Question 6

$$P(0 \leq T \leq t)$$

$$= \int_0^t P(T_1 - T_2 = x)\, dx$$

$$= \int_0^t \left[\int_0^\infty f_{T_2}(T_2 = y)\, f_{T_1}(T_1 = x+y)\, dy \right] dx$$

$\downarrow$ Since independent

$$= \int_0^t \left[\int_0^\infty \lambda e^{-\lambda y}\, \lambda e^{-(x+y)\lambda}\, dy \right] dx$$

$$= \lambda^2 \left[\int_0^t \left(\int_0^\infty e^{-2\lambda y}\, dy \right) e^{-\lambda x}\, dx \right]$$

$$= \lambda^2 \left[\int_0^t \frac{1}{2\lambda} e^{-\lambda x}\, dx \right]$$

$$= \frac{\lambda}{2} \left[\int_0^t e^{-\lambda x} dx \right] = \frac{\lambda}{2} \left[\frac{e^{-\lambda t} - 1}{-\lambda} \right]$$

$$\boxed{F_T(t) = 1 - \frac{e^{-\lambda t}}{2}}$$

$\left(\because F_T(t) = \boxed{P(-\infty < T \leq 0)} + P(0 \leq T < t) \right)$

this is $1/2$

$$f_T(t) = \text{differentiate CDF}$$

$$\Rightarrow f_T(t) = \frac{\lambda e^{-\lambda t}}{2}$$

(b) Since the integral is symmetric on both sides of y-axis

$$t<0 \Rightarrow P(t \le T \le 0) = \frac{1 - e^{\lambda t}}{2}$$

the same as $P(0 \le T < t)$

$$\Rightarrow P(-\infty < T \le t) = \frac{1}{2} - P(t \le T \le 0)$$

$$\Rightarrow \boxed{F_T(t) = \frac{e^{\lambda t}}{2}} \quad t<0$$

$$\Rightarrow f_T(t) = \frac{\lambda}{2} e^{\lambda t}$$

(c)
$$f_T(t) = \begin{cases} \frac{\lambda}{2} e^{-\lambda t} & t \ge 0 \\ \frac{\lambda}{2} e^{\lambda t} & t<0 \end{cases}$$

$$\boxed{= \frac{\lambda}{2} e^{-\lambda |t|}}$$

(d) It's called the Laplace PDF (with center at $x=0$)