Date
03/10/2017

Lecture 18

Odd periodic extension

Similarly, from eqn (6), we obtain

(ie, $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L}x \, dx).$$

$$b_n = \frac{8k}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right).$$

Hence, the other half-range expansion of $f(x)$ is

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin \frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - \dots\right)$$

This series

represents the odd periodic extension of $f(x)$, of period $2L$, shown in Fig 3(b).

Note :— Half-range series, are, as the name implies, series defined over half of the normal range. That is, for standard trigonometric Fourier series, the f^n $f(x)$ is defined only in $[0, \pi]$ instead of $[-\pi, \pi]$. The value that $f(x)$ takes in the other half of the interval $[-\pi, 0]$ is free to be defined.

If we take $f(x) = f(-x)$, ie, $f(x)$ is even then the Fourier series for $f(x)$ can be entirely expressed in terms of even f^ns ie, cosine entirely. If on the other hand, $f(x) = -f(-x)$, ie, $f(x)$ is an odd f^n & the Fourier series is correspondingly odd & consists only of sine terms. We are not defining the same f^n as two different Fourier series, for $f(x)$ is different, at least over half the range.

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Complex Fourier Series

The Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \rightarrow (1)$$

can be written in complex form, which sometimes simplifies calculations.

we know that,

$$e^{inx} = \cos nx + i \sin nx \rightarrow (2)$$

$$e^{-inx} = \cos nx - i \sin nx \rightarrow (3)$$

By addition of (2) & (3), we get

$$\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right) \rightarrow (4)$$

$$\sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right) \rightarrow (5)$$

From this, using $\frac{1}{i} = -i$,

we have from eqn (1)

$$a_n \cos nx + b_n \sin nx$$

$$= \frac{1}{2} a_n \left(e^{inx} + e^{-inx}\right)$$

$$+ \frac{1}{2i} b_n \left(e^{inx} - e^{-inx}\right)$$

$$= \frac{1}{2}\left(a_n - ib_n\right) e^{inx}$$

$$+ \frac{1}{2}\left(a_n + ib_n\right) e^{-inx}$$

We insert this into eqn (1),

writing

$$a_0 = c_0, \quad \frac{a_n - ib_n}{2} = c_n$$

$$\& \quad \frac{a_n + ib_n}{2} = k_n$$

hen (1) becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{inx} + k_n e^{-inx} \right) \longrightarrow (6)$$

For the co-efficients, $c_1, c_2, \ldots$

& $k_1, k_2, \ldots,$ are obtain

from (2), (3) & the Euler

formulas (5),

$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx - i \sin nx) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$k_n = \frac{a_n + i b_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i \sin nx)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx$$

$\rightarrow (7)$

Finally, we can combine the two formulas in (7) into one by the trick of writing $k_n = c_{-n}$.

Then (6), (7) together with (6a) [i.e., $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$] give

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx,$$

$$n = 0, \pm 1, \pm 2, \ldots \longrightarrow (8)$$

This is the so-called complex form of the Fourier series or, more briefly, the complex Fourier series of $f(x)$. The c_n are called the complex Fourier co-efficients of $f(x)$.

For a function of period

2L our reasoning gives the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L},$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L} dx.$$

$\longrightarrow (9)$

This complex form of Fourier series is true when $f(x)$ is continuous at x & the Dirichlet's cond^n are satisfied.

If $f(x)$ is discontinuous at x, the L.H.S of $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$

is replaced by $\dfrac{f(x+0)+f(x-0)}{2}$

i.e., $f(x) \longrightarrow \dfrac{f(x+0)+f(x-0)}{2}$.

Laplace Transform

Methods of finding Inverse Laplace Transform

e.g., $\dfrac{2s-5}{(3s-4)(2s+1)^3} = \dfrac{A}{3s-4} + \dfrac{B}{(2s+1)^3} + \dfrac{C}{(2s+1)^2} + \dfrac{D}{(2s+1)}$

(1) Partial fractions Method.

$\longrightarrow$ Heaviside expansion formula.

(2) Series method.

$$F(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} \cdots$$

e.g., $L^{-1}\left[e^{-1/s}/s\right]$.

$$f(t) = a_0 + a_1 t + a_2 t^2/2! + \cdots$$

(3) Method of differential eq^n :-

Eg. $\mathcal{L}^{-1}\left\{e^{-\sqrt{s}}\right\}$

Let $y = e^{-\sqrt{s}}$

Then $4sy'' + 2y' - y = 0$

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(4) Diff. w.r.t. a parameter

Eg. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

Hint :- $\dfrac{d}{da}\left(\dfrac{s}{s^2+a^2}\right) = \dfrac{-2as}{(s^2+a^2)^2}$.

(5) Use of Tables

(6) The Complex Inversion formula

This formula which supplies a powerful direct method for finding inverse L.T uses complex variable theory - (Find it?)

(7) → Linked with Partial Fraction.

The Heaviside Expansion Formula.

5) Use of Tables

6) The Complex Inversion formula

This formula which supplies a powerful direct method for finding inverse L.T uses complex variable theory. (Find it?)

7) → Linked with Partial Fraction.

The Heaviside Expansion Formula.

Let $P(s)$ & $Q(s)$ be polynomials where $P(s)$ has degree less than that of $Q(s)$.

Suppose that $Q(s)$ has n distinct zeros.

α_k, $k = 1, 2, \ldots, n$. Then

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}.$$

This is often called Heaviside's expansion theorem or formula which can be extended for repeated case.

Q) Find $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

Sol^n:- We have,

$$P(s) = 2s^2 - 4$$

$$Q(s) = (s+1)(s-2)(s-3)$$

$$= s^3 - 4s^2 + s + 6.$$

$$Q'(s) = 3s^2 - 8s + 1.$$

$$\alpha_1 = -1,\ \alpha_2 = 2,\ \alpha_3 = 3.$$

$\therefore$ the reqd. inverse is

$$\frac{P(-1)}{Q'(-1)}e^{-t} + \frac{P(2)}{Q'(2)}e^{2t} + \frac{P(3)}{Q'(3)}e^{3t}$$

$$= \left(\frac{-2}{12}\right)e^{-t} + \frac{4}{(-3)}e^{2t} + \frac{14}{4}e^{3t}$$

$= -\frac{1}{6} e^{-t} - \frac{4}{3} e^{2t} + \frac{7}{2} e^{3t}$

Check by Partial Fractions

H.W

Q) Find $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$

Hint: Here, $P(s) = 3s+1$, $Q(s) = (s-1)(s^2+1) = s^3 - s^2 + s - 1$, $Q'(s) = 3s^2 - 2s + 1$

$\alpha_1 = 1, \alpha_2 = i, \alpha_3 = -i$

$\therefore$ the reqd. inverse is $2e^t - 2\cos t + \sin t$.

H.W.

EX/ Suppose that $F(s) = \dfrac{P(s)}{Q(s)}$ where $P(s)$ & $Q(s)$ are polynomials as earlier, but that $Q(s) = 0$ has a repeated root of multiplicity m, while the remaining roots

$b_1, b_2, \ldots, b_n$ do not repeat

(a) show that

$$F(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{(s-a)^m} + \frac{A_2}{(s-a)^{m-1}} + \cdots + \frac{A_m}{(s-a)} + \frac{B_1}{(s-b_1)} + \frac{B_2}{(s-b_2)} + \cdots + \frac{B_n}{(s-b_n)}$$

Hint:- Use Partial Fraction

b) show that

$$A_k = \lim_{s \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left\{ (s-a)^m F(s) \right\},$$

$k = 1, 2, \ldots, m$

c) show that

$$\mathcal{L}^{-1}\{F(s)\}$$

$$= e^{at}\left[\frac{A_1 t^{m-1}}{(m-1)!} + \frac{A_2 t^{m-2}}{(m-2)!} + \cdots + A_m\right]$$

$$+ B_1 e^{b_1 t} + \cdots + B_n e^{b_n t}.$$

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eg, i) $\mathcal{L}^{-1}\left\{\dfrac{2s^2 - 9s + 19}{(s-1)^2 (s+3)}\right\}$

ii) $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{(s+1)^2 (s+2)^2}\right\}$

Proof of Heaviside's expansion formula

Since $Q(s)$ is a polynomial with n distinct zeros $\alpha_1, \alpha_2, \dots, \alpha_n$, we can write according to the method of partial fractions

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \dots + \frac{A_k}{s-\alpha_k} + \dots + \frac{A_n}{s-\alpha_n} \quad \longrightarrow (1)$$

Multiplying both sides by $(s-\alpha_k)$ & letting $s \to \alpha_k$, we find using L' Hospital's rule

$$A_k = \lim_{s \to \alpha_k} \frac{P(s)}{Q(s)} (s-\alpha_k)$$

$$= \lim_{s \to \alpha_k} \left\{ P(s) \cdot \left[\frac{(s-\alpha_k)}{Q(s)} \right] \right\}$$

$$= \lim_{s \to \alpha_k} P(s) \cdot \lim_{s \to \alpha_k} \left\{ \frac{s-\alpha_k}{Q(s)} \right\} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= P(\alpha_k) \cdot \lim_{s \to \alpha_k} \left\{ \frac{1}{Q'(s)} \right\} = \frac{P(\alpha_k)}{Q'(\alpha_k)}$$

Thus, (1) can be written as

$$\frac{P(s)}{Q(s)} = \frac{P(\alpha_1)}{Q'(\alpha_1)} \cdot \frac{1}{(s-\alpha_1)} + \dots + \frac{P(\alpha_k)}{Q'(\alpha_k)} \cdot \frac{1}{(s-\alpha_k)} + \dots + \frac{P(\alpha_n)}{Q'(\alpha_n)} \cdot \frac{1}{(s-\alpha_n)}.$$

Then, taking the inverse Laplace transform, we have as required

$$\mathcal{L}^{-1}\left\{ \frac{P(s)}{Q(s)} \right\} = \frac{P(\alpha_1)}{Q'(\alpha_1)} \cdot e^{\alpha_1 t} + \dots + \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t} + \dots + \frac{P(\alpha_n)}{Q'(\alpha_n)} e^{\alpha_n t}$$

$$= \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}.$$

—x—