Simple Evaluation

$$\pi_{fst} = \text{GD}\ \frac{\texttt{fst = fun t -> match t with (a,b) -> a}}{\texttt{fst} \Rightarrow \texttt{fun t -> match t with (a,b) -> a}}$$

$$\pi_{snd} = \text{GD}\ \frac{\texttt{snd = fun t -> match t with (a,b) -> b}}{\texttt{snd} \Rightarrow \texttt{fun t -> match t with (a,b) -> b}}$$

$$\pi_{rev} = \text{GD}\ \frac{\texttt{rev = fun acc l -> match l with [] -> acc | x::xs -> rev (x::acc) xs}}{\texttt{rev} \Rightarrow \texttt{fun acc l -> match l with [] -> acc | x::xs -> rev (x::acc) xs}}$$

$$\pi_1 = \texttt{fun a -> 2+a*3} \Rightarrow \texttt{fun a -> 2+a*3}$$

$$\pi_2 = \texttt{(9, fun x -> x*4)} \Rightarrow \texttt{(9, fun x -> x*4)}$$

$$\pi_3 = \text{APP}\ \frac{\pi_{snd}\quad \pi_2 \quad \text{PM}\ \dfrac{\pi_2 \quad \texttt{fun x -> x*4} \Rightarrow \texttt{fun x -> x*4}}{\texttt{match (9, fun x -> x*4) with (a,b) -> b} \Rightarrow \texttt{fun x -> x*4}}}{\texttt{snd (9, fun x -> x*4)} \Rightarrow \texttt{fun x -> x*4}}$$

$$\pi_4 = \text{APP}\ \frac{\pi_{fst}\quad \pi_2 \quad \text{PM}\ \dfrac{\pi_2 \quad 9 \Rightarrow 9}{\texttt{match (9, fun x -> x*4) with (a,b) -> a} \Rightarrow 9}}{\texttt{fst (9, fun x -> x*4)} \Rightarrow 9}$$

$$\text{LD}\ \frac{\pi_1 \quad \text{OP}\ \dfrac{8 \Rightarrow 8 \quad \text{APP}\ \dfrac{\pi_1 \quad 1 \Rightarrow 1 \quad \text{OP}\ \dfrac{2 \Rightarrow 2 \quad \text{OP}\ \dfrac{1 \Rightarrow 1 \quad 3 \Rightarrow 3 \quad 1*3 \Rightarrow 3}{1*3 \Rightarrow 3} \quad 2+3 \Rightarrow 5}{2+1*3 \Rightarrow 5}}{\texttt{(fun a -> 2+a*3) 1} \Rightarrow 5} \quad 8-5 \Rightarrow 3}{\texttt{8 - (fun a -> 2+a*3) 1} \Rightarrow 3}}{\texttt{let f = fun a -> 2+a*3 in 8 - f 1} \Rightarrow 3}$$

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$$\text{LD}\ \frac{\text{TU}\ \dfrac{\text{OP}\ \dfrac{3 \Rightarrow 3 \quad 6 \Rightarrow 6 \quad 3+6 \Rightarrow 9}{3+6 \Rightarrow 9} \quad \texttt{fun x -> x*4} \Rightarrow \texttt{fun x -> x*4}}{\texttt{(3+6, fun x -> x*4)} \Rightarrow \texttt{(9, fun x -> x*4)}} \quad \text{APP}\ \dfrac{\pi_3 \quad \pi_4 \quad \text{OP}\ \dfrac{9 \Rightarrow 9 \quad 4 \Rightarrow 4 \quad 9*4 \Rightarrow 36}{9*4 \Rightarrow 36}}{\texttt{(snd (9, fun x -> x*4)) (fst (9, fun x -> x*4))} \Rightarrow 36}}{\texttt{let x = (3+6, fun x -> x*4) in (snd x) (fst x)} \Rightarrow 36}$$

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Induction Proofs

We first prove that `impl n a` terminates with $a * n! * n!$: We omit the global definitions of π_{impl} and π_{bar} of `impl` and `bar` here for simplicity.

- Base case: $n = 0$:

$$\text{APP'}\ \frac{\pi_{impl} \quad \text{PM}\ \dfrac{}{\texttt{match 0 with 0 -> a | ...} \Rightarrow a}}{\texttt{impl 0 a} \Rightarrow a}$$

- Inductive step: We assume `impl n a` terminates with $a * n! * n!$ for an input $n \geq 0$. Now, we show that it terminates with $a * (n+1)! * (n+1)!$ for input $n+1$:

$$\text{APP'}\ \frac{\pi_{impl} \quad \text{PM}\ \dfrac{\text{APP'}\ \dfrac{\text{by I.H.}}{\texttt{impl (n+1-1) (a * (n+1) * (n+1))} \Rightarrow (a * (n+1) * (n+1)) * n! * n!}}{\texttt{match n+1 with ... | \_ -> impl (n+1-1) (a * (n+1) * (n+1))} \Rightarrow (a * (n+1) * (n+1)) * n! * n!}}{\texttt{impl (n+1) a} \Rightarrow a * (n+1)! * (n+1)!}$$

Now, that we have shown that `impl n a` $\Rightarrow a * n! * n!$ (1), we prove:

$$\text{APP}\ \frac{\pi_{bar} \quad \text{APP'}\ \dfrac{\text{by (1)}}{\texttt{impl n 1} \Rightarrow n! * n!}}{\texttt{bar n} \Rightarrow n! * n!}$$

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