Charne's Penalty Method:

(This method is also known as Big M-method)

If in a starting simplex tableau, we don’t have an identity sub-matrix (i.e. an obvious starting BFS), then we introduce artificial variables to have a starting BFS. This is known as artificial variable technique. There are two methods to find the starting BFS and solve the problem – the Big-M method and two-phase method. Today we will discuss about the Big- M method.

Suppose a constraint equation i does not have a slack variable. i.e. there is no ith unit vector column in the LHS of the constraint equations. (This happens for example when the ith constraint in the original LPP is either $\geq$ or $=$.) Then we augment the equation with an artificial variable R_i to form the ith unit vector column. However as the artificial variable is an extra variable to the given LPP, we use a mechanism in which the optimization process automatically attempts to drive out these variables to zero level. This is achieved by giving a large penalty to the coefficient of the artificial variable in the objective function as follows:

Artificial variable objective coefficient

= - M in a maximization problem,
= M in a *minimization* problem

where M is a very large positive number.

Consider the LPP:

Minimize $z = 2x_1 + x_2$

Subject to the constraints

$$3x_1 + x_2 \geq 9$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

Putting this in the standard form, the LPP is:

Minimize $z = 2x_1 + x_2$

Subject to the constraints

$$
\begin{aligned}
3x_1 + x_2 - s_1 \qquad &= 9 \\
x_1 + x_2 \qquad - s_2 &= 6 \\
x_1, x_2, s_1, s_2 &\geq 0
\end{aligned}
$$

Here s_1, s_2 are surplus variables.

Note that we do not have a 2 by 2 identity submatrix in the LHS.

Introducing the artificial variables, R_1, R_2, the LPP is modified as follows:

Minimize $z = 2x_1 + x_2 + MR_1 + MR_2$

Subject to the constraints

$$\begin{aligned} 3x_1 + x_2 - s_1 \quad + R_1 \quad &= 9 \\ x_1 + x_2 \quad - s_2 \quad + R_2 &= 6 \\ x_1, x_2, s_1, s_2, R_1, R_2 &\geq 0 \end{aligned}$$

Note that we now have a 2 by 2 identity submatrix in the coefficient matrix of the constraint equations.

Now we solve the above LPP by the Simplex method. Express the objective function z in terms of the artificial variables and the non-basic variables. In Max. type problems we select the most negative element. In Min. type problem we select the most positive element. Other steps remain same.

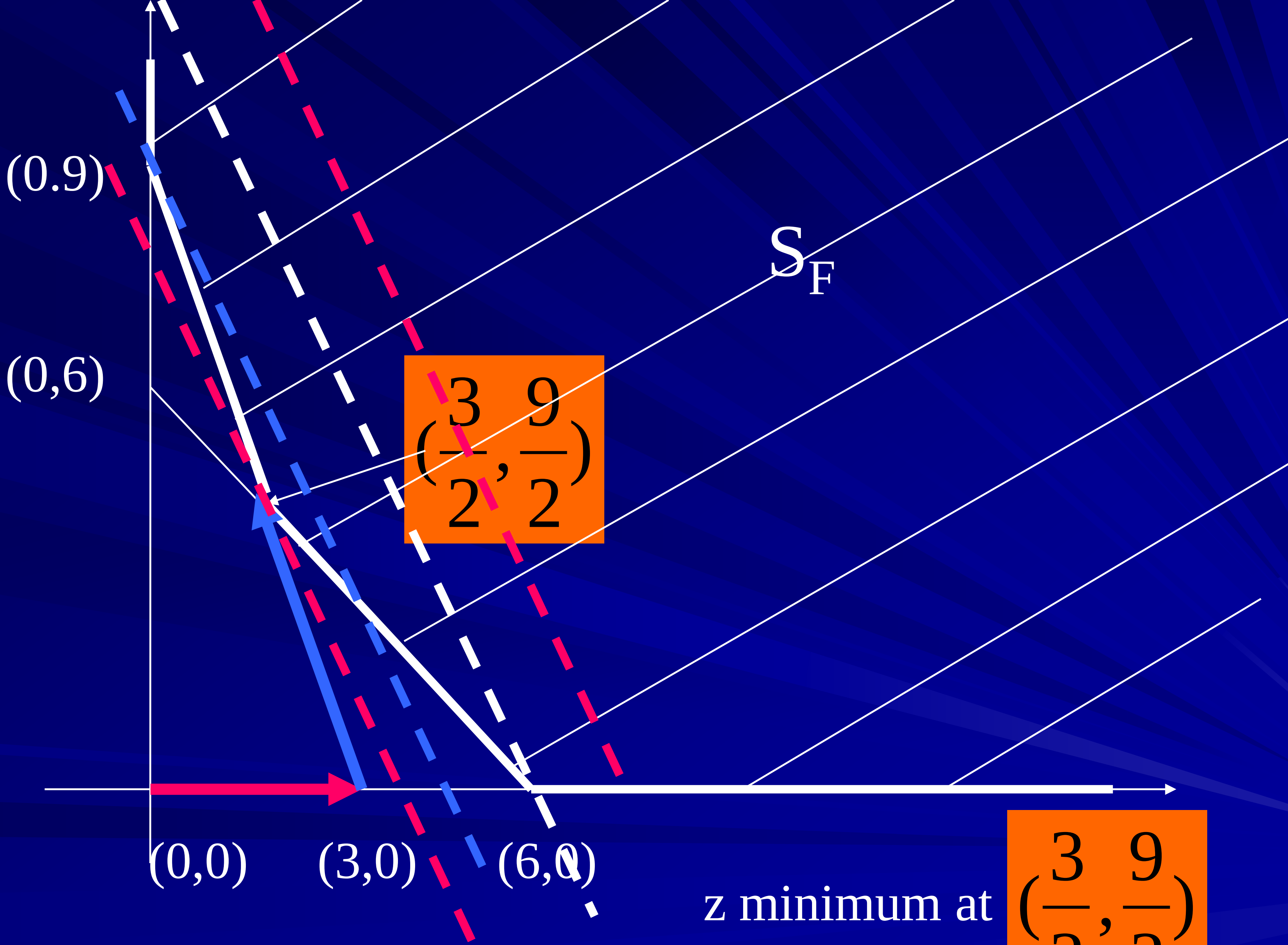
Basic	z	x1	x2	s1	s2	R1	R2	Sol.
z	1	-2+4M ↓ ~~-2~~	-1+2M ~~-1~~	-M ~~0~~	-M ~~0~~	0 ~~-M~~	0 ~~-M~~	15M ~~0~~
← R1	0	$\boxed{3}$	1	-1	0	1	0	9
R2	0	1	1	0	-1	0	1	6
z	1	0	-1/3+2M/3 ↓	-2/3+M/3	-M	2/3-4M/3	0	6+3M
x1	0	1	1/3	-1/3	0	1/3	0	3
← R2	0	0	$\boxed{2/3}$	1/3	-1	-1/3	1	3
z	1	0	0	-1/2	-1/2	1/2–M	1/2-M	15/2
x1	0	1	0	-1/2	1/2	1/2	-1/2	3/2
x2	0	0	1	1/2	-3/2	-1/2	3/2	9/2

Note that we have obtained the optimal solution to the given problem as :

$$x_1 = \frac{3}{2}, \; x_2 = \frac{9}{2}$$

Minimum value is $$z = \frac{15}{2}$$

We also find the optimal solution by Graphical Method.



z minimum at $(\frac{3}{2},\frac{9}{2})$

Maximize $z = 2x_1 + 3x_2 - 5x_3$

Subject to the constraints

$$x_1 + x_2 + x_3 = 7$$

$$2x_1 - 5x_2 + x_3 \geq 10$$

$$x_1, x_2, x_3 \geq 0$$

Introducing surplus and artificial variables, s_2, R_1 and R_2, the LPP is modified as follows:

Maximize

$$z = 2x_1 + 3x_2 - 5x_3 - MR_1 - MR_2$$

Subject to the constraints

$$\begin{aligned} x_1 + x_2 + x_3 \qquad\qquad + R_1 \qquad &= 7 \\ 2x_1 - 5x_2 + x_3 - s_2 \qquad + R_2 &= 10 \\ x_1, x_2, x_3, s_2, R_1, R_2 \geq 0 \end{aligned}$$

Now we solve the above LPP by the Simplex method.

Basic	z	x1	x2	x3	s2	R1	R2	Sol.
z	1	-2-3M ~~-2~~	-3+4M ~~-3~~	5-2M ~~5~~	M ~~0~~	0 ~~M~~	0 ~~M~~	-17M ~~0~~
R1	0	1	1	1	0	1	0	7
R2	0	$\boxed{2}$	-5	1	-1	0	1	10
z	1	0	-8 - 7M/2	6 - M/2	-1 - M/2	0	1 + 3M/2	10 - 2M
R1	0	0	$\boxed{7/2}$	1/2	1/2	1	-1/2	2
x1	0	1	-5/2	1/2	-1/2	0	1/2	5
z	1	0	0	50/7	1/7	16/7 + M	-1/7 +M	102/7
x2	0	0	1	1/7	1/7	2/7	-1/7	4/7
x1	0	1	0	6/7	-1/7	5/7	1/7	45/7

The Maximum value of

$$z = 102/7$$

where $x_1 = 45/7,\ x_2 = 4/7,\ x_3 = 0.$

Remarks

- If in any iteration, there is a tie for entering variable between an artificial variable and other variable (decision, surplus or slack), we must prefer the non-artificial variable to enter the basis.
- If in any iteration, there is a tie for leaving variable between an artificial variable and other variable (decision, surplus or slack), we must prefer the *artificial* variable to leave the basis.

- If in the final optimal tableau, an artificial variable is present in the basis at a non-zero level, this means our original problem has *no feasible solution*.

Maximize: $z = 5x_1 + 6x_2$

Subject to the constraints

$$
\begin{aligned}
-2x_1 + 3x_2 &= 3 \\
x_1 + 2x_2 &\le 5 \\
6x_1 + 7x_2 &\le 3 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

Introducing slack and artificial variables, s_2, s_3, and R_1, the LPP is modified as follows:

Maximize $z = 5x_1 + 6x_2 - MR_1$

Subject to the constraints

$$\begin{aligned}
-2x_1 + 3x_2 + R_1 &= 3 \\
x_1 + 2x_2 + s_2 &= 5 \\
6x_1 + 7x_2 + s_3 &= 3 \\
x_1, x_2, R_1, s_2, s_3 &\geq 0
\end{aligned}$$

Basic	z	x1	x2	R1	s2	s3	Sol
z	1	-5+2M ~~-5~~	-6-3M ~~-6~~	0 ~~M~~	0 ~~0~~	0 ~~0~~	-3M ~~0~~
R1	0	-2	3	1	0	0	3
s2	0	1	2	0	1	0	5
s3	0	6	$\boxed{7}$	0	0	1	3
z	1	1/7+ 32M/7	0	0	0	6/7+ 3M/7	18/7- 12M/7
R1	0	-32/7	0	1	0	-3/7	12/7
s2	0	-12/7	0	0	1	-2/7	29/7
x2	0	6/7	1	0	0	1/7	3/7

This is the optimal tableau. As R_1 is not zero, there is NO feasible solution

Minimize $z = 4x_1 + 6x_2$

Subject to the constraints

$$
\begin{aligned}
-2x_1 + 3x_2 &= 3 \\
4x_1 + 5x_2 &\geq 10 \\
4x_1 + 8x_2 &\geq 5 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Introducing the surplus and artificial variables, R_1, R_2, the LPP is modified as follows:

Minimize $z = 4x_1 + 6x_2 + M R_1 + M R_2 + M R_3$

Subject to the constraints

$$
\begin{aligned}
-2x_1 + 3x_2 \qquad\qquad + R_1 \qquad\qquad &= 3 \\
4x_1 + 5x_2 \quad - s_2 \quad + R_2 \qquad &= 10 \\
4x_1 + 8x_2 \qquad\quad - s_3 \quad + R_3 &= 5 \\
x_1, x_2, s_2, s_3, R_1, R_2, R_3 &\geq 0
\end{aligned}
$$

Basic	z	x1	x2	s2	s3	R1	R2	R3	Sol.
z	1	-4+6M ~~-4~~	-6+16M ~~-6~~	-M ~~0~~	-M ~~0~~	0 ~~-M~~	0 ~~-M~~	0 ~~-M~~	18M ~~0~~
R1	0	-2	3	0	0	1	0	0	3
R2	0	4	5	-1	0	0	1	0	10
R3	0	4	$\boxed{8}$	0	-1	0	0	1	5
z	0	-1-2M	0	-M	-3/4 +M	0	0	3/4 -2M	15/4 +8M
R1	0	-7/2	0	0	3/8	1	0	-3/8	9/8
R2	0	3/2	0	-1	5/8	0	1	-5/8	55/8
x2	0	1/2	1	0	-1/8	0	0	1/8	5/8

Basic	z	x1	x2	s2	s3	R1	R2	R3	Sol.
z	1	-1-2M	0	-M	-3/4 +M	0	0	3/4 -2M	15/4 +8M
R1	0	-7/2	0	0	$\boxed{3/8}$	1	0	-3/8	9/8
R2	0	3/2	0	-1	5/8	0	1	-5/8	55/8
x2	0	1/2	1	0	-1/8	0	0	1/8	5/8
z	1	-8 + 22M/3	0	-M	0	2 -8M/3	0	-M	6 +5M
s3	0	-28/3	0	0	1	8/3	0	-1	3
R2	0	22/3	0	-1	0	-5/3	1	0	5
x2	0	-2/3	1	0	0	1/3	0	0	1

Basic	z	x1	x2	s2	s3	R1	R2	R3	Sol.
z	1	-8 + 22M/3	0	-M	0	2 -8M/3	0	-M	6 +5M
s3	0	-28/3	0	0	1	8/3	0	-1	3
R2	0	$\boxed{22/3}$	0	-1	0	-5/3	1	0	5
x2	0	-2/3	1	0	0	1/3	0	0	1
z	1	0	0	-6/11	0	2/11	12/11 - M	-M -M	$\frac{126}{11}$
s3	0	0	0	-14/11	1	6/11	14/11	-1	$\frac{103}{11}$
x1	0	1	0	-3/22	0	-5/22	3/22	0	$\frac{15}{22}$
x2	0	0	1	-1/11	0	2/11	1/11	0	$\frac{16}{11}$

This is the optimal Tableau.

The Optimal solution is:

$$\boxed{x_1 = \frac{15}{22}, \;\; x_2 = \frac{16}{11}}$$

Min z = $\boxed{\dfrac{126}{11}}$