Modeling the Two-Wheeled Self-Balancing Robot

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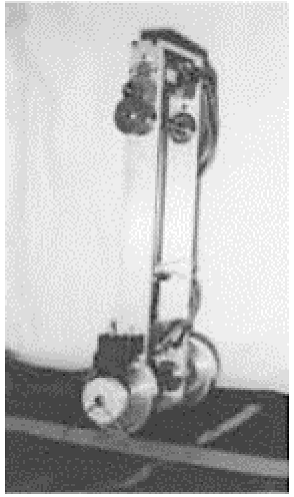
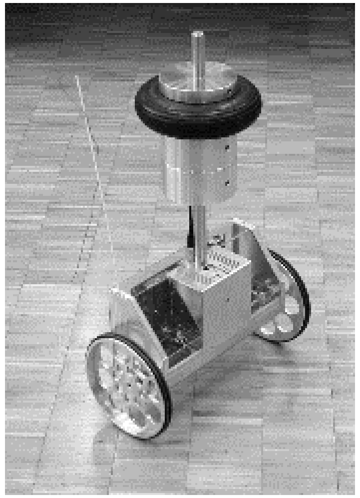
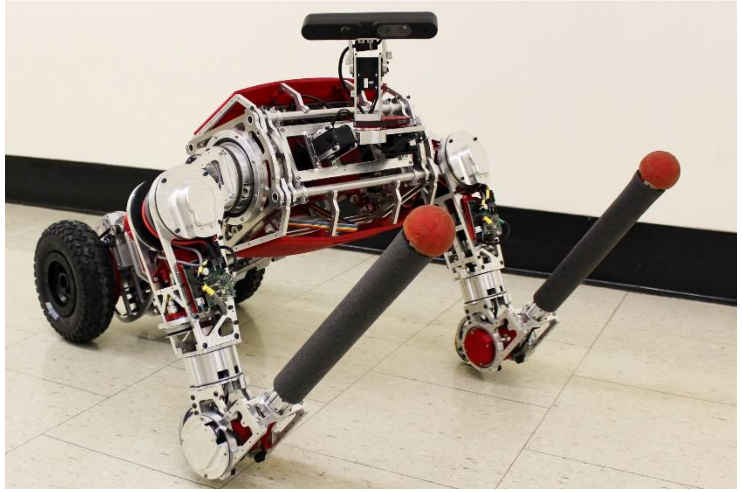
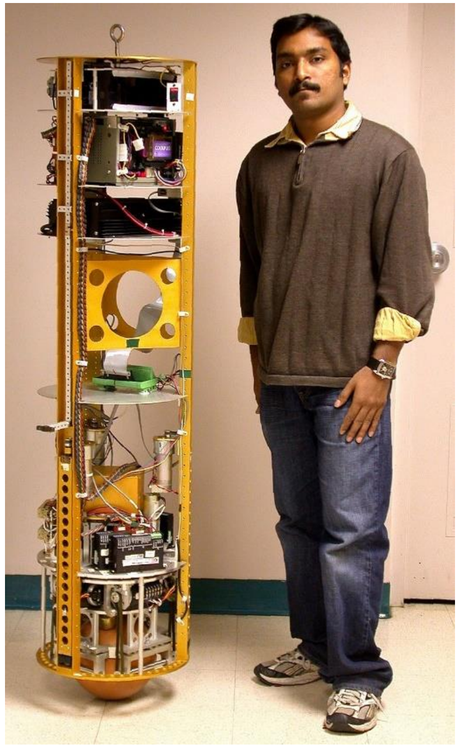
1/25/19

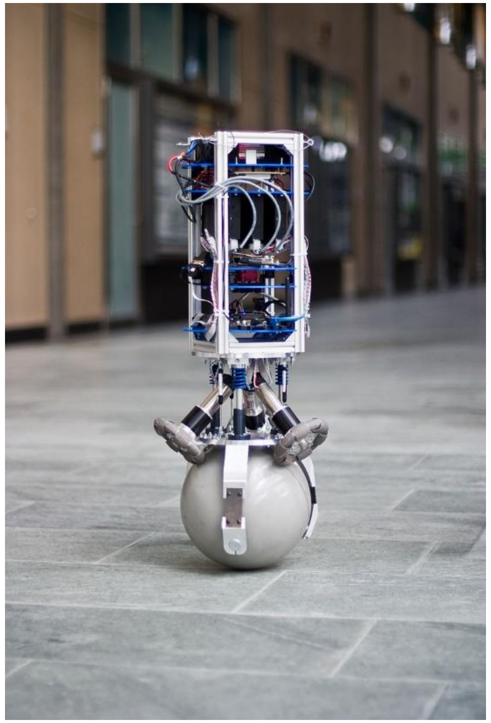
Hall of Fame

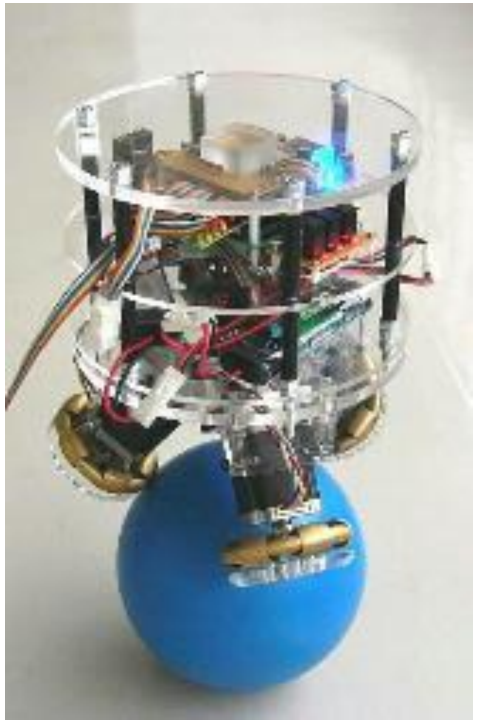






Ball of Fame



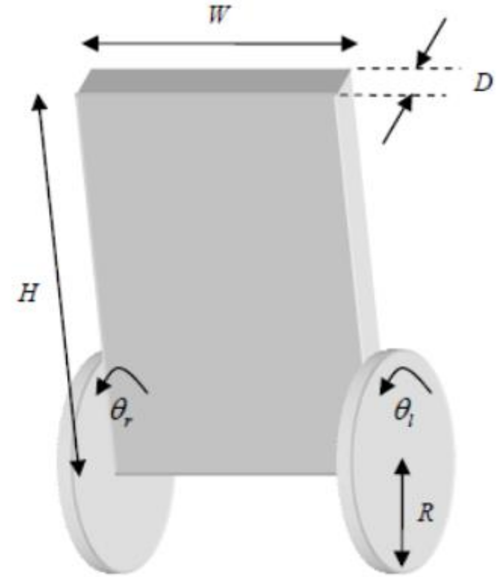




Underactuated Systems

A system is said to be **underactuated** if it possesses fewer actuators than degrees of freedom.

- The WIP has four degrees of freedom and two actuators, having 3 degrees of planar motion and a tilt-angular angle.

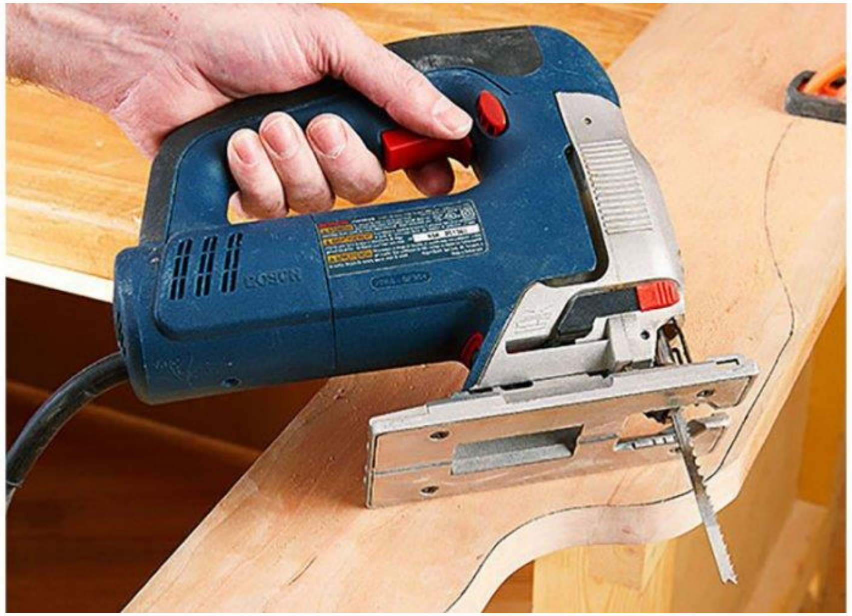


Nonholonomic Systems

- When a system is **nonholonomic**, constraints exist that constrain the velocities of particles present on the system but not their position

Example: a saw's edge can cut arbitrary shapes but the blade's change is constrained to linear motion

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0$$



Nonholonomic Systems

Holonomic Constraint

There exists a function, f, of generalized coordinates,

$$\boldsymbol{q} = [q_1 \quad q_2 \quad \cdots \quad q_n] \text{ ,}$$

(including time) that can be expressed as

$$f(\boldsymbol{q})=\boldsymbol{0}.$$

Nonholonomic Constraint

There exists a function, f, of the change of generalized coordinates,

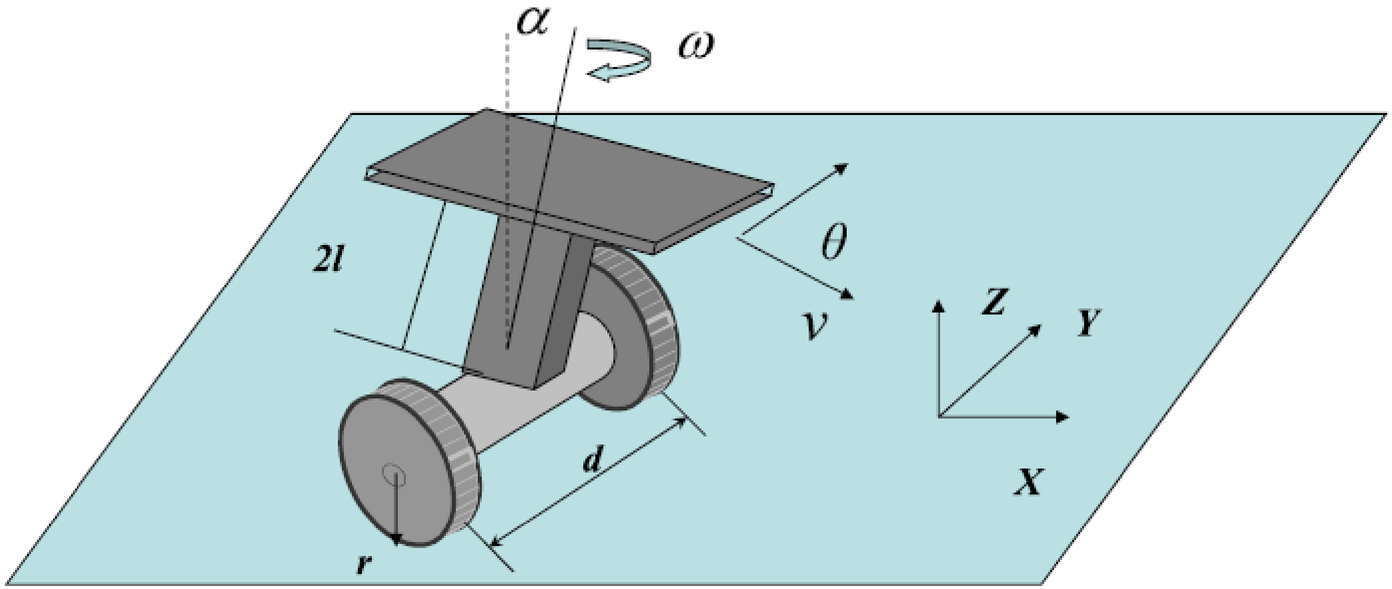
$$\dot{\boldsymbol{q}} = [\dot{q}_1 \quad \dot{q}_2 \quad \cdots \quad \dot{q}_n] \text{ ,}$$

(including dt) that can be expressed as

$$f(\dot{\boldsymbol{q}})=\boldsymbol{0}.$$

Kinematics of TWIP

- Relate angular velocity of the wheels to generalized coordinates of the entire mobile platform



$$\begin{bmatrix} \dot{\theta}_r \\ \dot{\theta}_l \end{bmatrix} \xrightarrow{\boldsymbol{s}} \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{\theta} \\ \dot{\theta}_r \\ \dot{\theta}_l \end{bmatrix}$$

Kinematics of TWIP

- Approach:
  - Define a coordinate system
  $$\boldsymbol{q} = [x_0 \quad y_0 \quad \theta \quad \theta_r \quad \theta_l], \boldsymbol{v}(t) = [\dot{\theta}_r \quad \dot{\theta}_l]$$
  - Utilize the nonholonomic constraint
  $$\boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = 0, \boldsymbol{A} \in \mathbb{R}^{3\times 5}$$
  - Define a transformation, $\boldsymbol{S} \in \mathbb{R}^{5\times 2}$, whose column vectors are members of $\mathrm{Nul}\{\boldsymbol{A}(\boldsymbol{q})\}$
  $$\boldsymbol{S}^T(\boldsymbol{q})\boldsymbol{A}^T(\boldsymbol{q}) = 0$$
  - It is evident, then, that
  $$\dot{\boldsymbol{q}} = \mathbf{S}(\mathbf{q})\mathbf{v}(\mathrm{t})$$

Kinematics of TWIP

- Assume that the robot cannot slip, being only able to travel in the direction θ

$$\dot{y}_0\cos(\theta) - \dot{x}_0 \sin(\theta) = 0$$

- The change of the distance displaced by the wheels is equal to change of the change of the position of COM and proportional to the change of the robot's heading angle

$$\dot{x}_0\cos(\theta) + \dot{y}_0 \sin(\theta) + \frac{\mathrm{d}}{2}\dot{\theta} = r\dot{\theta}_r$$

$$\dot{x}_0\cos(\theta) + \dot{y}_0 \sin(\theta) - \frac{\mathrm{d}}{2}\dot{\theta} = r\dot{\theta}_l$$

Kinematics of TWIP

$$A(q) = \begin{bmatrix} -\sin(\theta) & cos(\theta) & 0 & 0 & 0 \\ \cos(\theta) & \sin(\theta) & \frac{d}{2} & -r & 0 \\ \cos(\theta) & \sin(\theta) & -\frac{d}{2} & 0 & -r \end{bmatrix}$$

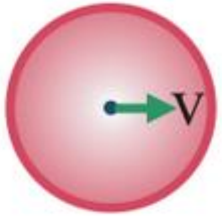
Kinematics of TWIP

$$\dot{\boldsymbol{q}} = \mathbf{S}(\mathbf{q})\mathbf{v}(\mathrm{t}) = \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{\theta} \\ \dot{\theta}_r \\ \dot{\theta}_l \end{bmatrix} = \begin{bmatrix} \frac{r}{2}\cos(\theta) & \frac{r}{2}\cos(\theta) \\ \frac{r}{2}\sin(\theta) & \frac{r}{2}\sin(\theta) \\ \frac{r}{d} & -\frac{r}{d} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_r \\ \dot{\theta}_l \end{bmatrix}$$

Energy Expressions

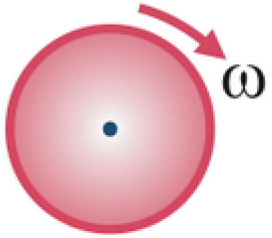
1. Translational Kinetic Energy

$$K = \frac{1}{2} M v^2$$

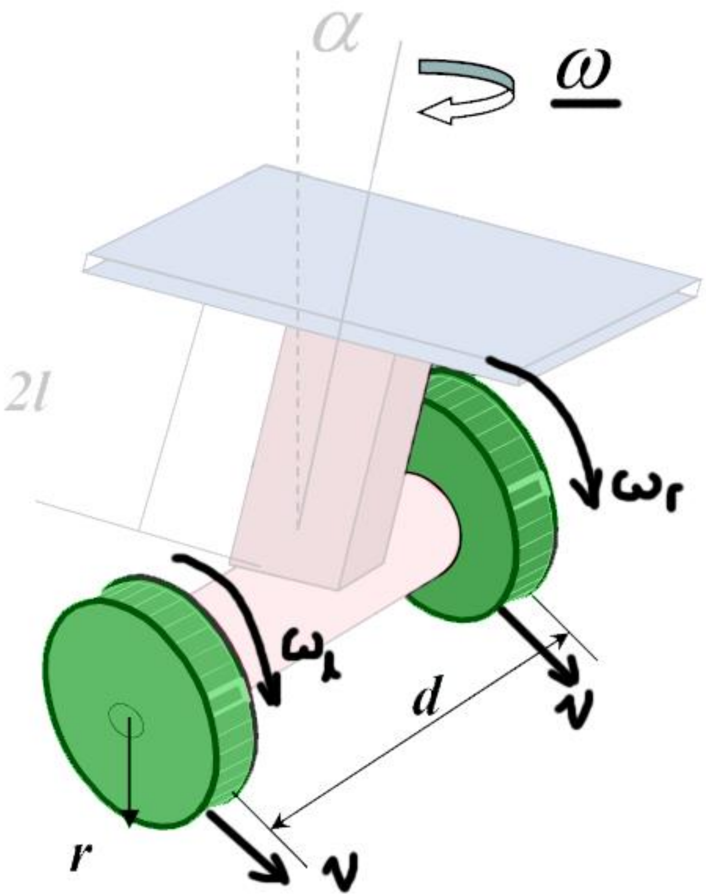


2. Rotational Kinetic Energy

$$K = \frac{1}{2} I \omega^2 = \frac{1}{2} \omega^2 \iiint_M r^2(m)\, \mathrm{d}m$$



Energy Expressions - Wheels



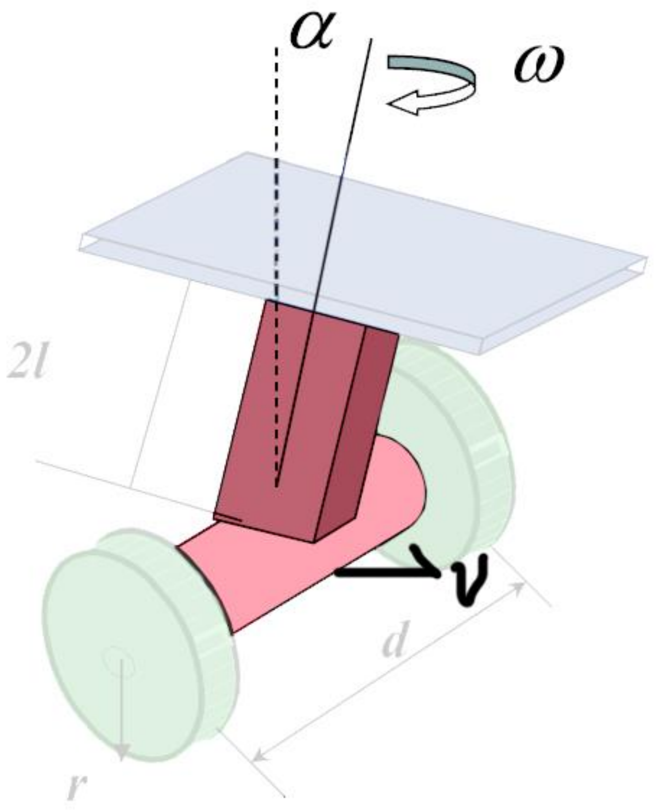
Mass of one wheel: M_w

Rotational Moment of Inertia one wheel about x-axis: I_w

Rotational Moment of Inertia of Platform about z-axis: I_M

$$K_w = M_w v^2 + I_w \left(\frac{v}{r}\right)^2 + 2\left(M_w + \left(\frac{I_w}{r^2}\right)\right) d^2 \omega^2$$

Energy Expressions - Platform



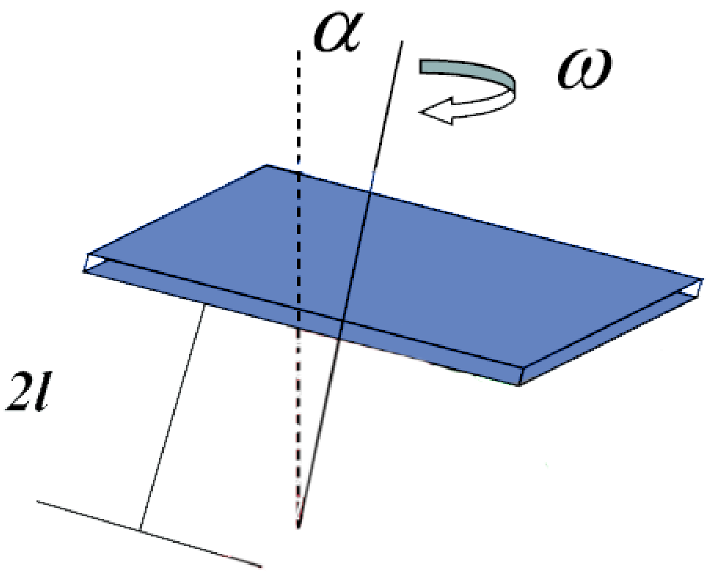
Mass of Platform: M

Rotational Moment of Inertia of Platform about y-axis: I_p

Rotational Moment of Inertia of Platform about z-axis: I_M

$$K_p = \frac{1}{2}Mv^2 + \frac{1}{2}I_M\dot{\alpha}^2 + \frac{1}{2}I_p\omega^2$$

Energy Expressions - Pendulum



Mass of Pendulum: m

$$K_l = \frac{1}{2}m(v + l\cos(\alpha)\,\dot{\alpha})^2 + \frac{1}{2}m(-\sin(\alpha)\,\dot{\alpha})^2$$

Energy Expressions - TWIP

$$K = M_w v^2 + \frac{1}{2} M v^2 + \frac{1}{2} m(v + l\cos(\alpha)\,\dot{\alpha})^2 + \frac{1}{2} m(-\sin(\alpha)\,\dot{\alpha})^2 + \frac{1}{2} I_M \dot{\alpha}^2 + \frac{1}{2} I_p \omega^2 + I_w \left(\frac{v}{r}\right)^2 + 2\left(M_w + \left(\frac{I_w}{r^2}\right)\right) d^2 \omega^2$$

$$U = mgl(1 - \cos(\alpha))$$

Dynamics of the TWIP

Dynamic analysis finds the relationship between a set of generalized coordinates, q, and the generalized forces involved, τ. In general,

$$\tau = M(q)\ddot{q} + h(q, \dot{q})$$

Where $M(q) \in \mathbb{R}^{n \times n}$ is known as the mass matrix and $h(q, \dot{q}) \in \mathbb{R}^{n}$is are a superposition of forces representing the centripetal, Coriolis, gravity and friction terms.

Typically, dynamics can be determined in two ways:

- Newton-Euler Formulation
- Lagrangian Dynamics

Lagrangian Method

Allow a set of independent coordinates, $q \in \mathbb{R}^n$, that describes a system configuration. A **Lagrangian function** is

$$\mathcal{L}(q, \dot{q}) = K(q, \dot{q}) - U(q)$$

The equations of motion, $\tau \in \mathbb{R}^n$, can be expressed in terms of the Lagrangian as

$$\tau = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q}$$

Lagrangian Method

Generalized coordinates, q, should be chosen so that $\dot{q}$ is dual with τ, namely that $\tau^T\dot{q}$ corresponds to system power. Using consistent notation, allow $q = [x, \theta, \alpha]$ and $\dot{q} = [v, \omega, \dot{\alpha}]$ serve as the generalized coordinates. The system dynamics, in terms of the Lagrangian $\mathcal{L}(q, \dot{q})$, can be expressed as

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{x}} - \dfrac{\partial \mathcal{L}}{\partial x} = \dfrac{\tau_l}{r} + \dfrac{\tau_r}{r} + d_l + d_r \\ \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} - \dfrac{\partial \mathcal{L}}{\partial \theta} = 2d\ \left(\dfrac{\tau_l}{r} - \dfrac{\tau_r}{r} + d_l - d_r\right) \\ \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\alpha}} - \dfrac{\partial \mathcal{L}}{\partial \alpha} = 0 \end{cases}$$

Lagrangian Method

Substitution of the energy expressions yields a system of second order ODEs

$$\left(M + 2M_w + m + 2\frac{I_w}{r^2}\right)\dot{v} + ml\ddot{\alpha}\cos(\alpha) - ml\dot{\alpha}^2 sin(\alpha) = \frac{\tau_l}{r} + \frac{\tau_r}{r} + d_l + d_r$$

$$\left(I_p + 2\left(M_w + \frac{I_w}{r^2}\right)d^2\right)\dot{\omega} = 2d\ \left(\frac{\tau_l}{r} - \frac{\tau_r}{r} + d_l - d_r\right)$$

$$ml\dot{v}\cos(\alpha) + (ml^2 + I_M)\ddot{\alpha} - mgl\sin(\alpha) = 0$$

DE Solutions

These equations, describing a system of non-linear second order differential equations, do not readily present themselves to a solution.

System Conversion can be used to facilitate the implementation of this system for *simulation*. For convention, consider the form

$$M(t, y)\dot{y} = f(t, y)$$

where $M(t, y)$ is a mass matrix from the equations of motion.

MATLAB's `ode45()` and other numerical solver's accept this form.

DE System Conversion

Second order DE's can be converted to a system of first order differential equations. Namely, consider the spring-mass-damper system,

$$M\ddot{x} + k_d\dot{x} + kx = F(t)$$

This is comparable to

$$\begin{cases} \dot{x_1} = x_2 \\ M\dot{x_2} + k_d x_2 + kx_1 = F(t) \end{cases}$$

In matrix form,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{k_d}{M} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

Where x_1 describes the position of a linear motion spring-mass-damper system.

Example MATLAB Implementation

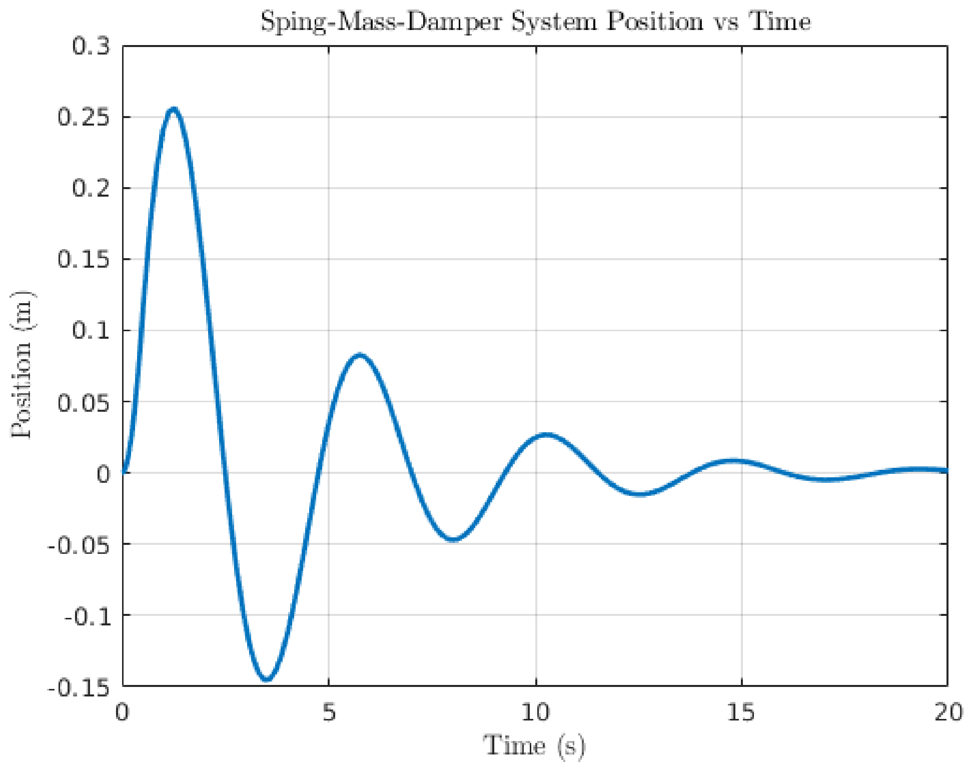
```
function smdsys
    %% System parameters
    % M     -  Mass of system
    % kd    -  Damping coefficient
    % k     -  Spring constant
    M = 1;
    kd = 0.5;
    k = 2;

    %% ODE Simulation
    % tspan     -   setup the simulation time period
    tspan = [0, 20];
    ode45(@(t,x) dsys(t, x, M, kd, k), tspan, [0; 0]);

end

%% Forcing function acting on the spring
function ft = F(t)
    ft = rectangularPulse(t);
end

%% Differential calculation
function dxdt = dsys(t, x, M, kd, k)
    dxdt = [0, 1; -k/M, -kd/M]*x + [0; F(t)];
End
```



DE System Conversion for TWIP

For conversion, consider the coordinates,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} \alpha \\ \dot{\alpha} \\ \theta \\ \omega \\ x \\ \dot{v} \end{bmatrix}$$

Note that these coordinates are Lagrangian coordinates, as the derivatives also relate to power of the system.

DE System Conversion for TWIP

In the case of our wheeled IP problem,

$$
\begin{cases}
\dot{y}_1 = y_2 \\
\dot{y}_2 = \dfrac{\cos(y_1)\, I_m \left(-mly_2^2 \sin(y_1) - \frac{\tau_l}{r} - \frac{\tau_l}{r} - d_l - d_r\right) + mgl \sin(y_1) \left(M + 2M_w + m + \frac{2I_w}{r}\right)}{(l^2 m + I_M)\left(M + 2M_w + m + \frac{2I_w}{r^2}\right) - (\cos(y_1))^2 l^2 m^2} \\
\dot{y}_3 = y_4 \\
\dot{y}_4 = \dfrac{2d\, \left(\frac{\tau_l}{r} - \frac{\tau_r}{r} + d_l - d_r\right)}{I_p + 2\left(M_w + \frac{I_w}{r^2}\right) d^2} \\
\dot{y}_5 = y_6 \\
\dot{y}_6 = \dfrac{(l^2 m + I_m)\left(-mly_2^2 \sin(y_1) - \frac{\tau_l}{r} - \frac{\tau_l}{r} - d_l - d_r\right) + m^2 g\, l^2 \sin(y_1) \cos(y_1)}{(l^2 m + I_M)\left(M + 2M_w + m + \frac{2I_w}{r^2}\right) - (\cos(y_1))^2 l^2 m^2}
\end{cases}
$$

Controllability

The forcing functions in the differential system are τ_r, τ_l, being the torque realized by the motors at the wheels. Also, another actuator opportunity is present in the system, being the external forces on the wheels d_l, d_r.

Before proceeding to controller design, a more rigorous description of system *controllability* should be defined.

Controllability Definition

Consider a nonlinear system described by the linear state space representation,

$$\dot{x} = f(x) + \sum_{i=1}^{M} g_i(x)u_i \,, x \in \Omega_x \subset \mathbb{R}^N$$

where $u = [u_1, u_2, \dots, u_n]^T \in \Omega_u \subset \mathbb{R}^M$ is the system input. The system is said to be *controllable* if there exists an input $u(t)$ such that the system can converge from an initial state $x(t_0) = x_0 \in \Omega_x$ to a final state $x(t_f) = x_f \in \Omega_x$within a finite time interval $t_f - t_0$.

- What other ways can we express this concept?
- How can we determine the controllability of a system analytically?

Controllability – Lie Algebra

An important algebraic abstraction in formulating controllability is a *Lie Algebra*. Recall the definition:

A Lie Algebra over a field $\mathbb{R}$ is a vector space $\mathbb{G}$ for which the bilinear map $[,]: \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}$ satisfies the conditions

$$[X, Y] = -[Y, X]$$
$$\left[X, [Y, Z]\right] = \left[Y, [Z, X]\right] = \left[Z, [X, Y]\right] = 0, X, Y, Z \in \mathbb{G}$$

The latter case being called the *Jacobi Identity*. The operator is known as the *Lie Bracket*.

Recursive Lie Algebra

Once a Lie bracket is introduced, it is possible to act on the Lie Bracket with another Lie Bracket. That is,

$$adj_f^n g = \left[f, \dots, [f, g]\right]$$

That is, each order of the Lie Bracket produces a vector space that is either linearly dependent or independent of all lower order Lie Brackets. Consider the distribution, Δ, which is the vector space created from all orders of Lie Brackets.

$$span\{\Delta\} = \{adj_f^1 g, adj_f^2 g, \dots, adj_f^n g\}$$

Controllability Theorem

If

$$\dim\{\Delta\} = n, \mathbb{L}\{\Delta\} \in \mathbb{R}^n$$

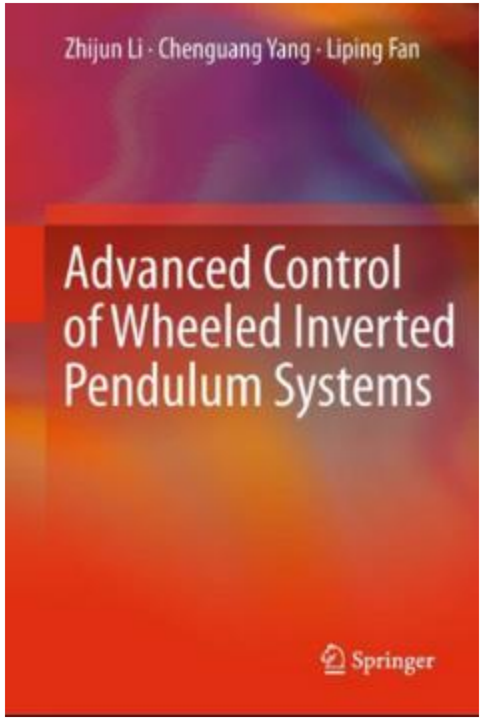
The system is controllable.

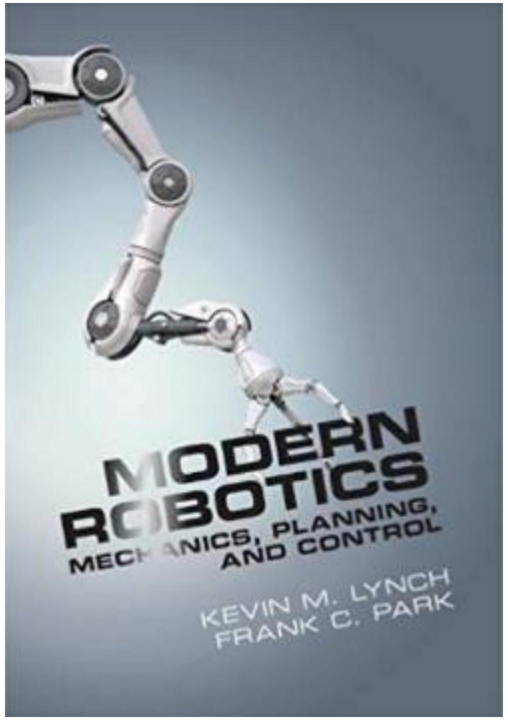
Lie Algebra Example

Consider the *chained-form system* with the state space representation. We observe that there are up to three linearly independent vector fields. How many lie brackets are possible in the given system?

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_2$$

Resources





A mathematically rigorous treatment of linear and non-linear controller design for TWIP systems

Li, Z., Yang, C., & Fan, L. (2012). *Advanced control of wheeled inverted pendulum systems*. Springer Science & Business Media.

An overview of robotic modeling techniques having the background information explained informally

Lynch, K., Park, F. (2017). *Modern Robotics: Mechanics, Planning, and Control*. Cambridge University Press.