A Brief Introduction of Natural Transformation

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These are my notes of basic category theory.

Definition 1. A *category* $\mathcal{C}$ consists of:

- A collection $\mathrm{Ob}(\mathcal{C})$ of *objects* $X, Y, Z \cdots$
- A collection
$$\mathrm{Mor}(\mathcal{C}) = \bigcup_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{dom}_{\mathcal{C}}(X) = \bigcup_{Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{cod}_{\mathcal{C}}(Y) = \bigcup_{X,Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$
of *morphisms* $f, g, h \cdots$ with a binary operation "$\circ$" which is defined on the subclass of $\mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C})$, where $\mathrm{dom}_{\mathcal{C}}(X)$ is the *domain* of its elements as well as $\mathrm{cod}_{\mathcal{C}}(Y)$ is the *codomain* of its elements, and *homomorphism* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the intersection of both. [*] For $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \wedge g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, the binary operation that defines *composite morphism* $g \circ^{\mathcal{C}} f$ (which is abbreviated as $g \circ f$) satisfies:
  1. $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$;
  2. $\forall h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)[h \circ (g \circ f) = (h \circ g) \circ f]$;
  3. $\forall h \in \mathrm{Hom}_{\mathcal{C}}(Y, X) \exists \mathbf{1}^{\mathcal{C}}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)[f \circ \mathbf{1}^{\mathcal{C}}_X = f \wedge \mathbf{1}^{\mathcal{C}}_X \circ h = h]$.

It's easy to verify that the *identity morphism* $\mathbf{1}^{\mathcal{C}}_X$ (which is abbreviated as $\mathbf{1}_X$) is unique for all $X \in \mathrm{Ob}(\mathcal{C})$.

In commutative diagram,

$$X \xrightarrow{f} Y \quad \text{means} \quad f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), \qquad \begin{tikzcd} X \arrow[d, "f"] \arrow[dr, "h"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd} \quad \text{means} \quad g \circ f = h.$$

Example 1 (ZF)**.** Consider a category Rel:

- Objects are all sets.
- Homomorphism between any sets X, Y is the power set of binary relations $\mathscr{P}(X \times Y)$.
- The composition of morphisms is the composition of binary relations.
- Identity morphism $\mathbf{1}_X$ is the identity mapping id_X.

Obviously, it's indeed a category. This example shows that the morphisms are not only mappings, they may have looser structures. Compared to this, morphisms are more like binary relations.

Definition 2. $\mathcal{C}'$ is the *subcategory* of category $\mathcal{C}$ if:

- $\mathcal{C}'$ is a category;
- $\mathrm{Ob}(\mathcal{C}') \subseteq \mathrm{Ob}(\mathcal{C})$;
- $\forall X, Y \in \mathrm{Ob}(\mathcal{C}')[\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)]$;
- $\forall f, g \in \mathrm{Mor}(\mathcal{C}')[f \circ^{\mathcal{C}'} g = f \circ^{\mathcal{C}} g]$ (if $f \circ^{\mathcal{C}} g$ is defined);
- for all $X \in \mathrm{Ob}(\mathcal{C}')$, the identity morphism $\mathbf{1}_X$ in $\mathcal{C}'$ is also that in $\mathcal{C}$.

In particular, if $\forall X, Y \in \mathrm{Ob}(\mathcal{C}')[\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)]$, then we say $\mathcal{C}'$ is the *full subcategory* of $\mathcal{C}$.

A *opposite* category $\mathcal{C}^{\mathrm{op}}$ of category $\mathcal{C}$ satisfies:

[*] A few textbooks may require that the different homomorphisms in the same category are disjoint, however, it will cause a problem in defining functor category, we will explain this in detail after the definition of functor category.

- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$;
- $\forall X, Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})[\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)]$;
- $\forall f, g \in \mathrm{Mor}(\mathcal{C}^{\mathrm{op}})[f \circ^{\mathrm{op}} g = g \circ^{\mathcal{C}} f]$ (if $g \circ^{\mathcal{C}} f$ is defined);
- for all $X \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, the identity morphism $\mathbf{1}_X$ in $\mathcal{C}^{\mathrm{op}}$ is also that in $\mathcal{C}$.

It's easy to verify that $\mathcal{C}$ is also a category, and we have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$. $\mathcal{C}^{\mathrm{op}}$ has the symmetric algebraic properties as $\mathcal{C}$.

A category $\mathcal{C}$ is called *small* if both $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets in ZFC but not proper class, * and *large* otherwise. A *locally small* category is a category such that for all objects X and Y, $\mathrm{Hom}(X, Y)$ is a set in ZFC, called a *homset*.

Definition 3. A *functor* $F : \mathcal{C} \to \mathcal{D}$, between category $\mathcal{C}$ and $\mathcal{D}$, consists the following data:

- Mapping $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
- Mapping $F : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, which satisfies:
  1. $F[\mathrm{Hom}_{\mathcal{C}}(X, Y)] \subseteq \mathrm{Hom}_{\mathcal{D}}(F(X), G(Y))$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$;
  2. For all $f, g \in \mathrm{Mor}(\mathcal{C})$, if $g \circ f$ is defined in $\mathrm{Mor}(\mathcal{C})$, then $F(g \circ f) = F(g) \circ F(f)$;
  3. $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$.

In commutative diagram,

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

means that F is the functor between $\mathcal{C}$ and $\mathcal{D}$.

For functors $F : \mathcal{C}_1 \to \mathcal{C}_2, G : \mathcal{C}_2 \to \mathcal{C}_3$, the composition $GF : \mathcal{C}_1 \to \mathcal{C}_3$ between both satisfies:

$$GF(X) = G(F(X)) \text{ and } GF(f) = G(F(f)) \text{ for all } X \in \mathrm{Ob}(\mathcal{C}_1), f \in \mathrm{Mor}(\mathcal{C}_1).$$

It's trivial to verify that the composition is also a functor and it satisfy associative law.

For any category $\mathcal{C}$, there exists a *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that satisfies

$$\mathrm{id}_{\mathcal{C}}(X) = X \text{ and } \mathrm{id}_{\mathcal{C}}(f) = f \text{ for all } X \in \mathrm{Ob}(\mathcal{C}_1), f \in \mathrm{Mor}(\mathcal{C}_1).$$

It's easy to verify that $F\mathrm{id}_{\mathcal{C}} = F$ and $\mathrm{id}_{\mathcal{C}}G = G$ for all functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.

Definition 4. The *natural transformation* θ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a mapping from $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Mor}(\mathcal{D})$ whose each value satisfies

$$\theta_X := \theta(X) \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$$

and the commutative diagram below:

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"'] & G(Y), \end{tikzcd} \tag{1}$$

where $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. In other words, we can record the above natural transformation as $\theta : F \Rightarrow G$, or in such a commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D}. \end{tikzcd}$$

We may use symbol "$F(\theta)_X$" instead of "$F((\theta)_X)$" in some particular case (such as there are more than one symbols of natural transformations in the brackets).

For any functor $F : \mathcal{C} \to \mathcal{D}$, there exists a *identity transformation* $\mathbf{id}^F : F \to F$ that satisfies

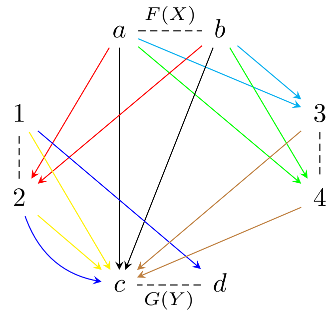
$$\forall X \in \mathrm{Ob}(\mathcal{C})[\mathbf{id}^F_X = \mathbf{1}_{F(X)}].$$

Example 2. Consider two *finite* categories $\mathcal{C}, \mathcal{D}$ whose $\mathrm{Ob}(\mathcal{C}) = \{X, Y\}$ and $\mathrm{Ob}(\mathcal{D}) = \{\{a, b\}, \{1, 2\}, \{3, 4\}, \{c, d\}\}$. There are three morphisms in $\mathcal{C}$: $\mathbf{1}_X$, $\mathbf{1}_Y$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Consider four functors $F, F', G, G' : \mathcal{C} \to \mathcal{D}$ such that

$$F(X) = \{a, b\} = F'(X), G(Y) = \{c, d\} = G'(Y).$$

* "X is a set in ZFC" has two meanings: we can prove X exists in ZFC, i.e., $\mathsf{ZFC} \vdash_{\mathbf{H}} \exists X$; or the existence of X in ZFC is consistent with ZFC, i.e., $\vdash_{\mathbf{H}} \mathrm{Con}(\mathsf{ZFC}) \to \mathrm{Con}(\mathsf{ZFC} + \exists X)$, where $\mathbf{H}$ means the *Frege-Hilbert first-order logic axiomatic system*. The meaning in the text is the former. Of course, to prove the consistency, we often need to add extra axioms. The provability of ZFC is limited, so we can only define the set in the model $(V_\kappa, \in)$, where κ is the least strongly inaccessible cardinal, but that's enough.

And consider two natural transformations $\theta, \psi : F \Rightarrow G$, and all the morphisms (mappings) in $\mathcal{D}$ except identity morphisms are shown below:



Where the arrow with different color means different mapping, balck arrows mean the morphism k, and the elements connected by one dashed line belong to the same set. There are 7 isomorphisms (except identity morphisms) in $\mathcal{D}$ in total, it's easy to see that $\mathcal{D}$ is indeed a category (we just need to verify that the compositions of any morphisms in $\mathcal{D}$ are also morphisms in it).

- Consider the following combination of objects and morphisms:

$$G(X) = \{1, 2\} = G'(X), F(Y) = \{3, 4\} = F'(Y);$$

$$\text{red:}\theta_X,\ \text{yellow:}G(f),\ \text{blue:}G'(f),\ \text{cyan:}F(f),\ \text{grenn:}F'(f),\ \text{brown:}\theta_Y.$$

  The four functors are indeed functors. What's more, it's trivial to verify that

$$\theta_Y \circ F'(f) = \theta_Y \circ F(f) = k = G(f) \circ \theta_X = G'(f) \circ \theta_X,$$

  thus we know θ is indeed a natural transformation, and obviously θ have more that one "sources" and "targets".

- Consider the following combination of objects and morphisms:

$$G(X) = \{3, 4\}, F(Y) = \{1, 2\};$$

$$\text{red:}F(f),\ \text{yellow:}\theta_Y,\ \text{blue:}\psi_Y,\ \text{cyan:}\theta_X,\ \text{grenn:}\psi_X,\ \text{brown:}G(f).$$

  The two functors are indeed functors. What's more, it's trivial to verify that

$$\theta_Y \circ F(f) = \psi_Y \circ F(f) = k = G(f) \circ \theta_X = G(f) \circ \psi_X,$$

  thus we know θ and ψ are indeed natural transformations between F and G.

These examples show us that one natural transformation can rely on different functors, and there may be different natural transformations between two functors.

Definition 5. For functors $F, G, H : \mathcal{C} \to \mathcal{D}$, natural transformations $\theta : F \Rightarrow G$ and $\psi : G \Rightarrow H$, the element of *vertical composition* of the natural transformations is defined as $(\psi \odot \theta)_X = \psi_X \circ \theta_X$. In commutative diagrams forms,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=4, "\Downarrow\theta"] \arrow[r, phantom, shift right=4, "\Downarrow\psi"] & \mathcal{D} \end{tikzcd} \quad \text{means} \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow\psi\odot\theta"] & \mathcal{D}. \end{tikzcd}$$

Actually, we need to prove that the definition is well-defined, i.e., to verify that $(\psi \odot \theta)_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), H(X))$ for all $X \in \mathrm{Ob}(\mathcal{C})$, it's easy to do so.

Definition 6. For functors $F, F' : \mathcal{C}_1 \to \mathcal{C}_2$ and $G, G' : \mathcal{C}_2 \to \mathcal{C}_3$, natural transformations $\theta : F \Rightarrow F'$ and $\psi : G \Rightarrow G'$, the element of *horizontal composition* of natural transformations $(\psi \ominus \theta)_X$ is defined as $G'(\theta_X) \circ \psi_{F(X)} = \psi_{F'(X)} \circ G(\theta_X)$. In commutative diagrams forms,

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}_2 \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{C}_3 \end{tikzcd} \quad \textit{means} \quad \begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "G'F'"'] \arrow[r, phantom, "\Downarrow\psi\ominus\theta"] & \mathcal{C}_3, \end{tikzcd}$$

which satisfy

$$\begin{tikzcd} GF(X) \arrow[r, "G(\theta_X)"] \arrow[d, "\psi_{F(X)}"'] & GF'(X) \arrow[d, "\psi_{F'(X)}"] \\ G'F(Y) \arrow[r, "G'(\theta_X)"'] & G'F'(Y). \end{tikzcd} \tag{2}$$

Actually, we need to prove that the definition is well-defined, i.e., to verify the commutative diagram and that $(\psi \ominus \theta)_X \in \mathrm{Hom}_{\mathcal{C}_3}(GF(X), G'F'(X))$ for all $X \in \mathrm{Ob}(\mathcal{C}_1)$, it's easy to do so observing commutative diagram 1.

Theorem 1. The vertical and horizontal compositions of natural transformations are natural transformations.

Proof. For

$$
\begin{tikzcd}
\mathcal{C}_1 \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=60, "H"'{name=H}] \arrow[r, phantom, shift left=4, "\Downarrow\theta"] \arrow[r, phantom, shift right=4, "\Downarrow\psi"] & \mathcal{C}_2 \arrow[r, bend left=60, "F'"] \arrow[r, "G'" description] \arrow[r, bend right=60, "H'"'] \arrow[r, phantom, shift left=4, "\Downarrow\theta'"] \arrow[r, phantom, shift right=4, "\Downarrow\psi'"] & \mathcal{C}_3
\end{tikzcd},
$$

we need to verify the following commutative diagrams:

$$
(a)\quad
\begin{tikzcd}
F(X) \arrow[r, "(\psi\odot\theta)_X"] \arrow[d, "F(f)"'] & H(X) \arrow[d, "H(f)"] \\
F(y) \arrow[r, "(\psi\odot\theta)_X"'] & H(y)
\end{tikzcd}
\quad \text{and (b)} \quad
\begin{tikzcd}
GF(X) \arrow[r, "(\theta'\ominus\theta)_X"] \arrow[d, "GF(f)"] & G'F'(X) \arrow[d, "G'F'(f)"] \\
GF(Y) \arrow[r, "(\theta'\ominus\theta)_Y"'] & G'F'(Y)
\end{tikzcd}.
$$

From (a), we have

$$
\begin{aligned}
&H(f) \circ (\psi \odot \theta)_X \\
=&H(f) \circ \psi_X \circ \theta_X && \text{(Def: vertical composition)} \\
=&(\psi_Y \circ G(f)) \circ \theta_X && \text{(Property of natural transformation } \psi) \\
=&\psi_Y \circ \theta_Y \circ F(f) && \text{(Property of natural transformation } \theta) \\
=&(\psi \odot \theta)_X \circ F(f), && \text{(Def: vertical composition)}
\end{aligned}
$$

thus $(\psi \odot \theta)$ is natural transformation.

From (b), we have

$$
\begin{aligned}
&G'F'(f) \circ (\theta' \ominus \theta)_X \\
=&G'F'(f) \circ \theta'_{F'(X)} \circ G(\theta_X) && \text{(Def: horizontal composition)} \\
=&\theta'_{F'(Y)} \circ GF'(f) \circ G(\theta_X) && \text{(Property of natural transformation } \theta') \\
=&\theta'_{F'(Y)} \circ G(F'(f) \circ G(\theta_X)) && \text{(Property of functor } G) \\
=&\theta'_{F'(Y)} \circ G(\theta_Y \circ F(f)) && \text{(Property of natural transformation } \theta) \\
=&\theta'_{F'(Y)} \circ G(\theta_Y) \circ GF(f) && \text{(Property of functor } G) \\
=&(\theta' \ominus \theta)_Y \circ GF(f), && \text{(Def: horizontal composition)}
\end{aligned}
$$

thus $(\psi \ominus \theta)$ is natural transformation.

What's more, we can prove

$$
(\psi \odot \theta) \ominus (\psi' \odot \theta') = (\psi' \ominus \psi) \odot (\theta' \ominus \theta) \tag{3}
$$

in the below step:

$$
\begin{aligned}
&((\psi' \odot \theta') \ominus (\psi \odot \theta))_X \\
=&(\psi' \odot \theta')_{H(X)} \circ F'(\psi \odot \theta)_X && \text{(Def: horizontal composition)} \\
=&\psi'_{H(X)} \circ \theta_{H(X)} \circ F'(\theta_X) \circ F'(\theta_X) && \text{(Def: vertical composition, Property of functor } F') \\
=&\psi'_{H(X)} \circ (G'(\psi_X) \circ \theta'_{G(X)}) \circ F'(\theta_X) && \text{(Commutative diagram 2)} \\
=&(\psi' \ominus \psi)_X \circ (\theta' \ominus \theta)_X && \text{(Def: horizontal composition)} \\
=&((\psi' \ominus \psi) \odot (\theta' \ominus \theta))_X, && \text{(Def: vertical composition)}
\end{aligned}
$$

where $X \in \mathrm{Ob}(\mathcal{C})$. $\square$

Theorem 2. Both vertical and horizontal compositions of natural transformations satisfy associative law.

Proof. For vertical composition, observe the following commutative diagram and natural transformations:

For

$$\begin{tikzcd}[row sep=small] \mathcal{C} \arrow[rr, bend left=60, "F"{name=F}] \arrow[rr, bend left=15, "G"{name=G, description}] \arrow[rr, bend right=15, "H"{name=H, description}] \arrow[rr, bend right=60, "K"'{name=K}] & & \mathcal{D}, \arrow[from=F, to=G, Rightarrow, shorten=2pt, "\theta"] \arrow[from=G, to=H, Rightarrow, shorten=2pt, "\psi"] \arrow[from=H, to=K, Rightarrow, shorten=2pt, "\phi"] \end{tikzcd}$$

it's trivial to prove that $((\phi \odot \psi) \odot \theta)_X = (\phi \odot (\psi \odot \theta))_X$ for all $X \in \mathrm{Ob}(\mathcal{C})$, thus the vertical composition satisfies associative law.

For

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "F'"'{name=F'}] & \mathcal{C}_2 \arrow[r, bend left=50, "G"{name=G}] \arrow[r, bend right=50, "G'"'{name=G'}] & \mathcal{C}_3 \arrow[r, bend left=50, "H"{name=H}] \arrow[r, bend right=50, "H'"'{name=H'}] & \mathcal{C}_4, \arrow[from=F, to=F', Rightarrow, shorten=2pt, "\theta"] \arrow[from=G, to=G', Rightarrow, shorten=2pt, "\psi"] \arrow[from=H, to=H', Rightarrow, shorten=2pt, "\phi"] \end{tikzcd} \tag{4}$$

we have

$$\begin{aligned} & (\phi \ominus (\psi \ominus \theta))_X \\ =& \phi_{G'F'(X)} \circ H(\psi \ominus \theta)_X && \text{(Def: horizontal composition)} \\ =& \phi_{G'F'(X)} \circ H(\psi_{F'(X)} \circ G(\theta_X)) && \text{(Ditto)} \\ =& \phi_{G'F'(X)} \circ H(\psi_{F'(X)}) \circ HG(\theta_X) && \text{(Property of functor } H) \\ =& (\phi \ominus \psi)_{F'(X)} \circ HG(\theta_X) && \text{(Def: horizontal composition)} \\ =& ((\phi \ominus \psi) \ominus \theta)_X, && \text{(Ditto)} \end{aligned}$$

thus the horizontal composition satisfies associative law. □

Lemma 1 (ZF)**.** For any sets A, B and mapping $f : A \to B$, we have

$$\exists g : B \to A[g \circ f = \mathrm{id}_A] \iff f \text{ is a injection } \iff \forall C \forall h, h' : C \to A[f \circ h = f \circ h' \implies h = h'],$$

$$\exists g : B \to A[f \circ g = \mathrm{id}_B] \implies f \text{ is a surjection } \iff \forall C \forall h, h' : B \to C[h \circ f = h' \circ f \implies h = h'],$$

and f is a surjection $\implies \exists g : B \to A[f \circ g = \mathrm{id}_B]$ can be proved in ZFC. It's easy to see that f is bijection if and only if it has both left and right inversal mappings, and obviously the two inversal mappings are the same one, which is unique.

Definition 7. Consider $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. If there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ that $f \circ g = \mathbf{1}_Y$, then we say g is the *right inverse* of it; if $g \circ f = \mathbf{1}_X$, then we say g is the *left inverse* of it. If f has both inverses, then we say f is a *isomorphism*, it's easy to verify that the two inverses are the same one, which is unique as well, so we say $f^{-1} := g$ is the *inverse* (*inversal morphism*) of it. If there exists an isomorphism between two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, then we say they are isomorphic and record it as $X \overset{\mathrm{M}}{\simeq} Y$.

What's more, it's easy to verify that the composition of isomorphisms is also an isomorphism (see Theorem 3), so we can find that the collection of *automorphisms* $\mathrm{Aut}_{\mathcal{C}}(X) := \{f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid f \text{ is an isomorphism}\}$ is a group $\langle \mathrm{Aut}_{\mathcal{C}}(X), \circ, \mathbf{1}_X \rangle$. Therefore if f has a left inverse, then f is a *injective morphism*, which satisfies the *left cancellation law*: $\forall Z \in \mathrm{Ob}(\mathcal{C}) \forall g, h \in \mathrm{Hom}_{\mathcal{C}}(Z, X)[f \circ g = f \circ h \iff g = h]$; if f has a right inverse, then f is a *surjective morphism*, which satisfies the *right cancellation law*: $\forall Z \in \mathrm{Ob}(\mathcal{C}) \forall g, h \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)[g \circ f = h \circ f \iff g = h]$.

Consider a functor $F : \mathcal{C} \to \mathcal{D}$, we can define the inverse of functor in the same way: If there exists a functor $G : \mathcal{D} \to \mathcal{C}$ that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$, then we say F is an *isomorphism functor* between $\mathcal{C}$ and $\mathcal{D}$, and $F^{-1} := G$ is the unique inverse (*inversal functor*) of F. From Lemma 1 we know that F is isomorphic if and only if $F \restriction_{\mathrm{Ob}(\mathcal{C})}$ and $F \restriction_{\mathrm{Mor}(\mathcal{C})}$ are both bijection. If there exists an isomorphism functor between two categories $\mathcal{C}$ and $\mathcal{D}$, we say they are *isomorphic* and record it as $\mathcal{C} \overset{\mathrm{F}}{\simeq} \mathcal{D}$.

(Please see formula 6 first) Consider the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\theta \qquad \psi\Uparrow"] & \mathcal{D}. \end{tikzcd}$$

If $\psi \odot \theta = \mathbf{id}^F$ and $\theta \odot \psi = \mathbf{id}^G$, then we say θ is an *isomorphism transformation* between F and G, and $\theta^{-1} := \psi$ is the unique inverse (*inversal transformation*) of θ, we record it as $\theta : F \overset{\sim}{\Rightarrow} G$. It's trivial to prove that θ is an *isomorphism transformation* if and only if θ_X is an isomorphism for each $X \in \mathrm{Ob}(\mathcal{C})$, thus we have $(\theta^{-1})_X = (\theta_X)^{-1}$, we abbreviate it as "$\theta_X^{-1}$". If there exists an isomorphism transformation between two functors F and G, we say they are *isomorphic* and record it as $F \overset{\mathrm{T}}{\simeq} G$.

For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, if there exists an isomorphic $GF \overset{\mathrm{T}}{\simeq} \mathrm{id}_{\mathcal{C}}$ and $FG \overset{\mathrm{T}}{\simeq} \mathrm{id}_{\mathcal{D}}$, then we say G is the *quasi-inverse* of F, and F is a *equivalence* between $\mathcal{C}$ and $\mathcal{D}$. If there exists an equivalence between two categories $\mathcal{C}$ and $\mathcal{D}$, we say they are *equivalent* and record it as $\mathcal{C} \sim \mathcal{D}$.

If there is no confusion in some context, we abbrviate $\overset{\mathrm{M}}{\simeq}, \overset{\mathrm{F}}{\simeq}, \overset{\mathrm{T}}{\simeq}$ as $\simeq$ and refer to "isomorphism", "isomorphism functor", "isomorphism transformation" as "isomorphism" uniformly.

Lemma 2. Observe commutative diagram 4, we have

$$G(\mathbf{id}_X^F) = \mathbf{id}_{F(X)}^G = \mathbf{id}_X^{GF}, \tag{5}$$

$$\psi \odot \mathbf{id}^G = \psi = \mathbf{id}^{G'} \odot \psi, \tag{6}$$

$$(\psi \ominus \mathbf{id}^F)_X = \psi_{F(X)}, \tag{7}$$

$$(\mathbf{id}^H \ominus \psi)_Y = H(\psi_Y), \tag{8}$$

$$\mathbf{id}^G \ominus \mathbf{id}^F = \mathbf{id}^{GF}, \tag{9}$$

$$\text{if } \psi \text{ and } \phi \text{ are isomorphisms, then } \phi \ominus \psi \text{ is also, and } (\phi \ominus \psi)^{-1} = \phi^{-1} \ominus \psi^{-1}, \tag{10}$$

$$[F \simeq F' \wedge G \simeq G' \wedge H \simeq H'] \Longrightarrow [GF \simeq G'F' \wedge HG \simeq H'G'], \tag{11}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1), Y \in \mathrm{Ob}(\mathcal{C}_2)$.

Proof. For (5), using the definition of identity morphism, properties of morphism and functor we have

$$\mathbf{id}_X^{GF} = \mathbf{1}_{GF(X)} = G(\mathbf{1}_{F(X)}) = \mathbf{id}_{F(X)}^G = G(\mathbf{1}_{F(X)}) = G(\mathbf{id}_X^F)$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

It's trivial to prove (6) and using definition of identity transformation and vertical composition.

For (7), we have

$$\begin{aligned}
&(\psi \ominus \mathbf{id}^F)_X \\
=&\psi_{F(X)} \circ G(\mathbf{id}_X^F) && \text{(Def: horizontal composition)} \\
=&\psi_{F(X)} \circ \mathbf{id}_{F(X)}^G && \text{(Formula 5)} \\
=&\psi_{F(X)} && \text{(Formula 6)}
\end{aligned}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

For (8), we have

$$\begin{aligned}
&(\mathbf{id}^H \ominus \psi)_Y \\
=&H(\psi_Y) \circ \mathbf{id}_{G(Y)}^H && \text{(Def: horizontal composition)} \\
=&H(\psi_Y) \circ H(\mathbf{id}_Y^G) && \text{(Formula 5)} \\
=&H(\psi_Y \circ \mathbf{id}_Y^G) && \text{(Property of functor } H\text{)} \\
=&H(\psi_Y) && \text{(Formula 6)}
\end{aligned}$$

for all $Y \in \mathrm{Ob}(\mathcal{C}_2)$.

For (9), we have

$$\begin{aligned}
&(\mathbf{id}^G \ominus \mathbf{id}^F)_X \\
=&G(\mathbf{id}_X^F) \circ \mathbf{id}_{F(X)}^G && \text{(Def: horizontal composition)} \\
=&\mathbf{id}_X^{GF} \circ \mathbf{id}_X^{GF} && \text{(Formula 5)} \\
=&\mathbf{id}_X^{GF} && \text{(Formula 6)}
\end{aligned}$$

for all $X \in \mathrm{Ob}(\mathcal{C}_1)$.

For (10), we have

$$\begin{aligned}
&(\phi \ominus \psi) \odot (\phi^{-1} \ominus \psi^{-1}) \\
=&(\phi^{-1} \odot \phi) \ominus (\psi^{-1} \odot \psi) && \text{(Formula 3)} \\
=&\mathbf{id}^H \ominus \mathbf{id}^G && \text{(Property of inverse)} \\
=&\mathbf{id}^{HG}. && \text{(Formula 9)}
\end{aligned}$$

Similarly, we can prove $(\phi^{-1} \ominus \psi^{-1}) \odot (\phi \ominus \psi) = \mathbf{id}^{H'G'}$.

For (11), we suppose that the three natural transformations are all isomorphisms, we claim that $\psi \ominus \theta : GF \overset{\sim}{\Rightarrow} G'F'$ and $\phi \ominus \psi : HG \overset{\sim}{\Rightarrow} H'G'$, it's trivial to prove using Formula 10. $\square$

Theorem 3. The composition of any morphisms/functors/transformations which are isomorphisms is an isomorphism, and the composition of any equivalences are equivalence. Therefore isomorphic objects, categories, functors and equivalent categories satisfy transitivity.

Proof. Because the compositions of isomorphic morphisms, functors and transformations have some similar algebraic properties, we just need to prove the composition of isomorphic morphisms is isomorphic:
Suppose isomorphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, we claim that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X,Z)$:

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ \mathbf{1}_Y \circ g^{-1} = g \circ g^{-1} = \mathbf{1}_Z,$$

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ \mathbf{1}_Y \circ f = f^{-1} \circ f = \mathbf{1}_X.$$

Observe the following diagram:

$$\begin{tikzcd} \mathcal{C}_1 \arrow[loop left, "\mathrm{id}_{\mathcal{C}_1}"] \arrow[r, bend left, "F"] & \mathcal{C}_2 \arrow[l, bend left, "F'"] \arrow[r, bend left, "G"] & \mathcal{C}_3 \arrow[l, bend left, "G'"] \arrow[loop right, "\mathrm{id}_{\mathcal{C}_3}"] \end{tikzcd}.$$

We need to prove that if F and G are equivalence then GF is also, we now assume that $F'F \simeq \mathrm{id}_{\mathcal{C}_1}$, $FF' \simeq \mathrm{id}_{\mathcal{C}_2}$, $G'G \simeq \mathrm{id}_{\mathcal{C}_2}$ and $GG' \simeq \mathrm{id}_{\mathcal{C}_3}$. Using formula 11 we have

$$\mathrm{id}_{\mathcal{C}_2} \simeq G'G \Longrightarrow \mathrm{id}_{\mathcal{C}_1} \simeq F'F = F'\mathrm{id}_{\mathcal{C}_2}F \simeq F'(G'G)F = (F'G')(GF),$$

$$\mathrm{id}_{\mathcal{C}_2} \simeq F'F \Longrightarrow \mathrm{id}_{\mathcal{C}_3} \simeq GG' = G\mathrm{id}_{\mathcal{C}_2}G' \simeq G(FF')G' = (GF)(F'G').$$

Using transitivity of isomorphic functors, we have $\mathrm{id}_{\mathcal{C}_1} \simeq (F'G')(GF)$ and $\mathrm{id}_{\mathcal{C}_3} \simeq (GF)(F'G')$. Thus GF is equivalence. □

Corollary 1. If functors G, G' are quasi-inverses of equivalence $F : \mathcal{C} \to \mathcal{D}$, then $G \simeq G'$.

Proof. Using formula 11, we have

$$G'F \simeq \mathrm{id}_{\mathcal{C}} \wedge FG \simeq \mathrm{id}_{\mathcal{D}} \Longrightarrow G' = G'\mathrm{id}_{\mathcal{D}} \simeq G'(FG) = (G'F)G \simeq \mathrm{id}_{\mathcal{C}}G = G,$$

thus we have $G' \simeq G$ using transitivity of isomorphic functors. □

Definition 8. For functor $F : \mathcal{C} \to \mathcal{D}$,

- F is *essentially surjective* if $\forall Y \in \mathrm{Ob}(\mathcal{D}) \exists X \in \mathrm{Ob}(\mathcal{C})[F(X) \overset{\mathrm{M}}{\simeq} F(Y)]$;
- F is *faithful* if for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, $F \restriction_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \to \mathrm{Hom}_{\mathcal{C}}(F(X), F(Y))$ is injective;
- F is *full* if for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, $F \restriction_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \to \mathrm{Hom}_{\mathcal{C}}(F(X), F(Y))$ is surjective.

Definition 9. A full subcategory $\mathcal{C}'$ of category $\mathcal{C}$ is a *skeleton* of $\mathcal{C}$ if $\forall X \in \mathrm{Ob}(\mathcal{C}) \exists! Y \in \mathrm{Ob}(\mathcal{C}')[X \overset{\mathrm{M}}{\simeq} Y]$. If $\mathcal{C}$ is the skeleton of itself, then we say it's a *skeletal* category.