Problems 2

① Let $f:(a,b)\to\mathbb{R}$ be a function such that f' exists & satisfies $|f'(x)|\le M$ $\forall x\in(a,b)$ for some $M\geq 0$. Show that for every $E\subseteq(a,b)$, we have $m^*(f(E))\le m^*(E)\cdot M$.

Solution:- Suppose $M=0$.

$$\Rightarrow f'(x)=0 \quad \forall x\in(a,b)$$

$$\Rightarrow f(x)=c$$

$E\subset(a,b)$,

$\therefore f(E)=\{c\}$. $\therefore m^*(f(E))=0\le 0$

Assume $M>0$.

Cauchy MVT: for any $x,y\in(a,b)$, $x<y$. Then there exists $c\in(x,y)$ such that

$$(f(y)-f(x))=f'(c)(y-x).$$

$$\therefore \boxed{\begin{aligned}|f(y)-f(x)|&=|f'(c)|\,|y-x|\\&\le M\,|y-x|.\end{aligned}} \quad \forall x,y\in(a,b)$$

$$\Rightarrow \boxed{l(f(I_n))\le M\,l(I_n)} \quad I_n \text{ intervals in } [a,b]$$

Let $E\subseteq(a,b)$.

$$m^*(E)=\inf_{\substack{E\subseteq\cup I_n\\ I_n \text{ finite intervals.}}}\left(\sum_{n=1}^{\infty} l(I_n)\right)$$

Let $I_n=(a_n,b_n)$

$(\ ,\ )\ ,\ a_n, b_n \in (a,b)$

$$\underline{|f(b_n) - f(a_n)|} \leq M\ |b_n - a_n| \qquad \forall n$$

$$\forall n.$$

$$\Rightarrow \sum_{n=1}^{\infty} |f(b_n) - f(a_n)| \leq M \sum_{n=1}^{\infty} |b_n - a_n|.$$

$$\Rightarrow m^*(f(E)) \leq \inf\Big(\sum_{n=1}^{\infty} \underbrace{|f(b_n) - f(a_n)|}_{l(f(I_n))}\Big) \leq M \inf\Big(\sum_{n=1}^{\infty} (b_n - a_n)\Big).$$

infimum is taken over all coverings of E.

$$\Rightarrow m^*(f(E)) \leq M\ m^*(E).$$

$$f(E) = \{ y \mid y = f(x)\ \exists x \in E \}.$$

$$E \subseteq \bigcup_{n=1}^{\infty} I_n \Rightarrow \underline{f(E)} = \underline{\bigcup_{n=1}^{\infty} f(I_n)}.$$

$$\sum_{n=1}^{\infty} l(f(I_n)) \leq M \sum_{n=1}^{\infty} l(I_n)$$

$$\forall\ E \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$\Rightarrow m^*(f(E)) \leq M\ m^*(E). \qquad f(E) \subseteq \bigcup_{n=1}^{\infty} f(I_n)$$

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(2). Let $f: E \to \mathbb{R}$ be a function & $E \subseteq \mathbb{R}$ measurable

Then show that if $\{x \in E \mid f(x) < r\}$ is measurable in $\mathbb{R}$, for every $r \in \mathbb{Q}$, then f is measurable.

Sol: To show: f is measurable.

i.e. For $\alpha \in \mathbb{R}$,

to show: $\{x \in E \mid f(x) < \alpha\}$ is measurable.

$$\{x \in E \mid f(x) < \alpha\} = \bigcap_{i=1}^{\infty} \{x \in E \mid f(x) < r_i\}$$

α irrational.

Say $r_1, r_2, \ldots, r_n, \ldots \in \mathbb{Q}$ such that

$r_n \to \alpha$. say $r_n \downarrow \alpha$.

OR

Let $I = \{r \in \mathbb{Q} \mid r < \alpha\} \subseteq \mathbb{Q}$

I is countable.

$$\{x \in E \mid f(x) < \alpha\} = \underbrace{\bigcup_{r \in I} \underbrace{\{x \in E \mid f(x) < r\}}_{\text{measurable}}}_{\text{measurable}}.$$

(3) Let $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

$f : \mathbb{R} \to \mathbb{R}$,

Does f measurable ??

Sol:- For $\alpha \in \mathbb{R}$

$$\{x \in \mathbb{R} \mid f(x) > \alpha\}$$

For $\alpha \geq 1$: $\{x \in \mathbb{R} \mid f(x) > \alpha\} = \phi$

$0 < \alpha \leq 1$: $\{x \in \mathbb{R} \mid f(x) > \alpha\} = \mathbb{R} \setminus \mathbb{Q}$.

$\alpha \leq 0$: $\{x \in \mathbb{R} \mid f(x) > \alpha\} = \mathbb{R}$.

measurable

$\therefore$ f is measurable.

OR. Define $g(x) = 1 \quad \forall x \in \mathbb{R}$.

Then $f(x) = g(x)$ a.e.

$$\{x \in \mathbb{R} \mid f(x) \neq g(x)\} = \{x \in \mathbb{R} \mid f(x) = 0\} = \mathbb{Q}.$$

has measure 0.

& g is measurable $\Rightarrow$ f is measurable.

④. Let $g(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$

Does g measurable?

Sol:- Like above.

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⑤ Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ be a simple function.

canonical representation.

Let: $A, B \subseteq \mathbb{R}$. Show that

(i) $\chi_{A \cap B} = \chi_A \cdot \chi_B$

(ii) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$.

(iii) $\chi_{A^c} = 1 - \chi_A$.

Sol:- (i) $\chi_{A \cap B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \\ 0 & \text{if } x \in (A \cap B)^c. \end{cases}$

$\chi_{A \cap B}(x) = 1 \iff x \in A \;\&\; x \in B.$

$\iff \chi_A(x) = 1 \;\&\; \chi_B(x) = 1.$

$\Longleftrightarrow \quad (\chi_A \cdot \chi_B)(x) = 1.$

i.e., $\chi_A \cdot \chi_B\,(x) = 1$ iff $x \in A \cap B.$

if $x \in \underset{\overset{\shortparallel}{A^c \cup B^c}}{(A \cap B)^c} \Longleftrightarrow \chi_{A\cap B}(x) = 0.$

$\Longleftrightarrow \quad x \in A^c$ or $x \in B^c$

$\Longleftrightarrow \quad \chi_A(x) = 0$ or $\chi_B(x) = 0.$

$\Longleftrightarrow \quad (\chi_A \cdot \chi_B)(x) = 0.$

$\therefore \quad \chi_{A\cap B} = \chi_A \cdot \chi_B.$

(ii) To show: $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B.$

$\chi_{A\cup B}(x) = 1 \Longleftrightarrow x \in A \cup B.$

if $x \in \underset{\overset{\cap\!\shortmid}{A\cup B}}{A \cap B}$, then $\chi_A(x) + \chi_B(x) - (\chi_A \cdot \chi_B)(x)$

$= 1 + 1 - 1 = 1.$

Also $\chi_{A\cup B}(x) = 1$

if $x \notin A \cap B$, then $x \in A \setminus B$ or
& $x \in A \cup B$

$x \in B \setminus A$

$$\Rightarrow \quad \chi_A(x) + \chi_B(x) = 1$$

$$\& \quad (\chi_A \cdot \chi_B)(x) = 0.$$

Thus, ~~if~~ Let $x \in A \cup B$,

(i) if $x \in A \cap B$, then $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - (\chi_A \chi_B)(x)$
$= 1$

(ii) if $x \notin (A \cap B)^C$ " "

Thus $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B$.

(iii). To show: $\chi_{A^c} = 1 - \chi_A$.

Check it!