Question 7:

(Writeup credits to Shashwat Goel)

I couldn't fully appreciate tut q7 (triangles) so I wrote my own explanation that I think gives a direct way to calculate no. of integer length triangles of perimeter p. Please check:

Hint for Observation:
f(p) can sometimes be equal to f(p+3), try creating a one-one map between them
Observation:
The one-one map is (a, b, c) -> (a+1, b+1, c+1). The triangle inequality is always preserved under this

Remaining part:
To Think: So when are they not equal/mappable like this?
Answer:
a) When one of a+1, b+1, or c+1 is 1. Also for p>3, only one of these can be 1 (otherwise triangle inequality is violated). b) when a+b=c, so a+1 + b+1 > c+1

Next step hint: Now think about parities
Next step: For a+b=c, you require p to be odd. Also, for a=1, b=c and therefore p needs to be odd again. Therefore f(p+3)=f(p) for odd p

Finally, can you count using this analysis, i.e. solve for even p?
Answer: Yes! as a+b=c and a+b+c=p, c=p/2. No. of valid unordered pairs s.t. a+b=c is floor(c/2). Therefore, no. of extra triangles from a+b=c case for even p is floor(p/4). Moreover, exactly one triangle comes from the a+1=1 case, where b+1, c+1 = (p+3-1)/2. So f(p+3) = f(p) + floor(p/4) + 1 (edited)

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finally giving this nice closed form

f(p)
= f(p-3) for even p
= f(p-3) + floor ((p-3)/4) + 1 for odd p

base case: f(3) = 1, f(4)=0, f(5)=1

Points to note in the final solution:

- **Bijections are useful to show whether something is increasing, decreasing or staying the same.**
- The number of triangles increase linearly, so the probability mass function is quadratic.
- This problem has a brute force pattern where you try to count for each perimeter the number of triangles. (eg. if the perimeter is 2000, one side can be 999 and other be from 2 to 999, if it's 998 then the range is 3 to 998 and so on, sum the series and try this yourself, but it's an ugly method yet obvious in an exam)

Question 8:

This is the required probability.

EXAMPLE 9. *From the set of all permutations of* $\{1, 2, 3, \ldots, n\}$ *select a permutation at random, assuming equal likelihood of all permutations. What is the probability that (a) the cycle containing 1 has length k?; (b) 1 and 2 belong to the same cycle?*

SOLUTION. (*a*) Let us count the permutations in which 1 is contained in a cycle of length k.

There are $\binom{n-1}{k-1}$ possible ways of choosing the elements of this cycle.

There are $(k-1)!$ ways of writizng them as a cycle and $(n-k)!$ ways of permuting the rest of the numbers. Thus we get

$$\binom{n-1}{k-1}(k-1)!\ (n-k)! = (n-1)!$$

ways of having 1 in a cycle of length k. So the desired probability is

$$\frac{(n-1)!}{n!} = \frac{1}{n}.$$

Note that **the answer is independent of** k. This is an interesting surprise!

(*b*) Let us count the permutations in which 1 and 2 belong to distinct cycles. If the cycle containing 1 (but not 2) has length k, there are $\binom{n-2}{k-1}$ ways of choosing its elements, $(k-1)!$ ways of writing them as a cycle with 1 and $(n-k)!$ ways of permuting the rest (which includes 2). Summing this product

$$\binom{n-2}{k-1}(k-1)!(n-k)!$$

for values of k from $k = 1$ to $k = n-1$, we get the total number of permutations in which 1 belongs to a cycle distinct from that of 2, as

$$(n-2)! \sum_{k=1}^{n-1} (n-k) = (n-2)! \times \frac{n(n-1)}{2} = \frac{n!}{2}.$$

Note here that the summation we have done uses methods from Chapter 15. Thus the number of permutations in which 1 and 2 belong to the same cycle is $n! - (n!/2) = n!/2$.

The desired probability is then $(n!/2) \div n! = 1/2$.

... of all onto functions from $A = \{a_1, a_2, \ldots, a_n\}$ to