Suppose we choose two numbers at random from the interval [0, ∞) with an *exponential* density with parameter λ. What is the density of their sum? Let X, Y , and Z = X + Y denote the relevant random variables, and f_X, f_Y,and f_Z their densities. Then

$$f_X(x) = f_Y(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{7.2.9}$$

and so, if z > 0,

$$rcl f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy \tag{7.2.10}$$

$$= \int_0^z \lambda e^{-\lambda(z-y)} \lambda e^{-\lambda y} dy \tag{7.2.11}$$

$$= \int_0^z \lambda^2 e^{-\lambda z} dy \tag{7.2.12}$$

$$= \lambda^2 z e^{-\lambda z} \tag{7.2.13}$$

while if z < 0, $f_Z(z) = 0$ (see Figure 7.3). Hence,

$$f_Z(z) = \begin{cases} \lambda^2 z^{-\lambda z}, & \text{if } z \geq 0, \\ 0, & \text{otherwise} \end{cases} \tag{7.2.14}$$

Suppose we choose independently two numbers at random from the interval [0, 1] with uniform probability density. What is the density of their sum? Let X and Y be random variables describing our choices and Z = X + Y their sum. Then we have

$$f_X(x) = f_Y(y) = \begin{matrix} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{matrix} \tag{7.2.3}$$

and the density function for the sum is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy. \tag{7.2.4}$$

Since $f_Y(y) = 1 if 0 \leq y \leq 1$ and 0 otherwise, this becomes

$$f_Z(z) = \int_0^1 f_X(z-y) dy. \tag{7.2.5}$$

Now the integrand is 0 unless 0 ≤ z − y ≤ 1 (i.e., unless z − 1 ≤ y ≤ z) and then it is 1. So if 0 ≤ z ≤ 1, we have

$$f_Z(z) = \int_0^z dy = z, \tag{7.2.6}$$

while if 1 < z ≤ 2, we have

$$f_Z(z) = \int_{z-1}^1 dy = 2 - z, \tag{7.2.7}$$

and if z < 0 or z > 2 we have $_fZ(z) = 0$ (see Figure 7.2). Hence,

$$f_Z(z) = \begin{cases} z & \text{if } 0 \leq z \leq 1 \\ 2-z, & \text{if} 1 < z \leq 2 \\ 0, & \text{otherwise} \end{cases} \tag{7.2.8}$$