A Brief Introduction of Natural Transformation

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Definition 1. A category C consists of:

- A collection Ob(C) of objects $X, Y, Z \cdots$
- A collection

$$\operatorname{Mor}(\mathcal{C}) = \bigsqcup_{X \in \operatorname{Ob}(\mathcal{C})} \operatorname{dom}_{\mathcal{C}}(X) = \bigsqcup_{Y \in \operatorname{Ob}(\mathcal{C})} \operatorname{cod}_{\mathcal{C}}(Y) = \bigsqcup_{X,Y \in \operatorname{Ob}(\mathcal{C})} \operatorname{Hom}_{\mathcal{C}}(X,Y)$$

of morphisms $f, g, h \cdots$ with a binary operation " \circ " which is defined on the subclass of $\operatorname{Mor}(\mathcal{C}) \times \operatorname{Mor}(\mathcal{C})$, where $\operatorname{dom}_{\mathcal{C}}(X)$ is the domain of its elements as well as $\operatorname{cod}_{\mathcal{C}}(Y)$ is the codomain of its elements, and $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is the intersection of both. For $W, X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ and $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \wedge g \in \operatorname{Hom}_{\mathcal{C}}(Y,Z)$, the binary operation satisfies:

- 1. $g \circ f \in \operatorname{Hom}_{\mathcal{C}}(X, Z);$
- 2. $\forall h \in \text{Hom}_{\mathcal{C}}(Z, W)[h \circ (g \circ f) = (h \circ g) \circ f];$
- 3. $\forall h \in \operatorname{Hom}_{\mathcal{C}}(Y, X) \exists \mathbf{1}_X \in \operatorname{Hom}_{\mathcal{C}}(X, X) [f \circ \mathbf{1}_X = f \wedge \mathbf{1}_X \circ h = h].$

It's easy to verify that the identity morphism $\mathbf{1}_X$ is unique for all $X \in \mathrm{Ob}(\mathcal{C})$. In commutative diagram,

$$X \xrightarrow{f} Y \quad means \quad f \in \operatorname{Hom}_{\mathcal{C}}(X,Y), \qquad \downarrow^{f} \qquad means \quad g \circ f = h.$$

$$Y \xrightarrow{g} Z$$

Definition 2. A functor $F: \mathcal{C} \to \mathcal{D}$, between category \mathcal{C} and \mathcal{D} , consists the following data:

- $Mapping F : Ob(\mathcal{C}) \to Ob(\mathcal{D}).$
- Mapping $F : \operatorname{Mor}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{D})$, which satisfies:
 - 1. $F[\operatorname{Hom}_{\mathcal{C}}(X,Y)] \subseteq \operatorname{Hom}_{\mathcal{D}}(F(X),G(Y))$ for all $X,Y \in \operatorname{Ob}(\mathcal{C})$;
 - 2. For all $f, g \in \operatorname{Mor}(\mathcal{C})$, if $g \circ f$ is defined in $\operatorname{Mor}(\mathcal{C})$, then $F(g \circ f) = F(g) \circ F(f)$;
 - 3. $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$ for all $X \in Ob(\mathcal{C})$.

In commutative diagram,

$$\mathcal{C} \stackrel{F}{\longrightarrow} \mathcal{D}$$

means that F is the functor between C and D.

For functors $F: \mathcal{C}_1 \to \mathcal{C}_2, G: \mathcal{C}_2 \to \mathcal{C}_3$, the composition $GF: \mathcal{C}_1 \to \mathcal{C}_3$ between both satisfies:

$$GF(X) = G(F(X))$$
 and $GF(f) = G(F(f))$ for all $X \in Ob(\mathcal{C}_1)$, $f \in Mor(\mathcal{C}_1)$.

For any category C, there exists a identity functor $id_C: C \to C$ that satisfies

$$id_{\mathcal{C}}(X) = X \text{ and } id_{\mathcal{C}}(f) = f \text{ for all } X \in Ob(\mathcal{C}_1), f \in Mor(\mathcal{C}_1).$$

It's easy to verify that $\operatorname{Fid}_{\mathcal{C}} = F$ and $\operatorname{id}_{\mathcal{C}} G = G$ for all functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$.

Definition 3. The natural transformation θ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a mapping from $Ob(\mathcal{C})$ to $Mor(\mathcal{D})$ whose each value satisfies

$$\theta(X) = \theta_X \in \operatorname{Hom}_{\mathcal{D}}(F(X), G(X))$$

and the commutative diagram below:

$$F(X) \xrightarrow{\theta_X} G(X)$$

$$F(f) \downarrow \qquad \qquad \downarrow G(f)$$

$$F(Y) \xrightarrow{\theta_Y} G(Y),$$

$$(1)$$

where $X, Y \in \text{Ob}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(X, Y)$. In other words, we can record the above natural transformation as $\theta : F \Rightarrow G$, or in such a commutative diagram:

$$\mathcal{C} \underbrace{ \left(\begin{array}{c} F \\ \theta \end{array} \right) \mathcal{D}}_{G}$$

We may use symbol " $F(\theta)_X$ " instead of " $F((\theta)_X)$ " in some particular case (such as there are more than one symbols of natural transformations in the brackets).

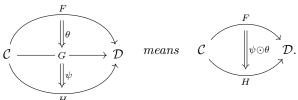
For any functor $F: \mathcal{C} \to \mathcal{D}$, there exists a identity transformation $id^F: F \to F$ that satisfies

$$\forall X \in \mathrm{Ob}(\mathcal{C})[\mathrm{id}_X^F = \mathbf{1}_{F(X)}].$$

For functors $F: \mathcal{C}_1 \to \mathcal{C}_2, G: \mathcal{C}_2 \to \mathcal{C}_3$, it's trivial to prove the following formula using the definition of identity morphism and transformation:

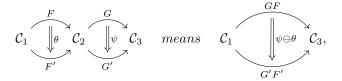
$$\forall X \in \mathrm{Ob}(\mathcal{C}_1)[G(\mathrm{id}_X^F) = \mathrm{id}_{F(X)}^G]. \tag{2}$$

Definition 4. For functors $F, G, H : \mathcal{C} \to \mathcal{D}$, natural transformations $\theta : F \Rightarrow G$ and $\psi : G \Rightarrow H$, the element of longitudinal composition of the natural transformations is defined as $(\psi \odot \theta)_X = \psi_X \circ \theta_X$. In commutative diagrams forms,



Actually, we need to prove that the definition is well-defined, i.e., to verify that $(\psi \odot \theta)_X \in \operatorname{Hom}_{\mathcal{D}}(F(X), H(X))$ for all $X \in \operatorname{Ob}(\mathcal{C})$, it's easy to do so.

Definition 5. For functors $F, F' : \mathcal{C}_1 \to \mathcal{C}_2$ and $G, G' : \mathcal{C}_2 \to \mathcal{C}_3$, natural transformations $\theta : F \Rightarrow F'$ and $\psi : G \Rightarrow G'$, the element of horizontal composition of natural transformations $(\psi \ominus \theta)_X$ is defined as $G'(\theta_X) \circ \psi_{F(X)} = \psi_{F'(X)} \circ G(\theta_X)$. In commutative diagrams forms,



which satisfy

$$GF(X) \xrightarrow{G(\theta_X)} GF'(X)$$

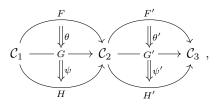
$$\psi_{F(X)} \downarrow \qquad \qquad \downarrow \psi_{F'(X)}$$

$$G'F(Y) \xrightarrow{G'(\theta_X)} G'F'(Y).$$
(3)

Actually, we need to prove that the definition is well-defined, i.e., to verify the commutative diagram and that $(\psi \ominus \theta)_X \in \operatorname{Hom}_{\mathcal{C}_3}(GF(X), G'F'(X))$ for all $X \in \operatorname{Ob}(\mathcal{C}_1)$, it's easy to do so observing Commutative diagram (1).

Theorem 1. The longitudinal composition of natural transformations is natural transformation.

Proof. For



we need to verify the following commutative diagrams:

$$F(X) \xrightarrow{(\psi \odot \theta)_X} H(X) \qquad GF(X) \xrightarrow{(\theta' \ominus \theta)_X} G'F'(X)$$

$$(a)_{F(f)} \downarrow \qquad \downarrow_{H(f)} \text{ and (b)} \qquad \downarrow_{GF(f)} \qquad \downarrow_{G'F'(f)}.$$

$$F(y) \xrightarrow{(\psi \odot \theta)_X} H(y) \qquad GF(Y) \xrightarrow{(\theta' \ominus \theta)_Y} G'F'(Y)$$

From (a), we have

$$H(f) \circ (\psi \odot \theta)_{X} \qquad \qquad \text{(Assumption)}$$

$$= H(f) \circ \psi_{X} \circ \theta_{X} \qquad \qquad \text{(Def: longitudinal composition)}$$

$$= (\psi_{Y} \circ G(f)) \circ \theta_{X} \qquad \qquad \text{(Property of natural transformation } \psi)$$

$$= \psi_{Y} \circ \theta_{Y} \circ F(f) \qquad \qquad \text{(Property of natural transformation } \theta)$$

$$= (\psi \odot \theta)_{X} \circ F(f), \qquad \qquad \text{(Def: longitudinal composition)}$$

thus $(\psi \odot \theta)$ is natural transformation.

From (b), we have

thus $(\psi \ominus \theta)$ is natural transformation.

What's more, we can prove that $(\psi \odot \theta) \ominus (\psi' \odot \theta') = (\psi' \ominus \psi) \odot (\theta' \ominus \theta)$:

$$\begin{array}{ll} ((\psi'\odot\theta')\ominus(\psi\odot\theta))_X & \text{(Assumption)} \\ = (\psi'\odot\theta')_{H(X)}\circ F'(\psi\odot\theta)_X & \text{(Def: horizontal composition)} \\ = \psi'_{H(X)}\circ\theta_{H(X)}\circ F'(\theta_X)\circ F'(\theta_X) & \text{(Def: longitudinal composition, Property of functor } F') \\ = \psi'_{H(X)}\circ(G'(\psi_X)\circ\theta'_{G(X)})\circ F'(\theta_X) & \text{(Commutative diagram (3))} \\ = (\psi'\ominus\psi)_X\circ(\theta'\ominus\theta)_X & \text{(Def: horizontal composition)} \\ = ((\psi'\ominus\psi)\odot(\theta'\ominus\theta))_X, & \text{(Def: longitudinal composition)} \end{array}$$

where $X \in \text{Ob}(\mathcal{C})$.

Theorem 2. Both longitudinal and horizontal compositions of natural transformations satisfy associative law.

 ${\it Proof.}$ For longitudinal composition, observe the following commutative diagram and natural transformations: For

$$\begin{array}{ccc}
& F \\
C & D \\
& H & \uparrow
\end{array}$$
 as well as
$$\begin{cases}
\theta : F \Rightarrow G, \\
\psi : G \Rightarrow H, \\
\phi : H \Rightarrow K,
\end{cases}$$

it's trivial to prove that $((\phi \odot \psi) \odot \theta)_X = (\phi \odot (\psi \odot \theta))_X$ for all $X \in \text{Ob}(\mathcal{C})$, thus the longitudinal composition satisfies associative law.

For

we have

thus the horizontal composition satisfies associative law.

Theorem 3. Observe the following natural commutative:

$$C_1 \xrightarrow{F} C_2 \underbrace{\downarrow \theta}_H C_3 \xrightarrow{K} C_4,$$

we have

$$\theta \odot id^G = \theta = id^H \odot \theta, \tag{4}$$

$$(\theta \ominus \mathrm{id}^F)_X = \theta_{F(X)},\tag{5}$$

$$(\mathrm{id}^K \ominus \theta)_Y = K(\theta_Y), \tag{6}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1), Y \in \mathrm{Ob}(\mathcal{C}_2)$.

Proof. It's trivial to prove (4) and using definition of identity transformation and longitudinal composition. For (5), we have

$$\begin{array}{ll} (\theta\ominus\operatorname{id}^F)_X & \text{(Assumption)} \\ =&\theta_{F(X)}\circ G(\operatorname{id}_X^F) & \text{(Def: horizontal composition)} \\ =&\theta_{F(X)}\circ\operatorname{id}_{F(X)}^G & \text{(formula(2))} \\ =&\theta_{F(X)} & \text{(formula(4))} \end{array}$$

for all $X \in \text{Ob}(\mathcal{C}_1)$. For (6), we have

$$\begin{array}{ll} (\operatorname{id}^K\ominus\theta)_Y & (\operatorname{Assumption}) \\ = K(\theta_Y)\circ\operatorname{id}_{G(Y)}^K & (\operatorname{Def: horizontal composition}) \\ = K(\theta_Y)\circ K(\operatorname{id}_Y^G) & (\operatorname{formula}(2)) \\ = K(\theta_Y\circ\operatorname{id}_Y^G) & (\operatorname{Property of functor }K) \\ = K(\theta_Y) & (\operatorname{formula}(4)) \end{array}$$

for all $Y \in \text{Ob}(\mathcal{C}_2)$.