Problems 4

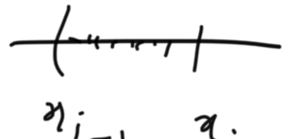
① Let $f:[0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

Then show that f is not Riemann integrable but f is Lebesgue integrable

Sol:-

Let P be any partition of $[a,1]$

Let $x_0 = a < x_1 < \dots < x_n = b$.



$$U(P,f) = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]}(f)\,(x_i - x_{i-1}).$$

$$= \sum_{i=1}^{n} 1\,(x_i - x_{i-1})$$

$$= 1 \quad \forall P$$

$$L(P,f) = \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]}(f)\,(x_i - x_{i-1}).$$

$$= \sum_{i=1}^{n} 0\,(x_i - x_{i-1})$$

$$= 0.$$

$$\therefore \quad \underline{\int_0^1} f(x)\,dx = \sup_P \Big(L(P,f) \Big) = 0$$

$$\neq$$

$$\overline{\int^1} f(x)\,dx = \inf_P \Big(U(P,f) \Big) = 1$$

0 1

$\therefore$ f is not Riemann integrable.

Note that $f = \chi_{(\mathbb{R}\setminus\mathbb{Q})\cap[0,1]}$

$(\mathbb{R}\setminus\mathbb{Q})\cap[0,1]$ is measurable. f is a simple function,

& $$\int_{[0,1]} f = \int_{[0,1]} \chi_{(\mathbb{R}\setminus\mathbb{Q})\cap[0,1]}$$

$$= m\left((\mathbb{R}\setminus\mathbb{Q})\cap[0,1]\right)$$

$$= m([0,1]) - m(\mathbb{Q}\cap[0,1])$$

$$= 1 - 0$$

$$= 1$$

$\therefore$ f is Lebesgue integrable.

---

(2) Let $f: E \to \mathbb{R}$ be a measurable function, where E is a measurable set. Let $M_1, M_2 \in \mathbb{R}$, $M_1 < M_2$, let the truncation of f at M_1 & M_2 be defined by

$$g(x) = \begin{cases} M_1, & \text{if } f(x) < M_1, \\ f(x), & \text{if } M_1 \le f(x) \le M_2 \\ M_2, & \text{if } f(x) > M_2. \end{cases}$$

Show that g is measurable on E.

Sol:- To Show: for any $\alpha \in \mathbb{R}$,

$I = \{x \in E \mid g(x) > \alpha\}$ is measurable.

Case 1: Suppose $\alpha \geq M_2$, then $I = \phi$. which is measurable.

Case 2: Suppose $\alpha < M_1$, then $I = E \in \mathcal{M}$

Case 3: Suppose $M_1 < \alpha \leq M_2$, then

$I = \{x \in E \mid f(x) > \alpha\} \in \mathcal{M}$.

---

(3) Let $f: E \to \mathbb{R}$ be a bounded & measurable function, where E is a measurable set & $m(E) < \infty$. Suppose $|f(x)| \leq M \quad \forall x \in E$ for some $M > 0$

(i) Show that if $\int_E f = M\, m(E)$, then

$f = M$ a.e on E.

(ii) Show that if $f < M$ a.e on E & if $m(E) > 0$, then $\int_E f < M\, m(E)$.

Sol:- (i) Given that $\int_E f = M\, m(E) \geq 0$

For any $n \in \mathbb{N}$, let

$$E_n = \left\{ x \in E \,/\, f(x) < M - \frac{1}{n} \right\}.$$

Then
$$\int_E f = \int_{E_n} f + \int_{E \setminus E_n} f.$$

$$\leq \left(M - \frac{1}{n}\right) \int_{E_n} 1 + M \int_{E \setminus E_n} 1.$$

$$= \left(M - \frac{1}{n}\right) m(E_n) + M\, \underset{m(E) - m(E_n)}{\underset{\shortparallel}{m(E \setminus E_n)}}.$$

$$= \left(M - \frac{1}{n}\right) m(E_n) + M \left(m(E) - m(E_n) \right)$$

$$= M\, m(E) - \frac{1}{n}\, m(E_n).$$

$$\therefore 0 \le \int_E f \le M\, m(E) - \frac{1}{n}\, m(E_n)$$

$$\Rightarrow \quad 0 \le M\, m(E) \le M\, m(E) - \frac{1}{n}\, m(E_n)$$

$$\Rightarrow \qquad \le M\, m(E)$$

$$M\, \cancel{m}(E) = M\, \cancel{m}(E) - \frac{1}{n}\, m(E_n)$$

$$\Rightarrow \quad \frac{1}{n}\, m(E_n) = 0$$

$$\Rightarrow \quad m(E_n) = 0.$$

true $\forall n$.

Let $\quad F = \bigcup_{n=1}^{\infty} E_n = \{x \in E \mid f(x) < M\}.$

$$\Rightarrow \quad m(F) \le \sum_{n=1}^{\infty} m(E_n) = \sum 0 = 0.$$

$\therefore \quad m(F) = 0.$

on $E \setminus F$, $\quad f = M$

Thus $f = M$ a.e on E.

This proves (i).

(ii) Given that $|f| \le M$ on E

$\therefore \int_E f \leq \left| \int_E f \right| \leq \int_E |f| \leq M\, m(E).$

$\therefore \int_E f \leq M\, m(E).$

Suppose $\int_E f = M\, m(E).$

Then by (i), $f = M$ a.e on E.

This is a contradiction because $f < M$ a.e on E.

$\therefore \int_E f < M\, m(E).$

---

④ Let $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $\forall x \in [0,1]$ $\forall n \geq 1$.

(i) Show that $\{f_n\}$ is bounded on $[0,1]$ & evaluate $\lim\limits_{n \to \infty} \int_{[0,1]} f_n(x)$.

(ii) Show that $\{f_n\}$ does not converge uniformly on $[0,1]$.

Sol:- To show:- There exists $M > 0$ such that
$|f_n(x)| \leq M \quad \forall x \in [0,1], \forall n.$

For any $n \geq 1$, $\forall x \in [0,1]$,

$$(1-nx)^2 = 1 + n^2x^2 - 2nx \geq 0$$

$$\Rightarrow \quad 1 + n^2x^2 \geq 2nx \geq 0$$

$$\& \quad 1 + n^2x^2 > 0.$$

$$\Rightarrow \quad 0 \leq f_n(x) = \frac{nx}{1+n^2x^2} \leq \frac{1}{2}$$

$\therefore$ $\{f_n(x)\}$ is bounded on $[0,1]$.

Each $f_n(x)$ is continuous $\forall n$, $\forall x \in [0,1]$

$\Rightarrow$ Each f_n is measurable. & Riemann integrable on $[0,1]$

$$\therefore \quad \mathcal{L}\int_{[0,1]} f_n = \mathcal{R}\int_0^1 f_n(x)\, dx.$$

Consider $\displaystyle\int_0^1 f_n(x)\, dx = \int_0^1 \frac{nx}{1+n^2x^2}\, dx.$

$$= \left(\frac{1}{2n} \ln\left(1+n^2x^2\right)\right)\Big|_{x=0}^{1}$$

$$= \frac{1}{2n}\left(\ln(1+n^2)\right)$$

$$\lim_{n\to\infty} \int_{[0,1]}^{L} f_n = \lim_{n\to\infty} \frac{\ln(1+n^2)}{2n}.$$

$$= 0$$

(ii). To show: $f_n \not\to 0$ uniformly on $[0,1]$.

Suffices to show there exists a sequence $\{x_n\}$ in $[0,1]$ such that $x_n \to 0$ & $f_n(x_n) \not\to f(0) = 0$, as $n \to \infty$, where $f \equiv 0$.

Take $x_n = \frac{1}{n} \in [0,1]$, $x_n \to 0$

$$f_n(x_n) = \frac{n x_n}{1+n^2 x_n^2} = \frac{n\left(\frac{1}{n}\right)}{1+n^2\left(\frac{1}{n^2}\right)}$$

$$= \frac{1}{2} \quad \forall n$$

$\therefore f_n(x_n) \not\to f(0) = 0$, as $n \to \infty$.

$\therefore f_n$ is not uniformly convergent to $f \equiv 0$ on $[0,1]$.
