Let X have the probability density function given by

$$f_X(x) = \begin{cases} \frac{1}{2}\, x,\ 0 \le x \le 2 \\ 0\ elsewhere \end{cases}$$

Find the density function of Y = Φ(X) = 6X - 3.

Notice that f_X(x) is positive for all x such that $0 \le x \le 1$. The function Φ is increasing for all X. We can then find the inverse function Φ^{-1} as follows

$$\begin{aligned} y &= 6x - 3 \\ \Rightarrow 6x &= y + 3 \\ \Rightarrow x &= \frac{y+3}{6} = \Phi^{-1}(y) \end{aligned}$$

We can then find the derivative of Φ^{-1} with respect to y as

$$\begin{aligned} \frac{d\Phi^{-1}}{dy} &= \frac{d}{dy}\left(\frac{y+3}{6}\right) \\ &= \frac{1}{6} \end{aligned}$$

The density of y is then

$$\begin{aligned} g(y) = f_Y(y) &= f_X\left[\Phi^{-1}(y)\right] \cdot \left|\frac{d\,\Phi^{-1}(y)}{dy}\right| \\ &= \left(\frac{1}{2}\right)\left(\frac{3+y}{6}\right)\left|\frac{1}{6}\right|, \quad 0 \le \frac{3+y}{6} \le 2 \end{aligned}$$

For all other values of y, g(y) = 0. Simplifying the density and the bounds we obtain

$$g(y) = f_Y(y) = \begin{cases} \frac{3+y}{72},\ -3 \le y \le 9 \\ 0\ \ elsewhere \end{cases}$$

Let X have the probability density function given by

$$f_X(x) = \begin{cases} e^{-x}, \; 0 \le x \le \infty \\ 0 \; elsewhere \end{cases}$$

Find the density function of $Y = X^{1/2}$.

Notice that $f_X(x)$ is positive for all x such that $0 \le x \le \infty$. The function Φ is increasing for all X. We can then find the inverse function Φ^{-1} as follows

$$\begin{aligned} y &= x^{\frac{1}{2}} \\ \Rightarrow y^2 &= x \\ \Rightarrow x &= \Phi^{-1}(y) = y^2 \end{aligned}$$

We can then find the derivative of Φ^{-1} with respect to y as

$$\begin{aligned} \frac{d\Phi^{-1}}{dy} &= \frac{d}{dy} y^2 \\ &= 2y \end{aligned}$$

The density of y is then

$$\begin{aligned} f_Y(y) &= f_X\left[\Phi^{-1}(y)\right] \cdot \left| \frac{d\Phi^{-1}(y)}{dy} \right| \\ &= e^{-y^2} \; |2y| \end{aligned}$$