

# Calculus


*Better Explained*

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A Guide to Developing  
Lasting Intuition

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$$(f+g)' = f' + g'$$
$$(f \cdot g)' = f \cdot dg + g \cdot df$$
$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

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## 1 MINUTE CALCULUS: X-RAY AND TIME-LAPSE VISION

We usually take shapes, formulas, and situations at face value. Calculus gives us two superpowers to dig deeper:

- **X-Ray Vision:** You see the hidden pieces inside a pattern. You don't just see the tree, you know it's made of rings, with another growing as we speak.



- **Time-Lapse Vision:** You see the future path of an object laid out before you (cool, right?). “Hey, there’s the moon. For the next few days it’ll be white, but on the sixth it’ll be low in the sky, in a color I like. I’ll take a photo then.”





So why is Calculus useful? Well, just imagine having X-Ray or Time-Lapse vision to use at will. That object over there, how was it put together? What will happen to it?

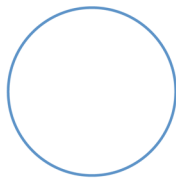
*(Strangely, my letters to Marvel about Calculus-man have been ignored to date.)*

### 1.1 Calculus In 10 Minutes: See Patterns Step-By-Step

What do X-Ray and Time-Lapse vision have in common? They examine patterns step-by-step. An X-Ray shows the individual slices inside, and a time-lapse puts each future state next to the other.

This seems pretty abstract. Let's look at a few famous patterns:

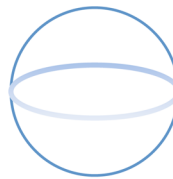
## Circle and Sphere Fun Facts



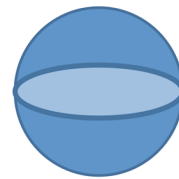
$$2\pi r$$



$$\pi r^2$$



$$4\pi r^2$$



$$\frac{4}{3}\pi r^3$$

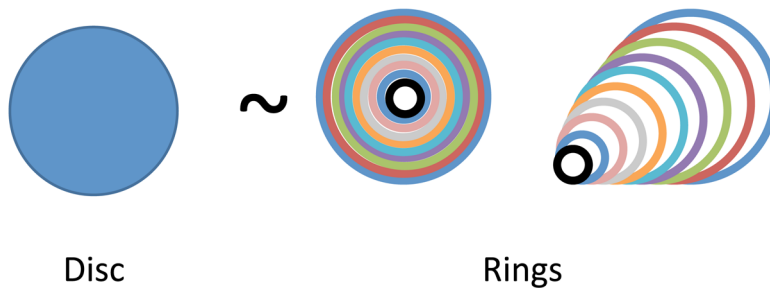
We have a vague feeling these formulas are connected, right?

Let's turn our X-Ray vision and see where this leads. Suppose we know the equation for circumference ( $2\pi r$ ) and want to figure out the equation for area. What can we do?

This is a tough question. Squares are easy to measure, but what can we do with an ever-curving shape?

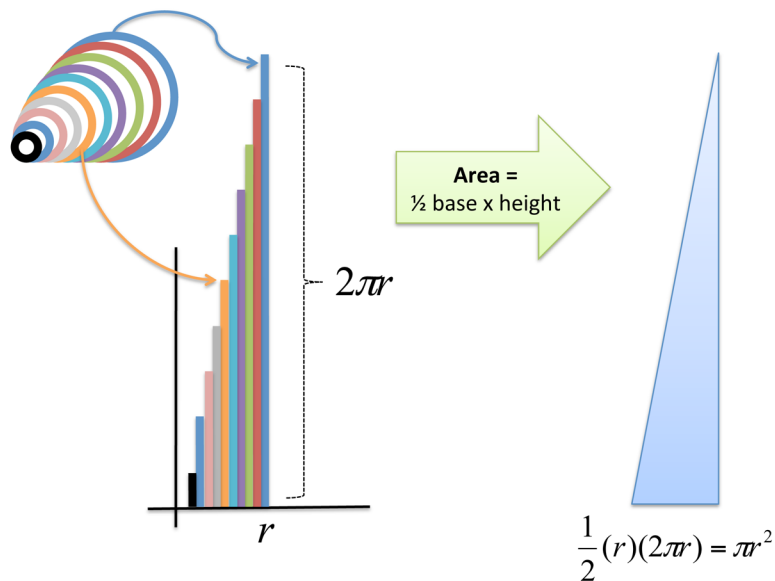
Calculus to the rescue. Let's use our X-Ray vision to realize a disc is really just a bunch of rings put together. Similar to a tree trunk, here's a "step-by-step" view of a filled-in circle:

## Dissecting a Circle



Why does this viewpoint help? Well, let's unroll those curled-up rings so they're easier to measure:

## Unroll the Rings



Whoa! We have a bunch of straightened rings that form a triangle, which is much easier to measure (Wikipedia has an [animation](#)).

The height of the largest ring is the full circumference ( $2\pi r$ ), and each ring gets smaller. The height of each ring depends on its original distance from the center; the ring 3 inches from the center has a height of  $2\pi \cdot 3$  inches. The smallest ring is a pinpoint, more or less, without any height at all.

And because triangles are easier to measure than circles, finding the area isn't too much trouble. The area of the "ring triangle"  $= \frac{1}{2} \text{base} \cdot \text{height} = \frac{1}{2} r(2\pi r) = \pi r^2$ , which is the formula for a circle's area!

Our X-Ray vision revealed a simple, easy-to-measure structure within a curvy shape. We realized a circle and a set of glued-together rings were really the same. From another perspective, a filled-in disc is the "time lapse" of a ring that got larger and larger.

Remember learning arithmetic? We learned a few things to do with numbers (and equations): add/subtract, multiply/divide, and use exponents/roots. Not a bad start.

Calculus gives us two new options: split apart and glue together. A key epiphany of calculus is that our existing patterns can be seen as a bunch of glued-together pieces. It's like staring at a building and knowing it was made brick-by-brick.

## 1.2 So... What Can I Do With Calculus?

It depends. What can you do with arithmetic?

Technically, we don't need numbers. Our caveman ancestors did fine (well, for some version of "fine"). But having an idea of quantity makes the world a lot easier. You don't have a "big" and "small" pile of rocks: you have an exact count. You don't need to describe your mood as "good" or "great" – you can explain your mood from 1 to 10.

Arithmetic provides universal metaphors that go beyond computation, and once seen, they're difficult to give up.

Calculus is similar: it's a step-by-step view of the world. Do we need X-Ray and Time Lapse vision all the time? Nope. But they're nice perspectives to "turn on" when we face a puzzling situation. What steps got us here? What's the next thing that will happen? Where does our future path lead?

There are specific rules to calculus, just like there are rules to arithmetic. And they're quite useful when you're cranking through an equation. But don't forget there are general notions of "quantity" and "step-by-step thinking" that we can bring everywhere.

Our primary goal is to *feel* what a calculus perspective is like (*What Would Archimedes Do?*). Over time, we'll work up to using the specific rules ourselves.