Group Theory

lecture 2

Defn A gp is a non-empty set G together with a law of composition which is associative and has identity elt and every elt of G has an inverse.

Remark (1) If the law of composition is written additively then

$$na = \underbrace{a + a \cdots + a}_{n\text{-times}}$$ and then

inverse of a is denoted by $-a$.

(2) If the law of composition is written multiplicatively then

$$a^n = \underbrace{a \cdot a \cdots a}_{n\text{-times}}$$ and the

inverse of a is denoted by a^{-1}.

Symmetric Group:

$$S_n = \left\{ f : [n] \longrightarrow [n] \;\middle|\; \begin{array}{l} f \text{ is } 1\text{-}1 \\ \text{and onto} \end{array} \right\}.$$

$$[n] = \{1, 2, \dots, n\}$$

$|S_n|$ = cardinality of $S_n = n!$.

S_n forms a group wrt composition operation.

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\,2)(3) = (1\,2)$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\,2\,3)$$

Consider $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 6 & 1 & 2 \end{pmatrix} \in S_6$

we will write it as $\sigma = (1\,5)(2\,4\,6)$

The identity is denoted by (1)

The elements of S_n are called permutations and S_n is called the symmetric group on $[n]$.

<u>In S_3</u>

$(1\,2\,3)\cdot(2\,3) = \underline{\underline{(1\,2)}}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$f_2 \in S_3 \qquad f_1 \in S_3$

$\underline{\underline{(2\,3)}}\;\underline{\underline{(1\,2\,3)}} = \underline{\underline{(1\,3)}}$

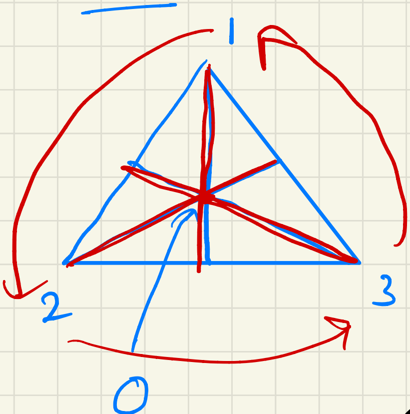
S_n is not an abelian group for $n \geq 3$.

$S_2 = \{(1), (12)\}$

$(12)(12) = (1)$.

$\underline{\underline{S_3}}$

Symmetry means invariance under transformation

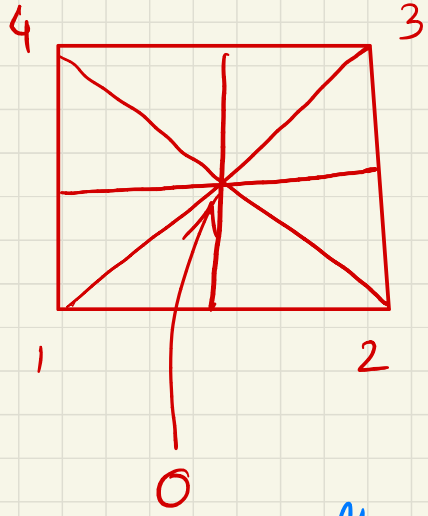


The three rotations about the centre O through $0^\circ, 120^\circ, 240^\circ$ are symmetries.

Also the three reflections along three bisectors are symmetries.

0°	rotation	is	(1)	Refletions gives
120°	"	"	$(1\,2\,3)$	$(2\,3)$,
240°	"	"	$(1\,3\,2)$	$(1\,3)$, $(1\,2)$

Dihedral group D_4.



Consider the following transformations of the square to itself

(1) The four rotation about the centre O through 0°, 90°, 180°, 270° anticlock-wise direction.

(2) The two reflections along the diagonals and the two reflections along the horizontal & vertical bisectors.

The above eight transformations forms a group called Dihedral group.

$$D_4 = \left\{ (1),\ \underset{a}{(1\,2\,3\,4)},\ \underset{a^2}{(1\,3)(2\,4)},\ \underset{a^3}{(1\,4\,3\,2)},\ \underset{b}{(2\,4)},\ \underset{ba^2}{(1\,3)},\ \underset{ba^3}{(12)(3\,4)},\ \underset{ba}{(14)(23)} \right\}.$$

check $ab = ba^3$

$$\boxed{D_4 = \langle a, b \mid a^4 = Id,\ b^2 = Id,\ ab = ba^3 \rangle}$$

$$ba^2 = (24)\ (1\,3)(2\,4)$$

$$= (13)$$

$$ba^4 = b$$

$$a^4 = (1).$$

$$b^2 = (1)$$