

Notes for  
**HUMPHREYS'**  
GTM 9

Take note: Gau Syu

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# Chapter I

## Basic Concepts

### 1 Definition and first examples

**Definition 1.1.** A **Lie algebra** is a vector space with an skew-symmetric bilinear operation satisfying Jacobi identity.

**Example 1.1** ( $A_l$ ). Let  $\dim V = l + 1$ , the set of endomorphisms of  $V$  having trace 0, usually denoted by  $\mathfrak{sl}(V)$  or  $\mathfrak{sl}(l + 1, F)$ , is called **special linear algebra**.

$\dim A_l = (l + 1)^2 - 1$ , basis:

$$\begin{aligned} h_i &= e_{ii} - e_{i+1, i+1} \quad (1 \leq i \leq l) \\ e_{ij} &\quad (i \neq j) \end{aligned}$$

**Example 1.2** ( $C_l$ ). Let  $\dim V = 2l$ ,  $f$  be the symplectic form on  $V$ , the set of all endomorphisms  $x$  of  $V$  satisfying  $f(x(v), w) = -f(v, x(w))$ , usually denoted by  $\mathfrak{sp}(V)$  or  $\mathfrak{sp}(2l, F)$ , is called the **symplectic algebra**.

$$\begin{aligned} \mathfrak{sp}(2l, F) &= \{x \in \mathfrak{gl}(2l, F) \mid sx = -x^t s\} \quad s = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A, B, C \in \mathfrak{gl}(l, F), B = B^t, C = C^t \right\} \end{aligned}$$

$\dim C_l = 2l^2 + l$ , basis:

$$\begin{aligned} e_{ij} - e_{l+j, l+i} &\quad (1 \leq i, j \leq l) \\ e_{i, l+j} + e_{j, l+i} &\quad (1 \leq i < j \leq l) \\ e_{l+i, j} + e_{l+j, i} &\quad (1 \leq i < j \leq l) \\ e_{i, l+i} &\quad (1 \leq i \leq l) \\ e_{l+i, i} &\quad (1 \leq i \leq l) \end{aligned}$$

**Example 1.3** ( $B_l$ ). Let  $\dim V = 2l + 1$ ,  $f$  be the symmetric bilinear form on  $V$  whose matrix is  $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$ , the set of all endomorphisms  $x$  of  $V$  satisfying  $f(x(v), w) = -f(v, x(w))$ , usually denoted by  $\mathfrak{o}(V)$  or  $\mathfrak{o}(2l + 1, F)$ , is called the **orthogonal algebra**.

$$\begin{aligned} \mathfrak{o}(2l + 1, F) &= \{x \in \mathfrak{gl}(2l + 1, F) \mid sx = -x^t s\} \quad s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} 0 & b & c \\ -c^t & m & p \\ -b^t & q & -m^t \end{pmatrix} \mid p = -p^t, q = -q^t \right\} \end{aligned}$$

$\dim B_l = 2l^2 + l$ , basis:

$$\begin{aligned} e_{1+i, 1+j} - e_{1+l+j, 1+l+i} &\quad (1 \leq i, j \leq l) \\ e_{1+i, 1+l+j} - e_{1+j, 1+l+i} &\quad (1 \leq i < j \leq l) \\ e_{1+l+i, 1+j} - e_{1+l+j, 1+i} &\quad (1 \leq i < j \leq l) \\ e_{1, 1+i} - e_{1+l+i, 1} &\quad (1 \leq i \leq l) \\ e_{1, 1+l+i} - e_{1+i, 1} &\quad (1 \leq i \leq l) \end{aligned}$$

**Example 1.4** ( $D_l$ ). Let  $\dim V = 2l$ ,  $f$  be the symmetric bilinear form on  $V$  whose matrix is  $s = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$ , the set of all endomorphisms  $x$  of  $V$  satisfying  $f(x(v), w) = -f(v, x(w))$ , usually denoted by  $\mathfrak{o}(V)$  or  $\mathfrak{o}(2l, F)$ , is also called the **orthogonal algebra**.

$$\begin{aligned} \mathfrak{o}(2l, F) &= \{x \in \mathfrak{gl}(2l, F) \mid sx = -x^t s\} \quad s = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A, B, C \in \mathfrak{gl}(l, F), B = -B^t, C = -C^t \right\} \end{aligned}$$

$\dim D_l = 2l^2 - l$ , basis:

$$\begin{aligned} e_{ij} - e_{l+j, l+i} & \quad (1 \leq i, j \leq l) \\ e_{i, l+j} - e_{j, l+i} & \quad (1 \leq i < j \leq l) \\ e_{l+i, j} - e_{l+j, i} & \quad (1 \leq i < j \leq l) \end{aligned}$$

*Remark.* The Lie algebra corresponding to Lie groups  $O(n, F)$  and  $SO(n, F)$  consists of the skew-symmetric  $n \times n$  matrices, with the Lie bracket  $[\cdot, \cdot]$  given by the commutator. One Lie algebra corresponds to both groups. It is often denoted by  $\mathfrak{o}(n, F)$  or  $\mathfrak{so}(n, F)$ , and called the **orthogonal Lie algebra** or **special orthogonal Lie algebra**.

**Example 1.5.** The set of **upper triangular matrices**  $\mathfrak{t}(n, F)$ ; the set of **strictly upper triangular matrices**  $\mathfrak{n}(n, F)$ ; the set of all **diagonal matrices**  $\mathfrak{d}(n, F)$ .

$$[\mathfrak{d}(n, F), \mathfrak{n}(n, F)] = \mathfrak{n}(n, F) \quad (1.1)$$

$$[\mathfrak{t}(n, F), \mathfrak{t}(n, F)] = \mathfrak{n}(n, F) \quad (1.2)$$

**Example 1.6.** The only 2-dimensional non-Abelian Lie algebra has a basis  $x, y$  with commutation:

$$[x, y] = x$$

- Derivation

- Inner derivation
- Jacobi identity is equivalent to say all  $\text{ad } x$  are derivations.
- Adjoint representation

- Structure constants

**Exercise 1.1.** Let  $L$  be the real vector space  $\mathbb{R}^3$ . Define  $[xy] = x \times y$  (cross product of vectors) for  $x, y \in L$ , and verify that  $L$  is a Lie algebra. Write down the structure constants relative to the usual basis of  $\mathbb{R}^3$ .

**Solution.** Let  $e_1, e_2, e_3$  be the basis of  $L$ , then  $e_i \times e_j = e_k$  for  $(ijk)$  a cycle of  $(123)$ . To verify that  $L$  is a Lie algebra, it suffices to verify the Jacobi identity.

The structure constants are  $a_{12}^3 = 1, a_{23}^1 = 1, a_{13}^2 = -1$ . □

**Exercise 1.2.** Verify that the following equations and those implied by skew-symmetric bilinear define a Lie algebra structure on a three dimensional vector space with basis  $x, y, z$ :  $[xy] = z, [xz] = y, [yz] = 0$ .

**Solution.** The structure constants are  $a_{12}^3 = 1, a_{23}^1 = 0, a_{13}^2 = 1$ . □

**Exercise 1.3.** Let  $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  be an ordered basis for  $\mathfrak{sl}(2, F)$ . Compute the matrices of  $\text{ad } x, \text{ad } h, \text{ad } y$  relative to this basis.

**Solution.**  $\text{ad } x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ad } h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \text{ad } y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$   $\square$

**Exercise 1.4.** Find a linear Lie algebra isomorphic to the nonabelian two dimensional algebra constructed in Example 1.6.

**Solution.** Consider the adjoint representation  $\text{ad } x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ad } y = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$   $\square$

**Exercise 1.5.** Verify the assertions made in example 1.5, and compute dimension of each algebra, by exhibiting bases.

**Solution.**  $\dim \mathfrak{t}(n, F) = \frac{l^2+l}{2}$ , basis:  $e_{ij}(1 \leq i \leq j \leq l)$ ;  $\dim \mathfrak{n}(n, F) = \frac{l^2-l}{2}$ , basis:  $e_{ij}(1 \leq i < j \leq l)$ ;  $\dim \mathfrak{d}(n, F) = l$ , basis:  $e_{ii}(1 \leq i \leq l).$   $\square$

**Exercise 1.6.** Let  $x \in \mathfrak{gl}(n, F)$  have  $n$  distinct eigenvalues  $a_1, \dots, a_n$  in  $F$ . Prove that the eigenvalues of  $\text{ad } x$  are precisely the  $n^2$  scalars  $a_i - a_j(1 \leq i, j \leq n)$ , which of course need not be distinct.

**Solution.** Choose a basis for  $F^n$  so that  $x$  is a diagonal matrix whose entries are  $a_1, \dots, a_n$ . Then the matrices  $e_{ij}$  are eigenvalues of  $x$  since  $xe_{ij} - e_{ij}x = a_i e_{ij} - a_j e_{ij}.$   $\square$

**Exercise 1.7.** Let  $\mathfrak{s}(n, F)$  denote the **scalar matrices** in  $\mathfrak{gl}(n, F)$ . If  $\text{char } F$  is 0 or else a prime not dividing  $n$ , prove that  $\mathfrak{gl}(n, F) = \mathfrak{sl}(n, F) \oplus \mathfrak{s}(n, F)$ , with  $[\mathfrak{s}(n, F), \mathfrak{gl}(n, F)] = 0$ .

**Solution.** Choose  $x \in \mathfrak{gl}(n, F)$ . Let  $s$  be the scale  $\text{tr}(x)/n$ . Then  $x - s \in \mathfrak{sl}(n, F)$  and  $s \in \mathfrak{s}(n, F)$ , so  $\mathfrak{sl}(n, F)$  and  $\mathfrak{s}(n, F)$  generate  $\mathfrak{gl}(n, F)$ . Since the sum of their dimensions is  $n^2$ ,  $\mathfrak{gl}(n, F) = \mathfrak{sl}(n, F) + \mathfrak{s}(n, F)$  is a direct sum. Since scalar matrices commute with all other matrices, we also get  $[\mathfrak{s}(n, F), \mathfrak{gl}(n, F)] = 0.$   $\square$

**Exercise 1.8.** Verify the stated dimension of  $D_l$ .

**Exercise 1.9.** When  $\text{char } F = 0$ , show that each classical algebra  $L = A_l, B_l, C_l$  or  $D_l$  is equal to  $[LL]$ . (This shows again that each algebra consists of trace 0 matrices.)

**Solution.** It is sufficient to show  $L \subset [L, L]$ .

- $A_1$ :

$$\begin{aligned} e_{12} &= \frac{1}{2}[h, e_{12}] \\ e_{21} &= \frac{1}{2}[e_{21}, h] \\ h &= [e_{12}, e_{21}] \end{aligned}$$

- $A_l(l \geq 2)$ :

$$\begin{aligned} e_{ij} &= [e_{ik}, e_{kj}], & k \neq i, j; i \neq j \\ h_i &= [e_{ij}, e_{ji}], & j \neq i \end{aligned}$$

- $B_l(l \geq 2)$ :

$$\begin{aligned}
e_{1,l+i+1} - e_{i+1,1} &= [e_{1,j+1} - e_{l+j+1,1}, e_{j+1,l+i+1} - e_{i+1,l+j+1}] \\
e_{1,i+1} - e_{l+i+1,1} &= [e_{1,l+j+1} - e_{j+1,1}, e_{l+j+1,i+1} - e_{l+i+1,j+1}] \\
e_{i+1,i+1} - e_{l+i+1,l+i+1} &= [e_{i+1,1} - e_{1,l+i+1}, e_{1,i+1} - e_{l+i+1,1}] \\
e_{i+1,j+1} - e_{l+i+1,l+j+1} &= [e_{i+1,1} - e_{1,l+i+1}, e_{1,j+1} - e_{l+j+1,1}] \\
e_{i+1,l+j+1} - e_{j+1,l+i+1} &= [e_{i+1,i+1} - e_{l+i+1,l+i+1}, e_{i+1,l+j+1} - e_{j+1,l+i+1}] \\
e_{l+i+1,j+1} - e_{j+l+1,i+1} &= [e_{l+i+1,l+i+1} - e_{i+1,i+1}, e_{l+i+1,j+1} - e_{j+l+1,i+1}]
\end{aligned}$$

where  $1 \leq i \neq j \leq l$ .

- $C_l(l \geq 3)$ :

$$\begin{aligned}
e_{ii} - e_{l+i,l+i} &= [e_{i,l+i}, e_{l+i,i}] \\
e_{ij} - e_{l+j,l+i} &= [e_{ii} - e_{l+i,l+i}, e_{ij} - e_{l+j,l+i}], \quad i \neq j \\
e_{i,l+j} + e_{j,l+i} &= [e_{ii} - e_{l+i,l+i}, e_{i,l+j} + e_{j,l+i}] \\
e_{l+i,j} + e_{l+j,i} &= [e_{l+i,l+i} - e_{ii}, e_{l+i,j} + e_{l+j,i}]
\end{aligned}$$

- $D_l(l \geq 2)$ :

$$\begin{aligned}
e_{ii} - e_{l+i,l+i} &= \frac{1}{2}[e_{ij} - e_{l+j,l+i}, e_{ji} - e_{l+i,l+j}] \\
&\quad + \frac{1}{2}[e_{i,l+j} - e_{j,l+i}, e_{l+j,i} - e_{l+i,j}] \\
e_{ij} - e_{l+j,l+i} &= [e_{ii} - e_{l+i,l+i}, e_{ij} - e_{l+j,l+i}] \\
e_{i,l+j} - e_{j,l+i} &= [e_{ii} - e_{l+i,l+i}, e_{i,l+j} - e_{j,l+i}] \\
e_{l+i,j} - e_{l+j,i} &= [e_{l+i,l+i} - e_{ii}, e_{l+i,j} - e_{l+j,i}]
\end{aligned}$$

□

**Exercise 1.10.** For small values of  $l$ , isomorphisms occur among certain of the classical algebras. Show that  $A_1, B_1, C_1$  are all isomorphic, while  $D_1$  is the one dimensional Lie algebra. Show that  $B_2$  is isomorphic to  $C_2$ ,  $D_3$  to  $A_3$ . What can you say about  $D_2$ ?

**Solution.** The isomorphism of  $A_1, B_1, C_1$  is given as follows:

$$\begin{array}{ccccc}
A_1 & \rightarrow & B_1 & \mapsto & C_1 \\
e_{11} - e_{22} & \mapsto & 2(e_{22} - e_{33}) & \mapsto & e_{11} - e_{22} \\
e_{12} & \mapsto & 2(e_{13} - e_{21}) & \mapsto & e_{12} \\
e_{21} & \mapsto & 2(e_{12} - e_{31}) & \mapsto & e_{21}
\end{array}$$

For  $B_2, C_2$  we first calculate the eigenvectors for  $h_1 = e_{22} - e_{44}, h_2 = e_{33} - e_{55}$  and  $h'_1 = e_{11} - e_{33}, h'_2 = e_{22} - e_{44}$  respectively. We denote  $\lambda = (\lambda(h_1), \lambda(h_2))$  for the eigenvalue of  $h_1, h_2$ ,  $\lambda'$  is similar. See the following table:

$B_2$		$C_2$	
$\alpha = (1, 0)$	$e_{21} - e_{14}$	$\alpha' = (-1, 1)$	$e_{21} - e_{34}$
$-\alpha = (-1, 0)$	$e_{12} - e_{41}$	$-\alpha' = (1, -1)$	$e_{12} - e_{43}$
$\beta = (-1, 1)$	$e_{32} - e_{45}$	$\beta' = (2, 0)$	$e_{13}$
$-\beta = (1, -1)$	$e_{23} - e_{54}$	$-\beta' = (-2, 0)$	$e_{31}$
$\alpha + \beta = (0, 1)$	$e_{15} - e_{31}$	$\alpha' + \beta' = (1, 1)$	$e_{14} + e_{23}$
$-(\alpha + \beta) = (0, -1)$	$e_{13} - e_{51}$	$-(\alpha' + \beta') = (-1, -1)$	$e_{41} + e_{32}$
$2\alpha + \beta = (1, 1)$	$e_{25} - e_{34}$	$2\alpha' + \beta' = (0, 2)$	$e_{24}$
$-(2\alpha + \beta) = (-1, -1)$	$e_{43} - e_{52}$	$-(2\alpha' + \beta') = (0, -2)$	$e_{42}$

We make a linear transformation

$$\tilde{h}_1' = -\frac{1}{2}h_1' + \frac{1}{2}h_2', \tilde{h}_2' = \frac{1}{2}h_1' + \frac{1}{2}h_2'$$

Then  $\alpha(h_1) = \alpha'(\tilde{h}_1')$ ,  $\alpha(h_2) = \alpha'(\tilde{h}_2')$ ,  $\beta(h_1) = \beta'(\tilde{h}_1')$ ,  $\beta(h_2) = \beta'(\tilde{h}_2')$ . So the isomorphism of  $B_2, C_2$  is given as follows:

$$\begin{array}{ll}
B_2 & \rightarrow \\
e_{22} - e_{44} & \mapsto -\frac{1}{2}(e_{11} - e_{33}) + \frac{1}{2}(e_{22} - e_{44}) \\
e_{33} - e_{55} & \mapsto \frac{1}{2}(e_{11} - e_{33}) + \frac{1}{2}(e_{22} - e_{44}) \\
e_{12} - e_{41} & \mapsto \frac{\sqrt{2}}{2}(e_{12} - e_{43}) \\
e_{21} - e_{14} & \mapsto \frac{\sqrt{2}}{2}(e_{21} - e_{34}) \\
e_{32} - e_{45} & \mapsto e_{13} \\
e_{23} - e_{54} & \mapsto e_{31} \\
e_{15} - e_{31} & \mapsto \frac{\sqrt{2}}{2}(e_{14} + e_{23}) \\
e_{13} - e_{51} & \mapsto \frac{\sqrt{2}}{2}(e_{32} + e_{41}) \\
e_{25} - e_{34} & \mapsto e_{24} \\
e_{43} - e_{52} & \mapsto e_{42}
\end{array}$$

For  $A_3$  and  $D_3$ , we calculate the eigenvalues and eigenvectors for  $h_1 = e_{11} - e_{22}$ ,  $h_2 = e_{22} - e_{33}$ ,  $h_3 = e_{33} - e_{44}$  and  $h_1' = e_{11} - e_{44}$ ,  $h_2' = e_{22} - e_{55}$ ,  $h_3' = e_{33} - e_{66}$  respectively.

$A_3$		$D_3$	
$\alpha = (1, 1, -1)$	$e_{13}$	$\alpha' = (0, 1, 1)$	$e_{26} - e_{35}$
$-\alpha = (-1, -1, 1)$	$e_{31}$	$-\alpha' = (0, -1, -1)$	$e_{62} - e_{53}$
$\beta = (-1, 1, 1)$	$e_{24}$	$\beta' = (0, 1, -1)$	$e_{23} - e_{65}$
$-\beta = (1, -1, -1)$	$e_{42}$	$-\beta' = (0, -1, 1)$	$e_{32} - e_{56}$
$\gamma = (-1, 0, -1)$	$e_{41}$	$\gamma' = (1, -1, 0)$	$e_{12} - e_{54}$
$-\gamma = (1, 0, 1)$	$e_{14}$	$-\gamma' = (-1, 1, 0)$	$e_{21} - e_{45}$
$\alpha + \gamma = (0, 1, -2)$	$e_{43}$	$\alpha' + \gamma' = (1, 0, 1)$	$e_{16} - e_{34}$
$-(\alpha + \gamma) = (0, -1, 2)$	$e_{34}$	$-(\alpha' + \gamma') = (-1, 0, -1)$	$e_{61} - e_{43}$
$\beta + \gamma = (-2, 1, 0)$	$e_{21}$	$\beta' + \gamma' = (1, 0, -1)$	$e_{13} - e_{64}$
$-(\beta + \gamma) = (2, -1, 0)$	$e_{12}$	$-(\beta' + \gamma') = (-1, 0, 1)$	$e_{31} - e_{46}$
$\alpha + \beta + \gamma = (-1, 2, -1)$	$e_{23}$	$\alpha' + \beta' + \gamma' = (1, 1, 0)$	$e_{15} - e_{24}$
$-(\alpha + \beta + \gamma) = (1, -2, 1)$	$e_{32}$	$-(\alpha' + \beta' + \gamma') = (-1, -1, 0)$	$e_{51} - e_{42}$

We take a linear transformation

$$\tilde{h}_1' = -h_1' + h_3', \tilde{h}_2' = h_1' + h_2', \tilde{h}_3' = -h_1' - h_3'$$



Then  $\alpha(h_i) = \alpha'(\tilde{h}_i')$ ,  $\beta(h_i) = \beta'(\tilde{h}_i')$ ,  $\gamma(h_i) = \gamma'(\tilde{h}_i')$ ,  $i = 1, 2, 3$ . The isomorphism of  $A_3$  and  $D_3$  can be given as follows:

$A_3$	$\rightarrow$	$D_3$
$e_{11} - e_{22}$	$\mapsto$	$-(e_{11} - e_{44}) + (e_{33} - e_{66})$
$e_{22} - e_{33}$	$\mapsto$	$(e_{11} - e_{44}) + (e_{22} - e_{55})$
$e_{33} - e_{44}$	$\mapsto$	$-(e_{11} - e_{44}) - (e_{33} - e_{66})$
$e_{13}$	$\mapsto$	$e_{26} - e_{35}$
$e_{31}$	$\mapsto$	$e_{62} - e_{53}$
$e_{24}$	$\mapsto$	$e_{23} - e_{65}$
$e_{42}$	$\mapsto$	$e_{32} - e_{56}$
$e_{41}$	$\mapsto$	$e_{12} - e_{54}$
$e_{14}$	$\mapsto$	$e_{21} - e_{45}$
$e_{43}$	$\mapsto$	$e_{16} - e_{34}$
$e_{34}$	$\mapsto$	$e_{61} - e_{43}$
$e_{21}$	$\mapsto$	$e_{13} - e_{64}$
$e_{12}$	$\mapsto$	$e_{31} - e_{46}$
$e_{23}$	$\mapsto$	$e_{15} - e_{24}$
$e_{32}$	$\mapsto$	$e_{51} - e_{42}$

□

*Remark.* We have

$$\begin{aligned}
\mathfrak{so}(2) &\cong S^1 \\
\mathfrak{so}(3) &\cong \mathfrak{sl}(2) \cong \mathfrak{sp}(1) \\
\mathfrak{so}(4) &\cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \\
\mathfrak{so}(5) &\cong \mathfrak{sp}(4) \\
\mathfrak{so}(6) &\cong \mathfrak{sl}(4)
\end{aligned}$$

*Remark.* When  $F = \mathbb{C}$ , there exist another isomorphism:  $\mathfrak{su}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$ , where the lie structure of  $\mathfrak{su}(2, \mathbb{C})$  is given by

$$[e_i, e_j] = \epsilon_{ijk} e_k$$

i.e.

$$\begin{aligned}
[e_1, e_2] &= e_3 \\
[e_2, e_3] &= e_1 \\
[e_3, e_1] &= e_2
\end{aligned}$$

However,  $\mathfrak{su}(2, \mathbb{R}) \not\cong \mathfrak{sl}(2, \mathbb{R})$ .

This example shows that isomorphic Lie algebras over  $\mathbb{C}$  may **not** be isomorphic over other fields.

**Exercise 1.11.** Verify that the commutator of two derivations of an  $F$ -algebra is again a derivation, whereas the ordinary product need not be.

**Solution.** Let  $D_1$  and  $D_2$  be derivations of an  $F$ -algebra  $R$ , and pick  $x, y \in R$ . we check that  $[D_1, D_2]$  is a derivation:

$$\begin{aligned}
[D_1, D_2](xy) &= D_1(D_2(xy)) - D_2(D_1(xy)) \\
&= D_1(D_2(x)y + xD_2(y)) - (D_2(D_1(x)y + xD_1(y))) \\
&= D_1(D_2(x))y + D_2(x)D_1(y) + D_1(x)D_2(y) + xD_1(D_2(y)) - \\
&\quad (D_2(D_1(x))y + D_1(x)D_2(y) + D_2(x)D_1(y) + xD_2(D_1(y))) \\
&= (D_1(D_2(x)) - D_2(D_1(x)))y + x(D_1(D_2(y)) - D_2(D_1(y))) \\
&= [D_1, D_2](x)y + x[D_1, D_2](y).
\end{aligned}$$

Now consider the  $F$ -algebra  $F[x, y]$  with derivations  $\delta = \frac{\partial}{\partial x}$  and  $\varepsilon = \frac{\partial}{\partial y}$ . Then their product is not a derivation. If it were, then  $\varepsilon(a)\delta(b) + \delta(a)\varepsilon(b) = 0$  for all  $a, b \in F[x, y]$ , but this is false by taking  $a = x, b = y$ .  $\square$

**Exercise 1.12.** Let  $L$  be a Lie algebra over an algebraically closed field and let  $x \in L$ . Prove that the subspace of  $L$  spanned by the eigenvectors of  $\text{ad } x$  is a subalgebra.

**Solution.** By definition, it is closed under addition. To see that it is closed under the Lie bracket, we need only do so for eigenvectors  $v$  and  $w$  of  $\text{ad } x$ . In particular, we have  $[x, v] = av$  and  $[x, w] = bw$  for some  $a, b \in F$ . Then

$$[x, [v, w]] = [[x, v], w] - [[x, w], v] = a[v, w] - b[w, v] = (a + b)[v, w]$$

so  $[v, w]$  is also an eigenvector of  $\text{ad } x$ . Hence the subspace of  $L$  spanned by eigenvectors of  $\text{ad } x$  is a subalgebra of  $L$ .  $\square$

## 2 Ideals and homomorphisms

- **Normalizer**  $N_L(K)$ : the largest Lie subalgebra of  $L$  in which  $K$  is a Lie ideal.

– A subalgebra  $K$  is called **self-normalizing** if  $N_L(K) = K$ .

- If a derivation  $\delta$  is nilpotent, then  $e^\delta \in \text{Aut}(L)$ .

– Leibnitz' rule:

$$\frac{\delta^n}{n!}(xy) = \sum_{i=0}^n \frac{\delta^i x}{i!} \frac{\delta^{n-i} y}{(n-i)!}$$

**Exercise 2.1.** Prove that the set of all inner derivations is an ideal of  $\text{Der } L$ .

**Solution.** For any  $\delta \in \text{Der } L, x \in L$ ,  $[\delta, \text{ad } x] = \text{ad } \delta(x)$  is an inner derivation.  $\square$

**Exercise 2.2.** Show that  $\mathfrak{sl}(n, F)$  is precisely the derived algebra of  $\mathfrak{gl}(n, F)$  (cf. Exercise 1.9).

**Solution.** It is easy to see

$$[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] \subset \mathfrak{sl}(n, F)$$

Conversely, by exercise 1.9,

$$\mathfrak{sl}(n, F) = [\mathfrak{sl}(n, F), \mathfrak{sl}(n, F)] \subset [\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)]$$

$\square$

**Exercise 2.3.** Prove that the center of  $\mathfrak{gl}(n, F)$  equals  $\mathfrak{s}(n, F)$  (the scalar matrices). Prove that  $\mathfrak{sl}(n, F)$  has center 0, unless  $\text{char } F$  divides  $n$ , in which case the center is  $\mathfrak{s}(n, F)$ .

**Solution.** Clearly, we have  $\mathfrak{s}(n, F) \subset Z(\mathfrak{gl}(n, F))$ . Conversely, Let  $A = \sum_{i,j} a_{ij} e_{ij} \in Z(\mathfrak{gl}(n, F))$ , then for each  $e_{kl} \in \mathfrak{gl}(n, F)$ ,

$$\begin{aligned} [A, e_{kl}] &= \sum_{i,j} a_{ij} [e_{ij}, e_{kl}] \\ &= \sum_{i,j} a_{ij} (\delta_{jk} e_{il} - \delta_{li} e_{kj}) \\ &= \sum_{i=1}^n a_{ik} e_{il} - \sum_{j=1}^n a_{lj} e_{kj} \\ &= (a_{kk} - a_{ll}) e_{kl} + \sum_{\substack{i=1 \\ i \neq k}}^n a_{ik} e_{il} - \sum_{\substack{j=1 \\ j \neq l}}^n a_{lj} e_{kj} \end{aligned}$$

So

$$a_{kk} = a_{ll}, a_{ij} = 0, i \neq j$$

i.e.

$$A \in \mathfrak{s}(n, F)$$

For  $\mathfrak{sl}(n, F)$ , if  $c \in Z(\mathfrak{sl}(n, F))$ ,  $\forall x \in \mathfrak{sl}(n, F)$ ,  $[x, c] = 0$ . But we know  $\mathfrak{gl}(n, F) = \mathfrak{sl}(n, F) + \mathfrak{s}(n, F)$  and  $\mathfrak{s}(n, F)$  is the center of  $\mathfrak{gl}(n, F)$ . Hence  $c \in Z(\mathfrak{gl}(n, F)) = \mathfrak{s}(n, F)$ . We have

$$Z(\mathfrak{sl}(n, F)) = \mathfrak{sl}(n, F) \cap \mathfrak{s}(n, F)$$

If  $\text{char } F$  does not divide  $n$ , each  $aI \in \mathfrak{s}(n, F)$ ,  $a \neq 0$  has trace  $na \neq 0$ , so  $aI \notin \mathfrak{sl}(n, F)$ . i.e.,  $Z(\mathfrak{sl}(n, F)) = \mathfrak{sl}(n, F) \cap \mathfrak{s}(n, F) = 0$ . Else if  $\text{char } F$  divides  $n$ , each  $aI \in \mathfrak{s}(n, F)$  has trace  $na = 0$ , in this case  $Z(\mathfrak{sl}(n, F)) = \mathfrak{sl}(n, F) \cap \mathfrak{s}(n, F) = \mathfrak{s}(n, F)$ .  $\square$

**Exercise 2.4.** Show that (up to isomorphism) there is a unique Lie algebra over  $F$  of dimension 3 whose derived algebra has dimension 1 and lies in  $Z(L)$ .

**Solution.** Let  $L_0$  be the 3-dimensional lie algebra over  $F$  with basis  $(x_0, y_0, z_0)$  and commutation:

$$[x_0, y_0] = z_0, [x_0, z_0] = [y_0, z_0] = 0.$$

Suppose  $L$  be any 3-dimensional lie algebra over  $F$  whose derived algebra has dimension 1 and lies in  $Z(L)$ . We can take a basis  $(x, y, z)$  of  $L$  such that  $z \in [LL] \subset Z(L)$ . By hypothesis,  $[x, y] = \lambda z, [x, z] = [y, z] = 0, \lambda \in F$ . Then  $L \rightarrow L_0, x \mapsto x_0, y \mapsto y_0, z \mapsto \lambda z_0$  is an isomorphism.  $\square$

**Exercise 2.5.** Suppose  $\dim L = 3, L = [LL]$ . Prove that  $L$  must be simple. [Observe first that any homomorphic image of  $L$  also equals its derived algebra.] Recover the simplicity of  $\mathfrak{sl}(2, F)$ ,  $\text{char } F \neq 2$ .

**Solution.** Let  $I$  be a proper ideal of  $L$ . It is clear from surjectivity of  $L \rightarrow L/I$  that  $[L/I, L/I] = L/I$ . From this, we rule out  $\dim I = 2$  because then  $L/I$  would have to be Abelian,  $[L/I, L/I] = 0 \neq L/I$ . Also,  $\dim I = 1$  implies that  $\dim L/I = 2$ , and the only

non-Abelian 2-dimensional Lie algebra is described in Example 1.6 and is not equal to its derived algebra. Hence  $I = 0$ , so  $L$  is simple.

Since  $[\mathfrak{sl}(2, F), \mathfrak{sl}(2, F)] = \mathfrak{sl}(2, F)$  if  $\text{char } F \neq 2$ , and  $\dim \mathfrak{sl}(2, F) = 3$ , we see that  $\mathfrak{sl}(2, F)$  is simple for  $\text{char } F \neq 2$ .  $\square$

**Exercise 2.6.** Prove that  $\mathfrak{sl}(3, F)$  is simple, unless  $\text{char } F = 3$  (cf. Exercise 2.3). [Use the standard basis  $h_1, h_2, e_{ij} (i \neq j)$ . If  $I \neq 0$  is an ideal, then  $I$  is the direct sum of eigenspaces for  $\text{ad } h_1$  or  $\text{ad } h_2$ , compare the eigenvalues of  $\text{ad } h_1, \text{ad } h_2$  acting on the  $e_{ij}$ .]

**Solution.** Let  $I \neq 0$  be an ideal, and  $V_0 = \text{span}\{h_1, h_2\}$ . It is easy to see that If one of  $h_1, h_2$  is contained in  $I$ , then  $I = L$ .

Let  $a \in I$  be an eigenvector of  $\text{ad } h_1$  with eigenvalue  $\lambda$ , then by compute the eigenvalues of  $\text{ad } h_1$ , we see that there exist an  $a' \in L$  having eigenvalue  $-\lambda$ , then  $[a, a'] \in I$ . On the other hand,

$$\text{ad } h_1([a, a']) = [\lambda a, a'] + [a, -\lambda a'] = 0$$

therefore  $[a, a'] \in V_0$ . Since  $I$  is the direct sum of eigenspaces,  $V_0 \subset I$ , which implies  $I = L$ . Hence  $L$  is simple.  $\square$

**Exercise 2.7.** Prove that  $\mathfrak{t}(n, F)$  and  $\mathfrak{d}(n, F)$  are self-normalizing subalgebras of  $\mathfrak{gl}(n, F)$ , whereas  $\mathfrak{n}(n, F)$  has normalizer  $\mathfrak{t}(n, F)$ .

**Solution.** Let  $a = \sum_{ij} a_{ij} e_{ij} \in \mathfrak{gl}(n, F)$ ,  $[a, \mathfrak{t}(n, F)] \subset \mathfrak{t}(n, F)$ . But

$$\begin{aligned} [a, e_{kk}] &= \sum_{ij} a_{ij} \delta_{jk} e_{ik} - \sum_{ij} a_{ij} \delta_{ki} e_{kj} \\ &= \sum_i a_{ik} e_{ik} - \sum_j a_{kj} e_{kj} \\ &\subset \mathfrak{t}(n, F) \end{aligned}$$

It must be  $a_{ik} = 0$  for  $i > k$ , and  $a_{kj} = 0$  for  $j < k$ . Hence  $a_{kl} = 0$  for all  $k > l$ . This implies  $a \in \mathfrak{t}(n, F)$ , i.e.,  $\mathfrak{t}(n, F)$  is the self-normalizing subalgebras of  $\mathfrak{gl}(n, F)$ .

Similarly for  $\mathfrak{d}(n, F)$ , let  $a = \sum_{ij} a_{ij} e_{ij} \in \mathfrak{gl}(n, F)$ ,  $[a, \mathfrak{d}(n, F)] \subset \mathfrak{d}(n, F)$ . But

$$\begin{aligned} [a, e_{kk}] &= \sum_{ij} a_{ij} \delta_{jk} e_{ik} - \sum_{ij} a_{ij} \delta_{ki} e_{kj} \\ &= \sum_i a_{ik} e_{ik} - \sum_j a_{kj} e_{kj} \\ &\subset \mathfrak{d}(n, F) \end{aligned}$$

It must be  $a_{ik} = 0$  for  $i \neq k$ , and  $a_{kj} = 0$  for  $j \neq k$ . Hence  $a_{kl} = 0$  for all  $k \neq l$ . This implies  $a \in \mathfrak{d}(n, F)$ , i.e.,  $\mathfrak{d}(n, F)$  is the self-normalizing subalgebras of  $\mathfrak{gl}(n, F)$ .  $\square$

**Exercise 2.8.** Prove that in each classical linear Lie algebra, the set of diagonal matrices is a self-normalizing subalgebra, when  $\text{char } F = 0$ .

**Exercise 2.9.** Prove the basic theorem for homomorphisms of Lie algebras.

**Exercise 2.10.** Let  $\sigma$  be the automorphism of  $\mathfrak{sl}(2, F)$  defined as

$$\sigma = \exp \operatorname{ad} x \cdot \exp \operatorname{ad}(-y) \cdot \exp \operatorname{ad} x$$

Verify that  $\sigma(x) = -y, \sigma(y) = -x, \sigma(h) = -h$ .

**Solution.**

$$\begin{aligned}\exp \operatorname{ad} x(x) &= x \\ \exp \operatorname{ad} x(h) &= h - 2x \\ \exp \operatorname{ad} x(y) &= y + h - x \\ \exp \operatorname{ad}(-y)(x) &= x + h - y \\ \exp \operatorname{ad}(-y)(h) &= h - 2y \\ \exp \operatorname{ad}(-y)(y) &= y\end{aligned}$$

$$\begin{aligned}\sigma(x) &= \exp \operatorname{ad} x \exp \operatorname{ad}(-y)(x) \\ &= \exp \operatorname{ad} x(x + h - y) \\ &= x + h - 2x - y - h + x \\ &= -y \\ \sigma(y) &= \exp \operatorname{ad} x \exp \operatorname{ad}(-y)(y + h - x) \\ &= \exp \operatorname{ad} x(y + h - 2y - x - h + y) \\ &= \exp \operatorname{ad} x(-x) \\ &= -x \\ \sigma(h) &= \exp \operatorname{ad} x \exp \operatorname{ad}(-y)(h - 2x) \\ &= \exp \operatorname{ad} x(h - 2y - 2(x + h - y)) \\ &= \exp \operatorname{ad} x(-h - 2x) \\ &= -h + 2x - 2x = -h\end{aligned}$$

□

**Exercise 2.11.** If  $L = \mathfrak{sl}(n, F), g \in \operatorname{GL}(n, F)$ , prove that the map of  $L$  to itself defined by  $x \mapsto -gx^t g^{-1}$  ( $x^t = \text{transpose of } x$ ) belongs to  $\operatorname{Aut} L$ . When  $n = 2, g = \text{identity matrix}$ , prove that this automorphism is inner.

**Solution.**  $g \in \operatorname{GL}(n, F)$  and  $\operatorname{Tr}(-gx^t g^{-1}) = -\operatorname{Tr}(x)$ , i.e.,  $\operatorname{Tr}(x) = 0$  if and only if so is  $\operatorname{Tr}(-gx^t g^{-1})$ . So the map  $x \mapsto -gx^t g^{-1}$  is a linear space automorphism of  $\mathfrak{sl}(n, F)$ . We just verify it is a homomorphism of lie algebras:

$$\begin{aligned}[-gx^t g^{-1}, -gy^t g^{-1}] &= gx^t y^t g^{-1} - gy^t x^t g^{-1} \\ &= -g((xy)^t - (yx)^t)g^{-1} \\ &= -g[x, y]^t g^{-1}\end{aligned}$$

When  $n = 2, g = \text{identity matrix}$ , the automorphism  $\sigma: x \mapsto -x^t$ , i.e.

$$\sigma(x) = -y, \sigma(y) = -x, \sigma(h) = -h$$

So  $\sigma = \exp \operatorname{ad} x \exp \operatorname{ad}(-y) \exp \operatorname{ad} x$  is an inner automorphism.

□

*Remark.* Warning: An inner automorphism is not exactly of form  $\exp \operatorname{adx}$  with  $\operatorname{adx}$  is nilpotent. It can be the composition of elements with this form.

**Exercise 2.12.** Let  $L$  be an orthogonal Lie algebra (type  $B_l$  or  $D_l$ ). If  $g$  is an **orthogonal matrix**, in the sense that  $g$  is invertible and  $g^t s g = s$ , prove that  $x \mapsto g x g^{-1}$  defines an automorphism of  $L$ .

**Solution.**  $x \in B_l$  or  $D_l$ ,  $s x = -x^t s$ . Hence

$$\begin{aligned} s g x g^{-1} &= (g^{-1})^t s x g^{-1} \\ &= -(g^{-1})^t x^t s g^{-1} \\ &= -(g^{-1})^t x^t g^t s \\ &= -(g x g^{-1})^t s \end{aligned}$$

So the map  $x \mapsto g x g^{-1}$  is a linear automorphism of  $B_l$  or  $C_l$ . We just verify it is a homomorphism of lie algebras:

$$[g x g^{-1}, g y g^{-1}] = g x y g^{-1} - g y x g^{-1} = g[x, y] g^{-1}$$

□

### 3 Solvable and nilpotent Lie algebras

**Exercise 3.1.** Let  $I$  be an ideal of  $L$ . Then each member of the derived series or descending central series of  $I$  is also an ideal of  $L$ .

**Solution.** For the derived series, it is enough to show that  $[I, I]$  is an ideal by induction. Pick  $x, y \in I$  and  $z \in L$ . Then  $[z, [x, y]] = -[y, [z, x]] - [x, [y, z]] \in [I, I]$  since  $[z, x], [y, z] \in I$ . For the descending central series of  $I$ , we have seen that  $I^1 = I^{(1)}$  is an ideal. So by induction, suppose that  $I^k$  is an ideal. Then pick  $x \in I, y \in I^k, z \in L$ . We have  $[z, [x, y]] = -[y, [z, x]] - [x, [y, z]] \in I^{k+1}$  since  $[z, x] \in I$  and  $[y, z] \in I^k$ . So  $I^{k+1}$  is also an ideal. □

**Exercise 3.2.** Prove that  $L$  is solvable if and only if there exists a chain of subalgebras  $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$  such that  $L_{i+1}$  is an ideal of  $L_i$  and such that each quotient  $L_i/L_{i+1}$  is abelian.

**Solution.** If  $L$  is solvable, take  $L_i = L^{(i)}$ . Then  $[L_i, L_i]$  is an ideal of  $L_i$  by the Jacobi identity, and  $L_i/[L_i, L_i]$  is abelian. Conversely, given such a chain of subalgebras, we see by induction that  $L^{(i)} \subset L_i$  because  $[L_i, L_i]$  is the smallest ideal  $I$  for which  $L_i/I$  is abelian. □

**Exercise 3.3.** Let  $\operatorname{char} F = 2$ . Prove that  $\mathfrak{sl}(2, F)$  is nilpotent.

**Solution.** Let  $(x, h, y)$  be the standard basis for  $\mathfrak{sl}(2, F)$ , then  $[hx] = 2x = 0, [xy] = h, [hy] = -2y = 0$ . Hence  $[\mathfrak{sl}(2, F), \mathfrak{sl}(2, F)] = Fh$ , then  $\mathfrak{sl}(2, F)$  is nilpotent. □

**Exercise 3.4.** Prove that  $L$  is solvable (resp. nilpotent) if and only if  $\operatorname{ad} L$  is solvable (resp. nilpotent).

**Solution.**  $\operatorname{ad} L$  is a homomorphic image of  $L$ , moreover,  $\operatorname{ad} L \cong L/Z(L)$ . □

**Exercise 3.5.** Prove that the non-abelian two dimensional algebra constructed in Example 1.6 is solvable but not nilpotent. Do the same for the algebra in Exercise 1.2.

**Solution.** The 2-dimensional non-abelian Lie algebra  $L$  has basis  $\{x, y\}$  such that  $[x, y] = x$ . Then it is clear that  $L^i$  is the subspace spanned by  $x$  for  $i > 0$ , so  $L$  is not nilpotent. However,  $L^{(1)} = \langle x \rangle$ , so  $L^{(2)} = 0$  and hence  $L$  is solvable.

The Lie algebra  $L$  of Exercise 1.2 has a basis  $\{x, y, z\}$  such that  $[x, y] = z, [x, z] = y$  and  $[y, z] = 0$ . Then  $L^i = \langle y, z \rangle$  for  $i > 0$ , so  $L$  is not nilpotent. However,  $L^{(1)} = \langle y, z \rangle$  and  $L^{(2)} = 0$ , so  $L$  is solvable.  $\square$

**Exercise 3.6.** Prove that the sum of two nilpotent ideals of a Lie algebra  $L$  is again a nilpotent ideal. Therefore,  $L$  possesses a unique maximal nilpotent ideal. Determine this ideal for each algebra in Exercise 3.5.

**Solution.** Let  $I, J$  are nilpotent ideals. We can deduce by induction that

$$(I + J)^n \subset \sum_{k=0}^n I^k \cap J^{n-k}$$

where  $I^0 = J^0 = L$ . Then  $I + J$  is clear a nilpotent ideal.

Taking the sum of all nilpotent ideals of  $L$  gives a unique maximal nilpotent ideal.

In the 2-dimensional algebra of Example 1.6, the maximal nilpotent ideal can have dimension at most 1 since it is not itself nilpotent, hence is the subspace spanned by  $x$ . Similarly, the unique maximal nilpotent ideal of the Lie algebra of Exercise 1.2 is the subspace spanned by  $y$  and  $z$ .  $\square$

**Exercise 3.7.** Let  $L$  be nilpotent,  $K$  a proper subalgebra of  $L$ . Prove that  $N_L(K)$  includes  $K$  properly.

**Solution.** Since  $K$  is a subalgebra of  $L$ ,  $\text{ad } K$  acts on  $L/K$  (quotient taken as vector spaces). Since  $K$  is a proper subalgebra of  $L$ , we have  $L/K \neq 0$ , so there exists a vector  $v \notin K$  such that  $[K, v] \subset K$ . In particular,  $v \in N_L(K)$ , so  $N_L(K)$  properly contains  $K$ .  $\square$

**Exercise 3.8.** Let  $L$  be nilpotent. Prove that  $L$  has an ideal of codimension 1.

**Solution.** If  $\dim L = 1$ , then  $0$  is a codimension 1 ideal. So suppose  $\dim L > 1$ . If  $L$  is abelian, then we take any codimension 1 subspace. Otherwise,  $0 < \dim L/[L, L] < \dim L$  since  $L$  is nilpotent. Since  $L/[L, L]$  is abelian, it has a codimension 1 ideal  $I$ . By the dimension formula for vector spaces the inverse image of  $I$  has codimension 1 in  $L$ .  $\square$

**Exercise 3.9.** Prove that every nilpotent Lie algebra  $L$  has an outer derivation, as follows: Write  $L = K + Fx$  for some ideal  $K$  of codimension one (Exercise 3.8). Then  $C_L(K) \neq 0$  (why?). Choose  $n$  so that  $C_L(K) \subset L^n, C_L(K) \not\subset L^{n+1}$ , and let  $z \in C_L(K) - L^{n+1}$ . Then the linear map  $\delta$  sending  $K$  to 0,  $x$  to  $z$ , is an outer derivation.

**Solution.** If  $K = 0$ , then  $C_L(K) = L$ . Otherwise,  $K$  is nonzero and nilpotent since it is a subalgebra of  $L$ , and hence  $Z(K) \neq 0$ . Since  $Z(K) \subset C_L(K)$ , we conclude that  $C_L(K) \neq 0$ .

For all  $k_1 + \lambda_1 x, k_2 + \lambda_2 x \in L$ ,  $[k_1 + \lambda_1 x, k_2 + \lambda_2 x] \in K$ , so  $\delta([k_1 + \lambda_1 x, k_2 + \lambda_2 x]) = 0$ . In the other hand,

$$\begin{aligned} & [\delta(k_1 + \lambda_1 x), k_2 + \lambda_2 x] + [k_1 + \lambda_1 x, \delta(k_2 + \lambda_2 x)] \\ &= [\lambda_1 z, k_2 + \lambda_2 x] + [k_1 + \lambda_1 x, \lambda_2 z] \\ &= \lambda_1 \lambda_2 [z, x] + \lambda_1 \lambda_2 [x, z] \\ &= 0 \end{aligned}$$

We conclude that  $\delta$  is a derivation. If  $\delta$  is a inner derivation,  $\delta = \text{ad } y$ , then  $[y, K] = \delta(K) = 0$ , so  $y \in C_L(K) \in L^n$ . Then we have  $[y, x] \in L^{n+1}$ . But  $[y, x] = \delta(x) = z \notin L^{n+1}$ . This is a contradiction. So  $\delta$  is a outer derivation.  $\square$

**Exercise 3.10.** Let  $L$  be a Lie algebra,  $K$  an ideal of  $L$  such that  $L/K$  is nilpotent and such that  $\text{ad } x|_K$  is nilpotent for all  $x \in L$ . Prove that  $L$  is nilpotent.

**Solution.** If  $L/K$  is nilpotent, say  $(L/K)^n = 0$ , then we know that  $L^n \subset K$ . The fact that  $\text{ad } x|_K$  is nilpotent for all  $x \in L$  then implies that  $\text{ad } x|_{L^n}$  is nilpotent for all  $x \in L$ , and hence  $L$  is nilpotent by Engel's theorem.  $\square$



## Chapter II

# Semisimple Lie Algebras

### 4 Theorems of Lie and Cartan

**Exercise 4.1.** Let  $L = \mathfrak{sl}(V)$ . Use Lie's Theorem to prove that  $\text{Rad } L = Z(L)$ , conclude that  $L$  is semisimple (cf. Exercise 2.3).

**Solution.** Observe that  $\text{Rad } L$  lies in each maximal solvable subalgebra  $B$  of  $L$ . Select a basis of  $V$  so that  $B = L \cap \mathfrak{t}(n, F)$ , and notice that the transpose of  $B$  is also a maximal solvable subalgebra of  $L$ . Conclude that  $\text{Rad } L \subset L \cap \mathfrak{d}(n, F)$ , then that  $\text{Rad } L = Z(L)$ .  $\square$

**Exercise 4.2.** Show that the proof of Theorem 4.1 still goes through in prime characteristic, provided  $\dim V$  is less than  $\text{char } F$ .

**Solution.** The only part in which  $\text{char } F = 0$  is used in the proof of Theorem 4.1 is to show that  $n\lambda([x, y]) = 0$  implies  $\lambda([x, y]) = 0$ . Here  $n < \dim V$ , so using our relaxed condition, this implication still holds.  $\square$

**Exercise 4.3.** This exercise illustrates the failure of Lie's Theorem when  $F$  is allowed to have prime characteristic  $p$ . Consider the  $p \times p$  matrices:

$$x = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad y = \text{diag}(0, 1, 2, 3, \dots, p-1)$$

Check that  $[x, y] = x$ , hence that  $x$  and  $y$  span a two dimensional solvable subalgebra  $L$  of  $\mathfrak{gl}(p, F)$ . Verify that  $x, y$  have no common eigenvector.

**Solution.**  $[x, y] = x$ . However, the eigenvectors of  $y$  are the standard basis vectors, and none of these are eigenvectors for  $x$  since it operates by shifting entries.  $\square$

**Exercise 4.4.** When  $p = 2$ , Exercise 3.3 show that a solvable Lie algebra of endomorphisms over a field of prime characteristic  $p$  need not have derived algebra consisting of nilpotent endomorphisms (cf. Corollary C of Theorem 4.1). For arbitrary  $p$ , construct a counterexample to Corollary C as follows: Start with  $L \subset \mathfrak{gl}(p, F)$  as in Exercise 4.3. Form the vector space direct sum  $M = L + F^p$ , and make  $M$  a Lie algebra by decreeing that  $F^p$  is abelian, while  $L$  has its usual product and acts on  $F^p$  in the given way. Verify that  $M$  is solvable, but that its derived algebra ( $= Fx + F^p$ ) fails to be nilpotent.

**Exercise 4.5.** If  $x, y \in \text{End } V$  commute, prove that  $(x + y)_s = x_s + y_s$ , and  $(x + y)_n = x_n + y_n$ . Show by example that this can fail if  $x, y$  fail to commute. [Hint: Show first that  $x, y$  semisimple (resp. nilpotent) implies  $x + y$  semisimple (resp. nilpotent).]

**Solution.** For a counterexample when  $x$  and  $y$  do not commute, take  $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ . Then both  $x$  and  $y$  are nilpotent, but  $x + y$  is not nilpotent because its eigenvalues are  $\pm 1$ .  $\square$

**Exercise 4.6.** Check formula

$$(\delta - (a + b).1)^n(xy) = \sum_{i=0}^n \binom{n}{i} ((\delta - a.1)^{n-i}x) \cdot ((\delta - b.1)^i y)$$

**Exercise 4.7.** Prove the converse of Theorem 4.3.

**Solution.** The converse of Theorem 4.3 says that if  $L$  is a solvable subalgebra of  $\mathfrak{gl}(V)$  where  $\dim V < \infty$ , then  $\text{Tr}(xy) = 0$  for all  $x \in [L, L]$  and  $y \in L$ . By Lie's theorem, we may choose a basis for  $V$  such that  $L$  consists of upper triangular matrices. Then  $x \in [L, L]$  is a strictly upper triangular matrix, and hence so is  $yx$ . Finally,  $\text{tr}(xy) = \text{tr}(yx) = 0$ , so we are done.  $\square$

**Exercise 4.8.** Note that it suffices to check the hypothesis of Theorem 4.3 (or its corollary) for  $x, y$  ranging over a basis of  $[L, L]$ , resp.  $L$ . For the example given in Exercise 1.2, verify solvability by using Cartan's criterion.

**Solution.** In the example given in Exercise 1.2,

$$\text{ad } x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \text{ad } y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \text{ad } z = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also,  $[L, L]$  is spanned by  $y$  and  $z$ . Hence  $\text{Tr}(\text{ad } x \text{ ad } y) = \text{Tr}(\text{ad } x \text{ ad } z) = \text{Tr}(\text{ad } y \text{ ad } z) = 0$ , so  $L$  is solvable.  $\square$

## 5 Killing form

**Exercise 5.1.** Prove that if  $L$  is nilpotent, the Killing form of  $L$  is identically zero.

**Solution.** Pick  $x, y \in L$ . Then  $\text{ad}([x, y])$  is nilpotent, and hence  $\text{Tr}(\text{ad}([x, y])) = 0$ . This implies  $\text{Tr}(\text{ad } x \text{ ad } y) = -\text{Tr}(\text{ad } y \text{ ad } x)$ , but  $\text{Tr}(\text{ad } x \text{ ad } y) = \text{Tr}(\text{ad } y \text{ ad } x)$  then gives that  $\text{Tr}(\text{ad } x \text{ ad } y) = 0$ , so the Killing form of  $L$  is identically zero.  $\square$

**Exercise 5.2.** Prove that  $L$  is solvable if and only if  $[LL]$  lies in the radical of the Killing form.

**Solution.** If  $L$  is solvable, then  $[L, L]$  lies in the radical of the Killing form by the corollary to Theorem 4.3. The converse is Exercise 4.7.  $\square$

**Exercise 5.3.** Let  $L$  be the two dimensional non-abelian Lie algebra of Exercise 1.6, which is solvable. Prove that  $L$  has nontrivial Killing form.

**Solution.** The image of the adjoint representation of  $L$  is a subalgebra of  $\mathfrak{gl}(2, F)$  with basis elements  $\text{ad } x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ad } y = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ . So  $\text{Tr}(\text{ad } x \text{ ad } x) = 1$ , and hence the Killing form of  $L$  is nontrivial.  $\square$

**Exercise 5.4.** Let  $L$  be the three dimensional solvable Lie algebra of Exercise 1.2. Compute the radical of its Killing form.

**Solution.** We compute the matrix of Killing Form  $\kappa$  relative to the basis  $(x, y, z)$ :

$$\kappa = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let  $s = ax + by + cz$  be any element in the radical  $S$  of  $\kappa$ . Then

$$(a, b, c) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

So  $a = 0$ , and we see that  $S$  is spanned by  $y$  and  $z$ . □

**Exercise 5.5.** Let  $L = \mathfrak{sl}(2, F)$ . Compute the basis of  $L$  dual to the standard basis, relative to the Killing form.

**Solution.** The matrix of the Killing form relative to the basis  $(x, h, y)$  is

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

The basis of  $L$  dual to the standard basis is  $(\frac{1}{4}y, \frac{1}{8}h, \frac{1}{4}x)$ . □

**Exercise 5.6.** Let  $\text{char } F = p \neq 0$ . Prove that  $L$  is semisimple if its Killing form is nondegenerate. Show by example that the converse fails. [Look at  $\mathfrak{sl}(3, F)$  modulo its center, when  $\text{char } F = 3$ .]

**Solution.** If  $\text{Rad}(L) \neq 0$ , the last nonzero term  $I$  in its derived series is a abelian subalgebra of  $L$ , and by Exercise 3.1,  $I$  is a ideal of  $L$ . In another words,  $L$  has a nonzero abelian ideal. It is suffice to prove any abelian ideal of  $L$  is zero.

Let  $S$  be the radical of the Killing form, which is nondegenerate. So  $S = 0$ . To prove that  $L$  is semisimple, it will suffice to prove that every abelian ideal  $I$  of  $L$  is included in  $S$ . Suppose  $x \in I, y \in L$ . Then  $\text{ad } x \text{ ad } y$  maps  $L \rightarrow L \rightarrow I$ , and  $(\text{ad } x \text{ ad } y)^2$  maps  $L$  into  $[II] = 0$ . This means that  $\text{ad } x \text{ ad } y$  is nilpotent, hence that  $0 = \text{Tr}(\text{ad } x \text{ ad } y) = \kappa(x, y)$ , so  $I \subset S = 0$ . □

**Exercise 5.7.** Relative to the standard basis of  $\mathfrak{sl}(3, F)$ , compute the determinant of  $\kappa$ . Which primes divide it?

**Solution.** We write down the matrix of  $\text{ad } x$  relative to basis

$$\{e_{11} - e_{22}, e_{22} - e_{33}, e_{12}, e_{13}, e_{21}, e_{23}, e_{31}, e_{33}\}$$

when  $x$  runs over this basis.

$$\begin{aligned} \text{ad}(e_{11} - e_{22}) &= \text{diag}(0, 0, 2, 1, -2, -1, -1, 1) \\ \text{ad}(e_{22} - e_{33}) &= \text{diag}(0, 0, -1, 1, 1, 2, -1, -2) \end{aligned}$$



The matrix of the Killing form relative to this basis is

$$\kappa = \begin{pmatrix} 12 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \end{pmatrix}$$

Its determinant is  $\det(\kappa) = 2^8 3^9$ , so prime 2 and 3 divide the determinant of  $\kappa$ .  $\square$

**Exercise 5.8.** Let  $L = L_1 \oplus \cdots \oplus L_t$  be the decomposition of a semisimple Lie algebra  $L$  into its simple ideals. Show that the semisimple and nilpotent parts of  $x \in L$  are the sums of the semisimple and nilpotent parts in the various  $L_i$  of the components of  $x$ .

**Solution.** Write  $x = x_1 + \cdots + x_t$  where  $x_i \in L_i$ . We can decompose each  $x_i$  as  $x_{i,s} + x_{i,n}$  where  $x_{i,s}$  is semisimple and  $x_{i,n}$  is nilpotent. Note that  $\text{ad } x_i$  and  $\text{ad } x_j$  commute since  $[x_i, x_j] = 0$ . Hence  $\text{ad } x_{i,s}$  and  $\text{ad } x_{j,s}$  commute, as well as  $\text{ad } x_{i,n}$  and  $\text{ad } x_{j,n}$ . This means that  $x_{1,s} + \cdots + x_{t,s}$  is semisimple and that  $x_{1,n} + \cdots + x_{t,n}$  is nilpotent, so by uniqueness of Jordan-Chevalley decomposition, we conclude that  $x_s = x_{1,s} + \cdots + x_{t,s}$  and that  $x_n = x_{1,n} + \cdots + x_{t,n}$ .  $\square$

## 6 Complete reducibility of representations

**Exercise 6.1.** Using the standard basis for  $L = \mathfrak{sl}(2, F)$ , write down the Casimir element of the adjoint representation of  $L$  (cf. Exercise 5.5). Do the same thing for the usual (3-dimensional) representation of  $\mathfrak{sl}(3, F)$ , first computing dual bases relative to the trace form.

**Solution.** For the adjoint representation of  $L = \mathfrak{sl}(2, F)$ , The matrix of  $\beta$  respect to basis  $(x, h, y)$  is

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

we can deduce the dual basis of  $(x, h, y)$  is  $(\frac{1}{4}y, \frac{1}{8}h, \frac{1}{4}x)$ . So the Casimir element of this representation is

$$c_{\text{ad}} = \frac{1}{4} \text{ad } x \text{ad } y + \frac{1}{8} \text{ad } h \text{ad } h + \frac{1}{4} \text{ad } y \text{ad } x$$

For the usual representation of  $L = \mathfrak{sl}(3, F)$ , The matrix of  $\beta$  respect to basis

$$\{e_{11} - e_{22}, e_{22} - e_{33}, e_{12}, e_{13}, e_{21}, e_{23}, e_{31}, e_{33}\}$$

is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We can deduce the dual basis is

$$\frac{2}{3}e_{11} - \frac{1}{3}e_{22} - \frac{1}{3}e_{33}, \frac{1}{3}e_{11} + \frac{1}{3}e_{22} - \frac{2}{3}e_{33}, e_{21}, e_{31}, e_{32}, e_{12}, e_{13}, e_{23}$$

So

$$c_\varphi = \sum_x xx' = \begin{pmatrix} \frac{8}{3} & 0 & 0 \\ 0 & \frac{8}{3} & 0 \\ 0 & 0 & \frac{8}{3} \end{pmatrix}$$

□

**Exercise 6.2.** Let  $V$  be an  $L$ -module. Prove that  $V$  is a direct sum of irreducible submodules if and only if each  $L$ -submodule of  $V$  possesses a complement.

**Solution.** If each  $L$ -submodule of  $V$  possesses a complement, then we can write  $V$  as a direct sum of irreducible submodules by induction on  $\dim V$ .

Conversely, suppose that  $V$  is a direct sum of irreducible submodules  $V_1 \oplus \cdots \oplus V_r$ , and let  $W$  be a proper  $L$ -submodule of  $V$ . The map  $V \rightarrow V/W$  is surjective, and hence there is some  $i$  such that  $V_i \rightarrow V/W$  is a nonzero map. Since  $V_i$  is irreducible, it must be injective, which means that  $V_i \cap W = 0$ . By induction on codimension,  $V_i + W$  has a direct sum complement  $W''$ . Set  $W' = W'' + V_i$ . Then  $W \cap (V_i + W'') = 0$  and  $V = W \oplus W'$ . □

**Exercise 6.3.** If  $L$  is solvable, every irreducible representation of  $L$  is one dimensional.

**Solution.** Let  $V$  be a representation of  $L$ . By Lie's theorem, there is a basis for  $V$  such that  $L$  acts by upper triangular matrices. Then the subspace of  $V$  spanned by the first basis vector is invariant under  $L$ . Hence if  $V$  is irreducible, it must be 1-dimensional. □

**Exercise 6.4.** Use Weyl's Theorem to give another proof that for  $L$  semisimple,  $\text{ad } L = \text{Der } L$  (Theorem 5.3). [If  $\delta \in \text{Der } L$ , make the direct sum  $F + L$  into an  $L$ -module via the rule  $x.(a, y) = (0, a\delta(x) + [xy])$ . Then consider a complement to the submodule  $L$ .]

**Solution.** Clearly,  $L$  is a submodule of  $F + L$ . By Weyl's Theorem, it has a complement of dimension 1. Let  $(a_0, x_0), a_0 \neq 0$  be its basis. Then  $L$  acts on it trivially. Hence

$$0 = x.(a_0, x_0) = (0, a_0\delta(x) + [x, x_0])$$

i.e.

$$\delta(x) = [\frac{1}{a_0}x_0, x] = \text{ad } \frac{1}{a_0}x_0(x)$$

So  $\delta \in \text{Int } L$ .

□

**Exercise 6.5.** A Lie algebra  $L$  for which  $\text{Rad } L = Z(L)$  is called **reductive**. (Examples:  $L$  abelian,  $L$  semisimple,  $L = \mathfrak{gl}(n, F)$ .)

1. If  $L$  is reductive, then  $L$  is a completely reducible  $\text{ad } L$ -module. [If  $\text{ad } L \neq 0$ , use Weyl's Theorem.] In particular,  $L$  is the direct sum of  $Z(L)$  and  $[LL]$ , with  $[LL]$  semisimple.
2. If  $L$  is a classical linear Lie algebra, then  $L$  is semisimple. [Cf. Exercise 1.9.]
3. If  $L$  is a completely reducible  $\text{ad } L$ -module, then  $L$  is reductive.
4. If  $L$  is reductive, then all finite dimensional representations of  $L$  in which  $Z(L)$  is represented by semisimple endomorphisms are completely reducible.

**Solution.** 1. Let  $L$  be reductive. If  $L$  is abelian, then it is clearly completely reducible as an  $\text{ad } L$ -module ( $\text{ad } L = 0$ ). So assume  $\text{ad } L \neq 0$ . Since  $\text{ad } L \cong L/Z(L) = L/\text{Rad } L$ , we see that  $\text{ad } L$  is semisimple. So by Weyl's theorem,  $L$  is a completely reducible  $\text{ad } L$ -module.

$L/Z(L)$  is semisimple, so  $[LL]/Z(L) \cong [L/Z(L), L/Z(L)] \cong L/Z(L)$ , hence

$$L = Z(L) + [LL]$$

On the other hand,  $Z(L)$  is a  $\text{ad } L$ -submodule of  $L$  and  $L$  is a completely reducible  $\text{ad } L$ -module. So  $Z(L)$  has a component  $M$  in  $L$ .

$$L = M \oplus Z(L)$$

where  $M$  is a ideal of  $L$ .

$$[LL] \subset [M \oplus Z(L), M \oplus Z(L)] \subset [M, M] \subset M$$

We conclude that

$$L = [LL] \oplus Z(L)$$

Hence  $[LL] \cong L/Z(L)$  is semisimple.

2. If  $L$  is a classical linear Lie algebra, by Exercise 4.1,  $\text{Rad } L = Z(L)$ . And by Exercise 1.9,  $Z(L) = 0$ , so  $L = [LL]$  is semisimple.
3.  $L$  is a completely reducible  $\text{ad } L$ -module. Clearly  $Z(L)$  is a submodule. So

$$L = Z(L) \oplus M$$

where  $M$  is a direct sum of some simple ideal of  $L$ . So  $M$  is semisimple.  $L/Z(L) \cong M$  is semisimple. Hence  $0 = \text{Rad}(L/Z(L)) = \text{Rad } L/Z(L)$ . Hence  $\text{Rad } L \subset Z(L)$ .

On the other hand,  $Z(L) \subset \text{Rad } L$  is clearly. We conclude that  $\text{Rad } L = Z(L)$ ,  $L$  is reductive.

4. let  $L$  be reductive and let  $\varphi: L \rightarrow \mathfrak{gl}(V)$  be a finite-dimensional representation of  $L$  in which  $Z(L)$  is represented by semisimple endomorphisms. Since  $Z(L)$  is abelian, we may simultaneously diagonalize the elements of  $\varphi(Z(L))$  to get an eigenspace decomposition of  $V$ . Since  $[L, L]$  commutes with  $Z(L)$ , each eigenspace is invariant under  $[L, L]$ . On each such eigenspace  $W$ , each element of  $\varphi(Z(L))$  acts as a scalar, so  $L$ -submodules of  $W$  coincide with  $[L, L]$ -submodules of  $W$ . We conclude complete reducibility from Weyl's theorem for semisimple Lie algebras.

□

**Exercise 6.6.** Let  $L$  be a simple Lie algebra. Let  $\beta(x, y)$  and  $\gamma(x, y)$  be two symmetric associative bilinear forms on  $L$ . If  $\beta, \gamma$  are nondegenerate, prove that  $\beta$  and  $\gamma$  are proportional. [Use Schur's Lemma.]

**Solution.**  $L$  is a irreducible  $L$ -module by  $\text{ad}$ , and  $L^*$  is a  $L$ -module. We can define a linear map  $\phi: L \rightarrow L^*; x \mapsto \beta_x$ , where  $\beta_x \in L^*$  defined by  $\beta_x(y) = \beta(x, y)$ . Then it is easy to check that  $\phi$  is a module homomorphism of  $L$ -module.

Similarly, we can define a linear map  $\psi: L^* \rightarrow L; f \mapsto x_f$ , where  $x_f$  defined by  $f(z) = \gamma(x_f, z)$  for all  $z \in L$ . This  $x_f$  exists because  $\gamma$  is non-degenerate. Then  $\psi$  is also a homomorphism of  $L$ -modules.

So  $\psi \circ \phi$  is a homomorphism from  $L$  to  $L$ , i.e,  $\psi \circ \phi$  is a endomorphism of  $L$  which commutative with all  $\text{ad } x, x \in L$ , and  $L$  is a irreducible  $L$ -module. By Schur's lemma we have

$$\psi \circ \phi = \lambda \text{id}$$

So

$$\beta(x, y) = \beta_x(y) = \gamma(x_{\beta_x}, y) = \gamma(\psi \circ \phi(x), y) = \gamma(\lambda x, y) = \lambda \gamma(x, y) \quad \forall x, y \in L$$

□

**Exercise 6.7.** It will be seen later on that  $\mathfrak{sl}(n, F)$  is actually simple. Assuming this and using Exercise 6.6, prove that the Killing form  $\kappa$  on  $\mathfrak{sl}(n, F)$  is related to the ordinary trace form by  $\kappa(x, y) = 2n \text{Tr}(xy)$ .

**Solution.** Clearly  $\text{Tr}(xy)$  is a nonzero symmetric associative bilinear form on  $\mathfrak{sl}(n, F)$ , its radical is a ideal of  $\mathfrak{sl}(n, F)$ , hence is equal to 0, and  $\text{Tr}(xy)$  is nondegenerate. By Exercise 6.6,  $\kappa(x, y) = \lambda \text{Tr}(xy)$ .

We can only compute it for  $x = y = e_{11} - e_{22}$ . In this case,  $\text{Tr}(xy) = 2$ . The matrix of  $\text{ad}(e_{11} - e_{22})$  relative to the standard basis of  $\mathfrak{sl}(n, F)$  is a diagonal matrix

$$\text{diag}(\underbrace{0, \dots, 0}_{n-1}, 2, -2, \underbrace{1, \dots, 1}_{2n-4}, \underbrace{-1, \dots, -1}_{2n-4}, 0, \dots, 0)$$

$$\text{Hence } \kappa(x, y) = \text{Tr}(\text{ad } x \text{ ad } y) = 4 + 4 + 2(2n - 4) = 4n = 2n \text{Tr}(xy).$$

□

**Exercise 6.8.** If  $L$  is a Lie algebra, then  $L$  acts (via  $\text{ad}$ ) on  $(L \otimes L)^*$ , which may be identified with the space of all bilinear forms  $\beta$  on  $L$ . Prove that  $\beta$  is associative if and only if  $L.\beta = 0$ .

**Solution.** By definition,

$$\begin{aligned} z.\beta(x \otimes y) &= -\beta(z.(x \otimes y)) \\ &= -\beta(z.x \otimes y + x \otimes z.y) \\ &= \beta([x, z] \otimes y) - \beta(x \otimes [z, y]) \end{aligned}$$

Hence

$$L.\beta = 0 \iff \beta([x, z] \otimes y) = \beta(x \otimes [z, y]), \forall x, y, z \in L$$

which means  $\beta$  is associative.

□

**Exercise 6.9.** Let  $L'$  be a semisimple subalgebra of a semisimple Lie algebra  $L$ . If  $x \in L'$ , its Jordan decomposition in  $L'$  is also its Jordan decomposition in  $L$ .

**Solution.** This follows from the corollary to Theorem 6.4 by using the adjoint representation of  $L$  and noting that it is injective.

□



## 7 Representations of $\mathfrak{sl}(2, F)$

In these exercises,  $L = \mathfrak{sl}(2, F)$ .

**Exercise 7.1.** Use Lie's Theorem to prove the existence of a maximal vector in an arbitrary finite dimensional  $L$ -module. [Look at the subalgebra  $B$  spanned by  $h$  and  $x$ .]

**Solution.**  $\phi: L \rightarrow \mathfrak{gl}(V)$  is a representation. Let  $B$  be the subalgebra of  $L$  spanned by  $h$  and  $x$ . Then  $\phi(B)$  is a solvable subalgebra of  $\mathfrak{gl}(V)$ . And  $\phi(x)$  is a nilpotent endomorphism of  $V$ . By Lie's theorem, there is a common eigenvector  $v$  for  $B$ . So  $h.v = \lambda v, x.v = 0, v$  is a maximal vector.  $\square$

**Exercise 7.2.**  $M = \mathfrak{sl}(3, F)$  contains a copy of  $L$  in its upper left-hand  $2 \times 2$  position. Write  $M$  as direct sum of irreducible  $L$ -submodules ( $M$  viewed as  $L$ -module via the adjoint representation):  $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$ .

**Solution.** Let  $h = e_{11} - e_{22}, x = e_{12}, y = e_{21}$ .

First, we know  $\text{ad } h.e_{12} = 2e_{12}, \text{ad } x.e_{12} = 0$ . So  $e_{12}$  is a maximal vector with highest weight 2. It can generate a irreducible module isomorphic to  $V(2)$ . Let

$$v_0 = e_{12}, v_1 = [e_{21}, e_{12}] = -(e_{11} - e_{22}), v_2 = [e_{21}, -(e_{11} - e_{22})] = -e_{21}.$$

So  $V(2) \cong \text{span}\{e_{12}, e_{11} - e_{22}, e_{21}\}$ .

$\text{ad } h.e_{13} = e_{13}, \text{ad } x.e_{13} = 0$ . So  $e_{13}$  is a maximal vector with weight 1. It can generate a irreducible module isomorphic to  $V(1)$ .

$$[e_{21}, e_{13}] = e_{23}.$$

We have  $V(1) \cong \text{span}\{e_{13}, e_{23}\}$ .

$\text{ad } h.e_{32} = e_{32}, \text{ad } x.e_{32} = 0$ . So  $e_{32}$  is a maximal vector with weight 1. It can generate a irreducible module isomorphic to  $V(1)$ .

$$[e_{21}, e_{32}] = -e_{31}.$$

We have another  $V(1) \cong \text{span}\{e_{31}, e_{32}\}$ .

At last, we have a 1-dimensional irreducible submodule of  $V(0) \cong \text{span}\{e_{22} - e_{33}\}$ . Then

$$M \cong V(0) \oplus V(1) \oplus V(1) \oplus V(2),$$

$\square$

**Exercise 7.3.** Verify that formulas (a)-(c) of Lemma 7.2 do define an irreducible representation of  $L$ . [Hint: To show that they define a representation, it suffices to show that the matrices corresponding to  $x, y, h$  satisfy the same structural equations as  $x, y, h$ .]

**Solution.**

$$\begin{aligned}
[h, x].v_i &= 2x.v_i = 2(\lambda - i + 1)v_{i-1} \\
h.x.v_i - x.h.v_i &= (\lambda - i + 1)h.v_{i-1} - (\lambda - 2i)x.v_i \\
&= (\lambda - i + 1)(\lambda - 2i + 2)v_{i-1} - (\lambda - 2i)(\lambda - i + 1)v_{i-1} \\
&= 2(\lambda - i + 1)v_{i-1} \\
[h, y].v_i &= -2y.v_i = -2(i + 1)v_{i+1} \\
h.y.v_i - y.h.v_i &= (i + 1)h.v_{i-1} - (\lambda - 2i)y.v_i \\
&= (i + 1)(\lambda - 2i - 2)v_{i+1} - (\lambda - 2i)(i + 1)v_{i+1} \\
&= -2(i + 1)v_{i+1} \\
[x, y].v_i &= hv_i = (\lambda - 2i)v_i \\
x.y.v_i - y.x.v_i &= (i + 1)x.v_{i+1} - (\lambda - i + 1)y.v_{i-1} \\
&= (i + 1)(\lambda - i)v_i - (\lambda - i + 1)iv_i \\
&= (\lambda - 2i)v_i
\end{aligned}$$

□

**Exercise 7.4.** The irreducible representation of  $L$  of highest weight  $m$  can also be realized “naturally”, as follows. Let  $X, Y$  be a basis for the two dimensional vector space  $F^2$ , on which  $L$  acts as usual. Let  $\mathcal{R} = F[X, Y]$  be the polynomial algebra in two variables, and extend the action of  $L$  to  $\mathcal{R}$  by the derivation rule:  $z.fg = (z.f)g + f(z.g)$ , for  $z \in L, f, g \in \mathcal{R}$ . Show that this extension is well defined and that  $\mathcal{R}$  becomes an  $L$ -module. Then show that the subspace of homogeneous polynomials of degree  $m$ , with basis  $X^m, X^{m-1}Y, \dots, XY^{m-1}, Y^m$ , is invariant under  $L$  and irreducible of highest weight  $m$ .

**Exercise 7.5.** Suppose  $\text{char } F = p > 0, L = \mathfrak{sl}(2, F)$ . Prove that the representation  $V(m)$  of  $L$  constructed as in Exercise 7.3 or 7.4 is irreducible so long as the highest weight  $m$  is strictly less than  $p$ , but reducible when  $m = p$ .

**Solution.** When  $m < p$ , conditions (a)-(c) of Lemma 7.2 still imply the irreducibility of  $V(m)$ . However, when  $m = p$ , the submodule spanned by  $\{v_0, \dots, v_{m-1}\}$  is invariant under  $L$ , so  $V(m)$  is reducible. □

**Exercise 7.6.** Decompose the tensor product of the two  $L$ -modules  $V(3), V(7)$  into the sum of irreducible submodules:  $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$ . Try to develop a general formula for the decomposition of  $V(m) \otimes V(n)$ .

**Solution.** In general, for  $V = V(m) \otimes V(n)$ . We suppose  $m \geq n$ .  $u_i, i = 0, \dots, m$  is the basis of  $V(m)$  and  $v_j, j = 1, \dots, n$  is the basis of  $V(n)$ .

$$h.(u_i \otimes v_j) = (m + n - 2(i + j))u_i \otimes v_j$$

Hence

$$V_{m+n-2k} = \text{span}\{u_i \otimes v_j, i + j = k\}$$

For  $k = 0, \dots, m$ , suppose  $w = \sum_{i=0}^k \lambda_i u_i \otimes v_{k-i} \in V_{m+n-2k}$  is a maximal vector. Then

$$\begin{aligned} x.w &= \sum_{i=0}^k \lambda_i ((x.u_i) \otimes v_{k-i} + u_i \otimes (x.v_{k-i})) \\ &= \sum_{i=1}^k \lambda_i (m-i+1) u_{i-1} \otimes v_{k-i} + \sum_{i=0}^{k-1} \lambda_i (n-k+i+1) u_i \otimes v_{k-i-1} \\ &= \sum_{i=1}^k (\lambda_i (m-i+1) + \lambda_{i-1} (n-k+i)) u_{i-1} \otimes v_{k-i} \\ &= 0 \end{aligned}$$

Therefore

$$\lambda_i (m-i+1) + \lambda_{i-1} (n-k+i) = 0$$

We conclude that

$$\lambda_i = (-1)^i \frac{(n-k+i)!(m-i)!}{(n-k)!m!} \lambda_0$$

Let  $\lambda_0 = 1$ , then  $w = \sum_{i=0}^k \lambda_i u_i \otimes v_{k-i}$  is a maximal vector with weight  $m+n-2k$ . It generates a irreducible submodule of  $V$  isomorphic to  $V(m+n-2k)$ . So

$$\bigoplus_{k=0}^n V(m+n-2k) \subset V(m) \otimes V(n).$$

Compare the dimensional of two sides, we have the decomposition  $V(m) \otimes V(n) \cong V(m-n) \oplus V(m-n+2) \oplus \dots \oplus V(m+n)$ .  $\square$

**Solution.** To find a decomposition of  $V(m) \otimes V(n)$ , it is enough to count the dimensions of eigenspaces of  $h$ . In particular, note that since  $h.(v \otimes w) = h.v \otimes w + v \otimes h.w$ , if  $v \in V_\lambda \subset V(m)$  and  $w \in V_\mu \subset V(n)$ , then  $v \otimes w \in V_{\lambda+\mu} \subset V(m) \otimes V(n)$ .

Hence the dimension of  $V_\lambda$  in  $V(m) \otimes V(n)$  is the number of ways to write  $\lambda$  as a sum of elements from the two sets  $\{m, m-2, \dots, -m\}$  and  $\{n, n-2, \dots, -n\}$ . Without loss of generality, assume that  $m \geq n$ . Then  $\lambda$  can be written as such a sum  $\min(\frac{m+n-|\lambda|}{2} + 1, \frac{m+n-(m-n)}{2} + 1)$  ways if  $m+n-|\lambda|$  is even and 0 ways otherwise.

Hence we have the decomposition  $V(m) \otimes V(n) \cong V(m-n) \oplus V(m-n+2) \oplus \dots \oplus V(m+n)$ .  $\square$

**Exercise 7.7.** In this exercise we construct certain infinite dimensional  $L$ -modules. Let  $\lambda \in F$  be an arbitrary scalar. Let  $Z(\lambda)$  be a vector space over  $F$  with countably infinite basis  $(v_0, v_1, v_2, \dots)$ .

1. Prove that formulas (a)-(c) of Lemma 7.2 define an  $L$ -module structure on  $Z(\lambda)$ , and that every nonzero  $L$ -submodule of  $Z(\lambda)$  contains at least one maximal vector.
2. Suppose  $\lambda + 1 = i$  is a nonnegative integer. Prove that  $v_i$  is a maximal vector (e.g.,  $\lambda = -1, i = 0$ ). This induces an  $L$ -module homomorphism  $Z(\mu) \xrightarrow{\phi} Z(\lambda), \mu = \lambda - 2i$ , sending  $v_0$  to  $v_i$ . Show that  $\phi$  is a monomorphism, and that  $\text{im } \phi, Z(\lambda)/\text{im } \phi$  are both irreducible  $L$ -modules (but  $Z(\lambda)$  fails to be completely reducible when  $i > 0$ ).

3. Suppose  $\lambda + 1$  is not a nonnegative integer. Prove that  $Z(\lambda)$  is irreducible.

**Solution.** 1. We need to verify that  $[xy] = h$ ,  $[hx] = 2x$ ,  $[hy] = -2y$  as linear transformations on  $Z(\lambda)$ .

It suffices to check these on the basis  $(v_0, v_1, v_2, \dots)$ . Given  $v_i$ , we have

$$\begin{aligned}
[xy].v_i &= (i+1)x.v_{i+1} - (\lambda - i + 1)y.v_{i-1} \\
&= (i+1)(\lambda - i)v_i - (\lambda - i + 1)iv_i \\
&= (\lambda - 2i)v_i = h.v_i; \\
[hx].v_i &= (\lambda - i + 1)h.v_{i-1} - (\lambda - 2i)x.v_i \\
&= (\lambda - i + 1)(\lambda - 2i + 2)v_{i-1} - (\lambda - 2i)(\lambda - i + 1)v_{i-1} \\
&= 2(\lambda - i + 1)v_{i-1} = 2x.v_i; \\
[hy].v_i &= (i+1)h.v_{i+1} - (\lambda - 2i)y.v_i \\
&= (i+1)(\lambda - 2i - 2)v_{i+1} - (\lambda - 2i)(i+1)v_{i+1} \\
&= -2(i+1)v_{i+1} = -2y.v_i;
\end{aligned}$$

So  $Z(\lambda)$  is an  $L$ -module.

Let  $U$  be an arbitrary nonzero  $L$ -submodule of  $Z(\lambda)$ . For any nonzero  $v \in U$  write  $v$  as a linear combination of basis:  $v = \sum_{i \in I} a_i v_i$ , where all  $a_i \neq 0$ . We have

$$h.v = \sum_{i \in I} a_i (\lambda - 2i)v_i \in U$$

This implies all  $v_i \in U$ . So

$$U = \text{span}\{v_j, j \in J\}$$

Let  $k = \min J$ , then  $v_k \in U$ , and  $v_{k-1} \notin U$ , so  $x.v_k = (\lambda - k + 1)v_{k-1} = 0$ . We conclude that  $v_k$  is a maximal vector in  $U$ .

2.  $x.v_i = (\lambda - i + 1)v_{i-1} = 0$ , hence  $v_i$  is a maximal vector.

To see  $\phi$  is a monomorphism, it suffices to show it is injective on basis. In deed,

$$\phi(v_k) = \binom{k+i}{i} v_{k+i}$$

We prove this by induction on  $k$

When  $k = 0$ ,  $\phi(v_0) = v_i$ . If we already have  $\phi(v_{k-1}) = \binom{k-1+i}{i} v_{k-1+i}$ , then

$$\begin{aligned}
\phi(v_k) &= \phi\left(\frac{1}{k}y.v_{k-1}\right) = \frac{1}{k}y.\phi(v_{k-1}) \\
&= \frac{1}{k} \binom{k-1+i}{i} y.v_{k-1+i} = \frac{k+i}{k} \binom{k-1+i}{i} v_{k+i} \\
&= \binom{k+i}{i} v_{k+i}
\end{aligned}$$

$\text{im } \phi \cong Z(\mu)$  is a submodule of  $Z(\lambda)$  and by (1) it has a maximal vector of form  $v_s$ . But

$$x.v_s = (\mu - s + 1)v_s = -(i + s)v_{s-1} = 0$$

From  $i + s > 0$ , we have  $v_{s-1} = 0$ . So  $v_0$  is the unique maximal vector in  $Z(\mu)$  and  $Z(\mu)$  is irreducible.  $Z(\lambda)/\text{im } \phi \cong V(i-1)$  is a irreducible module.

Next we show  $Z(\lambda)$  is not completely reducible. If  $Z(\lambda)$  is completely reducible, then we have an  $L$ -module decomposition  $Z(\lambda) = \text{im } \phi \oplus V$ . Then there exists a  $w \in \text{im } \phi$  such that  $v_0 + w \in V$ . But

$$y^i.(v_0 + w) = v_i + y^i.w \in \text{im } \phi$$

which contradicts with the fact that  $V$  is an  $L$ -module.

3. If  $Z(\lambda)$  reducible, it has a proper nonzero submodule  $U$ . By (1)  $U$  has a maximal vector  $v_k$  with  $k > 0$ .

$$x.v_k = (\lambda - k + 1)v_{k-1} = 0$$

Hence  $\lambda + 1 = k$  is a positive integer. We get a contradiction. □

## 8 Root space decomposition

**Exercise 8.1.** If  $L$  is a classical linear Lie algebra of type  $A_l, B_l, C_l$  or  $D_l$ , prove that the set of all diagonal matrices in  $L$  is a maximal toral subalgebra, of dimension  $l$  (Cf. Exercise 2.8.)

**Solution.** Since a toral subalgebra is abelian, any toral subalgebra containing the set of diagonal matrices in  $L$  must contain only diagonal matrices because commuting semisimple matrices are simultaneously diagonalizable. Its dimension can be immediately verified in the four cases to be  $l$ . □

**Exercise 8.2.** For each algebra in Exercise 8.1, determine the roots and root spaces. How are the various  $h_\alpha$  expressed in terms of the basis for  $H$  given in section 1?

**Exercise 8.3.** If  $L$  is of classical type, compute explicitly the restriction of the Killing form to the maximal toral subalgebra described in Exercise 8.1.

**Exercise 8.4.** If  $L = \mathfrak{sl}(2, F)$ , prove that each maximal toral subalgebra is one dimensional.

**Solution.**  $\mathfrak{h}$  is a maximal toral subalgebra of  $L$ ,  $L = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} L_\alpha$ ,  $\dim L_\alpha = 1, \alpha \in \Phi$  then  $-\alpha \in \Phi$ . This implies  $\text{Card}(\Phi)$  is even and nonzero. So  $\dim \mathfrak{h} = 1$ . □

**Exercise 8.5.** If  $L$  is semisimple,  $H$  a maximal toral subalgebra, prove that  $H$  is self-normalizing (i.e.,  $H = N_L(H)$ ).

**Solution.**  $L = H \oplus \sum_{\alpha \in \Phi} L_\alpha$ . For  $x \in N_L(H)$ ,  $x = h_0 + \sum_{\alpha \in \Phi} x_\alpha, x_\alpha \in L_\alpha$ . Choose  $h \in H$  such that  $\alpha(h) \neq 0, \forall \alpha \in \Phi$ , then

$$[h, x] = \sum_{\alpha \in \Phi} \alpha(h)x_\alpha \in H$$

Hence  $x_\alpha = 0, \forall \alpha \in \Phi$ .  $x = h_0 \in H$ . i.e,  $N_L(H) = H$ . □

**Exercise 8.6.** Compute the basis of  $\mathfrak{sl}(n, F)$  which is dual (via the Killing form) to the standard basis. (Cf. Exercise 5.5.)

**Solution.** The dual of  $e_{ij} (i \neq j)$  via the Killing form is  $e_{ji}$ , the dual of  $h_i$  via the Killing form is  $e_{ii} - \frac{1}{n}I_n$   $\square$

**Exercise 8.7.** Let  $L$  be semisimple,  $H$  a maximal toral subalgebra. If  $h \in H$ , prove that  $C_L(h)$  is reductive (in the sense of Exercise 6.5). Prove that  $H$  contains elements  $h$  for which  $C_L(h) = H$ ; for which  $h$  in  $\mathfrak{sl}(n, F)$  is this true?

**Solution.**  $L$  is semisimple. We have a decomposition  $L = H \oplus \sum_{\alpha \in \Phi} L_\alpha$

$$\begin{aligned} x &= h_0 + \sum_{\alpha \in \Phi} x_\alpha \in C_L(h) \\ \iff [h, x] &= \sum_{\alpha \in \Phi} \alpha(h)x_\alpha = 0 \\ \iff \alpha(h) &= 0 \quad \text{or} \quad x_\alpha = 0 \end{aligned}$$

Hence

$$C_L(h) = H \oplus \sum_{\substack{\alpha \in \Phi \\ \alpha(h)=0}} L_\alpha$$

Denote  $\Phi_h = \{\alpha \in \Phi \mid \alpha(h) = 0\}$ . Now we claim that

$$Z(C_L(h)) = \{h' \in H \mid \alpha(h') = 0, \forall \alpha \in \Phi_h\}$$

Let  $x = h_0 + \sum_{\alpha \in \Phi_h} x_\alpha \in Z(C_L(h))$ . We can find a  $h' \in H$  such that  $\alpha(h') \neq 0, \forall \alpha \in \Phi_h$ . Then  $[h', x] = \sum_{\alpha \in \Phi_h} \alpha(h')x_\alpha = 0$ . It implies  $x_\alpha = 0$ . We have  $x = h_0 \in H$ . Next we take  $0 \neq x_\alpha \in L_\alpha, \forall \alpha \in \Phi_h$ , then  $[x, x_\alpha] = \alpha(h_0)x_\alpha = 0$ . Hence  $\alpha(x) = \alpha(h_0) = 0, \forall \alpha \in \Phi_h$ .

Next we show  $Z(C_L(h)) = \text{Rad}(C_L(h))$ . Clearly  $Z(C_L(h))$  is a solvable ideal of  $C_L(h)$ , it is enough to show it is a maximal solvable ideal.

If  $x = h_0 + \sum_{\alpha \in \Phi_h} x_\alpha \in \text{Rad}(C_L(h)) \setminus Z(C_L(h))$ . We have a  $h' \in H$  such that  $\alpha(h') \neq 0$  and  $\alpha(h') \neq \beta(h'), \forall \alpha \neq \beta \in \Phi_h$ . Then  $[h', x] = \sum_{\alpha \in \Phi_h} \alpha(h')x_\alpha \in \text{Rad}(C_L(h))$ . Hence  $h_0, x_\alpha \in \text{Rad}(C_L(h)), \alpha \in \Phi_h$ . If there is a  $\alpha \in \Phi_h$  such that  $x_\alpha \neq 0$ , then  $h_\alpha = [x_\alpha, y_\alpha] \in \text{Rad}(C_L(h)), 2y_\alpha = -[h_\alpha, y_\alpha] \in \text{Rad}(C_L(h))$ . Hence  $\mathfrak{sl}(2, F) \cong S_\alpha \subset \text{Rad}(C_L(h))$  which contradict with the solvability of  $\text{Rad}(C_L(h))$ .

Now we get  $x = h_0 \in \text{Rad}(C_L(h)) \setminus Z(C_L(h))$ . So there is a  $\alpha \in \Phi_h$  such that  $\alpha(h_0) \neq 0$ . Then  $[x, x_\alpha] = \alpha(h_0)x_\alpha \in \text{Rad}(C_L(h)), [x, y_\alpha] = -\alpha(h_0)y_\alpha \in \text{Rad}(C_L(h))$ . We also have  $S_\alpha \subset \text{Rad}(C_L(h))$  which contradict with the solvability of  $\text{Rad}(C_L(h))$ .

All of the above show that  $Z(C_L(h)) = \text{Rad}(C_L(h))$ . i.e.,  $C_L(h)$  is reductive. We know there is a  $h \in H, \alpha(h) \neq 0, \forall \alpha \in \Phi$ . In this case,  $C_L(h) = H$ .

In  $\mathfrak{sl}(n, F)$ , for  $e_{ij} (i \neq j) \in L_\alpha$  and  $h = \sum a_k h_k \in H$ , we have

$$[h, e_{ij}] = \alpha(h)e_{ij} = \begin{cases} (4a_i - a_{i+1} - a_{i-1})e_{i,i+1} & j = i+1 \\ (a_{j+1} + a_{j-1} - 4a_j)e_{j+1,j} & i = j+1 \\ (a_i - a_j - a_{i-1} + a_{j-1})e_{ij} & |i-j| > 1 \end{cases}$$

Then these  $h$  for which  $C_L(h) = H$  is these satisfying

$$\begin{cases} 4a_i - a_{i+1} - a_{i-1} \neq 0 & 1 \leq i \leq n \\ a_i - a_j - a_{i-1} + a_{j-1} \neq 0 & |i - j| > 1 \end{cases}$$

□

**Exercise 8.8.** For  $\mathfrak{sl}(n, F)$  (and other classical algebras), calculate explicitly the root strings and Cartan integers. In particular, prove that all Cartan integers  $2\frac{(\alpha, \beta)}{(\beta, \beta)}$ ,  $\alpha \neq \pm\beta$ , for  $\mathfrak{sl}(n, F)$  are 0,  $\pm 1$ .

**Exercise 8.9.** Prove that every three dimensional semisimple Lie algebra has the same root system as  $\mathfrak{sl}(2, F)$ , hence is isomorphic to  $\mathfrak{sl}(2, F)$ .

**Solution.** This is a direct consequence of Proposition 8.3(f). □

**Exercise 8.10.** Prove that no four, five or seven dimensional semisimple Lie algebras exist.

**Solution.** Let  $L$  is a semisimple Lie algebra with a maximal toral subalgebra  $H$ . We have  $L = H \oplus \sum_{\alpha \in \Phi} L_\alpha$ . Since  $\alpha \in \Phi$  implies  $-\alpha \in \Phi$ ,  $\sum_{\alpha \in \Phi} L_\alpha$  has dimensional  $2k$  with  $k > 1$ . Therefore

$$\dim H = \dim L - \dim\left(\sum_{\alpha \in \Phi} L_\alpha\right) = \dim L - 2k$$

In the other hands,  $\Phi = \{\pm\alpha_1, \dots, \pm\alpha_k\}$  span  $H^*$ , then

$$\dim H = \dim H^* \leq k$$

We conclude

$$\frac{\dim L}{3} \leq k < \frac{\dim L}{2}$$

If  $\dim L = 4$ , we can not find a integer  $k$  satisfying it.

If  $\dim L = 5, k = 2$ . Then  $\dim H = 1$ , i.e,  $\Phi$  spans a 1-dimensional space.  $\alpha_2 = m\alpha_1$  with  $m = \pm 1$ . We get a contradiction.

If  $\dim L = 7, k = 3$ . Then  $\dim H = 1$ . We can deduce a contradiction as the case  $\dim L = 5$ . Hence, there is no four, five or seven dimensional semisimple Lie algebra. □

**Exercise 8.11.** If  $(\alpha, \beta) > 0$ , and  $\alpha \neq \pm\beta$ , prove that  $\alpha - \beta \in \Phi$  ( $\alpha, \beta \in \Phi$ ). Is the converse true?

**Solution.** We have  $\beta - \frac{2(\alpha, \beta)}{(\beta, \beta)}\alpha \in \Phi$  since  $\alpha, \beta \in \Phi$ .

Let the  $\beta$  string through  $\alpha$  is  $\alpha - r\beta, \dots, \alpha, \dots, \alpha + q\beta$ . We have  $r > 0$  since  $(\alpha, \beta) > 0$ . Hence  $\alpha - \beta$  appears in the string. It is a root. □

## Chapter III

# Root System

### 9 Axiomatics

Unless otherwise specified,  $\Phi$  denotes a root system in  $E$ , with Weyl group  $\mathcal{W}$ .

**Exercise 9.1.** Let  $E'$  be a subspace of  $E$ . If a reflection  $\sigma_\alpha$  leaves  $E'$  invariant, prove that either  $\alpha \in E'$  or else  $E' \subset P_\alpha$ .

**Solution.** Suppose  $E' \not\subset P_\alpha$ . Let  $\lambda \in E' \setminus P_\alpha$ , then  $\sigma_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha \in E'$ . Since  $\lambda \notin P_\alpha$ ,  $\langle \lambda, \alpha \rangle \neq 0$ . Hence  $\alpha \in E'$ .  $\square$

**Exercise 9.2.** Prove that  $\Phi^\vee$  is a root system in  $E$ , whose Weyl group is naturally isomorphic to  $\mathcal{W}$ ; show also that  $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$ , and draw a picture of  $\Phi^\vee$  in the cases  $A_1, A_2, B_2, G_2$ .

**Solution.** By definition,  $\Phi^\vee$  consists of  $\alpha^\vee = 2\alpha/(\alpha, \alpha)$  for  $\alpha \in \Phi$ . If  $\Phi$  is finite, spans  $E$ , and does not contain 0, it is obvious that  $\Phi^\vee$  also satisfies these conditions. Since  $(c\alpha)^\vee = c(\alpha)^\vee$  for  $c \in \mathbb{R}$ , it follows that the only multiples of  $\alpha^\vee$  in  $\Phi^\vee$  are  $\pm\alpha^\vee$ . Also, for  $\alpha, \beta \in \Phi$ ,

$$\sigma_{\alpha^\vee} \beta^\vee = \beta^\vee - \frac{2(\beta^\vee, \alpha^\vee)}{(\alpha^\vee, \alpha^\vee)} \alpha^\vee = \frac{2\beta}{(\beta, \beta)} - \frac{4(\beta, \alpha)}{(\beta, \beta)(\alpha, \alpha)} \alpha$$

and

$$(\sigma_\alpha(\beta))^\vee = \left( \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha \right)^\vee = \frac{2\beta - 4\frac{(\beta, \alpha)}{(\alpha, \alpha)} \alpha}{(\beta, \beta) - 4\frac{(\beta, \alpha)^2}{(\alpha, \alpha)} + 44\frac{(\beta, \alpha)^2}{(\alpha, \alpha)}} = \frac{2\beta}{(\beta, \beta)} - \frac{4(\beta, \alpha)}{(\beta, \beta)(\alpha, \alpha)} \alpha$$

so  $\sigma_{\alpha^\vee} \beta^\vee = (\sigma_\alpha(\beta))^\vee$ , and hence  $\Phi^\vee$  is invariant under  $\sigma_{\alpha^\vee}$ . Finally,

$$\langle \alpha^\vee, \beta^\vee \rangle = \frac{2(\alpha^\vee, \beta^\vee)}{(\beta^\vee, \beta^\vee)} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = \langle \beta, \alpha \rangle \in \mathbb{Z}$$

so all of the axioms for a root system are satisfied for  $\Phi^\vee$ .

Finally, by the above calculations, the bijection  $\alpha \mapsto \alpha^\vee$  induces an isomorphism  $\sigma_\alpha \mapsto \sigma_{\alpha^\vee}$  (thinking of the Weyl groups as subgroups of the symmetric group on  $\Phi$  and  $\Phi^\vee$ ).  $\square$

**Exercise 9.3.** In Table 1, show that the order of  $\sigma_\alpha \sigma_\beta$  in  $\mathcal{W}$  is (respectively) 2, 3, 4, 6 when  $\theta = \pi/2, \pi/3$  (or  $2\pi/3$ ),  $\pi/4$  (or  $3\pi/4$ ),  $\pi/6$  (or  $5\pi/6$ ). [Note that  $\sigma_\alpha \sigma_\beta =$  rotation through  $2\theta$ .]

**Exercise 9.4.** Prove that the respective Weyl groups of  $A_1 \times A_1, A_2, B_2, G_2$  are dihedral of order 4, 6, 8, 12. If  $\Phi$  is any root system of rank 2, prove that its Weyl group must be one of these.

**Solution.**  $\sigma_\alpha$  and  $\sigma_\alpha \sigma_\beta$  generate the Weyl group, then the conclusions follow from Exercise 9.3.  $\square$



**Solution.** There are only two reflections for the Weyl group of  $A_1 \times A_1$ , and they commute with each other, so its Weyl group is isomorphic to  $Z/2 \times Z/2$ , which is the dihedral group of order 4.

Picking alternating chambers of  $A_2$ , draw a regular triangle. The reflections of  $A_2$  are symmetries of this triangle, so its Weyl group is  $\mathcal{S}_3$ , the dihedral group of order 6.

Drawing a square with vertices on the diagonal vectors of  $B_2$ , we see that all reflections preserve this square, and since the symmetries of the square are generated by reflections, the Weyl group of  $B_2$  is  $D_4$ , the dihedral group of order 8.

Finally, the reflections of  $G_2$  preserve a regular hexagon whose vertices are on the short vectors. So the Weyl group of  $G_2$  is a subgroup of  $D_6$ , but since the reflections generate it, we get the whole group.

The fact that the Weyl group of every rank 2 root system must be dihedral of order 4, 6, 8 or 12 follows from the possibilities allowed in Table 1.  $\square$

**Exercise 9.5.** Show by example that  $\alpha - \beta$  may be a root even when  $(\alpha, \beta) \leq 0$  (cf. Lemma 9.4).

**Solution.** This can be seen in the root system  $G_2$ . Using the labels in Figure 9.1, we have that  $\alpha$  and  $\alpha + \beta$  form an obtuse angle, i.e.,  $(\alpha, \alpha + \beta) \leq 0$ , but that  $\beta$  is a root.  $\square$

**Exercise 9.6.** Prove that  $\mathcal{W}$  is a normal subgroup of  $\text{Aut } \Phi$  (= group of all isomorphisms of  $\Phi$  onto itself).

**Solution.** Any element of  $\mathcal{W}$  can be written  $\sigma_{\alpha_1} \cdots \sigma_{\alpha_r}$  for  $\alpha_i \in \Phi$ . Then for  $\tau \in \text{Aut } \Phi$ , we have

$$\tau(\sigma_{\alpha_1} \cdots \sigma_{\alpha_r})\tau^{-1} = (\tau\sigma_{\alpha_1}\tau^{-1}) \cdots (\tau\sigma_{\alpha_r}\tau^{-1}) = \sigma_{\tau(\alpha_1)} \cdots \sigma_{\tau(\alpha_r)} \in \mathcal{W};$$

where the last equality is Lemma 9.2.  $\square$

**Exercise 9.7.** Let  $\alpha, \beta \in \Phi$  span a subspace  $E'$  of  $E$ . Prove that  $E' \cap \Phi$  is a root system in  $E'$ . Prove similarly that  $\Phi \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$  is a root system in  $E'$  (must this coincide with  $E' \cap \Phi$ ?). More generally, let  $\Phi'$  be a nonempty subset of  $\Phi$  such that  $\Phi' = -\Phi'$ , and such that  $\alpha, \beta \in \Phi', \alpha + \beta \in \Phi$  implies  $\alpha + \beta \in \Phi'$ . Prove that  $\Phi'$  is a root system in the subspace of  $E$  it spans. [Use Table 1].

**Exercise 9.8.** Compute root strings in  $G_2$  to verify the relation  $r - q = \langle \beta, \alpha \rangle$ .

**Exercise 9.9.** Let  $\Phi$  be a set of vectors in a euclidean space  $E$ , satisfying only (R1), (R3), (R4). Prove that the only possible multiples of  $\alpha \in \Phi$  which can be in  $\Phi$  are  $\pm \frac{1}{2}\alpha, \pm\alpha, \pm 2\alpha$ . Verify that  $\{\alpha \in \Phi \mid 2\alpha \notin \Phi\}$  is a root system.

**Exercise 9.10.** Let  $\alpha, \beta \in \Phi$ . Let the  $\alpha$ -string through  $\beta$  be  $\beta - r\alpha, \dots, \beta + q\alpha$ , and let the  $\beta$ -string through  $\alpha$  be  $\alpha - r'\beta, \dots, \alpha + q'\beta$ . Prove that  $\frac{q(r+1)}{(\beta, \beta)} = \frac{q'(r'+1)}{(\alpha, \alpha)}$ .

**Exercise 9.11.** Let  $c$  be a positive real number. If  $\Phi$  possesses any roots of squared length  $c$ , prove that the set of all such roots is a root system in the subspace of  $E$  it spans. Describe the possibilities occurring in Figure 1.

## 10 Simple roots and Weyl group

**Exercise 10.1.** Let  $\Phi^\vee$  be the dual system of  $\Phi$ ,  $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$ . Prove that  $\Delta^\vee$  is a base of  $\Phi^\vee$ . [Compare Weyl chambers of  $\Phi$  and  $\Phi^\vee$ .]

**Exercise 10.2.** If  $\Delta$  is a base of  $\Phi$ , prove that the set  $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi$  ( $\alpha \neq \beta \in \Delta$ ) is a root system of rank 2 in the subspace of  $E$  spanned by  $\alpha, \beta$  (cf. Exercise 9.7). Generalize to an arbitrary subset of  $\Delta$ .

**Exercise 10.3.** Prove that each root system of rank 2 is isomorphic to one of those listed in (9.3).

**Exercise 10.4.** Verify the Corollary of Lemma 10.2A directly for  $G_2$ .

**Solution.**  $a + b + b + b + a, a + b + b + b, a + b + b, a + b, a, b$ . □

**Exercise 10.5.** If  $\sigma \in \mathcal{W}$  can be written as a product of  $t$  simple reflections, prove that  $t$  has the same parity as  $l(\sigma)$ .

**Solution.** It is enough to prove that if the identity is written as  $t$  simple reflections, then  $t$  is even. To see this, first note that the number of negative roots in the set  $\sigma_1 \cdots \sigma_t(\Delta)$  has the same parity of  $t$ . This follows by induction since each  $\sigma_i = \sigma_{\alpha_i}$  fixes the sign of  $\alpha_j$  if  $\alpha_j \neq \alpha_i$ , and changes the sign of  $\alpha_i$ . So if  $1 = \sigma_1 \cdots \sigma_t$ , then  $t$  is even because  $\Delta$  has no negative roots. □

**Exercise 10.6.** Define a function  $sn: \mathcal{W} \rightarrow \{\pm 1\}$  by  $sn(\sigma) = (-1)^{l(\sigma)}$ . Prove that  $sn$  is a homomorphism (cf. the case  $A_2$ , where  $\mathcal{W}$  is isomorphic to the symmetric group  $\mathcal{S}_3$ ).

**Solution.** This is immediate from Exercise 10.5: given  $\sigma, \tau \in \mathcal{W}$ , we have that  $l(\sigma\tau) = l(\sigma) + l(\tau) \pmod{2}$ . □

**Exercise 10.7.** Prove that the intersection of “positive” open half-spaces associated with any basis  $\gamma_1, \dots, \gamma_l$  of  $E$  is nonvoid. [If  $\delta_i$  is the projection of  $\gamma_i$  on the orthogonal complement of the subspace spanned by all basis vectors except  $\gamma_i$ , consider  $\gamma = \sum r_i \delta_i$  when all  $r_i > 0$ .]

**Solution.**  $(\gamma, \delta_j) = \sum r_i (\delta_i, \delta_j) = (\gamma_j, \delta_j) > 0$ . □

**Exercise 10.8.** Let  $\Delta$  be a base of  $\Phi$ ,  $\alpha \neq \beta$  simple roots,  $\Phi_{\alpha\beta}$  the rank 2 root system in  $E_{\alpha\beta} = \mathbb{R}\alpha + \mathbb{R}\beta$  (see Exercise 10.2 above). The Weyl group  $\mathcal{W}_{\alpha\beta}$  of  $\Phi_{\alpha\beta}$  is generated by the restrictions  $\tau_\alpha, \tau_\beta$  to  $E_{\alpha\beta}$  of  $\sigma_\alpha, \sigma_\beta$ , and  $\mathcal{W}_{\alpha\beta}$  may be viewed as a subgroup of  $\mathcal{W}$ . Prove that the “length” of an element of  $\mathcal{W}_{\alpha\beta}$  (relative to  $\tau_\alpha, \tau_\beta$ ) coincides with the length of the corresponding element of  $\mathcal{W}$ .

**Exercise 10.9.** Prove that there is a unique element  $\sigma$  in  $\mathcal{W}$  sending  $\Phi^+$  to  $\Phi^-$  (relative to  $\Delta$ ). Prove that any reduced expression for  $\sigma$  must involve all  $\sigma_\alpha$  ( $\alpha \in \Delta$ ). Discuss  $l(\sigma)$ .

**Solution.** Note that  $-\Delta = \{-\alpha \mid \alpha \in \Delta\}$  is also a base for  $\Phi$ , so since  $\mathcal{W}$  acts transitively on bases of  $\Phi$  (Theorem 10.3), there is a  $\sigma \in \mathcal{W}$  such that  $\sigma(\Delta) = -\Delta$ . Then  $\sigma$  necessarily takes positive roots of  $\Phi$  to negative roots of  $\Phi$  (relative to  $\Delta$ ). If  $\tau \in \mathcal{W}$  also has this property, then  $\sigma\tau$  takes a positive base to another positive base. By definition, two bases can be positive with respect to  $\Delta$  only if they are equal, so since  $\mathcal{W}$  acts simply transitively

on bases,  $\sigma\tau = 1$ , so  $\tau = \sigma$  because  $\sigma$  has order 2 (for the same reason just discussed). Hence  $\sigma$  is unique.

Let  $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$  be a reduced expression for  $\sigma(\alpha_i \in \Delta)$ . Suppose  $\beta \in \Delta$  is not in this expression. Since  $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ , it is clear that  $\sigma_{\alpha_s} \cdots \sigma_{\alpha_t}$  cannot take  $\beta$  to another simple root. Since each  $\sigma_{\alpha_i}$  permutes  $\Phi^+ \setminus \{\alpha_i\}$  (Lemma 10.2B),  $\sigma(\beta) \notin \Phi^-$ , which is a contradiction. Hence a reduced expression for  $\sigma$  must involve all  $\sigma_\alpha(\alpha \in \Delta)$ .

Since  $\sigma(\Phi^+) = \Phi^-$ ,  $l(\sigma) = n(\sigma) = \#(\Phi^+) = \#(\Phi)/2$ .  $\square$

**Exercise 10.10.** Given  $\Delta = \{\alpha_1, \dots, \alpha_l\}$  in  $\Phi$ , let  $\lambda = \sum_{i=1}^l k_i \alpha_i$  ( $k_i \in \mathbb{Z}$ , all  $k_i \geq 0$  or all  $k_i \leq 0$ ). Prove that either  $\lambda$  is a multiple (possibly 0) of a root, or else there exists  $\sigma \in \mathcal{W}$  such that  $\sigma\lambda = \sum_{i=1}^l k'_i \alpha_i$ , with some  $k'_i > 0$  and some  $k'_i < 0$ .

**Solution.** If  $\lambda$  is not a multiple of any root, then the hyperplane  $P_\lambda$  orthogonal to  $\lambda$  is not included in  $\bigcup_{\alpha \in \Phi} P_\alpha$ . Take  $\mu \in P_\lambda - \bigcup_{\alpha \in \Phi} P_\alpha$ . Then find  $\sigma \in \mathcal{W}$  for which all  $(\alpha, \sigma\mu) > 0$ .

It follows that  $0 = (\lambda, \mu) = (\sigma\lambda, \sigma\mu) = \sum_{i=1}^l k_i(\alpha_i, \sigma\mu)$ .  $\square$

**Exercise 10.11.** Let  $\Phi$  be irreducible. Prove that  $\Phi^\vee$  is also irreducible. If  $\Phi$  has all roots of equal length, so does  $\Phi^\vee$  (and then  $\Phi^\vee$  is isomorphic to  $\Phi$ ). On the other hand, if  $\Phi$  has two root lengths, then so does  $\Phi^\vee$ ; but if  $\alpha$  is long, then  $\alpha^\vee$  is short (and vice versa). Use this fact to prove that  $\Phi$  has a unique maximal short root (relative to the partial order  $\prec$  defined by  $\Delta$ ).

**Solution.** Since  $({}^\vee\Phi^\vee) = \Phi$ , to prove that  $\Phi$  irreducible implies  $\Phi^\vee$  irreducible, it is enough to prove that if  $\Phi$  is reducible, then so is  $\Phi^\vee$ . But this is obvious because  $(\alpha^\vee, \beta^\vee) = 0$  if and only if  $(\alpha, \beta) = 0$ .

Also, if all roots of  $\Phi$  have the same length, then  $(\alpha, \alpha)$  is a constant  $C$  for  $\alpha \in \Phi$ , so  $\beta^\vee = \frac{2}{C}\beta$  for all  $\beta \in \Phi$ , which means all root lengths are the same in  $\Phi^\vee$ . Multiplication by this nonzero scalar gives an isomorphism between  $\Phi$  and  $\Phi^\vee$ .

If instead  $\Phi$  has 2 root lengths, then so does  $\Phi^\vee$ . This must hold because if  $\Phi^\vee$  had one root length, then so would  $({}^\vee\Phi^\vee) = \Phi$ . Since the length of  $\beta^\vee$  gets shorter the longer  $\beta$  is, it is clear that short roots of  $\Phi$  correspond to long roots of  $\Phi^\vee$ , and vice versa. Finally, a maximal root of  $\Phi^\vee$  is long (Lemma 10.4D), so corresponds to a maximal short root of  $\Phi$  (heights are preserved in passing to duals).  $\square$

**Exercise 10.12.** Let  $\lambda \in \mathfrak{C}(\Delta)$ . If  $\sigma\lambda = \lambda$  for some  $\sigma \in \mathcal{W}$ , then  $\sigma = 1$ .

**Solution.** Note that  $\mathcal{W}$  sends chambers of  $\Phi$  to other chambers, so if  $\sigma\lambda = \lambda$ , then  $\sigma$  fixes a chamber of  $\Phi$ . Then the exercise is a consequence of Theorem 10.3(e) and the fact that the set of chambers of  $\Phi$  and the set of bases of  $\Phi$  are isomorphic as  $\mathcal{W}$ -sets (10.1).  $\square$

**Exercise 10.13.** The only reflections in  $\mathcal{W}$  are those of the form  $\sigma_\alpha(\alpha \in \Phi)$ . [A vector in the reflecting hyperplane would, if orthogonal to no root, be fixed only by the identity in  $\mathcal{W}$ .]

**Solution.** Let  $\tau \in \mathcal{W}$  be some reflection. If the reflecting hyperplane of  $\tau$  is not orthogonal to a root of  $\Phi$ , then let  $\gamma$  be a vector in this hyperplane. Then  $\gamma$  is contained in  $\mathfrak{C}(\Delta)$  for some base  $\Delta$ . By Exercise 10.12, the only element of  $\mathcal{W}$  fixing  $\gamma$  is the identity, so  $\tau = 1$ .  $\square$

**Exercise 10.14.** Prove that each point of  $E$  is  $\mathcal{W}$ -conjugate to a point in the closure of the fundamental Weyl chamber relative to a base  $\Delta$ . [Enlarge the partial order on  $E$  by defining  $\mu \prec \lambda$  iff  $\lambda - \mu$  is a nonnegative  $\mathbb{R}$ -linear combination of simple roots. If  $\mu \in E$ , choose  $\sigma \in \mathcal{W}$  for which  $\lambda = \sigma\mu$  is maximal in this partial order.]

**Solution.** This is similar to the proof of Theorem 10.3(a). Here we replace a regular element  $\gamma$  with any point of  $E$ . The difference is that  $(\sigma(\gamma), \alpha)$  may be 0 for some  $\alpha$ . This will imply only that  $(\sigma(\gamma), \alpha) \geq 0$  for all  $\alpha \in \Delta$ , which is to say that  $\sigma(\gamma) \in \overline{\mathfrak{C}(\Delta)}$ .  $\square$

## 11 Classification

**Exercise 11.1.** Verify the Cartan matrices (Table 1).

**Exercise 11.2.** Calculate the determinants of the Cartan matrices (using induction on  $l$  for types  $A_l - D_l$ ), which are as follows:

$$A_l : l + 1; \quad B_l : 2; \quad C_l : 2; \quad D_l : 4; \quad E_6 : 3; \quad E_7 : 2; \quad E_8, F_4, G_2 : 1$$

**Solution.** By expanding along the first row, we get that  $\det A_l = 2 \det A_{l-1} - \det A_{l-2}$  (the second matrix needs to again be expanded along the first column). Similar relations hold for  $B_l, C_l$  and  $D_l$ .

By inspection,  $\det A_1 = 2$  and  $\det A_2 = 3$ , so by induction,  $\det A_l = l + 1$ .

Also,  $\det B_1 = \det B_2 = 2$ , and  $\det C_1 = \det C_2 = 2$ , so  $\det B_l = 2, \det C_l = 2$ .

Furthermore  $D_2 = A_1 \times A_1$  and  $D_3 = A_3$ , so  $\det D_2 = \det D_3 = 4$ , which gives  $\det D_l = 4$ .

The determinants of  $E_6, E_7, E_8, F_4$  and  $G_2$  can all be done via row reduction.  $\square$

**Exercise 11.3.** Use the algorithm of (11.1) to write down all roots for  $G_2$ . Do the same for  $C_3$  :

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

**Solution.** Using the algorithm of (11.1), we start with simple roots for  $G_2$ , a short root  $\alpha$ , and a long root  $\beta$ . We know that  $\langle \alpha, \beta \rangle = -1$  and  $\langle \beta, \alpha \rangle = -3$ . This means that we have a root string

$$\{\beta, \beta + \alpha, \beta + 2\alpha, \beta + 3\alpha\}$$

Also, if  $2\beta + k\alpha$  is to be a root, we need  $r - q = \langle \beta + k\alpha, \beta \rangle = 2 - k$  to be negative, i.e.,  $k \geq 3$ . This means  $2\beta + 3\alpha$  is also a root, and we have listed all positive roots of  $G_2$  (cf. p. 44).

In the case of  $C_3$ , we have 3 simple roots,  $\alpha$  and  $\beta$  (which are short), and  $\gamma$  (which is long). We immediately see that  $\{\alpha + \beta, \gamma + \beta, \gamma + 2\beta\}$  are roots. Since  $\langle \alpha + \beta, \gamma \rangle = -1$ , we also get that  $\alpha + \beta + \gamma$  is a root. Also,  $\langle \gamma + 2\beta, \alpha \rangle = -2$ , so  $\gamma + 2\beta + \alpha$  and  $\gamma + 2\beta + 2\alpha$  are also roots. All other combinations of roots result in nonnegative brackets  $\langle, \rangle$ , so these are all of the positive roots of  $C_3$ . To summarize, the positive roots are:

$$\{\alpha, \beta, \gamma, \alpha + \beta, \beta + \gamma, 2\beta + \gamma, \alpha + \beta + \gamma, \alpha + 2\beta + \gamma, 2\alpha + 2\beta + \gamma\}$$

$\square$

**Exercise 11.4.** Prove that the Weyl group of a root system  $\Phi$  is isomorphic to the direct product of the respective Weyl groups of its irreducible components.

**Solution.** By induction on the number of components, we need only show this in the case that  $\Phi$  is partitioned into two orthogonal components  $\Phi_1$  and  $\Phi_2$ . Let  $\mathcal{W}_1$  and  $\mathcal{W}_2$  be their respective Weyl groups, and let  $\mathcal{W}$  be the Weyl group of  $\Phi$ .

Then  $\mathcal{W}_1$  and  $\mathcal{W}_2$  are commuting subgroups of  $\mathcal{W}$  because of the orthogonality condition. Since  $\mathcal{W}$  is generated by reflections in both  $\mathcal{W}_1$  and  $\mathcal{W}_2$ , and  $\mathcal{W}_1 \cap \mathcal{W}_2 = 1$ , we see that  $\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2$ .  $\square$

**Exercise 11.5.** Prove that each irreducible root system is isomorphic to its dual, except that  $B_l, C_l$  are dual to each other.

**Solution.** Since  $\langle \beta^\vee, \alpha^\vee \rangle = \langle \alpha, \beta \rangle$  Exercise 9.2, the Cartan matrix of  $\Phi^\vee$  is the transpose of the Cartan matrix of  $\Phi$ . In terms of Dynkin diagrams, this corresponds to reversing the directions of the arrows. In the case of  $A_l, D_l, E_6, E_7$  and  $E_8$ , nothing happens, so they are self-dual. In the cases of  $F_4$  and  $G_2$ , one can find an isomorphism to their duals by reordering the simple roots. Finally,  $B_l$  and  $C_l$  become one another under this correspondence, so they are dual to one another.  $\square$

**Exercise 11.6.** Prove that an inclusion of one Dynkin diagram in another (e.g.,  $E_6$  in  $E_7$  or  $E_7$  in  $E_8$ ) induces an inclusion of the corresponding root systems.

**Solution.** An inclusion of Dynkin diagrams  $D_1 \hookrightarrow D_2$  corresponds to the Cartan matrix of  $D_1$  being a submatrix of the Cartan matrix of  $D_2$ .  $\square$

## 12 Construction of root systems and automorphisms

**Definition 12.1.** Let  $\Gamma = \{\sigma \in \text{Aut } \Phi \mid \sigma(\Delta) = \Delta\}$ , then  $\text{Aut } \Phi = \mathcal{W} \rtimes \Gamma$ . This  $\Gamma$  is usually viewed as the group of **diagram automorphisms** or **graph automorphisms**.

**Example 12.1** ( $A_l (l \geq 1)$ ).  $E = \text{span}\{\varepsilon_1 + \cdots + \varepsilon_{l+1}\}^\perp \subset \mathbb{R}^{l+1}$ ,  $I' = I \cap E$ .

$\Phi = \{\alpha \in I' \mid (\alpha, \alpha) = 2\} = \{\varepsilon_i - \varepsilon_j, i \neq j\}$ ,  $\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq l\}$ .

Weyl group:  $\mathcal{W} \cong \mathcal{S}_{l+1}$  by  $\sigma_{\alpha_i} \mapsto (i, i+1)$ .  $\Gamma \cong \mathbb{Z}/2$  when  $l \geq 2$ .

**Example 12.2** ( $B_l (l \geq 2)$ ).  $E = \mathbb{R}^l$ .  $\Phi = \{\alpha \in I \mid (\alpha, \alpha) = 1, 2\} = \{\pm \varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j), i \neq j\}$ ,

$\Delta = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_l\}$ .

Weyl group:  $\mathcal{W} \cong (\mathbb{Z}/2)^l \rtimes \mathcal{S}_l$  by corresponding  $\sigma_{\varepsilon_i}$  to sign changes.  $\Gamma = 1$ .

**Example 12.3** ( $C_l (l \geq 3)$ ).  $E = \mathbb{R}^l$ .  $C_l$  is dual to  $B_l$ , hence

$\Phi = \{\pm 2\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j), i \neq j\}$ ,

$\Delta = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{l-1} - \varepsilon_l, 2\varepsilon_l\}$ .

Weyl group:  $\mathcal{W} \cong (\mathbb{Z}/2)^l \rtimes \mathcal{S}_l$  same as  $B_l$ .  $\Gamma = 1$ .

**Example 12.4** ( $D_l (l \geq 4)$ ).  $E = \mathbb{R}^l$ .  $\Phi = \{\alpha \in I \mid (\alpha, \alpha) = 2\} = \{\pm(\varepsilon_i \pm \varepsilon_j), i \neq j\}$ ,

$\Delta = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_{l-1} + \varepsilon_l\}$ .

Weyl group:  $\mathcal{W} \cong (\mathbb{Z}/2)^{l-1} \rtimes \mathcal{S}_l$  where  $\sigma_{\varepsilon_i + \varepsilon_j} \sigma_{\varepsilon_i - \varepsilon_j}$  is corresponding to the sign change of  $i, j$  position.

$\Gamma = \mathcal{S}_3$  when  $l = 4$ , and  $\mathbb{Z}/2$  when  $l > 4$ .

**Example 12.5** ( $E_6, E_7, E_8$ ). It suffices to construct  $E_8$ .

$E = \mathbb{R}^8, I' = I + \mathbb{Z}((\varepsilon_1 + \cdots + \varepsilon_8)/2)$ ,  $I'' =$  subgroup of  $I'$  consisting of all elements  $\sum c_i \varepsilon_i + \frac{c}{2}(\varepsilon_1 + \cdots + \varepsilon_8)$  for which  $\sum c_i$  is an even integer.

$\Phi = \{\alpha \in I'' \mid (\alpha, \alpha) = 2\} = \{\pm(\varepsilon_i \pm \varepsilon_j), i \neq j\} \cup \{\frac{1}{2} \sum (-1)^{k_i} \varepsilon_i\}$  (where the  $k_i = 0, 1$ , add up to an even integer).

$\Delta = \{\frac{1}{2}(\varepsilon_1 + \varepsilon_8 - (\varepsilon_2 + \cdots + \varepsilon_7)), \varepsilon_1 + \varepsilon_2, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_6\}$ .

Weyl group has order  $2^{14} 3^5 5^2 7 = 696729600$ .

**Example 12.6** ( $F_4$ ).  $E = \mathbb{R}^4, I' = I + \mathbb{Z}((\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2)$ .

$\Phi = \{\alpha \in I' \mid (\alpha, \alpha) = 1, 2\} = \{\pm \varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j), i \neq j\} \cup \{\pm \frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$ .

$\Delta = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$ .

Weyl group has order 1152.

**Example 12.7** ( $G_2$ ).  $E = \text{span}\{\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}^\perp \subset \mathbb{R}^3, I' = I \cap E$ .

$\Phi = \{\alpha \in I' \mid (\alpha, \alpha) = 2, 6\} = \pm\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3, 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3, 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\}$ .

$\Delta = \{\varepsilon_1 - \varepsilon_2, -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$ .

Weyl group:  $\mathcal{W} \cong D_6$ .

**Exercise 12.1.** Verify the details of the constructions in (12.1).

**Exercise 12.2.** Verify Table 2.

Type	Long	Short
$A_l$	$\alpha_1 + \alpha_2 + \cdots + \alpha_l$	
$B_l$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_l$	$\alpha_1 + \alpha_2 + \cdots + \alpha_l$
$C_l$	$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$
$D_l$	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$	
$E_6$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	
$E_7$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	
$E_8$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	
$F_4$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
$G_2$	$3\alpha_1 + 2\alpha_2$	$2\alpha_1 + \alpha_2$

**Exercise 12.3.** Let  $\Phi \subset E$  satisfy (R1), (R3), (R4), but not (R2), cf. Exercise 9.9. Suppose moreover that  $\Phi$  is irreducible, in the sense of Section 11. Prove that  $\Phi$  is the union of root systems of type  $B_n, C_n$  in  $E$  ( $n = \dim E$ ), where the long roots of  $B_n$  are also the short roots of  $C_n$ . (This is called the non-reduced root system of type  $BC_n$  in the literature.)

**Exercise 12.4.** Prove that the long roots in  $G_2$  form a root system in  $E$  of type  $A_2$ .

**Solution.** Let  $\alpha$  be a short simple root of  $G_2$ , and let  $\beta$  be a long simple root. The long positive roots of  $G_2$  are  $\{\beta, 3\alpha + \beta, 3\alpha + 2\beta\}$  Exercise 11.3. It is clear from this description that the long roots form a root system, and that  $\{\beta, -3\alpha - 2\beta\}$  forms a base. Using the Cartan matrix for  $G_2$ , one deduces that the Cartan matrix for this base is the same as that of  $A_2$ .  $\square$

**Exercise 12.5.** In constructing  $C_l$ , would it be correct to characterize  $\Phi$  as the set of all vectors in  $I$  of squared length 2 or 4? Explain.

**Solution.** No, this would give vectors such as  $\pm 4\varepsilon_i$ . But ignoring that problem, one would also have vectors like  $2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ . The resulting set of vectors would be much larger than  $\Phi$ .  $\square$

**Exercise 12.6.** Prove that the map  $\alpha \mapsto -\alpha$  is an automorphism of  $\Phi$ . Try to decide for which irreducible  $\Phi$  this belongs to the Weyl group.

**Solution.** It is immediate that  $\alpha \mapsto -\alpha$  is an automorphism of  $\Phi$  since

$$\langle -\alpha, -\beta \rangle = \frac{2(-\alpha, -\beta)}{(-\beta, -\beta)} = \frac{2(\alpha, \beta)}{(\beta, \beta)} = \langle \alpha, \beta \rangle.$$

Since  $\text{Aut } \Phi$  is a semidirect product of  $\mathcal{W}$  and the subgroup  $\Gamma$ , it is immediate that  $\alpha \mapsto -\alpha$  is an element of  $\mathcal{W}$  for  $B_l, C_l, E_7, E_8, F_4$  and  $G_2$  by Table 12.1.

For  $A_l$ ,  $\alpha \mapsto -\alpha$  is not an element of  $\mathcal{W}$  if  $l > 1$ . To see this, we use the description of  $A_l$  as the set of vectors  $\{\varepsilon_i - \varepsilon_j, i \neq j\}$  in  $\mathbb{R}^{l+1}$ . Then the reflections  $\sigma_{\alpha_i}$  where  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  permutes the vectors  $\varepsilon_i$  and  $\varepsilon_{i+1}$  and leaves the other standard vectors fixed. From this, it is clear that, for example, one cannot send  $(\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3)$  to  $(\varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_1)$  because this would require a permutation which swapped 1 with 2 and swapped 1 with 3. Of course it is clear that  $A_1$  has  $\alpha \mapsto -\alpha$  as an element of its Weyl group.

Similarly,  $D_l$  for  $l \geq 4$  does not have  $\alpha \mapsto -\alpha$  in its Weyl group for  $l$  odd, but does for  $l$  even. It can be described as the set of vectors  $\{\pm \varepsilon_i \pm \varepsilon_j, i \neq j\}$  in  $\mathbb{R}^l$ , and its Weyl group consists of permutations of the  $\varepsilon_i$  along the sign changes that involve an even number of sign changes.

Not sure about  $E_6$ .  $\square$

**Exercise 12.7.** Describe  $\text{Aut } \Phi$  when  $\Phi$  is not irreducible

**Solution.** Write  $\Phi = \Phi_1 \cup \dots \cup \Phi_r$  (disjoint) where the  $\Phi_i$  are irreducible root systems and  $(\Phi_i, \Phi_j) = 0$ . Any automorphism  $\sigma$  of  $\Phi$  must satisfy  $\sigma(\Phi_i) \subset \Phi_j$  because irreducibility is an invariant of isomorphism of root systems, i.e., if  $\sigma(\Phi_i)$  were contained in two or more components of  $\Phi$ , then we could write it as a disjoint union of pairwise orthogonal sets. Since  $\Phi$  is finite, it follows from a counting argument that  $\sigma(\Phi_i) = \Phi_j$ . Let  $S$  be the subgroup of permutations  $\sigma$  of  $\{1, \dots, r\}$  such that  $\Phi_i$  is isomorphic to  $\Phi_{\sigma(i)}$ . Then  $\text{Aut } \Phi$  is the semidirect product of  $S$  with  $\text{Aut } \Phi_1 \times \dots \times \text{Aut } \Phi_r$ .  $\square$

## 13 Abstract theory of weights

**Example 13.1** ( $\mathfrak{sl}(3, F)$ ).  $h_1 = e_{11} - e_{22}, h_2 = e_{22} - e_{33}, e_{12}, e_{13}, e_{23}, e_{21}, e_{31}, e_{32}$ .

$$\text{ad } h_1 = \text{diag}(0, 0, 2, 1, -1, -2, -1, 1), \text{ad } h_2 = \text{diag}(0, 0, -1, 1, 2, 1, -1, -2)$$

$$\alpha_1 = (2, -1), \alpha_1 + \alpha_2 = (1, 1), \alpha_2 = (-1, 2)$$

The Cartan matrix is  $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ .

$$\alpha_1 = 2\lambda_1 - \lambda_2, \alpha_2 = -\lambda_1 + 2\lambda_2;$$

$$\lambda_1 = \frac{1}{3}(2\alpha_1 + \alpha_2), \lambda_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$$

$\text{ord}(\lambda_1) = \text{ord}(\lambda_2) = 3$ , the fundamental group is  $\text{Cycle}(3)$ .

$$\delta = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_1 + \alpha_2) = (1, 1) = \lambda_1 + \lambda_2$$

The standard set with highest weight must be like the follow:

$$\Pi = \{m, m-2, \dots, -m\}$$

with highest weight  $m$ .

**Exercise 13.1.** Let  $\Phi = \Phi_1 \cup \dots \cup \Phi_t$  be the decomposition of  $\Phi$  into its irreducible components, with  $\Delta = \Delta_1 \cup \dots \cup \Delta_t$ . Prove that  $\Lambda$  decomposes into a direct sum  $\Lambda_1 \oplus \dots \oplus \Lambda_t$ ; what about  $\Lambda^+$ ?

**Solution.** Given any element  $\lambda \in \Lambda$ , let  $\lambda_i$  be defined by  $\langle \lambda_i, \alpha \rangle = \langle \lambda, \alpha \rangle$  if  $\alpha \in \Phi_i$  and 0 otherwise. In other words, let  $\lambda_i$  be the orthogonal projection of  $\lambda$  onto the subspace spanned by  $\Phi_i$ . Then  $\lambda = \lambda_1 + \dots + \lambda_t$ , so  $\Lambda = \Lambda_1 + \dots + \Lambda_t$ . It is clear that  $\Lambda_i \cap \Lambda_j = 0$  if  $i \neq j$ , so the sum is direct.

However, we cannot say the same thing about  $\Lambda^+$ . For example, consider the root system  $A_1 \times A_1$  with base  $\{(1, 0), (0, 1)\}$ . Write  $A_1 \times A_1 = \Phi_1 \cup \Phi_2$  where  $\Phi_1 = \{(\pm 1, 0)\}$  and  $\Phi_2 = \{(0, \pm 1)\}$ . Then  $(2, -1) \in \Lambda^+$ , but  $(0, -1) \notin \Lambda_2^+$ , so we do not have  $\Lambda^+$  generated by  $\Lambda_1^+$  and  $\Lambda_2^+$ .  $\square$

**Exercise 13.2.** Show by example (e.g., for  $A_2$ ) that  $\lambda \notin \Lambda^+, \alpha \in \Delta, \lambda - \alpha \in \Lambda^+$  is possible.

**Solution.** In  $A_2$ ,  $\lambda = 3\lambda_1 - \lambda_2 \notin \Lambda^+, \alpha = 2\lambda_1 - \lambda_2 \in \Delta, \lambda - \alpha = \lambda_1 \in \Lambda^+$ .  $\square$

**Exercise 13.3.** Verify some of the data in Table 1, e.g., for  $F_4$ .

**Exercise 13.4.** Using Table 1, show that the fundamental group of  $A_l$  is cyclic of order  $l+1$ , while that of  $D_l$  is isomorphic to  $\mathbb{Z}/4$  ( $l$  odd), or  $\mathbb{Z}/2 \times \mathbb{Z}/2$  ( $l$  even). (It is easy to remember which is which, since  $A_3 = D_3$ .)

**Solution.** The first is clear because the element  $\lambda_i$  listed for  $A_l$  in Table 1 does not have order smaller than  $l+1$ . If it did, then  $\frac{j(l-i+1)}{l+1}$  would be an integer for  $j < l+1$  which is not true for  $i = 1$ , for example. Since the determinant of the Cartan matrix for  $A_l$  is  $l+1$ , we conclude that its fundamental group is  $\mathbb{Z}/(l+1)$ .

The same considerations show that if  $l$  is even, then every element  $\lambda_i$  listed for  $D_l$  has order 2, whereas if  $l$  is odd, then  $\lambda_l$  has order 4. Hence the respective fundamental groups are  $\mathbb{Z}/2 \times \mathbb{Z}/2$  for  $l$  even, and  $\mathbb{Z}/4$  for  $l$  odd since the Cartan matrix for  $D_l$  has determinant 4.  $\square$

**Exercise 13.5.** If  $\Lambda'$  is any subgroup of  $\Lambda$  which includes  $\Lambda_r$ , prove that  $\Lambda'$  is  $\mathcal{W}$ -invariant. Therefore, we obtain a homomorphism  $\phi: \text{Aut } \Phi/\mathcal{W} \rightarrow \text{Aut}(\Lambda/\Lambda_r)$ . Prove that  $\phi$  is injective, then deduce that  $-1 \in \mathcal{W}$  if and only if  $\Lambda_r \supset 2\Lambda$  (cf. Exercise 12.6). Show that  $-1 \in \mathcal{W}$  for precisely the irreducible root systems  $A_1, B_l, C_l, D_l$  ( $l$  even),  $E_7, E_8, F_4, G_2$ .

**Solution.**  $\sigma_I \lambda_j = \lambda_j - \delta_{ij} \alpha_i$ , hence  $\sigma(\lambda + \Lambda_r) = \lambda + \Lambda_r$ . Then  $\Lambda'$  is  $\mathcal{W}$ -invariant since  $\Lambda' = \coprod_{\lambda} \lambda + \Lambda_r$ .

$$\ker \phi = \{\sigma \mid \sigma(\lambda + \Lambda_r) = \lambda + \Lambda_r, \forall \lambda \in \Lambda\} \quad \text{Aut } \Phi/\mathcal{W} = \Gamma$$

$\square$



**Exercise 13.6.** Prove that the roots in  $\Phi$  which are dominant weights are precisely the highest long root and (if two root lengths occur) the highest short root (cf. (10.4) and Exercise 10.11), when  $\Phi$  is irreducible.

**Exercise 13.7.** If  $\varepsilon_1, \dots, \varepsilon_l$  is an **obtuse** basis of the euclidean space  $E$  (i.e., all  $(\varepsilon_i, \varepsilon_j) \leq 0$  for  $i \neq j$ ), prove that the dual basis is **acute** (i.e., all  $(\varepsilon_i^*, \varepsilon_j^*) \geq 0$  for  $i \neq j$ ). [Reduce to the case  $l = 2$ .]

**Exercise 13.8.** Let  $\Phi$  be irreducible. Without using the data in Table 1, prove that each  $\lambda_i$  is of the form  $\sum_j q_{ij} \alpha_j$ , where all  $q_{ij}$  are positive rational numbers. [Deduce from Exercise 13.7 that all  $q_{ij}$  are nonnegative. From  $(\lambda_i, \lambda_j) > 0$ . Then show that if  $q_{ij} > 0$  and  $(\alpha_j, \alpha_k) < 0$ , then  $q_{ik} > 0$ .]

**Exercise 13.9.** Let  $\lambda \in \Lambda^+$ . Prove that  $\sigma(\lambda + \delta) - \delta$  is dominant only for  $\sigma = 1$ .

**Exercise 13.10.** If  $\lambda \in \Lambda^+$ , prove that the set  $\Pi$  consisting of all dominant weights  $\mu \prec \lambda$  and their  $\mathcal{W}$ -conjugates is saturated, as asserted in (13.4).

**Exercise 13.11.** Prove that each subset of  $\Lambda$  is contained in a unique smallest saturated set, which is finite if the subset in question is finite.

**Exercise 13.12.** For the root system of type  $A_2$ , write down the effect of each element of the Weyl group on each of  $\lambda_1, \lambda_2$ . Using this data, determine which weights belong to the saturated set having highest weight  $\lambda_1 + 3\lambda_2$ . Do the same for type  $G_2$  and highest weight  $\lambda_1 + 2\lambda_2$ .

**Exercise 13.13.** Call  $\lambda \in \Lambda^+$  **minimal** if  $\mu \in \Lambda^+, \mu \prec \lambda$  implies that  $\mu = \lambda$ . Show that each coset of  $\Lambda_r$  in  $\Lambda$  contains precisely one minimal  $\Lambda$ . Prove that  $\lambda$  is minimal if and only if the  $\mathcal{W}$ -orbit of  $\lambda$  is saturated (with highest weight  $\lambda$ ), if and only if  $\lambda \in \Lambda^+$  and  $\langle \lambda, \alpha \rangle = 0, 1, -1$  for all roots  $\alpha$ . Determine (using Table 1) the nonzero minimal  $\lambda$  for each irreducible  $\Phi$ , as follows:

$$A_l: \lambda_1, \dots, \lambda_l$$

$$B_l: \lambda_l$$

$$C_l: \lambda_1$$

$$D_l: \lambda_1, \lambda_{l-1}, \lambda_l$$

$$E_6: \lambda_1, \lambda_6$$

$$E_7: \lambda_7$$

## Chapter IV

# Isomorphism and Conjugacy Theorem

## 14 Isomorphism theorem

**Exercise 14.1.** Generalize Theorem 14.2 to the case:  $L$  semisimple.

**Exercise 14.2.** Let  $L = \mathfrak{sl}(2, F)$ . If  $H, H'$  are any two maximal toral subalgebras of  $L$ , prove that there exists an automorphism of  $L$  mapping  $H$  onto  $H'$ .

**Exercise 14.3.** Prove that the subspace  $M$  of  $L \times L'$  introduced in the proof of Theorem 14.2 will actually equal  $D$ , if  $x$  and  $x'$  are chosen carefully.

**Exercise 14.4.** Let  $\sigma$  be as in Proposition 14.3. Is it necessarily true that  $\sigma(x_\alpha) = -y_\alpha$  for nonsimple  $\alpha$ , where  $[x_\alpha, y_\alpha] = h_\alpha$ ?

**Exercise 14.5.** Consider the simple algebra  $\mathfrak{sl}(3, F)$  of type  $A_2$ . Show that the subgroup of  $\text{Int } L$  generated by the automorphisms  $\tau_\alpha$  in (14.3) is strictly larger than the Weyl group (here  $\mathcal{S}_3$ ). [View  $\text{Int } L$  as a matrix group and compute  $\tau_\alpha^2$  explicitly.]

**Exercise 14.6.** Use Theorem 14.2 to construct a subgroup  $\Gamma(L)$  of  $\text{Aut } L$  isomorphic to the group of all graph automorphisms (12.2) of  $\Phi$ .

**Exercise 14.7.** For each classical algebra (1.2), show how to choose elements  $h_\alpha \in H$  corresponding to a base of  $\Phi$  (cf. Exercise 8.2).

## 15 Cartan subalgebras

Cartan subalgebra will be abbreviated as CSA.

**Exercise 15.1.** A semisimple element of  $\mathfrak{sl}(n, F)$  is regular if and only if its eigenvalues are all distinct (i.e., if and only if its minimal and characteristic polynomials coincide).

**Exercise 15.2.** Let  $L$  be semisimple ( $\text{char } F = 0$ ). Deduce from Exercise 8.7 that the only solvable Engel subalgebras of  $L$  are the CSA's.

**Exercise 15.3.** Let  $L$  be semisimple ( $\text{char } F = 0$ ),  $x \in L$  semisimple. Prove that  $x$  is regular if and only if  $x$  lies in exactly one CSA.

**Exercise 15.4.** Let  $H$  be a CSA of a Lie algebra  $L$ . Prove that  $H$  is maximal nilpotent, i.e., not properly included in any nilpotent subalgebra of  $L$ . Show that the converse is false.

**Exercise 15.5.** Show how to carry out the proof of Lemma A of (15.2) if the field  $F$  is only required to be of cardinality exceeding  $\dim L$ .

**Exercise 15.6.** Let  $L$  be semisimple ( $\text{char } F = 0$ ),  $L'$  a semisimple subalgebra. Prove that each CSA of  $L'$  lies in some CSA of  $L$ . [Cf. Exercise 6.9.]

## 16 Conjugacy theorems

**Exercise 16.1.** *Prove that  $\mathcal{E}(L)$  has order one if and only if  $L$  is nilpotent.*

**Exercise 16.2.** *Let  $L$  be semisimple,  $H$  a CSA,  $\Delta$  a base of  $\Phi$ . Prove that any subalgebra of  $L$  consisting of nilpotent elements, and maximal with respect to this property, is conjugate under  $\mathcal{E}(L)$  to  $N(\Delta)$ , the derived algebra of  $B(\Delta)$ .*

**Exercise 16.3.** *Let  $\Psi$  be a set of roots which is **closed root set** [closed] ( $\alpha, \beta \in \Psi, \alpha + \beta \in \Phi$  implies  $\alpha + \beta \in \Psi$ ) and satisfies  $\Psi \cap -\Psi = \emptyset$ . Prove that  $\Psi$  is included in the set of positive roots relative to some base of  $\Phi$ . [Use Exercise 16.2.] (This exercise belongs to the theory of root systems, but is easier to do using Lie algebras.)*

**Exercise 16.4.** *How does the proof of Theorem 16.4 simplify in case  $L = \mathfrak{sl}(2, F)$ ?*

**Exercise 16.5.** *Let  $L$  be semisimple. If a semisimple element of  $L$  is regular, then it lies in only finitely many Borel subalgebras. (The converse is also true, but harder to prove, and suggests a notion of “regular” for elements of  $L$  which are not necessarily semisimple.)*

## Chapter V

# Existence Theorem

### 17 Universal enveloping algebras

**Exercise 17.1.** *Prove that if  $\dim L < \infty$ , then  $\mathfrak{U}(L)$  has no zero divisors. [Hint: Use the fact that the associated graded algebra  $\mathfrak{G}$  is isomorphic to a polynomial algebra.]*

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