Assignment-2

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5 a) $P(\text{loss}) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

So PMF is same as geometric random variable

$$PMF = \left(\frac{8}{9}\right)^{x-1} \left(\frac{1}{9}\right) \quad x \geq 1$$

b) $$PMF = {}^{k-1}C_2 \left(\frac{2}{3}\right)^{k-3} \left(\frac{1}{3}\right)^3 \quad k \geq 3$$

$$= {}^{\frac{z}{2}-1}C_2 \left(\frac{2}{3}\right)^{\frac{z}{2}-3} \left(\frac{1}{3}\right)^3 \quad z = 6, 8, 10 \ldots$$

c) A_1: Both win first round

A_2: Only Vinay wins first round

A_3: Only Mahesh wins first round

A_4: Both loose first round

Let N denote number of rounds untill each one of them won atleast once.

$E(N|A_2) = 1 + \frac{1}{2/3}$. Because when event A_2 (A_3) occurs, the distribution on time untill Mahesh (Vinay) wins is a geometric variable with expectation $1/p = \frac{1}{2/3}$.

$$E(N) = E(N|A_1) \cdot P(A_1) + E(N|A_2) \cdot P(A_2) + E(N|A_3) P(A_3) + E(N|A_4) P(A_4)$$

$$= 1 \cdot \left(\frac{2}{3} \times \frac{2}{3}\right) + \left(1 + \frac{1}{2/3}\right) \left[\left(\frac{1}{3} \cdot \frac{2}{3}\right) + \left(\frac{1}{3} \cdot \frac{2}{3}\right)\right] + (1 + E(N)) \left(\frac{1}{3} \cdot \frac{1}{3}\right)$$

$$= \frac{4}{9} + \frac{10}{9} + (1 + E(N)) \cdot \frac{1}{9}$$

$$\Rightarrow E(N) = 15/8$$

16. a) It's an independent event.

So probability of same result is $\left(\frac{1}{2}\right)^z \left(\frac{1}{2}\right)^z$

$$= \frac{1}{2^{2z}}$$

b) i) suppose, 2^{nd} success at $(k+1)^{th}$ trail then 1^{st} success can happen at any of the k trails which can be selected by ${}^kC_1$

Probability such that all N coins land in same way

$$= \left(\frac{1}{2}\right)^z + \left(\frac{1}{2}\right)^z = \left(\frac{1}{2}\right)^{z-1} \quad (\because \text{1 for heads another for trails})$$

$\therefore P(E) = \left(\frac{1}{2}\right)^{z-1} \qquad P(E^c) = 1 - \left(\frac{1}{2}\right)^{z-1}$

where E = All coins have same face up

$$PMF(k) = {}^kC_1 \left(\left(\frac{1}{2}\right)^{z-1}\right)^2 \left[1 - \left(\frac{1}{2}\right)^{z-1}\right]^{k-1} \quad k \geq 1$$

$$= {}^kC_1 \left(\frac{1}{2}\right)^{2z-2} \left[1 - \left(\frac{1}{2}\right)^{z-1}\right]^{k-1}$$

ii) Number of tails before first success, M, can be written as

$$M = X_1 + X_2 + \cdots X_N$$

where X_i is number of tails that occur on unsuccessful trail i

N is number of unsuccessful trails

$E(X) = 1\cdot\left(\frac{1}{2}\right) + 2\cdot\left(\frac{1}{2}\right) = 3/2$ ($\because$ in an unsuccessful trail both are equiprobable)

$\text{var}(X) = E(X^2) - [E(X)]^2$

$= 1^2\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{2}\right) - \left(\frac{3}{2}\right)^2 = \frac{1}{4}$

N is a shifted geometric random variable : $N = B - 1$
where B is random variable with parameter $1/4$

$\therefore E(N) = E(B) - 1 = 4 - 1 = 3$

$\text{Var}(N) = \text{Var}(B) = \frac{1 - 1/4}{(1/4)^2} = 12$

$E[M] = \sum_n E[X_1 + X_2 + \cdots X_N | N = n] \cdot P(N = n)$

$= \sum_n n\, E(X_1 | N = n] P(N = n)$

$= E[X_1] \sum_n n P(N = n) = E[X_1] E(N)$

$= \left(\frac{3}{2}\right)(4 - 1) = 9/2$

$\text{var}(M) = E[\text{var}(M|N)] + \text{var}(E(M|N))$

$\text{var}(M|N) = E[M^2|N] - [E(M|N)]^2$

$E(\text{var}(M|N)) = E[E(M^2|N)] - E[\{E(M|N)\}^2]$

$= E(M^2) - E[\{E(M|N)\}^2]$ —(1)

$\text{var}(E(M|N)) = E[\{E(M|N)\}^2] - [E[E(M|N)]]^2$

$= E[\{E(M|N)\}^2] - [E(M)]^2$ —(2)

(1) + (2) $\Rightarrow$ $E[M^2] - [E(M)]^2 = \text{Var}(M)$

$\therefore \text{Var}(M) = E[\text{var}(M|N)] + \text{Var}(E(M|N))$

$\text{Var}(M|N = n) = \text{var}(X_1 + \cdots + X_n | N = n) = \text{Var}(X_1 + \cdots X_n)$

$= n \text{Var}(X_1)$

$E[\text{var}(M|N)] = \sum_n P(N = n) E[M|N = n]$

$= \sum_n P(N = n)\, n\, \text{var}(X_1) = \text{Var}(X_1) E(N)$

$\text{Var}(E(M|N)) = \text{Var}(N E(X)) = [E(X)]^2 \text{Var}(N)$ $\because E(X)$ is constant

$\Rightarrow \text{Var}(M) = E(N)\text{Var}(X) + E(X)^2 \text{Var}(N)$

$= (4-1)\left(\frac{1}{4}\right) + \left(\frac{3}{2}\right)^2 (12)$

$= \frac{111}{4}$

c) Since it's a geometric Random variable

$E[X] = E[X_m] + E[X_{m-1}] + \cdots E[X_2]$

where x is number of trails in Sandeep's experiment

x_i represents the random variable when i coins are present

$E[x_i] = 2^{i-1}$ since x_i is also a geometric random variable with parameter $\frac{1}{2^{i-1}}$

$E[x] = 2^{M-1} + 2^{M-2} + \dots 2 \quad = 2^M - 2$