prove that it is faithful of full, we now show that F is essentially surjective: Consider $G : \mathcal{D} \to \mathcal{E}$ is also essentially surjective, then we have

$$\forall e \in \mathrm{Ob}(\mathcal{E})\exists d \in \mathrm{Ob}(\mathcal{D})[G(d) \simeq e].$$

What's more, there exists c in $\mathcal{C}$ that $F(c) \simeq d$, using proposition 3 we have $GF(c) \simeq G(d) \simeq e$. Thus we have $GF(c) \simeq e$ by the transitivity of equivalence. $\square$

Definition 9. A full subcategory $\mathcal{C}'$ of category $\mathcal{C}$ is a *skeleton* of $\mathcal{C}$ if $\forall X \in \mathrm{Ob}(\mathcal{C})\exists! Y \in \mathrm{Ob}(\mathcal{C}')[X \overset{\mathrm{M}}{\simeq} Y]$. If $\mathcal{C}$ is the skeleton of itself, then we say it's a *skeletal* category.

Lemma 4. For category $\mathcal{C}$, we have:

1. Suppose the Axiom of Choice, [*] we have every nonempty category has at least one skeleton.
2. Every inclusion functor $\iota : \mathcal{C}' \to \mathcal{C}$ is an equivalence, where $\mathcal{C}'$ is a skeleton of $\mathcal{C}$; and we can find a quasi-inverse called *skeletal functor* $\kappa : \mathcal{C} \to \mathcal{C}'$ of ι, which gives the skeleton of $\mathcal{C}$.
3. Any two skeletons of a category are equivalent.
4. $\mathcal{C}$ is a skeletal category if and only if, for all the isomorphism $f \in \mathrm{Mor}(\mathcal{C})$, $\mathrm{dom}(f) = \mathrm{cod}(f)$.
5. Every faithful, full and essentially surjective functor between two skeletal categories is an isomorphism.

Proof. Assume two categories $\mathcal{C}$, $\mathcal{D}$ and a functor $F : \mathcal{C} \to \mathcal{D}$.

1. Because the isomorphism relation among objects satisfies reflexivity, symmetry and transitivity, we can divide them into an equivalence (not functor, but a simple relation) class $\mathcal{C}/\overset{\mathrm{M}}{\simeq}$. Using the Axiom of Choice, we can construct a choice function which selects representative element in each $[X]$ for all $X \in \mathrm{Ob}(\mathcal{C})$. By preserving the representative elements and all the morphisms between any two of them, they form a full subcategory $\mathcal{C}'$ of category $\mathcal{C}$, and it's easy to see that $\mathcal{C}'$ is the skeleton of $\mathcal{C}$.

2. Construct:

   - $\kappa \restriction_{\mathrm{Ob}(\mathcal{C})}$ is the choice function mentioned in Proposition 1.
   - A natural transformation (we will verify this) $\theta : \mathrm{id}_{\mathcal{C}} \Rightarrow \iota\kappa$ such that $\theta_X \in \mathrm{Hom}_{\mathcal{C}}(X, \iota\kappa(X))$ is the isomorphism (it's uniquely exists) for all $X \in \mathrm{Ob}(X)$.
   - For all $X, Y \in \mathrm{Ob}(X, Y)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, because θ_X is an isomorphism, we have

   $$\kappa(X) = \iota\kappa(X) \xrightarrow{\theta_X^{-1}} \mathrm{id}_{\mathcal{C}}(X) \xrightarrow{\mathrm{id}_{\mathcal{C}}(f)} \mathrm{id}_{\mathcal{C}}(Y) \xrightarrow{\theta_Y} \iota\kappa(Y) = \kappa(Y),$$

   then we can define $\kappa(f) := \theta_Y \circ f \circ \theta_X^{-1} \in \mathrm{Hom}_{\mathcal{C}'}(\kappa(X), \kappa(Y))$.

   Verify:

   - κ is indeed a functor, i.e., it satisfies the three factors of functor on morphisms. It's trivial.
   - θ is indeed a natural transformation , i.e., we have the following commutative diagram:

   $$\begin{tikzcd}
   \mathrm{id}_{\mathcal{C}}(X) \arrow[r, "\theta_X"] \arrow[d, "\mathrm{id}_{\mathcal{C}}(f)=f"'] & \iota\kappa(X) \arrow[d, "\iota\kappa(f)=\kappa(f)"] \\
   \mathrm{id}_{\mathcal{C}}(Y) \arrow[r, "\theta_Y"'] & \iota\kappa(Y).
   \end{tikzcd}$$

   We can easily verify the diagram by substituting the definition of $\kappa(f)$.
   - $\theta : id_{\mathcal{C}} \overset{\sim}{\Rightarrow} \iota\kappa$, i.e., θ is an isomorphism, obviously.

   Thus $\mathrm{id}_{\mathcal{C}} \simeq \iota\kappa$. On the other hand, because

   $$\forall X \in \mathrm{Ob}(\mathcal{C}')[\kappa\iota(X) = X = \mathrm{id}_{\mathcal{C}'}(X)] \wedge \forall f \in \mathrm{Mor}(\mathcal{C}')[\kappa\iota(f) = f = \mathrm{id}_{\mathcal{C}'}(f)],$$

   then we have $\kappa\iota = \mathrm{id}_{\mathcal{C}'}$, that implies $\kappa\iota \simeq \mathrm{id}_{\mathcal{C}'}$. Thus $\mathcal{C}' \sim \mathcal{C}$.

3. It's easy to prove so using Proposition 2 and the transitivity of equivalence (functor).

4. We can easily prove so by assuming contradiction.

5. Suppose $\mathcal{C}$ and $\mathcal{D}$ are skeletal categories, and functor $F : \mathcal{C} \to \mathcal{D}$ is faithful, full and essentially surjective.

   - $F \restriction_{\mathrm{Ob}(\mathcal{C})} \to \mathrm{Ob}(\mathcal{D})$ **is surjective:** Because F is essentially surjective, for all $Y \in \mathrm{Ob}(\mathcal{D})$ there exists $X \in \mathrm{Ob}(\mathcal{C})[F(X) \simeq Y]$. Since $\mathcal{D}$ is a skeletal category, through Proposition 4, we know that $F(X) = Y$.
   - $F \restriction_{\mathrm{Ob}(\mathcal{C})} \to \mathrm{Ob}(\mathcal{D})$ **is injective:** For any $X, Y \in \mathrm{Ob}(\mathcal{C})$, because of faithful and full functor F, if $F(X) = F(Y)$, then we have $X \simeq Y$ using Lemma 3.3. Since $\mathcal{D}$ is a skeletal category, through Proposition 4, we have $X = Y$.

[*] The Axiom of Choice in ZFC provides choice functions based on sets, but not proper classes, so the categories in this proposition refer to the small categories. If you want this proposition to be true in any large categories, you need stronger axiom of choice, which is based on any class. However, it can't be discussed in ZFC.

- $F\restriction_{\mathrm{Mor}(\mathcal{C})}\to \mathrm{Mor}(\mathcal{D})$ **is bijective:** Since F is full and faithful, we have $F\restriction_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}\to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ is bijective for each $X,Y\in \mathrm{Ob}(\mathcal{C})$. What's more, because $F\restriction_{\mathrm{Ob}(\mathcal{C})}\to \mathrm{Ob}(\mathcal{D})$ is bijective, it's easy to prove that $F\restriction_{\mathrm{Mor}(\mathcal{C})}\to \mathrm{Mor}(\mathcal{D})$ is bijective, too.

Because $F\restriction_{\mathrm{Ob}(\mathcal{C})}\to \mathrm{Ob}(\mathcal{D})$ and $F\restriction_{\mathrm{Mor}(\mathcal{C})}\to \mathrm{Mor}(\mathcal{D})$ are bijective, F is an isomorphism functor.

□

Theorem 4. A functor between two categories is an equivalence if and only if it's faithful, full and essentially surjective.

Proof. Let $F:\mathcal{C}\to\mathcal{D}$ be a functor.

($\Longrightarrow$) Let F be a equivalence and $G:\mathcal{D}\to\mathcal{C}$ be its quasi-inverse, so that there exist isomorphism transformations $\theta : GF \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{C}}$ and $\psi : FG \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$, and we assume that $X,Y\in \mathrm{Ob}(\mathcal{C})$.

Notice the following commutative diagram:

$$\text{(a)}\quad \begin{tikzcd} GF(X) \arrow[r, "\theta_X"] \arrow[d, "GF(f)=GF(g)"'] & \mathrm{id}_{\mathcal{C}}(X) \arrow[d, bend right, "\mathrm{id}_{\mathcal{C}}(g)"'] \arrow[d, bend left, "\mathrm{id}_{\mathcal{C}}(f)"] \\ GF(Y) \arrow[r, "\theta_Y"'] & \mathrm{id}_{\mathcal{C}}(Y) \end{tikzcd} \quad\text{and (b)}\quad \begin{tikzcd} GF(X) \arrow[r, "\theta_X"] \arrow[d, "GF(t)"'] & \mathrm{id}_{\mathcal{C}}(X) \arrow[d, "\mathrm{id}_{\mathcal{C}}(t)=t"] \\ GF(Y) \arrow[r, "\theta_Y"'] & \mathrm{id}_{\mathcal{C}}(Y). \end{tikzcd}$$

- F **is faithful:** Let $f,g\in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $F(f)=F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$, so $GF(f)=GF(g)$. Since $\theta : GF \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{C}}$, we have diagram (a), that means

$$\mathrm{id}_{\mathcal{C}}(g)\circ\theta_X=\theta_Y\circ GF(g)=\theta_Y\circ GF(f)=\mathrm{id}_{\mathcal{C}}(f)\circ\theta_X.$$

  Because θ is an isomorphism transformation, using Corollary 1.2 we have θ_X is an isomorphism, so it's a bimorphism (by Corollary 3.1), then we can cancel it. Thus we have $g=\mathrm{id}_{\mathcal{C}}(g)=\mathrm{id}_{\mathcal{C}}(f)=f$. So $F\restriction_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}\to \mathrm{Hom}_{\mathcal{C}}(F(X),F(Y))$ is injective, that means F is faithful. For future reference, we observe that a similar proof shows that G is faithful as well.

- F **is full:** For all $\varphi\in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$, we have $G(\varphi)\in \mathrm{Hom}_{\mathcal{D}}(GF(X),GF(Y))$. Since θ is an isomorphism, we have

$$X=\mathrm{id}_{\mathcal{C}}(X)\xrightarrow{\theta_X^{-1}} GF(X)\xrightarrow{G(\varphi)} GF(Y)\xrightarrow{\theta_Y}\mathrm{id}_{\mathcal{C}}(Y)=Y.$$

  so we can assume that $t=\theta_Y\circ G(\varphi)\circ\theta_X^{-1}\in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, we claim that $F(t)=\varphi$, so that $F\restriction_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}\to \mathrm{Hom}_{\mathcal{C}}(F(X),F(Y))$ is surjective, that means F is full:
  It's easy to verify the diagram (b), thus we have

$$\theta_Y\circ G(\varphi)=\theta_Y\circ G(\varphi)\circ\theta_X^{-1}\circ\theta_X=t\circ\theta_X=\theta_Y\circ GF(t).$$

  It's obvious that θ_Y is a bimorphism, then we can cancel it, thus we have $G(\varphi)=GF(t)$. Because φ and $F(t)$ are in the same homology set, and G is faithful, we have $F(t)=\varphi$.

- F **is essentially surjective:** For any $W\in \mathrm{Ob}(\mathcal{D})$, since $\psi : FG \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$, we have $\psi_W\in \mathrm{Hom}_{\mathcal{D}}(FG(W),\mathrm{id}_{\mathcal{D}}(W))$ is an isomorphism, so $F(G(W))\overset{\mathrm{M}}{\simeq}\mathrm{id}_{\mathcal{D}}(W)=W$. Hence, F is essentially surjective.

($\Longleftarrow$) (Need the Axiom of Choice) Consider the skeleton $\mathcal{C}'$ of category $\mathcal{C}$, the skeleton $\mathcal{D}'$ of category $\mathcal{D}$, inclusion functors $\iota_{\mathcal{C}}:\mathcal{C}'\to\mathcal{C}$ and $\iota_{\mathcal{D}}:\mathcal{D}'\to\mathcal{D}$, skeletal functors $\kappa_{\mathcal{C}}:\mathcal{C}\to\mathcal{C}'$ and $\kappa_{\mathcal{D}}:\mathcal{D}\to\mathcal{D}'$ (see the proof of Lemma 4.2 for the the methods of construction). We define $F':=\kappa_{\mathcal{D}}F\iota_{\mathcal{C}}:\mathcal{C}'\to\mathcal{D}'$.

Through Lemma 4.2 we know that $\kappa_{\mathcal{C}}$ and $\kappa_{\mathcal{D}}$ are equivalence, so they are faithful, full and essentially surjective (we just proved this). Because F and $\iota_{\mathcal{C}}$ are faithful, full and essentially surjective as well, the functor F' between skeletal categories is, too (by Lemma 3.4). Using Lemma 4.5 we have F' is an isomorphism functor. Suppose $G:=\iota_{\mathcal{C}}{F'}^{-1}\kappa_{\mathcal{D}}:\mathcal{D}\to\mathcal{C}$, we claim that G is the quasi-inverse of F:

$$GF=\underbrace{\underline{(\iota_{\mathcal{C}}{F'}^{-1}\kappa_{\mathcal{D}})F}\mathrm{id}_{\mathcal{C}}\simeq \underline{\iota_{\mathcal{C}}{F'}^{-1}(\kappa_{\mathcal{D}}F}\iota_{\mathcal{C}})\kappa_{\mathcal{C}}}_{\text{since } \mathrm{id}_{\mathcal{C}}\simeq\iota_{\mathcal{C}}\kappa_{\mathcal{C}}}=\iota_{\mathcal{C}}{F'}^{-1}F'\kappa_{\mathcal{C}}=\iota_{\mathcal{C}}\kappa_{\mathcal{C}}\simeq\mathrm{id}_{\mathcal{C}},$$

$$FG=\underbrace{\mathrm{id}_{\mathcal{D}}\underline{F(\iota_{\mathcal{C}}{F'}^{-1}\kappa_{\mathcal{D}})}\simeq\iota_{\mathcal{D}}(\kappa_{\mathcal{D}}\underline{F\iota_{\mathcal{C}}){F'}^{-1}\kappa_{\mathcal{D}}}}_{\text{since } \mathrm{id}_{\mathcal{D}}\simeq\iota_{\mathcal{D}}\kappa_{\mathcal{D}}}=\iota_{\mathcal{D}}F'{F'}^{-1}\kappa_{\mathcal{D}}=\iota_{\mathcal{D}}\kappa_{\mathcal{D}}\simeq\mathrm{id}_{\mathcal{D}}.$$

Hence, F is an equivalence. □

Corollary 4. Two skeletal categories are equivalent if and only if they are isomorphic, therefore two categories are equivalent if and only if they have isomorphic skeletons.

Proof. For the first half of the proposition, it's easy to prove so using Theorem 4 and Lemma 4.5. For the last half of the proposition, it's easy to prove so using the first half of the proposition, Lemma 4.3 and the transitivity of equivalence. □

References

[1] Category theory: Lecture notes and online books - logic matterslogic matters. https://www.logicmatters.net/categories/.

[2] Comma category - wikipedia. https://en.wikipedia.org/wiki/Comma_category.

[3] Arjuna. 类型论驿站写作计划-知乎. https://zhuanlan.zhihu.com/p/32182423. 范畴论学习笔记.

[4] Steve Awodey. *Category Theory*, volume 52 of *Oxford Logic Guides*. Oxford University Press, New York, 2010. Second Edition. ISBN: 978-0-19-958736-0. URL: https://b-ok.cc/book/1299760/dc958a.

[5] Herbert B. Enderton. *Elements of Set Theory*. Academic Press, New York, 1977. ISBN: 0-12-238440-7. URL: https://b-ok.cc/book/1247629/9b7166.

[6] Randall R. Holmes. Category theory. URL: https://web.auburn.edu/holmerr/8970/Textbook/CategoryTheory.pdf, October 2019.

[7] Thomas Jech. *Set Theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The Third Millennium Edition, revised and expanded. ISBN: 3-540-44085-2. URL: https://lost-contact.mit.edu/afs/adrake.org/usr/rkh/Books/books/Jech%20-%20Set%20theory.pdf.

[8] Serge Lang. *Algebra*, volume 211 of *Graudate Texts in Mathematics*. Springer-Verlag, New York, 2002. Revised Third Edition. ISBN: 0-387-95385-X. URL: https://math24.files.wordpress.com/2013/02/algebra-serge-lang.pdf.

[9] Tom Leinster. *Basic Category Theory*, volume 143 of *Cambridge Studies in Advanced Mathmatics*. Cambridge University Press, Cambridge, 2014. ISBN: 978-1-107-04424-1. URL: https://b-ok.cc/book/2470627/243ab6.

[10] Rmily Riehl. Category theory in context. URL: https://b-ok.cc/book/2778051/bff65a, 2016.

[11] 李文威. *代数学方法（第一卷）：基础架构*, volume 67 of *现代数学基础*. 高等教育出版社, 北京, 2019. 第1版. ISBN: 9787040507256. URL: https://github.com/wenweili/AlJabr-1.

[12] 贺伟. *范畴论*. 科学出版社, 北京, 2006. ISBN: 7-03-017096-2. URL: https://vdisk.weibo.com/s/zi1L5by5mdx5j.