A Brief Introduction of Natural Transformation

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Definition 1. A *category* $\mathcal{C}$ consists of:

- A collection $\mathrm{Ob}(\mathcal{C})$ of *objects* $X, Y, Z \cdots$
- A collection
$$\mathrm{Mor}(\mathcal{C}) = \bigcup_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{dom}_{\mathcal{C}}(X) = \bigcup_{Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{cod}_{\mathcal{C}}(Y) = \bigcup_{X,Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$
of *morphisms* $f, g, h \cdots$ with a binary operation " $\circ$" which is defined on the subclass of $\mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C})$, where $\mathrm{dom}_{\mathcal{C}}(X)$ is the *domain* of its elements as well as $\mathrm{cod}_{\mathcal{C}}(Y)$ is the *codomain* of its elements, and $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the intersection of both. For $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \wedge g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, the binary operation that defines *composite morphism* satisfies:
  1. $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$;
  2. $\forall h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)[h \circ (g \circ f) = (h \circ g) \circ f]$;
  3. $\forall h \in \mathrm{Hom}_{\mathcal{C}}(Y, X) \exists \mathbf{1}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)[f \circ \mathbf{1}_X = f \wedge \mathbf{1}_X \circ h = h]$.

It's easy to verify that the *identity morphism* $\mathbf{1}_X$ is unique for all $X \in \mathrm{Ob}(\mathcal{C})$.
In commutative diagram,

$$X \xrightarrow{f} Y \quad \text{means} \quad f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), \qquad \begin{tikzcd} X \arrow[d, "f"] \arrow[dr, "h"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd} \quad \text{means} \quad g \circ f = h.$$

Definition 2. A *functor* $F : \mathcal{C} \to \mathcal{D}$, between category $\mathcal{C}$ and $\mathcal{D}$, consists the following data:

- Mapping $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
- Mapping $F : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, which satisfies:
  1. $F[\mathrm{Hom}_{\mathcal{C}}(X, Y)] \subseteq \mathrm{Hom}_{\mathcal{D}}(F(X), G(Y))$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$;
  2. For all $f, g \in \mathrm{Mor}(\mathcal{C})$, if $g \circ f$ is defined in $\mathrm{Mor}(\mathcal{C})$, then $F(g \circ f) = F(g) \circ F(f)$;
  3. $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$.

In commutative diagram,
$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$
means that F is the functor between $\mathcal{C}$ and $\mathcal{D}$.

For functors $F : \mathcal{C}_1 \to \mathcal{C}_2, G : \mathcal{C}_2 \to \mathcal{C}_3$, the composition $GF : \mathcal{C}_1 \to \mathcal{C}_3$ between both satisfies:
$$GF(X) = G(F(X)) \text{ and } GF(f) = G(F(f)) \text{ for all } X \in \mathrm{Ob}(\mathcal{C}_1), f \in \mathrm{Mor}(\mathcal{C}_1).$$
It's trivial to verify that the composition is also a functor and it satisfy associative law.

For any category $\mathcal{C}$, there exists a *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that satisfies
$$\mathrm{id}_{\mathcal{C}}(X) = X \text{ and } \mathrm{id}_{\mathcal{C}}(f) = f \text{ for all } X \in \mathrm{Ob}(\mathcal{C}_1), f \in \mathrm{Mor}(\mathcal{C}_1).$$
It's easy to verify that $F\mathrm{id}_{\mathcal{C}} = F$ and $\mathrm{id}_{\mathcal{C}}G = G$ for all functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.

Definition 3. The *natural transformation* θ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a mapping from $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Mor}(\mathcal{D})$ whose each value satisfies
$$\theta_X := \theta(X) \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$$

and the commutative diagram below:

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"'] & G(Y), \end{tikzcd} \tag{1}$$

where $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. In other words, we can record the above natural transformation as $\theta : F \Rightarrow G$, or in such a commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D}. \arrow[from=U, to=D, Rightarrow, "\theta"] \end{tikzcd}$$

We may use symbol "$F(\theta)_X$" instead of "$F((\theta)_X)$" in some particular case (such as there are more than one symbols of natural transformations in the brackets).

For any functor $F : \mathcal{C} \to \mathcal{D}$, there exists a *identity transformation* $\mathbf{id}^F : F \to F$ that satisfies

$$\forall X \in \mathrm{Ob}(\mathcal{C})[\mathbf{id}^F_X = \mathbf{1}_{F(X)}].$$

For functors $F : \mathcal{C}_1 \to \mathcal{C}_2, G : \mathcal{C}_2 \to \mathcal{C}_3$, it's trivial to prove the following formula using the definition of identity morphism and transformation:

$$\forall X \in \mathrm{Ob}(\mathcal{C}_1)[G(\mathbf{id}^F_X) = \mathbf{id}^G_{F(X)}]. \tag{2}$$

Definition 4. For functors $F, G, H : \mathcal{C} \to \mathcal{D}$, natural transformations $\theta : F \Rightarrow G$ and $\psi : G \Rightarrow H$, the element of *vertical composition* of the natural transformations is defined as $(\psi \odot \theta)_X = \psi_X \circ \theta_X$. In commutative diagrams forms,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=60, "H"'{name=D}] & \mathcal{D} \arrow[from=U, to=M, Rightarrow, "\theta"] \arrow[from=M, to=D, Rightarrow, "\psi"] \end{tikzcd} \quad \text{means} \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "H"'{name=D}] & \mathcal{D}. \arrow[from=U, to=D, Rightarrow, "\psi\odot\theta"] \end{tikzcd}$$

Actually, we need to prove that the definition is well-defined, i.e., to verify that $(\psi \odot \theta)_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), H(X))$ for all $X \in \mathrm{Ob}(\mathcal{C})$, it's easy to do so.

Definition 5. For functors $F, F' : \mathcal{C}_1 \to \mathcal{C}_2$ and $G, G' : \mathcal{C}_2 \to \mathcal{C}_3$, natural transformations $\theta : F \Rightarrow F'$ and $\psi : G \Rightarrow G'$, the element of *horizontal composition* of natural transformations $(\psi \ominus \theta)_X$ is defined as $G'(\theta_X) \circ \psi_{F(X)} = \psi_{F'(X)} \circ G(\theta_X)$. In commutative diagrams forms,

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "F"{name=U1}] \arrow[r, bend right, "F'"'{name=D1}] & \mathcal{C}_2 \arrow[r, bend left, "G"{name=U2}] \arrow[r, bend right, "G'"'{name=D2}] & \mathcal{C}_3 \arrow[from=U1, to=D1, Rightarrow, "\theta"] \arrow[from=U2, to=D2, Rightarrow, "\psi"] \end{tikzcd} \quad \textit{means} \quad \begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "GF"{name=U}] \arrow[r, bend right, "G'F'"'{name=D}] & \mathcal{C}_3, \arrow[from=U, to=D, Rightarrow, "\psi\ominus\theta"] \end{tikzcd}$$

which satisfy

$$\begin{tikzcd} GF(X) \arrow[r, "G(\theta_X)"] \arrow[d, "\psi_{F(X)}"'] & GF'(X) \arrow[d, "\psi_{F'(X)}"] \\ G'F(Y) \arrow[r, "G'(\theta_X)"'] & G'F'(Y). \end{tikzcd} \tag{3}$$

Actually, we need to prove that the definition is well-defined, i.e., to verify the commutative diagram and that $(\psi \ominus \theta)_X \in \mathrm{Hom}_{\mathcal{C}_3}(GF(X), G'F'(X))$ for all $X \in \mathrm{Ob}(\mathcal{C}_1)$, it's easy to do so observing commutative diagram (1).

Theorem 1. The vertical and horizontal compositions of natural transformations are natural transformations.

Proof. For

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left=60, "F"{name=U1}] \arrow[r, "G"{name=M1, description}] \arrow[r, bend right=60, "H"'{name=D1}] & \mathcal{C}_2 \arrow[r, bend left=60, "F'"{name=U2}] \arrow[r, "G'"{name=M2, description}] \arrow[r, bend right=60, "H'"'{name=D2}] & \mathcal{C}_3 \arrow[from=U1, to=M1, Rightarrow, "\theta"] \arrow[from=M1, to=D1, Rightarrow, "\psi"] \arrow[from=U2, to=M2, Rightarrow, "\theta'"] \arrow[from=M2, to=D2, Rightarrow, "\psi'"] \end{tikzcd} \ ,$$

we need to verify the following commutative diagrams:

$$
(a)\ \begin{tikzcd}
F(X) \arrow[r, "(\psi\odot\theta)_X"] \arrow[d, "F(f)"'] & H(X) \arrow[d, "H(f)"] \\
F(y) \arrow[r, "(\psi\odot\theta)_X"'] & H(y)
\end{tikzcd}
\quad \text{and (b)} \quad
\begin{tikzcd}
GF(X) \arrow[r, "(\theta'\ominus\theta)_X"] \arrow[d, "GF(f)"] & G'F'(X) \arrow[d, "G'F'(f)"] \\
GF(Y) \arrow[r, "(\theta'\ominus\theta)_Y"'] & G'F'(Y)
\end{tikzcd}.
$$

From (a), we have

$$
\begin{aligned}
&H(f)\circ(\psi\odot\theta)_X && \text{(Assumption)}\\
=&H(f)\circ\psi_X\circ\theta_X && \text{(Def: vertical composition)}\\
=&(\psi_Y\circ G(f))\circ\theta_X && \text{(Property of natural transformation } \psi)\\
=&\psi_Y\circ\theta_Y\circ F(f) && \text{(Property of natural transformation } \theta)\\
=&(\psi\odot\theta)_X\circ F(f), && \text{(Def: vertical composition)}
\end{aligned}
$$

thus $(\psi\odot\theta)$ is natural transformation.

From (b), we have

$$
\begin{aligned}
&G'F'(f)\circ(\theta'\ominus\theta)_X && \text{(Assumption)}\\
=&G'F'(f)\circ\theta'_{F'(X)}\circ G(\theta_X) && \text{(Def: horizontal composition)}\\
=&\theta'_{F'(Y)}\circ GF'(f)\circ G(\theta_X) && \text{(Property of natural transformation } \theta')\\
=&\theta'_{F'(Y)}\circ G(F'(f)\circ G(\theta_X)) && \text{(Property of functor } G)\\
=&\theta'_{F'(Y)}\circ G(\theta_Y\circ F(f)) && \text{(Property of natural transformation } \theta)\\
=&\theta'_{F'(Y)}\circ G(\theta_Y)\circ GF(f) && \text{(Property of functor } G)\\
=&(\theta'\ominus\theta)_Y\circ GF(f), && \text{(Def: horizontal composition)}
\end{aligned}
$$

thus $(\psi\ominus\theta)$ is natural transformation.

What's more, we can prove that $(\psi\odot\theta)\ominus(\psi'\odot\theta')=(\psi'\ominus\psi)\odot(\theta'\ominus\theta)$:

$$
\begin{aligned}
&((\psi'\odot\theta')\ominus(\psi\odot\theta))_X && \text{(Assumption)}\\
=&(\psi'\odot\theta')_{H(X)}\circ F'(\psi\odot\theta)_X && \text{(Def: horizontal composition)}\\
=&\psi'_{H(X)}\circ\theta_{H(X)}\circ F'(\theta_X)\circ F'(\theta_X) && \text{(Def: vertical composition, Property of functor } F')\\
=&\psi'_{H(X)}\circ(G'(\psi_X)\circ\theta'_{G(X)})\circ F'(\theta_X) && \text{(Commutative diagram (3))}\\
=&(\psi'\ominus\psi)_X\circ(\theta'\ominus\theta)_X && \text{(Def: horizontal composition)}\\
=&((\psi'\ominus\psi)\odot(\theta'\ominus\theta))_X, && \text{(Def: vertical composition)}
\end{aligned}
$$

where $X\in\mathrm{Ob}(\mathcal{C})$. □

Theorem 2. Both vertical and horizontal compositions of natural transformations satisfy associative law.

Proof. For vertical composition, observe the following commutative diagram and natural transformations:
For

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend left=20, "G"] \arrow[r, bend right=20, "H"'] \arrow[r, bend right=60, "K"'] & \mathcal{D}
\end{tikzcd}
\quad \text{as well as} \quad
\begin{cases}
\theta: F\Rightarrow G,\\
\psi: G\Rightarrow H,\\
\phi: H\Rightarrow K,
\end{cases}
$$

it's trivial to prove that $((\phi\odot\psi)\odot\theta)_X=(\phi\odot(\psi\odot\theta))_X$ for all $X\in\mathrm{Ob}(\mathcal{C})$, thus the vertical composition satisfies associative law.
For

$$
\begin{tikzcd}
\mathcal{C}_1 \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}_2 \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{C}_3 \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "H'"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{C}_4,
\end{tikzcd}
$$

we have

$$\begin{aligned}
&(\phi \ominus (\psi \ominus \theta))_X && \text{(Assumption)}\\
=&\phi_{G'F'(X)} \circ H(\psi \ominus \theta)_X && \text{(Def: horizontal composition)}\\
=&\phi_{G'F'(X)} \circ H(\psi_{F'(X)} \circ G(\theta_X)) && \text{(Ditto)}\\
=&\phi_{G'F'(X)} \circ H(\psi_{F'(X)}) \circ HG(\theta_X) && \text{(Property of functor } H\text{)}\\
=&(\phi \ominus \psi)_{F'(X)} \circ HG(\theta_X) && \text{(Def: horizontal composition)}\\
=&((\phi \ominus \psi) \ominus \theta)_X, && \text{(Ditto)}
\end{aligned}$$

thus the horizontal composition satisfies associative law. □

Theorem 3. Observe the following natural commutative:

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, "F"] & \mathcal{C}_2 \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}_3 \arrow[r, "K"] & \mathcal{C}_4,
\end{tikzcd}$$

we have

$$\theta \odot \mathbf{id}^G = \theta = \mathbf{id}^H \odot \theta, \tag{4}$$
$$(\theta \ominus \mathbf{id}^F)_X = \theta_{F(X)}, \tag{5}$$
$$(\mathbf{id}^K \ominus \theta)_Y = K(\theta_Y), \tag{6}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1), Y \in \mathrm{Ob}(\mathcal{C}_2)$.

Proof. It's trivial to prove (4) and using definition of identity transformation and vertical composition.

For (5), we have

$$\begin{aligned}
&(\theta \ominus \mathbf{id}^F)_X && \text{(Assumption)}\\
=&\theta_{F(X)} \circ G(\mathbf{id}^F_X) && \text{(Def: horizontal composition)}\\
=&\theta_{F(X)} \circ \mathbf{id}^G_{F(X)} && \text{(formula 2)}\\
=&\theta_{F(X)} && \text{(formula 4)}
\end{aligned}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

For (6), we have

$$\begin{aligned}
&(\mathbf{id}^K \ominus \theta)_Y && \text{(Assumption)}\\
=&K(\theta_Y) \circ \mathbf{id}^K_{G(Y)} && \text{(Def: horizontal composition)}\\
=&K(\theta_Y) \circ K(\mathbf{id}^G_Y) && \text{(formula 2)}\\
=&K(\theta_Y \circ \mathbf{id}^G_Y) && \text{(Property of functor } K\text{)}\\
=&K(\theta_Y) && \text{(formula 4)}
\end{aligned}$$

for all $Y \in \mathrm{Ob}(\mathcal{C}_2)$. □

Lemma 1 (ZF)**.** For any sets A, B and mapping $f : A \to B$, we have

$$\exists g : B \to A[g \circ f = \mathrm{id}_A] \iff f \text{ is a injection} \iff \forall C \forall h, h' : C \to A[f \circ h = f \circ h' \implies h = h'],$$

$$\exists g : B \to A[f \circ g = \mathrm{id}_B] \implies f \text{ is a surjection} \iff \forall C \forall h, h' : B \to C[h \circ f = h' \circ f \implies h = h'],$$

and f is a surjection $\implies \exists g : B \to A[f \circ g = \mathrm{id}_B]$ can be proved in ZFC. It's easy to see that f is bijection if and only if it has both left and right inversal mappings, and obviously the two inversal mappings are the same one, which is unique.

Definition 6. Consider $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. If there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ that $f \circ g = \mathbf{1}_Y$, then we say g is the *right inverse* of it; if $g \circ f = \mathbf{1}_X$, then we say g is the *left inverse* of it. If f has both inverses, then we say f is *isomorphic*, it's easy to verify that the two inverses are the same one, which is unique as well, so we say $f^{-1} := g$ is the *inverse* (*inversal morphism*) of it.

What's more, it's trivial to verify that the composition of isomorphic morphisms are also isomorphic, so we can find that the collection of *automorphisms* $\mathrm{Aut}_{\mathcal{C}}(X) := \{f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid f \text{ is isomorphic}\}$ is a group $\langle \mathrm{Aut}_{\mathcal{C}}(X), \circ, \mathbf{1}_X \rangle$. Therefore if f has a left inverse, then f is a *injective morphism*, which satisfies the *left cancellation law*: $\forall Z \in \mathrm{Ob}(C) \forall g, h \in \mathrm{Hom}_{\mathcal{C}}(Z, X)[f \circ g = f \circ h \iff g = h]$; if f has a right inverse, then f is a *surjective morphism*, which satisfies the *right cancellation law*: $\forall Z \in \mathrm{Ob}(C) \forall g, h \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)[g \circ f = h \circ f \iff g = h]$.

Consider a functor $F : \mathcal{C} \to \mathcal{D}$, we can define the inverse of functor in the same way: If there exists a functor $G : \mathcal{D} \to \mathcal{C}$ that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$, then we say F is isomorphic (F is a *isomorphism* between $\mathcal{C}$ and $\mathcal{D}$), and $F^{-1} := G$ is the unique inverse (*inversal functor*) of F. From Lemma 1 we know that F is isomorphic if and only if $F\restriction_{\mathrm{Ob}(\mathcal{C})}$ and $F\restriction_{\mathrm{Mor}(\mathcal{C})}$ are both bijection.

Consider the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\theta \quad \psi\Uparrow"] & \mathcal{D}. \end{tikzcd}$$

If $\psi \odot \theta = \mathbf{id}^F$ and $\theta \odot \psi = \mathbf{id}^G$, then we say θ is isomorphic between F and G, and $\theta^{-1} := \psi$ is the unique inverse (*inversal transformation*) of θ, we record it as $\theta : F \overset{\sim}{\Rightarrow} G$. It's trivial to prove that θ is isomorphic if and only if θ_X is isomorphic for each $X \in \mathrm{Ob}(\mathcal{C})$, thus we have $(\theta^{-1})_X = (\theta_X)^{-1}$, we abbreviate it as "$\theta_X^{-1}$".

For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, if there exist isomorphic $\theta : FG \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$ and $\psi : GF \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{C}}$, then we say G is the *quasi-inverse* of F, and F is a *equivalence* between $\mathcal{C}$ and $\mathcal{D}$. If there exist an equivalence between two categories $\mathcal{C}$ and $\mathcal{D}$, we say they are *equivalent* and record it as $\mathcal{C} \simeq \mathcal{D}$.

It's tirvial to verify that the composition of any isomorphic morphisms/functors/transformations is isomorphic, and the composition of any equivalences are equivalence.