Sets in $\mathbb{R}$

Intervals

The order relation in $\mathbb{R}$ determines a natural collection of subsets, namely 'intervals'. There are four types of bounded intervals:-

(a) open interval:-

$$(a,b) := \{x \in \mathbb{R} : a < x < b\}.$$

(b) closed interval:-

$$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

(c), (d) Half-open (or half-closed) intervals:-

$$(a,b] = \{x \in \mathbb{R} : a < x \leq b\},$$

$$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}.$$

There are five types of unbounded intervals:-

(a) $(a,\infty) := \{x \in \mathbb{R} : x > a\},$

(b) $(-\infty, b) := \{x \in \mathbb{R} : x < b\},$

(c) $[a,\infty) := \{x \in \mathbb{R} : a \leq x\},$

(d) $(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}.$

(e) $(-\infty,\infty) := \mathbb{R}.$

Characterization Theorem:-

If S is a subset of $\mathbb{R}$ that contains at least two points and has the property:-

if $x, y \in S$ and $x < y$, then $[x,y] \subset S$,

then S is an interval.

Nested intervals:-

If $\{I_n : n \in \mathbb{N}\}$ be a family of intervals such that $I_{n+1} \subset I_n \; \forall n \in \mathbb{N}$, then the family $\{I_n\}$ is said to be a family of nested intervals.

Example:-

(i) Let $I_n := [0, \frac{1}{n}]$.

Then $I_1 = [0, 1]$, $I_2 = [0, \frac{1}{2}]$, — ,

Observe that $I_1 \supset I_2 \supset I_3 \supset$ — — and

$$\bigcap_{n=1}^{\infty} I_n = \{0\}.$$

(ii) Let $I_n := (0, \frac{1}{n})$

Then $I_1 \supset I_2 \supset \cdots \supset I_n \supset$ — — and

$\bigcap_{n=1}^{\infty} I_n = \Phi$ as if $x \in \bigcap_{n=1}^{\infty} I_n$, then $x > 0$, so by Archimedean property, $\exists$ a natural number n_x such that $\frac{1}{n_x} < x$, so $x \notin (0, \frac{1}{n_x})$.

So, $x \notin \bigcap I_n$.

(iii) Let $I_n := (n, \infty)$.

Then $I_1 \supset I_2 \supset$ — — $\supset I_n \supset$ — — and each I_n is unbounded. Here $\bigcap I_n = \Phi$ (again by Archimedean property)

(iv) Let $I_n := [n, \infty)$.

Then I_n's are nested intervals and

$$\bigcap I_n = \Phi.$$

~~Result~~.

Nested Interval Property :-

If $I_n := [a_n, b_n], n \in \mathbb{N}$, is a family of nested closed and bounded intervals, then

$\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

Proof :- Here $I_n \subseteq I_1$, so $a_n \in I_n \subseteq I_1$

$\Rightarrow a_n \leq b_1 \ \forall n \in \mathbb{N}$.

So, as $\{a_n\}_{n \in \mathbb{N}}$ is a non-empty subset, which is bounded above, it has a supremum, namely ξ.

Claim :- $\xi \leq b_n \ \forall n \in \mathbb{N}$.

Proof :- Observe that $a_k \leq b_n \ \forall k \in \mathbb{N}$.

To prove this, we consider two cases :-

(a) $n \leq k$, then $a_k \leq b_k \leq b_n$.

(b) $n > k$, then $a_k \leq a_n \leq b_n$.

So, each b_n is an upper bound of $\{a_k\}_{k \in \mathbb{N}}$.

Also, ξ is the supremum of $\{a_k \mid k \in \mathbb{N}\}$, so $\xi \leq b_n \ \forall n \in \mathbb{N}$. So, $a_n \leq \xi \leq b_n \ \forall n \in \mathbb{N}$.

So, $\xi \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. So, $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

Theorem :- If $I_n := [a_n, b_n], n \in \mathbb{N}$, is a family of nested closed and bounded intervals such that $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0$, then there is one and only one point x in $\bigcap_{n=1}^{\infty} I_n$.

Proof: Let $\eta := \inf\{b_n : n \in \mathbb{N}\}$.

Then we can show similarly that
$$\eta \in \bigcap_{n=1}^{\infty} I_n .$$

So, $[\xi, \eta] \subset I_n \ \forall n \in \mathbb{N}$ as each I_n is an interval. So, $[\xi, \eta] \subset \bigcap_{n=1}^{\infty} I_n$. We can show that $[\xi, \eta] = \bigcap_{n=1}^{\infty} I_n$.

$\therefore 0 \leq \eta - \xi \leq b_n - a_n \ \forall n \in \mathbb{N}$.

Since $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, so for $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $b_m - a_m < \varepsilon$.

$\therefore 0 \leq \eta - \xi < \varepsilon$.

$\therefore \eta = \xi$ as otherwise can take ε to be $\eta - \xi$.

So, $x = \eta = \xi$.

<u>Application of Nested Interval Property:</u>–

The set $\mathbb{Q}$ of all rational numbers is countable. We will use Nested Interval Property to prove that $\mathbb{R}$ is uncountable. So, the set of irrational numbers, i.e. $\mathbb{R} - \mathbb{Q}$ is also uncountable. This proof was first given by Cantor in 1874. He later gave another proof using decimal representations of real numbers.

<u>Theorem:</u> The set $\mathbb{R}$ of real numbers is uncountable.

<u>Proof:</u> It is enough to show that the interval $I := [0,1]$ is uncountable. As $\mathbb{R}$ is

countable implies the subset I would be countable too.

We shall prove that I is countable by method of contradiction. If possible, assume that I is countable and we can enumerate the set as $I = \{x_1, x_2, \ldots, x_n, \ldots\}$.

First, select a closed subinterval I_1 of I such that $x_1 \notin I_1$. Then select a closed subinterval I_2 of I_1 s.t $x_2 \notin I_2$. This way we obtain non empty closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that $I_n \subseteq I$ and $x_n \notin I_n$ By Nested Interval Property, there exists a point $\xi \in I_n \; \forall n$. So, $\xi \neq x_n \; \forall n \in \mathbb{N}$

So, the enumeration of I is not the complete listing of the elements of I, a contradiction.

Open Sets

Neighbourhood:-

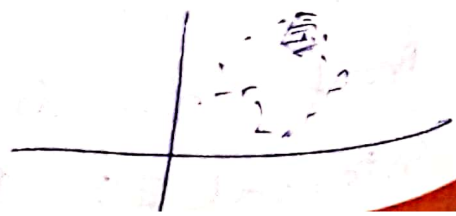
We can think of points within ε of x approximate x to an accuracy of degree ~~ε~~ ε. Here $\varepsilon > 0$ but as ε becomes smaller and smaller, one obtains points that approximate x to a higher and higher degree of accuracy. These points are interval $(x-\varepsilon, x+\varepsilon)$ or we denote by $N_\varepsilon(x)$. This open interval $(x-\varepsilon, x+\varepsilon)$ is said to be the ε-neighbourhood of x.

Let $c \in \mathbb{R}$ and $S \subset \mathbb{R}$. We say that S is a neighbourhood of c if ~~thee~~ there exists an open ~~interval~~ interval (a,b) such that $c \in (a,b) \subset S$.

Interior point:- Let $S \subseteq \mathbb{R}$. A point x in $\mathbb{R}$ is ~~Interi~~ said to be an interior point of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

The set of all interior points of S is said to be the interior of S and is denoted by $\operatorname{int} S$ or S°.

Note that $S^\circ \subseteq S$.



Example:-

(a) Let $S = \{x \in \mathbb{R} : -1 < x < 1\} = (-1, 1)$

By definition, $(-1, 1)$ is a neighbourhood of each point x of S. So, $\text{int } S = S$

(b) Let $S = [0, 1]$. By the previous argument, $(0, 1) \subset \text{int } S$. Our claim is $(0, 1) = \text{int } S$.

We know that $\text{int } S \subset S$.

If 0 is an interior point of S, then there is a neighbourhood (a, b) such that $0 \in (a, b) \subset [0, 1]$, which is not possible. So, 0 is not an interior point. Similarly, 1 is not an interior point.

So, $\text{int } S = (0, 1)$.

(c) Let $S = \mathbb{N}$. Let $n \in S$. If n is an interior point of S, then $\exists$ an open interval (a, b) such that $a \geq n-1$, $b \leq n+1$ and $n \in (a, b) \subset \mathbb{N}$.

Here $\frac{a+n}{2} \in (a, b)$ but $\frac{a+n}{2} \notin \mathbb{N}$.

So, $(a, b) \not\subset \mathbb{N}$, so n is not an interior point of S. So, $\text{int } S = \Phi$.

(d) Let $S = \bar{\mathbb{Q}}$. S has no interior point. So, $\text{int } S = \Phi$.

__Open set:__ Let $S \subseteq \mathbb{R}$. S is said to be an open set if each point of S is an interior point, i.e. $S = \text{int } S$.

Examples:-

(a) $S = (-1, 1)$ is an open set ~~and~~ as here $S = \text{int } S$.
In fact each open ~~finite~~ interval is an open set.

(b) $S = \mathbb{R}$ is an open set as for each real number x, $x \in (x-1, x+1) \subset \mathbb{R}$.

(c) $S = \Phi$. S contains no points. Therefore the requirement in the definition of an open set is vacuously satisfied. So, S is an open set.

(d) $S = [0, 1]$. Here $0 \in [0, 1]$ but 0 is not an interior point of S, so S is not an open set.
~~(e) $S = \mathbb{N}$. Similarly~~, In fact each closed and bounded interval is not an open set.

(e) $S = \mathbb{N}$. We have seen that no point of $\mathbb{N}$ is an interior point of $\mathbb{N}$, so $\mathbb{N}$ is not an open set.

The next two results will characterize the interior of a set.

Theorem:- Let $S \subseteq \mathbb{R}$. Then $\text{int } S$ is an open set.

Proof:-

Case:-1 Let $\text{int } S = \Phi$. Then $\text{int } S = \Phi$ is an open set.

Case:-2 Let $\text{int } S \neq \Phi$. Let $x \in \text{int } S$. Then x is an interior point of S. Then there is ~~an an~~ a neighbourhood $N(x)$ of x such that $x \in N(x) \subset S$.
i.e. there is an open interval (a, b) such that

$x \in (a,b) \subset S$.

Then, $y \in (a,b) \Rightarrow y \in S$ and $y \in (a,b) \subset S$. So, each y is an interior point of S.

Theorem:- Let $S \subset \mathbb{R}$. Then int S is the largest open set contained in S.

Proof:- By the previous theorem, int S is an open set and int $S \subset S$.

Let P be an open set contained in S.

Let $x \in P$. So, x is an interior point of P. Therefore there exists ~~an~~ a neighbourhood $N(x)$ of x such that $N(x) \subset P$. So, $N(x) \subset S$. So, x is an interior point of S. So, $x \in$ int S. So, $P \subset$ int S.

So, int S is the largest open set contained in S.

Note:- Int S is the union of all open sets contained in S.

Properties of open sets:-

Theorem:- The union of finite number of open sets in $\mathbb{R}$ is open.

Proof:- Let $G_1, G_2, -, G_m$ be m open ~~sets~~ sets in $\mathbb{R}$.

Let $G = G_1 \cup - - \cup G_m$.

Let $x \in G$. $\therefore x \in G_i$ for some i. $\therefore x$ is an interior point of G_i.

So, $\exists$ a neighbourhood $N(x)$ such that

$$x \in N(x) \subset G_i$$

$$\therefore x \in N(x) \subset G.$$

So, ~~G is an open set~~ x is an interior point of G.

So, G is an open set.

Similarly, we can prove that

Theorem:-

The union of an arbitary callection of open sets in $\mathbb{R}$ is an open set.

Theorem:- The intersection of a finite number of open sets in $\mathbb{R}$ is an open set.

Proof:- Let $G_1, G_2, -, G_m$ be m open sets in $\mathbb{R}$.

Let $G = G_1 \cap G_2 \cap \cdots \cap G_m$.

Case-1 $G = \Phi$. Then G is an open set.

Case-2 $G \neq \Phi$. Let $x \in G$. $\therefore x \in G_i \ \forall i = 1, 2, -, m$.

Since G_i is an open set, there exists an open interval $(x-\varepsilon_i, x+\varepsilon_i)$ such that

$$x \in (x-\varepsilon_i, x+\varepsilon_i) \subset G_i.$$

Let $\varepsilon = \min\{\varepsilon_i \mid \cancel{i \in \mathbb{N}} \ i = 1, 2, -, m\}$.

Then, $x \in (x-\varepsilon, x+\varepsilon) \subset G_i \ \forall i = 1, 2, -, m$.

$\therefore x \in (x-\varepsilon, x+\varepsilon) \subset G$.

So, x is an Interior point of G.

So, G is an open set.

Note:- The intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set. For example:- let $G_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$.

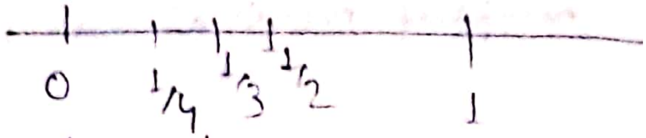
Then $\bigcap_{i \in \mathbb{N}} G_i = \{0\}$ is not an open set.

Limit and Isolated points

We start with ~~an~~ examples.

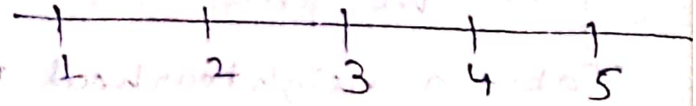
Examples:-

a) Let $S = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.



We can see that though the real number $0 \notin S$ but 0 is closed to ~~a~~ S, i.e., if we take ~~any~~ any ε-neighbourhood of x, that $N_\varepsilon(x)$ contains some points of S.

b) Let $S = \{n \mid n \in \mathbb{N}\}$



Here 1 is isolated in S as if we take the $\frac{1}{2}$-neighbourhood of 1, that $N_{1/2}(1)$ contains no other points of ~~a~~ S.

Def:-

1) Limit point:

Let $S \subseteq \mathbb{R}$ and $S \neq \Phi$. A point p in $\mathbb{R}$ is said to be a limit point (or an accumulation point or a cluster point) of S if every neighbourhood of p contains a point of S other than p.

Therefore, p is a limit point if for each $\varepsilon > 0$,

$(N_\varepsilon(x) - \{p\}) \cap S \neq \Phi$. For a neighbourhood $N(p)$ of p, $N(p) - \{p\}$ is called a ~~dotel~~ deleted neighbourhood of p.

2) Isolated point:-

Let $S \subseteq \mathbb{R}$ and $S \neq \phi$. A point x in S is said to be an isolated point of S if x is not a limit point of S i.e. there exists a neighbourhood ~~N(p) of~~ $N(x)$ of x such that $N(x) \cap S = \{x\}$.

! Remark:- A limit point of S may or may not belong to S.

Examples of limit points and isolated points:-

a) Let $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

First we prove that 0 is a limit point of S.
Take a neighbourhood $\cancel{N(p)}$ $N(0)$ of 0. $N(0)$ contains an open interval (a,b) such that $0 \in (a,b) \subset N(0)$.
By Archimedean property of $\mathbb{R}$, there exists a natural number m such that $0 < \frac{1}{m} < b$ (as $b > 0$). So, $\frac{1}{m} \in \{N(0) - 0\} \cap S$.

We can see that each point of S is an isolated point, i.e. not ~~a~~ limit point of S.
Take $\frac{1}{n} \in S$ $(n \neq 1)$. Take ~~δ~~ $\varepsilon = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n-1} - \frac{1}{n}$.

Then, $N_\varepsilon(\frac{1}{n}) \cap S = \{\frac{1}{n}\}$, so $\frac{1}{n}$ is not a limit point of S. In fact, 0 is the only limit point of S.

b) $S = \mathbb{N}$. We have shown earlier that each point $n \in \mathbb{N}$ is not a limit point (i.e. an isolated ~~point~~ point) of S. In fact, no real number is a limit point of $\mathbb{N}$. If $x \in \mathbb{R} - \mathbb{N}$, then two cases arise—

Case-1. $x < \overset{1}{\cancel{0}}$, then choose $\varepsilon := |x| \cancel{= -x}$ ~~Then~~
$1-x$.

Then, $N_\varepsilon(x) \cap S = \Phi$. So, x is not a limit point.

Case-2 $x > 1$. Then ~~the~~ there is a natural number n_x such that $n_x - 1 < x < n_x$.

Choose $\varepsilon := \min\{n_x - x, x-(n_x - 1)\}$.

Then, $N_\varepsilon(x) \cap S = \Phi$

So, x is not a limit point.

c) $S = \mathbb{Q}$. Let $x \in \mathbb{R}$. Choose any neighbourhood of x of the form $N_\varepsilon(x)$. Now, by density property of x, there exists a rational number r such that

$$x - \varepsilon < r < x$$

So, $r \in (N_\varepsilon(x) - \{x\}) \cap S$.

So, x is a limit point of S.

d) Similarly, each real number is a limit point of the set $\mathbb{R}$ of real numbers.

From the above examples, we observe that all the sets are infinite. Though $\mathbb{N}$ is an infinite set, it has no limit points. Consider the example $S = \{0,1\}$.

$0 \qquad 1$

Clearly, $\{0,1\}$ has no limit point. In fact, Theorem:- A finite set has no limit point.

This result follows immediately from the following proposition.

Proposition:- Let $S \subset \mathbb{R}$ and p be a limit point of S. Then every neighbourhood of p contains infinitely many elements of S.

Proof:- ~~Let $\epsilon > 0$.~~ Let $N(p)$ be a neighbourhood of p.

So, $(N(p) - \{p\}) \cap S \neq \Phi$.

Let $B := (N(p) - \{p\}) \cap S$.

Claim:- B is an infinite set.

If not, let $B = \{s_1, s_2, \dots, s_m\}$.

Take $\epsilon_i := |s_i - p|$. Let $\epsilon := \min\{\epsilon_i \mid i = 1, 2, \dots, m\}$.

Then $\epsilon > 0$ and it is obvious that no a_i belongs to ~~$N_\epsilon(p)$, so~~ $N_\epsilon(p) - \{p\}$.

But $N_\epsilon(p) - \{p\} \subset N(p) - \{p\}$, so

$(N_\epsilon(p) - \{p\}) \cap S \subset (N(p) - \{p\}) \cap S = B$.

So, $(N_\epsilon(p) - \{p\}) \cap S = \Phi$, contradicting to the fact that p is a limit point of S. So, B is an infinite set.

We have seen an example $\mathbb{N}$, which is an infinite set but doesnot have any limit point. On the other hand, $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is an infinite set which has a limit point. Here note that $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is bounded but $\mathbb{N}$ is not bounded. In fact, we have the following generalised result:—

Theorem: (Balzano - Weierstrass ~~Theo~~ Theorem):—

Every bounded infinite subset of $\mathbb{R}$ has at least one limit point.

Proof: Let S be a bounded infinite subset of $\mathbb{R}$.

Since S is ~~bounded~~, bounded, S has a supremum u and an infimum, l.

Let $H := \{x \in \mathbb{R} : x$ is greater than infinitely many elements of $S\}$

Since S is an infinite set, so $u \in H$. So, H is a non-empty set. Also, every elt of H is greater than l. So, H is bounded below.

So, by completeness axiom, H has an infimum, say y.

Claim: y is a limit point of S.

Proof: Let ~~$N(p)$~~ $N(y)$ be a neighbourhood of y. Then there exists $\varepsilon > 0$ such that

$$(y-\varepsilon, y+\varepsilon) \subset N(y).$$

Since $y+\varepsilon > y$, so by definition of infimum, $\exists\ h \in H$ such that $y < h < y+\varepsilon$. So, h exceeds infinitely many elements of S.

Since $y-\varepsilon \notin H$, so $y-\varepsilon$ can exceed at most a finitely many elements of S. So, $(y-\varepsilon, h)$ contains infinitely many elements of S. So, y is a limit point of S.

Closed Set

Defn: Let $S \subseteq \mathbb{R}$. S is said to be a closed set if S contains all its limit points.

Examples:

1) Let $S := \{x \mid a \leq x \leq b\}$ where $a<b$.

We shall prove that the set of limit point of $S = [a,b]$ is S. So, S is a closed set.

Case-1. Let $x<a$. Choose ε to be $\frac{a-x}{2} > 0$

Then, $N_\varepsilon(x) = \cancel{(a-\varepsilon,} (x-\varepsilon, x+\varepsilon)$, and

$x+\varepsilon = a-\varepsilon < a$. So, $N_\varepsilon(x) \cap S = \Phi$.

So, x is not a limit point of S.

Case-2 Let $x=a$. Let $N(a)$ be a neighbourhood of x which contains $N_\varepsilon(a) = (a-\varepsilon, a+\varepsilon)$

Let $\delta := \min\{\varepsilon, b-a\}$. Then $\delta > 0$ and

$a < a+\frac{\delta}{2} < a+\delta < a+\varepsilon$, so $a+\delta/2 \in N_\varepsilon(a) - \{a\}$.

Also, $a < a+\delta/2 < a+\delta \leq b$. $\therefore a+\delta/2 \in S$.

$\therefore a+\delta/2 \in (N_\varepsilon(a) - \{a\}) \cap S$. So, a is a limit point of S.

Case-3 Let $x \in (a,b)$. Let $N(x)$ be a neighbourhood of x which contains $N_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$

Let $\delta := \frac{1}{2}\min\{x-a, b-x\}$. Then $N_\delta(x) \subset S$.

If $\varepsilon \leq \delta$, then $x+\varepsilon/2 \in (N_\varepsilon(x) - \{x\}) \cap S$.

If $\varepsilon > \delta$, then $x+\delta/2 \in (N_\varepsilon(x) - \{x\}) \cap S$.

So, x is a limit point of S.

Case-4 $x=b$.

~~Proof~~ That x is a limit point of S, the proof is similar to case-2.

Case-5 $x>b$.

Proof similar to case-1 shows that x is not a limit point of S.

2) Let $S := \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$.

It has been proved that 0 is a limit point of S, but $0 \notin S$. So, S is not a ~~limit point of S~~ closed set.

3) Let $S := \{ x \mid a < x < b \}$.

Similar to example-1, we can show that the ~~these~~ set of limit points of S is $[a,b]$. So, S is not a closed set.

4) Let $S := \mathbb{N}$. We have shown earlier that the set of limit points of S is empty $\subset S$. So, S is a closed set.

5) Every finite set is a closed set.

6) Let $S := \mathbb{Q}$. ~~So~~ Here the set of limit points of S is $\mathbb{R}$. So, $\mathbb{Q}$ is not a closed set.

In the next theorem, we shall discuss the interrelation between the collection of open sets and collection of closed sets.

Theorem: Let $G \subseteq \mathbb{R}$. Then G is open if and only if G^c is closed.

Proof: We shall prove one ~~dire~~ direction, i.e. G is open if G^c is closed. The other direction can be proved similarly.

Case-1: Let $G^c = \mathbb{R}$ is a closed set. Then, $G = \Phi$. Because, G doesn't contain any point, the requirement in the definition of an open set is vacuously satisfied. So, G is an open set.

Case-2: Let G^c be a proper subset of $\mathbb{R}$. Then $G \neq \Phi$. Let $x \in G$. Since G^c is closed, so x can't be a limit point of G. So there exists a neighbourhood $N(x)$ such that $(N(x) - \{x\}) \cap G^c = \Phi$.

So, $N(x) - \{x\} \subset (G^c)^c = G$. Also, $x \in G$

So, $N(x) \subset G$. So, x is an interior point of G. Since, x is an arbitary point of G, so G is ~~open~~ open.

The previous result shows that a set is open if and only if its complement is closed. Similarly, closed ~~subsets~~ subsets of $\mathbb{R}$ have some properties, which are opposite to properties of open sets.

Theorem: The union of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Proof: Let $F_1, F_2, \ldots, F_m$ be closed sets and

$$F := F_1 \cup \cdots \cup F_m.$$

Let x be a limit point of F. Let $N(x)$ be a neighbourhood of x, Then, $(N(x)-\{x\}) \cap F \neq \Phi$.

Let $N(x)$ contain one interval $(x-\epsilon, x+\epsilon)$.

Claim: There exists a F_i, $i=1,2,\ldots,m$ such that

$$((x-\epsilon, x+\epsilon)-\{x\}) \cap F_i \neq \Phi \quad \forall \epsilon.$$

Proof: We shall prove by method of contradiction.

Suppose, if possible, for each i, $\exists$ F_i s.t.

$$((x-\epsilon_i, x+\epsilon_i)-\{x\}) \cap F_i = \Phi.$$

Let $\epsilon := \min\{\epsilon_i \mid i=1,2,\ldots,m\}$.

Then, $(x-\epsilon, x+\epsilon) \subset (x-\epsilon_i, x+\epsilon_i)$ $\forall i$ and so

$$((x-\epsilon, x+\epsilon)-\{x\}) \cap F_i = \Phi \quad \forall i$$

$\Rightarrow ((x-\epsilon, x+\epsilon)-\{x\}) \cap F = \Phi$, which is a contradiction to the fact that x is a limit point of F.

So, x is a limit point of F_i $\therefore$ $x \in F_i \Rightarrow x \in F$.

So, F is a closed subset of $\mathbb{R}$.

Theorem: The intersection of arbitary collection of closed sets in $\mathbb{R}$ is a closed set.

Proof: Let $\{F_i\}$ be an arbitary collection of closed sets in $\mathbb{R}$. Let $F = \bigcap_{i \in I} F_i$. Let x be a limit point of F.

Then for each nbhd $N(x)$, $(N(x)-\{x\}) \cap F \neq \Phi$.

So, $(N(x)-\{x\}) \cap F_i \neq \Phi$

$\Rightarrow x$ is a limit pt. of F_i $\forall i \in I$

$\Rightarrow$ $x \in F_i$ $\forall i$ as each F_i is a closed set

$\Rightarrow x \in F$.

$\therefore$ F is a closed set.

<u>Note:-</u> The union of an infinite ~~collecti~~ collection of closed sets is not necessarily a closed set. For example:- let $F_n = \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$ $\forall n \geq 1$.

Then, $\bigcup_{n=1}^{\infty} F_n = (-1, 1)$, but $(-1, 1)$ is not a closed set.