Discussion of Tutorial 4

Lecture 18

Ex Describe conjugacy classes of S_4.

Let $\sigma \in S_4$.

$\tau \sigma \tau^{-1}$ where $\tau \in S_4$.

let $\sigma = (1\ 2\ 3\ 4)$

$\tau \sigma \tau^{-1}$ will be a 4-cycles.

Propn. Two elts of S_n are conjugates in S_n iff they have the same cycle type.

$(1\ 2)(3\ 4) \longleftrightarrow (1\ 3)(2\ 4)$

Defn. If $\sigma \in S_n$ is the product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$ with $n_1 \leq n_2 \leq \cdots \leq n_r$ (including 1-cycle) then the integers $n_1, n_2, \ldots, n_r$ is called a cycle type of σ.

S_4.

cycle type σ	an elt of that type σ	The conjugacy class of σ	$|G|$
$1+1+1+1$	e	e	1
$1+1+2$	(12)	$(12), (23), (13), (14)$, $(24), (34)$	6
$1+3$	(123)	$(123), (124), (234)$, $(134), (132), (142)$, $(143), (243)$	8
$2+2$	$(12)(34)$	$(12)(34), (13)(24)$, $(14)(23)$	3
4	(1234)	$(1234), (1243)$, $(1324), (1342)$, $(1423), (1432)$	6

$$|S_4| = 1 + 6 + 8 + 3 + 6.$$

<u>Remark</u>: If σ is an m-cycle then the no. of conjugates of σ (i.e the number of m-cycles) is

$$\frac{n(n-1)(n-2)\cdots(n-m+1)}{m}.$$

Q4 $\quad |G| = 1 \Rightarrow G = \{e\}$

If G has one conjugacy class then

Let G be a gp with 2 conjugacy clas

$$|G| = 1 + x.$$

$$\Rightarrow x \mid |G| \Rightarrow x \mid (1+x)$$

$$\Rightarrow x = 1.$$

$\therefore |G| = 2. \quad G \cong \mathbb{Z}_2.$

Let G be a gp with 3 conjugacy clas

$$|G| = 1 + x + y, \quad \text{where } 1 \le x \le y,$$

$$\Rightarrow x \mid |G| \ \& \ y \mid |G|.$$

Since $\quad |G| - x = 1+y$

$\Rightarrow \quad x \mid (1+y)$ similarly $y \mid (1+x)$.

$$x \leq (1+y) \leq 2+x.$$

$$\Rightarrow \quad x \leq y \leq 1+x.$$

So either $\quad y = x \quad$ or $\quad y = 1+x$.

<u>case 1</u>. $\quad y = x$. then $|G| = 1+2x$

$\because \quad x \mid |G| \Rightarrow x \mid 1 \quad$ So $x = y = 1$.

and $|G| = 3$.

$\therefore \quad G \cong \mathbb{Z}_3$.

<u>case 2</u>. $\quad y = x+1$, $\quad$ then $|G| = 2x+2$.

$\because x \mid |G| \Rightarrow x \mid 2 \quad \Rightarrow x = 1$ or 2.

If $x = 1$, $\quad y = 2$, $\quad \Rightarrow |G| = 4$.

But every gp of order 4 is abelian and so has 4 conjugacy class.

$\therefore$ $|G| = 4$ is not possible.

If $x = 2$, then $y = 3$.

$\Rightarrow |G| = 6$.

G has to be a non-abelian gp of order 6. Thus $G \cong S_3$.