A Brief Introduction of Basic Category Theory

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These are my notes of basic category theory. The category theory which talk about in this text is based on *Frege-Hilbert first-order logic axiomatic system* and ZF *axiomatic set theory* (sometimes including *the Axiom of Choice* (AC)), as for *metacategory*, please see Def:Metacategory - ProofWiki. The main sources of this text come from [18, 第二章].

The newest edition of this note (including `pdf.` file and the source code) can be downloaded from `https://github.com/Fungus-00/Mathematical-Notes/`. This note is seriously unfinished, for reference only.

Definition 1. A *category* $\mathcal{C}$ consists of:

- A collection $\mathrm{Ob}(\mathcal{C})$ of *objects* $X, Y, Z \cdots$
- A collection
$$\mathrm{Mor}(\mathcal{C}) = \bigsqcup_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{scr}_{\mathcal{C}}(X) = \bigsqcup_{Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{tar}_{\mathcal{C}}(Y) = \bigsqcup_{X,Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X,Y)$$
of *morphisms* $f, g, h \cdots$ with a binary operation "$\circ$" which is defined on the subclass of $\mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C})$, where $\mathrm{scr}_{\mathcal{C}}(X)$ is the *domain (source)* of its elements as well as $\mathrm{tar}_{\mathcal{C}}(Y)$ is the *codomain (target)* of its elements, and *hom-class* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is the intersection of both. What's more, we define $\mathrm{dom}_{\mathcal{C}}(f) = X$ and $\mathrm{cod}_{\mathcal{C}}(f) = Y$ if $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. [*] For $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \wedge g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, the binary operation that defines *composite morphism* $g \circ^{\mathcal{C}} f$ (which is abbreviated as $g \circ f$) satisfies:
  1. $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X,Z)$;
  2. $\forall h \in \mathrm{Hom}_{\mathcal{C}}(Z,W)[h \circ (g \circ f) = (h \circ g) \circ f]$;
  3. $\forall h \in \mathrm{Hom}_{\mathcal{C}}(Y,X) \exists 1_X^{\mathcal{C}} \in \mathrm{Hom}_{\mathcal{C}}(X,X)[f \circ 1_X^{\mathcal{C}} = f \wedge 1_X^{\mathcal{C}} \circ h = h]$.

It's easy to verify that the *identity morphism* $1_X^{\mathcal{C}}$ (which is abbreviated as 1_X) is unique for all $X \in \mathrm{Ob}(\mathcal{C})$. In addition, we abbreviate $\mathrm{Hom}_{\mathcal{C}}(X,X)$ as $\mathrm{End}_{\mathcal{C}}(X)$, it's easy to see that $\langle \mathrm{End}_{\mathcal{C}}(X), \circ, 1_X \rangle$ is a monoid group.

In commutative diagram,

$$X \xrightarrow{f} Y \quad \text{means} \quad f \in \mathrm{Hom}_{\mathcal{C}}(X,Y), \qquad \begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "h"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd} \quad \text{means} \quad g \circ f = h.$$

Mark. (Read the annotation of this page first) Sometimes we have to define a mapping that is also a morphis, for instance, for sets Y and $Z \subseteq X$, and a mapping $f : X \to Y$, we define a new mapping $g := \{\langle x, f(x) \rangle \,|\, x \in Z\}$, strictly speaking, we have defined a 3-tuple $\langle Z, g, Y \rangle$ if the mapping is also a morphism, we now abbreviate it as "$[Z \ni x \mapsto f(x)]$". What's more, we will abbreviate the restriction $\{\langle x, f(x) \rangle \,|\, x \in Z\}$ as "$f \restriction_Z$".

Notice that the meanings of the above two markers are different, a classical example is, for a morphism β and collection M of some morphisms, $[M \ni \alpha \mapsto \alpha \circ \beta]$ means a new mapping (even morphism) created, but restriction $f \restriction_Z$ is only a reduction of the original mapping.

Definition 2. $\mathcal{C}'$ is a *subcategory* of category $\mathcal{C}$ if:

- $\mathcal{C}'$ is a category;
- $\mathrm{Ob}(\mathcal{C}') \subseteq \mathrm{Ob}(\mathcal{C})$;

[*] Strictly speaking, the morphism is composed by a 3-tuple $\langle X, f, Y \rangle$, otherwise, it will cause confusion. For instance, in set category Set (see Example 3.1), if we don't discriminate the same mapping in different homology class, i.e., which have different codomains (such as $f : \{0\} \to \{0\}$ and $g : \{0\} \to \{0,1\}$ which satisfy $f(0) = g(0) = 0$), then they are the same morphism, it will contradict the disjoint of homology class. Of course, for convenience, we will omit the 3-tuples when describing morphisms.

- $\forall X, Y \in \mathrm{Ob}(\mathcal{C}')[\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)]$;
- $\forall f, g \in \mathrm{Mor}(\mathcal{C}')[f \circ^{\mathcal{C}'} g = f \circ^{\mathcal{C}} g]$ (if $\mathrm{dom}_{\mathcal{C}}(f) = \mathrm{cod}_{\mathcal{C}}(g)$);
- for all $X \in \mathrm{Ob}(\mathcal{C}')$, the identity morphism $\mathit{1}_X$ in $\mathcal{C}'$ is also that in $\mathcal{C}$.

In particular, if $\forall X, Y \in \mathrm{Ob}(\mathcal{C}')[\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)]$, then we say $\mathcal{C}'$ is the *full subcategory* of $\mathcal{C}$.

A *opposite* category $\mathcal{C}^{\mathrm{op}}$ of category $\mathcal{C}$ satisfies:

- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$;
- $\forall X, Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})[\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)]$;
- $\forall f, g \in \mathrm{Mor}(\mathcal{C})[g \circ^{\mathrm{op}} f = f \circ^{\mathcal{C}} g]$ (if $\mathrm{dom}_{\mathcal{C}}(f) = \mathrm{cod}_{\mathcal{C}}(g)$);
- for all $X \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, the identity morphism $\mathit{1}_X$ in $\mathcal{C}^{\mathrm{op}}$ is also that in $\mathcal{C}$.

It's easy to verify that $\mathcal{C}^{\mathrm{op}}$ is also a category, and we have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$. $\mathcal{C}^{\mathrm{op}}$ has the symmetric algebraic properties as $\mathcal{C}$.

A category $\mathcal{C}$ is called *small* if both $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets in ZFC but not proper class, [*] and *large* otherwise. A *locally small* category is a category such that for all objects X and Y, $\mathrm{Hom}(X, Y)$ is a set in ZFC, called *hom-set*.

Example 1. Here are some examples of category.

1. Consider a category Rel in ZF:
   - Objects are all sets.
   - Homomorphism between any sets X, Y is the power set of binary relations $\mathscr{P}(X \times Y)$.
   - The composition of morphisms is the composition of binary relations.
   - Identity morphism $\mathit{1}_X$ is the identity mapping $\mathrm{id}_X = \{\langle x, x\rangle \mid x \in X\}$.

   Obviously, it's indeed a category. This example shows that the morphisms are not only mappings, they may have looser structures. Compared to this, morphisms are more like binary relations.

2. Consider a set S, we can use it to construct a *discrete category* $\mathsf{Disc}(S)$, which $\mathrm{Ob}(\mathsf{Disc}(S)) = S$ and $\mathrm{Mor}(\mathsf{Disc}(S)) = \{\mathit{1}_x | \, x \in S\}$. A category without any objects and morphisms is called *zero* category $\mathbf{0}$. A discrete category which has exactly one object is written as $\mathbf{1}$.

3. There are some classic examples of *concrete* categories, which the objects are sets with (possible) structures, the morphisms are mappings (Cat is a little special) that preserve the structures, the composition of morphisms is the composition of mappings, and the identity morphisms are identity mappings:

Symbol	**Object**	**Morphism**
Set	Set	Mapping
Ord	Preordered set	Order-preserving mapping
On	Ordinal number	
Cpt		Computable function
Mon	Monoid group	Group homomorphism
Grp	Group	
Ab	Abelian group	
Rng	Ring	Ring homomorphism
Top	Topological space	Continuous mapping
Met	Metric space	
${}_R$Mod	Left module over the ring R	R-homomorphism
Mod$_R$	Right module over the ring R	
Vect$_{\Bbbk}$	Vector space over the field $\Bbbk$	$\Bbbk$-Linear mapping
fVect$_{\Bbbk}$	Finite vector space over the field $\Bbbk$	
Man	Smooth manifolds	Smooth mapping
Com	Complex	Simplicial mapping
Str$_{\mathcal{L}}$	Structure given by the language $\mathcal{L}$	$\mathcal{L}$-Elementary embedding
Cat	Small category	Functor

   Evidently, they are all large and locally small categories. And On/Grp/Ab/Met/fVect$_{\Bbbk}$ is the full subcategory of Ord/Mon/Grp/Top/Vect$_{\Bbbk}$.

4. Consider a category $\mathcal{C}$ and its object I, the *slice* category $\mathcal{C}/I$ satisfies:

[*]"X is a set in ZFC" has two meanings: we can prove X exists in ZFC, i.e., ZFC $\vdash_{\mathbf{H}} \exists X$; or the existence of X in ZFC is consistent with ZFC, i.e., $\vdash_{\mathbf{H}}$ Con(ZFC) $\to$ Con(ZFC $+ \exists X$), where $\mathbf{H}$ means the Frege-Hilbert first-order logic axiomatic system. The meaning in the text is the former. Of course, to prove the consistency, we often need to add extra axioms. The provability of ZFC is limited, so we can only define the set in the model $(V_\kappa, \in)$, where κ is the least strongly inaccessible cardinal, but that's enough. See [8] or [13] for more details.

- Objects are all the morphisms $f \in \mathrm{Mor}(\mathcal{C})$ which satisfy $f \in \mathrm{Hom}_{\mathcal{C}}(X, I)$;
- $\mathrm{Hom}_{\mathcal{C}/I}(f,g) = \{j \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{dom}_{\mathcal{C}}(f), \mathrm{dom}_{\mathcal{C}}(g)) \mid g \circ^{\mathcal{C}} j = f\}$;
- $\mathit{1}_f^{\mathcal{C}/I} = \mathit{1}^{\mathcal{C}}_{\mathrm{dom}_{\mathcal{C}}(f)}$;
- if $\mathrm{cod}_{\mathcal{C}/I}(f) = \mathrm{dom}_{\mathcal{C}/I}(g)$ then $k \circ^{\mathcal{C}/I} j = k \circ^{\mathcal{C}} j$.

It's easy to verify that $\mathcal{C}/I$ is indeed a category.
Similarly, we can define *coslice* category $I/\mathcal{C}$, whose the objects are $f \in \mathrm{Mor}(\mathcal{C})$ which satisfy $f \in \mathrm{Hom}_{\mathcal{C}}(I, X)$, and $\mathrm{Hom}_{I/\mathcal{C}}(f,g) = \{j \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{cod}_{\mathcal{C}}(f), \mathrm{cod}_{\mathcal{C}}(g)) \mid j \circ^{\mathcal{C}} f = g\}$.

Definition 3. A *functor* $F : \mathcal{C} \to \mathcal{D}$, between category $\mathcal{C}$ and $\mathcal{D}$, consists the following data:

- Mapping $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
- Mapping $F : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, which satisfies:
  1. $F[\mathrm{Hom}_{\mathcal{C}}(X,Y)] \subseteq \mathrm{Hom}_{\mathcal{D}}(F(X), G(Y))$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$;
  2. For all $f, g \in \mathrm{Mor}(\mathcal{C})$, if $\mathrm{cod}_{\mathcal{C}}(f) = \mathrm{dom}_{\mathcal{C}}(g)$, then $F(g \circ^{\mathcal{C}} f) = F(g) \circ^{\mathcal{D}} F(f)$;
  3. $F(\mathit{1}^{\mathcal{C}}_X) = \mathit{1}^{\mathcal{D}}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$.

In commutative diagram,

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

means that F is the functor between $\mathcal{C}$ and $\mathcal{D}$.

For functors $F : \mathcal{C}_1 \to \mathcal{C}_2, G : \mathcal{C}_2 \to \mathcal{C}_3$, the composition $GF : \mathcal{C}_1 \to \mathcal{C}_3$ between both satisfies:

$$GF(X) = G(F(X)) \text{ and } GF(f) = G(F(f)) \text{ for all } X \in \mathrm{Ob}(\mathcal{C}_1), f \in \mathrm{Mor}(\mathcal{C}_1).$$

It's trivial to verify that the composition is also a functor and it satisfy associative law.

For any category $\mathcal{C}$, there exists a *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that satisfies

$$\mathrm{id}_{\mathcal{C}}(X) = X \text{ and } \mathrm{id}_{\mathcal{C}}(f) = f \text{ for all } X \in \mathrm{Ob}(\mathcal{C}_1), f \in \mathrm{Mor}(\mathcal{C}_1).$$

It's easy to verify that for all functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and $C : \mathcal{C} \to \mathcal{C}$, $FC = F \wedge CG = G$ if and only if $C = \mathrm{id}_{\mathcal{C}}$.

Definition 4. The *natural transformation* θ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a mapping from $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Mor}(\mathcal{D})$ whose each value satisfies $\theta_X := \theta(X) \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ and the commutative diagram below:

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"'] & G(Y), \end{tikzcd} \tag{1}$$

where $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. In other words, we can record the above natural transformation as $\theta : F \Rightarrow G$, [*] or in such a commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D}. \end{tikzcd}$$

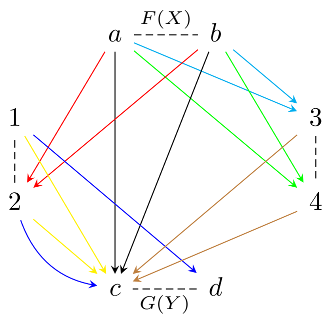
We may use symbol "$F(\theta)_X$" instead of "$F((\theta)_X)$" in some particular case (such as there are more than one symbols of natural transformations in the brackets).

For functor $F : \mathcal{C} \to \mathcal{D}$, there exists a *identity transformation* $\mathbf{id}^F : F \to F$ that satisfies $\forall X \in \mathrm{Ob}(\mathcal{C})[\mathbf{id}^F_X = \mathit{1}_{F(X)}]$.

Example 2. Consider two *finite* categories $\mathcal{C}, \mathcal{D}$ where $\mathrm{Ob}(\mathcal{C}) = \{X, Y\}$ and $\mathrm{Ob}(\mathcal{D}) = \{\{a,b\}, \{1,2\}, \{3,4\}, \{c,d\}\}$. There are three morphisms in $\mathcal{C}$: $\mathit{1}_X$, $\mathit{1}_Y$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Consider four functors $F, F', G, G' : \mathcal{C} \to \mathcal{D}$ such that

$$F(X) = \{a,b\} = F'(X), G(Y) = \{c,d\} = G'(Y).$$

And consider two natural transformations $\theta, \psi : F \Rightarrow G$, and all the morphisms (mappings) in $\mathcal{D}$ except identity morphisms are shown below:



[*]Like morphism, strictly speaking, functor $F : \mathcal{C} \to \mathcal{D}$ and natural transformation $\theta : F \to G$ are also composed by 3-tuples $\langle \mathcal{C}, F, \mathcal{D} \rangle$ and $\langle F, \theta, G \rangle$, rather than simple "mappings". For instance, consider a category $\mathcal{C}$ and its subcategory $\mathcal{C}'$, category $\mathcal{D}$, functor $F : \mathcal{D} \to \mathcal{C}'$, inclusion functor (see Example 4.1) $\iota : \mathcal{C}' \to \mathcal{C}$. Then the composition $\iota F : \mathcal{D} \to \mathcal{C}$ is different from single functor $F : \mathcal{D} \to \mathcal{C}'$, although they are the same if you regard them as mappings.

Where the arrows with different colors mean the different mappings, balck arrows mean the morphism k, and the elements connected by one dashed line belong to the same set. There are 7 isomorphisms (except identity morphisms) in $\mathcal{D}$ in total, it's easy to see that $\mathcal{D}$ is indeed a category (we just need to verify that the compositions of any morphisms in $\mathcal{D}$ are also morphisms in it).

- Consider the following combination of objects and morphisms:

$$G(X) = \{1, 2\} = G'(X), F(Y) = \{3, 4\} = F'(Y);$$

red:θ_X, yellow:$G(f)$, blue:$G'(f)$, cyan:$F(f)$, grenn:$F'(f)$, brown:θ_Y.

  The four functors are indeed functors. What's more, it's trivial to verify that

$$\theta_Y \circ F'(f) = \theta_Y \circ F(f) = k = G(f) \circ \theta_X = G'(f) \circ \theta_X,$$

  thus we know θ is indeed a natural transformation, and obviously θ have more than one "sources" and "targets".

- Consider the following combination of objects and morphisms:

$$G(X) = \{3, 4\}, F(Y) = \{1, 2\};$$

red:$F(f)$, yellow:θ_Y, blue:ψ_Y, cyan:θ_X, grenn:ψ_X, brown:$G(f)$.

  The two functors are indeed functors. What's more, it's trivial to verify that

$$\theta_Y \circ F(f) = \psi_Y \circ F(f) = k = G(f) \circ \theta_X = G(f) \circ \psi_X,$$

  thus we know θ and ψ are indeed natural transformations between F and G.

These examples show us that one natural transformation can rely on different functors, and there may be different natural transformations "between" two functors. Hence it is necessary to label the natural transformations as 3-tuples.

Definition 5. For functors $F, G, H : \mathcal{C} \to \mathcal{D}$, natural transformations $\theta : F \Rightarrow G$ and $\psi : G \Rightarrow H$, the element of *vertical composition* of the natural transformations is defined as $(\psi \odot \theta)_X = \psi_X \circ \theta_X$. In commutative diagrams forms,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=70, "H"'{name=H}] & \mathcal{D}
\arrow[Rightarrow, from=F, to=G, "\theta"] \arrow[Rightarrow, from=G, to=H, "\psi"]
\end{tikzcd}
\quad \text{means} \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "H"'{name=H}] & \mathcal{D}.
\arrow[Rightarrow, from=F, to=H, "\psi\odot\theta"]
\end{tikzcd}$$

Actually, we need to prove that the definition is well-defined, i.e., to verify that $(\psi \odot \theta)_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), H(X))$ for all $X \in \mathrm{Ob}(\mathcal{C})$, it's easy to do so.

Definition 6. For functors $F, F' : \mathcal{C}_1 \to \mathcal{C}_2$ and $G, G' : \mathcal{C}_2 \to \mathcal{C}_3$, natural transformations $\theta : F \Rightarrow F'$ and $\psi : G \Rightarrow G'$, the element of *horizontal composition* of natural transformations $(\psi \ominus \theta)_X$ is defined as $G'(\theta_X) \circ \psi_{F(X)} = \psi_{F'(X)} \circ G(\theta_X)$. In commutative diagrams forms,

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "F'"'{name=F2}] & \mathcal{C}_2 \arrow[r, bend left=50, "G"{name=G}] \arrow[r, bend right=50, "G'"'{name=G2}] & \mathcal{C}_3
\arrow[Rightarrow, from=F, to=F2, "\theta"] \arrow[Rightarrow, from=G, to=G2, "\psi"]
\end{tikzcd}
\quad \textit{means} \quad
\begin{tikzcd}
\mathcal{C}_1 \arrow[r, bend left=50, "GF"{name=A}] \arrow[r, bend right=50, "G'F'"'{name=B}] & \mathcal{C}_3,
\arrow[Rightarrow, from=A, to=B, "\psi\ominus\theta"]
\end{tikzcd}$$

which satisfy

$$\begin{tikzcd}
GF(X) \arrow[r, "G(\theta_X)"] \arrow[d, "\psi_{F(X)}"'] & GF'(X) \arrow[d, "\psi_{F'(X)}"] \\
G'F(Y) \arrow[r, "G'(\theta_X)"'] & G'F'(Y).
\end{tikzcd} \tag{2}$$

Actually, we need to prove that the definition is well-defined, i.e., to verify the commutative diagram and that $(\psi \ominus \theta)_X \in \mathrm{Hom}_{\mathcal{C}_3}(GF(X), G'F'(X))$ for all $X \in \mathrm{Ob}(\mathcal{C}_1)$, it's easy to do so observing commutative diagram 1.

Theorem 1. The vertical and horizontal compositions of natural transformations are natural transformations, and the natural transformations satisfy the *interchange law* (formula 3).

Proof. For

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, bend left=70, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=70, "H"'{name=H}] & \mathcal{C}_2 \arrow[r, bend left=70, "F'"{name=F2}] \arrow[r, "G'"{name=G2, description}] \arrow[r, bend right=70, "H'"'{name=H2}] & \mathcal{C}_3
\arrow[Rightarrow, from=F, to=G, "\theta"] \arrow[Rightarrow, from=G, to=H, "\psi"]
\arrow[Rightarrow, from=F2, to=G2, "\theta'"] \arrow[Rightarrow, from=G2, to=H2, "\psi'"]
\end{tikzcd},$$

we need to verify the following commutative diagrams:

$$
\text{(a)}\quad
\begin{tikzcd}
F(X) \arrow[r, "(\psi\odot\theta)_X"] \arrow[d, "F(f)"'] & H(X) \arrow[d, "H(f)"] \\
F(y) \arrow[r, "(\psi\odot\theta)_X"'] & H(y)
\end{tikzcd}
\quad \text{and (b)} \quad
\begin{tikzcd}
GF(X) \arrow[r, "(\theta'\ominus\theta)_X"] \arrow[d, "GF(f)"] & G'F'(X) \arrow[d, "G'F'(f)"] \\
GF(Y) \arrow[r, "(\theta'\ominus\theta)_Y"'] & G'F'(Y)
\end{tikzcd}.
$$

From (a), we have

$$
\begin{aligned}
& H(f)\circ(\psi\odot\theta)_X \\
=& H(f)\circ\psi_X\circ\theta_X && \text{(Def: vertical composition)} \\
=& (\psi_Y\circ G(f))\circ\theta_X && \text{(Property of natural transformation } \psi) \\
=& \psi_Y\circ\theta_Y\circ F(f) && \text{(Property of natural transformation } \theta) \\
=& (\psi\odot\theta)_X\circ F(f), && \text{(Def: vertical composition)}
\end{aligned}
$$

thus $(\psi\odot\theta)$ is natural transformation.

From (b), we have

$$
\begin{aligned}
& G'F'(f)\circ(\theta'\ominus\theta)_X \\
=& G'F'(f)\circ\theta'_{F'(X)}\circ G(\theta_X) && \text{(Def: horizontal composition)} \\
=& \theta'_{F'(Y)}\circ GF'(f)\circ G(\theta_X) && \text{(Property of natural transformation } \theta') \\
=& \theta'_{F'(Y)}\circ G(F'(f)\circ G(\theta_X)) && \text{(Property of functor } G) \\
=& \theta'_{F'(Y)}\circ G(\theta_Y\circ F(f)) && \text{(Property of natural transformation } \theta) \\
=& \theta'_{F'(Y)}\circ G(\theta_Y)\circ GF(f) && \text{(Property of functor } G) \\
=& (\theta'\ominus\theta)_Y\circ GF(f), && \text{(Def: horizontal composition)}
\end{aligned}
$$

thus $(\psi\ominus\theta)$ is natural transformation.

For interchange law

$$
(\psi\odot\theta)\ominus(\psi'\odot\theta') = (\psi'\ominus\psi)\odot(\theta'\ominus\theta), \tag{3}
$$

we can prove it in the below step:

$$
\begin{aligned}
& ((\psi'\odot\theta')\ominus(\psi\odot\theta))_X \\
=& (\psi'\odot\theta')_{H(X)}\circ F'(\psi\odot\theta)_X && \text{(Def: horizontal composition)} \\
=& \psi'_{H(X)}\circ\theta_{H(X)}\circ F'(\theta_X)\circ F'(\theta_X) && \text{(Def: vertical composition, Property of functor } F') \\
=& \psi'_{H(X)}\circ(G'(\psi_X)\circ\theta'_{G(X)})\circ F'(\theta_X) && \text{(Commutative diagram 2)} \\
=& (\psi'\ominus\psi)_X\circ(\theta'\ominus\theta)_X && \text{(Def: horizontal composition)} \\
=& ((\psi'\ominus\psi)\odot(\theta'\ominus\theta))_X, && \text{(Def: vertical composition)}
\end{aligned}
$$

where $X\in\mathrm{Ob}(\mathcal{C})$. □

Theorem 2. Both vertical and horizontal compositions of natural transformations satisfy associative law.

Proof. For vertical composition, observe the following commutative diagram and natural transformations:

For

$$
\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=70, "F"] \arrow[r, bend left=15, "G" description] \arrow[r, bend right=15, "H" description] \arrow[r, bend right=70, "K"'] \arrow[r, phantom, shift left=5, "\Downarrow\theta"] \arrow[r, phantom, "\Downarrow\psi"] \arrow[r, phantom, shift right=5, "\Downarrow\phi"] & \mathcal{D},
\end{tikzcd}
$$

it's trivial to prove that $((\phi\odot\psi)\odot\theta)_X = (\phi\odot(\psi\odot\theta))_X$ for all $X\in\mathrm{Ob}(\mathcal{C})$, thus the vertical composition satisfies associative law.

For

$$
\begin{tikzcd}
\mathcal{C}_1 \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}_2 \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{C}_3 \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "H'"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{C}_4,
\end{tikzcd}
\tag{4}
$$

we have

$$
\begin{aligned}
&(\phi \ominus (\psi \ominus \theta))_X \\
=&\phi_{G'F'(X)} \circ H(\psi \ominus \theta)_X && \text{(Def: horizontal composition)} \\
=&\phi_{G'F'(X)} \circ H(\psi_{F'(X)} \circ G(\theta_X)) && \text{(Ditto)} \\
=&\phi_{G'F'(X)} \circ H(\psi_{F'(X)}) \circ HG(\theta_X) && \text{(Property of functor } H\text{)} \\
=&(\phi \ominus \psi)_{F'(X)} \circ HG(\theta_X) && \text{(Def: horizontal composition)} \\
=&((\phi \ominus \psi) \ominus \theta)_X, && \text{(Ditto)}
\end{aligned}
$$

thus the horizontal composition satisfies associative law. □

Lemma 1 (ZF)**.** For any nonempty sets A, B and mapping $f : A \to B$, we have

$$\exists g : B \to A[g \circ f = \mathrm{id}_A] \iff f \text{ is a injection } \iff \forall C \neq \varnothing \forall h, h' : C \to A[f \circ h = f \circ h' \implies h = h'],$$

$$\exists g : B \to A[f \circ g = \mathrm{id}_B] \implies f \text{ is a surjection } \iff \forall C \neq \varnothing \forall h, h' : B \to C[h \circ f = h' \circ f \implies h = h'],$$

and f is a surjection $\implies \exists g : B \to A[f \circ g = \mathrm{id}_B]$ can be proved using AC. It's easy to see that f is a bijection if and only if it has both left and right inversal mappings, and obviously the two inversal mappings are the same one, which is unique.

Proof. The proofs are shown in [6, Theorem 3J] (left parts) and [5, (§2.1) Proposition 2.2 & Example 2.3] (right parts). □

Definition 7. Consider $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. If there exists a morphism $g \circ f = 1_X$, then we say f is a *section*, and g is the *left inverse* of it; if $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ that $f \circ g = 1_Y$, then we say f is a *retraction*, and g is the *right inverse* of it. If f has both inverses, then we say f is a *isomorphism*, it's easy to verify that the two inverses are the same one, which is unique as well, so we say $f^{-1} := g$ is the *inverse* (*inversal morphism*) of it. If there exists an isomorphism between two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, then we say they are *isomorphic* and record it as $X \overset{\mathrm{M}}{\simeq} Y$.

What's more, it's easy to verify that the composition of isomorphisms is also an isomorphism (see Theorem 3), so we can find that the collection of *automorphisms* $\mathrm{Aut}_{\mathcal{C}}(X) := \{f \in \mathrm{Hom}_{\mathcal{C}}(X, X) \mid f \text{ is an isomorphism}\}$ is a group $\langle \mathrm{Aut}_{\mathcal{C}}(X), \circ, 1_X \rangle$. If f satisfies the *left cancellation law*: $\forall Z \in \mathrm{Ob}(\mathcal{C}) \forall g, h \in \mathrm{Hom}_{\mathcal{C}}(Z, X)[f \circ g = f \circ h \iff g = h]$, then we say f is *monic* (or f is a *monomorphism*); similarly, if f satisfies the *right cancellation law*, then we say f is *epic* (or f is a *epimorphism*). If f is monic as well as epic, then we say f is a *bimorphism*. We call a category *groupoid* if all the morphisms in it are isomorphisms, and call it is *balanced* if all the bimorphisms in $\mathrm{Mor}(\mathcal{C})$ are isomorphisms.

Consider a functor $F : \mathcal{C} \to \mathcal{D}$, we can define the inverse of functor in the same way: If there exists a functor $G : \mathcal{D} \to \mathcal{C}$ that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$, then we say F is an *functorial isomorphism* between $\mathcal{C}$ and $\mathcal{D}$ and denote it as $F : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$, and $F^{-1} := G$ is the unique inverse (*inversal functor*) of F. From Lemma 1 we know that F is a functorial isomorphism if and only if $F \restriction_{\mathrm{Ob}(\mathcal{C})} \to \mathrm{Ob}(\mathcal{D})$ and $F \restriction_{\mathrm{Mor}(\mathcal{C})} \to \mathrm{Mor}(\mathcal{D})$ are both bijection. If there exists a functorial isomorphism between two categories $\mathcal{C}$ and $\mathcal{D}$, we say they are *isomorphic* and record it as $\mathcal{C} \overset{\mathrm{F}}{\simeq} \mathcal{D}$.

(Please see Corollary 1.1 first) Consider the following diagram:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\theta \quad \psi\Uparrow"] & \mathcal{D}.
\end{tikzcd}
\tag{5}
$$

If $\psi \odot \theta = \mathbf{id}^F$ and $\theta \odot \psi = \mathbf{id}^G$, then we say θ is a *natural isomorphism* between F and G, and $\theta^{-1} := \psi$ is the unique inverse (*inversal transformation*) of θ, we record it as $\theta : F \overset{\sim}{\Rightarrow} G$. If there exists a natural isomorphism between two functors F and G, we say they are *isomorphic* and record it as $F \overset{\mathrm{T}}{\simeq} G$.

For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, if $GF \overset{\mathrm{T}}{\simeq} \mathrm{id}_{\mathcal{C}}$ and $FG \overset{\mathrm{T}}{\simeq} \mathrm{id}_{\mathcal{D}}$, then we say G is the *quasi-inverse* of F, and F is a *equivalence* between $\mathcal{C}$ and $\mathcal{D}$. If there exists an equivalence between two categories $\mathcal{C}$ and $\mathcal{D}$, we say they are *equivalent* and record it as $\mathcal{C} \sim \mathcal{D}$. We say categories $\mathcal{C}$ and $\mathcal{D}$ are *dual equivalent* if $\mathcal{C}^{\mathrm{op}} \sim \mathcal{D}$.

If there is no confusion in some context, we abbrviate $\overset{\mathrm{M}}{\simeq}, \overset{\mathrm{F}}{\simeq}, \overset{\mathrm{T}}{\simeq}$ as $\simeq$ and refer to "isomorphism", "functorial isomorphism" and "natural isomorphism" as "isomorphism" uniformly.

Corollary 1. Observe diagram 5, we have:

1. (For any θ, ψ and $\phi : F \Rightarrow F$) $\theta \odot \phi = \theta \wedge \phi \odot \psi = \psi$ if and only if $\phi = \mathbf{id}^F$.
2. θ is a natural isomorphism if and only if θ_X is an isomorphism for each $X \in \mathrm{Ob}(\mathcal{C})$.
3. If θ is an isomorphism, then $(\theta^{-1})_X = (\theta_X)^{-1}$ for all $X \in \mathrm{Ob}(\mathcal{C})$, thus we abbreviate it as θ_X^{-1}.

Proof. It's easy to prove so using the definitions of identity transformation, vertical composition and identity morphism. □

Lemma 2. Observe commutative diagram 4, we have

$$G(\mathbf{id}^F_X) = \mathbf{id}^G_{F(X)} = \mathbf{id}^{GF}_X, \tag{6}$$
$$(\psi \ominus \mathbf{id}^F)_X = \psi_{F(X)}, \tag{7}$$
$$(\mathbf{id}^H \ominus \psi)_Y = H(\psi_Y), \tag{8}$$
$$\mathbf{id}^G \ominus \mathbf{id}^F = \mathbf{id}^{GF}, \tag{9}$$
$$\text{if } \psi \text{ and } \phi \text{ are isomorphisms, then } \phi \ominus \psi \text{ is also, and } (\phi \ominus \psi)^{-1} = \phi^{-1} \ominus \psi^{-1}, \tag{10}$$
$$F \simeq F' \implies [G \simeq G' \implies GF \simeq G'F'] \wedge [H \simeq H' \implies \wedge HG \simeq H'G'], \tag{11}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1), Y \in \mathrm{Ob}(\mathcal{C}_2)$.

Proof. For (6), using the definition of identity morphism, properties of morphism and functor we have

$$\mathbf{id}^{GF}_X = \mathit{1}_{GF(X)} = G(\mathit{1}_{F(X)}) = \mathbf{id}^G_{F(X)} = G(\mathit{1}_{F(X)}) = G(\mathbf{id}^F_X)$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

For (7), we have

$$\begin{aligned} & (\psi \ominus \mathbf{id}^F)_X \\ =& \psi_{F(X)} \circ G(\mathbf{id}^F_X) && \text{(Def: horizontal composition)} \\ =& \psi_{F(X)} \circ \mathbf{id}^G_{F(X)} && \text{(Formula 6)} \\ =& \psi_{F(X)} && \text{(Corollary 1.1)} \end{aligned}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

For (8), we have

$$\begin{aligned} & (\mathbf{id}^H \ominus \psi)_Y \\ =& H(\psi_Y) \circ \mathbf{id}^H_{G(Y)} && \text{(Def: horizontal composition)} \\ =& H(\psi_Y) \circ H(\mathbf{id}^G_Y) && \text{(Formula 6)} \\ =& H(\psi_Y \circ \mathbf{id}^G_Y) && \text{(Property of functor } H\text{)} \\ =& H(\psi_Y) && \text{(Corollary 1.1)} \end{aligned}$$

for all $Y \in \mathrm{Ob}(\mathcal{C}_2)$.

For (9), we have

$$\begin{aligned} & (\mathbf{id}^G \ominus \mathbf{id}^F)_X \\ =& G(\mathbf{id}^F_X) \circ \mathbf{id}^G_{F(X)} && \text{(Def: horizontal composition)} \\ =& \mathbf{id}^{GF}_X \circ \mathbf{id}^{GF}_X && \text{(Corollary 1.1)} \\ =& \mathbf{id}^{GF}_X && \text{(Formula 6)} \end{aligned}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

For (10), we have

$$\begin{aligned} & (\phi \ominus \psi) \odot (\phi^{-1} \ominus \psi^{-1}) \\ =& (\phi^{-1} \odot \phi) \ominus (\psi^{-1} \odot \psi) && \text{(Interchange law)} \\ =& \mathbf{id}^H \ominus \mathbf{id}^G && \text{(Property of inverse)} \\ =& \mathbf{id}^{HG}. && \text{(Formula 9)} \end{aligned}$$

Similarly, we can prove $(\phi^{-1} \ominus \psi^{-1}) \odot (\phi \ominus \psi) = \mathbf{id}^{H'G'}$.

For (11), we suppose that the three natural transformations are all isomorphisms, we claim that $\psi \ominus \theta : GF \overset{\sim}{\Rightarrow} G'F'$ and $\phi \ominus \psi : HG \overset{\sim}{\Rightarrow} H'G'$, it's trivial to prove using formula 10. ☐

Theorem 3. If two particular morphisms/functors/transformations are isomorphic, than the isomorphism between them is unique. The composition of any isomorphisms/functorial isomorphism/natural isomorphism is also an isomorphism, and the composition of any equivalences are equivalence. Therefore isomorphic objects, categories, functors and equivalent categories satisfy transitivity.

Proof. The uniqueness is trivial to prove.

Because the compositions of isomorphisms, functorial isomorphisms and natural isomorphisms have similar algebraic properties, we just need to prove the composition of isomorphism is an isomorphism:
Suppose isomorphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, we claim that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$:

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ \mathit{1}_Y \circ g^{-1} = g \circ g^{-1} = \mathit{1}_Z,$$

$$(f^{-1}\circ g^{-1})\circ(g\circ f)=f^{-1}\circ(g^{-1}\circ g)\circ f=f^{-1}\circ 1_Y\circ f=f^{-1}\circ f=1_X.$$

Observe the following diagram:

$$\begin{tikzcd} \mathrm{id}_{\mathcal{C}_1} \circlearrowright \mathcal{C}_1 \arrow[r, bend left, "F"] & \mathcal{C}_2 \arrow[l, bend left, "F'"] \arrow[r, bend left, "G"] & \mathcal{C}_3 \circlearrowleft \mathrm{id}_{\mathcal{C}_3}. \arrow[l, bend left, "G'"] \end{tikzcd}$$

We need to prove that if F and G are equivalence then GF is also, we now assume that $F'F\simeq \mathrm{id}_{\mathcal{C}_1}$, $FF'\simeq\mathrm{id}_{\mathcal{C}_2}$, $G'G\simeq\mathrm{id}_{\mathcal{C}_2}$ and $GG'\simeq\mathrm{id}_{\mathcal{C}_3}$. Using formula 11 we have

$$\mathrm{id}_{\mathcal{C}_2}\simeq G'G\Longrightarrow \mathrm{id}_{\mathcal{C}_1}\simeq F'F=F'\mathrm{id}_{\mathcal{C}_2}F\simeq F'(G'G)F=(F'G')(GF),$$

$$\mathrm{id}_{\mathcal{C}_2}\simeq F'F\Longrightarrow \mathrm{id}_{\mathcal{C}_3}\simeq GG'=G\mathrm{id}_{\mathcal{C}_2}G'\simeq G(FF')G'=(GF)(F'G').$$

Using transitivity of functorial isomorphisms, we have $\mathrm{id}_{\mathcal{C}_1}\simeq(F'G')(GF)$ and $\mathrm{id}_{\mathcal{C}_3}\simeq(GF)(F'G')$. Thus GF is equivalence. □

Corollary 2. If functors G, G' are quasi-inverses of equivalence $F:\mathcal{C}\to\mathcal{D}$, then $G\simeq G'$.

Proof. Using formula 11, we have

$$G'F\simeq\mathrm{id}_{\mathcal{C}}\wedge FG\simeq\mathrm{id}_{\mathcal{D}}\Longrightarrow G'=G'\mathrm{id}_{\mathcal{D}}\simeq G'(FG)=(G'F)G\simeq\mathrm{id}_{\mathcal{C}}G=G,$$

thus we have $G'\simeq G$ using transitivity of functorial isomorphisms. □

Corollary 3. Consider two morphisms f and g in $\mathcal{C}$ which satisfy $\mathrm{cod}_{\mathcal{C}}(f)=\mathrm{dom}_{\mathcal{C}}(g)$.

1. Every section is monic, and every retraction is epic.
2. The composition of monomorphisms is monic, and the composition of epimorphisms is epic.
3. If $g\circ f$ is monic then f is monic, if $g\circ f$ is epic then g is epic.
4. The following propositions are equivalent:
   - f is an isomorphism.
   - f is a monomorphism as well as a retraction.
   - f is an epimorphism as well as a section.

Proof. The proofs are trivial, the details are shown in [20, §1.4]. □

Example 3. There are some examples of isomorphisms.

1. Consider two monomorphisms $f\in\mathrm{Hom}_{\mathcal{C}}(X,Z)$ and $g\in\mathrm{Hom}_{\mathcal{C}}(Y,Z)$, if f and g are *factor through* each other each other, i.e., $\exists i\in\mathrm{Hom}_{\mathcal{C}}(X,Y)\exists j\in\mathrm{Hom}_{\mathcal{C}}(Y,X)[f=g\circ i\wedge g=f\circ j]$, then the *factors* i and j are both isomorphisms and they are inverses to each other. In commutative diagram:

$$\begin{tikzcd} X \arrow[rr, shift right, "i"'] \arrow[dr, "f"'] & & Y \arrow[ll, shift right, "j"'] \arrow[dl, "g"] \\ & Z. & \end{tikzcd}$$

2. For any concrete categories, all the morphisms in it satisfy the following implications:

$$\begin{tikzcd} \text{section} \arrow[d, Rightarrow] & \text{isomorphism} \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{retraction} \arrow[d, Rightarrow] \\ \text{injection} \arrow[d, Rightarrow] & \text{bijection} \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{surjection} \arrow[d, Rightarrow] \\ \text{monomorphism} & \text{bimorphism} \arrow[l, Rightarrow] \arrow[r, Rightarrow] & \text{epimorphism} \end{tikzcd}$$

3. Set, Grp are balanced category.(See [7, Theorem 2.5.2])
   However, consider a morphism, an inclusion ring homomorphism $f:\mathbb{Z}\to\mathbb{Q}$ in $\mathrm{Mor}(\mathsf{Rng})$. We will find that it is a epimorphism but not a retraction, because there is no inverse of it in $\mathrm{Mor}(\mathsf{Rng})$. So Rng isn't a balanced category. Actually, Top is not neither. (See [7, p19])
   There are more detailed examples of morphisms in [7]:
   - Monic but not injective [Example 2.1.2].
   - Injective but not *split monic* (a section) [Example 2.2.3, 2.2.4].
   - Epic but not surjective [Example 2.3.2].

- Surjective but not *split epic* (a retraction) [Example 2.4.3, 2.4.4].

Definition 8. For functor $F : \mathcal{C} \to \mathcal{D}$, we define:

- F is *essentially surjective* if $\forall Y \in \mathrm{Ob}(\mathcal{D}) \exists X \in \mathrm{Ob}(\mathcal{C})[F(X) \overset{\mathrm{M}}{\simeq} F(Y)]$.
- F is *faithful* if for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, $F \restriction_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective.
- F is *full* if for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, $F \restriction_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective.

Lemma 3. Consider a functor $F : \mathcal{C} \to \mathcal{D}$, for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ and morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we have the following propositions:

1. $F(f)$ is a section/retraction if f is a section/retraction; and the left/right inverse of $F(f)$ is $F(g)$, where g is the left/right inverse of f.
2. When F is faithful and full, we have f is a section/retraction if $F(f)$ is a section/retraction; and the left/right inverse of f is g, where $F(g)$ is the left/right inverse of $F(f)$.
3. $X \simeq Y \Longrightarrow F(X) \simeq F(Y)$; if F is faithful and full, we have $F(X) \simeq F(Y) \Longrightarrow X \simeq Y$.
4. The composition of faithful/full/essentially-surjective functors is also faithful/full/essentially-surjective.

Proof. Proof of **Proposition 1** is trivial.

For **Proposition 2**, we only suppose that $F(f)$ is a section, the proof that f is a retraction is similar: Because $F(f)$ has a left inverse in $\mathrm{Hom}_{\mathcal{C}}(F(Y), F(X))$ and F is full, we have $F \restriction_{\mathrm{Hom}_{\mathcal{C}}(Y,X)} \to \mathrm{Hom}_{\mathcal{C}}(F(Y), F(X))$ is surjective, so there exists $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ that $F(g)$ is the left inverse of $F(f)$, that is,

$$F(1_X^{\mathcal{C}}) = 1_{F(X)}^{\mathcal{D}} = F(g) \circ^{\mathcal{D}} F(f) = F(g \circ^{\mathcal{C}} f).$$

Because F is faithful, that means, $F \restriction_{\mathrm{Hom}_{\mathcal{C}}(X,X)}$ is injective, then we have $1_X^{\mathcal{C}} = g \circ^{\mathcal{C}} f$, thus f is a section.

Proof of **Proposition 3** is trivial using Propositions 1 and 2.

For **Proposition 4**, it's trivial to prove that it is faithful of full, we now show that F is essentially surjective: Consider $G : \mathcal{D} \to \mathcal{E}$ is also essentially surjective, then we have

$$\forall e \in \mathrm{Ob}(\mathcal{E}) \exists d \in \mathrm{Ob}(\mathcal{D})[G(d) \simeq e].$$

What's more, there exists c in $\mathcal{C}$ that $F(c) \simeq d$, using proposition 3 we have $GF(c) \simeq G(d) \simeq e$. Thus we have $GF(c) \simeq e$ by the transitivity of equivalence. □

Example 4. Observe the following example.

1. For category $\mathcal{C}$ and its subcategory $\mathcal{C}'$, there exists an *inclusion functor* $\iota := \mathrm{id}_{\mathcal{C}} \restriction_{\mathcal{C}'} : \mathcal{C}' \to \mathcal{C}$. It's obviously faithful, and ι is full if and only if $\mathcal{C}'$ is a full subcategory. There is a classical example: inclusion functor $F : \mathsf{Vect}_{\Bbbk} \to \mathsf{fVect}_{\Bbbk}$. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *contravariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. What's more, for functor $F : \mathcal{C} \to \mathcal{D}$, we can define $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.
2. There are some *forgetful functors* such as $\mathsf{Set} \to \mathsf{Rel}$, $\mathsf{Grp} \to \mathsf{Set}$, $\mathsf{Top} \to \mathsf{Set}$, $\mathsf{Ab} \to \mathsf{Grp}$, $\mathsf{Vect}(\Bbbk) \to \mathsf{Ab}$ (which forget field $\Bbbk$) that has a feature, that is, they forget some (order, algebraic, topological, etc.) structures. The functors above are all faithful but not full.
3. For vector space category $\mathsf{Vect}_{\Bbbk}$, we define a functor $D : \mathsf{Vect}_{\Bbbk}^{\mathrm{op}} \to \mathsf{Vect}_{\Bbbk}$:
   For all $V \in \mathsf{Vect}_{\Bbbk}$, we define the *dual vector space* of V: $D(V) := V^{\vee} := \mathrm{Hom}_{\Bbbk}(V, \Bbbk) = \{\Bbbk - \text{linear mapping } V \to \Bbbk\}$, see Algebraic dual space - Dual space - Wikipedia for the operation of it. It's trivial to verify that the dual vector space is indeed a vector space.
   For all $f \in \mathrm{Hom}_{\Bbbk}(V, U)$ we define $D(f) := f^{\vee} = [U^{\vee} \ni \lambda \mapsto \lambda \circ f] \in \mathrm{Hom}_{\Bbbk}(U^{\vee}, V^{\vee})$, in other words, $f^{\vee}$ is a mapping from $U^{\vee}$ to $V^{\vee}$ which satisfies $f^{\vee}(\lambda) = \lambda \circ f$ for all $\lambda \in U^{\vee}$. It's trivial to verify that f is linear.
   We can easily verify that D is indeed a functor. Consider $D^{\mathrm{op}} : \mathsf{Vect}_{\Bbbk} \to (\mathsf{Vect}_{\Bbbk}^{\mathrm{op}})^{\mathrm{op}} = \mathsf{Vect}_{\Bbbk}$, then we can define the *dual space contravariant functor* $FF^{\mathrm{op}} : \mathsf{Vect}_{\Bbbk} \to \mathsf{Vect}_{\Bbbk}$. Similarly, we can define the finite dual space contravariant functor $DD^{\mathrm{op}} : \mathsf{fVect}_{\Bbbk} \to \mathsf{fVect}_{\Bbbk}$.

Corollary 4. For $\mathcal{C}_1 \xrightarrow{F} \mathcal{C}_2 \xrightarrow{G} \mathcal{C}_3$, we have:

1. $F : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2 \Longrightarrow F^{\mathrm{op}} : \mathcal{C}_1^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}_2^{\mathrm{op}}$;
2. $(F^{\mathrm{op}})^{\mathrm{op}} = F$;
3. $(FG)^{\mathrm{op}} = F^{\mathrm{op}} G^{\mathrm{op}}$;
4. $F \simeq G \Longrightarrow F^{\mathrm{op}} \simeq G^{\mathrm{op}}$;
5. $\mathrm{id}_{\mathcal{C}_1}^{\mathrm{op}} = \mathrm{id}_{\mathcal{C}_1^{\mathrm{op}}}$.

Definition 9. A full subcategory $\mathcal{C}'$ of category $\mathcal{C}$ is a *skeleton* of $\mathcal{C}$ if $\forall X \in \mathrm{Ob}(\mathcal{C}) \exists! Y \in \mathrm{Ob}(\mathcal{C}')[X \overset{\mathrm{M}}{\simeq} Y]$. If $\mathcal{C}$ is the skeleton of itself, then we say it's a *skeletal* category.

Example 5. Consider a topological space $(X, \mathcal{T})$, let $I = [0, 1]$. We now define:

- $f : I \to X$ is a *path* from $x \in X$ to $y \in X$ if f is a continuous mapping and $f(x) = 0$, $f(y) = 1$. We denote the collection of the paths in X from x to y as $\mathrm{P}_X(x, y)$.
- For $f \in \mathrm{P}_X(x, y)$ and $g \in \mathrm{P}(y, z)$, the composition of paths $f * g : I \to X$ is defined as follow:
$$f * g(t) = \begin{cases} f(2t), & 0 \le t \le \dfrac{1}{2}, \\ g(2t-1), & \dfrac{1}{2} \le t \le 1. \end{cases}$$
It's easy to see that the $f * g$ is also a path.
- The identity path $\mathrm{Id}_x := [I \ni t \mapsto x] \in \mathrm{P}(x, x)$. Note that $\mathrm{Id}_x * f = f = f * \mathrm{Id}_x$ is not always true.
- The inverse of path f from x is defined as $f^{-1} := [I \ni t \mapsto f(1-t)]$. Note that $f * f^{-1} = \mathrm{Id}_x = f^{-1} * f$ is not always true.
- We call that the path f and g from x to y is homotopy, if there exists a continuous mapping $F : I^2 \to X$ satisfies for all $s, t \in I$ that
$$F(s, 0) = f(s) \wedge F(s, 1) = g(s) \wedge F(0, t) = x \wedge F(1, t) = y.$$
We denote it as "$f \overset{\mathrm{P}}{\simeq} g$".

We can verify that the homotopy relation of paths is an equivalence relation (see [19, 定理 10.1.1]), then we can construct a category $\Pi_1(X)$ called *fundamental groupoid*:

- $\mathrm{Ob}(\Pi_1(X)) := X$.
- $\mathrm{Hom}_{\Pi_1(X)}(x, y) := \mathrm{P}_X(x, y)/\overset{\mathrm{P}}{\simeq}$.
- For $f \in \mathrm{P}_X(x, y)$ and $g \in \mathrm{P}(y, z)$, we define $[g] \circ^{\Pi_1(X)} [f] := [g * f]$.
- $\mathit{1}_x^{\Pi_1(X)} := [\mathrm{Id}_x]$ for all $x \in X$.
- $[f]^{-1} := [f^{-1}]$ for all $f \in \mathrm{Mor}(\Pi_1(X))$.

We can verify the definition is well. All the details of the content above are shown in [19, §10.1]. Therefore, $\Pi_1(X)$ is indeed a groupoid. Moreover, note that the *loop* $\mathrm{Aut}(x) = \mathrm{Hom}(x, x)$ is the fundamental group $\pi_1(X, x)$, then we can immediately find that the fundamental group $\pi_1(X, x)$ is the skeleton of $\Pi_1(X)$ for each $x \in X$. See [10, Chapter 2] for more details.

Lemma 4. For category $\mathcal{C}$, we have:

1. Suppose AC, * we have every nonempty category has at least one skeleton.
2. Every inclusion functor $\iota : \mathcal{C}' \to \mathcal{C}$ is an equivalence, where $\mathcal{C}'$ is a skeleton of $\mathcal{C}$; and we can find a quasi-inverse called *skeletal functor* $\kappa : \mathcal{C} \to \mathcal{C}'$ of ι, which gives the skeleton of $\mathcal{C}$.
3. Any two skeletons of a category are equivalent.
4. $\mathcal{C}$ is a skeletal category if and only if, for all the isomorphism $f \in \mathrm{Mor}(\mathcal{C})$, $\mathrm{dom}(f) = \mathrm{cod}(f)$.
5. Every faithful, full and essentially surjective functor between two skeletal categories is an isomorphism.

Proof. Assume two categories $\mathcal{C}$, $\mathcal{D}$ and a functor $F : \mathcal{C} \to \mathcal{D}$.

1. Because the isomorphism relation among objects satisfies reflexivity, symmetry and transitivity, we can divide them into an equivalence (not functor, but a simple relation) class $\mathcal{C}/\overset{\mathrm{M}}{\simeq}$. Using AC, we can construct a choice function which selects representative element in each $[X]$ for all $X \in \mathrm{Ob}(\mathcal{C})$. By preserving the representative elements and all the morphisms between any two of them, they form a full subcategory $\mathcal{C}'$ of category $\mathcal{C}$, and it's easy to see that $\mathcal{C}'$ is the skeleton of $\mathcal{C}$.
2. Construct:
   - $\kappa \upharpoonright_{\mathrm{Ob}(\mathcal{C})}$ is the choice function mentioned in Proposition 1.

*The Axiom of Choice in ZFC provides choice functions based on sets, but not proper classes, so the categories in this proposition refer to the small categories. If you want this proposition to be true in any large categories, you need stronger axiom of choice, which is based on any class. However, it can't be discussed in ZFC.

- A natural transformation (we will verify this) $\theta : \mathrm{id}_{\mathcal{C}} \Rightarrow \iota\kappa$ such that $\theta_X \in \mathrm{Hom}_{\mathcal{C}}(X, \iota\kappa(X))$ is the isomorphism (it's uniquely exists) for all $X \in \mathrm{Ob}(X)$.
- For all $X, Y \in \mathrm{Ob}(X, Y)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, because θ_X is an isomorphism, we have

$$\kappa(X) = \iota\kappa(X) \xrightarrow{\theta_X^{-1}} \mathrm{id}_{\mathcal{C}}(X) \xrightarrow{\mathrm{id}_{\mathcal{C}}(f)} \mathrm{id}_{\mathcal{C}}(Y) \xrightarrow{\theta_Y} \iota\kappa(Y) = \kappa(Y),$$

  then we can define $\kappa(f) := \theta_Y \circ f \circ \theta_X^{-1} \in \mathrm{Hom}_{\mathcal{C}'}(\kappa(X), \kappa(Y))$.

Verify:

- κ is indeed a functor, i.e., it satisfies the three factors of functor on morphisms. It's trivial.
- θ is indeed a natural transformation , i.e., we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{C}}(X) \arrow[r, "\theta_X"] \arrow[d, "\mathrm{id}_{\mathcal{C}}(f)=f"'] & \iota\kappa(X) \arrow[d, "\iota\kappa(f)=\kappa(f)"] \\
\mathrm{id}_{\mathcal{C}}(Y) \arrow[r, "\theta_Y"'] & \iota\kappa(Y).
\end{tikzcd}$$

  We can easily verify the diagram by substituting the definition of $\kappa(f)$.
- $\theta : id_{\mathcal{C}} \stackrel{\sim}{\Rightarrow} \iota\kappa$, i.e., θ is an isomorphism, obviously.

Thus $\mathrm{id}_{\mathcal{C}} \simeq \iota\kappa$. On the other hand, because

$$\forall X \in \mathrm{Ob}(\mathcal{C}')[\kappa\iota(X) = X = \mathrm{id}_{\mathcal{C}'}(X)] \wedge \forall f \in \mathrm{Mor}(\mathcal{C}')[\kappa\iota(f) = f = \mathrm{id}_{\mathcal{C}'}(f)],$$

then we have $\kappa\iota = \mathrm{id}_{\mathcal{C}'}$, that implies $\kappa\iota \simeq \mathrm{id}_{\mathcal{C}'}$. Thus $\mathcal{C}' \sim \mathcal{C}$.

3. It's easy to prove so using Proposition 2 and the transitivity of equivalence (functor).
4. We can easily prove so by assuming contradiction.
5. Suppose $\mathcal{C}$ and $\mathcal{D}$ are skeletal categories, and functor $F : \mathcal{C} \to \mathcal{D}$ is faithful, full and essentially surjective.
   - $F \restriction_{\mathrm{Ob}(\mathcal{C})} \to \mathrm{Ob}(\mathcal{D})$ **is surjective:** Because F is essentially surjective, for all $Y \in \mathrm{Ob}(\mathcal{D})$ there exists $X \in \mathrm{Ob}(\mathcal{C})[F(X) \simeq Y]$. Since $\mathcal{D}$ is a skeletal category, through Proposition 4, we know that $F(X) = Y$.
   - $F \restriction_{\mathrm{Ob}(\mathcal{C})} \to \mathrm{Ob}(\mathcal{D})$ **is injective:** For any $X, Y \in \mathrm{Ob}(\mathcal{C})$, because of faithful and full functor F, if $F(X) = F(Y)$, then we have $X \simeq Y$ using Lemma 3.3. Since $\mathcal{D}$ is a skeletal category, through Proposition 4, we have $X = Y$.
   - $F \restriction_{\mathrm{Mor}(\mathcal{C})} \to \mathrm{Mor}(\mathcal{D})$ **is bijective:** Since F is full and faithful, we have $F \restriction_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is bijective for each $X, Y \in \mathrm{Ob}(\mathcal{C})$. What's more, because $F \restriction_{\mathrm{Ob}(\mathcal{C})} \to \mathrm{Ob}(\mathcal{D})$ is bijective, it's easy to prove that $F \restriction_{\mathrm{Mor}(\mathcal{C})} \to \mathrm{Mor}(\mathcal{D})$ is bijective, too.

   Because $F \restriction_{\mathrm{Ob}(\mathcal{C})} \to \mathrm{Ob}(\mathcal{D})$ and $F \restriction_{\mathrm{Mor}(\mathcal{C})} \to \mathrm{Mor}(\mathcal{D})$ are bijective, F is a functorial isomorphism.

□

Theorem 4. A functor between two categories is an equivalence if and only if it's faithful, full and essentially surjective.

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

($\Longrightarrow$) Let F be a equivalence and $G : \mathcal{D} \to \mathcal{C}$ be its quasi-inverse, so that there exist natural isomorphisms $\theta : GF \stackrel{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{C}}$ and $\psi : FG \stackrel{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$, and we assume that $X, Y \in \mathrm{Ob}(\mathcal{C})$.

Notice the following commutative diagram:

$$\text{(a)} \quad \begin{tikzcd}
GF(X) \arrow[r, "\theta_X"] \arrow[d, "GF(f)=GF(g)"'] & \mathrm{id}_{\mathcal{C}}(X) \arrow[d, bend right, "\mathrm{id}_{\mathcal{C}}(g)"'] \arrow[d, bend left, "\mathrm{id}_{\mathcal{C}}(f)"] \\
GF(Y) \arrow[r, "\theta_Y"'] & \mathrm{id}_{\mathcal{C}}(Y)
\end{tikzcd} \quad \text{and (b)} \quad \begin{tikzcd}
GF(X) \arrow[r, "\theta_X"] \arrow[d, "GF(t)"'] & \mathrm{id}_{\mathcal{C}}(X) \arrow[d, "\mathrm{id}_{\mathcal{C}}(t)=t"] \\
GF(Y) \arrow[r, "\theta_Y"'] & \mathrm{id}_{\mathcal{C}}(Y).
\end{tikzcd}$$

- F **is faithful:** Let $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $F(f) = F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, so $GF(f) = GF(g)$. Since $\theta : GF \stackrel{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{C}}$, we have diagram (a), that means

$$\mathrm{id}_{\mathcal{C}}(g) \circ \theta_X = \theta_Y \circ GF(g) = \theta_Y \circ GF(f) = \mathrm{id}_{\mathcal{C}}(f) \circ \theta_X.$$

  Because θ is a natural isomorphism, using Corollary 1.2 we have θ_X is an isomorphism, so it's a bimorphism (by Corollary 3.1), then we can cancel it. Thus we have $g = \mathrm{id}_{\mathcal{C}}(g) = \mathrm{id}_{\mathcal{C}}(f) = f$. So $F \restriction_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \to \mathrm{Hom}_{\mathcal{C}}(F(X), F(Y))$ is injective, that means F is faithful. For future reference, we observe that a similar proof shows that G is faithful as well.

- **F is full:** For all $\varphi \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, we have $G(\varphi) \in \mathrm{Hom}_{\mathcal{D}}(GF(X), GF(Y))$. Since θ is a natural isomorphism, we have
$$X = \mathrm{id}_{\mathcal{C}}(X) \xrightarrow{\theta_X^{-1}} GF(X) \xrightarrow{G(\varphi)} GF(Y) \xrightarrow{\theta_Y} \mathrm{id}_{\mathcal{C}}(Y) = Y.$$
so we can assume that $t = \theta_Y \circ G(\varphi) \circ \theta_X^{-1} \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we claim that $F(t) = \varphi$, so that $F \upharpoonright_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \to \mathrm{Hom}_{\mathcal{C}}(F(X), F(Y))$ is surjective, that means F is full:
It's easy to verify the diagram (b), thus we have
$$\theta_Y \circ G(\varphi) = \theta_Y \circ G(\varphi) \circ \theta_X^{-1} \circ \theta_X = t \circ \theta_X = \theta_Y \circ GF(t).$$
It's obvious that θ_Y is a bimorphism, then we can cancel it, thus we have $G(\varphi) = GF(t)$. Because φ and $F(t)$ are in the same hom-class, and G is faithful, we have $F(t) = \varphi$.
- **F is essentially surjective:** For any $W \in \mathrm{Ob}(\mathcal{D})$, since $\psi : FG \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$, we have $\psi_W \in \mathrm{Hom}_{\mathcal{D}}(FG(W), \mathrm{id}_{\mathcal{D}}(W))$ is an isomorphism, so $F(G(W)) \overset{\mathrm{M}}{\simeq} \mathrm{id}_{\mathcal{D}}(W) = W$. Hence, F is essentially surjective.

($\Longleftarrow$) (Need AC) Consider the skeleton $\mathcal{C}'$ of category $\mathcal{C}$, the skeleton $\mathcal{D}'$ of category $\mathcal{D}$, inclusion functors $\iota_{\mathcal{C}} : \mathcal{C}' \to \mathcal{C}$ and $\iota_{\mathcal{D}} : \mathcal{D}' \to \mathcal{D}$, skeletal functors $\kappa_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}'$ and $\kappa_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}'$ (see the proof of Lemma 4.2 for the the methods of construction). We define $F' := \kappa_{\mathcal{D}} F \iota_{\mathcal{C}} : \mathcal{C}' \to \mathcal{D}'$.

Through Lemma 4.2 we know that $\kappa_{\mathcal{C}}$ and $\kappa_{\mathcal{D}}$ are equivalence, so they are faithful, full and essentially surjective (we just proved this). Because F and $\iota_{\mathcal{C}}$ are faithful, full and essentially surjective as well, the functor F' between skeletal categories is, too (by Lemma 3.4). Using Lemma 4.5 we have F' is a functorial isomorphism. Suppose $G := \iota_{\mathcal{C}} {F'}^{-1} \kappa_{\mathcal{D}} : \mathcal{D} \to \mathcal{C}$, we claim that G is the quasi-inverse of F:
$$GF = \underbrace{\underline{(\iota_{\mathcal{C}} {F'}^{-1} \kappa_{\mathcal{D}}) F} \mathrm{id}_{\mathcal{C}} \simeq \underline{\iota_{\mathcal{C}} {F'}^{-1} (\kappa_{\mathcal{D}} F \iota_{\mathcal{C}})} \kappa_{\mathcal{C}}}_{\text{since } \mathrm{id}_{\mathcal{C}} \simeq \iota_{\mathcal{C}} \kappa_{\mathcal{C}}} = \iota_{\mathcal{C}} {F'}^{-1} F' \kappa_{\mathcal{C}} = \iota_{\mathcal{C}} \kappa_{\mathcal{C}} \simeq \mathrm{id}_{\mathcal{C}},$$
$$FG = \underbrace{\mathrm{id}_{\mathcal{D}} \underline{F (\iota_{\mathcal{C}} {F'}^{-1} \kappa_{\mathcal{D}})} \simeq \iota_{\mathcal{D}} (\kappa_{\mathcal{D}} \underline{F \iota_{\mathcal{C}}) {F'}^{-1} \kappa_{\mathcal{D}}}}_{\text{since } \mathrm{id}_{\mathcal{D}} \simeq \iota_{\mathcal{D}} \kappa_{\mathcal{D}}} = \iota_{\mathcal{D}} F' {F'}^{-1} \kappa_{\mathcal{D}} = \iota_{\mathcal{D}} \kappa_{\mathcal{D}} \simeq \mathrm{id}_{\mathcal{D}}.$$
Hence, F is an equivalence. □

Corollary 5. (Without AC) Two skeletal categories are equivalent if and only if they are isomorphic, therefore two categories are equivalent if and only if they have isomorphic skeletons.

Proof. For the first half of the proposition, it's easy to prove so using Theorem 4 and Lemma 4.5. For the last half of the proposition, it's easy to prove so using the first half of the proposition, Lemma 4.3 and the transitivity of equivalence. □

Example 6. Consider vector space category $\mathsf{Vect}_{\Bbbk}$ and it's full subcategory $\mathsf{fVect}_{\Bbbk}$. We have defined the dual space contravariant functor DD^{op} in Example 4.3. We now define a mapping $\mathrm{ev}_V : V \to DD^{\mathrm{op}}(V) = (V^\vee)^\vee$ such that $\mathrm{ev}_V(x) = [V^\vee \ni \lambda \mapsto \lambda(x)]$ for all $x \in V$. It's easy to verify that ev_V is an injective linear mapping, and ev_V is bijective if and only if V is a finite dimensional (see [16, §1.3 定理 2]).

For any $V, U \in \mathsf{Vect}_{\Bbbk}$, we have the following diagram:
$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}_V"] \arrow[d, "f"'] & (V^\vee)^\vee \arrow[d, "(f^\vee)^\vee"] \\ U \arrow[r, "\mathrm{ev}_U"'] & (U^\vee)^\vee. \end{tikzcd}$$
To verify that, we just need to prove that $\forall x \in V[(f^\vee)^\vee \circ \mathrm{ev}_V(x) = \mathrm{ev}_U \circ f(x) \in \mathrm{Hom}_{\Bbbk}(U^\vee, \Bbbk)]$:
$$(f^\vee)^\vee \circ \mathrm{ev}_V(x) = (f^\vee)^\vee([(V^\vee)^\vee \ni \alpha \mapsto \alpha(x)]) = [(V^\vee)^\vee \ni \alpha \mapsto \alpha(x)] \circ f^\vee = [U^\vee \ni \beta \mapsto \underbrace{f^\vee(\beta)}_{=\beta \circ f}(x)],$$
$$\mathrm{ev}_U \circ f(x) = \mathrm{ev}_U(f(x)) = [U^\vee \ni \beta \mapsto \big(\beta \circ f\big)(x)].$$
Thus ev is a natural transformation, and when we restrict it to $\mathsf{fVect}_{\Bbbk}$, we have a natural isomorphism $\mathrm{ev} : \mathrm{id}_{\mathsf{fVect}_{\Bbbk}} \overset{\sim}{\Rightarrow} DD^{\mathrm{op}}$, using Corollary 4 we have $\mathrm{id}_{\mathsf{fVect}_{\Bbbk}^{\mathrm{op}}} \simeq D^{\mathrm{op}} D$. Thus $D : \mathsf{fVect}_{\Bbbk}^{\mathrm{op}} \to \mathsf{fVect}_{\Bbbk}$ is the equivalence and D^{op} is its quasi-inverse.

Definition 10. For category $\mathcal{C}$ and its objects X and Z,

- we say X is *initial* if $\forall Y \in \mathrm{Ob}(\mathcal{C}) \exists! f \in \mathrm{Mor}(\mathcal{C})[f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)]$;
- we say X is *terminal* if $\forall Y \in \mathrm{Ob}(\mathcal{C}) \exists! f \in \mathrm{Mor}(\mathcal{C})[f \in \mathrm{Hom}_{\mathcal{C}}(Y, X)]$;
- a initial and terminal object is called *zero object*;
- suppose $0 \in \mathrm{Ob}(\mathcal{C})$ is a zero object, if $f \in \mathrm{Hom}_{\mathcal{C}}(X, 0)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(0, Z)$, then we say $g \circ f$ is a *zero morphism* and denote it as "0_{XZ}".

Corollary 6. In a category $\mathcal{C}$, all the initial objects are isomorphic, and all the terminal objects are isomorphic.

Proof. If $X, Y \in \mathrm{Ob}(\mathcal{C})$ is initial, then $\mathrm{Hom}_{\mathcal{C}}(X, X) = \{\mathit{1}_X\}$ and $\mathrm{Hom}_{\mathcal{C}}(Y, Y) = \{\mathit{1}_Y\}$. Suppose $f \in \mathrm{Hom}_{\mathcal{C}}$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, there must be $g \circ f = \mathit{1}_X$ and $f \circ g = 1_Y$, thus f and g are the isomorphism.

The proof about terminal objects is similar. □

Example 7. Here are some the example of initial and terminal objects.

1. For any pre-ordered set $(S, \preccurlyeq)$, we can construct a category of it, which the objects are the element in S, and

$$\mathrm{Hom}_S(x, y) := \begin{cases} \{\langle x, y\rangle\}, & x \preccurlyeq y, \\ \varnothing, & \text{otherwise}, \end{cases}$$

$$\langle x, y\rangle \circ \langle y, z\rangle := \langle x, z\rangle\,.$$

   - If $(S, \preccurlyeq)$ is a partially ordered set, then the initial and terminal objects (if exist) in it are both unique.
   - If $(S, \preccurlyeq)$ is a linearly ordered set, it's easy to see that $x \in S$ is initial if and only if $x = \min(S)$, and x is terminal if and only if $x = \max(S)$.
   - (See [5, §7]) Consider a fliter $(\mathbb{F}, \subseteq)$ on F and a ideal $(\mathbb{I}, \subseteq)$ on I. Evidently, F is the only terminal object in $\mathbb{F}$, and $\varnothing$ is the only initial object of $\mathbb{I}$; a fliter/ideal has initial/terminal object if and only if it is principal. What's more, any topology with inclusion relation has both objects, it has a zero object if and only if it's empty.

2. In Set, the initial object is $\varnothing$, the terminal objects are the sets that have exact one element.

3. In Rel, $\varnothing$ is the unique initial and unique terminal object.

4. In Grp, trivial group e is a zero object, the homomorphism which the codomain is a trivial group is a zero morphism.

5. In $\mathsf{Vect}_{\Bbbk}$, the null space is a zero object, and zero mapping is a zero morphism.

6. For $X \in \mathrm{Ob}(\mathcal{C})$, $\mathit{1}_X$ the terminal object of slice category $\mathcal{C}/X$; and is the initial object of coslice category $X/\mathcal{C}$.

Definition 11. Consider a category $\mathcal{C}$ and let $\{X_i\}_{i\in I} \subseteq \mathrm{Ob}(\mathcal{C})$ (I is a set) be a family of objects in $\mathcal{C}$, and an object P in $\mathcal{C}$.

For a family of morphisms $\{p_i\}_{i\in I}$ where $p_i \in \mathrm{Hom}_{\mathcal{C}}(P, X_i)$ for each $i \in I$, we say the pair $\langle P, \{p_i\}_{i\in I}\rangle$ is the *product* of $\{X_i\}_{i\in I}$, if for all $Y \in \mathrm{Ob}(\mathcal{C})$, and a family of morphisms $\{f_i\}_{i\in I}$ where $f_i \in \mathrm{Hom}_{\mathcal{C}}(Y, X_i)$ for each $i \in I$, there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(Y, P)$ such that $\forall i \in I[f_i = p_i \circ u]$.
In commutative diagram (where $i, j \in I$):

$$\begin{tikzcd} & Y \arrow[dl, "f_i"'] \arrow[d, dashed, "\exists! u"] \arrow[dr, "f_j"] & \\ X_i & P \arrow[l, "p_i"] \arrow[r, "p_j"'] & X_j. \end{tikzcd}$$

Each p_i is called *projection.*

For a family of morphisms $\{q_i\}_{i\in I}$ where $q_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, P)$ for each $i \in I$, we say the pair $\langle P, \{q_i\}_{i\in I}\rangle$ is the *coproduct* of $\{X_i\}_{i\in I}$, if for all $Y \in \mathrm{Ob}(\mathcal{C})$, and a family of morphisms $\{g_i\}_{i\in I}$ where $g_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$ for each $i \in I$, there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(P, Y)$ such that $\forall i \in I[g_i = u \circ q_i]$.
In commutative diagram (where $i, j \in I$):

$$\begin{tikzcd} X_i \arrow[r, "q_i"] \arrow[dr, "g_i"'] & P \arrow[d, dashed, "\exists! u"] & X_j \arrow[l, "q_j"'] \arrow[dl, "g_j"] \\ & Y. & \end{tikzcd}$$

Each q_i is called *coprojection.*

Lemma 5. In category $\mathcal{C}$, let $\langle P, \{p_i\}_{i\in I}\rangle$ be the product of $\{X_i\}_{i\in I}$ and $k \in \mathrm{Hom}_{\mathcal{C}}(P, P)$ satisfies $\forall i \in I[p_i \circ k = p_i]$, then $k = \mathit{1}_P$. Similarly, if $\langle P, \{q_i\}_{i\in I}\rangle$ is the coproduct and $\forall i \in I[k \circ q_i = q_i]$, then $k = \mathit{1}_P$.

Proof. According to the definition of product, for P and $\{p_i\}_{i\in I}$ themselves, we have the following diagram:

$$\begin{tikzcd} P \arrow[d, dashed, "\exists! u"'] \arrow[dr, "p_i"] & \\ P \arrow[r, "p_i"'] & X_i. \end{tikzcd}$$

Because $\forall i \in I[p_i \circ k = p_i = p_i \circ \mathit{1}_P]$ and the morphism u is unique, we have $k = u = \mathit{1}_P$.

The proof about coproduct is similar. □

Theorem 5. Consider a category $\mathcal{C}$ and its objects P, Q. If $\langle P, \{p_i\}_{i\in I}\rangle$ and $\langle Q, \{q_i\}_{i\in I}\rangle$ are both the products/coproducts of the fixed family of objects $\{X_i\}_{i\in I}$, then $P \overset{\mathrm{M}}{\simeq} Q$.

Proof. According to the definition of product, for any $i \in I$ we have the following diagram:

$$\begin{tikzcd}
& P \arrow[dl, "p_i"'] \arrow[d, dashed, "\exists! u"] \\
X_i & Q \arrow[l, "q_i"] \arrow[d, dashed, "\exists! v"] \\
& P. \arrow[ul, "p_i"]
\end{tikzcd}$$

That is $p_i = q_i \circ u$ and $q_i = p_i \circ v$, substituting each other leads to

$$\forall i \in I [p_i = p_i \circ v \circ u \wedge q_i = q_i \circ u \circ v].$$

Using Proposition 1, we get $v \circ u = \mathit{1}_P$ and $u \circ v = \mathit{1}_Q$, thus u and v are the isomorphisms between P and Q.

The proof about coproduct is similar, just observe the following diagram:

$$\begin{tikzcd}
P \arrow[d, dashed, "\exists! u"'] & \\
Q \arrow[d, dashed, "\exists! v"'] & X_i. \arrow[ul, "p_i"'] \arrow[l, "q_i"] \arrow[dl, "p_i"] \\
P &
\end{tikzcd}$$

□

Definition 12. For a family of small categories $\{\mathcal{C}_i\}_{i\in I}$, we define:

Product category $\prod_{i\in I} \mathcal{C}_i$ (using Cartesian product):

$$\begin{aligned}
\mathrm{Ob}(\prod_{i\in I} \mathcal{C}_i) &:= \prod_{i\in I} \mathrm{Ob}(\mathcal{C}_i);\\
\mathrm{Hom}_{\prod_{i\in I}\mathcal{C}_i}(\{X_i\}_I, \{Y_i\}_I) &:= \prod_{i\in I} \mathrm{Hom}(X_i, Y_i);\\
\{f_i\}_I \circ^{\prod} \{g_i\}_I &:= \{f_i \circ^{\mathcal{C}_i} g_i\}_I;\\
\mathit{1}^{\prod}_{\{X_i\}_I} &:= \{1_{X_i}\}_I;
\end{aligned}$$

where $\{X_i\}_I, \{Y_i\}_I \in \mathrm{Ob}(\prod_{i\in I}\mathcal{C}_i)$ and $\{f_i\}_I, \{g_i\}_I \in \mathrm{Mor}(\prod_{i\in I}\mathcal{C}_i)$.

If $\{\mathrm{Ob}(\mathcal{C}_i)\}_{i\in I}$ is disjoint, we define the the *coproduct category* $\coprod_{i\in I}\mathcal{C}_i$ of $\{\mathcal{C}_i\}_{i\in I}$ as follow:

$$\begin{aligned}
\mathrm{Ob}(\coprod_{i\in I} \mathcal{C}_i) &:= \bigsqcup_{i\in I} \mathrm{Ob}(\mathcal{C}_i);\\
\mathrm{Hom}_{\coprod_{i\in I}\mathcal{C}_i}(X_j, Y_k) &:= \begin{cases} \mathrm{Hom}_{\mathcal{C}_j}(X_j, Y_k), & j = k,\\ \varnothing, & j \neq k, \end{cases}
\end{aligned}$$

where $i, j \in I$, and $X_j \in \mathrm{Ob}(\mathcal{C}_j)$, $Y_k \in \mathrm{Ob}(\mathcal{C}_k)$; the composition of morphisms is defined as the composition in each $\mathcal{C}_i$.

It's easy to verify that the product/coproduct category is indeed a category. We can define the *projection functor* $\mathbf{p}_j : \prod_{i\in I}\mathcal{C}_i \to \mathcal{C}_j$ for each $j \in I$ such that $\mathbf{p}_j(\{X_i\}_{i\in I}) = X_j$ and $\mathbf{p}_j(\{f_i\}_{i\in I}) = f_j$, and the definition of inclusion functor $\iota_j : \mathcal{C}_j \to \coprod_{i\in I}\mathcal{C}_i$ is obvious. Moreover, we denote $\prod\{\mathcal{C}_1, \mathcal{C}_2\}$ as $\mathcal{C}_1 \times \mathcal{C}_2$.

Corollary 7. In Cat, for a collection of small category $\{\mathcal{C}_i\}_{i\in I}$, the product category with a collection of projection functor $\left\langle \prod_{i\in I}\mathcal{C}_i, \{\mathbf{p}_i\}_{i\in I}\right\rangle$ is a product of $\{\mathcal{C}_i\}_{i\in I}$, and the coproduct category with a collection of inclusion functor $\left\langle \coprod_{i\in I}\mathcal{C}_i, \{\iota_i\}_{i\in I}\right\rangle$ is a coproduct of $\{\mathcal{C}_i\}_{i\in I}$.

Definition 13. For small categories $\mathcal{C}$ and $\mathcal{D}$, we now define *functor category* $\mathrm{Fct}(\mathcal{C}, \mathcal{D})$: The objects are the functors $F : \mathcal{C} \to \mathcal{D}$, the morphisms between F and G are the natural transformations $\theta : F \Rightarrow G$, the composition of morphisms is the vertical composition of natural transformations. [*]

Definition 14. Observe the following commutative diagram in category $\mathcal{C}$:

$$\begin{tikzcd}
& & X_1 \arrow[dr, "b_1"] \arrow[drr, bend left, "q_1"] & & \\
P \arrow[urr, bend left, "p_1"] \arrow[r, dashed, "\exists! u"] \arrow[drr, bend right, "p_2"'] & A \arrow[ur, "a_1"] \arrow[dr, "a_2"'] & & B \arrow[r, dashed, "\exists! v"] & Q \\
& & X_2 \arrow[ur, "b_2"'] \arrow[urr, bend right, "q_2"'] & &
\end{tikzcd}$$

[*]Note that the functor category $\mathrm{Fct}(\mathcal{C}, \mathcal{D})$ is also a small category. Because evidently $\mathrm{Ob}(\mathrm{Fct}(\mathcal{C}, \mathcal{D}))$ is a set, and for any functors $F, G : \mathcal{C} \to \mathcal{D}$, the family of the natural transformations between F and G is the subset of the set $\prod_{X\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$.

We say the pair $\langle A, a_1, a_2 \rangle$ is the *pullback* of b_1 and b_2, if $b_1 \circ a_1 = b_2 \circ a_2$ and satisfies:

$$\forall P \in \mathrm{Ob}(\mathcal{C}) \forall p_1 \in \mathrm{Hom}_{\mathcal{C}}(P, X_1) \forall p_2 \in \mathrm{Hom}(P, X_2) \exists! u \in \mathrm{Hom}(P, A)[a_1 \circ u = p_1 \land a_2 \circ u = p_2].$$

We say the pair $\langle B, b_1, b_2 \rangle$ is the *pushout* of a_1 and a_2, if $b_1 \circ a_1 = b_2 \circ a_2$ and satisfies

$$\forall Q \in \mathrm{Ob}(\mathcal{C}) \forall q_1 \in \mathrm{Hom}_{\mathcal{C}}(X_1, Q) \forall q_2 \in \mathrm{Hom}(X_2, Q) \exists! v \in \mathrm{Hom}(B, Q)[v \circ b_1 = q_1 \land v \circ b_2 = q_2].$$

Definition 15. For category $\mathcal{C}$ and its object A, we now define *hom-functor* as follow:
Covariant hom-functor $\mathsf{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathsf{Set}$ satisfies

$$\begin{aligned} \mathsf{Hom}_{\mathcal{C}}(A, X) &:= \mathrm{Hom}_{\mathcal{C}}(A, X), \\ \mathsf{Hom}_{\mathcal{C}}(A, f) &:= [\mathrm{Hom}_{\mathcal{C}}(A, \mathrm{dom}(f)) \ni g \mapsto f \circ g]. \end{aligned}$$

Contravariant hom-functor $\mathsf{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \to \mathsf{Set}$ satisfies

$$\begin{aligned} \mathsf{Hom}_{\mathcal{C}}(X, A) &:= \mathrm{Hom}_{\mathcal{C}}(X, A), \\ \mathsf{Hom}_{\mathcal{C}}(f, A) &:= [\mathrm{Hom}_{\mathcal{C}}(\mathrm{cod}(f), A) \ni g \mapsto g \circ f]. \end{aligned}$$

We immediately get $\mathsf{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, -) = \mathsf{Hom}_{\mathcal{C}}(-, A)$.

Theorem 6 (Yoneda Lemma)**.** Consider a category $\mathcal{C}$, we define:

$$\mathcal{C}^{\wedge} := \mathrm{Fct}(\mathcal{C}, \mathsf{Set}), \quad \mathcal{C}^{\vee} := \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}),$$

and *Yoneda embeddings*:

$$\begin{aligned} h_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{\wedge}, &\quad h_{\mathcal{C}}(X) := \mathsf{Hom}_{\mathcal{C}}(X, -); \\ k_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{\vee}, &\quad k_{\mathcal{C}}(X) := \mathsf{Hom}_{\mathcal{C}}(-, X). \end{aligned}$$

We now claim that:
For all $S \in \mathrm{Ob}(\mathcal{C})$ and $F \in \mathcal{C}^{\wedge}$, the mapping

$$\Theta : \mathrm{Hom}_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S), F) \to F(S) \phi \mapsto u_S := \phi_S(1_S)$$

is a bijection.
$\mathcal{C}^{\vee}$ and $k_{\mathcal{C}}$ have the similar property.

Proof. For any $T \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{c}a(S, T)$, we have

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(S, S) \arrow[rrr, "\mathsf{Hom}_{\mathcal{C}}(A,f)"'] \arrow[ddd, "\phi_S"] & & & \mathrm{Hom}_{\mathcal{C}}(S, T) \arrow[ddd, "\phi_T"'] \\
& 1_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& u \arrow[r, mapsto] & \big(F(f)\big)(u_S) = \phi_X(f) & \\
F(S) \arrow[rrr, "F(f)"] & & & F(T),
\end{tikzcd}$$

thus $\big(F(f)\big)(u_S) = \phi_X(f)$. It's obvious to see that it Θ is a bijection. □

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