ANSWERS FOR NUMERICAL METHODS

Exercise Set 1.2 (Page 000)

1. For each part, $f \in C[a,b]$ on the given interval. Since $f(a)$ and $f(b)$ are of opposite sign, the Intermediate Value Theorem implies a number c exists with $f(c) = 0$.

3. For each part, $f \in C[a,b]$, f' exists on (a,b), and $f(a) = f(b) = 0$. Rolle's Theorem implies that a number c exists in (a,b) with $f'(c) = 0$. For part (d), we can use $[a,b] = [-1,0]$ or $[a,b] = [0,2]$.

5. **a.** $P_2(x) = 0$

b. $R_2(0.5) = 0.125$; actual error $= 0.125$

c. $P_2(x) = 1 + 3(x-1) + 3(x-1)^2$

d. $R_2(0.5) = -0.125$; actual error $= -0.125$

7. Since

$$P_2(x) = 1 + x \quad \text{and} \quad R_2(x) = \frac{-2e^{\xi}(\sin\xi + \cos\xi)}{6}x^3$$

for some number ξ between x and 0, we have the following:

a. $P_2(0.5) = 1.5$ and $f(0.5) = 1.446889$. An error bound is 0.093222 and $|f(0.5) - P_2(0.5)| \leq 0.0532$

b. $|f(x) - P_2(x)| \leq 1.252$

c. $\int_0^1 f(x)\,dx \approx 1.5$

d. $|\int_0^1 f(x)\,dx - \int_0^1 P_2(x)\,dx| \leq \int_0^1 |R_2(x)|\,dx \leq 0.313$, and the actual error is 0.122.

9. The error is approximately 8.86×10^{-7}.

11. **a.** $P_3(x) = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{23}{648}x^3$

b. We have

$$f^{(4)}(x) = \frac{-199}{2592}e^{x/2}\sin\frac{x}{3} + \frac{61}{3888}e^{x/2}\cos\frac{x}{3},$$

so

$$|f^{(4)}(x)| \le |f^{(4)}(0.60473891)| \le 0.09787176 \quad \text{for } 0 \le x \le 1,$$

and

$$|f(x) - P_3(x)| \le \frac{|f^{(4)}(\xi)|}{4!}|x|^4 \le \frac{0.09787176}{24}(1)^4 = 0.004077990.$$

13. A bound for the maximum error is 0.0026.

15. **a.**

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}$$

Use this series to integrate

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

and obtain the result.

b.

$$\begin{aligned}\frac{2}{\sqrt{\pi}}e^{-x^2}\sum_{k=0}^{\infty}\frac{2^k x^{2k+1}}{1\cdot 3\cdots(2k+1)} &= \frac{2}{\sqrt{\pi}}\left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^7 + \frac{1}{24}x^8 + \cdots\right]\\ &\quad\cdot\left[x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + \frac{8}{105}x^7 + \frac{16}{945}x^9 + \cdots\right]\\ &= \frac{2}{\sqrt{\pi}}\left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 + \cdots\right] = \operatorname{erf}(x)\end{aligned}$$

c. 0.8427008 **d.** 0.8427069

e. The series in part (a) is alternating, so for any positive integer n and positive x we have the bound

$$\left| \operatorname{erf}(x) - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)k!} \right| < \frac{x^{2n+3}}{(2n+3)(n+1)!} .$$

We have no such bound for the positive term series in part (b).

Exercise Set 1.3 (Page 000)

1.	Absolute Error	Relative Error
a.	0.001264	4.025×10^{-4}
b.	7.346×10^{-6}	2.338×10^{-6}
c.	2.818×10^{-4}	1.037×10^{-4}
d.	2.136×10^{-4}	1.510×10^{-4}
e.	2.647×10^{1}	1.202×10^{-3}
f.	1.454×10^{1}	1.050×10^{-2}
g.	420	1.042×10^{-2}
h.	3.343×10^{3}	9.213×10^{-3}

3.	Approximation	Absolute Error	Relative Error
a.	134	0.079	5.90×10^{-4}
b.	133	0.499	3.77×10^{-3}
c.	2.00	0.327	0.195
d.	1.67	0.003	1.79×10^{-3}
e.	1.80	0.154	0.0786
f.	-15.1	0.0546	3.60×10^{-3}
g.	0.286	2.86×10^{-4}	10^{-3}
h.	0.00	0.0215	1.00

5.	Approximation	Absolute Error	Relative Error
a.	133.9	0.021	1.568×10^{-4}
b.	132.5	0.001	7.55×10^{-6}
c.	1.700	0.027	0.01614
d.	1.673	0	0
e.	1.986	0.03246	0.01662
f.	-15.16	0.005377	3.548×10^{-4}
g.	0.2857	1.429×10^{-5}	5×10^{-5}
h.	-0.01700	0.0045	0.2092

7.

	Approximation	Absolute Error	Relative Error
a.	3.14557613	3.983×10^{-3}	1.268×10^{-3}
b.	3.14162103	2.838×10^{-5}	9.032×10^{-6}

9. **b.** The first formula gives -0.00658 and the second formula gives -0.0100. The true three-digit value is -0.0116.

11. a. $39.375 \leq$ volume ≤ 86.625 **b.** $71.5 \leq$ surface area ≤ 119.5

Exercise Set 1.4 (Page 000)

1.	x_1	Absolute Error	Relative Error	x_2	Absolute Error	Relative Error
a.	92.26	0.01542	1.672×10^{-4}	0.005419	6.273×10^{-7}	1.157×10^{-4}
b.	0.005421	1.264×10^{-6}	2.333×10^{-4}	-92.26	4.580×10^{-3}	4.965×10^{-5}
c.	10.98	6.875×10^{-3}	6.257×10^{-4}	0.001149	7.566×10^{-8}	6.584×10^{-5}
d.	-0.001149	7.566×10^{-8}	6.584×10^{-5}	-10.98	6.875×10^{-3}	6.257×10^{-4}

3. **a.** -0.1000

b. -0.1010

c. Absolute error for part (a) is 2.331×10^{-3} with relative error 2.387×10^{-2}. Absolute error for part (b) is 3.331×10^{-3} with relative error 3.411×10^{-2}.

5.	Approximation	Absolute Error	Relative Error
a. and b.	3.743	1.011×10^{-3}	2.694×10^{-3}
c. and d,	3.755	1.889×10^{-4}	5.033×10^{-4}

7. **a.** The approximate sums are 1.53 and 1.54, respectively. The actual value is 1.549. Significant round-off error occurs earlier with the first method.

9.

	Approximation	Absolute Error	Relative Error
a.	2.715	3.282×10^{-3}	1.207×10^{-3}
b.	2.716	2.282×10^{-3}	8.394×10^{-4}
c.	2.716	2.282×10^{-3}	8.394×10^{-4}
d.	2.718	2.818×10^{-4}	1.037×10^{-4}

11. The rates of convergence are as follows.

a. $O(h^2)$ **b.** $O(h)$ **c.** $O(h^2)$ **d.** $O(h)$

13. Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x$ and $x_{n+1} = 1 + \frac{1}{x_n}$, we have $x = 1 + \frac{1}{x}$. This implies that $x = (1+\sqrt{5})/2$. This number is called the *golden ratio*. It appears frequently in mathematics and the sciences.

15. a. $n = 50$ **b.** $n = 500$

c. An accuracy of 10^{-4} cannot be obtained with `Digits` set to 10 in some earlier versions of Maple. However, in Release 7 we get $n = 5001$.

Exercise Set 2.2 (Page 000)

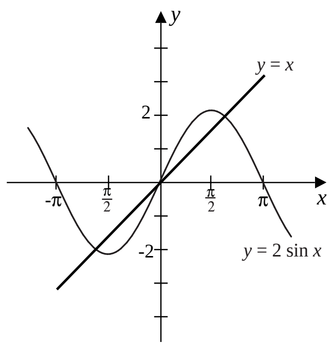
1. $p_3 = 0.625$

3. The Bisection method gives the following.

a. $p_7 = 0.5859$ **b.** $p_8 = 3.002$ **c.** $p_7 = 3.419$

5. a.

Note: New Figure



b. With $[1, 2]$, we have $p_7 = 1.8984$.

7. a. 2 **b.** -2 **c.** -1 **d.** 1

9. $\sqrt{3} \approx p_{14} = 1.7320$ using $[1, 2]$

11. A bound is $n \geq 12$, and $p_{12} = 1.3787$.

13. Since $-1 < a < 0$ and $2 < b < 3$, we have $1 < a + b < 3$ or $1/2 < 1/2(a + b) < 3/2$ in all cases. Further,

$$f(x) < 0, \quad \text{for } -1 < x < 0 \quad \text{and} \quad 1 < x < 2;$$

$$f(x) > 0, \quad \text{for } 0 < x < 1 \quad \text{and} \quad 2 < x < 3.$$

Thus, $a_1 = a$, $f(a_1) < 0$, $b_1 = b$, and $f(b_1) > 0$.

a. Since $a + b < 2$, we have $p_1 = \frac{a+b}{2}$ and $1/2 < p_1 < 1$. Thus, $f(p_1) > 0$. Hence, $a_2 = a_1 = a$ and $b_2 = p_1$. The only zero of f in $[a_2, b_2]$ is $p = 0$, so the convergence will be to 0.

b. Since $a + b > 2$, we have $p_1 = \frac{a+b}{2}$ and $1 < p_1 < 3/2$. Thus, $f(p_1) < 0$. Hence, $a_2 = p_1$ and $b_2 = b_1 = b$. The only zero of f in $[a_2, b_2]$ is $p = 2$, so the convergence will be to 2.

c. Since $a + b = 2$, we have $p_1 = \frac{a+b}{2} = 1$ and $f(p_1) = 0$. Thus, a zero of f has been found on the first iteration. The convergence is to $p = 1$.

Exercise Set 2.3 (Page 000)

1. a. $p_3 = 2.45454$ **b.** $p_3 = 2.44444$

3. Using the endpoints of the intervals as p_0 and p_1, we have the following.

a. $p_{11} = 2.69065$ **b.** $p_7 = -2.87939$ **c.** $p_6 = 0.73909$ **d.** $p_5 = 0.96433$

5. Using the endpoints of the intervals as p_0 and p_1, we have the following.

a. $p_{16} = 2.69060$ **b.** $p_6 = -2.87938$ **c.** $p_7 = 0.73908$ **d.** $p_6 = 0.96433$

7. For $p_0 = 0.1$ and $p_1 = 3$ we have $p_7 = 2.363171$.
For $p_0 = 3$ and $p_1 = 4$ we have $p_7 = 3.817926$.
For $p_0 = 5$ and $p_1 = 6$ we have $p_6 = 5.839252$.
For $p_0 = 6$ and $p_1 = 7$ we have $p_9 = 6.603085$.

9. For $p_0 = 1$ and $p_1 = 2$, we have $p_5 = 1.73205068$, which compares to 14 iterations of the Bisection method.

11 . For $p_0 = 0$ and $p_1 = 1$, the Secant method gives $p_7 = 0.589755$. The closest point on the graph is $(0.589755, 0.347811)$.

13 . **a.** For $p_0 = -1$ and $p_1 = 0$, we have $p_{17} = -0.04065850$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_9 = 0.9623984$.

b. For $p_0 = -1$ and $p_1 = 0$, we have $p_5 = -0.04065929$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_{12} = -0.04065929$. The Secant method fails to find the zero in $[0, 1]$.

15 . For $p_0 = \frac{1}{2}$, $p_1 = \frac{\pi}{4}$, and tolerance of 10^{-100}, the Secant method required 11 iterations, giving the 100-digit answer

$p_{11} = .7390851332151606416553120876738734040134117589007574649656806357732 8$
$4654883547594599376106931766531 9.$

17 . For $p_0 = 0.1$ and $p_1 = 0.2$, the Secant method gives $p_3 = 0.16616$, so the depth of the water is $1 - p_3 = 0.83385$ ft.

Exercise Set 2.4 (Page 000)

1. $p_2 = 2.60714$

3. **a.** For $p_0 = 2$, we have $p_5 = 2.69065$.

b. For $p_0 = -3$, we have $p_3 = -2.87939$.

c. For $p_0 = 0$, we have $p_4 = 0.73909$.

d. For $p_0 = 0$, we have $p_3 = 0.96434$.

5. Newton's method gives the following approximations:

With $p_0 = 1.5, p_6 = 2.363171$; with $p_0 = 3.5, p_5 = 3.817926$;

With $p_0 = 5.5, p_4 = 5.839252$; with $p_0 = 7, p_5 = 6.603085$.

7. Newton's method gives the following:

a. For $p_0 = 0.5$ we have $p_{13} = 0.567135$.

b. For $p_0 = -1.5$ we have $p_{23} = -1.414325$.

c. For $p_0 = 0.5$ we have $p_{22} = 0.641166$.

d. For $p_0 = -0.5$ we have $p_{23} = -0.183274$.

9. With $p_0 = 1.5$, we have $p_3 = 1.73205081$ which compares to 14 iterations of the Bisection method and 5 iterations of the Secant method.

11. **a.** $p_{10} = 13.655776$ **b.** $p_6 = 0.44743154$

c. With $p_0 = 0$, Newton's method did not converge in 10 iterations. The initial approximation $p_0 = 0.48$ is sufficiently close to the solution for rapid convergence.

13 . Newton's method gives $p_{15} = 1.895488$ for $p_0 = \frac{\pi}{2}$, and $p_{19} = 1.895489$ for $p_0 = 5\pi$. The sequence does not converge in 200 iterations for $p_0 = 10\pi$. The results do not indicate the fast convergence usually associated with Newton's method.

15 . Using $p_0 = 0.75$, Newton's method gives $p_4 = 0.8423$.

17 . The minimal interest rate is 6.67%.

19. a. $\dfrac{e}{3}, t = 3$ hours **b.** 11 hours and 5 minutes **c.** 21 hours and 14 minutes

Exercise Set 2.5 (Page 000)

1. The results are listed in the following table.

	a.	b.	c.	d.
q_0	0.258684	0.907859	0.548101	0.731385
q_1	0.257613	0.909568	0.547915	0.736087
q_2	0.257536	0.909917	0.547847	0.737653
q_3	0.257531	0.909989	0.547823	0.738469
q_4	0.257530	0.910004	0.547814	0.738798
q_5	0.257530	0.910007	0.547810	0.738958

3. Newton's Method gives $p_6 = -0.1828876$, and the improved value is $q_6 = -0.183387$.

5. a. (i) Since $|p_{n+1} - 0| = \frac{1}{n+1} < \frac{1}{n} = |p_n - 0|$, the sequence $\left\{\frac{1}{n}\right\}$ converges linearly to 0. (ii) We need $\frac{1}{n} \leq 0.05$ or $n \geq 20$. (iii) Aitken's Δ^2 method gives $q_{10} = 0.0\overline{45}$.

b. (i) Since $|p_{n+1} - 0| = \frac{1}{(n+1)^2} < \frac{1}{n^2} = |p_n - 0|$, the sequence $\left\{\frac{1}{n^2}\right\}$ converges linearly to 0. (ii) We need $\frac{1}{n^2} \leq 0.05$ or $n \geq 5$. (iii) Aitken's Δ^2 method gives $q_2 = 0.0363$.

7. a. Since

$$\frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1,$$

the sequence is quadratically convergent.

b. Since

$$\frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \frac{10^{-(n+1)^k}}{(10^{-n^k})^2} = \frac{10^{-(n+1)^k}}{10^{-2n^k}} = 10^{2n^k - (n+1)^k}$$

diverges, the sequence $p_n = 10^{-n^k}$ does not converge quadratically.

Exercise Set 2.6 (Page 000)

1. a. For $p_0 = 1$, we have $p_{22} = 2.69065$.

b. For $p_0 = 1$, we have $p_5 = 0.53209$; for $p_0 = -1$, we have $p_3 = -0.65270$, and for $p_0 = -3$, we have $p_3 = -2.87939$.

c. For $p_0 = 1$, we have $p_4 = 1.12412$; and for $p_0 = 0$, we have $p_8 = -0.87605$.

d. For $p_0 = 0$, we have $p_{10} = 1.49819$.

3. The following table lists the initial approximation and the roots.

	p_0	p_1	p_2	Approximated Roots	Complex Conjugate Roots
a.	-1	0	1	$p_7 = -0.34532 - 1.31873i$	$-0.34532 + 1.31873i$
	0	1	2	$p_6 = 2.69065$	
b.	0	1	2	$p_6 = 0.53209$	
	1	2	3	$p_9 = -0.65270$	
	-2	-3	-2.5	$p_4 = -2.87939$	
c.	0	1	2	$p_5 = 1.12412$	
	2	3	4	$p_{12} = -0.12403 + 1.74096i$	$-0.12403 - 1.74096i$
	-2	0	-1	$p_5 = -0.87605$	
d.	0	1	2	$p_6 = 1.49819$	
	-1	-2	-3	$p_{10} = -0.51363 - 1.09156i$	$-0.51363 + 1.09156i$
	1	0	-1	$p_8 = 0.26454 - 1.32837i$	$0.26454 + 1.32837i$

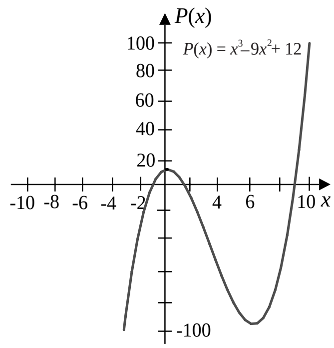
5. a. The roots are 1.244, 8.847, and -1.091. The critical points are 0 and 6.

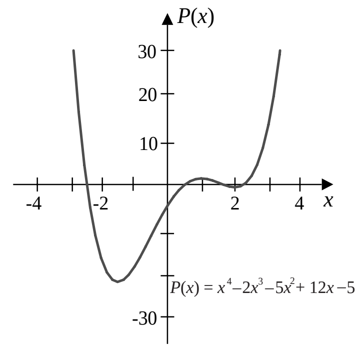
FIGURE 0 PLACED HERE

for Exercise 5a

b. The roots are 0.5798, 1.521, 2.332, and -2.432, and the critical points are 1, 2.001, and -1.5.

Note: New Figures





7. Let $c_1 = (2 + \frac{2}{9}\sqrt{129})^{-1/3}$ and $c_2 = (2 + \frac{2}{9}\sqrt{129})^{1/3}$. The roots are $c_2 - \frac{4}{3}c_1$, $-\frac{1}{2}c_2 + \frac{2}{3}c_1 + \frac{1}{2}\sqrt{3}(c_2 + \frac{4}{3}c_1)i$, and $-\frac{1}{2}c_2 + \frac{2}{3}c_1 - \frac{1}{2}\sqrt{3}(c_2 + \frac{4}{3}c_1)i$.

9. **a.** For $p_0 = 0.1$ and $p_1 = 1$ we have $p_{14} = 0.23233$.

b. For $p_0 = 0.55$ we have $p_6 = 0.23235$.

c. For $p_0 = 0.1$ and $p_1 = 1$ we have $p_8 = 0.23235$.

d. For $p_0 = 0.1$ and $p_1 = 1$ we have $p_{88} = 0.23035$.

e. For $p_0 = 0$, $p_1 = 0.25$, and $p_2 = 1$ we have $p_6 = 0.23235$.

11. The minimal material is approximately 573.64895 cm^2.

Exercise Set 3.2 (Page 000)

1. **a.** (i) $P_1(x) = -0.29110731x + 1$; $P_1(0.45) = 0.86900171$; $|\cos 0.45 - P_1(0.45)| = 0.03144539$; (ii) $P_2(x) = -0.43108687x^2 - 0.03245519x + 1$; $P_2(0.45) = 0.89810007$; $|\cos 0.45 - P_2(0.45)| = 0.0023470$

b. (i) $P_1(x) = 0.44151844x + 1$; $P_1(0.45) = 1.1986833$; $|\sqrt{1.45} - P_1(0.45)| = 0.00547616$; (ii) $P_2(x) = -0.070228596x^2 + 0.483655598x + 1$; $P_2(0.45) = 1.20342373$; $|\sqrt{1.45} - P_2(0.45)| = 0.00073573$

c. (i) $P_1(x) = 0.78333938x$; $P_1(0.45) = 0.35250272$; $|\ln 1.45 - P_1(0.45)| = 0.01906083$; (ii) $P_2(x) = -0.23389466x^2 + 0.92367618x$; $P_2(0.45) = 0.36829061$; $|\ln 1.45 - P_2(0.45)| = 0.00327294$

d. (i) $P_1(x) = 1.14022801x$; $P_1(0.45) = 0.0.51310260$; $|\tan 0.45 - P_1(0.45)| = 0.03004754$; (ii) $P_2(x) = 0.86649261x^2 + 0.62033245x$; $P_2(0.45) = 0.45461436$; $|\tan 0.45 - P_2(0.45)| = 0.02844071$

3. **a.**

n	$x_0, x_1, \ldots, x_n$	$P_n(8.4)$
1	8.3, 8.6	17.87833
2	8.3, 8.6, 8.7	17.87716
3	8.3, 8.6, 8.7, 8.1	17.87714

b.

n	$x_0, x_1, \ldots, x_n$	$P_n(-\frac{1}{3})$
1	−0.5, −0.25	0.21504167
2	−0.5, −0.25, 0.0	0.16988889
3	−0.5, −0.25, 0.0, −0.75	0.17451852

c.

n	$x_0, x_1, \ldots, x_n$	$P_n(0.25)$
1	0.2, 0.3	−0.13869287
2	0.2, 0.3, 0.4	−0.13259734
3	0.2, 0.3, 0.4, 0.1	−0.13277477

d.

n	$x_0, x_1, \ldots, x_n$	$P_n(0.9)$
1	0.8, 1.0	0.44086280
2	0.8, 1.0, 0.7	0.43841352
3	0.8, 1.0, 0.7, 0.6	0.44198500

5. $\sqrt{3} \approx P_4\left(\frac{1}{2}\right) = 1.708\overline{3}$

7. **a.**

n	Actual Error	Error Bound
1	0.00118	0.00120
2	1.367×10^{-5}	1.452×10^{-5}

b.

n	Actual Error	Error Bound
1	4.0523×10^{-2}	4.5153×10^{-2}
2	4.6296×10^{-3}	4.6296×10^{-3}

c.

n	Actual Error	Error Bound
1	5.9210×10^{-3}	6.0971×10^{-3}
2	1.7455×10^{-4}	1.8128×10^{-4}

d.

n	Actual Error	Error Bound
1	2.7296×10^{-3}	1.4080×10^{-2}
2	5.1789×10^{-3}	9.2215×10^{-3}

9. $f(1.09) \approx 0.2826$. The actual error is 4.3×10^{-5}, and an error bound is 7.4×10^{-6}. The discrepancy is due to the fact that the data are given to only four decimal places and only four-digit arithmetic is used.

11. $y = 4.25$

13. The largest possible step size is 0.004291932, so 0.004 would be a reasonable choice.

15. The difference between the actual value and the computed value is $\frac{2}{3}$.

17. **a.**

x	erf(x)
0.0	0
0.2	0.2227
0.4	0.4284
0.6	0.6039
0.8	0.7421
1.0	0.8427

b. Linear interpolation with $x_0 = 0.2$ and $x_1 = 0.4$ gives $\text{erf}(\frac{1}{3}) \approx 0.3598$. Quadratic interpolation with $x_0 = 0.2, x_1 = 0.4$, and $x_2 = 0.6$ gives $\text{erf}(\frac{1}{3}) \approx 0.3632$. Since $\text{erf}(1/3) \approx 0.3626$, quadratic interpolation is more accurate.

Exercise Set 3.3 (Page 000)

1. Newton's interpolatory divided-difference formula gives the following:

a. $P_1(x) = 16.9441 + 3.1041(x - 8.1); P_1(8.4) = 17.87533$

$P_2(x) = P_1(x) + 0.06(x - 8.1)(x - 8.3); P_2(8.4) = 17.87713$

$P_3(x) = P_2(x) + -0.00208333(x - 8.1)(x - 8.3)(x - 8.6); P_3(8.4) = 17.87714$

b. $P_1(x) = -0.1769446 + 1.9069687(x - 0.6); P_1(0.9) = 0.395146$

$P_2(x) = P_1(x) + 0.959224(x - 0.6)(x - 0.7); P_2(0.9) = 0.4526995$

$P_3(x) = P_2(x) - 1.785741(x - 0.6)(x - 0.7)(x - 0.8); P_3(0.9) = 0.4419850$

3. In the following equations we have $s = \frac{1}{h}(x - x_n)$.

a. $P_1(s) = 1.101 + 0.7660625s; \quad f(-\frac{1}{3}) \approx P_1(-\frac{4}{3}) = 0.07958333$

$P_2(s) = P_1(s) + 0.406375s(s + 1)/2; \quad f(-\frac{1}{3}) \approx P_2(-\frac{4}{3}) = 0.1698889$

$P_3(s) = P_2(s) + 0.09375s(s + 1)(s + 2)/6; \quad f(-\frac{1}{3}) \approx P_3(-\frac{4}{3}) = 0.1745185$

b. $P_1(s) = 0.2484244 + 0.2418235s; \quad f(0.25) \approx P_1(-1.5) = -0.1143108$

$P_2(s) = P_1(s) - 0.04876419s(s + 1)/2; \quad f(0.25) \approx P_2(-1.5) = -0.1325973$

$P_3(s) = P_2(s) - 0.00283891s(s+1)(s+2)/6; \quad f(0.25) \approx P_3(-1.5) = -0.1327748$

5. **a.** $f(0.05) \approx 1.05126$

b. $f(0.65) \approx 1.91555$

7. $\Delta^3 f(x_0) = -6$ and $\Delta^4 f(x_0) = \Delta^5 f(x_0) = 0$, so the interpolating polynomial has degree 3.

9. $\Delta^2 P(10) = 1140$.

11. The approximation to $f(0.3)$ should be increased by 5.9375.

13. $f[x_0] = f(x_0) = 1$, $f[x_1] = f(x_1) = 3$, $f[x_0, x_1] = 5$

Exercise Set 3.4 (Page 000)

1. The coefficients for the polynomials in divided-difference form are given in the following tables. For example, the polynomial in part (a) is

$$H_3(x) = 17.56492 + 3.116256(x-8.3) + 0.05948(x-8.3)^2 - 0.00202222(x-8.3)^2(x-8.6).$$

a.	**b.**	**c.**	**d.**
17.56492	0.022363362	−0.02475	−0.62049958
3.116256	2.1691753	0.751	3.5850208
0.05948	0.01558225	2.751	−2.1989182
−0.00202222	−3.2177925	1	−0.490447
		0	0.037205
		0	0.040475
			−0.0025277777
			0.0029629628

3. **a.** We have $\sin 0.34 \approx H_5(0.34) = 0.33349$.

b. The formula gives an error bound of 3.05×10^{-14}, but the actual error is 2.91×10^{-6}. The discrepancy is due to the fact that the data are given to only five decimal places.

c. We have $\sin 0.34 \approx H_7(0.34) = 0.33350$. Although the error bound is now 5.4×10^{-20}, the accuracy of the given data dominates the calculations. This result is actually less accurate than the approximation in part (b), since $\sin 0.34 = 0.333487$.

5. For 2(a) we have an error bound of 5.9×10^{-8}. The error bound for 2(c) is 0 since $f^{(n)}(x) \equiv 0$ for $n > 3$.

7. The Hermite polynomial generated from these data is

$$\begin{aligned} H_9(x) = {} & 75x + 0.222222x^2(x-3) - 0.0311111x^2(x-3)^2 \\ & - 0.00644444x^2(x-3)^2(x-5) + 0.00226389x^2(x-3)^2(x-5)^2 \\ & - 0.000913194x^2(x-3)^2(x-5)^2(x-8) + 0.000130527x^2(x-3)^2(x-5)^2(x-8)^2 \\ & - 0.0000202236x^2(x-3)^2(x-5)^2(x-8)^2(x-13). \end{aligned}$$

a. The Hermite polynomial predicts a position of $H_9(10) = 743$ ft and a speed of $H_9'(10) = 48$ ft/s. Although the position approximation is reasonable, the low-speed prediction is suspect.

b. To find the first time the speed exceeds 55 mi/h $= 80.\overline{6}$ ft/s, we solve for the smallest value of t in the equation $80.\overline{6} = H_9'(x)$. This gives $x \approx 5.6488092$.

c. The estimated maximum speed is $H_9'(12.37187) = 119.423$ ft/s ≈ 81.425 mi/h.

Exercise Set 3.5 (Page 000)

1. $S(x) = x$ on $[0, 2]$

3. The equations of the respective free cubic splines are given by

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$ and the coefficients in the following tables.

a.

i	a_i	b_i	c_i	d_i
0	17.564920	3.13410000	0.00000000	0.00000000

b.

i	a_i	b_i	c_i	d_i
0	0.22363362	2.17229175	0.00000000	0.00000000

c.

i	a_i	b_i	c_i	d_i
0	−0.02475000	1.03237500	0.00000000	6.50200000
1	0.33493750	2.25150000	4.87650000	−6.50200000

d.

i	a_i	b_i	c_i	d_i
0	−0.62049958	3.45508693	0.00000000	−8.9957933
1	−0.28398668	3.18521313	−2.69873800	−0.94630333
2	0.00660095	2.61707643	−2.98262900	9.9420966

5. The equations of the respective clamped cubic splines are given by

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$ and the coefficients in the following tables.

a.

i	a_i	b_i	c_i	d_i
0	17.564920	3.1162560	0.0600867	−0.00202222

b.

i	a_i	b_i	c_i	d_i
0	0.22363362	2.1691753	0.65914075	−3.2177925

c.

i	a_i	b_i	c_i	d_i
0	−0.02475000	0.75100000	2.5010000	1.0000000
1	0.33493750	2.18900000	3.2510000	1.0000000

d.

i	a_i	b_i	c_i	d_i
0	−0.62049958	3.5850208	−2.1498407	−0.49077413
1	−0.28398668	3.1403294	−2.2970730	−0.47458360
2	0.006600950	2.6666773	−2.4394481	−0.44980146

7. **a.** The equation of the spline is

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

on the interval $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.0	−0.7573593	0.0	−.6.627417
0.25	0.7071068	−2.0	−4.970563	6.627417
0.5	0.0	−3.242641	0.0	6.627417
0.75	−0.7071068	−2.0	4.970563	−6.627417

b. $\int_0^1 S(x)\, dx = 0.000000$ **c.** $S'(0.5) = -3.24264$, and $S''(0.5) = 0.0$

9. **a.** The equation of the spline is

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

on the interval $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.0	0.0	−5.193321	2.028118
0.25	0.7071068	−2.216388	−3.672233	4.896310
0.5	0.0	−3.134447	0.0	4.896310
0.75	−0.7071068	−2.216388	3.672233	2.028118

b. $\int_0^1 s(x)\ dx = 0.000000$ **c.** $s'(0.5) = -3.13445,$ and $s''(0.5) = 0.0.$

11. $a = 2$, $b = -1$, $c = -3$, $d = 1$

13. $B = \frac{1}{4}$, $D = \frac{1}{4}$, $b = -\frac{1}{2}$, $d = \frac{1}{4}$

15. Let $f(x) = a + bx + cx^2 + dx^3$. Clearly, f satisfies properties (a), (c), (d), (e) of the definition and f interpolates itself for any choice of $x_0, \ldots, x_n$. Since (ii) of (f) in the definition holds, f must be its own clamped cubic spline. However, $f''(x) = 2c + 6dx$ can be zero only at $x = -c/3d$. Thus, part (i) of (f) in the definition cannot hold at two values x_0 and x_n, and f cannot be a natural cubic spline.

17.

x_i	a_i	b_i	c_i	d_i
1940	132165	1651.85	0.00000	2.64248
1950	151326	2444.59	79.2744	−4.37641
1960	179323	2717.16	−52.0179	2.00918
1970	203302	2279.55	8.25746	−0.381311
1980	226542	2330.31	−3.18186	0.106062

$S(1930) = 113004$, $S(1965) = 191860$, and $S(2010) = 296451$.

19. **a.** $S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$ on $[x_i, x_{i+1}]$, where

x_i	a_i	b_i	c_i	d_i
0	0	88.8	0	12.8
0.25	22.4	91.2	9.6	0
0.5	45.8	96.0	9.6	−4.8
1.0	95.6	102.0	2.4	−3.2
1.25				

b. 1:10 $\frac{13}{40}$

c. Starting speed ≈ 40.54 mi/h. Ending speed ≈ 35.09 mi/h.

Exercise Set 3.6 (Page 000)

1. a. $x(t) = -10t^3 + 14t^2 + t, \quad y(t) = -2t^3 + 3t^2 + t$

b. $x(t) = -10t^3 + 14.5t^2 + 0.5t, \quad y(t) = -3t^3 + 4.5t^2 + 0.5t$

c. $x(t) = -10t^3 + 14t^2 + t, \quad y(t) = -4t^3 + 5t^2 + t$

d. $x(t) = -10t^3 + 13t^2 + 2t, \quad y(t) = 2t$

3. a. $x(t) = -11.5t^3 + 15t^2 + 1.5t + 1, \quad y(t) = -4.25t^3 + 4.5t^2 + 0.75t + 1$

b. $x(t) = -6.25t^3 + 10.5t^2 + 0.75t + 1, \quad y(t) = -3.5t^3 + 3t^2 + 1.5t + 1$

c. For t between $(0, 0)$ and $(4, 6)$ we have

$$x(t) = -5t^3 + 7.5t^2 + 1.5t, \quad y(t) = -13.5t^3 + 18t^2 + 1.5t,$$

and for t between $(4, 6)$ and $(6, 1)$ we have

$$x(t) = -5.5t^3 + 6t^2 + 1.5t + 4, \quad y(t) = 4t^3 - 6t^2 - 3t + 6.$$

d. For t between $(0, 0)$ and $(2, 1)$ we have

$$x(t) = -5.5t^3 + 6t^2 + 1.5t, \quad y(t) = -0.5t^3 + 1.5t,$$

for t between $(2,1)$ and $(4,0)$ we have

$$x(t) = -4t^3 + 3t^2 + 3t + 2, \quad y(t) = -t^3 + 1,$$

and for t between $(4,0)$ and $(6,-1)$ we have

$$x(t) = -8.5t^3 + 13.5t^2 - 3t + 4, \quad y(t) = -3.25t^3 + 5.25t^2 - 3t.$$

Exercise Set 4.2 (Page 000)

1. The Midpoint rule gives the following approximations.

a. 0.1582031 **b.** – 0.2666667 **c.** 0.1743309 **d.** 0.1516327

e. – 0.6753247 **f.** – 0.1768200 **g.** 0.1180292 **h.** 1.8039148

3. The Trapezoidal rule gives the following approximations.

a. 0.265625 **b.** – 0.2678571 **c.** 0.2280741 **d.** 0.1839397

e. – 0.8666667 **f.** – 0.1777643 **g.** 0.2180895 **h.** 4.1432597

5. Simpson's rule gives the following approximations.

a. 0.1940104 **b.** – 0.2670635 **c.** 0.1922453 **d.** 0.16240168

e. – 0.7391053 **f.** – 0.1768216 **g.** 0.1513826 **h.** 2.5836964

7. Formula (1) gives the following approximations.

a. 0.19386574 **b.** – 0.26706310 **c.** 0.19225309 **d.** 0.16140992

e. – 0.73642770 **f.** – 0.17682071 **g.** 0.15158524 **h.** 2.5857891

9. $f(1) = \frac{1}{2}$

11. $c_0 = \frac{1}{4}$, $c_1 = \frac{3}{4}$, and $x_1 = \frac{2}{3}$

13.

	(i) Midpoint rule	(ii) Trapezoidal rule	(iii) Simpson's rule
a.	4.83393	5.43476	5.03420
b.	-7.2×10^{-7}	1.6×10^{-6}	5.3×10^{-8}

Exercise Set 4.3 (Page 000)

1. The Composite Trapezoidal rule approximations are as follows.

a. 0.639900 **b.** 31.3653 **c.** 0.784241 **d.** – 6.42872

e. – 13.5760 **f.** 0.476977 **g.** 0.605498 **h.** 0.970926

3. The Composite Midpoint rule approximations are as follows.

a. 0.633096 **b.** 11.1568 **c.** 0.786700 **d.** – 6.11274

e. – 14.9985 **f.** 0.478751 **g.** 0.602961 **h.** 0.947868

5. **a.** The Composite Trapezoidal rule requires $h < 0.000922295$ and $n \geq 2168$.

b. The Composite Simpson's rule requires $h < 0.037658$ and $n \geq 54$.

c. The Composite Midpoint rule requires $h < 0.00065216$ and $n \geq 3066$.

7. **a.** The Composite Trapezoidal rule requires $h < 0.04382$ and $n \geq 46$. The approximation is 0.405471.

b. The Composite Simpson's rule requires $h < 0.44267$ and $n \geq 6$. The approximation is 0.405466.

c. The Composite Midpoint rule requires $h < 0.03098$ and $n \geq 64$. The approximation is 0.405460.

9. $\alpha = 1.5$

11. **a.** 0.95449101, obtained using $n = 14$ in Composite Simpson's rule.

b. 0.99729312, obtained using $n = 20$ in Composite Simpson's rule.

13. The length of the track is approximately 9858 ft.

15. **a.** For $p_0 = 0.5$ we have $p_6 = 1.644854$ with $n = 20$.

b. For $p_0 = 0.5$ we have $p_6 = 1.645085$ with $n = 40$.

Exercise Set 4.4 (Page 000)

1. Romberg integration gives $R_{3,3}$ as follows:

a. 0.1922593 **b.** 0.1606105 **c.** -0.1768200 **d.** 0.08875677

e. 2.5879685 **f.** -0.7341567 **g.** 0.6362135 **h.** 0.6426970

3. Romberg integration gives the following values:

a. 0.19225936 with $n = 4$ **b.** 0.16060279 with $n = 5$

c. -0.17682002 with $n = 4$ **d.** 0.088755284 with $n = 5$

e. 2.5886286 with $n = 6$ **f.** -0.73396918 with $n = 6$

g. 0.63621335 with $n = 4$ **h.** 0.64269908 with $n = 5$

5. $R_{33} = 11.5246$

7. $f(2.5) \approx 0.43457$

9. $R_{31} = 5$

11. Let $N_2(h) = N\left(\frac{h}{3}\right) + \frac{1}{8}\left(N\left(\frac{h}{3}\right) - N(h)\right)$ and $N_3(h) = N_2\left(\frac{h}{3}\right) + \frac{1}{80}\left(N_2\left(\frac{h}{3}\right) - N_2(h)\right)$. Then $N_3(h)$ is an $O(h^6)$ approximation to M.

13. **a.** L'Hôpital's Rule gives

$$\lim_{h\to 0} \frac{\ln(2+h) - \ln(2-h)}{h} = \lim_{h\to 0} \frac{D_h(\ln(2+h) - \ln(2-h))}{D_h(h)} = \lim_{h\to 0}\left(\frac{1}{2+h} + \frac{1}{2-h}\right) = 1,$$

so

$$\lim_{h\to 0}\left(\frac{2+h}{2-h}\right)^{1/h} = \lim_{h\to 0} e^{\frac{1}{h}[\ln(2+h)-\ln(2-h)]} = e^1 = e.$$

b. $N(0.04) = 2.718644377221219$, $N(0.02) = 2.718372444800607$, $N(0.01) = 2.718304481241685$

c. Let $N_2(h) = 2N\left(\frac{h}{2}\right) - N(h)$, $N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{3}[N_2\left(\frac{h}{2}\right) - N_2(h)]$. Then $N_2(0.04) = 2.718100512379995$, $N_2(0.02) = 2.718236517682763$, and $N_3(0.04) = 2.718281852783685$. $N_3(0.04)$ is an $O(h^3)$ approximation satisfying $|e - N_3(0.04)| \leq 0.5 \times 10^{-7}$.

d.

$$N(-h) = \left(\frac{2-h}{2+h}\right)^{1/-h} = \left(\frac{2+h}{2-h}\right)^{1/h} = N(h)$$

e. Let

$$e = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots .$$

Replacing h by $-h$ gives

$$e = N(-h) - K_1 h + K_2 h^2 - K_3 h^3 + \cdots ,$$

but $N(-h) = N(h)$, so

$$e = N(h) - K_1 h + K_2 h^2 - K_3 h^3 + \cdots .$$

Thus,

$$K_1 h + K_3 h^3 + \cdots = -K_1 h - K_3 h^3 \cdots ,$$

and it follows that $K_1 = K_3 = K_5 = \cdots = 0$ and

$$e = N(h) + K_2 h^2 + K_4 h^4 + \cdots .$$

f. Let

$$N_2(h) = N\left(\frac{h}{2}\right) + \frac{1}{3}\left(N\left(\frac{h}{2}\right) - N(h)\right)$$

and

$$N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{15}\left(N_2\left(\frac{h}{2}\right) - N_2(h)\right).$$

Then

$$N_2(0.04) = 2.718281800660402, N_2(0.02) = 2.718281826722043$$

and

$$N_3(0.04) = 2.718281828459487.$$

$N_3(0.04)$ is an $O(h^6)$ approximation satisfying

$$|e - N_3(0.04)| \leq 0.5 \times 10^{-12}.$$

Exercise Set 4.5 (Page 000)

1. Gaussian quadrature gives the following.

a. 0.1922687 **b.** 0.1594104 **c.** – 0.1768190 **d.** 0.08926302

e. 2.5913247 **f.** – 0.7307230 **g.** 0.6361966 **h.** 0.6423172

3. Gaussian quadrature gives the following.

a.	0.1922594	**b.**	0.1606028	**c.**	-0.1768200	**d.**	0.08875529
e.	2.5886327	**f.**	-0.7339604	**g.**	0.6362133	**h.**	0.6426991

5. $a = 1$, $b = 1$, $c = \frac{1}{3}$, $d = -\frac{1}{3}$

Exercise Set 4.6 (Page 000)

1. Simpson's rule gives the following.

a. $S(1, 1.5) = 0.19224530$, $S(1, 1.25) = 0.039372434$, $S(1.25, 1.5) = 0.15288602$, and the actual value is 0.19225935.

b. $S(0, 1) = 0.16240168$, $S(0, 0.5) = 0.028861071$, $S(0.5, 1) = 0.13186140$, and the actual value is 0.16060279.

c. $S(0, 0.35) = -0.17682156$, $S(0, 0.175) = -0.087724382$, $S(0.175, 0.35) = -0.089095736$, and the actual value is -0.17682002.

d. $S(0, \frac{\pi}{4}) = 0.087995669$, $S(0, \frac{\pi}{8}) = 0.0058315797$, $S(\frac{\pi}{8}, \frac{\pi}{4}) = 0.082877624$, and the actual value is 0.088755285.

e. $S(0, \frac{\pi}{4}) = 2.5836964$, $S(0, \frac{\pi}{8}) = 0.33088926$, $S(\frac{\pi}{8}, \frac{\pi}{4}) = 2.2568121$, and the actual value is 2.5886286.

f. $S(1, 1.6) = -0.73910533$, $S(1, 1.3) = -0.26141244$, $S(1.3, 1.6) = -0.47305351$, and the actual value is -0.73396917.

g. $S(3, 3.5) = 0.63623873$, $S(3, 3.25) = 0.32567095$, $S(3.25, 3.5) = 0.31054412$, and the actual value is 0.63621334.

h. $S(0, \frac{\pi}{4}) = 0.64326905$, $S(0, \frac{\pi}{8}) = 0.37315002$, $S(\frac{\pi}{8}, \frac{\pi}{4}) = 0.26958270$, and the actual value is 0.64269908.

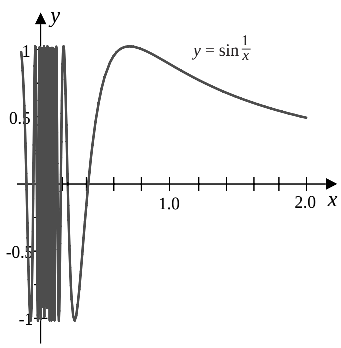
3. Adaptive quadrature gives the following.

a. 108.555281 **b.** – 1724.966983 **c.** – 15.306308 **d.** – 18.945949

5. Adaptive quadrature gives the following.

$$\int_{0.1}^{2} \sin\frac{1}{x}\, dx = 1.1454 \quad \text{and} \quad \int_{0.1}^{2} \cos\frac{1}{x}\, dx = 0.67378.$$

Note: New Figures



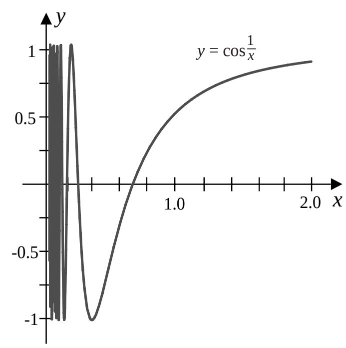


FIGURE 0 PLACED HERE

for Exercise 5 (i) and (ii)

7. $\int_0^{2\pi} u(t)\, dt \approx 0.00001$

9.

t	$c(t)$	$s(t)$
0.1	0.0999975	0.000523589
0.2	0.199921	0.00418759
0.3	0.299399	0.0141166
0.4	0.397475	0.0333568
0.5	0.492327	0.0647203
0.6	0.581061	0.110498
0.7	0.659650	0.172129
0.8	0.722844	0.249325
0.9	0.764972	0.339747
1.0	0.779880	0.438245

Exercise Set 4.7 (Page 000)

1. Composite Simpson's rule with $n = m = 4$ gives these values.

a. 0.3115733 **b.** 0.2552526 **c.** 16.50864 **d.** 1.476684

3. Composite Simpson's rule first with $n = 4$ and $m = 8$, then with $n = 8$ and $m = 4$, and finally with $n = m = 6$ gives the following.

a. 0.5119875, 0.5118533, 0.5118722

b. 1.718857, 1.718220, 1.718385

c. 1.001953, 1.000122, 1.000386

d. 0.7838542, 0.7833659, 0.7834362

e. −1.985611, −1.999182, −1.997353

f. 2.004596, 2.000879, 2.000980

g. 0.3084277, 0.3084562, 0.3084323

h. −22.61612, −19.85408, −20.14117

5. Gaussian quadrature with $n = m = 2$ gives the following.

a. 0.3115733 **b.** 0.2552446 **c.** 16.50863 **d.** 1.488875

7. Gaussian quadrature with $n = m = 3$, $n = 3$ and $m = 4$, $n = 4$ and $m = 3$, and $n = m = 4$ gives the following.

a. 0.5118655, 0.5118445, 0.5118655, 0.5118445, 2.1×10^{-5}, 1.3×10^{-7}, 2.1×10^{-5}, 1.3×10^{-7}

b. 1.718163, 1.718302, 1.718139, 1.718277, 1.2×10^{-4}, 2.0×10^{-5}, 1.4×10^{-4}, 4.8×10^{-6}

c. 1.000000, 1.000000, 1.0000000, 1.000000, 0, 0, 0, 0

d. 0.7833333, 0.7833333, 0.7833333, 0.7833333, 0, 0, 0, 0

e. -1.991878, -2.000124, -1.991878, -2.000124, 8.1×10^{-3}, 1.2×10^{-4}, 8.1×10^{-3}, 1.2×10^{-4}

f. 2.001494, 2.000080, 2.001388, 1.999984, 1.5×10^{-3}, 8×10^{-5}, 1.4×10^{-3}, 1.6×10^{-5}

g. 0.3084151, 0.3084145, 0.3084246, 0.3084245, 10^{-5}, 5.5×10^{-7}, 1.1×10^{-5}, 6.4×10^{-7}

h. -12.74790, -21.21539, -11.83624, -20.30373, 7.0, 1.5, 7.9, 0.564

9. Gaussian quadrature with $n = m = p = 2$ gives the first listed value. The second is the exact result.

a. 5.204036, $e(e^{0.5} - 1)(e - 1)^2$ **b.** 0.08429784, $\dfrac{1}{12}$

c. $0.08641975,\ \dfrac{1}{14}$ **d.** $0.09722222,\ \dfrac{1}{12}$

e. $7.103932,\ 2+\dfrac{1}{2}\pi^2$ **f.** $1.428074,\ \dfrac{1}{2}(e^2+1)-e$

11. Composite Simpson's rule with $n=m=14$ gives 0.1479103 and Gaussian quadrature with $n=m=4$ gives 0.1506823.

13. The area approximations are **a.** 1.0402528 and **b.** 1.0402523.

15. Gaussian quadrature with $n=m=p=4$ gives 3.0521250. The exact result is 3.0521249.

Exercise Set 4.8 (Page 000)

1. Composite Simpson's rule gives the following.

a. 0.5284163 **b.** 4.266654 **c.** 0.4329748 **d.** 0.8802210

3. Composite Simpson's rule gives the following.

a. 0.4112649 **b.** 0.2440679 **c.** 0.05501681 **d.** 0.2903746

5 . The escape velocity is approximately 6.9450 mi/s.

Exercise Set 4.9 (Page 000)

1. From the two-point formula we have the following approximations:

a. $f'(0.5) \approx 0.8520$, $f'(0.6) \approx 0.8520$, $f'(0.7) \approx 0.7960$

b. $f'(0.0) \approx 3.7070$, $f'(0.2) \approx 3.1520$, $f'(0.4) \approx 3.1520$

3. For the endpoints of the tables we use the three-point endpoint formula. The other approximations come from the three-point midpoint formula.

a. $f'(1.1) \approx 17.769705$, $f'(1.2) \approx 22.193635$, $f'(1.3) \approx 27.107350$, $f'(1.4) \approx 32.150850$

b. $f'(8.1) \approx 3.092050$, $f'(8.3) \approx 3.116150$, $f'(8.5) \approx 3.139975$, $f'(8.7) \approx 3.163525$

c. $f'(2.9) \approx 5.101375$, $f'(3.0) \approx 6.654785$, $f'(3.1) \approx 8.216330$, $f'(3.2) \approx 9.786010$

d. $f'(2.0) \approx 0.13533150$, $f'(2.1) \approx -0.09989550$, $f'(2.2) \approx -0.3298960$, $f'(2.3) \approx -0.5546700$

5. a. The five-point endpoint formula gives $f'(2.1) \approx 3.899344$, $f'(2.2) \approx 2.876876$, $f'(2.5) \approx 1.544210$, and $f'(2.6) \approx 1.355496$. The five-point midpoint formula gives $f'(2.3) \approx 2.249704$ and $f'(2.4) \approx 1.837756$.

b. The five-point endpoint formula gives $f'(-3.0) \approx -5.877358$, $f'(-2.8) \approx -5.468933$, $f'(-2.2) \approx -4.239911$, and $f'(-2.0) \approx -3.828853$. The five-point midpoint formula gives $f'(-2.6) \approx -5.059884$ and $f'(-2.4) \approx -4.650223$.

7. The approximation is -4.8×10^{-9}. $f''(0.5) = 0$. The error bound is 0.35874. The method is very accurate since the function is symmetric about $x = 0.5$.

9. $f'(3) \approx \frac{1}{12}[f(1) - 8f(2) + 8f(4) - f(5)] = 0.21062$ with an error bound given by

$$\max_{1 \le x \le 5} \frac{|f^{(5)}(x)|h^4}{30} \le \frac{23}{30} = 0.7\overline{6}.$$

11. The optimal $h = 2\sqrt{\varepsilon/M}$, where $M = \max |f''(x)|$.

13. Since $e'(h) = -\varepsilon/h^2 + hM/3$, we have $e'(h) = 0$ if and only if $h = \sqrt[3]{3\varepsilon/M}$. Also, $e'(h) < 0$ if $h < \sqrt[3]{3\varepsilon/M}$ and $e'(h) > 0$ if $h > \sqrt[3]{3\varepsilon/M}$, so an absolute minimum for $e(h)$ occurs at $h = \sqrt[3]{3\varepsilon/M}$.

15. Using three-point formulas gives the following table:

Time	0	3	5	8	10	13
Speed	79	82.4	74.2	76.8	69.4	71.2

Exercise Set 5.2 (Page 000)

1. Euler's method gives the approximations in the following tables.

a.

i	t_i	w_i	$y(t_i)$
1	0.500	0.0000000	0.2836165
2	1.000	1.1204223	3.2190993

b.

i	t_i	w_i	$y(t_i)$
1	2.500	2.0000000	1.8333333
2	3.000	2.6250000	2.5000000

c.

i	t_i	w_i	$y(t_i)$
1	1.250	2.7500000	2.7789294
2	1.500	3.5500000	3.6081977
3	1.750	4.3916667	4.4793276
4	2.000	5.2690476	5.3862944

d.

i	t_i	w_i	$y(t_i)$
1	0.250	1.2500000	1.3291498
2	0.500	1.6398053	1.7304898
3	0.750	2.0242547	2.0414720
4	1.000	2.2364573	2.1179795

3. Euler's method gives the approximations in the following tables.

a.

i	t_i	w_i	$y(t_i)$
2	1.2	1.0082645	1.0149523
4	1.4	1.0385147	1.0475339
6	1.6	1.0784611	1.0884327
8	1.8	1.1232621	1.1336536
10	2.0	1.1706516	1.1812322

b.

i	t_i	w_i	$y(t_i)$
2	1.4	0.4388889	0.4896817
4	1.8	1.0520380	1.1994386
6	2.2	1.8842608	2.2135018
8	2.6	3.0028372	3.6784753
10	3.0	4.5142774	5.8741000

c.

i	t_i	w_i	$y(t_i)$
2	0.4	−1.6080000	−1.6200510
4	0.8	−1.3017370	−1.3359632
6	1.2	−1.1274909	−1.1663454
8	1.6	−1.0491191	−1.0783314
10	2.0	−1.0181518	−1.0359724

d.

i	t_i	w_i	$y(t_i)$
2	0.2	0.1083333	0.1626265
4	0.4	0.1620833	0.2051118
6	0.6	0.3455208	0.3765957
8	0.8	0.6213802	0.6461052
10	1.0	0.9803451	1.0022460

5.

a.

i	t_i	w_i	$y(t_i)$
1	0.50	0.12500000	0.28361652
2	1.00	2.02323897	3.21909932

b.

i	t_i	w_i	$y(t_i)$
1	2.50	1.75000000	1.83333333
2	3.00	2.42578125	2.50000000

c.

i	t_i	w_i	$y(t_i)$
1	1.25	2.78125000	2.77892944
2	1.50	3.61250000	3.60819766
3	1.75	4.48541667	4.47932763
4	2.00	5.39404762	5.38629436

d.

i	t_i	w_i	$y(t_i)$
1	0.25	1.34375000	1.32914981
2	0.50	1.77218707	1.73048976
3	0.75	2.11067606	2.04147203
4	1.00	2.20164395	2.11797955

7.

a.

i	t_i	w_i	$y(t_i)$
2	1.2	1.0149771	1.0149523
4	1.4	1.0475619	1.0475339
6	1.6	1.0884607	1.0884327
8	1.8	1.1336811	1.1336536
10	2.0	1.1812594	1.1812322

b.

i	t_i	w_i	$y(t_i)$
2	1.4	0.4896141	0.4896817
4	1.8	1.1993085	1.1994386
6	2.2	2.2132495	2.2135018
8	2.6	3.6779557	3.6784753
10	3.0	5.8729143	5.8741000

c.

i	t_i	w_i	$y(t_i)$
2	0.4	−1.6201137	−1.6200510
4	0.8	−1.3359853	−1.3359632
6	1.2	−1.1663295	−1.1663454
8	1.6	−1.0783171	−1.0783314
10	2.0	−1.0359674	−1.0359724

d.

i	t_i	w_i	$y(t_i)$
2	0.2	0.1627236	0.1626265
4	0.4	0.2051833	0.2051118
6	0.6	0.3766352	0.3765957
8	0.8	0.6461246	0.6461052
10	1.0	1.0022549	1.0022460

9.

a.

i	t_i	w_i	$y(t_i)$
1	1.05	−0.9500000	−0.9523810
2	1.10	−0.9045353	−0.9090909
11	1.55	−0.6263495	−0.6451613
12	1.60	−0.6049486	−0.6250000
19	1.95	−0.4850416	−0.5128205
20	2.00	−0.4712186	−0.5000000

b. Linear interpolation gives

(i) $y(1.052) \approx -0.9481814$, (ii) $y(1.555) \approx -0.6242094$, (iii) $y(1.978) \approx -0.4773007$.

The actual values are $y(1.052) = -0.9505703$, $y(1.555) = -0.6430868$, $y(1.978) = -0.5055612$.

c.

i	t_i	w_i	$y(t_i)$
1	1.05	−0.9525000	−0.9523810
2	1.10	−0.9093138	−0.9090909
11	1.55	−0.6459788	−0.6451613
12	1.60	−0.6258649	−0.6250000
19	1.95	−0.5139781	−0.5128205
20	2.00	−0.5011957	−0.5000000

d. Linear interpolation gives

(i) $y(1.052) \approx -0.9507726$, (ii) $y(1.555) \approx -0.6439674$, (iii) $y(1.978) \approx -0.5068199$.

e.

i	t_i	w_i	$y(t_i)$
1	1.05	−0.9523813	−0.9523810
2	1.10	−0.9090914	−0.9090909
11	1.55	−0.6451629	−0.6451613
12	1.60	−0.6250017	−0.6250000
19	1.95	−0.5128226	−0.5128205
20	2.00	−0.5000022	−0.5000000

f. Hermite interpolation gives

(i) $y(1.052) \approx -0.9505706$, (ii) $y(1.555) \approx -0.6430884$, (iii) $y(1.978) \approx -0.5055633$.

11. b. $w_{50} = 0.10430 \approx p(50)$

c. Since $p(t) = 1 - 0.99e^{-0.002t}$, $p(50) = 0.10421$.

Exercise Set 5.3 (Page 000)

1. a.

i	t	w_i	$y(t_i)$
1	0.5	0.2646250	0.2836165
2	1.0	3.1300023	3.2190993

b.

i	t	w_i	$y(t_i)$
1	2.5	1.7812500	1.8333333
2	3.0	2.4550638	2.5000000

c.

i	t	w_i	$y(t_i)$
1	1.25	2.7777778	2.7789294
2	1.50	3.6060606	3.6081977
3	1.75	4.4763015	4.4793276
4	2.00	5.3824398	5.3862944

d.

i	t	w_i	$y(t_i)$
1	0.25	1.3337962	1.3291498
2	0.50	1.7422854	1.7304898
3	0.75	2.0596374	2.0414720
4	1.00	2.1385560	2.1179795

3. a.

i	t	w_i	$y(t_i)$
1	0.5	0.5602111	0.2836165
2	1.0	5.3014898	3.2190993

b.

i	t	w_i	$y(t_i)$
1	2.5	1.8125000	1.8333333
2	3.0	2.4815531	2.5000000

c.

i	t	w_i	$y(t_i)$
1	1.25	2.7750000	2.7789294
2	1.50	3.6008333	3.6081977
3	1.75	4.4688294	4.4793276
4	2.00	5.3728586	5.3862944

d.

i	t	w_i	$y(t_i)$
1	0.25	1.3199027	1.3291498
2	0.50	1.7070300	1.7304898
3	0.75	2.0053560	2.0414720
4	1.00	2.0770789	2.1179795

5. **a.** $1.0221167 \approx y(1.25) = 1.0219569, \;\; 1.1640347 \approx y(1.93) = 1.1643901$

b. $1.9086500 \approx y(2.1) = 1.9249616, \;\; 4.3105913 \approx y(2.75) = 4.3941697$

c. $-1.1461434 \approx y(1.3) = -1.1382768, \;\; -1.0454854 \approx y(1.93) = -1.0412665$

d. $0.3271470 \approx y(0.54) = 0.3140018, \;\; 0.8967073 \approx y(0.94) = 0.8866318$

7. **a.** $1.0225530 \approx y(1.25) = 1.0219569, \;\; 1.1646155 \approx y(1.93) = 1.1643901$

b. $1.9132167 \approx y(2.1) = 1.9249616, \;\; 4.3246152 \approx y(2.75) = 4.3941697$

c. $-1.1441775 \approx y(1.3) = -1.1382768,\ \ -1.0447403 \approx y(1.93) = -1.0412665$

d. $0.3251049 \approx y(0.54) = 0.3140018,\ \ 0.8945125 \approx y(0.94) = 0.8866318$

9. a. $1.0227863 \approx y(1.25) = 1.0219569,\ \ 1.1649247 \approx y(1.93) = 1.1643901$

b. $1.9153749 \approx y(2.1) = 1.9249616,\ \ 4.3312939 \approx y(2.75) = 4.3941697$

c. $-1.1432070 \approx y(1.3) = -1.1382768,\ \ -1.0443743 \approx y(1.93) = -1.0412665$

d. $0.3240839 \approx y(0.54) = 0.3140018,\ \ 0.8934152 \approx y(0.94) = 0.8866318$

11. a. The Runge-Kutta method of order 4 gives the results in the following tables.

i	t	w_i	$y(t_i)$
2	1.2	1.0149520	1.0149523
4	1.4	1.0475336	1.0475339
6	1.6	1.0884323	1.0884327
8	1.8	1.1336532	1.1336536
10	2.0	1.1812319	1.1812322

b.

i	t	w_i	$y(t_i)$
2	1.4	0.4896842	0.4896817
4	1.8	1.1994320	1.1994386
6	2.2	2.2134693	2.2135018
8	2.6	3.6783790	3.6784753
10	3.0	5.8738386	5.8741000

c.

i	t	w_i	$y(t_i)$
2	0.4	−1.6200576	−1.6200510
4	0.8	−1.3359824	−1.3359632
6	1.2	−1.1663735	−1.1663454
8	1.6	−1.0783582	−1.0783314
10	2.0	−1.0359922	−1.0359724

d.

i	t	w_i	$y(t_i)$
2	0.2	0.1627655	0.1626265
4	0.4	0.2052405	0.2051118
6	0.6	0.3766981	0.3765957
8	0.8	0.6461896	0.6461052
10	1.0	1.0023207	1.0022460

13. With $f(t, y) = -y + t + 1$ we have

$$\begin{aligned} w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right) =& w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))] \\ =& w_i + \frac{h}{4}\left[f(t_i, w_i) + 3f\left(t_i + \frac{2}{3}h, w_i + \frac{2}{3}hf(t_i, w_i)\right)\right] \\ =& w_i\left(1 - h + \frac{h^2}{2}\right) + t_i\left(h - \frac{h^2}{2}\right) + h. \end{aligned}$$

15. In 0.2 s we have approximately 2099 units of KOH.

Exercise Set 5.4 (Page 000)

1. The Adams-Bashforth methods give the results in the following tables.

a.

i	t_i	2-step	3-step	4-step	5-step	$y(t_i)$
1	0.2	0.0268128	0.0268128	0.0268128	0.0268128	0.0268128
2	0.4	0.1200522	0.1507778	0.1507778	0.1507778	0.1507778
3	0.6	0.4153551	0.4613866	0.4960196	0.4960196	0.4960196
4	0.8	1.1462844	1.2512447	1.2961260	1.3308570	1.3308570
5	1.0	2.8241683	3.0360680	3.1461400	3.1854002	3.2190993

b.

i	t_i	2-step	3-step	4-step	5-step	$y(t_i)$
1	2.2	1.3666667	1.3666667	1.3666667	1.3666667	1.3666667
2	2.4	1.6750000	1.6857143	1.6857143	1.6857143	1.6857143
3	2.6	1.9632431	1.9794407	1.9750000	1.9750000	1.9750000
4	2.8	2.2323184	2.2488759	2.2423065	2.2444444	2.2444444
5	3.0	2.4884512	2.5051340	2.4980306	2.5011406	2.5000000

c.

i	t_i	2-step	3-step	4-step	5-step	$y(t_i)$
1	1.2	2.6187859	2.6187859	2.6187859	2.6187859	2.6187859
2	1.4	3.2734823	3.2710611	3.2710611	3.2710611	3.2710611
3	1.6	3.9567107	3.9514231	3.9520058	3.9520058	3.9520058
4	1.8	4.6647738	4.6569191	4.6582078	4.6580160	4.6580160
5	2.0	5.3949416	5.3848058	5.3866452	5.3862177	5.3862944

d.

i	t_i	2-step	3-step	4-step	5-step	$y(t_i)$
1	0.2	1.2529306	1.2529306	1.2529306	1.2529306	1.2529306
2	0.4	1.5986417	1.5712255	1.5712255	1.5712255	1.5712255
3	0.6	1.9386951	1.8827238	1.8750869	1.8750869	1.8750869
4	0.8	2.1766821	2.0844122	2.0698063	2.0789180	2.0789180
5	1.0	2.2369407	2.1115540	2.0998117	2.1180642	2.1179795

3. The Adams-Bashforth methods give the results in the following tables.

a.

i	t_i	2-step	3-step	4-step	5-step	$y(t_i)$
2	1.2	1.0161982	1.0149520	1.0149520	1.0149520	1.0149523
4	1.4	1.0497665	1.0468730	1.0477278	1.0475336	1.0475339
6	1.6	1.0910204	1.0875837	1.0887567	1.0883045	1.0884327
8	1.8	1.1363845	1.1327465	1.1340093	1.1334967	1.1336536
10	2.0	1.1840272	1.1803057	1.1815967	1.1810689	1.1812322

b.

i	t_i	2-step	3-step	4-step	5-step	$y(t_i)$
2	1.4	0.4867550	0.4896842	0.4896842	0.4896842	0.4896817
4	1.8	1.1856931	1.1982110	1.1990422	1.1994320	1.1994386
6	2.2	2.1753785	2.2079987	2.2117448	2.2134792	2.2135018
8	2.6	3.5849181	3.6617484	3.6733266	3.6777236	3.6784753
10	3.0	5.6491203	5.8268008	5.8589944	5.8706101	5.8741000

c.

i	t_i	2-step	3-step	4-step	5-step	$y(t_i)$
5	0.5	−1.5357010	−1.5381988	−1.5379372	−1.5378676	−1.5378828
10	1.0	−1.2374093	−1.2389605	−1.2383734	−1.2383693	−1.2384058
15	1.5	−1.0952910	−1.0950952	−1.0947925	−1.0948481	−1.0948517
20	2.0	−1.0366643	−1.0359996	−1.0359497	−1.0359760	−1.0359724

d.

i	t_i	2-step	3-step	4-step	5-step	$y(t_i)$
2	0.2	0.1739041	0.1627655	0.1627655	0.1627655	0.1626265
4	0.4	0.2144877	0.2026399	0.2066057	0.2052405	0.2051118
6	0.6	0.3822803	0.3747011	0.3787680	0.3765206	0.3765957
8	0.8	0.6491272	0.6452640	0.6487176	0.6471458	0.6461052
10	1.0	1.0037415	1.0020894	1.0064121	1.0073348	1.0022460

5. The Adams Fourth-order Predictor-Corrector method gives the results in the following tables.

a.

i	t_i	w_i	$y(t_i)$
2	1.2	1.0149520	1.0149523
4	1.4	1.0475227	1.0475339
6	1.6	1.0884141	1.0884327
8	1.8	1.1336331	1.1336536
10	2.0	1.1812112	1.1812322

b.

i	t_i	w_i	$y(t_i)$
2	1.4	0.4896842	0.4896817
4	1.8	1.1994245	1.1994386
6	2.2	2.2134701	2.2135018
8	2.6	3.6784144	3.6784753
10	3.0	5.8739518	5.8741000

c.

i	t_i	w_i	$y(t_i)$
5	0.5	−1.5378788	−1.5378828
10	1.0	−1.2384134	−1.2384058
15	1.5	−1.0948609	−1.0948517
20	2.0	−1.0359757	−1.0359724

d.

i	t_i	w_i	$y(t_i)$
2	0.2	0.1627655	0.1626265
4	0.4	0.2048557	0.2051118
6	0.6	0.3762804	0.3765957
8	0.8	0.6458949	0.6461052
10	1.0	1.0021372	1.0022460

7. Milne-Simpson's Predictor-Corrector method gives the results in the following tables.

a.

i	t_i	w_i	$y(t_i)$
2	1.2	1.01495200	1.01495231
5	1.5	1.06725997	1.06726235
7	1.7	1.11065221	1.11065505
10	2.0	1.18122584	1.18123222

b.

i	t_i	w_i	$y(t_i)$
2	1.4	0.48968417	0.48968166
5	2.0	1.66126150	1.66128176
7	2.4	2.87648763	2.87655142
10	3.0	5.87375555	5.87409998

c.

i	t_i	w_i	$y(t_i)$
5	0.5	−1.53788255	−1.53788284
10	1.0	−1.23840789	−1.23840584
15	1.5	−1.09485532	−1.09485175
20	2.0	−1.03597247	−1.03597242

d.

i	t_i	w_i	$y(t_i)$
2	0.2	0.16276546	0.16262648
5	0.5	0.27741080	0.27736167
7	0.7	0.50008713	0.50006579
10	1.0	1.00215439	1.00224598

Exercise Set 5.5 (Page 000)

1. $y_{22} = 0.14846014$ approximates $y(0.1) = 0.14846010$.

3. The Extrapolation method gives the results in the following tables.

a.

i	t_i	w_i	h_i	k	y_i
1	1.05	1.10385729	0.05	2	1.10385738
2	1.10	1.21588614	0.05	2	1.21588635
3	1.15	1.33683891	0.05	2	1.33683925
4	1.20	1.46756907	0.05	2	1.46756957

b.

i	t_i	w_i	h_i	k	y_i
1	0.25	0.25228680	0.25	3	0.25228680
2	0.50	0.51588678	0.25	3	0.51588678
3	0.75	0.79594460	0.25	2	0.79594458
4	1.00	1.09181828	0.25	3	1.09181825

c.

i	t_i	w_i	h_i	k	y_i
1	1.50	−1.50000055	0.50	5	−1.50000000
2	2.00	−1.33333435	0.50	3	−1.33333333
3	2.50	−1.25000074	0.50	3	−1.25000000
4	3.00	−1.20000090	0.50	2	−1.20000000

d.

i	t_i	w_i	h_i	k	y_i
1	0.25	1.08708817	0.25	3	1.08708823
2	0.50	1.28980537	0.25	3	1.28980528
3	0.75	1.51349008	0.25	3	1.51348985
4	1.00	1.70187009	0.25	3	1.70187005

5. $P(5) \approx 56,751$.

Exercise Set 5.6 (Page 000)

1. **a.** $w_1 = 0.4787456 \approx y(t_1) = y(0.2966446) = 0.4787309$

b. $w_4 = 0.31055852 \approx y(t_4) = y(0.2) = 0.31055897$

3. The Runge-Kutta-Fehlberg method gives the results in the following tables.

a.

i	t_i	w_i	h_i	y_i
1	1.0500000	1.1038574	0.0500000	1.1038574
2	1.1000000	1.2158864	0.0500000	1.2158863
3	1.1500000	1.3368393	0.0500000	1.3368393
4	1.2000000	1.4675697	0.0500000	1.4675696

b.

i	t_i	w_i	h_i	y_i
1	0.2500000	0.2522868	0.2500000	0.2522868
2	0.5000000	0.5158867	0.2500000	0.5158868
3	0.7500000	0.7959445	0.2500000	0.7959446
4	1.0000000	1.0918182	0.2500000	1.0918183

c.

i	t_i	w_i	h_i	y_i
1	1.1382206	−1.7834313	0.1382206	−1.7834282
3	1.6364797	−1.4399709	0.3071709	−1.4399551
5	2.6364797	−1.2340532	0.5000000	−1.2340298
6	3.0000000	−1.2000195	0.3635203	−1.2000000

d.

i	t_i	w_i	h_i	y_i
1	0.2	1.0571819	0.2	1.0571810
2	0.4	1.2014801	0.2	1.2014860
3	0.6	1.3809214	0.2	1.3809312
4	0.8	1.5550243	0.2	1.5550314
5	1.0	1.7018705	0.2	1.7018701

5. The Adams Variable Step-Size Predictor-Corrector method gives the results in the following tables.

a.

i	t_i	w_i	h_i	y_i
1	1.05000000	1.10385717	0.05000000	1.10385738
2	1.10000000	1.21588587	0.05000000	1.21588635
3	1.15000000	1.33683848	0.05000000	1.33683925
4	1.20000000	1.46756885	0.05000000	1.46756957

b.

i	t_i	w_i	h_i	y_i
1	0.20000000	0.20120278	0.20000000	0.20120267
2	0.40000000	0.40861919	0.20000000	0.40861896
3	0.60000000	0.62585310	0.20000000	0.62585275
4	0.80000000	0.85397394	0.20000000	0.85396433
5	1.00000000	1.09183759	0.20000000	1.09181825

c.

i	t_i	w_i	h_i	y_i
5	1.16289739	−1.75426113	0.03257948	−1.75426455
10	1.32579477	−1.60547206	0.03257948	−1.60547731
15	1.57235777	−1.46625721	0.04931260	−1.46626230
20	1.92943707	−1.34978308	0.07694168	−1.34978805
25	2.47170180	−1.25358275	0.11633076	−1.25358804
30	3.00000000	−1.19999513	0.10299186	−1.20000000

d.

i	t_i	w_i	h_i	y_i
1	0.06250000	1.00583097	0.06250000	1.00583095
5	0.31250000	1.13099427	0.06250000	1.13098105
10	0.62500000	1.40361751	0.06250000	1.40360196
12	0.81250000	1.56515769	0.09375000	1.56514800
14	1.00000000	1.70186884	0.09375000	1.70187005

7. The current after 2 s is approximately $i(2) = 8.693$ amperes.

Exercise Set 5.7 (Page 000)

1. The Runge-Kutta for Systems method gives the results in the following tables.

a.

i	t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}
1	0.200	2.12036583	2.12500839	1.50699185	1.51158743
2	0.400	4.44122776	4.46511961	3.24224021	3.26598528
3	0.600	9.73913329	9.83235869	8.16341700	8.25629549
4	0.800	22.67655977	23.00263945	21.34352778	21.66887674
5	1.000	55.66118088	56.73748265	56.03050296	57.10536209

b.

i	t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}
1	0.500	0.95671390	0.95672798	−1.08381950	−1.08383310
2	1.000	1.30654440	1.30655930	−0.83295364	−0.83296776
3	1.500	1.34416716	1.34418117	−0.56980329	−0.56981634
4	2.000	1.14332436	1.14333672	−0.36936318	−0.36937457

c.

i	t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}	w_{3i}	u_{3i}
1	0.5	0.70787076	0.70828683	−1.24988663	−1.25056425	0.39884862	0.39815702
2	1.0	−0.33691753	−0.33650854	−3.01764179	−3.01945051	−0.29932294	−0.30116868
3	1.5	−2.41332734	−2.41345688	−5.40523279	−5.40844686	−0.92346873	−0.92675778
4	2.0	−5.89479008	−5.89590551	−8.70970537	−8.71450036	−1.32051165	−1.32544426

d.

i	t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}	w_{3i}	u_{3i}
2	0.2	1.38165297	1.38165325	1.00800000	1.00800000	−0.61833075	−0.61833075
5	0.5	1.90753116	1.90753184	1.12500000	0.12500000	−0.09090565	−0.09090566
7	0.7	2.25503524	2.25503620	1.34300000	1.34000000	0.26343971	0.26343970
10	1.0	2.83211921	2.83212056	2.00000000	2.00000000	0.88212058	0.88212056

3. First use the Runge-Kutta method of order four for systems to compute all starting values:

$$w_{1,0}, w_{2,0}, \ldots, w_{m,0}$$

$$w_{1,1}, w_{2,1}, \ldots, w_{m,1}$$

$$w_{1,2}, w_{2,2}, \ldots, w_{m,2}$$

$$w_{1,3}, w_{2,3}, \ldots, w_{m,3}.$$

Then for each $j = 3, 4, \ldots N-1$, compute, for each $i = 1, \ldots, m$, the predictor values

$$\begin{aligned} w_{i,j+1}^{(0)} =& w_{i,j} + \frac{h}{24}[55 f_i(t_i, w_{1,j}, \ldots, w_{m,j}) - 59 f_i(t_{j-1}, w_{1,j-1}, \ldots, w_{m,j-1}) \\ & + 37 f_i(t_{j-2}, w_{1,j-2}, \ldots, w_{m,j-2}) - 9 f_i(t_{j-3}, w_{1,j-3}, \ldots, w_{m,j-3})], \end{aligned}$$

and then the corrector values

$$\begin{aligned} w_{i,j+1} =& w_{i,j} + \frac{h}{24}[9 f_i(t_{j+1}, w_{i,j+1}^{(0)}, \ldots, w_{m,j+1}^{(0)}) + 19 f_i(t_i, w_{1,j}, \ldots, w_{m,j}) \\ & - 5 f_i(t_{j-1}, w_{1,j-1}, \ldots, w_{m,j-1}) + f_i(t_{j-2}, w_{1,j-2}, \ldots, w_{m,j-2})]. \end{aligned}$$

5. The predicted number of prey, x_{1i}, and predators, x_{2i}, are given in the following table.

i	t_i	x_{1i}	x_{2i}
10	1.0	4393	1512
20	2.0	288	3175
30	3.0	32	2042
40	4.0	25	1258

A stable solution is $x_1 = 833.\overline{3}$ and $x_2 = 1500$.

Exercise Set 5.8 (Page 000)

1. Euler's method gives the results in the following tables.

a.

i	t_i	w_i	$y(t_i)$
2	0.200	0.027182818	0.4493290
5	0.500	0.000027183	0.0301974
7	0.700	0.000000272	0.0049916
10	1.000	0.000000000	0.0003355

b.

i	t_i	w_i	$y(t_i)$
2	0.200	0.373333333	0.0461052
5	0.500	−0.933333333	0.2500151
7	0.700	0.146666667	0.4900003
10	1.000	1.333333333	1.0000000

c.

i	t_i	w_i	$y(t_i)$
2	0.500	16.47925	0.4794709
4	1.000	256.7930	0.8414710
6	1.500	4096.142	0.9974950
8	2.000	65523.12	0.9092974

d.

i	t_i	w_i	$y(t_i)$
2	0.200	6.128259	1.000000001
5	0.500	−378.2574	1.000000000
7	0.700	−6052.063	1.000000000
10	1.000	387332.0	1.000000000

3. The Adams Fourth-Order Predictor-Corrector method gives the results in the following tables.

a.

i	t_i	w_i	$y(t_i)$
2	0.200	0.4588119	0.4493290
5	0.500	−0.0112813	0.0301974
7	0.700	0.0013734	0.0049916
10	1.000	0.0023604	0.0003355

b.

i	t_i	w_i	$y(t_i)$
2	0.200	0.0792593	0.0461052
5	0.500	0.1554027	0.2500151
7	0.700	0.5507445	0.4900003
10	1.000	0.7278557	1.0000000

c.

i	t_i	w_i	$y(t_i)$
2	0.500	188.3082	0.4794709
4	1.000	38932.03	0.8414710
6	1.500	9073607	0.9974950
8	2.000	2115741299	0.9092974

d.

i	t_i	w_i	$y(t_i)$
2	0.200	−215.7459	1.000000000
5	0.500	−682637.0	1.000000000
7	0.700	−159172736	1.000000000
10	1.000	−566751172258	1.000000000

5. The following tables list the results of the Backward Euler method applied to the problems in Exercise 1.

a.

i	t_i	w_i	k	$y(t_i)$
2	0.20	0.75298666	2	0.44932896
5	0.50	0.10978082	2	0.03019738
7	0.70	0.03041020	2	0.00499159
10	1.00	0.00443362	2	0.00033546

b.

i	t_i	w_i	k	$y(t_i)$
2	0.20	0.08148148	2	0.04610521
5	0.50	0.25635117	2	0.25001513
7	0.70	0.49515013	2	0.49000028
10	1.00	1.00500556	2	1.00000000

c.

i	t_i	w_i	k	$y(t_i)$
2	0.50	0.50495522	2	0.47947094
4	1.00	0.83751817	2	0.84147099
6	1.50	0.99145076	2	0.99749499
8	2.00	0.90337560	2	0.90929743

d.

i	t_i	w_i	k	$y(t_i)$
2	0.20	1.00348713	3	1.00000001
5	0.50	1.00000262	2	1.00000000
7	0.70	1.00000002	1	1.00000000
10	1.00	1.00000000	1	1.00000000

Exercise Set 6.2 (Page 000)

1. **a.** Intersecting lines with solution $x_1 = x_2 = 1$.

b. Intersecting lines with solution $x_1 = x_2 = 0$.

c. One line, so there are an infinite number of solutions with $x_2 = \frac{3}{2} - \frac{1}{2}x_1$.

d. Parallel lines, so there is no solution.

e. One line, so there are an infinite number of solutions with $x_2 = -\frac{1}{2}x_1$.

f. Three lines in the plane that do not intersect at a common point.

g. Intersecting lines with solution $x_1 = \frac{2}{7}$ and $x_2 = -\frac{11}{7}$.

h. Two planes in space that intersect in a line with $x_1 = -\frac{5}{4}x_2$ and $x_3 = \frac{3}{2}x_2 + 1$.

3. Gaussian elimination gives the following solutions.

a. $x_1 = 1.1875, x_2 = 1.8125, x_3 = 0.875$ with one row interchange required.

b. $x_1 = -1, x_2 = 0, x_3 = 1$ with no interchange required.

c. $x_1 = 1.5, x_2 = 2, x_3 = -1.2, x_4 = 3$ with no interchange required.

d. $x_1 = \frac{22}{9}, x_2 = -\frac{4}{9}, x_3 = \frac{4}{3}, x_4 = 1$ with one row interchange required.

e. No unique solution.

f. $x_1 = -1, x_2 = 2, x_3 = 0, x_4 = 1$ with one row interchange required.

5. **a.** When $\alpha = -1/3$, there is no solution.

b. When $\alpha = 1/3$, there are an infinite number of solutions with $x_1 = x_2 + 1.5$, and x_2 is arbitrary.

c. If $\alpha \neq \pm 1/3$, then the unique solution is

$$x_1 = \frac{3}{2(1+3\alpha)} \quad \text{and} \quad x_2 = \frac{-3}{2(1+3\alpha)}.$$

7. a. There is sufficient food to satisfy the average daily consumption.

b. We could add 200 of species 1, or 150 of species 2, or 100 of species 3, or 100 of species 4.

c. Assuming none of the increases indicated in part (b) was selected, species 2 could be increased by 650, or species 3 could be increased by 150, or species 4 could be increased by 150.

d. Assuming none of the increases indicated in parts (b) or (c) were selected, species 3 could be increased by 150, or species 4 could be increased by 150.

Exercise Set 6.3 (Page 000)

1. a. None

b. Interchange rows 2 and 3.

c. None

d. Interchange rows 1 and 2.

3. a. Interchange rows 1 and 3, then interchange rows 2 and 3.

b. Interchange rows 2 and 3.

c. Interchange rows 2 and 3.

d. Interchange rows 1 and 3, then interchange rows 2 and 3.

5. Gaussian elimination with three-digit chopping arithmetic gives the following results.

a. $x_1 = 30.0, x_2 = 0.990$

b. $x_1 = 1.00, x_2 = 9.98$

c. $x_1 = 0.00, x_2 = 10.0, x_3 = 0.142$

d. $x_1 = 12.0, x_2 = 0.492, x_3 = -9.78$

e. $x_1 = 0.206, x_2 = 0.0154, x_3 = -0.0156, x_4 = -0.716$

f. $x_1 = 0.828, x_2 = -3.32, x_3 = 0.153, x_4 = 4.91$

7. Gaussian elimination with partial pivoting and three-digit chopping arithmetic gives the following results.

a. $x_1 = 10.0, x_2 = 1.00$

b. $x_1 = 1.00, x_2 = 9.98$

c. $x_1 = -0.163, x_2 = 9.98, x_3 = 0.142$

d. $x_1 = 12.0, x_2 = 0.504, x_3 = -9.78$

e. $x_1 = 0.177, x_2 = -0.0072, x_3 = -0.0208, x_4 = -1.18$

f. $x_1 = 0.777, x_2 = -3.10, x_3 = 0.161, x_4 = 4.50$

9. **a.** $\alpha = 6$

Exercise Set 6.4 (Page 000)

1. **a.**

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 9 & 5 & 1 \end{bmatrix}$$

b.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 7 \\ -2 & 1 & -5 \end{bmatrix}$$

c.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -7 & -2 & 1 \end{bmatrix}$$

d.

$$\begin{bmatrix} 6 & -7 & 15 \\ 0 & -1 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

3. a. Singular, $\det A = 0$

b. $\det A = -8,\ \det A^{-1} = -0.125$

c. Singular, $\det A = 0$

d. Singular, $\det A = 0$

e. $\det A = 28,\ \det A^{-1} = \dfrac{1}{28}$

f. $\det A = 3,\ \det A^{-1} = \dfrac{1}{3}$

5. a. Not true. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}. \quad \text{Then} \quad AB = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

is not symmetric.

b. True. Let A be a nonsingular symmetric matrix. From the properties of transposes and inverses we have $(A^{-1})^t = (A^t)^{-1}$. Thus $(A^{-1})^t = (A^t)^{-1} = A^{-1}$, and A^{-1} is symmetric.

c. Not true. Use the matrices A and B from part (a).

7. a. The solution is $x_1 = 0, x_2 = 10$, and $x_3 = 26$.

b. We have $D_1 = -1, D_2 = 3, D_3 = 7$, and $D = 0$, and there are no solutions.

c. We have $D_1 = D_2 = D_3 = D = 0$, and there are infinitely many solutions.

9. **a.** For each $k = 1, 2, \ldots, m$, the number a_{ik} represents the total number of plants of type v_i eaten by herbivores in the species h_k. The number of herbivores of types h_k eaten by species c_j is b_{kj}. Thus, the total number of plants of type v_i ending up in species c_j is $a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} = (AB)_{ij}$.

b. We first assume $n = m = k$ so that the matrices will have inverses. Let $x_1, ..., x_n$ represent the vegetations of type $v_1, ..., v_n$, let $y_1, ..., y_n$ represent the number of herbivores of species $h_1, ..., h_n$, and let $z_1, ..., z_n$ represent the number of carnivores of species $c_1, ..., c_n$.

If
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{then} \quad \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = A^{-1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$
Thus, $(A^{-1})_{i,j}$ represents the amount of type v_j plants eaten by a herbivore of species h_i. Similarly, if
$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = B \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad \text{then} \quad \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = B^{-1} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$
Thus $(B^{-1})_{i,j}$ represents the number of herbivores of species h_j eaten by a carnivore of species c_i. If $x = Ay$ and $y = Bz$, then $x = ABz$ and $z = (AB)^{-1}x$. But, $y = A^{-1}x$ and $z = B^{-1}y$, so $z = B^{-1}A^{-1}x$.

11. **a.** In component form:

$$(a_{11}x_1 - b_{11}y_1 + a_{12}x_2 - b_{12}y_2) + (b_{11}x_1 + a_{11}y_1 + b_{12}x_2 + a_{12}y_2)i = c_1 + id_1$$

$$(a_{21}x_1 - b_{21}y_1 + a_{22}x_2 - b_{22}y_2) + (b_{21}x_1 + a_{21}y_1 + b_{22}x_2 + a_{22}y_2)i = c_2 + id_2,$$

so

$$a_{11}x_1 + a_{12}x_2 - b_{11}y_1 - b_{12}y_2 = c_1$$
$$b_{11}x_1 + b_{12}x_2 + a_{11}y_1 + a_{12}y_2 = d_1$$
$$a_{21}x_1 + a_{22}x_2 - b_{21}y_1 - b_{22}y_2 = c_2$$
$$b_{21}x_1 + b_{22}x_2 + a_{21}y_1 + a_{22}y_2 = d_2$$

b. The system

$$\begin{bmatrix} 1 & 3 & 2 & -2 \\ -2 & 2 & 1 & 3 \\ 2 & 4 & -1 & -3 \\ 1 & 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \\ -1 \end{bmatrix}$$

has the solution $x_1 = -1.2, x_2 = 1, y_1 = 0.6$, and $y_2 = -1$.

Exercise Set 6.5 (Page 000)

1. a. $x_1 = -3$, $x_2 = 3$, $x_3 = 1$

b. $x_1 = \frac{1}{2}$, $x_2 = \frac{-9}{2}$, $x_3 = \frac{7}{2}$

3. a.

$$P^tLU = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$$

b.

$$P^tLU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

c.

$$P^tLU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

d.

$$P^tLU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -3 & -1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise Set 6.6 (Page 000)

1. (**i**) The symmetric matrices are in (a), (b), and (f).

(**ii**) The singular matrices are in (e) and (h).

(**iii**) The strictly diagonally dominant matrices are in (a), (b), (c), and (d).

(**iv**) The positive definite matrices are in (a) and (f).

3. Choleski factorization gives the following results.

a.

$$L = \begin{bmatrix} 1.414213 & 0 & 0 \\ -0.7071069 & 1.224743 & 0 \\ 0 & -0.8164972 & 1.154699 \end{bmatrix}$$

b.

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0.5 & 1.658311 & 0 & 0 \\ 0.5 & -0.7537785 & 1.087113 & 0 \\ 0.5 & 0.4522671 & 0.08362442 & 1.240346 \end{bmatrix}$$

c.

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0.5 & 1.658311 & 0 & 0 \\ -0.5 & -0.4522671 & 2.132006 & 0 \\ 0 & 0 & 0.9380833 & 1.766351 \end{bmatrix}$$

d.

$$L = \begin{bmatrix} 2.449489 & 0 & 0 & 0 \\ 0.8164966 & 1.825741 & 0 & 0 \\ 0.4082483 & 0.3651483 & 1.923538 & 0 \\ -0.4082483 & 0.1825741 & -0.4678876 & 1.606574 \end{bmatrix}$$

5. Crout factorization gives the following results.

a. $x_1 = 0.5, x_2 = 0.5, x_3 = 1$ **b.** $x_1 = -0.9999995, x_2 = 1.999999, x_3 = 1$

c. $x_1 = 1, x_2 = -1, x_3 = 0$

d. $x_1 = -0.09357798, x_2 = 1.587156, x_3 = -1.167431, x_4 = 0.5412844$

7. a. No, consider $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

b. Yes, since $A = A^t$.

c. Yes, since $\mathbf{x}^t(A+B)\mathbf{x} = \mathbf{x}^t A\mathbf{x} + \mathbf{x}^t B\mathbf{x}$.

d. Yes, since $\mathbf{x}^t A^2 \mathbf{x} = \mathbf{x}^t A^t A\mathbf{x} = (A\mathbf{x})^t(A\mathbf{x}) \geq 0$, and because A is nonsingular, equality holds only if $\mathbf{x} = \mathbf{0}$.

e. No, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

9. a. Since $\det A = 3\alpha - 2\beta$, A is singular if and only if $\alpha = 2\beta/3$.

b. $|\alpha| > 1, |\beta| < 1$

c. $\beta = 1$

d. $\alpha > \frac{2}{3}, \beta = 1$

11. a. Mating male i with female j produces offspring with the same wing characteristics as mating male j with female i.

b. No. Consider, for example, $\mathbf{x} = (1, 0, -1)^t$.

Exercise Set 7.2 (Page 000)

1. a. We have $||\mathbf{x}||_\infty = 4$ and $||\mathbf{x}||_2 = 5.220153$.

b. We have $||\mathbf{x}||_\infty = 4$ and $||\mathbf{x}||_2 = 5.477226$.

c. We have $||\mathbf{x}||_\infty = 2^k$ and $||\mathbf{x}||_2 = (1 + 4^k)^{1/2}$.

d. We have $||\mathbf{x}||_\infty = 4/(k+1)$ and $||\mathbf{x}||_2 = (16/(k+1)^2 + 4/k^4 + k^4 e^{-2k})^{1/2}$.

3. a. We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (0, 0, 0)^t$.

b. We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (0, 1, 3)^t$.

c. We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (0, 0, \frac{1}{2})^t$.

d. We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (1, -1, 1)^t$.

5. a. We have $||\mathbf{x} - \hat{\mathbf{x}}||_\infty = 8.57 \times 10^{-4}$ and $||A\hat{\mathbf{x}} - \mathbf{b}||_\infty = 2.06 \times 10^{-4}$.

b. We have $||\mathbf{x} - \hat{\mathbf{x}}||_\infty = 0.90$ and $||A\hat{\mathbf{x}} - \mathbf{b}||_\infty = 0.27$.

c. We have $||\mathbf{x} - \hat{\mathbf{x}}||_\infty = 0.5$ and $||A\hat{\mathbf{x}} - \mathbf{b}||_\infty = 0.3$.

d. We have $||\mathbf{x} - \hat{\mathbf{x}}||_\infty = 6.55 \times 10^{-2}$, and $||A\hat{\mathbf{x}} - \mathbf{b}||_\infty = 0.32$.

7. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $\|AB\|_\otimes = 2$, but $\|A\|_\otimes \cdot \|B\|_\otimes = 1$.

9. It is not difficult to show that (i) holds. If $\|A\| = 0$, then $\|A\mathbf{x}\| = 0$ for all vectors $\mathbf{x}$ with $\|\mathbf{x}\| = 1$. Using $\mathbf{x} = (1, 0, \dots, 0)^t$, $\mathbf{x} = (0, 1, 0, \dots, 0)^t, \dots$, and $\mathbf{x} = (0, \dots, 0, 1)^t$ successively implies that each column of A is zero. Thus, $\|A\| = 0$ if and only if $A = 0$. Moreover,

$$\|\alpha A\| = \max_{\|\mathbf{x}\|=1} \|(\alpha A\mathbf{x})\| = |\alpha| \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = |\alpha| \cdot \|A\|,$$
$$\|A + B\| = \max_{\|\mathbf{x}\|=1} \|(A + B)\mathbf{x}\| \le \max_{\|\mathbf{x}\|=1} (\|A\mathbf{x}\| + \|B\mathbf{x}\|),$$

so

$$||A+B|| \leq \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| + \max_{\|\mathbf{x}\|=1} \|B\mathbf{x}\| = \|A\| + \|B\|$$

and

$$\|AB\| = \max_{\|\mathbf{x}\|=1} \|(AB)\mathbf{x}\| = \max_{\|\mathbf{x}\|=1} \|A(B\mathbf{x})\|,$$

so

$$||AB|| \leq \max_{\|\mathbf{x}\|=1} \|A\| \; \|B\mathbf{x}\| = \|A\| \max_{\|\mathbf{x}\|=1} \|B\mathbf{x}\| = \|A\| \; \|B\|.$$

Exercise Set 7.3 (Page 000)

1. a. The eigenvalue $\lambda_1 = 3$ has the eigenvector $\mathbf{x}_1 = (1, -1)^t$, and the eigenvalue $\lambda_2 = 1$ has the eigenvector $\mathbf{x}_2 = (1, 1)^t$.

b. The eigenvalue $\lambda_1 = \frac{1+\sqrt{5}}{2}$ has the eigenvector $\mathbf{x}_1 = (1, \frac{1+\sqrt{5}}{2})^t$, and the eigenvalue $\lambda_2 = \frac{1-\sqrt{5}}{2}$ has the eigenvector $\mathbf{x}_2 = \left(1, \frac{1-\sqrt{5}}{2}\right)^t$.

c. The eigenvalue $\lambda_1 = \frac{1}{2}$ has the eigenvector $\mathbf{x}_1 = (1, 1)^t$ and the eigenvalue $\lambda_2 = -\frac{1}{2}$ has the eigenvector $\mathbf{x}_2 = (1, -1)^t$.

d. The eigenvalue $\lambda_1 = 0$ has the eigenvector $\mathbf{x}_1 = (1, -1)^t$ and the eigenvalue $\lambda_2 = -1$ has the eigenvector $\mathbf{x}_2 = (1, -2)^t$.

e. The eigenvalue $\lambda_1 = \lambda_2 = 3$ has the eigenvectors $\mathbf{x}_1 = (0, 0, 1)^t$ and $\mathbf{x}_2 = (1, 1, 0)^t$, and the eigenvalue $\lambda_3 = 1$ has the eigenvector $\mathbf{x}_3 = (1, -1, 0)^t$.

f. The eigenvalue $\lambda_1 = 7$ has the eigenvector $\mathbf{x}_1 = (1, 4, 4)^t$, the eigenvalue $\lambda_2 = 3$ has the eigenvector $\mathbf{x}_2 = (1, 2, 0)^t$, and the eigenvalue $\lambda_3 = -1$ has the eigenvector $\mathbf{x}_3 = (1, 0, 0)^t$.

g. The eigenvalue $\lambda_1 = \lambda_2 = 1$ has the eigenvectors $\mathbf{x}_1 = (-1, 1, 0)^t$ and $\mathbf{x}_2 = (-1, 0, 1)^t$, and the eigenvalue $\lambda_3 = 5$ has the eigenvector $\mathbf{x}_3 = (1, 2, 1)^t$.

h. The eigenvalue $\lambda_1 = 3$ has the eigenvector $\mathbf{x}_1 = (-1, 1, 2)^t$, the eigenvalue $\lambda_2 = 4$ has the eigenvector $\mathbf{x}_2 = (0, 1, 2)^t$, and the eigenvalue $\lambda_3 = -2$ has the eigenvector $\mathbf{x}_3 = (-3, 8, 1)^t$.

3. Since

$$A_1^k = \begin{bmatrix} 1 & 0 \\ \frac{2^k-1}{2^{k+1}} & 2^{-k} \end{bmatrix}, \text{ we have } \lim_{k\to\infty} A_1^k = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Also

$$A_2^k = \begin{bmatrix} 2^{-k} & 0 \\ \frac{16k}{2^{k-1}} & 2^{-k} \end{bmatrix}, \text{ so } \lim_{k\to\infty} A_2^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

5. **a.** 3 **b.** 1.618034 **c.** 0.5 **d.** 3.162278

e. 3 **f.** 8.224257 **g.** 5.203527 **h.** 5.601152

7. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $\rho(A) = \rho(B) = 1$ and $\rho(A + B) = 3$.

9. **a.** Since

$$\det(A - \lambda I) = \det((A - \lambda I)^t) = \det(A^t - \lambda I^t) = \det(A^t - \lambda I),$$

λ is an eigenvalue of A if and only if λ is an eigenvalue of A^t.

b. If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. By induction we have $A^n\mathbf{x} = \lambda^n\mathbf{x}$ for each positive integer n.

c. If $A\mathbf{x} = \lambda\mathbf{x}$ and A^{-1} exists, then $\mathbf{x} = \lambda A^{-1}\mathbf{x}$. Also, since A^{-1} exists, zero is not an eigenvalue of A, so $\lambda \neq 0$ and $\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$. So $1/\lambda$ is an eigenvalue of A^{-1}.

d. Since $A\mathbf{x} = \lambda\mathbf{x}$, we have $(A - \alpha I)\mathbf{x} = (\lambda - \alpha)\mathbf{x}$, and since $(A - \alpha I)^{-1}$ exists and $\alpha \neq \lambda$, we have

$$\frac{1}{\lambda - \alpha}\mathbf{x} = (A - \alpha I)^{-1}\mathbf{x}.$$

Exercise Set 7.4 (Page 000)

1. Two iterations of Jacobi's method give the following results.

a. $(0.1428571, -0.3571429, 0.4285714)^t$

b. $(0.97, 0.91, 0.74)^t$

c. $(-0.65, 1.65, -0.4, -2.475)^t$

d. $(-0.5208333, -0.04166667, -0.2166667, 0.4166667)^t$

e. $(1.325, -1.6, 1.6, 1.675, 2.425)^t$

f. $(0.6875, 1.125, 0.6875, 1.375, 0.5625, 1.375)^t$

3. Jacobi's Method gives the following results.

a. $\mathbf{x}^{(10)} = (0.03507839, -0.2369262, 0.6578015)^t$

b. $\mathbf{x}^{(6)} = (0.9957250, 0.9577750, 0.7914500)^t$

c. $\mathbf{x}^{(22)} = (-0.7975853, 2.794795, -0.2588888, -2.251879)^t$

d. $\mathbf{x}^{(14)} = (-0.7529267, 0.04078538, -0.2806091, 0.6911662)^t$

e. $\mathbf{x}^{(12)} = (0.7870883, -1.003036, 1.866048, 1.912449, 1.985707)^t$

f. $\mathbf{x}^{(17)} = (0.9996805, 1.999774, 0.9996805, 1.999840, 0.9995482, 1.999840)^t$

5. **a.** A is not strictly diagonally dominant.

b.

$$T_j = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0.25 \\ -1 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad \rho(T_j) = 0.97210521.$$

Since T_j is convergent, the Jacobi method will converge.

c. With $\mathbf{x}^{(0)} = (0, 0, 0)^t$, $\mathbf{x}^{(187)} = (0.90222655, -0.79595242, 0.69281316)^t$.

d. $\rho(T_j) = 1.39331779371$. Since T_j is not convergent, the Jacobi method will not converge.

7. $T_j = (t_{ik})$ has entries given by

$$t_{ik} = \begin{cases} 0, & i = k \text{ for } 1 \le i \le n, \text{ and } 1 \le k \le n \\ -\frac{a_{ik}}{a_{ii}}, & i \ne k \text{ for } 1 \le i \le n, \text{ and } 1 \le k \le n. \end{cases}$$

Thus,

$$\|T_j\|_\infty = \max_{1 \le i \le n} \sum_{\substack{k=1 \\ k \ne i}}^{n} \left|\frac{a_{ik}}{a_{ii}}\right| < 1,$$

since A is strictly diagonally dominant.

Exercise Set 7.5 (Page 000)

1. Two iterations of the SOR method give the following results.

a. $(0.05410079, -0.2115435, 0.6477159)^t$

b. $(0.9876790, 0.9784935, 0.7899328)^t$

c. $(-0.71885, 2.818822, -0.2809726, -2.235422)^t$

d. $(-0.6604902, 0.03700749, -0.2493513, 0.6561139)^t$

e. $(1.079675, -1.260654, 2.042489, 1.995373, 2.049536)^t$

f. $(0.8318750, 1.647766, 0.9189856, 1.791281, 0.8712129, 1.959155)^t$

3. The tridiagonal matrices are in parts (b) and (c).

b. For $\omega = 1.012823$ we have $\mathbf{x}^{(4)} = (0.9957846, 0.9578935, 0.7915788)^t$.

c. For $\omega = 1.153499$ we have $\mathbf{x}^{(7)} = (-0.7977651, 2.795343, -0.2588021, -2.251760)^t$.

5. a. The system was reordered so that the diagonal of the matrix had nonzero entries.

b. (i) The solution vector is $(-6.27212290601165 \times 10^{-3}, -2.36602456112022 \times 10^4, -1.36602492324141 \times 10^4, -3.34606444633457 \times 10^4, 2.36602456112022 \times 10^4, 1.00000000000000 \times 10^4, -2.73205026462435 \times 10^4, 2.36602492324141 \times 10^4)^t$, using 29 iterations with tolerance 1.00×10^{-2}.

(ii) The solution vector is $(-9.89308239877573 \times 10^{-3}, -2.36602492321617 \times 10^4, -1.36602492324141 \times 10^4, -3.34606444633457 \times 10^4, 2.36602456107651 \times 10^4, 1.00000000000000 \times 10^4, -2.73205026459521 \times 10^4, 2.36602456112022 \times 10^4)^t$, using 57 iterations with tolerance 1.00×10^{-2}.

(iii) The solution vector is $(-2.16147 \times 10^{-3}, -2.366025403900 \times 10^4, -1.366025404100 \times 10^4, -3.346065215000 \times 10^4, 2.366025411100 \times 10^4, 1.000000000000 \times 10^4, -2.732050807600 \times 10^4, 2.366025403600 \times 10^4)^t$, using 19 iterations with tolerance 1.00×10^{-2} and parameter 1.25.

Exercise Set 7.6 (Page 000)

1. The $||\cdot||_\infty$ condition number is as follows.

a. 50	**b.** 241.37	**c.** 600, 002
d. 339, 866	**e.** 12	**h.** 198.17

3. The matrix is ill-conditioned since $K_\infty = 60002$. For the new system we have $\tilde{\mathbf{x}} = (-1.0000, 2.0000)^t$.

5. **a.** (i) $(-10.0, 1.01)^t$, (ii) $(10.0, 1.00)^t$

b. (i) $(12.0, 0.499, -1.98)^t$, (ii) $(1.00, 0.500, -1.00)^t$

c. (i) $(0.185, 0.0103, -0.0200, -1.12)^t$, (ii) $(0.177, 0.0127, -0.0207, -1.18)^t$

d. (i) $(0.799, -3.12, 0.151, 4.56)^t$, (ii) $(0.758, -3.00, 0.159, 4.30)^t$

7. **a.** We have $\tilde{\mathbf{x}} = (188.9998, 92.99998, 45.00001, 27.00001, 21.00002)^t$.

b. The condition number is $K_\infty = 80$.

c. The exact solution is $\mathbf{x} = (189, 93, 45, 27, 21)^t$.

9. **a.**

$$\hat{H}^{-1} = \begin{bmatrix} 8.968 & -35.77 & 29.77 \\ -35.77 & 190.6 & -178.6 \\ 29.77 & -178.6 & 178.6 \end{bmatrix}$$

b.

$$\hat{H} = \begin{bmatrix} 0.9799 & 0.4870 & 0.3238 \\ 0.4860 & 0.3246 & 0.2434 \\ 0.3232 & 0.2433 & 0.1949 \end{bmatrix}$$

c. $\|H - \hat{H}\|_\infty = 0.04260$

Exercise Set 7.7 (Page 000)

Note: All the material in this section is new

1. **a.** $(0.18, 0.13)^t$

b. $(0.19, 0.10)^t$

c. Gaussian elimination gives the best answer since $\mathbf{v}^{(2)} = (0, 0)^t$ in the conjugate gradient method.

d. $(0.13, 0.21)^t$. There is no improvement, although $\mathbf{v}^{(2)} \neq \mathbf{0}$.

3. **a.** $(1.00, -1.00, 1.00)^t$

b. $(0.827, 0.0453, -0.0357)^t$

c. The partial pivoting and scaled partial pivoting also give $(1.00, -1.00, 1.00)^t$.

d. $(0.776, 0.238, -0.185)^t$;

The residual from (3b) is $(-0.0004, -0.0038, 0.0037)^t$, and the residual from part

(3d) is $(0.0022, -0.0038, 0.0024)^t$.

There does not appear to be much improvement, if any. Rounding error is more prevalent because of the increase in the number of matrix multiplications.

5. **a.** $\mathbf{x}^{(2)} = (0.1535933456, -0.1697932117, 0.5901172091)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 0.221$.

b. $\mathbf{x}^{(2)} = (0.9993129510, 0.9642734456, 0.7784266575)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 0.144$.

c. $\mathbf{x}^{(2)} = (-0.7290954114, 2.515782452, -0.6788904058, -2.331943982)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 2.2$.

d. $\mathbf{x}^{(2)} = (-0.7071108901, -0.0954748881, -0.3441074093, 0.5256091497)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 0.39$.

e. $\mathbf{x}^{(2)} = (0.5335968381, 0.9367588935, 1.339920949, 1.743083004, 1.743083004)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 1.3$.

f. $\mathbf{x}^{(2)} = (1.022375671, 1.686451893, 1.022375671, 2.060919568, 0.8310997764, 2.060919568)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 1.13$.

7. **a.** $\mathbf{x}^{(3)} = (0.06185567013, -0.1958762887, 0.6185567010)^t$, $\|\mathbf{r}^{(3)}\|_\infty = 0.4 \times 10^{-9}$.

b. $\mathbf{x}^{(3)} = (0.9957894738, 0.9578947369, 0.7915789474)^t$, $\|\mathbf{r}^{(3)}\|_\infty = 0.1 \times 10^{-9}$.

c. $\mathbf{x}^{(4)} = (-0.7976470579, 2.795294120, -0.2588235305, -2.251764706)^t$, $\|\mathbf{r}^{(4)}\|_\infty = 0.39 \times 10^{-7}$.

d. $\mathbf{x}^{(4)} = (-0.7534246575, 0.04109589039, -0.2808219179, 0.6917808219)^t$, $\|\mathbf{r}^{(4)}\|_\infty = 0.11 \times 10^{-9}$.

e. $\mathbf{x}^{(5)} = (0.4516129032, 0.7096774197, 1.677419355, 1.741935483, 1.806451613)^t$, $\|\mathbf{r}^{(5)}\|_\infty = 0.2 \times 10^{-9}$.

f. $\mathbf{x}^{(4)} = (1.000000000, 2.000000000, 1.000000000, 2.000000000, 0.9999999997, 2.000000000)^t$, $\|\mathbf{r}^{(4)}\|_\infty = 0.44 \times 10^{-9}$.

9.

a.	Jacobi 49 iterations	Gauss-Seidel 28 iterations	SOR ($\omega = 1.3$) 13 iterations	Conjugate Gradient 9 iterations
x_1	0.93406183	0.93406917	0.93407584	0.93407713
x_2	0.97473885	0.97475285	0.97476180	0.97476363
x_3	1.10688692	1.10690302	1.10691093	1.10691243
x_4	1.42346150	1.42347226	1.42347591	1.42347699
x_5	0.85931331	0.85932730	0.85933633	0.85933790
x_6	0.80688119	0.80690725	0.80691961	0.80692197
x_7	0.85367746	0.85370564	0.85371536	0.85372011
x_8	1.10688692	1.10690579	1.10691075	1.10691250
x_9	0.87672774	0.87674384	0.87675177	0.87675250
x_{10}	0.80424512	0.80427330	0.80428301	0.80428524
x_{11}	0.80688119	0.80691173	0.80691989	0.80692252
x_{12}	0.97473885	0.97475850	0.97476265	0.97476392
x_{13}	0.93003466	0.93004542	0.93004899	0.93004987
x_{14}	0.87672774	0.87674661	0.87675155	0.87675298
x_{15}	0.85931331	0.85933296	0.85933709	0.85933979
x_{16}	0.93406183	0.93407462	0.93407672	0.93407768

b.	Jacobi 60 iterations	Gauss-Seidel 35 iterations	SOR ($\omega = 1.2$) 23 iterations	Conjugate Gradient 11 iterations
x_1	0.39668038	0.39668651	0.39668915	0.39669775
x_2	0.07175540	0.07176830	0.07177348	0.07178516
x_3	−0.23080396	−0.23078609	−0.23077981	−0.23076923
x_4	0.24549277	0.24550989	0.24551535	0.24552253
x_5	0.83405412	0.83406516	0.83406823	0.83407148
x_6	0.51497606	0.51498897	0.51499414	0.51500583
x_7	0.12116003	0.12118683	0.12119625	0.12121212
x_8	−0.24044414	−0.24040991	−0.24039898	−0.24038462
x_9	0.37873579	0.37876891	0.37877812	0.37878788
x_{10}	1.09073364	1.09075392	1.09075899	1.09076341
x_{11}	0.54207872	0.54209658	0.54210286	0.54211344
x_{12}	0.13838259	0.13841682	0.13842774	0.13844211
x_{13}	−0.23083868	−0.23079452	−0.23078224	−0.23076923
x_{14}	0.41919067	0.41923122	0.41924136	0.41925019
x_{15}	1.15015953	1.15018477	1.15019025	1.15019425
x_{16}	0.51497606	0.51499318	0.51499864	0.51500583
x_{17}	0.12116003	0.12119315	0.12120236	0.12121212
x_{18}	−0.24044414	−0.24040359	−0.24039345	−0.24038462
x_{19}	0.37873579	0.37877365	0.37878188	0.37878788
x_{20}	1.09073364	1.09075629	1.09076069	1.09076341
x_{21}	0.39668038	0.39669142	0.39669449	0.39669775
x_{22}	0.07175540	0.07177567	0.07178074	0.07178516
x_{23}	−0.23080396	−0.23077872	−0.23077323	−0.23076923
x_{24}	0.24549277	0.24551542	0.24551982	0.24552253
x_{25}	0.83405412	0.83406793	0.83407025	0.83407148

c.	Jacobi 15 iterations	Gauss-Seidel 9 iterations	SOR ($\omega = 1.1$) 8 iterations	Conjugate Gradient 8 iterations
x_1	−3.07611424	−3.07611739	−3.07611796	−3.07611794
x_2	−1.65223176	−1.65223563	−1.65223579	−1.65223582
x_3	−0.53282391	−0.53282528	−0.53282531	−0.53282528
x_4	−0.04471548	−0.04471608	−0.04471609	−0.04471604
x_5	0.17509673	0.17509661	0.17509661	0.17509661
x_6	0.29568226	0.29568223	0.29568223	0.29568218
x_7	0.37309012	0.37309011	0.37309011	0.37309011
x_8	0.42757934	0.42757934	0.42757934	0.42757927
x_9	0.46817927	0.46817927	0.46817927	0.46817927
x_{10}	0.49964748	0.49964748	0.49964748	0.49964748
x_{11}	0.52477026	0.52477026	0.52477026	0.52477027
x_{12}	0.54529835	0.54529835	0.54529835	0.54529836
x_{13}	0.56239007	0.56239007	0.56239007	0.56239009
x_{14}	0.57684345	0.57684345	0.57684345	0.57684347
x_{15}	0.58922662	0.58922662	0.58922662	0.58922664
x_{16}	0.59995522	0.59995522	0.59995522	0.59995523
x_{17}	0.60934045	0.60934045	0.60934045	0.60934045
x_{18}	0.61761997	0.61761997	0.61761997	0.61761998
x_{19}	0.62497846	0.62497846	0.62497846	0.62497847
x_{20}	0.63156161	0.63156161	0.63156161	0.63156161
x_{21}	0.63748588	0.63748588	0.63748588	0.63748588
x_{22}	0.64284553	0.64284553	0.64284553	0.64284553
x_{23}	0.64771764	0.64771764	0.64771764	0.64771764
x_{24}	0.65216585	0.65216585	0.65216585	0.65216585
x_{25}	0.65624320	0.65624320	0.65624320	0.65624320
x_{26}	0.65999423	0.65999423	0.65999423	0.65999422
x_{27}	0.66345660	0.66345660	0.66345660	0.66345660
x_{28}	0.66666242	0.66666242	0.66666242	0.66666242
x_{29}	0.66963919	0.66963919	0.66963919	0.66963919
x_{30}	0.67241061	0.67241061	0.67241061	0.67241060
x_{31}	0.67499722	0.67499722	0.67499722	0.67499721
x_{32}	0.67741692	0.67741692	0.67741691	0.67741691
x_{33}	0.67968535	0.67968535	0.67968535	0.67968535
x_{34}	0.68181628	0.68181628	0.68181628	0.68181628
x_{35}	0.68382184	0.68382184	0.68382184	0.68382184
x_{36}	0.68571278	0.68571278	0.68571278	0.68571278
x_{37}	0.68749864	0.68749864	0.68749864	0.68749864
x_{38}	0.68918652	0.68918652	0.68918652	0.68918652
x_{39}	0.69067718	0.69067718	0.69067718	0.69067717
x_{40}	0.68363346	0.68363346	0.68363346	0.68363349

11. a.

Solution	Residual
2.55613420	0.00668246
4.09171393	−0.00533953
4.60840390	−0.01739814
3.64309950	−0.03171624
5.13950533	0.01308093
7.19697808	−0.02081095
7.68140405	−0.04593118
5.93227784	0.01692180
5.81798997	0.04414047
5.85447806	0.03319707
5.94202521	−0.00099947
4.42152959	−0.00072826
3.32211695	0.02363822
4.49411604	0.00982052
4.80968966	0.00846967
3.81108707	−0.01312902

This converges in 6 iterations with tolerance 5.00×10^{-2} in the l_∞ norm and $\|\mathbf{r}^{(6)}\|_\infty = 0.046$.

b.

Solution	Residual
2.55613420	0.00668246
4.09171393	−0.00533953
4.60840390	−0.01739814
3.64309950	−0.03171624
5.13950533	0.01308093
7.19697808	−0.02081095
7.68140405	−0.04593118
5.93227784	0.01692180
5.81798996	0.04414047
5.85447805	0.03319706
5.94202521	−0.00099947
4.42152959	−0.00072826
3.32211694	0.02363822
4.49411603	0.00982052
4.80968966	0.00846967
3.81108707	−0.01312902

This converges in 6 iterations with tolerance 5.00×10^{-2} in the l_∞ norm and $\|\mathbf{r}^{(6)}\|_\infty = 0.046$.

c. All tolerances lead to the same convergence specifications.

13. **a.** Let $\{\mathbf{v}^{(1)}, \ldots \mathbf{v}^{(n)}\}$ be a set of nonzero A-orthogonal vectors for the symmetric positive definite matrix A. Then $\langle \mathbf{v}^{(i)}, A\mathbf{v}^{(j)}\rangle = 0$, if $i \neq j$. Suppose

$$c_1\mathbf{v}^{(1)} + c_2\mathbf{v}^{(2)} + \cdots + c_n\mathbf{v}^{(n)} = \mathbf{0},$$

where not all c_i are zero. Suppose k is the smallest integer for which $c_k \neq 0$. Then

$$c_k\mathbf{v}^{(k)} + c_{k+1}\mathbf{v}^{(k+1)} + \cdots + c_n\mathbf{v}^{(n)} = \mathbf{0}.$$

We solve for $\mathbf{v}^{(k)}$ to obtain

$$\mathbf{v}^{(k)} = -\frac{c_{k+1}}{c_k}\mathbf{v}^{(k+1)} - \cdots - \frac{c_n}{c_k}\mathbf{v}^{(n)}.$$

Multiplying by A gives

$$A\mathbf{v}^{(k)} = -\frac{c_{k+1}}{c_k}A\mathbf{v}^{(k+1)} - \cdots - \frac{c_n}{c_k}A\mathbf{v}^{(n)},$$

so

$$\begin{aligned}
\mathbf{v}^{(k)t}A\mathbf{v}^{(k)} &= -\frac{c_{k+1}}{c_k}\mathbf{v}^{(k)t}A\mathbf{v}^{(k+1)} - \cdots - \frac{c_n}{c_k}\mathbf{v}^{(k)t}A\mathbf{v}^{(n)} \\
&= -\frac{c_{k+1}}{c_k}\langle \mathbf{v}^{(k)}, A\mathbf{v}^{(k+1)}\rangle - \cdots - \frac{c_n}{c_k}\langle \mathbf{v}^{(k)}, A\mathbf{v}^{(n)}\rangle \\
&= -\frac{c_{k+1}}{c_k}\cdot 0 - \cdots - \frac{c_n}{c_k}\cdot 0.
\end{aligned}$$

Since A is positive definite, $\mathbf{v}^{(k)} = \mathbf{0}$, which is a contradiction. Thus, all c_i must be zero, and $\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\}$ is linearly independent.

b. Let $\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\}$ be a set of nonzero A-orthogonal vectors for the symmetric positive definite matrix A, and let $\mathbf{z}$ be orthogonal to $\mathbf{v}^{(i)}$, for each $i = 1, \ldots, n$. From part (a), the set $\{\mathbf{v}^{(1)}, \ldots \mathbf{v}^{(n)}\}$ is linearly independent, so there is a collection of constants $\beta_1, \ldots, \beta_n$ with

$$\mathbf{z} = \sum_{i=1}^{n} \beta_i \mathbf{v}^{(i)}.$$

Hence,

$$\mathbf{z}^t\mathbf{z} = \sum_{i=1}^{n} \beta_i \mathbf{z}^t \mathbf{v}^{(i)} = \sum_{i=1}^{n} \beta_i \cdot 0 = 0,$$

and part (v) of the Inner Product Properties implies that $\mathbf{z} = \mathbf{0}$.

Exercise Set 8.2 (Page 000)

1. The linear least squares polynomial is $1.70784x + 0.89968$.

3. The least squares polynomials with their errors are:

$0.6208950 + 1.219621x$, with $E = 2.719 \times 10^{-5}$;

$0.5965807 + 1.253293x - 0.01085343x^2$, with $E = 1.801 \times 10^{-5}$;

$0.6290193 + 1.185010x + 0.03533252x^2 - 0.01004723x^3$, with $E = 1.741 \times 10^{-5}$.

5. **a.** The linear least squares polynomial is $72.0845x - 194.138$, with error of 329.

b. The least squares polynomial of degree 2 is $6.61821x^2 - 1.14352x + 1.23556$, with error of 1.44×10^{-3}.

c. The least squares polynomial of degree 3 is $-0.0136742x^3 + 6.84557x^2 - 2.37919x + 3.42904$, with error of 5.27×10^{-4}.

7. **a.** $k = 0.8996,\ \ E(k) = 0.407$

b. $k = 0.9052$, $E(k) = 0.486$

Part (b) best fits the total experimental data.

9. Point average = 0.101(ACT score) +0.487

Exercise Set 8.3 (Page 000)

1. The linear least squares approximations are as follows.

a. $P_1(x) = 1.833333 + 4x$

b. $P_1(x) = -1.600003 + 3.600003x$

c. $P_1(x) = 1.140981 - 0.2958375x$

d. $P_1(x) = 0.1945267 + 3.000001x$

e. $P_1(x) = 0.6109245 + 0.09167105x$

f. $P_1(x) = -1.861455 + 1.666667x$

3. The linear least squares approximations on $[-1, 1]$ are as follows.

a. $P_1(x) = 3.333333 - 2x$

b. $P_1(x) = 0.6000025x$

c. $P_1(x) = 0.5493063 - 0.2958375x$

d. $P_1(x) = 1.175201 + 1.103639x$

e. $P_1(x) = 0.4207355 + 0.4353975x$

f. $P_1(x) = 0.6479184 + 0.5281226x$

5. The errors for the approximations in Exercise 3 are as follows.

a. 0.177779	**b.** 0.0457206	**c.** 0.00484624
d. 0.0526541	**e.** 0.0153784	**f.** 0.00363453

7. The Gram-Schmidt process produces the following collections of polynomials.

a. $\phi_0(x) = 1, \phi_1(x) = x - 0.5$, $\phi_2(x) = x^2 - x + \frac{1}{6}$, and $\phi_3(x) = x^3 - 1.5x^2 + 0.6x - 0.05$

b. $\phi_0(x) = 1, \phi_1(x) = x - 1$, $\phi_2(x) = x^2 - 2x + \frac{2}{3}$, and $\phi_3(x) = x^3 - 3x^2 + \frac{12}{5}x - \frac{2}{5}$

c. $\phi_0(x) = 1, \phi_1(x) = x - 2$, $\phi_2(x) = x^2 - 4x + \frac{11}{3}$, and $\phi_3(x) = x^3 - 6x^2 + 11.4x - 6.8$

9. The least squares polynomials of degree 2 are as follows.

a. $P_2(x) = 3.833333\phi_0(x) + 4\phi_1(x) + 0.9999998\phi_2(x)$

b. $P_2(x) = 2\phi_0(x) + 3.6\phi_1(x) + 3\phi_2(x)$

c. $P_2(x) = 0.5493061\phi_0(x) - 0.2958369\phi_1(x) + 0.1588785\phi_2(x)$

d. $P_2(x) = 3.194528\phi_0(x) + 3\phi_1(x) + 1.458960\phi_2(x)$

e. $P_2(x) = 0.6567600\phi_0(x) + 0.09167105\phi_1(x) - 0.7375118\phi_2(x)$

f. $P_2(x) = 1.471878\phi_0(x) + 1.666667\phi_1(x) + 0.2597705\phi_2(x)$

11. **a.** $2L_0(x) + 4L_1(x) + L_2(x)$

b. $\frac{1}{2}L_0(x) - \frac{1}{4}L_1(x) + \frac{1}{16}L_2(x) - \frac{1}{96}L_3(x)$

c. $6L_0(x) + 18L_1(x) + 9L_2(x) + L_3(x)$

d. $\frac{1}{3}L_0(x) - \frac{2}{9}L_1(x) + \frac{2}{27}L_2(x) - \frac{4}{243}L_3(x)$

Exercise Set 8.4 (Page 000)

1. The interpolating polynomials of degree 2 are as follows.

a. $P_2(x) = 2.377443 + 1.590534(x - 0.8660254) + 0.5320418(x - 0.8660254)x$

b. $P_2(x) = 0.7617600 + 0.8796047(x - 0.8660254)$

c. $P_2(x) = 1.052926 + 0.4154370(x - 0.8660254) - 0.1384262x(x - 0.8660254)$

d. $P_2(x) = 0.5625 + 0.649519(x - 0.8660254) + 0.75x(x - 0.8660254)$

3. The interpolating polynomials of degree 3 are as follows.

a.

$$\begin{aligned}P_3(x) =&2.519044 + 1.945377(x - 0.9238795)\\&+ 0.7047420(x - 0.9238795)(x - 0.3826834)\\&+ 0.1751757(x - 0.9238795)(x - 0.3826834)(x + 0.3826834)\end{aligned}$$

b.

$$\begin{aligned}P_3(x) =&0.7979459 + 0.7844380(x - 0.9238795)\\&- 0.1464394(x - 0.9238795)(x - 0.3826834)\\&- 0.1585049(x - 0.9238795)(x - 0.3826834)(x + 0.3826834)\end{aligned}$$

c.

$$\begin{aligned}P_3(x) =&1.072911 + 0.3782067(x - 0.9238795)\\&- 0.09799213(x - 0.9238795)(x - 0.3826834)\\&+ 0.04909073(x - 0.9238795)(x - 0.3826834)(x + 0.3826834)\end{aligned}$$

d.

$$\begin{aligned}P_3(x) =&0.7285533 + 1.306563(x - 0.9238795)\\&+ 0.9999999(x - 0.9238795)(x - 0.3826834)\end{aligned}$$

5. The zeros of $\tilde{T}_3$ produce the following interpolating polynomials of degree 2.

a. $P_2(x) = 0.3489153 - 0.1744576(x - 2.866025) + 0.1538462(x - 2.866025)(x - 2)$

b. $P_2(x) = 0.1547375 - 0.2461152(x - 1.866025) + 0.1957273(x - 1.866025)(x - 1)$

c. $P_2(x) = 0.6166200 - 0.2370869(x - 0.9330127) - 0.7427732(x - 0.9330127)(x - 0.5)$

d. $P_2(x) = 3.0177125 + 1.883800(x - 2.866025) + 0.2584625(x - 2.866025)(x - 2)$

7. If $i > j$, then

$$\frac{1}{2}(T_{i+j}(x) + T_{i-j}(x)) = \frac{1}{2}\Big(\cos(i+j)\theta + \cos(i-j)\theta\Big) = \cos i\theta \cos j\theta = T_i(x)T_j(x).$$

Exercise Set 8.5 (Page 000)

1. The Padé approximations of degree 2 for $f(x) = e^{2x}$ are

$n = 2, m = 0 : r_{2,0}(x) = 1 + 2x + 2x^2,$

$n = 1, m = 1 : r_{1,1}(x) = (1 + x)/(1 - x),$

$n = 0, m = 2 : r_{0,2}(x) = (1 - 2x + 2x^2)^{-1}.$

i	x_i	$f(x_i)$	$r_{2,0}(x_i)$	$r_{1,1}(x_i)$	$r_{0,2}(x_i)$
1	0.2	1.4918	1.4800	1.5000	1.4706
2	0.4	2.2255	2.1200	2.3333	1.9231
3	0.6	3.3201	2.9200	4.0000	1.9231
4	0.8	4.9530	3.8800	9.0000	1.4706
5	1.0	7.3891	5.0000	undefined	1.0000

3. $r_{2,3}(x) = (1 + \frac{2}{5}x + \frac{1}{20}x^2)/(1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3)$

i	x_i	$f(x_i)$	$r_{2,3}(x_i)$
1	0.2	1.22140276	1.22140277
2	0.4	1.49182470	1.49182561
3	0.6	1.82211880	1.82213210
4	0.8	2.22554093	2.22563652
5	1.0	2.71828183	2.71875000

5. $r_{3,3}(x) = (x - \frac{7}{60}x^3)/(1 + \frac{1}{20}x^2)$

i	x_i	$f(x_i)$	6th Maclaurin Polynomial	$r_{3,2}(x_i)$
0	0.0	0.00000000	0.00000000	0.00000000
1	0.1	0.09983342	0.09966675	0.09938640
2	0.2	0.19866933	0.19733600	0.19709571
3	0.3	0.29552021	0.29102025	0.29246305
4	0.4	0.38941834	0.37875200	0.38483660
5	0.5	0.47942554	0.45859375	0.47357724

7. The Padé approximations of degree 5 are as follows.

a. $r_{0,5}(x) = (1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5)^{-1}$

b. $r_{1,4}(x) = (1 - \frac{1}{5}x)/(1 + \frac{4}{5}x + \frac{3}{10}x^2 + \frac{1}{15}x^3 + \frac{1}{120}x^4)$

c. $r_{3,2}(x) = (1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3)/(1 + \frac{2}{5}x + \frac{1}{20}x^2)$

d. $r_{4,1}(x) = (1 - \frac{4}{5}x + \frac{3}{10}x^2 - \frac{1}{15}x^3 + \frac{1}{120}x^4)/(1 + \frac{1}{5}x)$

i	x_i	$f(x_i)$	$r_{0,5}(x_i)$	$r_{1,4}(x_i)$	$r_{2,3}(x_i)$	$r_{4,1}(x_i)$
1	0.2	0.81873075	0.81873081	0.81873074	0.81873075	0.81873077
2	0.4	0.67032005	0.67032276	0.67031942	0.67031963	0.67032099
3	0.6	0.54881164	0.54883296	0.54880635	0.54880763	0.54882143
4	0.8	0.44932896	0.44941181	0.44930678	0.44930966	0.44937931
5	1.0	0.36787944	0.36809816	0.36781609	0.36781609	0.36805556

9. a. Since

$$\sin|x| = \sin(M\pi + s) = \sin M\pi \cos s + \cos M\pi \sin s = (-1)^M \sin s,$$

we have

$$\sin x = \text{ sign } x \sin|x| = \text{sign }(x)(-1)^M \sin s.$$

b. We have

$$\sin x \approx \left(s - \frac{31}{294}s^3\right) \Big/ \left(1 + \frac{3}{49}s^2 + \frac{11}{5880}s^3\right)$$

with $|\text{error}| \leq 2.84 \times 10^{-4}$.

c. Set $M = \text{round}(|x|/\pi)$; $s = |x| - M\pi$; $f_1 = \left(s - \frac{31}{294}s^3\right) \Big/ \left(1 + \frac{3}{49}s^2 + \frac{11}{5880}s^4\right)$. Then $f = (-1)^M f_1 \cdot x/|x|$ is the approximation.

d. Set $y = x + \frac{\pi}{2}$ and repeat part (c) with y in place of x.

Exercise Set 8.6 (Page 000)

1. $S_2(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x$

3. $S_3(x) = 3.676078 - 3.676078\cos x + 1.470431\cos 2x - 0.7352156\cos 3x + 3.676078\sin x - 2.940862\sin 2x$

5. $S_n(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{n-1} \frac{1-(-1)^k}{k}\sin kx$

7. The trigonometric least squares polynomials are as follows.

a. $S_2(x) = \cos 2x$

b. $S_2(x) = 0$

c. $S_3(x) = 1.566453 + 0.5886815\cos x - 0.2700642\cos 2x + 0.2175679\cos 3x + 0.8341640\sin x - 0.3097866\sin 2x$

d. $S_3(x) = -2.046326 + 3.883872\cos x - 2.320482\cos 2x + 0.7310818\cos 3x$

9. The trigonometric least squares polynomial is $S_3(x) = -0.4968929 + 0.2391965\cos x + 1.515393\cos 2x + 0.2391965\cos 3x - 1.150649\sin x$ with error $E(S_3) = 7.271197$.

11. Let $f(-x) = -f(x)$. The integral $\int_{-a}^{0} f(x)\,dx$ under the change of variable $t = -x$ transforms to

$$-\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-t)\,dt = -\int_{0}^{a} f(t)\,dt = -\int_{0}^{a} f(x)\,dx.$$

Thus,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0.$$

13. Representative integrations that establish the orthogonality are:

$$\int_{-\pi}^{\pi} [\phi_0(x)]^2\,dx = \frac{1}{2}\int_{-\pi}^{\pi} dx = \pi,$$

$$\int_{-\pi}^{\pi} [\phi_k(x)]^2\,dx = \int_{-\pi}^{\pi} (\cos kx)^2\,dx = \int_{-\pi}^{\pi} \left[\frac{1}{2} + \frac{1}{2}\cos 2kx\right] dx = \pi + \left[\frac{1}{4k}\sin 2kx\right]_{-\pi}^{\pi} = \pi,$$

$$\int_{-\pi}^{\pi} \phi_k(x)\phi_0(x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos kx\,dx = \frac{1}{2k}\sin kx\Big]_{-\pi}^{\pi} = 0,$$

and

$$\int_{-\pi}^{\pi} \phi_k(x)\phi_{n+j}(x)\,dx = \int_{-\pi}^{\pi} \cos kx \sin jx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\sin(k+j)x - \sin(k-j)x]\,dx = 0.$$

Exercise Set 8.7 (Page 000)

1. The trigonometric interpolating polynomials are as follows.

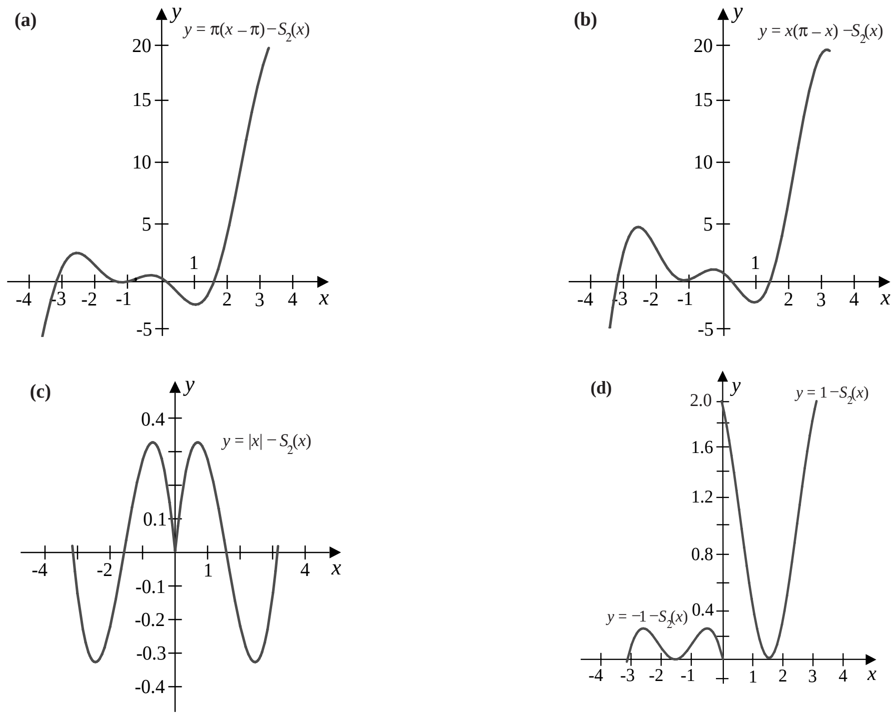
a. $S_2(x) = -12.33701 + 4.934802\cos x - 2.467401\cos 2x + 4.934802\sin x$

b. $S_2(x) = -6.168503 + 9.869604\cos x - 3.701102\cos 2x + 4.934802\sin x$

c. $S_2(x) = 1.570796 - 1.570796\cos x$

d. $S_2(x) = -0.5 - 0.5\cos 2x + \sin x$

Note: New Figures



3. The Fast Fourier Transform method gives the following trigonometric interpolating polynomials.

a. $S_4(x) = -11.10331 + 2.467401 \cos x - 2.467401 \cos 2x + 2.467401 \cos 3x - 1.233701 \cos 4x + 5.956833 \sin x - 2.467401 \sin 2x + 1.022030 \sin 3x$

b. $S_4(x) = 1.570796 - 1.340759 \cos x - 0.2300378 \cos 3x$

c. $S_4(x) = -0.1264264 + 0.2602724 \cos x - 0.3011140 \cos 2x + 1.121372 \cos 3x + 0.04589648 \cos 4x - 0.1022190 \sin x + 0.2754062 \sin 2x - 2.052955 \sin 3x$

d. $S_4(x) = -0.1526819 + 0.04754278 \cos x + 0.6862114 \cos 2x - 1.216913 \cos 3x + 1.176143 \cos 4x - 0.8179387 \sin x + 0.1802450 \sin 2x + 0.2753402 \sin 3x$

5.

	Approximation	Actual
a.	-69.76415	-62.01255
b.	9.869602	9.869604
c.	-0.7943605	-0.2739383
d.	-0.9593287	-0.9557781

Exercise Set 9.2 (Page 000)

1. **a.** The eigenvalues and associated eigenvectors are $\lambda_1 = 2,\ \mathbf{v}^{(1)} = (1,0,0)^t;\ \lambda_2 = 1,\ \mathbf{v}^{(2)} = (0,2,1)^t$; and $\lambda_3 = -1,\ \mathbf{v}^{(3)} = (-1,1,1)^t$. The set is linearly independent.

b. The eigenvalues and associated eigenvectors are $\lambda_1 = \lambda_2 = \lambda_3 = 1,\ \mathbf{v}^{(1)} = \mathbf{v}^{(2)} = (1,0,1)^t$ and $\mathbf{v}^{(3)} = (0,1,1)$. The set is linearly dependent.

c. The eigenvalues and associated eigenvectors are $\lambda_1 = 2,\ \mathbf{v}^{(1)} = (0,1,0)^t;\ \lambda_2 = 3,\ \mathbf{v}^{(2)} = (1,0,1)^t$; and $\lambda_3 = 1,\ \mathbf{v}^{(3)} = (1,0,-1)^t$. The set is linearly independent.

d. The eigenvalues and associated eigenvectors are $\lambda_1 = \lambda_2 = 3,\ \mathbf{v}^{(1)} = (1,0,-1)^t, \mathbf{v}^{(2)} = (0,1,-1)^t$; and $\lambda_3 = 0,\ \mathbf{v}^{(3)} = (1,1,1)^t$. The set is linearly independent.

e. The eigenvalues and associated eigenvectors are $\lambda_1 = 1, \mathbf{v}^{(1)} = (0,-1,1)^t; \lambda_2 = 1+\sqrt{2}, \mathbf{v}^{(2)} = (\sqrt{2},1,1)^t$; and $\lambda_3 = 1-\sqrt{2}, \mathbf{v}^{(3)} = (-\sqrt{2},1,1)^t$. The set is linearly independent.

f. The eigenvalues and associated eigenvectors are $\lambda_1 = 1,\ \mathbf{v}^{(1)} = (1,0,-1)^t; \lambda_2 = 1, \mathbf{v}^{(2)} = (1,-1,0)^t$; and $\lambda_3 = 4,\ \mathbf{v}^{(3)} = (1,1,1)^t$. The set is linearly independent.

3. **a.** The three eigenvalues are within $\{\lambda \mid |\lambda| \le 2\} \cup \{\lambda \mid |\lambda - 2| \le 2\}$.

b. The three eigenvalues are within $R_1 = \{\lambda \mid |\lambda - 4| \le 2\}$.

c. The three real eigenvalues satisfy $0 \le \lambda \le 6$.

d. The three real eigenvalues satisfy $1.25 \le \lambda \le 8.25$.

e. The four real eigenvalues satisfy $-8 \le \lambda \le 1$.

f. The four real eigenvalues are within $R_1 = \{\lambda \mid |\lambda - 2| \le 4\}$.

5. If $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, then for any $j = 1, 2, \ldots, k$, we have $c_1\mathbf{v}_j^t\mathbf{v}_1 + \cdots + c_k\mathbf{v}_j^t\mathbf{v}_k = \mathbf{0}$. But orthogonality gives $c_i\mathbf{v}_j^t\mathbf{v}_i = 0$ for $i \ne j$, so $c_j\mathbf{v}_j^t\mathbf{v}_j = 0$ and $c_j = 0$.

7. Since $\{\mathbf{v}_i\}_{i=1}^n$ is linearly independent in $\mathbb{R}^n$, there exist numbers $c_1, \ldots, c_n$ with

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

Hence, for any $j = 1, 2, \ldots, n,$

$$\mathbf{v}_j^t\mathbf{x} = c_1\mathbf{v}_j^t\mathbf{v}_1 + \cdots + c_n\mathbf{v}_j^t\mathbf{v}_n = c_j\mathbf{v}_j^t\mathbf{v}_j = c_j.$$

9. a. The eigenvalues are $\lambda_1 = 5.307857563$, $\lambda_2 = -0.4213112993$, $\lambda_3 = -0.1365462647$ with associated eigenvectors $(0.59020967, 0.51643129, 0.62044441)^t$, $(0.77264234, -0.13876278, -0.61949069)^t$, and $(0.23382978, -0.84501102, 0.48091581)^t$, respectively.

b. A is not positive definite, since $\lambda_2 < 0$ and $\lambda_3 < 0$.

Exercise Set 9.3 (Page 000)

1. The approximate eigenvalues and approximate eigenvectors are as follows.

a. $\mu^{(3)} = 3.666667$, $\mathbf{x}^{(3)} = (0.9772727, 0.9318182, 1)^t$

b. $\mu^{(3)} = 2.000000$, $\mathbf{x}^{(3)} = (1, 1, 0.5)^t$

c. $\mu^{(3)} = 5.000000$, $\mathbf{x}^{(3)} = (-0.2578947, 1, -0.2842105)^t$

d. $\mu^{(3)} = 5.038462$, $\mathbf{x}^{(3)} = (1, 0.2213741, 0.3893130, 0.4045802)^t$

e. $\mu^{(3)} = 7.531073$, $\mathbf{x}^{(3)} = (0.6886722, -0.6706677, -0.9219805, 1)^t$

f. $\mu^{(3)} = 4.106061$, $\mathbf{x}^{(3)} = (0.1254613, 0.08487085, 0.00922509, 1)^t$

3. The approximate eigenvalues and approximate eigenvectors are as follows.

a. $\mu^{(3)} = 3.959538$, $\mathbf{x}^{(3)} = (0.5816124, 0.5545606, 0.5951383)^t$

b. $\mu^{(3)} = 2.0000000$, $\mathbf{x}^{(3)} = (-0.6666667, -0.6666667, -0.3333333)^t$

c. $\mu^{(3)} = 7.189567$, $\mathbf{x}^{(3)} = (0.5995308, 0.7367472, 0.3126762)^t$

d. $\mu^{(3)} = 6.037037$, $\mathbf{x}^{(3)} = (0.5073714, 0.4878571, -0.6634857, -0.2536857)^t$

e. $\mu^{(3)} = 5.142562$, $\mathbf{x}^{(3)} = (0.8373051, 0.3701770, 0.1939022, 0.3525495)^t$

f. $\mu^{(3)} = 8.593142$, $\mathbf{x}^{(3)} = (-0.4134762, 0.4026664, 0.5535536, -0.6003962)^t$

5. The approximate eigenvalues and approximate eigenvectors are as follows.

a. $\mu^{(8)} = 4.000001$, $\mathbf{x}^{(8)} = (0.9999773, 0.99993134, 1)^t$

b. The method fails because of division by zero.

c. $\mu^{(7)} = 5.124890$, $\mathbf{x}^{(7)} = (-0.2425938, 1, -0.3196351)^t$

d. $\mu^{(15)} = 5.236112$, $\mathbf{x}^{(15)} = (1, 0.6125369, 0.1217216, 0.4978318)^t$

e. $\mu^{(10)} = 8.999890$, $\mathbf{x}^{(10)} = (0.9944137, -0.9942148, -0.9997991, 1)^t$

f. $\mu^{(11)} = 4.105317$, $\mathbf{x}^{(11)} = (0.11716540, 0.072853995, 0.01316655, 1)^t$

7. The approximate eigenvalues and approximate eigenvectors are as follows.

a. $\mu^{(9)} = 1.000015$, $\mathbf{x}^{(9)} = (-0.1999939, 1, -0.7999909)^t$

b. $\mu^{(12)} = -0.4142136$, $\mathbf{x}^{(12)} = (1, -0.7070918, -0.7071217)^t$

c. The method did not converge in 25 iterations. However, $\mu^{(42)} = 1.636636$, $\mathbf{x}^{(42)} = (-0.5706815, 0.3633636, 1)^t$.

d. $\mu^{(9)} = 1.381959$, $\mathbf{x}^{(9)} = (-0.3819400, -0.2361007, 0.2360191, 1)^t$

e. $\mu^{(6)} = 3.999997$, $\mathbf{x}^{(6)} = (0.9999939, 0.9999999, 0.9999940, 1)^t$

f. $\mu^{(3)} = 4.105293$, $\mathbf{x}^{(3)} = (0.06281419, 0.08704089, 0.01825213, 1)^t$

9. **a.** We have $|\lambda| \leq 6$ for all eigenvalues λ.

b. The approximate eigenvalue is $\mu^{(133)} = 0.69766854$, with the approximate eigenvector $\mathbf{x}^{(133)} = (1, 0.7166727, 0.2568099, 0.04601217)^t$.

c. Wielandt's deflation fails because λ_2 and λ_3 are complex numbers.

d. The characteristic polynomial is $P(\lambda) = \lambda^4 - \frac{1}{4}\lambda - \frac{1}{16}$ and the eigenvalues are $\lambda_1 = 0.6976684972$, $\lambda_2 = -0.2301775942 + 0.56965884i$, $\lambda_3 = -0.2301775942 - 0.56965884i$, and $\lambda_4 = -0.237313308$.

e. The beetle population should approach zero since A is convergent.

Exercise Set 9.4 (Page 000)

1. Householder's method produces the following tridiagonal matrices.

a.

$$\begin{bmatrix} 12.00000 & -10.77033 & 0.0 \\ -10.77033 & 3.862069 & 5.344828 \\ 0.0 & 5.344828 & 7.137931 \end{bmatrix}$$

b.

$$\begin{bmatrix} 2.0000000 & 1.414214 & 0.0 \\ 1.414214 & 1.000000 & 0.0 \\ 0.0 & 0.0 & 3.0 \end{bmatrix}$$

c.

$$\begin{bmatrix} 1.0000000 & -1.414214 & 0.0 \\ -1.414214 & 1.000000 & 0.0 \\ 0.0 & 0.0 & 1.000000 \end{bmatrix}$$

d.

$$\begin{bmatrix} 4.750000 & -2.263846 & 0.0 \\ -2.263846 & 4.475610 & -1.219512 \\ 0.0 & -1.219512 & 5.024390 \end{bmatrix}$$

3. a. Since $P = I - 2\mathbf{w}\mathbf{w}^t$, we have

$$P^t = (I - 2\mathbf{w}\mathbf{w}^t)^t = I^t - 2(\mathbf{w}\mathbf{w}^t)^t = I - 2(\mathbf{w}^t)^t\mathbf{w}^t = I - 2\mathbf{w}\mathbf{w}^t = P.$$

b. Using part (a) We have

$$P^tP = P^2 = \left(I - 2\mathbf{w}\mathbf{w}^t\right)^2 = I - 4\mathbf{w}\mathbf{w}^t + 4\mathbf{w}\mathbf{w}^t\mathbf{w}\mathbf{w}^t.$$

But $\mathbf{w}^t\mathbf{w} = 1$, so

$$P^tP = I - 4\mathbf{w}\mathbf{w}^t + 4\mathbf{w}\mathbf{w}^t = I \quad \text{and} \quad P^t = P = P^{-1}.$$

Exercise Set 9.5 (Page 000)

1. Two iterations of the QR method produce the following matrices.

a.

$$A^{(3)} = \begin{bmatrix} 0.6939977 & -0.3759745 & 0.0 \\ -0.3759745 & 1.892417 & -0.03039696 \\ 0.0 & -0.03039696 & 3.413585 \end{bmatrix}$$

b.

$$A^{(3)} = \begin{bmatrix} 4.535466 & 1.212648 & 0.0 \\ 1.212648 & 3.533242 & 3.83 \times 10^{-7} \\ 0.0 & 3.83 \times 10^{-7} & -0.06870782 \end{bmatrix}$$

c.

$$A^{(3)} = \begin{bmatrix} 4.679567 & -0.2969009 & 0.0 \\ -2.969009 & 3.052484 & -1.207346 \times 10^{-5} \\ 0.0 & -1.207346 \times 10^{-5} & 1.267949 \end{bmatrix}$$

d.

$$A^{(3)} = \begin{bmatrix} 0.3862092 & 0.4423226 & 0.0 & 0.0 \\ 0.4423226 & 1.787694 & -0.3567744 & 0.0 \\ 0.0 & -0.3567744 & 3.080815 & 3.116382 \times 10^{-5} \\ 0.0 & 0.0 & 3.116382 \times 10^{-5} & 4.745281 \end{bmatrix}$$

e.

$$A^{(3)} = \begin{bmatrix} -2.826365 & 1.130297 & 0.0 & 0.0 \\ 1.130297 & -2.429647 & -0.1734156 & 0.0 \\ 0.0 & -0.1734156 & 0.8172086 & 1.863997 \times 10^{-9} \\ 0.0 & 0.0 & 1.863997 \times 10^{-9} & 3.438803 \end{bmatrix}$$

f.

$$A^{(3)} = \begin{bmatrix} 0.2763388 & 0.1454371 & 0.0 & 0.0 \\ 0.1454371 & 0.4543713 & 0.1020836 & 0.0 \\ 0.0 & 0.1020836 & 1.174446 & -4.36 \times 10^{-5} \\ 0.0 & 0.0 & -4.36 \times 10^{-5} & 0.9948441 \end{bmatrix}$$

3. The matrices in Exercise 1 have the following eigenvalues, accurate to within 10^{-5}.

a. 3.414214, 2.000000, 0.58578644

b. $-0.06870782, 5.346462, 2.722246$

c. 1.267949, 4.732051, 3.000000

d. 4.745281, 3.177283, 1.822717, 0.2547188

e. $3.438803, 0.8275517, -1.488068, -3.778287$

f. 0.9948440, 1.189091, 0.5238224, 0.1922421

5. a. Let

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and $\mathbf{y} = P\mathbf{x}$. Show that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2$. Then use the relationship $x_1 + ix_2 = re^{i\alpha}$, where $r = \|\mathbf{x}\|_2$ and $\alpha = \tan^{-1}(x_2/x_1)$, and $y_1 + iy_2 = re^{i(\alpha+\theta)}$.

b. Let $\mathbf{x} = (1, 0)^t$ and $\theta = \pi/4$.

7. Jacobi's method produces the following eigenvalues, accurate to within the tolerance:

a. 3.414214, 0.5857864, 2.0000000; 3 iterations

b. 2.722246, 5.346462, -0.06870782; 3 iterations

c. 4.732051, $3, 1.267949$; 3 iterations

d. 0.2547188, 1.822717, 3.177283, 4.745281; 3 iterations

e. -1.488068, -3.778287, 0.8275517, 3.438803; 3 iterations

f. 0.1922421, 1.189091, 0.5238224, 0.9948440; 3 iterations

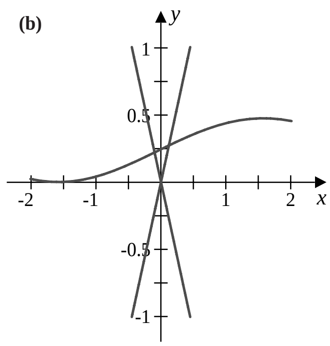
Exercise Set 10.2 (Page 000)

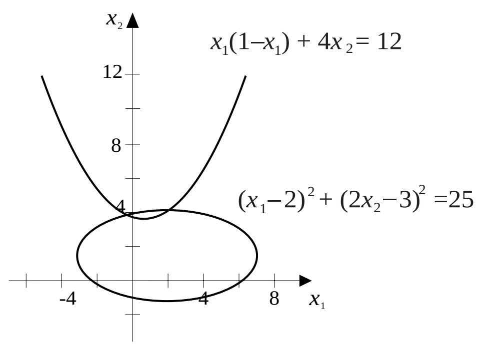
1. One example is $f(x_1, x_2) = \left(1, \dfrac{1}{|x_1 - 1| + |x_2|}\right)^t$.

3. a. $(-1, 3.5)^t$ and $(2.5, 4)^t$ **b.** $(0.11, 0.27)^t$ and $(-0.11, 0.23)^t$

c. $(1, 1, 1)^t$ **d.** $(1, -1, 1)^t$ and $(1, 1, -1)^t$

Note: New Figures





The graphs for parts (a) and (b) are shown with the approximate intersections. The three-dimensional graphs for parts (c) and (d) are not given since experimentation is needed in Maple to determine the approximate intersections.

5. a. With $\mathbf{x}^{(0)} = (-1, 3.5)^t$, $\mathbf{x}^{(1)} = (-1, 3.5)^t$, so $(-1, 3.5)^t$ is a solution. With $\mathbf{x}^{(0)} = (2.5, 4)^t$, $\mathbf{x}^{(3)} = (2.546947, 3.984998)^t$.

b. With $\mathbf{x}^{(0)} = (0.11, 0.27)^t$, $\mathbf{x}^{(6)} = (0.1212419, 0.2711051)^t$. With $\mathbf{x}^{(0)} = (-0.11, 0.23)^t$, $\mathbf{x}^{(4)} = (-0.09816344, 0.21950013)^t$.

c. With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, $\mathbf{x}^{(3)} = (1.036401, 1.085707, 0.9311914)^t$.

d. With $\mathbf{x}^{(0)} = (1, -1, 1)^t$, $\mathbf{x}^{(5)} = (0.9, -1, 0.5)^t$, and with $\mathbf{x}^{(0)} = (1, -1, 1)^t$, $\mathbf{x}^{(5)} = (0.5, 1, -0.5)^t$.

7. a. With $\mathbf{x}^{(0)} = (-0.5, -1, 1.5)^t$ we have $\mathbf{x}^{(5)} = (-0.66666667, -1.3333333, 1.3333333)^t$.

b. Adding the first two equations gives

$$4x_1 - 2x_2 = 0 \quad \text{so} \quad x_1 = \frac{x_2}{2}.$$

Subtracting the first two equations gives

$$-4x_2 + 2x_3 - 8 = 0 \quad \text{so} \quad x_3 = 2x_2 + 4.$$

c. Using the results of part (b) we have

$$\left(\frac{x_2}{2}\right)^2 + x_2^2 + (2x_2 + 4)^2 - 4 = 0 \quad \text{so} \quad 21x_2^2 + 64x_2 + 48 = 0.$$

d. The solutions to the quadratic equation in part (c) are $x_2 = -4/3$ and $x_2 = -12/7$.

e. The solution $x_2 = -4/3$ gives the complete solutions $(-2/3, -4/3, 4/3)^t$, and the solution $x_2 = -12/7$ gives the complete solutions $(-6/7, -12/7, 4/7)^t$. Thus we have

$$||(-2/3, -4/3, 4/3)^t - \mathbf{x}^{(0)}||_\infty = 0.16666667$$

and

$$||(-6/7, -12/7, 4/7)^t - \mathbf{x}^{(0)}||_\infty = 0.92857143,$$

so the initial approximation is closer to the solution $(-2/3, -4/3, 4/3)^t$.

9. a. Suppose $(x_1, x_2, x_3, x_4)^t$ is a solution to

$$\begin{aligned} 4x_1 - x_2 + x_3 &= x_1x_4, \\ -x_1 + 3x_2 - 2x_3 &= x_2x_4, \\ x_1 - 2x_2 + 3x_3 &= x_3x_4, \\ x_1^2 + x_2^2 + x_3^2 &= 1. \end{aligned}$$

Multiplying the first three equations by -1 and factoring gives

$$
\begin{aligned}
4(-x_1) - (-x_2) + (-x_3) =&(-x_1)x_4,\\
-(-x_1) + 3(-x_2) - 2(-x_3) =&(-x_2)x_4,\\
(-x_1) - 2(-x_2) + 3(-x_3) =&(-x_3)x_4,\\
(-x_1)^2 + (-x_2)^2 + (-x_3)^2 =&1.
\end{aligned}
$$

Thus, $(-x_1, -x_2, -x_3, x_4)^t$ is also a solution.

b. Using $\mathbf{x}^{(0)} = (1, 1, 1, 1)^t$ gives $\mathbf{x}^{(5)} = (0, 0.70710678, 0.70710678, 1)^t$.
Using $\mathbf{x}^{(0)} = (1, 0, 0, 0)^t$ gives $\mathbf{x}^{(6)} = (0.81649658, 0.40824829, -0.40824829, 3)^t$.
Using $\mathbf{x}^{(0)} = (1, -1, 1, -1)^t$ gives $\mathbf{x}^{(5)} = (0.57735027, -0.57735027, 0.57735027, 6)^t$.

The other three solutions, $(0, -0.70710678, -0.70710678, 1)^t$, $(-0.81649658, -0.40824829, 0.40824829, 3)^t$, and $(-0.57735027, 0.57735027, -0.57735027, 6)^t$ follow from part (a).

11. **a.** $k_1 = 8.77125, k_2 = 0.259690, k_3 = -1.37217$

b. Solving the equation $\frac{500}{\pi r^2} = k_1 e^{k_2 r} + k_3 r$ numerically gives $r = 3.18517$.

Exercise Set 10.3 (Page 000)

1. **a.** $\mathbf{x}^{(2)} = (0.4777920, 1.927557)^t$

b. $\mathbf{x}^{(2)} = (-0.3250070, -0.1386967)^t$

c. $\mathbf{x}^{(2)} = (0.5115893, -78.72872, -0.5120771)^t$

d. $\mathbf{x}^{(2)} = (-67.00583, 38.31494, 31.69089)^t$

3. **a.** $\mathbf{x}^{(9)} = (0.5, 0.8660254)^t$

b. $\mathbf{x}^{(8)} = (1.772454, 1.772454)^t$

c. $\mathbf{x}^{(9)} = (-1.456043, -1.664231, 0.4224934)^t$

d. $\mathbf{x}^{(5)} = (0.4981447, -0.1996059, -0.5288260)^t$

5. Using $\mathbf{x}^{(0)} = (1,1,1,1)^t$ gives $\mathbf{x}^{(6)} = (0, 0.70710678, 0.70710678, 1)^t$.
Using $\mathbf{x}^{(0)} = (1,0,0,0)^t$ gives $\mathbf{x}^{(15)} = (0.81649659, 0.40824821, -0.40824837, 3.00000004)^t$.

Using $\mathbf{x}^{(0)} = (1,-1,1,-1)^t$ gives $\mathbf{x}^{(11)} = (0.57735034, -0.57735022, 0.57735024, 6)^t$.
The other three solutions are $(0, -0.70710678, -0.70710678, 1)^t$,
$(-0.81649659, -0.40824821, 0.40824837, 3)^t$, and $(-0.57735034, 0.57735022, -0.57735024, 6)^t$.

7. We have

$$\begin{aligned}
\left[A^{-1} - \frac{A^{-1}\mathbf{x}\mathbf{y}^tA^{-1}}{1+\mathbf{y}^tA^{-1}\mathbf{x}}\right](A+\mathbf{x}\mathbf{y}^t) =& A^{-1}A - \frac{A^{-1}\mathbf{x}\mathbf{y}^tA^{-1}A}{1+\mathbf{y}^tA^{-1}\mathbf{x}} + A^{-1}\mathbf{x}\mathbf{y}^t - \frac{A^{-1}\mathbf{x}\mathbf{y}^tA^{-1}\mathbf{x}\mathbf{y}^t}{1+\mathbf{y}^tA^{-1}\mathbf{x}} \\
=& I - \frac{A^{-1}\mathbf{x}\mathbf{y}^t}{1+\mathbf{y}^tA^{-1}\mathbf{x}} + A^{-1}\mathbf{x}\mathbf{y}^t - \frac{A^{-1}\mathbf{x}\mathbf{y}^tA^{-1}\mathbf{x}\mathbf{y}^t}{1+\mathbf{y}^tA^{-1}\mathbf{x}} \\
=& I - \frac{A^{-1}\mathbf{x}\mathbf{y}^t - A^{-1}\mathbf{x}\mathbf{y}^t - \mathbf{y}^tA^{-1}\mathbf{x}A^{-1}\mathbf{x}\mathbf{y}^t + A^{-1}\mathbf{x}\mathbf{y}^tA^{-1}\mathbf{x}\mathbf{y}^t}{1+\mathbf{y}^tA^{-1}\mathbf{x}} \\
=& I + \frac{\mathbf{y}^tA^{-1}\mathbf{x}A^{-1}\mathbf{x}\mathbf{y}^t - \mathbf{y}^tA^{-1}\mathbf{x}(A^{-1}\mathbf{x}\mathbf{y}^t)}{1+\mathbf{y}^tA^{-1}\mathbf{x}} = I.
\end{aligned}$$

Exercise Set 10.4 (Page 000)

1. a. With $\mathbf{x}^{(0)} = (0,0)^t$, we have $\mathbf{x}^{(11)} = (0.4943541, 1.948040)^t$.

b. With $\mathbf{x}^{(0)} = (1,1)^t$, we have $\mathbf{x}^{(2)} = (0.4970073, 0.8644143)^t$.

c. With $\mathbf{x}^{(0)} = (2,2)^t$, we have $\mathbf{x}^{(1)} = (1.736083, 1.804428)^t$.

d. With $\mathbf{x}^{(0)} = (0,0)^t$, we have $\mathbf{x}^{(2)} = (-0.3610092, 0.05788368)^t$.

3. a. With $\mathbf{x}^{(0)} = (0,0,0)^t$, we have $\mathbf{x}^{(14)} = (1.043605, 1.064058, 0.9246118)^t$.

b. With $\mathbf{x}^{(0)} = (0,0,0)^t$, we have $\mathbf{x}^{(9)} = (0.4932739, 0.9863888, -0.5175964)^t$.

c. With $\mathbf{x}^{(0)} = (0,0,0)^t$, we have $\mathbf{x}^{(11)} = (-1.608296, -1.192750, 0.7205642)^t$.

d. With $\mathbf{x}^{(0)} = (0,0,0)^t$, we have $\mathbf{x}^{(1)} = (0, 0.00989056, 0.9890556)^t$.

5. a. With $\mathbf{x}^{(0)} = (0,0)^t$, we have $\mathbf{x}^{(8)} = (3.136548, 0)^t$ and $\mathbf{g}(\mathbf{x}^{(8)}) = 0.005057848$.

b. With $\mathbf{x}^{(0)} = (0,0)^t$, we have $\mathbf{x}^{(13)} = (0.6157412, 0.3768953)^t$ and $\mathbf{g}(\mathbf{x}^{(13)}) = 0.1481574$.

c. With $\mathbf{x}^{(0)} = (0,0,0)^t$, we have $\mathbf{x}^{(5)} = (-0.6633785, 0.3145720, 0.5000740)^t$ and $\mathbf{g}(\mathbf{x}^{(5)}) = 0.6921548$.

d. With $\mathbf{x}^{(0)} = (1,1,1)^t$, we have $\mathbf{x}^{(4)} = (0.04022273, 0.01592477, 0.01594401)^t$ and $\mathbf{g}(\mathbf{x}^{(4)}) = 1.010003$.

Exercise Set 10.5 (Page 000)

Note: All the material in this section is new

1. a. $(3, -2.25)^t$

b. $(0.42105263, 2.6184211)^t$

c. $(2.173110, -1.3627731)^t$

3. Using $\mathbf{x}(0) = \mathbf{0}$ in all parts gives:

a. $(0.44006047, 1.8279835)^t$

b. $(-0.41342613, 0.096669468)^t$

c. $(0.49858909, 0.24999091, -0.52067978)^t$

d. $(6.1935484, 18.532258, -21.725806)^t$

5. **a.** Using $\mathbf{x}(0) = (-1, 3.5)^t$ gives $(-1, 3.5)^t$.

Using $\mathbf{x}(0) = (2.5, 4.0)^t$ gives $(2.5469465, 3.9849975)^t$.

b. Using $\mathbf{x}(0) = (0.11, 0.27)^t$ gives $(0.12124195, 0.27110516)^t$.

c. Using $\mathbf{x}(0) = (1, 1, 1)^t$ gives $(1.0364005, 1.0857066, 0.93119144)^t$.

d. Using $\mathbf{x}(0) = (1, -1, 1)^t$ gives $(0.90016074, -1.0023801, 0.49661093)^t$.

Using $\mathbf{x}(0) = (1, 1, -1)^t$ gives $(0.50104035, 1.0023801, -0.49661093)^t$.

7. **a.** $(0.49998949, 0.86608576)^t$

b. $(1.7724820, 1.7722940)^t$

c. $(-1.4561027, -1.6642463, 0.42241506)^t$

d. $(0.49814392, -0.19960453, -0.52882611)^t$

9. $(0.50024553, 0.078230039, -0.52156996)^t$

Exercise Set 11.2 (Page 000)

1. The Linear Shooting method gives the following results.

a.

i	x_i	w_{1i}	$y(x_i)$
1	0.5	0.82432432	0.82402714

b.

i	x_i	w_{1i}	$y(x_i)$
1	0.25	0.3937095	0.3936767
2	0.50	0.8240948	0.8240271
3	0.75	1.337160	1.337086

3. The Linear Shooting method gives the following results.

a.

i	x_i	w_{1i}	$y(x_i)$
3	0.3	0.7833204	0.7831923
6	0.6	0.6023521	0.6022801
9	0.9	0.8568906	0.8568760

b.

i	x_i	w_{1i}	$y(x_i)$
5	1.25	0.1676179	0.1676243
10	1.50	0.4581901	0.4581935
15	1.75	0.6077718	0.6077740

c.

i	x_i	w_{1i}	$y(x_i)$
3	0.3	−0.5185754	−0.5185728
6	0.6	−0.2195271	−0.2195247
9	0.9	−0.0406577	−0.0406570

d.

i	x_i	w_{1i}	$y(x_i)$
3	1.3	0.0655336	0.06553420
6	1.6	0.0774590	0.07745947
9	1.9	0.0305619	0.03056208

5. **a.** The Linear Shooting method with $h = 0.1$ gives the following results.

i	x_i	w_{1i}
3	0.3	0.05273437
5	0.5	0.00741571
8	0.8	0.00038976

b. The Linear Shooting method with $h = 0.05$ gives the following results.

i	x_i	w_{1i}
6	0.3	0.04990547
10	0.5	0.00673795
16	0.8	0.00033755

7. **a.** The approximate potential is $u(3) \approx 36.66702$ using $h = 0.1$.

b. The actual potential is $u(3) = 36.66667$.

9. **a.** There are no solutions if b is an integer multiple of π and $B \neq 0$.

b. A unique solution exists whenever b is not an integer multiple of π.

c. There are infinitely many solutions if b is an multiple integer of π and $B = 0$.

Exercise Set 11.3 (Page 000)

1. The Linear Finite-Difference method gives the following results.

a.

i	x_i	w_{1i}	$y(x_i)$
1	0.5	0.83333333	0.82402714

b.

i	x_i	w_{1i}	$y(x_i)$
1	0.25	0.39512472	0.39367669
2	0.50	0.82653061	0.82402714
3	0.75	1.33956916	1.33708613

c.

$$\frac{4(0.82653061) - 0.83333333}{3} = 0.82426304$$

3. The Linear Finite-Difference method gives the following results.

a.

i	x_i	w_i	$y(x_i)$
2	0.2	1.018096	1.0221404
5	0.5	0.5942743	0.59713617
7	0.7	0.6514520	0.65290384

b.

i	x_i	w_i	$y(x_i)$
5	1.25	0.16797186	0.16762427
10	1.50	0.45842388	0.45819349
15	1.75	0.60787334	0.60777401

c.

i	x_i	w_{1i}	$y(x_i)$
3	0.3	−0.5183084	−0.5185728
6	0.6	−0.2192657	−0.2195247
9	0.9	−0.0405748	−0.04065697

d.

i	x_i	w_{1i}	$y(x_i)$
3	1.3	0.0654387	0.0655342
6	1.6	0.0773936	0.0774595
9	1.9	0.0305465	0.0305621

5. The Linear Finite-Difference method gives the following results.

i	x_i	$w_i(h = 0.1)$
3	0.3	0.05572807
6	0.6	0.00310518
9	0.9	0.00016516

i	x_i	$w_i(h = 0.05)$
6	0.3	0.05132396
12	0.6	0.00263406
18	0.9	0.00013340

7.

i	x_i	w_i
10	10.0	0.1098549
20	20.0	0.1761424
25	25.0	0.1849608
30	30.0	0.1761424
40	40.0	0.1098549

Exercise Set 11.4 (Page 000)

1. The Nonlinear Shooting method gives $w_1 = 0.405505 \approx \ln 1.5 = 0.405465$.

3. The Nonlinear Shooting method gives the following results.

a. 4 iterations required:

i	x_i	w_{1i}	$y(x_i)$
3	1.3	0.4347934	0.4347826
6	1.6	0.3846363	0.3846154
9	1.9	0.3448586	0.3448276

b.

6 iterations required:

i	x_i	w_{1i}	$y(x_i)$
3	1.3	2.069249	2.069231
6	1.6	2.225013	2.225000
9	1.9	2.426317	2.426316

c.

3 iterations required:

i	x_i	w_{1i}	$y(x_i)$
3	2.3	1.2676912	1.2676917
6	2.6	1.3401256	1.3401268
9	2.9	1.4095359	1.4095383

d.

7 iterations required:

i	x_i	w_{1i}	$y(x_i)$
5	1.25	0.4358290	0.4358272
10	1.50	1.3684496	1.3684447
15	1.75	2.9992010	2.9991909

5.

i	x_i	$w_{1i} \approx y(t_i)$	w_{2i}
3	0.6	0.71682963	0.92122169
5	1.0	1.00884285	0.53467944
8	1.6	1.13844628	-0.11915193

Exercise Set 11.5 (Page 000)

1. The Nonlinear Finite-Difference method gives $w_1 = 0.4067967 \approx \ln 1.5 = 0.4054651$.

3. The Nonlinear Finite-Difference method gives the following results.

a.

i	x_i	w_{1i}	$y(x_i)$
3	1.3	0.4347972	0.4347826
6	1.6	0.3846286	0.3846154
9	1.9	0.3448316	0.3448276

b.

i	x_i	w_{1i}	$y(x_i)$
3	1.3	2.0694081	2.0692308
6	1.6	2.2250937	2.2250000
9	1.9	2.4263387	2.4263158

c.

i	x_i	w_{1i}	$y(x_i)$
3	2.3	1.2677078	1.2676917
6	2.6	1.3401418	1.3401268
9	2.9	1.4095432	1.4095383

d.

i	x_i	w_{1i}	$y(x_i)$
5	1.25	0.4345979	0.4358273
10	1.50	1.3662119	1.3684447
15	1.75	2.9969339	2.9991909

5.

i	x_i	w_i
5	30	0.01028080
10	60	0.01442767
15	90	0.01028080

Exercise Set 11.6 (Page 000)

1. The Piecewise Linear method gives $\phi(x) = -0.07713274\phi_1(x) - 0.07442678\phi_2(x)$. This gives $\phi(x_1) = -0.07713274$ and $\phi(x_2) = -0.07442678$. The actual values are $y(x_1) = -0.07988545$ and $y(x_2) = -0.07712903$.

3. The Piecewise Linear method gives the following results.

a.

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	−0.212333	−0.21
6	0.6	−0.241333	−0.24
9	0.9	−0.090333	−0.09

b.

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	0.1815138	0.1814273
6	0.6	0.1805502	0.1804754
9	0.9	0.05936468	0.05934303

c.

i	x_i	$\phi(x_i)$	$y(x_i)$
5	0.25	−0.3585989	−0.3585641
10	0.50	−0.5348383	−0.5347803
15	0.75	−0.4510165	−0.4509614

d.

i	x_i	$\phi(x_i)$	$y(x_i)$
5	0.25	−0.1846134	−0.1845204
10	0.50	−0.2737099	−0.2735857
15	0.75	−0.2285169	−0.2284204

5. The Cubic Spline method gives the following results.

a.

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	−0.2100000	−0.21
6	0.6	−0.2400000	−0.24
9	0.9	−0.0900000	−0.09

b.

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	0.1814269	0.1814273
6	0.6	0.1804753	0.1804754
9	0.9	0.05934321	0.05934303

c.

i	x_i	$\phi(x_i)$	$y(x_i)$
5	0.25	−0.3585639	−0.3585641
10	0.50	−0.5347779	−0.5347803
15	0.75	−0.4509109	−0.4509614

d.

i	x_i	$\phi(x_i)$	$y(x_i)$
5	0.25	−0.1845191	−0.1845204
10	0.50	−0.2735833	−0.2735857
15	0.75	−0.2284186	−0.2284204

7.

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	1.0408182	1.0408182
6	0.6	1.1065307	1.1065306
9	0.9	1.3065697	1.3065697

9. A change in variable $w = (x - a)/(b - a)$ gives the boundary value problem

$$\begin{aligned} &-\frac{d}{dw}(p((b-a)w+a)y') + (b-a)^2 q((b-a)w+a)y \\ &= (b-a)^2 f((b-a)w+a), \end{aligned}$$

where $0 < w < 1$, $y(0) = \alpha$, and $y(1) = \beta$. Then Exercise 6 can be used.

11. Let $\mathbf{c} = (c_1, \ldots, c_n)^t$ be any vector and let $\phi(x) = \sum_{j=1}^{n} c_j \phi_j(x)$. Then

$$\begin{aligned}
\mathbf{c}^t A\mathbf{c} &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} c_i c_j = \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} a_{ij} c_i c_j \\
&= \sum_{i=1}^{n} \Bigg[\int_0^1 \{p(x) c_i \phi_i'(x) c_{i-1} \phi_{i-1}'(x) + q(x) c_i \phi_i(x) c_{i-1} \phi_{i-1}(x)\}\, dx \\
&\quad + \int_0^1 \{p(x) c_i^2 [\phi_i'(x)]^2 + q(x) c_i^2 [\phi_i'(x)]^2\}\, dx \\
&\quad + \int_0^1 \{p(x) c_i \phi_i'(x) c_{i+1} \phi_{i+1}'(x) + q(x) c_i \phi_i(x) c_{i+1} \phi_{i+1}(x)\}\, dx \Bigg] \\
&= \int_0^1 \{p(x)[\phi'(x)]^2 + q(x)[\phi(x)]^2\}\, dx.
\end{aligned}$$

So $\mathbf{c}^t A\mathbf{c} \geq 0$ with equality only if $\mathbf{c} = \mathbf{0}$. Since A is also symmetric, A is positive definite.

Exercise Set 12.2 (Page 000)

1. The Poisson Equation Finite-Difference method gives the following results.

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
1	1	0.5	0.5	0.0	0
1	2	0.5	1.0	0.25	0.25
1	3	0.5	1.5	1.0	1

3. The Poisson Equation Finite-Difference method gives the following results.

a. 30 iterations required:

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
2	2	0.4	0.4	0.1599988	0.16
2	4	0.4	0.8	0.3199988	0.32
4	2	0.8	0.4	0.3199995	0.32
4	4	0.8	0.8	0.6399996	0.64

b. 29 iterations required:

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
2	1	1.256637	0.3141593	0.2951855	0.2938926
2	3	1.256637	0.9424778	0.1830822	0.1816356
4	1	2.513274	0.3141593	−0.7721948	−0.7694209
4	3	2.513274	0.9424778	−0.4785169	−0.4755283

c. 126 iterations required:

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
4	3	0.8	0.3	1.2714468	1.2712492
4	7	0.8	0.7	1.7509419	1.7506725
8	3	1.6	0.3	1.6167917	1.6160744
8	7	1.6	0.7	3.0659184	3.0648542

d. 127 iterations required:

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
2	2	1.2	1.2	0.5251533	0.5250861
4	4	1.4	1.4	1.3190830	1.3189712
6	6	1.6	1.6	2.4065150	2.4064186
8	8	1.8	1.8	3.8088995	3.8088576

5. The approximate potential at some typical points is given in the following table.

i	j	x_i	y_j	$w_{i,j}$
1	4	0.1	0.4	88
2	1	0.2	0.1	66
4	2	0.4	0.2	66

Exercise Set 12.3 (Page 000)

1. The Heat Equation Backward-Difference method gives the following results.

a.

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
1	1	0.5	0.05	0.632952	0.652037
2	1	1.0	0.05	0.895129	0.883937
3	1	1.5	0.05	0.632952	0.625037
1	2	0.5	0.1	0.566574	0.552493
2	2	1.0	0.1	0.801256	0.781344
3	2	1.5	0.1	0.566574	0.552493

b.

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
1	1	1/3	0.05	1.59728	1.53102
2	1	2/3	0.05	−1.59728	−1.53102
1	2	1/3	0.1	1.47300	1.35333
2	2	2/3	0.1	−1.47300	−1.35333

3. The Forward-Difference method gives the following results.

a.

For $h = 0.4$ and $k = 0.1$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
2	5	0.8	0.5	3.035630	0
3	5	1.2	0.5	−3.035630	0
4	5	1.6	0.5	1.876122	0

For $h = 0.4$ and $k = 0.05$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
2	10	0.8	0.5	0	0
3	10	1.2	0.5	0	0
4	10	1.6	0.5	0	0

b.

For $h = \frac{\pi}{10}$ and $k = 0.05$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
3	10	0.94247780	0.5	0.4864823	0.4906936
6	10	1.88495559	0.5	0.5718943	0.5768449
9	10	2.82743339	0.5	0.1858197	0.1874283

c.

For $h = 0.2$ and $k = 0.04$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
4	10	0.8	0.4	1.166149	1.169362
8	10	1.6	0.4	1.252413	1.254556
12	10	2.4	0.4	0.4681813	0.4665473
16	10	3.2	0.4	−0.1027637	−0.1056622

d.

For $h = 0.1$ and $k = 0.04$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
3	10	0.3	0.4	0.5397009	0.5423003
6	10	0.6	0.4	0.6344565	0.6375122
9	10	0.9	0.4	0.2061474	0.2071403

5. The Crank-Nicolson method gives the following results.

a.

For $h = 0.4$ and $k = 0.1$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
2	5	0.8	0.5	8.2×10^{-7}	0
3	5	1.2	0.5	-8.2×10^{-7}	0
4	5	1.6	0.5	5.1×10^{-7}	0

For $h = 0.4$ and $k = 0.05$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i,t_j)$
2	10	0.8	0.5	-2.6×10^{-6}	0
3	10	1.2	0.5	2.6×10^{-6}	0
4	10	1.6	0.5	-1.6×10^{-6}	0

b.

For $h = \frac{\pi}{10}$ and $k = 0.05$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i,t_j)$
3	10	0.94247780	0.5	0.4926589	0.4906936
6	10	1.88495559	0.5	0.5791553	0.5768449
9	10	2.82743339	0.5	0.1881790	0.1874283

c.

For $h = 0.2$ and $k = 0.04$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i,t_j)$
4	10	0.8	0.4	1.171532	1.169362
8	10	1.6	0.4	1.256005	1.254556
12	10	2.4	0.4	0.4654499	0.4665473
16	10	3.2	0.4	−0.1076139	−0.1056622

d.

For $h = 0.1$ and $k = 0.04$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i,t_j)$
3	10	0.3	0.4	0.5440532	0.5423003
6	10	0.6	0.4	0.6395728	0.6375122
9	10	0.9	0.4	0.2078098	0.2071403

7. For the Modified Backward-Difference method, we have

i	j	x_i	t_j	w_{ij}
3	25	0.3	0.25	0.2883460
5	25	0.5	0.25	0.3468410
8	25	0.8	0.25	0.2169217

9. For the Modified Backward-Difference method, we have

i	j	x_i	t_j	w_{ij}(Backward-Difference)
2	10	0.3	0.225	1.207730
5	10	0.75	0.225	1.836564
9	10	1.35	0.225	0.6928342

Exercise Set 12.4 (Page 000)

1. The Wave Equation Finite-Difference method gives the following results.

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	4	0.25	1.0	-0.7071068	-0.7071068
3	4	0.50	1.0	-1.0000000	-1.0000000
4	4	0.75	1.0	-0.7071068	-0.7071068

3. a. The Finite-Difference method with $h = \frac{\pi}{10}$ and $k = 0.05$ gives the following results.

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	10	$\frac{\pi}{5}$	0.5	0.5163933	0.5158301
5	10	$\frac{\pi}{2}$	0.5	0.8785407	0.8775826
8	10	$\frac{4\pi}{5}$	0.5	0.5163933	0.5158301

b. The Finite-Difference method with $h = \frac{\pi}{20}$ and $k = 0.1$ gives the following results.

i	j	x_i	t_j	w_{ij}
4	5	$\frac{\pi}{5}$	0.5	0.5159163
10	5	$\frac{\pi}{2}$	0.5	0.8777292
16	5	$\frac{4\pi}{5}$	0.5	0.5159163

c. The Finite-Difference method with $h = \frac{\pi}{20}$ and $k = 0.05$ gives the following results.

i	j	x_i	t_j	w_{ij}
4	10	0.62831853	0.5	0.5159602
10	10	1.57079633	0.5	0.8778039
16	10	2.51327412	0.5	0.5159602

5. The Finite-Difference method gives the following results.

i	j	x_i	t_j	w_{ij}
2	5	0.2	0.5	-1
5	5	0.5	0.5	0
8	5	0.8	0.5	1

7. Approximate voltages and currents are given in the following table.

i	j	x_i	t_j	Voltage	Current
5	2	50	0.2	77.769	3.88845
12	2	120	0.2	104.60	-1.69931
18	2	180	0.2	33.986	-5.22995
5	5	50	0.5	77.702	3.88510
12	5	120	0.5	104.51	-1.69785
18	5	180	0.5	33.957	-5.22453

Exercise Set 12.5 (Page 000)

1. With $E_1 = (0.25, 0.75)$, $E_2 = (0, 1)$, $E_3 = (0.5, 0.5)$, and $E_4 = (0, 0.5)$, the basis functions are

$$\phi_1(x,y) = \begin{cases} 4x & \text{on } T_1 \\ -2+4y & \text{on } T_2 \end{cases}$$

$$\phi_2(x,y) = \begin{cases} -1-2x+2y & \text{on } T_1 \\ 0 & \text{on } T_2 \end{cases}$$

$$\phi_3(x,y) = \begin{cases} 0 & \text{on } T_1 \\ 1+2x-2y & \text{on } T_2 \end{cases}$$

$$\phi_4(x,y) = \begin{cases} 2-2x-2y & \text{on } T_1 \\ 2-2x-2y & \text{on } T_2 \end{cases}$$

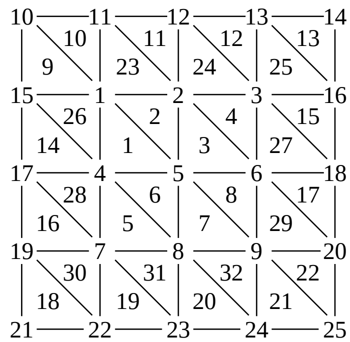
and $\gamma_1 = 0.323825, \gamma_2 = 0, \gamma_3 = 1.0000,$ and $\gamma_4 = 0.$

3. The Finite-Element method with $K = 8, N = 8, M = 32, n = 9, m = 25,$ and $NL = 0$ gives the following results.

$$\begin{array}{ll} \gamma_1 = 0.511023 & \gamma_2 = 0.720476 \\ \gamma_3 = 0.507899 & \gamma_4 = 0.720476 \\ \gamma_5 = 1.01885 & \gamma_6 = 0.720476 \\ \gamma_7 = 0.507896 & \gamma_8 = 0.720476 \\ \gamma_9 = 0.511023 & \gamma_i = 0, \quad 10 \le i \le 25 \end{array}$$

$$u(0.125, 0.125) \approx 0.614187, \qquad u(0.125, 0.25) \approx 0.690343,$$

$$u(0.25, 0.125) \approx 0.690343 \quad \text{and} \quad u(0.25, 0.25) \approx 0.720476.$$



5. The Finite-Element method with $K = 0, N = 12, M = 32, n = 20, m = 27$, and $NL = 14$ gives the following results.

γ_1=21.40335, γ_8 =24.19855, γ_{15}=20.23334, γ_{22}=15,

γ_2=19.87372, γ_9 =24.16799, γ_{16}=20.50056, γ_{23}=15,

γ_3=19.10019, γ_{10}=27.55237, γ_{17}=21.35070, γ_{24}=15,

γ_4=18.85895, γ_{11}=25.11508 γ_{18} =22.84663, γ_{25}=15,

γ_5=19.08533, γ_{12}=22.92824, γ_{19}=24.98178, γ_{26}=15,

γ_6=19.84115, γ_{13}=21.39741, γ_{20}=27.41907, γ_{27}=15,

γ_7=21.34694, γ_{14}=20.52179, γ_{21}=15.

$$u(1,0) \approx 22.92824, \quad u(4,0) \approx 22.84663, \quad u\left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right) \approx 18.85895.$$