Example 6.21

Let $X \sim Binomial(n, p)$. Using Chebyshev's inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$. Evaluate the bound for $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.

Solution

One way to obtain a bound is to write

$$\begin{aligned} P(X \geq \alpha n) &= P(X - np \geq \alpha n - np) \\ &\leq P\big(|X - np| \geq n\alpha - np\big) \\ &\leq \frac{Var(X)}{(n\alpha - np)^2} \\ &= \frac{p(1-p)}{n(\alpha - p)^2}. \end{aligned}$$

For $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$, we obtain

$$P(X \geq \frac{3n}{4}) \leq \frac{4}{n}.$$