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Article · March 2019

DOI: 10.1080/09720510.2018.1556409

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Taylor & Francis
Taylor & Francis Group

Journal of Statistics and Management Systems

ISSN: 0972-0510 (Print) 2169-0014 (Online) Journal homepage: https://www.tandfonline.com/loi/tsms20

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To cite this article: V. O. Akinsola & T. O. Oluyo (2019): Mathematical analysis with numerical solutions of the mathematical model for the complications and control of *diabetes mellitus*, Journal of Statistics and Management Systems, DOI: 10.1080/09720510.2018.1556409

To link to this article: https://doi.org/10.1080/09720510.2018.1556409

Published online: 22 Mar 2019.

Journal of Statistics & Management Systems
ISSN 0972-0510 (Print), ISSN 2169-0014 (Online)
DOI : 10.1080/09720510.2018.1556409

Mathematical analysis with numerical solutions of the mathematical model for the complications and control of *diabetes mellitus*

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Abstract

Introduction

A model is an abstraction that reduces a problem to its essential characteristics. Mathematical models are useful because they exemplify the mathematical core of a situation without extraneous information. Models help to explain a system and to study the effects of different components, and to make predictions about behaviour. Analysis of model via computational and applied mathematical methods are ways to deduce the consequences of the interactions. It is the analysis of mathematical models that allows us to formalize the cause and effect process and tie it to the biological observations. Furthermore, model analysis yields insights into why a system behaves the way it does, thus providing links between network structure and behaviour.

Methodology

Stability natures of the critical points of the models at various values of the model parameters were investigated to determine the behaviour of the model solution. Eigenvalue sensitivity and Eigenvalue elasticity analyses were carried out to identify key parameters of the model which drive the solutions and to figure out the effect of proportional changes in

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parameter values on population growth of both diabetics with complications and diabetics with and without complications. Mathematical algorithms were coded in MATLAB computational environment to achieve these. The numerical solutions of the model at various values of the parameters were performed using Euler method and Runge-Kutta method of order four and compared with the analytic solution. The algorithms were coded with Maple software package.

Result

The stability analysis showed that the models were asymptotically stable at specified parameter values hence suitable for their intended purposes. The Eigenvalue sensitivity and Eigenvalue elasticity analyses showed the rate at which complications were controlled is the most important parameter of the model hence the policy lever for effective control of the size of diabetics with complications. The solutions were represented graphically at various values of prominent parameters.

Conclusion

The population of diabetics will continue to increase for the time being, but the size of diabetics with complications can reduce drastically with comprehensive and concurrent treatment of *Diabetes mellitus* and its complications. Also with high rate of controlling complications of *Diabetes mellitus* and low probability of developing complications of the disease through interventions such as continuous education, reorientation, increase physical activities, balance nutrition, government and non-governmental support, the incidence of the disease reduce drastically.

Subject Classification: *(2010) 93A30, 65L20, 93B35, 34A30*

Keywords: *Mathematical model, Stability, Sensitivity analysis, Numerical solution, Diabetes mellitus.*

1. Introduction

Mathematical modeling is the process of using mathematical concepts like equations and graphs to represent real life situations. The model is an abstraction that reduces a problem to its essential characteristics. Models are designed to focus on certain aspects of the object of study; other aspects are abstracted away. Mathematical models are useful because they exemplify the mathematical core of a situation without extraneous information. Mathematical modeling is a powerful tool for understanding biologically observed phenomena which cannot be understood by verbal reasoning alone. The field of Mathematical Biology focus on interdisciplinary scientific problems in quantitative life sciences where Mathematical models are used to describe interaction between biological components. The act of constructing a model demands a critical consideration of the mechanisms that underlie a biological process. A model may help to explain a system and to study the effects of different components, and to make predictions about behaviour. The more direct approach is model simulation, in which the model is used to predict system behaviour (under given conditions). Simulations are carried out by numerical methods with software packages (Sampath and Wanbiao, 2005). Although model simulations will never replace laboratory experiments, a model allows one to probe system behaviour in ways that would not be possible in the laboratory. Simulations can be carried out quickly and incur no real cost. Every aspect of model behaviour can be observed at all time-points. Simulations can serve as valuable guides to experimental design, by indicating promising avenues for investigation, or by revealing inconsistencies between the understanding of a system (embodied in the model) and laboratory observations. Alternatively, models can be investigated directly, yielding general insight into their potential behaviour. These model analysis approaches sometimes involve sophisticated mathematical techniques. The pay-off for mastering these techniques is an insight into system behaviour that cannot be reached through simulation. While simulations indicate how a system behaves, model analysis reveals why a system behaves as it does. This analysis can reveal non-intuitive connections between the structure of a system and its consequent behaviour.

Analysis of the model via computational and applied mathematical methods are ways to deduce the consequences of the interactions (Gerda *et al.,* 2016). It is the analysis of mathematical models that allows us to formalize the cause and effect process and tie it to the biological observations. Furthermore, model analysis yields insights into why a system behaves the way it does, thus providing links between network structure and behaviour. In real world applications, there is not just one model that effectively describes a process but many possible models (Brian Ingalls, 2012). Since a model is a hypothesis, the results of model investigation are themselves hypotheses.

In our previous paper, Akinsola and Oluyo (2014), the work of Bouyayeb *et al.,* (2004) and Ibrahim *et al.,* (2011) were modified to obtain the proposed mathematical model of systems of differential equations for the complications and control of *Diabetes mellitus* with initial condition as

$$C'(t) = I - (\rho + \theta)C(t) + \rho N(t), \qquad C(0) = C_0 \tag{1}$$

$$N'(t) = 2I - (\nu + \delta)C(t) - \mu N(t), \qquad N(0) = N_0 \tag{2}$$

Which is represented thus in matrix form as

$$\begin{pmatrix} C(t) \\ N(t) \end{pmatrix}' = \begin{pmatrix} -(\rho+\theta) & \rho \\ -(\nu+\delta) & -\mu \end{pmatrix}\begin{pmatrix} C(t) \\ N(t) \end{pmatrix} + \begin{pmatrix} I \\ 2I \end{pmatrix}, \begin{pmatrix} C_0 \\ N_0 \end{pmatrix} \tag{3}$$

The analytic solution of the resulting system of equation obtained using elimination method with undetermined coefficient in Akinsola and Oluyo (2014) was

$$C(t) = K_1 e^{-\eta_1 t} + K_2 e^{-\eta_2 t} + \frac{\alpha}{\beta} I \tag{4}$$

$$\begin{aligned} N(t) = {} & K_1 e^{-\eta_1 t} + K_2 e^{-\eta_2 t} + \frac{\alpha}{\beta} I + \frac{\theta}{\rho} K_1 e^{-\eta_1 t} + \frac{\theta}{\rho} K_2 e^{-\eta_2 t} \\ & + \frac{\theta\alpha}{\pi\beta} I - \frac{I}{\rho} - \frac{I}{\rho}(\eta_1 K_1 e^{-\eta_1 t} + \eta_2 K_2 e^{-\eta_2 t}) \end{aligned} \tag{5}$$

ρ $\frac{1}{\rho}$

Where

$$\theta = \gamma + \mu + \nu + \delta \tag{6}$$

$$\eta_1 = \frac{1}{2}\left(\sigma - \sqrt{\sigma^2 - 4\beta}\right) \tag{7}$$

$$\eta_2 = \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 - 4\beta}\right) \tag{8}$$

$$\sigma = \rho + \theta + \mu \tag{9}$$

$$\beta = \rho\nu + \rho\delta + \rho\mu + \mu\theta \tag{10}$$

$$\alpha = 2\rho + \mu \tag{11}$$

$$K_1 = \frac{\beta(\rho + \theta - \eta_2)C_0 + I(\alpha\eta_2 - \beta) - \rho\beta N_0}{\beta(\eta_1 - \eta_2)} \tag{12}$$

$$K_2 = \frac{-\beta(\rho + \theta - \eta_1)C_0 + I(\beta - \alpha\eta_1) + \rho\beta N_0}{\beta(\eta_1 - \eta_2)} \tag{13}$$

The critical points of the model were found to be

$$C^*(t) = \frac{(2\rho + \mu)I}{\nu\rho + \mu\theta + \rho\delta + \rho\mu} \tag{14}$$

$$N^*(t) = \frac{(2(\rho+\theta)-(\mu+\delta))I}{v\rho+\mu\theta+\rho\delta+\rho\mu} \tag{15}$$

The characteristic equation was given as

$$\lambda^2+(\rho+\theta+\mu)\lambda+(\rho v+\rho\delta+\rho\mu+\mu\theta) = 0 \tag{16}$$

Applying Equation (9) and Equation (10) into Equation (16) yields

$$\lambda^2+\sigma\lambda+\beta = 0 \tag{17}$$

The roots of the quadratic Equation (17) give the Eigenvalues

$$\lambda_1 = \frac{-\sigma+\sqrt{\sigma^2-4\beta}}{2} \tag{18}$$

$$\lambda_2 = \frac{-\sigma-\sqrt{\sigma^2-4\beta}}{2} \tag{19}$$

Applying Equation (6) and (7) in Equation (17) and Equation (18) to obtain

$$\lambda_1 = -\eta_1 \tag{20}$$

$$\lambda_2 = -\eta_2 \tag{21}$$

The analyses and numerical solutions of the model were established using the parameter values as given in Table 1 except where stated otherwise. It should be noted that the definition used as the rate at which complications are cured in Akinsola and Oluyo (2014), Bouyayeb *et al.*, (2004) and Ibrahim *et al.*, (2011) has been re-defined as the rate at which complications are controlled due to the understanding that majority of chronic complications which are common and many are only controllable not curable unlike the acute ones which are medical emergencies.

Methodology

The model was analyzed quantitatively and qualitatively via stability, sensitivity and analytical and numerical solutions. Codes were written using Maple and Matlab which are computer- algebra packages that calculates with both number and symbols.

Table 1

Definition of Parameters, numerical values and sources

Parameter	Definition of parameters	Value	Source
$C(t)$	Numbers of people with *Diabetes mellitus* (diabetics) with complications.		
$D(t)$	Numbers of people with *Diabetes mellitus* (diabetics) without complications		
$N(t)$	Total population of people with *Diabetes mellitus* with or without complications		
δ	Mortality rate due to complications	0.05	Boutayeb *et al.*, 2004.
ρ	Probability of developing a complication	0.85	Akinsola and Oluyo, 2014.
v	Rate at which patients with complications become severely disabled	0.05	Boutayeb *et al.*, 2004.
γ	Rate at which complications are controlled	0.50	Informed from Stability Analysis.
μ	Natural mortality rate	0.02	Boutayeb *et al.*, 2004.
I	Incidence of *Diabetes mellitus*	6×10^6	Boutayeb *et al.*, 2004; Sarah *et al.*, 2004.

Stability Analysis

The notion of stability is central to any discussion on the behavior of systems of differential equation. The stability of a continuous or discrete-time system is determined by its response to inputs or disturbances. A good deal of information about a solution for such systems can be derived from analysis of the equation without calculating a solution.

Definitions and theorem on stability and nature of critical points

Consider the homogenous system of ordinary differential equation of the form:

$$x'(t) = a_{11}x + a_{12}y \tag{22}$$

$$y'(t) = a_{21}x + a_{22}y \tag{23}$$

Where the a_{ij} are real constants and determinant

$$\Delta = a_{11}a_{22} - a_{12}a_{21} \neq 0 \tag{24}$$

Let λ_1 and λ_2 be the roots of the quadratic equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0 \tag{25}$$

Equation (25) is called the **characteristic equation** or **auxiliary equation** and λ_1 and λ_2 are the **eigenvalues**. **Eigenvalues** are values of scalar, λ, for which Equation (26) holds

$$A.\boldsymbol{x} = \lambda.\boldsymbol{x} \tag{26}$$

Eigenvectors are solutions of $\boldsymbol{x}$ corresponding to particular values of eigenvalue λ.

Let $\boldsymbol{A}$ be a real $n \times n$ matrix when the system of equation (3) is represented in the vector compact form;

$$X = A(t)X + f(t),\ X(t_0) = X_0 \tag{27}$$

(a) If all the eigenvalues of A have nonpositive real parts and all those eigenvalues with zero real parts are simple, then the solution $X^* = 0$ of (27) is stable.

(b) If and only if all eigenvalues of $\boldsymbol{A}$ have negative real parts, the zero solution (critical points) of (27) is asymptotically stable.

(c) If one or more eigenvalues of A have a positive real part, the zero solution of (27) is unstable.

The results of the stability analysis of the model at indicated values of the parameters are in Table 2 and Table 3 in the result section of the paper.

Eigenvalue Sensitivity Analysis (*ESA*)

Eigenvalue Sensitivity with respect to a parameter is defined as the partial derivative of the eigenvalue with respect to that parameter. The eigenvalue sensitivity S_i ($i = 1, \cdots, N$ and N is the dimension of the state vector) with respect to the jth parameter of the system p_j. It is given in the form;

$$S_i(p_j) = \frac{\partial \lambda_i}{\partial p_j} = I_i^T \frac{\partial J}{\partial p_j} r_i \tag{28}$$

Where I_i and r_i are the left and right eigenvectors respectively and $\frac{\partial J}{\partial p_j}$ is the partial derivatives of the linearized Jacobian matrix with respect to a parameter.

The sensitivity of an eigenvalue to a design parameter can be calculated from the eigenvalue, the corresponding eigenvector, and the sensitivities of the stiffness and mass matrices to the design parameter (variable) (Binuyo, 2012).

Eigenvalue sensitivity analysis is a frame work in the field of structural design modification and optimization. It can predict the dynamic behavior of a structure after small design variable changes. There are two primary applications. In the first case sensitivity data are used solely as a qualitative indicator of the location and approximate scale of design changes to achieve a desired change in structural properties. The consequences of design changes would then be evaluated using exact methods. The second strategy uses the design sensitivities directly to predict the effect of proposed structural changes.

Sensitivity analysis is a useful tool in model building as well as in model evaluation. Formal model analysis tools are essential elements in understanding how structure drives behavior (Gonçalves, Lertpattarapong and Hines, 2000). Sensitivity analysis plays a central role in the investigation of mathematical models because model behaviour depends on parameter values. The aim of sensitivity analysis is to vary model parameters and assess the associated changes a certain parameter will have on the model outcomes. Sensitivity analysis can help the reviewer to determine which parameters are the key drivers of a model's results (Forrester, 2001). Sensitivity analysis helps to build confidence in the model by studying the uncertainties that are often associated with parameters in models. Sensitivity analysis allows the modeler to determine what level of accuracy is necessary for a parameter to make the model sufficiently useful and valid. Sensitivity analysis can also indicate which parameter values are reasonable to use in the model. If the model behaves as expected from real world observations, it gives some indication that the parameter values reflect, at least in part, the "real world." Sensitivity analysis is also useful for identifying weak points of the model. These can then be strengthened by experimentation, or simply noted and caution taken in any application. Specific parameter values can change the appearance of the graphs representing the behavior of the system. But significant changes in behavior do not occur for all parameters. This technique determines how sensitive model behaviour is to perturbations in the model parameters. Sensitivity analysis is also playing an increasing role in determining the analytical model itself. In the areas of system identification and analytical model improvement using test results, sensitivity analysis is of growing importance (Durbha and Haftka, 1986).

Eigenvalue Elasticity Analysis (*EEA*)

Eigenvalue Elasticity with respect to a parameter (matrix element) is defined as the partial derivative of the eigenvalue with respect to that parameter normalized for the size of the parameter and the size of the eigenvalue. The practice consists of linearizing the model under study (Jacobian), calculating its eigenvalues and then noting how the eigenvalues change as causal link gains change in the linearized model (Saleh, 2002). This could also be described as the product of the eigenvalue sensitivity and the ratio of the eigenvalue and parameter (Guneralp, 2005). In essence, elasticities are proportional sensitivities (Caswell, 2001).

Thus, it is given in the form;

$$E_i(P_j) = S_i(p_i)\cdot\frac{P_j}{\lambda_i} = I_i^{\,T}\frac{\partial J}{\partial p_j}r_i\cdot\frac{P_j}{\lambda_i} \tag{29}$$

Where I_i is left eigenvectors, r_i right eigenvectors with the partial derivatives of the linearized Jacobian matrix (*J*) with respect to a parameter p_j.

Elasticity analysis estimates the effect of a proportional change in the vital rates on population growth (Zhao, Chee and Pei, 2005). Eigenvalue elasticity is a convenient measure of transient-response sensitivity of the model to parameter changes (Saleh, 2002). The elasticity values are dimensionless, so they are comparable with each other. The elasticity of a complex conjugate eigenvalue pair is also a complex conjugate pair. In such a case, the real part of the elasticity gives the effect on the exponential envelope around oscillations, while the imaginary part gives the effect on the empirical frequency of oscillations (Saleh, 2002). Thus, EEA, by forming a connection between the model structure and behavior, provides a means to figure out the dominant structure in the model. The Jacobian matrix (J) can often be easily determined symbolically and the eigenvalues can be computed for particular parameters values both eigenvalue elasticity and sensitivity with respect to a parameter can be computed without the need to either compute closed form expressions for eigenvalues nor to perform numeric differentiation. If an eigenvalue elasticity with respect to one parameter is larger than others, it means that behaviour mode of that parameter is more sensitive to a certain proportional change in that parameter than to similar proportional changes in other parameters. Thus, a large elasticity might suggest that extra effort should be made to obtain a good estimate of the parameter and/or the parameter should be investigated as a possible policy lever (Binuyo, 2012). The right eigenvector measures the activity of the state variables of eigenvalue (mode) I and shows how observable an eigenvalue is among the state variables. The right eigenvectors are also known as the mode shapes of the system. (Makarov and Dong, 1998). The left eigenvector weighs the contribution of the activity of the state variables to eigenvalue (mode) I and shows how a state variable is able to influence an eigenvalue. The algorithms for ESA and EEA were coded using Matlab. The results are in Table 4.

Numerical solutions

Numerical analysis involves using mathematical techniques to generate numerical solutions to mathematical expressions. It involves creating, analyzing and implementation of computer algorithms for solving numerically the problems of continuous mathematics. A numerical method is said to be convergent if the numerical solution approaches the exact solution as the step size goes to zero, otherwise it diverges (Greenspan, 2006). Moreover, in addition to being convergent, for a method to be useful, it is of crucial importance that it is also stable in the sense that a small perturbation of the input data does not destroy the convergence and results in, at most, a small increase of the error. A method might converge on a problem but diverge on another, or converge with one set of starting values but not on another. It is essential that a method converges on all problems in a reasonably large class with all reasonable starting values for consistency (Akinsola and Oluyo, 2015). The Euler and Runge-Kutta methods were used to solve the resulting mathematical model of the complication and control of diabetes mellitus. The numerical solutions compares favourably with the exact solutions at the indicated values of the parameters. The algorithms were written in Maple computational environment.

The Euler Method

The Euler method is one of the simplest and most elementary method of solving single and systems of ordinary differential equations.

Given initial value system

$$\frac{dy}{dt} = f(t,x,y) \qquad y(0) = y_0 \tag{30}$$

$$\frac{dx}{dt} = g(t,x,y) \qquad x(0) = x_0 \tag{31}$$

The Euler's method approximate a solution (x, y) with step size h by:

$$x_{k+1} = x_k + h\,f(x_k,\,y_k) \tag{32}$$

$$y_{k+1} = y_k + h\,g(x_k,\,y_k) \tag{33}$$

The Runge- Kutta Method of order four

The Runge-Kutta method is a family of methods which depend on the order of derivatives involved. The Runge-Kutta method of order four is the classical form of the method in which four values of the derivatives are used for each iteration.

The Runge-Kutta formula of order four for system of two initial-value differential equation of the form of Equation (30) and Equation (31) is

$$x_{n+1} = x_n + 1/6(m_1 + 2m_2 + 2m_3 + m_4) \tag{34}$$

$$y_{n+1} = y_n + 1/6(k_1 + 2k_2 + 2k_3 + km_4) \tag{35}$$

Where

$$m_1 = h\,f(t_n,\,x_n,\,y_n) \tag{36}$$

$$k_1 = h\,g(t_n,\,x_n,\,y_n) \tag{37}$$

$$m_2 = h\,f(t_n + 1/2h,\; x_n + 1/2m_1, y_n + 1/2k_1) \tag{38}$$

$$k_2 = hg(t_n + 1/2h,\; x_n + 1/2m_{1,}y_n + 1/2k_1) \tag{39}$$

$$m_3 = h\,f(t_n + 1/2h,\; x_n + 1/2m_{2,}y_n + 1/2k_2) \tag{40}$$

$$k_3 = hg(t_n + 1/2h,\ x_n + 1/2m_{2,}\ y_n + 1/2k_2) \tag{41}$$

$$m_4 = h\,f(t_n + h,\ x_n + m_3,\ y_n + k_3) \tag{42}$$

$$k_4 = hg(t_n + h,\ x_n + m_3,\ y_n + m_3) \tag{43}$$

Result and Discussion

Table 2

Summary of the Eigenvalues and nature of stability of equilibrium point for different cases at various values of the rate at which complications are controlled (γ) in the model.

Cases	γ	λ_1	λ_2	Nature of Critical point
Case 1	0.0	–0.1200000000	–0.8700000000	Asymptotically Stable
Case 2	0.08	–0.1104708491	–0.9595291509	Asymptotically Stable
Case 3	0.5	–0.08120409763	–1.408795902	Asymptotically Stable
Case 4	1.0	–0.06461029670	–1.925389703	Asymptotically Stable
Case 5	1.5	–0.05519959657	–2.434800403	Asymptotically Stable
Case 6	2.0	–0.04910062591	–2.940899374	Asymptotically Stable
Case 7	2.5	–0.04481618641	–3.445183814	Asymptotically Stable

Table 3

Summary of the Eigenvalues and nature of stability of equilibrium points for different cases at various values of the probability of developing complications of *Diabetes mellitus* (ρ) in the model.

Cases	ρ	λ_1	λ_2	Nature of Critical point
Case 1	0.20	–0.04583426132	–0.7941657387	Asymptotically Stable
Case 2	0.35	–0.05839377009	–0.9316062299	Asymptotically Stable
Case 3	0.5	–0.06750621894	–1.072493781	Asymptotically Stable
Case 4	0.75	–0.07805186604	–1.311948134	Asymptotically Stable
Case 5	0.85	–0.08120409763	–1.408795902	Asymptotically Stable
Case 6	1.0	–0.08515307717	–1.554846923	Asymptotically Stable

Table 2 and Table 3 showed that the model is asymptotically stable at all the indicated values of the parameters hence the systems of differential equations are suitable for intended purpose.

Table 4 showed that the parameter for the rate at which complications are controlled (γ) has the highest value in the Sensitivity and Elasticity analysis hence it is the parameter that has the greatest impact on the formulated mathematical model. Therefore the rate at which complication are control is very important if the battle against *Diabetes mellitus* and its complications will be won. Policy makers and medical personnel's, diabetics and their care givers should also focus attention on the complications of the disease that patients are liable to have for effective control, treatment and intervention.

Table 4

Obtained values for the Eigenvalue Sensitivity and Eigenvalue Elasticity Analyses at indicated values of the rate at which complications are controlled in the Model.

Values of γ	Analysis	Parameters				
		ρ	μ	ν	γ	δ
0.0	ESA	0.0000	–1.0000	–1.0000	0.1333	–1.0000
	EEA	0.0000	–0.0230	–0.0575	0.0000	–0.0575
0.08	ESA	–0.0112	–1.0000	–0.8946	0.1066	–0.8946
	EEA	–0.0099	–0.0208	–0.0466	0.0089	–0.0466
0.5	ESA	–0.0292	–1.0000	–0.5942	0.0461	–0.5942
	EEA	–0.0176	–0.0142	–0.0211	0.0164	–0.0211
1.0	ESA	–0.0298	–1.0000	–0.4328	0.0240	–0.4328
	EEA	–0.0131	–0.0104	–0.0112	0.0125	–0.0112
1.5	ESA	–0.0272	–1.0000	–0.3424	0.0148	–0.3424
	EEA	–0.0095	–0.0082	–0.0070	0.0091	–0.0070
2.0	ESA	–0.0245	–1.0000	–0.2839	0.0101	–0.2839
	EEA	–0.0071	–0.0068	–0.0048	0.0068	–0.0048
2.5	ESA	–0.0221	–1.0000	–0.2427	0.0073	–0.2427
	EEA	–0.0055	–0.0058	–0.0035	0.0053	–0.0035

Table 5

Exact and Numerical solutions of Number of diabetics with complications at various indicated values of the rate at which complications are controlled.

Values of gamma, γ	Solution C(t)	Time 10	20	30	40	50	60	70	100
0.0	Exact	9.586344016 10^7	9.664846370 10^7	9.736168172 10^7	9.800915356 10^7	9.859643274 10^7	9.912860916 10^7	9.961034774 10^7	1.007939340 10^8
	Euler	9.586738088 10^7	9.665567931 10^7	9.737158995 10^7	9.802124635 10^7	9.861026792 10^7	9.914380302 10^7	9.962656830 10^7	1.008116748 10^8
	RK 4	9.586344012 10^7	9.664846365 10^7	9.736168169 10^7	9.800915356 10^7	9.859643282 10^7	9.912860929 10^7	9.961034788 10^7	1.007939344 10^8
0.08	Exact	9.513869413 10^7	9.526554564 10^7	9.538162875 10^7	9.548791948 10^7	9.558530468 10^7	9.567459014 10^7	9.575650802 10^7	9.596441070 10^7
	Euler	9.513928947 10^7	9.526662805 10^7	9.538310484 10^7	9.548970894 10^7	9.558733864 10^7	9.567680974 10^7	9.575886318 10^7	9.596694368 10^7
	RK 4	9.513869409 10^7	9.526554560 10^7	9.538162865 10^7	9.548791937 10^7	9.558530459 10^7	9.567459007 10^7	9.575650801 10^7	9.596441084 10^7
0.5	Exact	9.142581215 10^7	8.834510996 10^7	8.569285378 10^7	8.341255216 10^7	8.145513847 10^7	7.977799531 10^7	7.834410696 10^7	7.520104533 10^7
	Euler	9.140095672 10^7	8.830197237 10^7	8.563670714 10^7	8.334759790 10^7	8.138469735 10^7	7.970466654 10^7	7.826990071 10^7	7.513191957 10^7
	RK 4	9.142581217 10^7	8.834511002 10^7	8.569285392 10^7	8.341255236 10^7	8.145513873 10^7	7.977799561 10^7	7.8344 10728 10^7	7.520104572 10^7
1.0	Exact	8.720078615 10^7	8.080415595 10^7	7.556422795 10^7	7.127818536 10^7	6.777873376 10^7	6.492787984 10^7	6.261179963 10^7	5.801266872 10^7
	Euler	8.713002850 10^7	8.068755855 10^7	7.542013340 10^7	7.111990321 10^7	6.761574365 10^7	6.476676707 10^7	6.245698008 10^7	5.788917182 10^7
	RK 4	8.720078612 10^7	8.080415589 10^7	7.556422789 10^7	7.127818530 10^7	6.777873367 10^7	6.492787969 10^7	6.261179946 10^7	5.801266849 10^7
1.5	Exact	8.317808610 10^7	7.395435437 10^7	6.676708605 10^7	6.117595379 10^7	5.683578848 10^7	5.347601504 10^7	5.088453226 10^7	4.622908325 10^7
	Euler	8.304687671 10^7	7.374897907 10^7	6.652599594 10^7	6.092439421 10^7	5.658972245 10^7	5.324496514 10^7	5.067362581 10^7	4.608487887 10^7
	RK 4	8.317808612 10^7	7.395435440 10^7	6.676708601 10^7	6.117595372 10^7	5.683578839 10^7	5.34760148910 ^{7}	5.088453205 10^7	4.622908303 10^7
2.0	Exact	7.934793439 10^7	6.773116410 10^7	5.912130244 10^7	5.275198700 10^7	4.805211845 10^7	4.459610921 10^7	4.206680857 10^7	3.796148361 10^7
	Euler	7.914405096 10^7	6.742798443 10^7	5.878318399 10^7	5.241681270 10^7	4.774064265 10^7	4.431825134 10^7	4.182584521 10^7	3.782021736 10^7
	RK 4	7.934793447 10^7	6.773116425 10^7	5.912130263 10^7	5.275198721 10^7	4.805211868 10^7	4.459610947 10^7	4.206680881 10^7	3.796148384 10^7
2.5	Exact	7.570102767 10^7	6.207611673 10^7	5.24714056110 ^{7}	4.571503260 10^7	4.097666262 10^7	3.766795311 10^7	3.537202818 10^7	3.201249145 10^7
	Euler	7.541436191 10^7	6.167112855 10^7	5.204230250 10^7	4.531090738 10^7	3.736555722 10^7	3.512288067 10^7	3.188725192 10^7	4.061986382 10^7
	RK 4	7.570102786 10^7	6.207611700 10^7	5.24714058710 ^{7}	4.571503285 10^7	4.097666286 10^7	3.766795332 10^7	3.537202835 10^7	3.201249154 10^7

Table 6

Exact and Numerical solutions of the Number of diabetics with and without complications at various indicated values of the rate at which complications are controlled.

Values of gamma, γ.	Solution $N(t)$	10	20	30	40	50	60	70	100
0.0	Exact	1.120215915 10^8	1.120349116 10^8	1.120407268 10^8	1.120397387 10^8	1.120325895 10^8	1.120198670 10^8	1.120021093 10^8	1.119233410$10^8$
	Euler	1.120220073 10^8	1.120356663 10^8	1.120417529 10^8	1.120409774 10^8	1.120339900 10^8	1.120213852 10^8	1.120037070 10^8	1.119250001$10^8$
	RK 4	1.120215915 10^8	1.120349119 10^8	1.120407273 10^8	1.120397392 10^8	1.120325901 10^8	1.120198676 10^8	1.120021102 10^8	1.119233411$10^8$
0.08	Exact	1.120252707 10^8	1.120491654 10^8	1.120717999 10^8	1.120932791 10^8	1.121136987 10^8	1.121331458 10^8	1.121516995 10^8	1.122026882$10^8$
	Euler	1.120253398 10^8	1.120492920 10^8	1.120719738 10^8	1.120934918 10^8	1.121139428 10^8	1.121334149 10^8	1.121519880 10^8	1.122030101$10^8$
	RK 4	1.120252706 10^8	1.120491653 10^8	1.120717997 10^8	1.120932788 10^8	1.121136986 10^8	1.121331461 10^8	1.121517002 10^8	1.122026888$10^8$
0.5	Exact	1.120442735 10^8	1.121216408 10^8	1.122274386 10^8	1.123576188 10^8	1.125086676 10^8	1.126775358 10^8	1.128615772 10^8	1.134832605$10^8$
	Euler	1.120426058 10^8	1.121187768 10^8	1.122237560 10^8	1.123534180 10^8	1.125041862 10^8	1.126729589 10^8	1.128570476 10^8	1.134794174$10^8$
	RK 4	1.120442738 10^8	1.121216412 10^8	1.122274394 10^8	1.123576195 10^8	1.125086684 10^8	1.126775367 10^8	1.128615777 10^8	1.134832619$10^8$
1.0	Exact	1.120662271 10^8	1.122030029 10^8	1.123974426 10^8	1.126389216 10^8	1.129186791 10^8	1.132294922 10^8	1.135654064 10^8	1.146788353$10^8$
	Euler	1.120626667 10^8	1.121971880 10^8	1.123903339 10^8	1.126312155 10^8	1.129108713 10^8	1.132219266 10^8	1.135583127 10^8	1.146737994$10^8$
	RK 4	1.120662266 10^8	1.122030025 10^8	1.123974425 10^8	1.126389214 10^8	1.129186784 10^8	1.132294914 10^8	1.135654057 10^8	1.146788348$10^8$
1.5	Exact	1.120874789 10^8	1.122793754 10^8	1.125524455 10^8	1.128884721 10^8	1.132731784 10^8	1.136953770 10^8	1.141463016 10^8	1.156098667$10^8$
	Euler	1.120822042 10^8	1.122711865 10^8	1.125429332 10^8	1.128786799 10^8	1.132637659 10^8	1.136867367 10^8	1.141386439 10^8	1.156054070$10^8$
	RK 4	1.120874787 10^8	1.122793749 10^8	1.125524448 10^8	1.128884714 10^8	1.132731777 10^8	1.136953760 10^8	1.141463007 10^8	1.156098655$10^8$
2.0	Exact	1.121080543 10^8	1.123511071 10^8	1.126939660 10^8	1.131104092 10^8	1.135808999 10^8	1.140908828 10^8	1.146295150 10^8	1.163463271$10^8$
	Euler	1.121012318 10^8	1.123410412 10^8	1.126828586 10^8	1.130995557 10^8	1.135710093 10^8	1.140822933 10^8	1.146223373 10^8	1.163429996$10^8$
	RK 4	1.121080542 10^8	1.123511072 10^8	1.126939661 10^8	1.131104098 10^8	1.135809005 10^8	1.140908833 10^8	1.146295155 10^8	1.163463274$10^8$
2.5	Exact	1.121279776 10^8	1.124185205 10^8	1.128233574 10^8	1.133082888$10^8$	1.138490864 10^8	1.144285867 10^8	1.150346326 10^8	1.169377745$10^8$
	Euler	1.121197629 10^8	1.124070046 10^8	1.128112893 10^8	1.132971003 10^8	1.138394287 10^8	1.144206652 10^8	1.150284123 10^8	1.169356033$10^8$
	RK 4	1.121279776 10^8	1.124185205 10^8	1.128233574 10^8	1.133082889$10^8$	1.138490867 10^8	1.144285870 10^8	1.150346331 10^8	1.169377746$10^8$

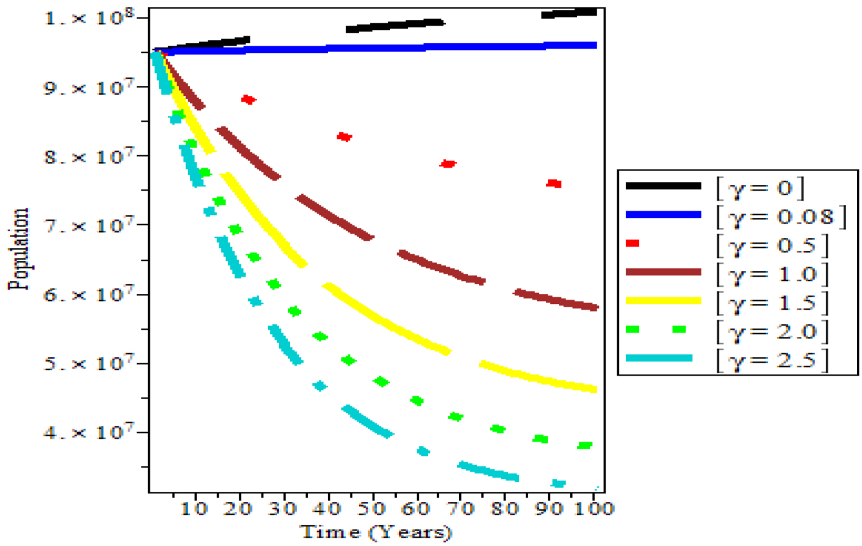

Figure 1

The number of diabetics with complications, $C(t)$, when the rate at which complications are controlled is at various values in the model.

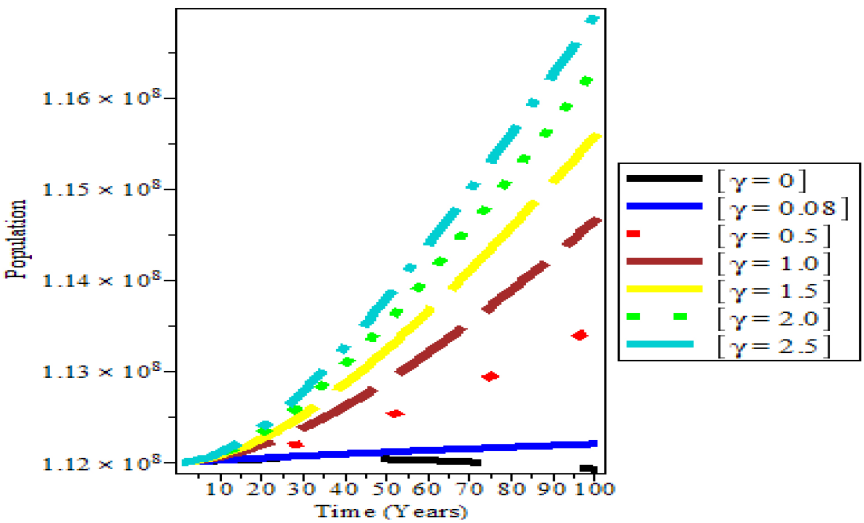

Figure 2

The number of diabetics with and without complications, $N(t)$, when the rate at which complications are controlled is at various values in the model.

From Table 5 and Table 6, it can be seen that the numerical solution with Euler method and Runge- Kutta method of order four compares favourably with the analytic solution. The tables show that the Runge-Kutta method of order four is more efficient than the Euler method.

The graphical illustration from Figure 1, showed that as the rate at which complications are controlled (γ) increases the number of diabetics with complications reduces.

A slight distinction from this is when the rate at which complications are controlled is zero, that is, when complications are not being controlled or uncontrollable, the number of diabetics with complications increases and reach a threshold where it reduces. This suggest that when complications of *Diabetes mellitus* are not well controlled it will eventually lead to untimely death of patients. Even with a small rate at which complications are controlled, that is 0.08, an appreciable effect was seen as the number of diabetics with complication was steady and even out. Hence effort to detect the complications that a diabetic patient suffers from is worthwhile and timely control of such is very important.

Figure 2, indicated that with increased rate at which complications are controlled, the number of diabetics with and without complication will continue to increase for the time being. This is in agreement with global reality and projection of increasing prevalence and incidence of *Diabetes mellitus*. The graphs showed that the higher the rate of control, the higher the number of diabetics with and without complications, it can be inferred that the number of diabetics without complications, D(t), is increasing since the number of diabetics with complications reduces. The case depicted when the rate at which complications are controlled is zero, is that the total number of diabetics will be reducing due to untimely death of patients. Concerted effort must be made towards reducing the incidence of *Diabetes mellitus*.

Due to the perceived link between the probability of developing a complication of diabetes mellitus (ρ) and the rate at which complications are controlled (γ), the values of the former were also varied at indicated values in the numerical solutions and the results were given in Table 7 and Table 8 and graphically in Figure 3 and Figure 4.

Table 7

Exact and Numerical solutions of Number of diabetics with complications at various indicated values of the probability of developing a complication.

Values of rho, ρ	Solution C(t)	Time 10	20	30	40	50	60	70	100
0.20	Exact	$9.024792575\ 10^7$	$8.587138263\ 10^7$	$8.184164252\ 10^7$	$7.813217079\ 10^7$	$7.471845888\ 10^7$	$7.157786958\ 10^7$	$6.868949419\ 10^7$	$6.135179050\ 10^7$
	Euler	$9.022906191\ 10^7$	$8.583654664\ 10^7$	$8.179339423\ 10^7$	$7.807277209\ 10^7$	$7.464990425\ 10^7$	$7.150191381\ 10^7$	$6.860767736\ 10^7$	$6.125980965\ 10^7$
	RK 4	$9.024792576\ 10^7$	$8.587138262\ 10^7$	$8.184164249\ 10^7$	$7.813217076\ 10^7$	$7.471845885\ 10^7$	$7.157786958\ 10^7$	$6.868949420\ 10^7$	$6.135179045\ 10^7$
0.35	Exact	$9.052622126\ 10^7$	$8.646837244\ 10^7$	$8.278935072\ 10^7$	$7.945535433\ 10^7$	$7.643558883\ 10^7$	$7.370199965\ 10^7$	$7.122902837\ 10^7$	$6.515115311\ 10^7$
	Euler	$9.050531407\ 10^7$	$8.643029639\ 10^7$	$8.273734411\ 10^7$	$7.939221490\ 10^7$	$7.636372657\ 10^7$	$7.362348369\ 10^7$	$7.114562855\ 10^7$	$6.506125736\ 10^7$
	RK 4	$9.052622124\ 10^7$	$8.646837241\ 10^7$	$8.278935072\ 10^7$	$7.945535433\ 10^7$	$7.643558881\ 10^7$	$7.370199964\ 10^7$	$7.122902836\ 10^7$	$6.515115311\ 10^7$
0.5	Exact	$9.080058736\ 10^7$	$8.704933786\ 10^7$	$8.370053214\ 10^7$	$8.071310139\ 10^7$	$7.805015439\ 10^7$	$7.567855261\ 10^7$	$7.356852862\ 10^7$	$6.855385581\ 10^7$
	Euler	$9.077804542\ 10^7$	$8.700886491\ 10^7$	$8.364603377\ 10^7$	$8.064787376\ 10^7$	$7.797696784\ 10^7$	$7.559972470\ 10^7$	$7.348598758\ 10^7$	$6.846863869\ 10^7$
	RK 4	$9.080058734\ 10^7$	$8.704933781\ 10^7$	$8.370053208\ 10^7$	$8.071310130\ 10^7$	$7.805015428\ 10^7$	$7.567855251\ 10^7$	$7.356852850\ 10^7$	$6.855385572\ 10^7$
0.75	Exact	$9.124928644\ 10^7$	$8.798321994\ 10^7$	$8.514202906\ 10^7$	$8.267329284\ 10^7$	$8.053103706\ 10^7$	$7.867494157\ 10^7$	$7.706964510\ 10^7$	$7.346703158\ 10^7$
	Euler	$9.122488515\ 10^7$	$8.794045470\ 10^7$	$8.508582004\ 10^7$	$8.260762684\ 10^7$	$8.045912284\ 10^7$	$7.8599341\ 11\ 10^7$	$7.699238434\ 10^7$	$7.339289664\ 10^7$
	RK 4	$9.124928644\ 10^7$	$8.798321995\ 10^7$	$8.514202907\ 10^7$	$8.267329286\ 10^7$	$8.053103700\ 10^7$	$7.867494149\ 10^7$	$7.706964502\ 10^7$	$7.346703156\ 10^7$
0.85	Exact	$9.142581215\ 10^7$	$8.834510996\ 10^7$	$8.569285378\ 10^7$	$8.341255216\ 10^7$	$8.145513847\ 10^7$	$7.977799531\ 10^7$	$7.834410696\ 10^7$	$7.520104533\ 10^7$
	Euler	$9.140095672\ 10^7$	$8.830197237\ 10^7$	$8.563670714\ 10^7$	$8.334759790\ 10^7$	$8.138469735\ 10^7$	$7.970466654\ 10^7$	$7.826990071\ 10^7$	$7.513191957\ 10^7$
	RK 4	$9.142581217\ 10^7$	$8.834511002\ 10^7$	$8.569285392\ 10^7$	$8.341255236\ 10^7$	$8.145513873\ 10^7$	$7.977799561\ 10^7$	$7.834410728\ 10^7$	$7.520104572\ 10^7$
1.0	Exact	$9.168748669\ 10^7$	$8.887583832\ 10^7$	$8.649272809\ 10^7$	$8.447624597\ 10^7$	$8.277339862\ 10^7$	$8.133882548\ 10^7$	$8.013369952\ 10^7$	$7.758590968\ 10^7$
	Euler	$9.166224657\ 10^7$	$8.883267397\ 10^7$	$8.643736912\ 10^7$	$8.441314143\ 10^7$	$8.270596750\ 10^7$	$8.126966140\ 10^7$	$8.006473793\ 10^7$	$7.752447829\ 10^7$
	RK 4	$9.168748669\ 10^7$	$8.887583832\ 10^7$	$8.649272809\ 10^7$	$8.447624597\ 10^7$	$8.277339863\ 10^7$	$8.133882548\ 10^7$	$8.013369948\ 10^7$	$7.758590956\ 10^7$

Table 8

Exact and Numerical solutions of Number of diabetics with and without complications at various indicated values of the probability of developing a complication.

		Time							
Values of Rho (ρ)	Solution N(t)	10	20	30	40	50	60	70	100
0.20	Exact	1.120500436 10^8	1.121455593 10^8	1.122828502 10^8	1.124585022 10^8	1.126693636 10^8	1.129125238 10^8	1.131852958 10^8	1.141574096 10^8
	Euler	1.120477590 10^8	1.121413621 10^8	1.122770695 10^8	1.124514290 10^8	1.126612539 10^8	1.129036030 10^8	1.131757615 10^8	1.141469856 10^8
	RK 4	1.120500437 10^8	1.121455596 10^8	1.122828502 10^8	1.124585025 10^8	1.126693638 10^8	1.129125243 10^8	1.131852962 10^8	1.141574097 10^8
0.35	Exact	1.120486906 10^8	1.121398665 10^8	1.122694757 10^8	1.124338286 10^8	1.126295653 10^8	1.128536261 10^8	1.131032253 10^8	1.139810043 10^8
	Euler	1.120465545 10^8	1.121360023 10^8	1.122642370 10^8	1.124275203 10^8	1.126224501 10^8	1.128459290 10^8	1.130951389 10^8	1.139726340 10^8
	RK 4	1.120486906 10^8	1.121398663 10^8	1.122694759 10^8	1.124338285 10^8	1.126295652 10^8	1.128536259 10^8	1.131032251 10^8	1.139810045 10^8
0.5	Exact	1.120473506 10^8	1.121342792 10^8	1.122564622 10^8	1.124100177 10^8	1.125914603 10^8	1.127976615 10^8	1.130258124 10^8	1.138180194 10^8
	Euler	1.120453591 10^8	1.121307323 10^8	1.122517290 10^8	1.124044092 10^8	1.125852380 10^8	1.127910435 10^8	1.130189801 10^8	1.138113366 10^8
	RK 4	1.120473501 10^8	1.121342788 10^8	1.122564615 10^8	1.124100170 10^8	1.125914599 10^8	1.127976611 10^8	1.130258122 10^8	1.138180193 10^8
0.75	Exact	1.120451456 10^8	1.121251959 10^8	1.122355448 10^8	1.123721548 10^8	1.125314875 10^8	1.127104417 10^8	1.129063001 10^8	1.135729302 10^8
	Euler	1.120433873 10^8	1.121221444 10^8	1.122315791 10^8	1.123675817 10^8	1.125265533 10^8	1.127053429 10^8	1.129011915 10^8	1.135684081 10^8
	RK 4	1.120451453 10^8	1.121251956 10^8	1.122355445 10^8	1.123721544 10^8	1.125314871 10^8	1.127104415 10^8	1.129062999 10^8	1.135729302 10^8
0.85	Exact	1.120442735 10^8	1.121216408 10^8	1.122274386 10^8	1.123576188 10^8	1.125086676 10^8	1.126775358 10^8	1.128615772 10^8	1.134832605 10^8
	Euler	1.120426058 10^8	1.121187768 10^8	1.122237560 10^8	1.123534180 10^8	1.125041862 10^8	1.126729589 10^8	1.128570476 10^8	1.134794174 10^8
	RK 4	1.120442738 10^8	1.121216412 10^8	1.122274394 10^8	1.123576195 10^8	1.125086684 10^8	1.126775367 10^8	1.128615777 10^8	1.134832619 10^8
1.0	Exact	1.120429757 10^8	1.121163899 10^8	1.122155489 10^8	1.123364375 10^8	1.124756214 10^8	1.126301633 10^8	1.127975515 10^8	1.133568328 10^8
	Euler	1.120429757 10^8	1.121163899 10^8	1.122155489 10^8	1.123364375 10^8	1.124756214 10^8	1.126301633 10^8	1.127975515 10^8	1.133568328 10^8
	RK 4	1.120414410 10^8	1.121137955 10^8	1.122122662 10^8	1.123327546 10^8	1.124717590 10^8	1.126262886 10^8	1.127937889 10^8	1.133538474 10^8
		1.120429757 10^8	1.121163902 10^8	1.122155488 10^8	1.123364376 10^8	1.124756215 10^8	1.126301629 10^8	1.127975511 10^8	1.133568324 10^8

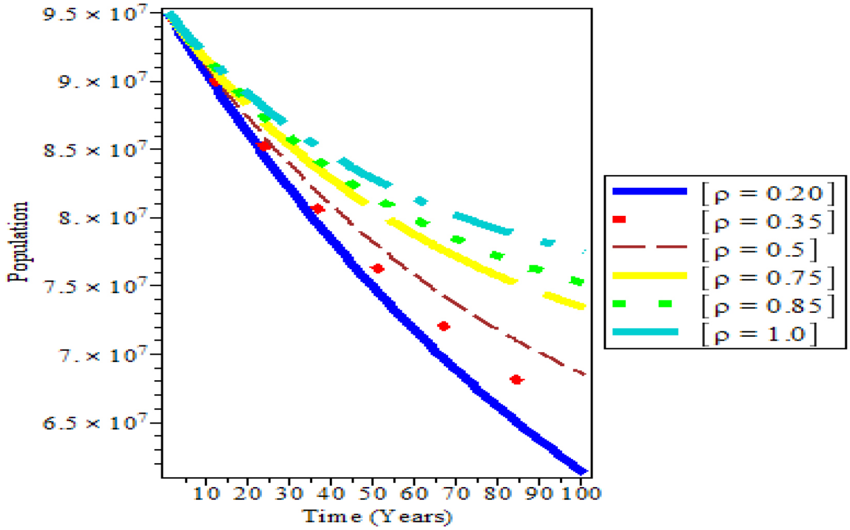

Figure 3

The number of diabetics with complications, $C(t)$, when the probability of developing a complication is at various values.

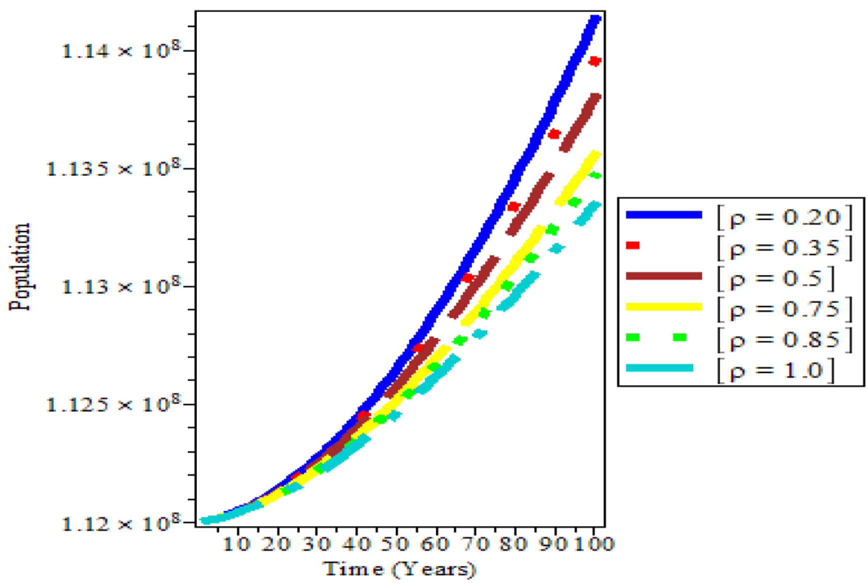

Figure 4

The number of diabetics with and without complications, $N(t)$, when the probability of developing complications is at various values.

In Tables 7 and 8, the numerical solution using the Euler method and Runge- Kutta method of order four gives an accurate approximate solution to the analytic solution.

Figure 3 points toward the fact that the higher the probability of developing a complication, the higher the number of diabetics with complications and the lower the probability of developing a complication, the lower the number of diabetics with complication. Hence measures to reduce the likelihood of diabetics developing complications should be a priority.

Figure 4 depicts that as the probability of developing a complication reduces, the number of diabetics with and without complications increase greatly. This suggests an increase in the population of diabetics without complication which could be as a result of complacency that can set in if the chance of developing a complication diminishes. This could be why diabetes epidemic is also increasing in developed countries where the probability of developing a complication is small and the rate at which complications are controlled is high.

Conclusion

The mathematical model of systems of ordinary differential equation for complications and control of *Diabetes mellitus* were analyzed and numerically solved. The population dynamics of diabetics with complications, $C(t)$ and diabetics with and without complications, $N(t)$, were examined. Parameter defined as the rate at which complications are cured in the previous works was redefined as the rate at which complications of *Diabetes mellitus* are controlled in the present research. Qualitative analyses of the models were obtained by determining the stability nature of the model at various indicated values of the parameters. The model solution was found to be asymptotically stable hence its suitability for the intended purpose ascertained.

The parameter with the greatest impact was determined to be the rate at which complication are controlled (γ) using Eigenvalue sensitivity analysis and Eigenvalue elasticity analysis.

Numerical solutions of the formulated models were obtained using Euler method and Runge- Kutta method of order four. Numerical simulations of the analytic solutions were also obtained at various values of the parameters.

It was found that number of diabetics with complications can be reduced drastically if the rate at which complications are controlled is high and if the probability of developing a complication is low. This point toward early detection and diagnosis of *Diabetes mellitus* before and after complications set in and provision of adequate medical care for diabetics.

Continuous improvement on the effectiveness and quality of care for diabetics will have significant impact on their life expectancy and quality of life.

The model validates the global projection of increasing incidence and prevalence of *Diabetes mellitus* as given by the International Diabetes Federation.

Efficient delivery of services for diabetes prevention and treatment need to be taken very serious by all and sundry. Addressing the major risk factors should be a top priority.

Health education is an essential component of any comprehensive intervention. An integrated approach to management at individual, community, and national levels incorporating prevention, early detection and diagnosis, treatment and rehabilitation should be provided. Broad partnership of government, private sector, medical experts, and public health specialist is essential if the war against the diabetes epidemic will be won.

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Received December, 2017

Revised July, 2018