2/1/20. **Integral Equation:-**

Initial value problem:

$$\frac{d^2y}{dx^2} = f(x) = y' = \int_0^x f(x) + y'(0)$$

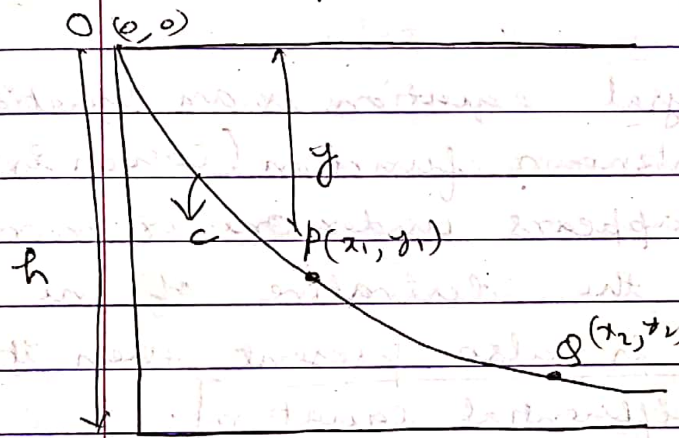
$$y(0) = 1, \quad y'(0) = -1$$

$$\Rightarrow y(x) = y(0) + \int y'(0)\,dx + \int_0^x\int_0^x f(x)\,dx\,dx$$

→ Brachistochrone Problem:

Greek word → Brachisto means shortest
chrone means time

Above problem was initially suggested by Abel.



Datum Level

Let us assume that the particles slides from its pos. of rest at the origin and reaches the $P(x_1, y_1)$ along the curve C. in time 't'. such that $\widehat{OP} = s$. Let the mass of the particle be M.

⇒ Total energy along the curve must be conserved. By conservation of energy, we may equate the sum of KE & PE at O to the sum at 'P' or 'Q' or anywhere else on curve C.

⇒ Taking 'h' as the height of the point P at the datum level. The PE's at O & P are mgh & $mg(h-y)$ respectively. The KE at O & P are 0 & $\frac{1}{2}mv^2$.

Then, $mgh = mg(h-y) + \frac{1}{2}mv^2$

$$\boxed{v = \sqrt{2gy}}$$

$$\frac{ds}{dt} = \sqrt{2gy}$$

$$\int_0^P dt = \int_0^P \frac{ds}{\sqrt{2gy}}$$

$$T = \int_0^y \frac{ds}{\sqrt{2gy}}$$

i.e., $$f(y) = \int_0^y \frac{ds}{\sqrt{2gy}} = \int_0^y \frac{u(\xi)\, d\xi}{\sqrt{2gy}}.$$

$\Rightarrow$ Let say 'C' is a parabola, $y^2 = 4ax$

$ds = v$

we take, $ds = u(\xi)\, d\xi$.

Definition:- An integral equation is an equation in which one unknown function (which is to be determined) appears under one or more integrals, and if the derivative of the unknown function is also present, then it is called $\boxed{\text{integro-differential equation}}$

Ex:- ① $u(x) = f(x) + \lambda \int_a^b \underbrace{K(x,\xi)}_{\text{Kernel}} \, u(\xi)\, d\xi$

↳ Non-homogenous linear eq^n. $\{f(x) \neq 0\}$

② $u(x) = \lambda \int_a^b K(x,\xi) \cdot u(\xi)\, d\xi$.

↳ Homogenous linear eq^n. $\{f(x) = 0\}$.

③ $u(x) = \lambda \int_a^b K(x,\xi) \cdot u^3(\xi)\, d\xi$

→ Non-linear equation.

$\Rightarrow$ Most general form of an I.E (integral eq^n) is:-

$$V(x) \cdot u(x) = f(x) + \lambda \int_a^b K(x,\xi) \cdot u^n(\xi)\, d\xi \quad , \; n \in Z^+.$$

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Some types of I.E's

① **Symmetric Kernel:**

The kernel is symmetric (or hermitian) if

$$K(x, \xi) = \overline{K}(\xi, x).$$

Ex:- $K(x, \xi) = x^2 + \xi^2$, $x + \xi$; $e^{\xi x}$.

② **Separable Kernel:-**

If $K(x, \xi) = \sum_{i=1}^{n} g(x) \cdot h_i(\xi)$.

means K can be expressed as the sum of a finite number of terms, each of which is a product of functions of x & ξ only, then the kernel is called separable or degenerate obviously, $h_i(\xi)$ and $g_i(x)$ are independent.

③ **Difference Kernel:-**

A kernel of the form $K(x - \xi)$ is called a difference kernel.

Ex:- $K(x, \xi) = (x - \xi)^2 = K(x - \xi)$.

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⇒ Classification of integral equations:-

An integral equation of types
$V(x).u(x) = f(x) + \lambda \int_a^b K(x,\xi).u(\xi)\,d\xi$, where — (1)
$u, v\ D \to R$, $K(x,\xi)$ is the kernel of (1), $\lambda \in F$ is called as Fredholm integral equation of third kind

(i) when $v(x) = 0\ \forall\ x \in D$. Then (1) reduces to
$$f(x) + \lambda \int_a^b K(x,\xi)\,u(\xi)\,d\xi = 0$$

(ii) where $V(x) = 1\ \forall\ x \in D$. Then (1) reduces to
$$u(x) = f(x) + \lambda \int_a^b K(x,\xi)\,u(\xi)\,d\xi$$

(iii) where $f(x) = 0\ \forall\ x \in D$. Then (1) reduces to
$$V(x).u(x) = \cancel{f(x)} + \lambda \int_a^b K(x,\xi)\,u(\xi)\,d\xi.$$

Homogenous Fredholm integral equation.

Note:- If we change $b \to x$ in all above limits then it is "Volterra integral equation".

⇒ Singular integral equation.
$$u(x) = \lambda \int_{-\infty}^{\infty} \underbrace{\frac{1}{|x-\xi|}}_{K(x-\xi)} u(\xi)\,d\xi$$

Let us take $K(x,\xi) = \frac{1}{|x-\xi|} = K(x-\xi)$.

convolution integral equation
$$u(x) = f + \lambda \int_a^b K(x-\xi)\,u(\xi)\,d\xi.$$

Ex: 1 Express $\frac{dy}{dx} = y.a_1(x) + b$ in terms of integral eq^n of volterra type and then find the solution.

Sol:- $\frac{dy}{dx} = y.a_1(x) + b$

$$\Rightarrow y(x) = \int_a^x a_1(x).y.dx + \int_a^x b.dx$$

$$= \int_a^x a_1(\xi).y(\xi)\,d\xi + \int_a^x b\,d\xi.$$

variation of parameters.

★ Eigenvalues and eigenfunctions of an integral eq^n:

Let us consider a homogenous F.I.E.

$$u(x) = \lambda \int_a^b K(x,\xi)\,u(\xi)\,d\xi \quad \text{——— (1)}$$

$u(x) \equiv 0$ is the trivial solution.

→ The value of parameter λ for which (1) posses non-trivial solution are called an eigenvalues of (1). The corresponding solutions $u(x)$ are called eigenfunctions or fundamental functions.

Ex: 1 Verify $u(x) = xe^x$ is a solution of the Volterra integral equation:

$$u(x) = \sin x + x\int_0^x \cos(x-\xi)\,u(\xi)\,d\xi$$

$$RHS \rightarrow \sin x + x\int_0^x \cos(x-\xi).\,e^{\xi}\,d\xi$$

corresponding Diff. eq^n s.t $u(x) = xe^x$ is a sol^n [Next page].

$$\rightarrow \sin x + xe^x - \left[\frac{e^x}{2} - \frac{1}{2}(\cos x - \sin x)\right]$$

$$- \left[\frac{e^x}{2}(-1) - \frac{1}{2}(-\sin x - \cos x)\right]$$

$\rightarrow x.e^x$

$\rightarrow$ LHS

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Small results:

1.) Leibneitz rule of differentiation:

$$\frac{d}{dx}\int_{P(x)}^{Q(x)} f(x,y)\,dy = f(x, Q(x))\frac{dQ}{dx} - f(x, P(x))\frac{dP}{dx} + \int_{P}^{Q} \frac{\partial f}{\partial x}\cdot dy$$

2.) Multiple integral of order 'n' into single integral of order one.

$$\underbrace{\int_a^x \int_a^x \cdots \int_a^x}_{n \text{ times}} u(\xi)\, \underbrace{d\xi\, d\xi \ldots, d\xi}_{n \text{ times}} = \int_a^x \frac{(x-\xi)^{n-1}}{(n-1)!}\cdot u(\xi)\, d\xi .$$

Ex:2. Show that $u(x) = \cos 2x$ is a solution of

$$u(x) = \cos x + 3\int_0^x K(x,\xi)\, u(\xi)\, d\xi,$$ where

$$K(x,\xi) = \sin x \cdot \cos \xi, \quad 0 \le x \le \xi$$
$$= \cos x \cdot \sin \xi, \quad \xi \le x \le \pi$$

$\rightarrow$ $$\text{RHS} = \cos x + 3\int_0^{\xi} K(x,\xi)\, u(\xi)\, d\xi + \int_{\xi}^{x} K(x,\xi)\, u(\xi)\, d\xi .$$

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Aim. Derive the integral equation sol. $y(x)$ is a solution.

Proof: Let $\frac{d^n y}{dx^n} = u(x)$, Then

$$\Rightarrow \frac{d^{n-1} y(x)}{dx^{n-1}} - \frac{d^{n-1} y(a)}{dx^{n-1}} = \int_a^x u(x)\,dx .$$

$$\Rightarrow y^{n-1}(x) = a_{n-2} + (x-a)\,a_{n-1} + \int_a^x (x-t).\,u(t)\,dt \quad \text{—— } (4)$$

Integrate (4) w.r.t x from a to x.

$$\rightarrow y^{n-3}(x) = y^{n-3}(a) + (x-a)\,a_{n-2} + \frac{(x-a)^2}{2!}.\,a_{n-1} + \int_a^x \int_a^x (x-t).\,u(t)\,dt .$$

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$$\Rightarrow y^{n-3}(a) + \frac{(x-a)}{3-1} a_{n-2} + \frac{(x-a)^2}{2!} a_{n-1}$$

$$+ \int_a^x \frac{(x-t)}{(3-1)!} \cdot u(t)\, dt. \quad \text{——— (5)}$$

Proceeding in the same way and integrating upto the form $y'(x)$.

$$\rightarrow \frac{dy}{dx} = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \cdot u(t)\, dt + a_{n-1}\frac{(x-a)^{n-2}}{(n-2)!} + a_{n-2}\frac{(x-a)^{n-3}}{(n-3)!}$$

$$+ \cdots + a_2(x-a) + a_1$$

$$\Rightarrow y(x) = \int_a^x \frac{(x-t)^n}{n!} \cdot u(t)\, dt + a_{n-1} \cdot \frac{(x-a)^{n-1}}{(n-1)!} + \cdots + a_1 (x-a) + a_0 \quad \longrightarrow (6)$$

$\rightarrow$ Substitute (3), (4), (5) and (6) in (1), then

$$\rightarrow U(x) + \alpha_1(x) \cdot a_{n-1} + \int_a^x u(x)\, dx + \alpha_2(x) \cdot \Big[a_{n-2} + (x-a)\, a_{n-1}.$$

$$+ \int_a^x (x-t) \cdot u(t)\, dt \Big] + \cdots + \alpha_n \cdot \left[\int_a^x \frac{(x-t)^n}{n!} \cdot u(t)\, dt + a_{n-1} \cdot \frac{(x-a)^{n-1}}{(n-1)!} + \cdots + a_0 \right] = 0$$

$$\rightarrow u(x) = -\alpha_1(x)\, a_{n-1} - \int_a^x u(x)\, dx - \alpha_2(x)\,[\;\_\_\_\;] - \cdots - \alpha_n[\;\_\_\_\;].$$

$$\rightarrow \psi(x) = \alpha_1(x) \cdot a_{n-1} + \{a_{n-2} + (x-a)\, a_{n-2}\}\, \alpha_2(x)$$

$$+ \left\{ a_0 + a_1(x-a) + \cdots + a_{n-1} \cdot \frac{(x-a)^{n-1}}{(n-1)!} \right\} \alpha_n(x) + \phi(x)$$

$$\rightarrow K(x,t) = -\left[\alpha_1(x) + (x-t) \cdot \alpha_2(x) + \cdots + \frac{(x-t)^{n-1}}{(n-1)} \alpha_n(x) \right]$$

$$\rightarrow u(x) = \psi(x) + \int_a^x K(x,t) \cdot u(t)\, dt$$

Ex:- Transform the IVP to Volterra integral equation:

$$\frac{d^2y}{dx^2} + xy = 1, \quad y(0) = 0, \quad y'(0) = 0$$

Sol:- Given, $y''(x) + xy = 1$

Let $y''(x) = u(x)$

$$\rightarrow \frac{dy}{dx} - \frac{d(y(0))}{dx} = \int_0^x u(x)\,dx$$

$$\rightarrow y'(x) = \int_0^x u(x)\,dx.$$

$$\rightarrow y(x) = y(0) + \int_0^x \int_0^x u(x)\,(dx)^2$$

$$= \int_0^x (x-t)\, u(t)\, dt$$

Putting all these in the given equation

$$u(x) + x\int_0^x (x-t).\, u(t)\,dt = 1$$

$\Rightarrow u(x)$

★ Conversion of BVP to Fredholm integral equations

→ IVP is converted to Volterra IE while,

BVP —— " —————— Fredholm IE.

Ex: Reduce the given BVP to FIE.

$$\frac{d^2u}{dx^2} + \lambda u = 0 \quad \text{with} \quad u(0) = 0, \quad u(l) = 0$$

Sol: The given differential function is

$$u''(x) = -\lambda u(x)$$

$$\Rightarrow \int_0^x u''(x)\,dx = -\int_0^x u(x)\,dx$$

$$\Rightarrow u'(x) - u'(0) = -\lambda \int_0^x u \, dx.$$

Let us take $u'(0) = c$, then

$$u'(x) = c - \lambda \int_0^x u(x) \, dx$$

$$\Rightarrow u(x) = u(0) + xc - \lambda \int_0^x \int_0^x u(x) \, dx$$

$$= cx - \lambda \int_0^x \int_0^x u(x) \, dx$$

15/1/20 Relation between eigen values problem and integral equations:

The Fredholm

$$\phi(x) = f(x) + \lambda \int_0^1 (xt^2 + x^2t^4) \, \phi(t) \, dt. \quad \text{——— } (1)$$

Where, ϕ is the unknown

Again, we multiply (II) by t^4 after taking $x=t$, then

$$c_2 = \int_0^1 t^4 \phi(t)\,dt = \underbrace{\int_0^1 t^4 f(t)\,dt}_{F_2} + \frac{\lambda c_1}{6} + \frac{\lambda c_2}{7} \quad \text{--- (4)}$$

We define, $F_1 = \int_0^1 t^2 f(t)\,dt$, $F_2 = \int_0^1 t^4 f(t)\,dt$.

Then, (III) and (IV) reduces to,

$$c_1 = F_1 + \frac{\lambda c_1}{4} + \frac{\lambda c_2}{5}$$

$$c_2 = F_2 + \frac{\lambda c_1}{6} + \frac{\lambda c_2}{7}$$

$$\Rightarrow \left(1 - \frac{\lambda}{4}\right) c_1 - \frac{\lambda c_2}{5} = F_1$$

$$-\frac{\lambda}{6} c_1 + \left(1 - \frac{\lambda}{7}\right) c_2 = F_2$$

$$\Rightarrow \begin{pmatrix} 1-\lambda/4 & -\lambda/5 \\ -\lambda/6 & 1-\lambda/7 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

$$\Rightarrow \hat{A} X = B.$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1/4 & 1/5 \\ 1/6 & 1/7 \end{pmatrix}$$

$$(I - \lambda . A).$$

$$\det(A_1 - \mu I) = 0$$

Define $\lambda = 1/\mu$. λ will be an eigenvalue.

Suppose A_1 is a matrix and μ is its eigenvalues.

The values of λ is called the eigenvalues of the kernel.

Ex. 1 Transform the BVP.

$$\frac{d^2y}{dx^2} + y = x \quad , \quad y(0) = 1, \quad y'(1) = 0$$

into Fredholm integral equations.

Sol:- The given equation is defined between $[0, 1]$.

Now, $\frac{d^2y}{dx^2} + y = x$.

$$\Rightarrow \int_0^x d\left(\frac{dy}{dx}\right) + \int_0^x y\,dx = \int_0^x x \cdot dx$$

$$= \frac{dy}{dx} - y'(0) = \frac{x^2}{2} - \int_0^x y \cdot dx$$

$$\Rightarrow \frac{dy}{dx} = \underbrace{y'(0)}_{c} + \frac{x^2}{2} - \int_0^x y \cdot dx.$$

$$\Rightarrow y' = c + \frac{x^2}{2} - \int_0^x y \cdot dx$$

integrate w.r.t x from 0 to x.

$$y(x) = \underbrace{y(0)}_{=1} + cx + \frac{x^3}{6} - \int_0^x (x - \xi) \cdot y(\xi)\, d\xi. \quad \text{—— (ii)}$$

$$\rightarrow \quad y(x) = 1 + cx + \frac{x^3}{6} - \int_0^x (x - \xi) \cdot y(\xi)\, d\xi.$$

Putting, $y'(1) = 0$ then

$$0 = c + \frac{1}{2} - \int_0^1 y\,dx$$

$$\Rightarrow \quad c = \int_0^1 y(t) \cdot dt - \frac{1}{2} \quad \text{—— (iii)}$$

combining (ii) and (iii):

$$y(x) = 1 + x\int_0^1 y(t)\,dt - \frac{x}{2} + \frac{x^3}{6} - \int_0^x (x-\xi)\cdot y(\xi)\,d\xi.$$

$$= 1 - \frac{x}{2} + \frac{x^3}{6} + x\int_0^1 y(t)\,dt - \int_0^x (x-\xi)\cdot y(\xi)\,d\xi.$$

$$= 1 - \frac{x}{2} + \frac{x^3}{6} + \int_0^1 x\cdot y(\xi)\,d\xi - \int_0^x (x-\xi)\cdot y(\xi)\,d\xi.$$

$$= 1 - \frac{x}{2} + \frac{x^3}{6} + \int_x^1 x\,y(\xi)\,d\xi + \int_0^x \xi\cdot y(\xi)\,d\xi.$$

$$= 1 - \frac{x}{2} + \frac{x^3}{6} + \int_0^1 K(x,\xi)\cdot y(\xi)\,d\xi.$$

where, $K(x,\xi) = \begin{cases} x, & x<\xi \\ \xi, & x>\xi. \end{cases}$

ch:3. of 2^{nd} kind

16/1/20 Solution of Fredholm integral equation:

characteristics values and characteristic function (eigen values and eigen functions).

Let us consider, $u(x) = \lambda \int_a^b K(x,t).u(t)\,dt.$ —— ①

obviously $u(x)=0$ will be a solution, trivial sol^n. To look for the solution of (1) s.t $u(x) \neq 0$.

For all such values of $\lambda \in R$ s.t $u(x) \neq 0$ is a sol^n is called eigenvalues/charac. of the kernel $K(x,t)$.

For the eigenvalues $\lambda \in R$ $\exists$ eigenfunctions ϕ s.t.

$$\phi_0(x) = \lambda_0 \int_a^b K(x,t).\ \phi_0(t)\,dt,$$

ϕ_0 would also be a solution of ①.

Remark:- (i) $\lambda \neq 0$, if $\lambda = 0 \Rightarrow u(x) = 0$. Not needed. {Always consider non zero eigen values.

(ii) if ϕ_0 is the eigenfunction corresponding to λ_0, $c\phi_0$ would also be an eigenfunction for ①, but

$$A\phi = \lambda\phi \ ; \ Ac\phi = c\lambda\phi.$$

$$\phi_1 = c\phi \rightarrow \phi_1 - c\phi = 0$$

linearly dependent.

(iii) A homogenous fredholm integral equation of type ① may not have any eigenvalues if the kernel is not symmetric.

Ex1 Calculate the eigenvalues and eigenfunctions of

$$g(x) = \lambda \int_0^1 e^{x+t} g(t)\,dt. \quad \text{—— ①}$$

Sol:- From ①, $g(x) = \lambda e^x \underbrace{\int_0^1 e^t g(t)\,dt}_{c}.$

Then, $g(x) = \lambda c e^x$, or, $g(t) = \lambda c e^t$.

Now, $c = \int_0^1 e^t . g(t)\,dt.$

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$$c = \int_0^1 \lambda c \cdot e^{2t}\, dt.$$

$$\Rightarrow 1 = \lambda \int_0^1 e^{2t}\, dt \quad , \quad c \neq 0 \quad \{ \text{otherwise it'll give trivial sol}^n \}.$$

$$\Rightarrow 1 = \lambda \cdot \frac{e^2 - 1}{2} \quad \Rightarrow \quad \boxed{\lambda = \frac{2}{e^2 - 1}}$$

Now, for $\lambda = \frac{2}{e^2-1}$, the corresponding eigenfunction

is $g(x) = \lambda c \cdot e^x = \frac{2c}{e^2-1} \cdot e^x = \hat{c} \cdot e^x$.

Ex:2 Find the eigenvalues of $\phi(x) = \lambda \int_0^1 (3x-2) \cdot t \cdot \phi(t)\, dt$

Sol^n: The given equation is $\phi(x) = \lambda (3x-2) \underbrace{\int_0^1 t \cdot \phi(t)\, dt}_{c}$

This given, $\phi(x) = \lambda c (3x-2)$

$\Rightarrow \phi(t) = \lambda c (3t-2)$

Now,

$$c = \int_0^1 t\,(\lambda c (3t-2))\, dt = \lambda c \int_0^1 (3t^2 - 2t)\, dt$$

$$c = \lambda c \left[t^3 - t^2 \right]_0^1$$

$$c = 0$$

$$\underline{c = 0}, \quad \Rightarrow \quad \phi(t) = 0$$

Hence, eigenvalues does not exist

trivial sol^n exists.

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★ Solution of a Fredholm integral equation with separable kernel.

1.) Orthogonality of two functions:
Let f and $g : [a,b] \to R$, then
$$\langle f, g \rangle = \int_a^b f(x) \cdot g(x)\, dx = 0,$$

implies f and g are orthogonal.
Lebesgue Integration, $f, g \in L^2(a,b)$, then $\int_a^b f(x) \cdot g(x) dx = 0$.

22/1/20 ★ Orthogonality of eigenfunctions:
f, g are said to be orthogonal on $[a,b]$ if
$$\int_a^b f \cdot g\, dx = 0$$
$\because$ L^2 is the Lebesgue space for above
$$\rightarrow \int_a^b f \cdot g\, dx \leq \left(\int_a^b f^2 dx\right)^{1/2} \cdot \left(\int_a^b g^2 \cdot dx\right)^{1/2}$$
$$= \|f\|_{L^2} \cdot \|g\|_{L^2}.$$

Prop. 1 The eigenfunctions of a symmetric kernel corresponding to two different eigenvalues are orthogonal.

proof: Let us consider an integral equation of type
$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt, \quad \text{where } K(x,t) = K(t,x). \longrightarrow (1)$$

Let λ_0 and λ_1 be the eigenvalues and ϕ_0 and ϕ_1 be the corresponding eigenfunctions respectively, where $\lambda_0 \neq \lambda_1$. Then we show that ϕ_0 and ϕ_1 are orthogonal, i.e., $\int_a^b \phi_0 \cdot \phi_1\, dx = 0$

By def^n: $$\phi_0(x) = \lambda_0 \int_a^b K(x,t)\, \phi_0(t)\, dt \quad \text{—— (II)}$$
$$\phi_1(x) = \lambda_1 \int_a^b K(x,t) \cdot \phi_1(t)\, dt \quad \text{—— (III)}$$

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$$\Rightarrow \int_a^b \phi_0(x)\cdot\phi_1(x)\,dx = \lambda_0\int_a^b \phi_1(x)\left\{\int_a^b K(x,t)\,\phi_0(t)\,dt\right\}dx.$$

$$= \lambda_0\int_a^b \phi_0(t)\left\{\int_a^b K(x,t)\cdot\phi_1(x)\,dx\right\}dt.$$

$$= \lambda_0\int_a^b \phi_0(t)\left\{\underbrace{\int_a^b K(t,x)\cdot\phi_1(x)\,dx}_{\text{put } x\to t,\ t\to x}\right\}dt.$$

$$= \lambda_0\int_a^b \phi_0(t)\left\{\int_a^b K(x,t),\ \phi_1(t)\,dt\right\}dt.$$

$$= \lambda_0\int_a^b \phi_0(t)\cdot\frac{\phi_1(x)}{\lambda_1}\cdot dt.$$

$$\int_a^b \phi_0(x)\cdot\phi_1(x)\,dx = \frac{\lambda_0}{\lambda_1}\int_a^b \underbrace{\phi_0(t)\cdot\phi_1(x)}_{t\to x}\,dt$$

$$\lambda_1\int_a^b \phi_0\,\phi_1\,dx = \lambda_0\int_a^b \phi_0\cdot\phi_1\,dx$$

$$(\lambda_1-\lambda_0)\cdot\int_a^b \phi_0\,\phi_1\,dx = 0$$

$\therefore$ Here, $\lambda_1-\lambda_0 \neq 0$ so $\int_a^b \phi_0\,\phi_1\,dx = 0$ Hence, ϕ_0, ϕ_1 are orthogonal.

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Ex: Show that the homogenous integral eqn.,

$$\phi(x) = \lambda \int_0^1 (t\cdot\sqrt{x} - x\sqrt{t})\,\phi(t)\,.dt \quad \text{——— (1)}$$

has no real eigenvalue and no eigenfunction.

Sol: The given equation (1) can be written as,

$$\phi(x) = \lambda \underbrace{\int_0^1 t\sqrt{x}\,\phi(t)\,dt}_{c_1} - \lambda \underbrace{\int_0^1 x\sqrt{t}\,.\,\phi(t)\,dt}_{c_2}$$

$$= \lambda c_1\sqrt{x} - \lambda c_2 . x \quad \text{——— (ii)}$$

where,

$$c_1 = \int_0^1 t\cdot\phi(t)\,dt \;;\quad c_2 = \int_0^1 \sqrt{t}\cdot\phi(t)\,dt \quad \text{——— (iii)}$$

Now from (iii a) and (ii),

$$c_1 = \int_0^1 t\,[\lambda c_1\sqrt{t} - \lambda c_2 t]\,dt = c_1\lambda.\left[\frac{t^{5/2}}{5/2}\right]$$

$$\Rightarrow c_1\left(1 - \frac{2\lambda}{5}\right) + \frac{\lambda}{3}.c_2 = 0 \quad \text{——— (iv)}$$

Similarly by (ii b) and (ii),

$$-\frac{\lambda}{2}.c_1 + \left(1 + \frac{2\lambda}{5}\right).c_2 = 0 \quad \text{——— (v)}$$

For non-zero c_1 and c_2,

$$\begin{vmatrix} 1 - \frac{2\lambda}{5} & \lambda/3 \\ -\lambda/2 & 1 + \frac{2\lambda}{5} \end{vmatrix} = 0$$

$$\Rightarrow 1 + \frac{\lambda^2}{150} = 0 \Rightarrow \lambda = \pm\sqrt{150}\; i$$

Ex:2 Find the eigenvalue of,

$$\phi(x) = \lambda \int_0^{\pi} (\cos^2 x \cdot \cos 2t + \cos 3x \cdot \cos^3 t) \cdot \phi(t) \cdot dt. \quad \text{--- (1)}$$

Sol: Here, $K(x,t) = \cos^2 x \cdot \cos 2t + \cos 3x \cdot \cos^3 t$.

$K(t,x) = \cos^2 t \cdot \cos 2x + \cos 3t \cdot \cos^3 x$

$\Rightarrow K(x,t) \neq K(t,x)$.

Eqⁿ (1) can be written as:

$$\phi(x) = \lambda \cos^2 x \cdot \underbrace{\int_0^{\pi} \cos 2t \cdot \phi(t)\, dt}_{c_1} + \lambda \cos 3x \cdot \underbrace{\int_0^{\pi} \cos^3 t \cdot \phi(t)\, dt}_{c_2}$$

i.e., $\phi(x) = \lambda c_1 \cos^2 x + \lambda c_2 \cos 3x$

i.e., $\phi(t) = \lambda c_1 \cos^2 t + \lambda c_2 \cos 3t$ --- (II)

Proceeding in the same way as in previous que:

$c_1 = \int_0^{\pi} \cos 2t \cdot \phi(t) dt \Rightarrow c_1 \left[1 - \frac{\lambda\pi}{4}\right] - 0 \cdot c_2 = 0$

$\Rightarrow c_1 \cdot \left[1 - \frac{\lambda\pi}{4}\right] = 0$ --- (III)

$c_2 = \int_0^{\pi} \cos^3 t \cdot \phi(t)\, dt \Rightarrow 0 \cdot c_1 + c_2 \cdot \left(1 - \frac{\lambda\pi}{8}\right) = 0$

$c_2 \cdot \left(1 - \frac{\lambda\pi}{8}\right) = 0$ --- (IV)

from (III) and (IV):

$\lambda = \frac{4}{\pi}, \quad \lambda = \frac{8}{\pi}$.

**. Symmetric implies real eigen value but vice-versa may not be true.

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Solution of FIE of 2^{nd} kind :

Let us consider a FIE of 2^{nd} kind. (1)

$u(x) = f(x) + \lambda \int_a^b K(x,t) \cdot u(t)\, dt$, where the kernel $(K(x,t))$ is separable i.e.,

$$K(x,t) = \sum_{i=1}^{N} f_i(x)\, g_i(t).$$

then, $u(x) = f(x) + \lambda \sum_{i=1}^{N} f_i(x) \int_a^b u(t) \cdot g_i(t)\, dt$ ——— (2)

Let us assume, $c_i = \int_a^b u(t) \cdot g_i(t)\, dt$, $\forall\ i = 1, 2, \ldots, n$ ——— (3)

By (2) & (3) :

$u(x) = f(x) + \lambda \sum_{i=1}^{N} f_i(x)\, c_i$ ——— (4)

To determine c_i's we multiply (4) by $g_1(x), g_2(x), \ldots, g_n(x)$. one by one respectively, and integrate.

(5)
$$\left\{\begin{array}{l} \int_a^b u(x) \cdot g_1(x)\, dx = \int_a^b f(x) \cdot g_1(x)\, dx + \sum_{i=1}^{N} c_i \int_a^b f_i(x)\, g_1(x)\, dx \\ \vdots \\ \int_a^b u(x) \cdot g_n(x)\, dx = \int_a^b f(x) \cdot g_n(x)\, dx + \lambda \sum_{i=1}^{N} c_i \int_a^b f_i(x)\, g_n(x)\, dx \end{array}\right.$$

Let us define, $\alpha_{ij} = \int_a^b g_j(x) \cdot f_i(x)\, dx$, $\forall\ i, j = 1, 2, \ldots, n$

$$\beta_j = \int_a^b f(x) \cdot g_j(x)\, dx \qquad \forall\ j = 1, 2, \ldots, n.$$

By equations (5),

$$c_1 = \beta_1 + \lambda \sum_{i=1}^{N} c_i \cdot \alpha_{i1}.$$

$$c_2 = \beta_2 + \lambda \sum_{i=1}^{N} c_i\, \alpha_{i2}$$

$$\vdots$$

$$c_n = \beta_n + \lambda \sum_{i=1}^{N} c_i\, \alpha_{in}.$$

$$(6) \; \begin{cases} (1-\lambda\alpha_{11})c_1 - \lambda c_2\alpha_{21} - \cdots - \lambda c_n\alpha_{n1} = \beta_1 \\ -\lambda c_1\alpha_{12} + (1-\lambda\alpha_{22})\cdot c_2 - \cdots - \lambda c_n\cdot\alpha_{n2} = \beta_2 \\ \vdots \\ -\lambda c_1\alpha_{1n} - \lambda c_2\alpha_{2n} \cdots\cdots + (1-\lambda\alpha_{nn})c_n = \beta_n \end{cases}$$

For the existence of solution (unique/infinitely many/no) we must calculate determinant of (6).

$$D(\lambda) = \begin{vmatrix} (1-\lambda c_{11}) & -\lambda\alpha_{21} & \cdots & -\lambda\alpha_{n1} \\ -\lambda\alpha_{12} & (1-\lambda c_{22}) & \cdots & -\lambda\alpha_{n2} \\ \vdots & \vdots & & \vdots \\ -\lambda\alpha_{1n} & -\lambda\alpha_{2n} & \cdots & (1-\lambda c_{nn}) \end{vmatrix} = 0$$

$\Rightarrow D(0) = 1$, i.e., $D(0) \neq 0 \Rightarrow \lambda = 0$ is not an eigenvalue.

Several cases may arise:

Case 1: when $f(x) = 0$, then $\beta_j = 0 \;\forall\; j = 1, 2, \ldots, n$.
Therefore, if (a) $D(\lambda) \neq 0$. Then will imply all $c_i = 0$, $i = 1, 2, \ldots, n$ which gives $u(x) = 0$.

(b). $D(\lambda) = 0$. Then any non-zero solution of (5) will give a solution of the IE (1).

Case 2: when $f(x) \neq 0$

Subcase (a): $f(x)$ and $g_j(x)$ are orthogonal i.e.,
$$\int_a^b f(x)\cdot g_j(x)\,dx = 0 \quad \forall\; j = 1, 2, \ldots n.$$

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If $D(\lambda) \neq 0$, then the unique sol^n of (5) is $c_i = 0$, $\forall i = 1, 2, \dots, n$. This gives $u(x) = F(x)$ as the solution.

If $D(\lambda) = 0$, then the system (5) provides infinite non-zero sol^n. Therefore the equation (1) has infinite non-zero solution together with $F(x)$.

(b)

~~Case 2~~: Atleast one of the $\beta_j \neq 0$ $\forall j = 1, 2, \dots, n$.

Then if $D(\lambda) \neq 0$, we get a non-zero unique solution of the system (6). This gives a non-zero solution of the IE (1).

If $D(\lambda) = 0$. No solution at all. There does not exist any solution to (1).

Ex 1: Solve $g(s) = f(s) + \lambda \int_0^1 st\, g(t)\, dt$. ——— (1)

Sol: Here $K(s,t) = s \cdot t = \phi(s) \cdot \psi(t)$.

$\phi(s) = s$; $\psi(t) = t$.

Let us take, $c = \int_0^1 t \cdot g(t)\, dt$, then the sol^n,

$$g(s) = f(s) + \lambda s c \quad \text{——— (2)}$$

$$\Rightarrow g(t) = f(t) + \lambda t c \quad \text{——— (3)}$$

Multiply (3) by (t) and integrate it:

$$\int_0^1 t \cdot g(t)\, dt = \int_0^1 t \cdot f(t)\, dt + \lambda c \cdot \int_0^1 t^2\, dt$$

$$\Rightarrow c = \int_0^1 t \cdot f(t)\, dt + \frac{\lambda c}{3}.$$

$$\Rightarrow c - \frac{\lambda c}{3} = \int_0^1 t f(t)\, dt$$

$$\Rightarrow c = \frac{3}{3-\lambda} \cdot \int_0^1 t \cdot f(t)\, dt, \quad \text{for } \lambda \neq 3$$

Therefore, the eigen value of eq^n (1) are $\lambda \in R$ s.t. $\lambda \neq 3$ and the sol^n is given by;

$$g(s) = f(s) + \frac{3\lambda s}{\lambda - 3} \int_0^1 t \cdot f(t)\, dt.$$

Ex:2 Solve $\phi(x) = \cos x + \lambda \int_0^{\pi} \sin x \cdot \phi(t)\, dt$

Solⁿ:

$K(x,t) = \sin x \cdot 1 \Rightarrow K(t,x) = \sin t \cdot 1.$

it is separable but not symmetric.

$\lambda = 1/2$, $c = \int_0^{\pi} \phi(t)\, dt$

$\Rightarrow c(1-2\lambda) = 0$

$\lambda = 1/2 \Rightarrow c \neq 0 \Rightarrow$ a unique solⁿ.

$\lambda \neq 1/2 \Rightarrow c = 0 \Rightarrow \phi(x) = \cos x.$