On the design of multiplex control to reject disturbances in nonlinear network systems affected by heterogeneous delays

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***Abstract*— We consider the problem of designing control protocols for nonlinear network systems affected by heterogeneous, time-varying delays and disturbances. For these networks, the goal is to reject polynomial disturbances affecting the agents and to guarantee the fulfilment of some desired network behaviour. To satisfy these requirements, we propose an integral control design implemented via a multiplex architecture. We give sufficient conditions for the desired disturbance rejection and stability properties by leveraging tools from contraction theory. We illustrate the effectiveness of the results via a numerical example that involves the control of a multi-terminal high-voltage DC grid.**

I. Introduction

Over the past decades, the size and complexity of network systems have considerably evolved thanks to the rapid development of computing and communication technologies. Much research efforts have been devoted to study collective behaviours such as consensus, synchronisation, formation control. A key challenge when designing the control protocols is to achieve desired behaviours despite imperfect communications, exogenous disturbances and delays.

In this context, we study the problem of designing distributed integral control protocols that guarantee the fulfilment of the desired network behaviour, while rejecting certain classes of disturbances. These requirements are captured via an Input-to-State Stability (ISS) property and we give sufficient conditions for this property based on non-Euclidean contraction theory.

Related works: the design of integral control protocols for network systems that are able to reject constant disturbances has been investigated in e.g. [1], [2]. In [3], a PI controller is delivered via multiplex architecture to achieve consensus. Recently, in [4], integral actions delivered by multiplex architecture with multiple layers are shown to be effective in rejecting higher order polynomial disturbances while guaranteeing a *scalability* property. The results in this paper are based on ideas from contraction theory [5], particularly leveraging the use of non-Euclidean norms [6], [7], [8]. We refer to [9], [10], [11] and references therein for details. In the context of delay-free networks, leveraging contraction theory, conditions for the synthesis of distributed controls using separable control contraction metrics are given in [12]; contracting recurrent network is introduced in [13] with guarantees of stability and robustness. For network systems affected by delays, [14] shows the preservation of contraction for a time-delayed network using Euclidean contraction metric and, in [4], conditions are given for networks with homogeneous delays.

Statement of Contributions: we present a distributed multiplex integral control design for nonlinear network systems affected by both heterogeneous time-varying delays and disturbances (possibly with polynomial components). The goal of the control protocol is to guarantee, for the network: (i) rejection of polynomial disturbances; (ii) the fulfilment of some desired behaviours. These properties are rigorously formalised in Section III. Specifically, our technical contributions can be summarised as follows: (i) we formalise the control problem as an Input-to-State Stability problem and give sufficient conditions to assess this property. While the results of this paper leverage some of the tools from [4], here, differently from [4], we consider a weaker stability property for networks with heterogeneous delays. These networks cannot be studied with the results in [4]; (ii) we show that our results can serve as design guidelines for the control protocol. The results are validated on the problem of designing a control protocol for multi-terminal high-voltage DC (MTDC) grid. Simulations confirm the effectiveness of the results.

II. Mathematical preliminaries

Given a norm $\|\cdot\|$, we denote by $\|A\|$ its induced matrix norm with respect to a $m \times m$ real matrix A and the corresponding matrix measure $\mu(A) = \lim_{h\to 0^+} \frac{\|I+hA\|-1}{h}$. For $\eta \in \mathbb{R}^n_{>0}$, we define a diagonal matrix by $[\eta] \in \mathbb{R}^{n\times n}$ with $[\eta]_{ii} = \eta_i$, $i \in \{1,\dots,n\}$. The diagonally weighted ℓ_∞-norm of $x \in \mathbb{R}^n$ is defined as $\|x\|_{\infty,[\eta]^{-1}} := \max_i\{|x_i|/\eta_i\}$ with the induced matrix norm $\|A\|_{\infty,[\eta]^{-1}} := \max_i\{\sum_j \frac{\eta_j}{\eta_i}|A_{ij}|\}$ and matrix measure $\mu_{\infty,[\eta]^{-1}}(A) := \max_i\{A_{ii} + \sum_{j\neq i} \frac{\eta_j}{\eta_i}|A_{ij}|\}$. We denote by $\mathbb{I}_n$ the $n\times n$ identity matrix, by $0_{m\times n}$ the $m\times n$ zero matrix (if $m = n$ we simply write 0_n) and by $\mathbf{1}_n \in \mathbb{R}^n$ the one vector. Let f be a sufficiently smooth function, we denote by $f^{(n)}$ the n-th derivative of f. We recall that a continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$.

The following results, originally introduced in [15], can be found in its current form in [16].

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Lemma 1. *Given r positive integers $n_1, \ldots, n_r$ such that $n_1 + \cdots + n_r = n$. Consider the vector $x := [x_1^\mathsf{T}, \ldots, x_r^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^n$, $x_i \in \mathbb{R}^{n_i}$. We let the composite norm $\|x\|_{cmpst} := \|[\|x_1\|_1, \ldots, \|x_r\|_r]\|_{agg}$, with $\|\cdot\|_i (\|\cdot\|_{agg})$ being local(aggregating) norms on $\mathbb{R}^{n_i}(\mathbb{R}^n)$, and induced matrix norm $\|\cdot\|_i(\|\cdot\|_{agg})$ and matrix measure $\mu_i(\cdot)(\mu_{agg}(\cdot))$. Finally, given an $n \times n$ block matrix A with $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, define:*

(i) the aggregate majorant $\lceil A \rceil$ with $(\lceil A \rceil)_{ij} = \|A_{ij}\|_{ij}$;
(ii) the aggregate Metzler majorant $\lceil A \rceil_M$ with

$$(\lceil A \rceil_M)_{ij} = \begin{cases} \|A_{ij}\|_{ij}, \text{if } j \neq i \\ \mu_i(A_{ii}), \text{if } j = i \end{cases}$$

where $\|A_{ij}\|_{ij} := \sup_{\|x_j\|_j = 1} \|A_{ij} x_j\|_i$.
Then,

(i) the composite norm $\|\cdot\|_{cmpst}$ is well-defined, i.e. satisfying the norm properties;
(ii) If the aggregating norm $\|\cdot\|_{agg}$ is monotonic, then:
   1) $\mu_{cmpst}(A) \leq \mu_{agg}(\lceil A \rceil_M)$;
   2) $\|A\|_{cmpst} \leq \|\lceil A \rceil\|_{agg}$.

If the norms $\|\cdot\|_i$, $\|\cdot\|_{\text{agg}}$ in Lemma 1 are p-norms (with the same p) then $\|\cdot\|_{\text{cmpst}}$ is again a p-norm defined on a larger space. The next lemma follows from Theorem 2.4 in [17].

Lemma 2. *let $u : [-\tau_0, +\infty) \to \mathbb{R}_{\geq 0}$, $\tau_0 < +\infty$ and assume that*

$$D^+ u(t) \leq a u(t) + b \sup_{t-\tau(t) \leq s \leq t} u(s) + c, \quad t \geq t_0$$

with: (i) $\tau(t)$ being bounded and non-negative, i.e. $0 \leq \tau(t) \leq \tau_0$, $\forall t$; (ii) $u(t) = \|\varphi(t)\|$, $\forall t \in [t_0 - \tau_0, t_0]$ where $\varphi(t)$ is bounded in $[t_0 - \tau_0, t_0]$; (iii) $a < 0$, $b \geq 0$ and $c \geq 0$. Assume that there exists some $\sigma > 0$ such that $a + b \leq -\sigma < 0, \forall t \geq t_0$. Then:

$$u(t) \leq \sup_{t_0 - \tau_0 \leq s \leq t_0} u(s) e^{-\hat{\lambda} t} + \frac{c}{\sigma},$$

where $\hat{\lambda} := \inf_{t \geq t_0} \{\lambda | \lambda(t) + a + b e^{\lambda(t)\tau(t)} = 0\}$ is positive.

III. System Description and Problem Formulation

Consider a network system comprised of N agents with the dynamics of the i-th agent described by

$$\dot{x}_i(t) = f_i(x_i(t), t) + u_i(t) + d_i(t), \ t \geq t_0 \tag{1}$$

$i \in \{1, \ldots, N\}$. In the above expression, $x_i(t) \in \mathbb{R}^n$ denotes the agent state, $u_i(t) \in \mathbb{R}^n$ denotes the control input and $f_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth function. The term $d_i(t) \in \mathbb{R}^n$ models the external disturbance of the form:

$$d_i(t) = w_i(t) + \bar{d}_i(t) := w_i(t) + \sum_{k=0}^{m-1} \bar{d}_{i,k} \cdot t^k \tag{2}$$

where $\bar{d}_i(t)$ represents the polynomial component of the disturbance of the order $m-1$ ($m \in \mathbb{Z}_{>0}$) with $\bar{d}_{i,k}$ being constant vectors and $w_i(t)$ is a piecewise continuous signal capturing *residual* terms in the disturbance that are not modelled with the polynomial. In order to reject polynomials up to order $m-1$, we design the control protocol $u_i(t)$ following [4] with integral actions delivered by m multiplex layers as illustrated in Figure 1.

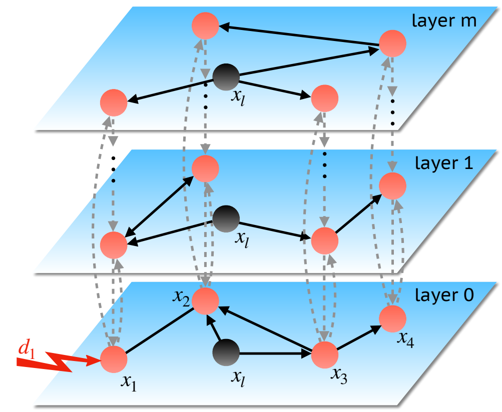

Fig. 1: An example of a network (layer 0) and the multiplex architecture (layer 1 to layer m). On each layer, the agents (orange nodes) can have different topologies and not necessarily every agent is connected to the possible leader (black node). Only one disturbance, e.g. d_1 on agent 1, is shown.

$$\begin{aligned} u_i(t) &= h_{i,0}(x_i, \{x_j\}_{j \in \mathcal{N}_i}, x_l, t) + h_{i,0}^{(\tau)}(x_i, \{x_j\}_{j \in \mathcal{N}_i}, x_l, t) \\ &\quad + r_{i,1}(t) \\ \dot{r}_{i,1}(t) &= h_{i,1}(x_i, \{x_j\}_{j \in \mathcal{N}_i}, x_l, t) + h_{i,1}^{(\tau)}(x_i, \{x_j\}_{j \in \mathcal{N}_i}, x_l, t) \\ &\quad + r_{i,2}(t) \\ &\ \vdots \\ \dot{r}_{i,m}(t) &= h_{i,m}(x_i, \{x_j\}_{j \in \mathcal{N}_i}, x_l, t) + h_{i,m}^{(\tau)}(x_i, \{x_j\}_{j \in \mathcal{N}_i}, x_l, t) \end{aligned} \tag{3}$$

where $\{x_j\}_{j \in \mathcal{N}_i}$ denotes the state of the neighbours of agent i, $x_l(t)$ is the state of possible leader, $h_{i,k}, h_{i,k}^{(\tau)}, \forall k$ are smooth coupling functions on k-th multiplex layer modelling delay-free and delayed communications from neighbours of agent i and the leader. For delayed communications, we consider time-varying delays $\tau_{ij}(t), \tau_{il}(t) \in (0, \tau_{\max}]$ when information is transmitted to agent i from agent j and from the leader, respectively. Note that in general, $\tau_{ij}(t) \neq \tau_{ji}(t)$. In what follows, time dependence inside these coupling functions are omitted for notational convenience.

Remark 1. *Protocols in (3) can be nonlinear and arise in a wide range of applications. For example, the classic diffusive-type protocol can be written as in (3) with $u_i(t) = h_{i,0}^{(\tau)}(x_i, \{x_j\}_{j \in \mathcal{N}_i}, x_l, t) = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t - \tau_{ij}) - x_i(t - \tau_{ij}))$, i.e. when all the communications are delayed and when there is no leader and no integral actions are applied.*

Let $x(t) = [x_1^\mathsf{T}(t), \ldots, x_N^\mathsf{T}(t)]^\mathsf{T}$, $u(t) = [u_1^\mathsf{T}(t), \ldots, u_N^\mathsf{T}(t)]^\mathsf{T}$ and $d(t) = w(t) + \bar{d}(t)$ where $w(t) = [w_1^\mathsf{T}(t), \ldots, w_N^\mathsf{T}(t)]^\mathsf{T}$, $\bar{d}(t) = [\bar{d}_1^\mathsf{T}(t), \ldots, \bar{d}_N^\mathsf{T}(t)]^\mathsf{T}$, the interconnected system (1) can then be written in a compact form as

$$\dot{x}(t) = f(x(t), t) + u(t) + d(t), \ t \geq t_0 \tag{4}$$

where $f(x(t), t) = [f_1^{\mathsf{T}}(x_1(t), t), \ldots, f_N^{\mathsf{T}}(x_N(t), t)]^{\mathsf{T}}$. Next we state the control goal in terms of the desired solution of the network when there are no disturbances and delays. Formally, the desired solution $x^*(t) = [x_1^{*\mathsf{T}}(t), \ldots, x_N^{*\mathsf{T}}(t)]^{\mathsf{T}}$ is a solution of the unperturbed dynamics satisfying $\dot{x}^* = f(x^*, t)$. Finally, let $r_k(t) = [r_{1,k}^{\mathsf{T}}(t), \ldots, r_{N,k}^{\mathsf{T}}(t)]^{\mathsf{T}}$, $k \in \{1, \ldots, m\}$ and we are ready to give the performance notion of Input-to-State Stability of system (4):

Definition 1. *The closed-loop system* (4) *affected by disturbance $d(t) = w(t) + \bar{d}(t)$ is Input-to-State Stable with respect to $w(t)$ if there exists class $\mathcal{KL}$ functions $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)$, a class $\mathcal{K}$ function $\gamma(\cdot)$, such that*

$$\begin{aligned}\|x(t) - x^*(t)\| \leq{}& \alpha\left(\sup_{t_0 - \tau_{\max} \leq s \leq t_0} \|x(s) - x^*(s)\|, t - t_0\right) \\ &+ \beta\left(\sup_{t_0 - \tau_{\max} \leq s \leq t_0} \sum_{k=1}^{m} \|r_k(s) + \bar{d}^{(k-1)}(s)\|, t - t_0\right) \\ &+ \gamma\left(\sup_t \|w(t)\|\right)\end{aligned}$$

holds $\forall t \geq t_0, \forall x(s), \forall r_k(s), k \in \{1, \ldots, m\}$, where $x(s) = \varphi(s), r_k(s) = \phi_k(s)$ with $\varphi(s), \phi_k(s)$ being continuous and bounded functions in $[t_0 - \tau_{\max}, t_0]$.

IV. Technical results

In this section, we give a sufficient condition guaranteeing that the closed-loop system (4) affected by disturbances of the form (2) is Input-to-State Stable. The results are stated in terms of: (i) a composite norm ([16, Section 2.4.4]) $\|x\|_{\text{cmpst}} = \|[\|x_1\|_p, \ldots, \|x_N\|_p]\|_{\infty, [\eta]^{-1}}$ where $[\eta]^{-1} \in \mathbb{R}^{N \times N}$ is a diagonal weighting matrix with $\eta \in \mathbb{R}^N_{>0}$; (ii) a block diagonal coordinate transformation matrix $T = \operatorname{diag}\{T_1, \ldots, T_N\}$. For the statement of our result, it is also useful to *relabel* the delays affecting the network. Specifically, we define $\tau_k(t), k = 1, \ldots, q$ with $q \leq N^2$, as an element of the set $\{\tau_{ij}(t) : i, j = 1, \ldots, N, i \neq j\} \bigcup \{\tau_{il}(t) : i = 1, \ldots, N\}$.

Proposition 1. *Consider the closed-loop system (1) affected by external disturbance (2). Assume that, $\forall t \geq t_0$ and for some matrices $T_i, T_j, i, j = 1, \ldots, N$, the following conditions are satisfied for some $0 < \underline{\sigma} < \bar{\sigma} < \infty$:*

- *C1* $h_{i,k}(x_i^*, \{x_j^*\}_{j \in \mathcal{N}_i}, x_l, t) = h_{i,k}^{(\tau)}(x_i^*, \{x_j^*\}_{j \in \mathcal{N}_i}, x_l, t) = 0$, $\forall i, k$;
- *C2* $\mu_p(T_i \tilde{A}_{ii}(t) T_i^{-1}) + \sum_j \frac{\eta_j}{\eta_i} \|T_i \tilde{A}_{ij}(t) T_j^{-1}\|_p \leq -\bar{\sigma}$, $\forall i$;
- *C3* $\sum_{k=1}^{q} \sum_j \frac{\eta_j}{\eta_i} \|T_i (\tilde{B}_k(t))_{ij} T_j^{-1}\|_p \leq \underline{\sigma}$, $\forall i$;

In the above expression,

$$\tilde{A}_{ii}(t) = \begin{bmatrix} \frac{\partial f_i}{\partial x_i} + \frac{\partial h_{i,0}}{\partial x_i} & \mathbb{I}_n & 0_n & \cdots & 0_n \\ \frac{\partial h_{i,1}}{\partial x_i} & 0_n & \mathbb{I}_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_{i,m-1}}{\partial x_i} & 0_n & 0_n & \cdots & \mathbb{I}_n \\ \frac{\partial h_{i,m}}{\partial x_i} & 0_n & 0_n & \cdots & 0_n \end{bmatrix},$$

$$\tilde{A}_{ij}(t) = \begin{bmatrix} \frac{\partial h_{i,0}}{\partial x_j} & 0_n & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_{i,m}}{\partial x_j} & 0_n & \cdots & 0_n \end{bmatrix},$$

and

$$(\tilde{B}_k)_{ii}(t) = \begin{bmatrix} \frac{\partial h_{i,0}^{(\tau)}}{\partial x_i} & 0_n & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_{i,m}^{(\tau)}}{\partial x_i} & 0_n & \cdots & 0_n \end{bmatrix},$$

$$(\tilde{B}_k)_{ij}(t) = \begin{bmatrix} \frac{\partial h_{i,0}^{(\tau)}}{\partial x_j} & 0_n & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_{i,m}^{(\tau)}}{\partial x_j} & 0_n & \cdots & 0_n \end{bmatrix}.$$

Then, the system is Input-to-State Stable. In particular,

$$\begin{aligned}&\|x(t) - x^*(t)\|_{cmpst} \leq \\ &\|T^{-1}\|_{cmpst} \|T\|_{cmpst} e^{-\lambda(t - t_0)} \left(\sup_{t_0 - \tau_{\max} \leq s \leq t_0} \|x(s) - x^*(s)\|_{cmpst}\right. \\ &+ \sup_{t_0 - \tau_{\max} \leq s \leq t_0} \sum_{k=1}^{m} \|\sum_{b=0}^{m-k} \frac{(m-1-b)!}{(m-k-b)!} \cdot \bar{d}_{m-1-b} \cdot s^{m-k-b} \\ &\left.+ r_k(s)\|_{cmpst}\right) + \frac{\|T^{-1}\|_{cmpst} \|T\|_{cmpst}}{\bar{\sigma} - \underline{\sigma}} \sup_t \|w(t)\|_{cmpst}\end{aligned}$$

where

$$\{\lambda | \lambda - \bar{\sigma} + \underline{\sigma} e^{\lambda \tau_{\max}} = 0\}. \tag{5}$$

Remark 2. *Condition $C1$ implies that $u_i(t) = 0$ at the desired solution which guarantees $x^*(t)$ is a solution of the unperturbed dynamics. This rather common condition is satisfied for any consensus type dynamics with diffusive couplings, see e.g [4], [18], [19]. Condition $C2$ and $C3$ give conditions on the Jacobian of the delay-free and delayed part of the agent dynamics, respectively. These two conditions set constraints for couplings from the neighbours for a given agent, which can be recast as constraints on the number of neighbours and the coupling strength during design progress.*

Remark 3. *The diagonal weighting matrix $[\eta]^{-1}$ in $\|x\|_{cmpst}$ can be carefully designed to achieve sharper bounds for the induced matrix norm and matrix measure, according to [16]. Hence, we could find such matrix $[\eta]^{-1}$ to minimise the upper bound in $C2$ or $C3$. On the other hand, the selection of the coordinate transformation matrix T is also of great significance as in many cases it turns out to be difficult to design controllers fulfilling conditions proposed for the original system. The conditions in Proposition 1 not only provides guideline for the design of the controller, but also for the computation of matrix T, see also [4].*

Remark 4. *Following Proposition 1, the closed-loop network has a convergence rate λ depending on $\tau_{\max}$. As is highlighted in* (5)*, larger $\tau_{\max}$ yields lower λ which means slower convergence towards the system desired solution. In particular, $\lambda = \bar{\sigma}$ when $\tau_{\max} = 0$, i.e. when there are no* *delays, and decreases as the delay increases. Hence* (5) *gives an implicit condition on communication delays for networks with desired convergence rates.*

Proof: Inspired by [4], we define

$$z_i(t) = [x_i^\top(t), \zeta_{i,1}^\top(t), \ldots, \zeta_{i,m}^\top(t)]^\top$$

where

$$\zeta_{i,k}(t) = r_{i,k}(t) + \sum_{b=0}^{m-k} \frac{(m-1-b)!}{(m-k-b)!} \cdot \bar{d}_{i,m-1-b} \cdot t^{m-k-b}$$

$k \in \{1, \cdots, m\}$. In these new coordinates the dynamics of the network system becomes

$$\dot{z}_i(t) = \tilde{f}_i(z_i, t) + \tilde{u}_i(t) + \tilde{w}_i(t) \tag{6}$$

$\forall i$, where $\tilde{f}_i(z_i, t) = [f_i^\top(x_i, t), 0_{1\times n}, \ldots, 0_{1\times n}]^\top$ and $\tilde{u}_i(t) = \tilde{h}_i(z, t) + \tilde{h}_i^{(\tau)}(z, t)$ with

$$\tilde{h}_i(z,t) = \begin{bmatrix} h_{i,0}(x_i, \{x_j\}_{j\in\mathcal{N}_i}, x_l, t) + \zeta_{i,1}(t) \\ h_{i,1}(x_i, \{x_j\}_{j\in\mathcal{N}_i}, x_l, t) + \zeta_{i,2}(t) \\ \vdots \\ h_{i,m-1}(x_i, \{x_j\}_{j\in\mathcal{N}_i}, x_l, t) + \zeta_{i,m}(t) \\ h_{i,m}(x_i, \{x_j\}_{j\in\mathcal{N}_i}, x_l, t) \end{bmatrix}$$

and

$$\tilde{h}_i^{(\tau)}(z,t) = \begin{bmatrix} h_{i,0}^{(\tau)}(x_i, \{x_j\}_{j\in\mathcal{N}_i}, x_l, t) \\ h_{i,1}^{(\tau)}(x_i, \{x_j\}_{j\in\mathcal{N}_i}, x_l, t) \\ \vdots \\ h_{i,m}^{(\tau)}(x_i, \{x_j\}_{j\in\mathcal{N}_i}, x_l, t) \end{bmatrix}$$

and $\tilde{w}_i(t) = [w_i^\top(t), 0_{1\times n}, \ldots, 0_{1\times n}]^\top$. In the augmented dynamics (6), the desired solution $z_i^*(t) = [x_i^{*\top}(t), 0_{1\times n}, \ldots, 0_{1\times n}]^\top$ satisfies $\dot{z}_i^*(t) = \tilde{f}_i(z_i^*, t)$. Hence the dynamics of the state deviation $e_i(t) := z_i(t) - z_i^*(t)$ follows

$$\dot{e}_i(t) = \tilde{f}_i(z_i, t) - \tilde{f}_i(z_i^*, t) + \tilde{h}_i(z, t) + \tilde{h}_i^{(\tau)}(z, t) + \tilde{w}_i(t)$$

Then following [20], let $\eta_i(\rho) = \rho z_i + (1-\rho) z_i^*$, $\eta(\rho) = [\eta_1^\top(\rho), \ldots, \eta_N^\top(\rho)]^\top$ and we can rewrite the error dynamics in a compact form as

$$\dot{e}(t) = A(t)e(t) + \sum_{k=1}^{q} B_k(t) e(t - \tau_k(t)) + \tilde{w}(t) \tag{7}$$

where $\tilde{w} = [\tilde{w}_1^\top(t), \ldots, \tilde{w}_N^\top(t)]^\top$. The Jacobian matrix $A(t)$ has entries $A_{ij}(t) = \int_0^1 \tilde{A}_{ij}(t) d\rho$ where $\tilde{A}_{ij} = J_{\tilde{f}_i}(\eta_j(\rho), t) + J_{\tilde{h}_i}(\eta_j(\rho), t)$ and $B_k(t)$ with entries $(B_k)_{ij}(t) = \int_0^1 (\tilde{B}_k)_{ij} d\rho$ where $(\tilde{B}_k)_{ij} = J_{\tilde{h}_i^{(\tau)}}(\eta_j(\rho), t)$. Then let $\tilde{e}(t) = Te(t)$ where $T = \text{diag}\{T_1, \ldots, T_N\}$, we have

$$\dot{\tilde{e}}(t) = TA(t)T^{-1}\tilde{e}(t) + \sum_{k=1}^{q} TB_k(t)T^{-1}\tilde{e}(t - \tau_k(t)) + T\tilde{w}(t)$$

Taking the Dini derivative of $\|\tilde{e}(t)\|_{\text{cmpst}}$, we obtain

$$\begin{aligned} D^+\|\tilde{e}(t)\|_{\text{cmpst}} &= \limsup_{h\to 0^+} \frac{1}{h} \left(\|\tilde{e}(t+h)\|_{\text{cmpst}} - \|\tilde{e}(t)\|_{\text{cmpst}} \right) \\ &\le \mu_{\text{cmpst}}(TA(t)T^{-1}) \|\tilde{e}(t)\|_{\text{cmpst}} + \sup_t \|T\tilde{w}(t)\|_{\text{cmpst}} \\ &\quad + \sum_{k=1}^{q} \|TB_k(t)T^{-1}\|_{\text{cmpst}} \sup_{t-\tau_{\max}\le s\le t} \|\tilde{e}(s)\|_{\text{cmpst}} \end{aligned}$$

In order to apply Lemma 2, we need to find upper bounds for $\mu_{\text{cmpst}}(TA(t)T^{-1})$ and $\sum_{k=1}^{q} \|TB_k(t)T^{-1}\|_{\text{cmpst}}$. We do this by leveraging sub-additivity of matrix measures and matrix norms, which imply the following bounds: $\mu_{\text{cmpst}}(TA(t)T^{-1}) \le \int_0^1 \mu_{\text{cmpst}}(T\tilde{A}(t)T^{-1}) d\rho$ and $\sum_{k=1}^{q} \|TB_k(t)T^{-1}\|_{\text{cmpst}} \le \int_0^1 \sum_{k=1}^{q} \|T\tilde{B}_k(t)T^{-1}\|_{\text{cmpst}} d\rho$. The fact that the aggregating norm $\|\cdot\|_{\infty,[\eta]^{-1}}$ is monotonic [16, Lemma 2.8] allows for applying Lemma 1, which yields

$$\begin{aligned} \mu_{\text{cmpst}}(T\tilde{A}(t)T^{-1}) &\le \mu_{\infty,[\eta]^{-1}}(\lceil T\tilde{A}(t)T^{-1} \rceil_{\mathbf{M}}) \\ &= \max_i \left\{ \mu_p(T_i\tilde{A}_{ii}(t)T_i^{-1}) + \sum_j \frac{\eta_j}{\eta_i} \|T_i\tilde{A}_{ij}(t)T_j\|_p \right\} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^{q} \|T\tilde{B}_k(t)T^{-1}\|_{\text{cmpst}} &\le \sum_{k=1}^{q} \|\lceil T\tilde{B}_k(t)T^{-1} \rceil\|_{\infty,[\eta]^{-1}} \\ &= \max_i \left\{ \sum_{k=1}^{q} \sum_j \frac{\eta_j}{\eta_i} \|T_i(\tilde{B}_k(t))_{ij}T_j\|_p \right\} \end{aligned}$$

Now, by means of $C2$ and $C3$ we have:

$$\max_i \left\{ \mu_p(T_i\tilde{A}_{ii}(t)T_i^{-1}) + \sum_{j\in\mathcal{N}_i} \frac{\eta_j}{\eta_i} \|T_i\tilde{A}_{ij}(t)T_j^{-1}\|_p \right\} \le -\bar{\sigma},$$

$$\max_i \left\{ \sum_{k=1}^{q} \sum_{j\in\mathcal{N}_i} \frac{\eta_j}{\eta_i} \|T_i(\tilde{B}_k(t))_{ij}T_j^{-1}\|_p \right\} \le \underline{\sigma}$$

for some $0 \le \underline{\sigma} < \bar{\sigma} < +\infty$. In turn, this implies that $\mu_{\text{cmpst}}(T\tilde{A}(t)T^{-1}) + \sum_{k=1}^{q} \|T\tilde{B}_k(t)T^{-1}\|_{\text{cmpst}} \le \bar{\sigma} + \underline{\sigma} := -\sigma$. Then, by means of Lemma 2 we get

$$\begin{aligned} \|\tilde{e}(t)\|_{\text{cmpst}} \le &\sup_{t_0-\tau_{\max}\le s\le t_0} \|\tilde{e}(s)\|_{\text{cmpst}} e^{-\lambda(t-t_0)} \\ &+ \frac{1}{\bar{\sigma} - \underline{\sigma}} \sup_t \|T\tilde{w}(t)\|_{\text{cmpst}} \end{aligned}$$

with λ satisfying $\lambda - \bar{\sigma} + \underline{\sigma} e^{\lambda\tau_{\max}} = 0$. Finally, notice that $\|e(t)\|_{\text{cmpst}} \le \|T^{-1}\|_{\text{cmpst}} \|\tilde{e}(t)\|_{\text{cmpst}}$ and $\|\tilde{e}(t)\|_{\text{cmpst}} \le \|T\|_{\text{cmpst}} \|e(t)\|_{\text{cmpst}}$, we obtain

$$\begin{aligned} \|e(t)\|_{\text{cmpst}} \le \Bigg(&\sup_{t_0-\tau_{\max}\le s\le t_0} \|e(s)\|_{\text{cmpst}} e^{-\lambda(t-t_0)} \\ &+ \frac{1}{\bar{\sigma} - \underline{\sigma}} \sup_t \|\tilde{w}(t)\|_{\text{cmpst}} \Bigg) \|T^{-1}\|_{\text{cmpst}} \|T\|_{\text{cmpst}} \end{aligned}$$

Also notice that $\|\tilde{w}(t)\|_{\text{cmpst}} = \|w(t)\|_{\text{cmpst}}$, $\|e_i(t)\|_p = \|[x_i^\top(t) - x_i^{*\top}(t), \zeta_{i,1}^\top(t), \ldots, \zeta_{i,m}^\top(t)]^\top\|_p \ge \|x_i(t) - x_i^*(t)\|_p$ and $\|e_i(t)\|_p \le \|x_i(t) - x_i^*(t)\|_p + \|\zeta_{i,1}(t)\|_p + \cdots + \|\zeta_{i,m}(t)\|_p$, by monotonicity of the aggregating norm

$\|\cdot\|_{\infty,[\eta]^{-1}}$ we get the upper bound of the state deviation of (1):

$$\|x(t)-x^*(t)\|_{\text{cmpst}}\leq$$
$$\|T^{-1}\|_{\text{cmpst}}\|T\|_{\text{cmpst}}e^{-\lambda(t-t_0)}\Bigg(\sup_{t_0-\tau_{\max}\leq s\leq t_0}\|x(s)-x^*(s)\|_{\text{cmpst}}$$
$$+\sup_{t_0-\tau_{\max}\leq s\leq t_0}\sum_{k=1}^{m}\|\sum_{b=0}^{m-k}\frac{(m-1-b)!}{(m-k-b)!}\cdot\bar{d}_{m-1-b}\cdot s^{m-k-b}$$
$$+r_k(s)\|_{\text{cmpst}}\Bigg)+\frac{\|T^{-1}\|_{\text{cmpst}}\|T\|_{\text{cmpst}}}{\bar{\sigma}-\underline{\sigma}}\sup_t\|w(t)\|_{\text{cmpst}}$$

$\square$

V. Application Example

We consider a MTDC grid model from [21], [22]. In our example, the grid has 5 terminals arranged as illustrated in Figure 2. The dynamics of the terminals are described by

$$c_i\dot{v}_i(t)=-\sum_{j\in\mathcal{N}_i}I_{ij}(t)+u_i(t)+d_i(t),\ i\in\{1,\ldots,5\} \quad (8)$$

where the current $I_{ij}(t)=\frac{1}{R_{ij}}(v_i(t)-v_j(t))$ according to Ohm's law and note that $v_j(t)$ is not delayed in this equation as communication is not required. In the above expression, v_i denotes the voltage deviation of terminal i from the nominal voltage v^{nom} which is assumed to be identical for all terminals [23], c_i is the capacitance of terminal i, R_{ij} is the line resistance between terminal i and terminal j satisfying $R_{ij}=R_{ji}$, u_i is the injected control current and d_i is the disturbance due to e.g. load changes at the terminal. The design of control protocols to reject constant disturbances for (8) was considered in [24] but without delays and in [23] with constant delays. We now leverage Proposition 1 to consider the case where delays are heterogeneous and the disturbances are polynomial.

A. Controller design

In this example, we consider a first order disturbance, which can model e.g. the rapid increase of the current in the terminal caused by fault [25]. To reject such disturbance before being diagnosed, we design the controller with 2 multiplex layers following:

$$\begin{aligned}
u_i(t)&=-k_0v_i(t)-\sum_{j\in\mathcal{N}_i}g_0(v_i(t-\tau_{ij}(t))-v_j(t-\tau_{ij}(t)))\\
&\quad+r_{i,1}(t)\\
\dot{r}_{i,1}(t)&=-k_1v_i(t)-\sum_{j\in\mathcal{N}_i}g_1(v_i(t-\tau_{ij}(t))-v_j(t-\tau_{ij}(t)))\\
&\quad+r_{i,2}(t)\\
\dot{r}_{i,2}(t)&=-k_2v_i(t)-\sum_{j\in\mathcal{N}_i}g_2(v_i(t-\tau_{ij}(t))-v_j(t-\tau_{ij}(t)))
\end{aligned} \quad (9)$$

In the above expression, the delay occurs because the voltage information needs to be transmitted from terminal j to terminal i via communication (see also [23]). For our design, we consider the composite norm $\|x\|_{\text{cmpst}}=$ $\|[\|x_1\|_2,\ldots,\|x_N\|_2]\|_{\infty,[\mathbf{1}_N]^{-1}}$ and the coordinate transformation matrix T with identical diagonal blocks

$$T_i=\begin{bmatrix}1&\alpha&0\\0&1&\beta\\0&0&1\end{bmatrix},\ \alpha,\beta\in\mathbb{R},\ \forall i.$$

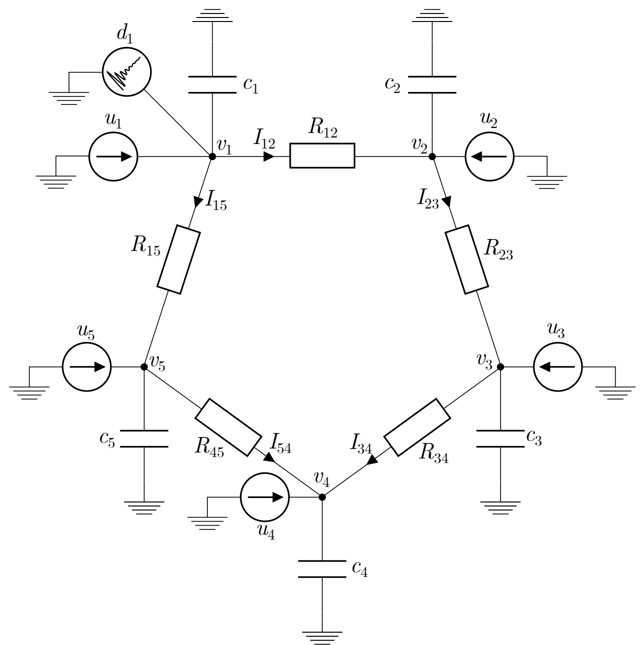

Fig. 2: Illustration of the MTDC composed of 5 terminals. Terminal 1 is affected by a disturbance d_1.

The desired solution of (8) is the solution $v_i^*(t)$ satisfying $\dot{v}_i^*(t)=0,\forall i$. Hence, $C1$ is guaranteed by design of the control (9). We recast the fulfilment of $C2$ and $C3$ as a convex optimisation problem[1] and solve it for the set of control parameters we will use: $k_0=0.8427, k_1=1.4093, k_2=0.5314, g_0=0.0031, g_1=0.0041, g_2=0.0026$, with corresponding $\alpha=-0.5$, $\beta=-1$.

B. Simulation

We let the capacitance $c_i=1$mF, $\forall i$ and the resistances $R_{ij}=20\Omega,\forall i,j$. We consider a disturbance acting on a random terminal, say terminal 1, which is $d_1(t)=3+t+e^{-0.2t}\sin t$. The heterogeneous delays are assumed to be constant and is randomly selected from $]0,0.3[$ seconds (see Appendix). The grid is assumed to be initiated at $v_i(0)\sim\mathcal{N}(0,1),\forall i$. Figure 3 (top panel) illustrates the voltage deviation of all the terminals from the nominal value. It shows that all the deviations finally reduce to 0 including the perturbed terminal 1. This is in accordance with the theoretical prediction as the designed controller injected a ramp current to compensate for the ramp component in the disturbance, as illustrated in the bottom panel.

[1]The code solving the optimisation problem can be found in `https://tinyurl.com/yc3frafb`.

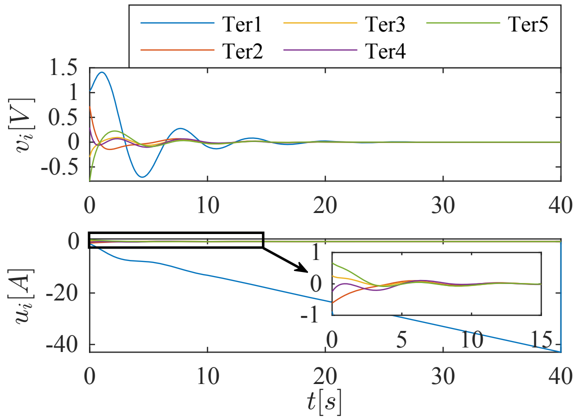

Fig. 3: Top panel: Voltage deviation from nominal voltage; Bottom panel: Injected control current.

VI. Conclusions and future work

We considered the problem of designing distributed multiplex integral control protocols for nonlinear networks affected by delays and disturbances. The designed control protocol, delivered via multiplex architecture and fulfilling certain conditions leveraging non-Euclidean contraction theory, is able to: (i) reject polynomial disturbances; (ii) achieve Input-to-State Stability for nonlinear networks affected by heterogeneous time-varying delays. We validated the results by considering the problem of controlling a MTDC grid and simulations confirmed the effectiveness of the results. Our future work will include studying a stronger scalability property for the class of networks considered here, which can be of interest for the application of e.g. smart grids and biological networks.

Appendix

The delays affecting the communication between terminals are listed below:

$$\begin{aligned}&\tau_{12}(t) = 0.043s; \tau_{15}(t) = 0.127s; \tau_{21}(t) = 0.275s;\\&\tau_{23}(t) = 0.238s; \tau_{32}(t) = 0.288s; \tau_{34}(t) = 0.197s;\\&\tau_{43}(t) = 0.011s; \tau_{45}(t) = 0.255s; \tau_{54}(t) = 0.204s;\\&\tau_{51}(t) = 0.281s.\end{aligned}$$

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