Integral eq^n s

1. The classical theory of integral eq^n s - S.M. Zemyan
2. Linear Integral eq^n - R.P. Kansal
3. Integral eq^n - Sharma & Goyal

$$\frac{d^2y}{dx^2} = f(x) \Rightarrow y' = \int_0^x f(x) + y'(0)$$

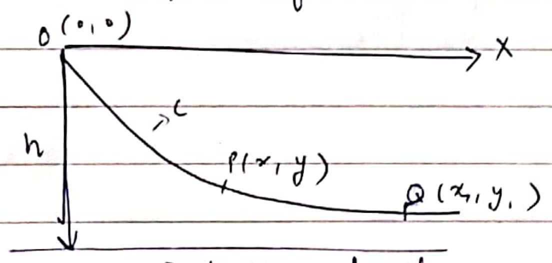
$$y(0)=1, \quad y'(0)=1 \Rightarrow \quad y(x) = y(0) + \int y'(0)\, dx + \int_0^x\int_0^x f(x)\,dx\,dx$$

$ **Brachistochrone Problem**

Greek word - brochi-to means shortest

chrone - time

Shortest time taken by the ball to reach the lowest pt. of the curve



Let us assume that the particle slides from its position of rest at the origin and reaches the point $P(x,y)$ along the curve C in time t s.t. $\widehat{OP} = s$. Let mass of the particle be m. By conservat^n of energy, we may equate the sum of K.E & P.E at O to the sum of K.E & P.E at P or Q. Taking h as the height of the pt P from the datum level.

The potential energies at O & P are mgh & $mg(h-y)$

The K.E at O & P are 0 & $\frac{1}{2}mv^2$, Then

$$mgh = mg(h-y) + \frac{1}{2}mv^2$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = mgy$$

$$\Rightarrow \quad v = \sqrt{2gy}$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{2gy}$$

$$\Rightarrow \quad \int_0^P dt = \int_0^P \frac{ds}{\sqrt{2gy}}$$

$$\Rightarrow \quad T = \int_0^y \frac{ds}{\sqrt{2gy}}$$

i.e. $\quad f(y) = \int_0^y \frac{ds}{\sqrt{2gy}}$

Let say C is parabola $y^2 = 4ax \Rightarrow s \checkmark \; d \checkmark$
we take $ds = u(\xi)\,d\xi$

$$\Rightarrow \quad f(y) = \int_0^y \frac{u(\xi)\,d\xi}{\sqrt{2gy}}.$$

§ Def^n: An integral eq^n is an eq^n in which one unknown f^n (which is to be determined) appears under one or more integerals. And if the derivatives of the unknown f^n is also present then its called integro-differential eq^n.

Ex:

1. $u(x) = f(x) + \lambda \int_a^b \underbrace{K(x,\xi)}_{\to \text{ kernel}} \cdot u(\xi)\,d\xi$ (non-homogeneous, linear)
2. $u(x) = \lambda \int_a^b K(x,\xi)\,u(\xi)\,d\xi$ (homo., linear)
3. $u(x) = \lambda \int_a^b K(x,\xi)\,u^3(\xi)\,d\xi$ (homo., non-linear)

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Most general form of an IE is

$$v(x)\,u(x) = f(x) + \lambda \int_a^b K(x,\xi)\,u^n(\xi)\,d\xi \,, \quad n \in \mathbb{Z}^+$$

Some types of IE :

1. Symmetric Kernel : The kernel is symmetric (or Hermitian) if

$$K(x,\xi) = \bar{K}(\xi, x)$$

eg. $K(x,\xi) = x^2 + \xi^2 \,,\ x+\xi \,,\ e^{\xi x}$

2. Separable Kernel : If $K(x,\xi) = \sum_{i=1}^{n} g_i(x)\,h_i(\xi)$ means K can be expressed as the sum of a finite number of terms, each of which is a product of functions of x & ξ only, then the kernel is called separable or degenerate. Obviously, $h_i(\xi)$ & $g_i(x)$ are independent.

3. Difference Kernel : A kernel of the form $K(x-\xi)$ is called difference kernel

Ex : $K(x,\xi) = (x-\xi)^2 = K(x-\xi)$

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8 Classificat^n of integral eq^{n}s.

An integral eq^n of type

$$v(x)\,u(x) = f(x) + \lambda \int_a^b K(x,\xi)\,u(\xi)\,d\xi \,, \text{ where } \quad \text{—(1)}$$

$u, v : D \to \mathbb{R}$, $K(x,\xi)$ is the kernel of (1) , $\lambda \in \mathbb{F}$ is called as Fredholm integral eq^n of third kind

i) when $v(x) = 0 \ \forall x \in D$. Then (1) reduces to

$$f(x) + \lambda \int_0^b K(x,\xi)\,u(\xi)\,d\xi = 0$$

ii.) when $v(x) = 1 \quad \forall x \in D$. Then ① reduces to

$$u(x) = f(x) + \lambda \int_a^b K(x,\xi)\, u(\xi)\, d\xi$$

iii.) when $f(x) = 0 \quad \forall x \in D$. Then ① reduces to

$$v(x)\, u(x) = \lambda \int_a^b K(x,\xi)\, u(\xi)\, d\xi$$

homogeneous Fredholm integral eq^n.

§ An integral of type,

$$v(x)\, u(x) = f(x) + \lambda \int_a^x K(x,\xi)\, u(\xi)\, d\xi \quad , \text{ where } - ① \quad a \le x \le b$$

$u, v : D \to \mathbb{R}$, $K(x,\xi)$ is the kernel of ①, $\lambda \in \mathbb{F}$ is called as Volterra integral eq^n of 3rd kind

i) when $v(x) = 0 \quad \forall x \in D$. Then ① reduces to

$$f(x) + \lambda \int_0^x K(x,\xi)\, u(\xi)\, d\xi = 0$$

ii) when $v(x) = 1 \quad \forall x \in D$. Then ① reduces to

$$u(x) = f(x) + \lambda \int_0^x K(x,\xi)\, u(\xi)\, d\xi.$$

iii.) when $f(x) = 0, \quad \forall x \in D$. Then ① reduces to

$$v(x)\, u(x) = \lambda \int_0^x K(x,\xi)\, u(\xi)\, d\xi$$

homogeneous Volterra int eq^n

§ Singular integral eq^n:

$$u(x) = \lambda \int_{-\infty}^{\infty} \frac{1}{|x-\xi|}\, u(\xi)\, d\xi.$$

Let us take $K(x,\xi) = \dfrac{1}{|x-\xi|} = K(x-\xi)$

convolut^n int eq^n : $u(x) = f + \lambda \int_a^b K(x_0 - \xi)\, u(\xi)\, d\xi$

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$x + l\bar{y}$ $\quad$ $x.\cancel{x} + l.x + x.\bar{y} + l.\bar{y}$ $\quad$ $x(l+\bar{y})$

x	l	y	$x + l\bar{y}$	$(x+l).(x+\bar{y})$
0	0	0	0.	0.
0	0	1		
0	1	0		
1	0	0		
1	1	0		
1	0	1		
0	1	1.		
1	1	1.		

X $\quad$ X $\quad$ X

Ex-1: Express $\frac{dy}{dx} = y\,a_1(x) + b$ in terms of integral eqⁿ of Volterra type & then find the solⁿ.

Solⁿ

$$y(x) = \int_a^x a_1(x)\, y\, dx + \int_a^x b\, dx$$

$$= \int_a^x a_1(\xi)\, y(\xi)\, d\xi + \int_a^x b\, d\xi$$

variatⁿ of Parameters

§ Eigenvalue & Eigenfⁿs of an integral eqⁿ:

Let us consider a homogenous F.I.E.

$$u(x) = \lambda \int_a^b K(x,\xi)\, u(\xi)\, d\xi \quad \text{--- ①}$$

$u(x) = 0$ is the trivial solⁿ.

The value of parameter λ for which ① possesses non-trivial solⁿ are called an eigenvalue of ①. The corresponding solⁿs $u(x)$ are called eigenfⁿ or fundamental fⁿ.

Ex. 1 Verify $u(x) = xe^x$ is a sol^n of the Volterra int eq^n.

$$u(x) = \sin x + 2\int_0^x \cos(x-\xi)\, u(\xi)\, d\xi$$

Sol^n

R.H.S. : $\sin x + 2\int_0^x \underbrace{\cos(x-\xi)}_{\text{II}}\, \underbrace{\xi e^\xi}_{\text{I}}\, d\xi$

$$= \left(\sin x + xe^x - \left[\frac{e^x}{2} - \frac{1}{2}(\cos x - \sin x)\right] - \left[\frac{e^x}{2}(-1) - \frac{1}{2}(-\sin x - \cos x)\right] \right) = xe^x = \text{L.H.S.}$$

Corresponding diff. eq s.t. $u(x) = xe^x$ is a sol^n.

Small Results:

1. **Leibnitz Rule of diff:**

$$\frac{d}{dx}\int_{P(x)}^{Q(x)} f(x,y)\, dy = f(x, Q(x))\frac{dQ}{dx} - f(x, P(x))\frac{dP}{dx} + \int_P^Q \underbrace{\frac{\partial f}{\partial x}}_{\text{must be continuous}} dy$$

2. Multiple integral of order n into single integral of order one.

$$\underbrace{\int_a^x \int_a^x \cdots \int_a^x}_{n} u(\xi)\, \underbrace{d\xi\, d\xi \cdots d\xi}_{n\text{-times}} = \int_a^x \frac{(x-\xi)^{n-1}}{(n-1)!}\, u(\xi)\, d\xi$$

Ex. 2 Show that $u(x) = \cos 2x$ is a sol^n of $u(x) = \cos x + 3\int_0^\pi k(x,\xi)\, u(\xi)\, d\xi$,

where $k(x,\xi) = \sin x \cos\xi, \quad 0 \le x \le \xi$

$= \cos x \sin\xi, \quad \xi \le x \le \pi$

Sol^n

R.H.S. : $\cos x + 3\int_0^\xi k(x,\xi)\, u(\xi)\, d\xi + \int_\xi^x k(x,\xi)\, u(\xi)\, d\xi$

Applicat^n of D.E's in IE's

1) Convers^n of an IVP to Volterra IES :

General n-th order ODE given by

$$\frac{d^n y}{dx^n} + \alpha_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \alpha_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \dots + \alpha_n(x)y = \phi(x) \quad \text{— (1)}$$

$a \le x \le b$.

$\bar{c}$ initial cond^n

$y(a) = a_0, \quad y'(a) = a_1, \dots, \quad y^{n-1}(a) = a_{n-1}$ where,
$a_i \ \forall i = 0, 1, \dots n-1$ is given.

Aim : Derive the integral eq^n s.t $y(x)$ is a sol^n

Proof Let $\frac{d^n y}{dx^n} = u(x)$. Then

$$\Rightarrow \frac{d^{n-1}(y(x))}{dx^{n-1}} - \frac{d^{n-1}y(a)}{dx^{n-1}} = \int_a^x u(x)\,dx$$

$$\Rightarrow y^{n-1}(x) = a_{n-1} + \int_a^x u(x)\,dx. \quad \text{— (3)}$$

Integrate again (3) from a to x,

$$y^{n-2}(x) = a_{n-2} + a_{n-1}\int_a^x dx + \int_a^x\int_a^x u(x)\,dx\,dx$$

$$= a_{n-2} + (x-a)a_{n-1} + \int_a^x (x-t)\,u(t)\,dt \quad \text{— (4)}$$

Integrate (4) w.r.t. x from a to x :

$$\Rightarrow y^{n-3}(x) = y^{n-3}(a) + (x-a)a_{n-2} + \frac{(x-a)^2 a_{n-1}}{2!} + \int_a^x\int_a^x (x-t)\,u(t)\,dt$$

$$= y^{n-3}(a) + (x-a)a_{n-2} + \frac{(x-a)^2}{2!}a_{n-1} + \int_a^x \frac{(x-t)^{3-1}}{(3-1)!}u(t)\,dt \quad \text{— (5)}$$

Proceeding in the same way & integrating upto the $y'(x)$

$$\frac{dy}{dx} = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} u(t)\,dt + a_{n-1}\frac{(x-a)^{n-2}}{(n-2)!} + a_{n-2}\frac{(x-a)^{n-3}}{(n-3)!} + \dots$$

$$\dots + a_2(x-a) + a_1$$

$$\Rightarrow \quad y(x) = \int_a^x \frac{(x-t)^n}{n!} u(t)\,dt + a_{n-1}\frac{(x-a)^{n-1}}{(n-1)!} + \dots + a_1(x-a) + a_2 \quad \text{—— (6)}$$

Subsitute (3), (4), (5) & (6) in (1), then

$$u(x) + \alpha_1(x) a_{n-1} + \int_a^x u(x)\,dx + \alpha_2(x)\Big[a_{n-2} + (x-a)a_{n-1} + \int_a^x (x-t)u(t)\,dt\Big] + \dots + \alpha_n(x)\Big[\int_a^x \frac{(x-t)^n}{n!} u(t)\,dt + a_{n-1}\frac{(x-a)^{n-1}}{(n-1)!} + \dots + a_0\Big] = \phi(x)$$

$$\Rightarrow \quad u(x) = -\alpha_1(x) a_{n-1} - \int_a^x u(x)\,dx - \alpha_2(x)[\,-\,] - \dots - \alpha_n[\dots] + \phi(x)$$

$$\Rightarrow \quad \psi(x) = \alpha_1(x) a_{n-1} + \{a_{n-2} + (x-a)a_{n-2}\}\alpha_2(x) + \{a_0 + a_1(x-a) + \dots + a_{n-1}\frac{(x-a)^{n-1}}{(n-1)!}\}\alpha_n(x) + \phi(x)$$

$$k(x,t) = -\Big[\alpha_1(x) + (x-t)\alpha_2(x) + \dots + \frac{(x-t)^{n-1}}{(n-1)}\alpha_n(x)\Big],$$

then,

$$\Rightarrow \quad u(x) = \underset{f(x)}{\underset{\vee}{\overset{\phi(x)-}{\psi(x)}}} + \int_a^x k(x,t)\,u(t)\,dt.$$

Ex Transform the$^{\text{I.V.P. to}}$ Volterra I.E's :

$$\frac{d^2y}{dx^2} + xy = 1, \quad y(0) = 0, \quad y'(0) = 0.$$

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Sol^n given $y''(x) + xy = 1$, let $y''(x) = u(x)$

$\Rightarrow \frac{dy}{dx} - \frac{dy(0)}{dx} = \int_0^x u(x)\,dx$

$\Rightarrow y'(x) = \int_0^x u(x)\,dx$

$\Rightarrow y(x) = y(0) + \int_0^x\int_0^x u(x)\,(dx)^2$

$= \int_0^x (x-t)\,u(t)\,dt$

Putting all these in the given eq^{n}s,

$u(x) + x\int_0^x (x-t)\,u(t)\,dt = 1$

$\Rightarrow u(x) = 1 - x\int_0^x (x-t)\,u(t)\,dt$

§ Convers^n of BVP to Fredholm Integral eq^{n}s.

Ex-1 Reduce the given BVP to FIE.

$\frac{d^2u}{dx^2} + \lambda u = 0$ $\bar{c}$ $u(0) = 0$, $u(l) = 0$

Sol^n The given DE is

$u''(x) = -\lambda u(x)$

$\Rightarrow \int_0^x u''(x)\,dx = -\lambda\int_0^x u(x)\,dx$

$\Rightarrow u''(x) - u''(0) = -\lambda\int_0^x u(x)\,dx$

let us take $u'(0) = c$, then

$u'(x) = c - \lambda\int u(x)\,dx$

$u(x) = u(0) + cx - \lambda\int_0^x\int_0^x u(x)\,dx$

$= cx - \lambda\int_0^x (x-t)\,u(t)\,dt$

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take $x = l$

$\Rightarrow \quad 0 = u(l) = cl - \lambda\int_0^l (l-t)\,u(t)\,dt$

$$c = \frac{\lambda}{l}\int_0^l (l-t)\,u(t)\,dt$$

$\Rightarrow \quad u(x) = \frac{\lambda x}{l}\int_0^l (l-t)\,u(t)\,dt - \lambda\int_0^x (x-t)\,u(t)\,dt$

$$= \frac{\lambda x}{l}\int_0^x (l-t)\,u(t)\,dt + \int_x^l (l-t)\,u(t)\,dt - \lambda\int_0^x (x-t)\,u(t)\,dt$$

$$= \frac{\lambda}{l}\int_0^x x(l-t)\,u(t)\,dt + \int_x^l \frac{\lambda x(l-t)}{l}\,u(t)\,dt - \lambda\int_0^x (x-t)\,u(t)\,dt$$

$\Rightarrow \quad u(x) = \lambda\left[\int_0^x \frac{t(l-x)}{l}\,u(t)\,dt + \int_x^l \frac{x(l-t)}{l}\,u(t)\,dt\right]$

$\Rightarrow \quad u(x) = \lambda\int_0^l K(x,t)\,u(t)\,dt,$

where $K(x,t) = \frac{t(l-x)}{l}$ if $0 < t < x$

$= \frac{x(l-t)}{l}$ if $0 < t < l.$

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~~The Fredholm eq^n is given by~~ :

§ Relat^n b/w EigenValue Problem & IE :

The Fredholm eq^n is given by

$$\phi(x) = f(x) + \lambda \int_0^1 (xt^2 + x^2t^4)\,\phi(t)\,dt \quad \text{—} (1)$$

where $\phi(x)$ is the unknown f^n & $K(x,t) = xt^2 + x^2t^4$ is the given kernel. If we set

$$c_1 = \int_0^1 t^2\phi(t)\,dt, \qquad c_2 = \int_0^1 t^4\,\phi(t)\,dt$$

Then (1) reduces to

$$\phi(x) = f(x) + \lambda c_1 x + \lambda c_2 x^2 \quad \text{—} (2)$$

where c_1 & c_2 needed to be determined.

Replacing x by t in (2) & multiplying both sides by t^2,

$$c_1 = \int_0^1 t^2\phi(t)\,dt = \int_0^1 f(t)t^2\,dt + \lambda c_1\int_0^1 t^3\,dt + \lambda c_2\int_0^1 t^4\,dt$$

$$= \int_0^1 t^2 f(t)\,dt + \frac{\lambda c_1}{4} + \frac{\lambda c_2}{5} \quad \text{—} (3)$$

Again we multiply (2) by 't^4' after taking '$x = t$' then

$$c_2 = \int_0^t t^4\phi(t)\,dt = \int_0^1 t^4 f(t)\,dt + \frac{\lambda c_1}{6} + \frac{\lambda c_2}{7} \quad \text{—} (4)$$

We define $F_1 = \int_0^1 t^2 f(t)\,dt$, $F_2 = \int_0^1 t^4 f(t)\,dt$, Then (3) & (4) reduce to :

$$c_1 = F_1 + \frac{\lambda c_1}{4} + \frac{\lambda c_2}{5}$$

$$c_2 = F_2 + \frac{\lambda c_1}{6} + \frac{\lambda c_2}{7}$$

$$\Rightarrow \left(1 - \frac{\lambda}{4}\right)c_1 - \frac{\lambda c_2}{5} = F_1 \qquad \Rightarrow \begin{pmatrix} 1-\lambda/4 & -\lambda/5 \\ -\lambda/6 & 1-\lambda/7 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

$$-\frac{\lambda}{6}c_1 + \left(1 - \frac{\lambda}{7}\right)c_2 = F_2$$

$$\Rightarrow \hat{A}\,X = B$$

$$\hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \lambda\underbrace{\begin{pmatrix} 1/4 & 1/5 \\ 1/6 & 1/7 \end{pmatrix}}_{A} \Rightarrow \hat{A} = I - \lambda A$$

$\det(A_1 - \mu I) = 0$.

fine $\lambda = \frac{1}{\mu}$. λ will be an eigen value

Suppose A_1 is a matrix & μ is its eigenvalue.

The value of λ is called the 'eigenvalues of the kernel'

Ex-1: Transform the BVP,

$$\frac{d^2y}{dx^2} + y = x \quad , y(0) = 1 \quad , y'(1) = 0$$

into Fredholm Integral eq^n.

Sol^n The given eq^n is defined b/w 0 to 1 $[0,1]$

Now,
$$\frac{d^2y}{dx^2} + y = x$$

$$\Rightarrow \int_0^x d\left(\frac{dy}{dx}\right) + \int_0^x y\,dx = \int_0^x x\,dx$$

$$\Rightarrow \frac{dy}{dx} - y'(0) = \frac{x^2}{2} - \int_0^x y\,dx$$

$$\Rightarrow \frac{dy}{dx} = \underbrace{y'(0)}_{c} + \frac{x^2}{2} - \int_0^x y\,dx$$

$$\Rightarrow y' = c + \frac{x^2}{2} - \int_0^x y\,dx \quad \text{— (i)}$$

Integrating w.r.t x from 0 to x

$$\Rightarrow y(x) = y(0) + cx + \frac{x^3}{6} - \int_0^x \int_0^x y\,dx\,dx$$

$$\Rightarrow y(x) = y(0) + cx + \frac{x^3}{6} - \int_0^x (x-\xi)\,y(\xi)\,d\xi \quad \text{— (ii)}$$

Putting $y(0) = 1$ in (ii),

$$1 = 1 + c\cdot 0 + 0 - 0$$

Putting $y'(1) = 0$ in (i)

$$0 = c + \frac{1}{2} - \int_0^1 y\,dx$$

$$\Rightarrow c = \int_0^1 y\,dt - \frac{1}{2} \quad \text{— (iii)}$$

Combining (ii) & (iii)

$$y(x) = 1 + x\int_0^1 y\,dt - \frac{1}{2}x + \frac{x^3}{6} - \int_0^x (x-\xi)\,y(\xi)\,d\xi$$

$$= 1 - \frac{x}{2} + \frac{x^3}{6} + \int_x^1 x\,y(\xi)\,d\xi + \int_0^x \xi\,y(\xi)\,d\xi$$

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$$= 1 - \frac{x}{2} + \frac{x^3}{6}\int_0^1 K(x,\xi)\,y(\xi)\,d\xi ,$$

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where , $K(x,\xi) = x , \quad x < \xi$

$= \xi , \quad x > \xi$

Chapter - 3 : Solⁿ of Fredholm Integral Eqⁿ of 2nd kind :

§ Characteristics values & characteristics fⁿ (eigenvalue & eigen fⁿ)

Let us consider,

$$u(x) = \lambda\int_a^b K(x,t)\,u(t)\,dt \quad \text{——} ①$$

obviously $u(x) = 0$ will be a solⁿ, trivial solⁿ. To look for the solⁿ of ① s.t. $u(x) \neq 0$. ∀ such values of $\lambda \in \mathbb{R}$ s.t. $u(x) \neq 0$. is a solⁿ is called eigenvalues / chara. of the kernel $K(x,t)$.

For the eigenvalues $\lambda \in \mathbb{R}$ ∃ Eigenfⁿs s.t.

$$\phi(x) = \lambda_0\int_a^b K(x,t)\,\phi_0(t)\,dt$$

ϕ_0 would also be a solⁿ of ①

$A\phi = \lambda\phi$

$Ac\phi = c\lambda\phi$

$\phi_1 = c\phi$

$\phi_1 - c\phi = 0$

Remark(i.) $\lambda \neq 0$. If $\lambda = 0 \Rightarrow u(x) = 0$. Not needed.

(ii.) ϕ_0 is the eigenfⁿ corresponding to λ_0, $c\phi_0$ would also be an eigenfⁿ for ① but linearly dependent.

(iii.) A homogeneous Fredholm I.E.s of type (1) may not have any eigenvalues if the kernel is not symmetric.

Ex-1 : Calculate the Eigenvalues & Eigenfⁿs of

$$g(x) = \lambda\int_0^1 e^{x+t}\,g(t)\,dt$$

Solⁿ: From ①, $g(x) = \lambda e^x \underbrace{\int_0^1 e^t g(t)\,dt}_{c}$; c is a constant

Then $g(x) = \lambda c e^x$ or $g(t) = \lambda c e^t$

Now, $c = \int_0^1 e^t g(t)\,dt = \int_0^1 e^t \lambda c e^t\,dt = \int_0^1 \lambda c e^{2t}\,dt$

$\Rightarrow \quad 1 = \lambda\int_0^1 e^{2t}\,dt , \quad c \neq 0$

$\Rightarrow \quad 1 = \dfrac{\lambda(e^2-1)}{2} \quad \Rightarrow \quad \lambda = \dfrac{2}{e^2-1}$

Now for $\lambda = \dfrac{2}{e^2-1}$, the corresponding eigen f^n is

$$g(x) = \lambda c e^x = \frac{2}{e^2-1} c e^x = \frac{2c}{e^2-1} e^x = \hat{c} e^x$$

Ex-2: Find the eigenvalues of $\phi(x) = \lambda \int_0^1 (3x-2) t \phi(t) dt$.

Sol^n The given eq^n: $\phi(x) = \lambda(3x-2) \underbrace{\int_0^1 t\phi(t) dt}_{= C}$

This gives $\phi(x) = \lambda(3x-2) C$

$\Rightarrow \quad \phi(t) = \lambda c(3t-2)$

Now, $\Rightarrow c = \int_0^1 t\phi(t) dt = \int_0^1 t\lambda c(3t-2) dt$

$$= \lambda c \int_0^1 (3t^2 - 2t) dt = \lambda c \left(t^3 - t^2 \right)\Big|_0^1$$

$$= 0$$

$\Rightarrow \quad c = 0$

trivial sol^n $\nexists$ any $\lambda \in \mathbb{R}$ s.t ① has an eigenvalue.

§ Sol^n of a Fredholm I.E $\bar{c}$ Separable Kernel:

1. Orthogonality of two f^ns:

Let f & g : $[a,b] \longrightarrow \mathbb{R}$, then

$\langle f, g \rangle = \int_a^b f(x) g(x) dx = 0 \Rightarrow f$ & g are orthogonal.

Lebesgue Integrat^n $f, g \in L^2(a,b)$, then $\int_a^b f(x) g(x) dx = 0$

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▫ Sol^n of Homogeneous / first kind Fredholm IE of Separable Kernel & degenerate Kernel :

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Let us start $\bar{c}$ homogeneous I.E.

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt \quad -(1)$$ where $\lambda \in \mathbb{R}$ is the eigen value & $K(x,t)$ is separable.

i.e. $$k(x,t) = \sum_{i=1}^{n} f_i(x)\, g_i(t), \text{ where } f_i : \mathbb{R} \to \mathbb{R},\ g_i : \mathbb{R} \to \mathbb{R} \quad -(2)$$

so by (1) & (2)

$$u(x) = \lambda \sum_{i=1}^{n} \int_a^b u(t) f_i(x) g_i(t)\, dt$$

$$= \lambda \sum_{i=1}^{n} f_i(x) \int_a^b u(t) g_i(t)\, dt \quad -(3)$$

We define $$c_i = \int_a^b u(t) g_i(t)\, dt \quad \forall\ i = 1, 2, \dots, n \quad -(4)$$

by (3) & (4)

$$u(x) = \lambda \sum_{i=1}^{n} f_i(x) c_i$$

multiplying (5) by $g_i(x)$ for $i = 1, 2, \dots, n$ succesively & integrating

$$\left.\begin{aligned}
\int_a^b u(x) g_1(x)\, dx &= \lambda \int_a^b \sum_{i=1}^{n} f_i(x) c_i g_1(x)\, dx \\
\int_a^b u(x) g_2(x)\, dx &= \lambda \sum_{i=1}^{n} \int_a^b f_i(x) c_i g_2(x)\, dx \\
&\ \ \vdots \\
\int_a^b u(x) g_n(x)\, dx &= \lambda \sum_{i=1}^{n} \int_a^b f_i(x) c_i g_n(x)\, dx
\end{aligned}\right] \quad -(6)$$

$$\therefore \cancel{\int_a^b u(x) g_j(x)\,dx = \lambda \sum_{i=1}^{n} \int_a^b f_i(x) g_j(x)\,dx \quad \forall j = 1,2,\dots,n}$$

$$\int_a^b u(x) g_j(x)\,dx = \lambda \sum_{i=1}^{n} c_i \int_a^b f_i(x) g_j(x)\,dx \quad \forall j=1,2,\dots,n \qquad (7)$$

$$\Rightarrow \int_a^b u(x) g_j(x)\,dx = \lambda \sum_{i=1}^{n} c_i \alpha_{ij} \quad \text{where} \quad \alpha_{ij} = \int_a^b f_i(x) g_j(x)\,dx, \quad \forall i,j = 1,2,\dots,n$$

for every j,

$$c_1 = \lambda \sum_{i=1}^{n} c_i \alpha_{i1} = \lambda c_1 \alpha_{11} + \lambda c_2 \alpha_{21} + \dots + \lambda c_n \alpha_{n1}$$

$$c_2 = \lambda \sum_{i=1}^{n} c_i \alpha_{i2} = \lambda c_1 \alpha_{12} + \lambda c_2 \alpha_{22} + \dots + \lambda c_n \alpha_{n2}$$

$$\vdots$$

$$c_n = \lambda \sum_{i=1}^{n} c_i \alpha_{in} = \lambda c_1 \alpha_{1n} + \lambda c_2 \alpha_{2n} + \dots + \lambda c_n \alpha_{nn}$$

$$\Rightarrow \left.\begin{aligned} &(1-\lambda\alpha_{11}) c_1 - \lambda \alpha_{21} c_2 - \dots - \lambda c_n \alpha_{n1} = 0 \\ &\cancel{(1-\lambda\alpha_{1n}) c_n - \lambda \alpha_{2n} c_2 - \dots - \lambda \alpha_{nn} c_n = 0} \\ &-\lambda c_1 \alpha_{12} + (1-\lambda\alpha_{22}) c_2 \dots - \lambda c_n \alpha_{n2} = 0 \\ &\vdots \\ &-\lambda c_1 \alpha_{1n} - \lambda c_2 \alpha_{2n} \dots - (1-\lambda\alpha_{nn}) = 0 \end{aligned}\right\} \qquad (8)$$

For the system of eq^{n}s (8), the determinant $D(\lambda)$ is given by

$$D(\lambda) = \begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{21} & \dots & -\lambda\alpha_{n1} \\ -\lambda\alpha_{12} & 1-\lambda\alpha_{22} & \dots & -\lambda\alpha_{n2} \\ \vdots & & & \\ -\lambda\alpha_{1n} & -\lambda\alpha_{2n} & \dots & +(1-\lambda\alpha_{nn}) \end{vmatrix}$$

For non-zero sol^n $D(\lambda) = 0$

$\Rightarrow \lambda = \lambda_1, \lambda_2, \dots, \lambda_n$

For each $\lambda_1 \to \phi_1 \to$ a sol^n to (1)

$\lambda_2 \to \phi_2 \to$ a sol^n to (1)

$\vdots$

$\lambda_n \to \phi_n \to$ " (1)

- Orthogonality of Eigen f^n:

f, g are said to be orthogonal on $[a,b]$ if

(L^2) $$\int_a^b f(x) g(x)\,dx = 0$$

$$\int_a^b fg\,dx \le \left(\int_a^b f^2 dx\right)^{1/2} \left(\int_a^b g^2 dx\right)^{1/2} = \|f\|_{L^2} \|g\|_{L^2}$$

Prop 1: The eigenf^n's of a symmetric kernel corresponding to two different eigen values are orthogonal.

Proof: Let us consider an integral eq^n of type

$$u(x) = \lambda \int_a^b K(x,t) u(t)\,dt, \quad \text{where } K(x,t) = K(t,x) \quad — ①$$

Let λ_0, λ_1 be the eigenvalues & ϕ_0, ϕ_1 be the corresponding eigenf^{n}s resp. where $\lambda_0 \ne \lambda_1$. Then we show that ϕ_0 & ϕ_1 are orthogonal.

By def^n,

$$\phi_0(x) = \phi_0 = \lambda_0 \int_a^b K(x,t) \phi_0(t)\,dt \quad — ②$$

$$\phi_1(x) = \phi_1 = \lambda_1 \int_a^b K(x,t) \phi_1(t)\,dt \quad — ③$$

$$\Rightarrow \int_a^b \phi_0(x)\phi_1(x)\,dx = \lambda_0 \int_a^b \phi_1(x) \left\{ \int_a^b K(x,t) \phi_0(t)\,dt \right\} dx.$$

$$= \lambda_0 \int_a^b \phi_0(t) \left\{ \int_a^b K(x,t) \phi_1(x)\,dx \right\} dt$$

$$= \lambda_0 \int_a^b \phi_0(t) \left\{ \underbrace{\int_a^b K(t,x) \phi_1(x)\,dx}_{\text{Put } x \to t,\ t \to x} \right\} dt$$

$$= \lambda_0 \int_a^b \phi_0(t) \left\{ \int_a^b K(x,t) \phi_1(t)\,dt \right\} dt$$

$$= \lambda_0 \int_a^b \phi_0(t) \frac{\phi_1(x)}{\lambda_1}\,dt$$

$$= \frac{\lambda_0}{\lambda_1} \int_a^b \underbrace{\phi_0(t)\phi_1(x)\,dt}_{t \to x}$$

$$\Rightarrow \lambda_1 \int_a^b \phi_0(x)\phi_1(x)\,dx = \lambda_0 \int_a^b \phi_0(x)\phi_1(x)\,dx$$

$$\Rightarrow (\lambda_1 - \lambda_0) \int_a^b \phi_0 \phi_1\,dx = 0$$

23/1/2020
Thursday

2.) The eigenvalue of a symmetric kernel is real.

Sol^n Let $\lambda_0 = \alpha + i\beta$ be an eigen value of I.F.

$$u(x) = \lambda \int_a^b K(x,t)\,u(t)\,dt \quad \text{—— ①}$$

& $\phi_0 = u + iv$ be the corresponding eigen f^n, then

$$\bar{\lambda}_0 = \alpha - i\beta \quad \& \quad \bar{\phi}_0(x) = u(x) - iv(x) \quad \text{— ②}$$

Also,

$$\phi_0(x) = \lambda_0 \int_a^b K(x,t)\,\phi_0(t)\,dt \quad \text{— (3a)}$$

$$\bar{\phi}_0(x) = \bar{\lambda}_0 \int_a^b K(x,t)\,\bar{\phi}_0(t)\,dt \quad \text{— (3b)}$$

Multiply (3a) by $\bar{\phi}_0(x)$ & integrate

$$\int_a^b \phi_0(x)\,\bar{\phi}_0(x)\,dx = \lambda_0 \int_a^b \left(\int_a^b K(x,t)\,\phi_0(t)\,dt \right) \bar{\phi}_0(x)\,dx$$

$$\int_a^b \phi_0(x)\, \bar{\phi}_0(x)\, dx = \lambda_0 \int_a^b \bar{\phi}_0(x) \left\{ \int_a^b K(x,t)\, \phi_0(t)\, dt \right\} dx$$

$$= \lambda_0 \int_a^b \phi_0(t) \left\{ \int_a^b K(x,t)\, \bar{\phi}_0(x)\, dx \right\} dt$$

$$= \lambda_0 \int_a^b \phi_0(t) \left\{ \int_a^b K(t,x)\, \bar{\phi}_0(x)\, dx \right\} dt \qquad \{\because K(x,t) = K(t,x)\}$$

interchanging $t \to x$ & $x \to t$ on the r.h.s.

$$\int_a^b \phi_0(x)\, \bar{\phi}_0(x)\, dx = \lambda_0 \int_a^b \phi_0(x) \left\{ \int_a^b K(x,t)\, \bar{\phi}_0(t)\, dt \right\} dx$$

$$= \lambda_0 \int_a^b \phi_0(x)\, \frac{\bar{\phi}_0(x)}{\bar{\lambda}_0}\, dx$$

$$\Rightarrow \quad (\lambda_0 - \bar{\lambda}_0) \int_a^b \phi_0(x)\, \bar{\phi}_0(x)\, dx = 0$$

$\because$ $\phi_0 \bar{\phi}_0 = u^2 + v^2$, the integral > 0. The only possibility is

$\lambda_0 - \bar{\lambda}_0 = 0 \quad \Rightarrow \quad 2i\beta = 0 \quad \Rightarrow \quad \beta = 0 \quad \Rightarrow \quad \lambda \in \mathbb{R}$

eg Show that the homogeneous integral eq^n

$$\phi(x) = \lambda \int_0^1 (t\sqrt{x} - x\sqrt{t})\, \phi(t)\, dt \qquad \text{— (1)}$$

has no real eigenvalue & no eigenf^n.

Sol^n The given eq^n (1) can be written as

$$\phi(x) = \lambda \underbrace{\int_0^1 t\sqrt{x}\, \phi(t)\, dt}_{c_1} - \lambda \underbrace{\int_0^1 x\sqrt{t}\, \phi(t)\, dt}_{c_2}$$

$$= \lambda c_1 \sqrt{x} - \lambda c_2 x, \text{ where} \qquad \text{— (2)}$$

$$c_1 = \int_0^1 t\, \phi(t)\, dt, \quad c_2 = \int_0^1 \sqrt{t}\, \phi(t)\, dt \qquad \text{— (3)}$$

Now, from (3) & (2)

$$c_1 = \int_0^1 t\left[\lambda c_1 \sqrt{t} - \lambda c_2 t\right] dt = c_1 \lambda \left[\frac{t^{5/2}}{5/2}\right] - \left[t^3/3\right] \lambda c_2$$

$$\Rightarrow \quad c_1\left(1 - \frac{2\lambda}{5}\right) + \frac{\lambda}{3} c_2 = 0 \qquad \text{— (4)}$$

Slly by ③ & ④

$$-\frac{\lambda}{2}c_1 + \left(1+\frac{2\lambda}{5}\right)c_2 = 0 \quad —⑤$$

For non-zero c_1 & c_2 :

$$\begin{vmatrix} 1-2\lambda/5 & \lambda/3 \\ -\lambda/2 & 1+2\lambda/5 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1 + 4\lambda^2/25 + \lambda^2/6 = 0$$ No real values of λ

$$\lambda = \pm i\sqrt{150}$$

eg Find the eigenvalue of

$$\phi(x) = \lambda\int_0^{\pi}(\cos^2 x \cos 2t + \cos 3x \cos^3 t)\,\phi(t)\,dt \quad —①$$

Sol^n Here $K(x,t) = \cos^2 x \cos 2t + \cos 3x \cos^3 t$

$$K(t,x) = \cos^2 t \cos 2x + \cos 3t \cos^3 x$$

$$\Rightarrow \quad K(x,t) \neq K(t,x)$$

The eq^n ① can be written as

$$\phi(x) = \lambda\cos^2 x\underbrace{\int_0^{\pi}\cos 2t\,\phi(t)\,dt}_{c_1} + \lambda\cos 3x\underbrace{\int_0^{\pi}\cos^3 t\,\phi(t)\,dt}_{c_2}$$

i.e., $\phi(x) = \lambda c_1 \cos^2 x + \lambda c_2 \cos 3x$

i.e., $\phi(t) = \lambda c_1 \cos^2 t + \lambda c_2 \cos 3t \quad —②$

Proceeding in the same way as in previous examples

$$c_1 = \int_0^{\pi}\cos 2t\,\phi(t)\,dt \Rightarrow c_1\left[1-\frac{\lambda\pi}{4}\right] - 0c_2 = 0$$

$$\Rightarrow c_1\left[1-\frac{\lambda\pi}{4}\right] = 0 \quad —③$$

$$c_2 = \int_0^{\pi}\cos^3 t\,\phi(t)\,dt \Rightarrow 0c_1 + c_2\left(1-\frac{\lambda\sqrt{\pi}}{8}\right) = 0$$

$$c_2\left(1-\frac{\lambda\sqrt{\pi}}{8}\right) = 0 \quad —④$$

From ③ & ④

$\lambda = \frac{4}{\lambda}$, $\lambda = \frac{8}{\lambda}$

30/01/2020

Thursday

real eigen value $\Leftarrow$ symmetric kernel but not vice-versa

Solⁿ of FIE of 2nd kind:

$u(x) = F(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt$, where kernel $K(x,t)$ is separable — ①

i.e. $K(x,t) = \sum_{i=1}^{n} f_i(x) g_i(t)$ — ②

$\therefore$

$$u(x) = F(x) + \lambda \sum_{i=1}^{n} f_i(x) \int_a^b u(t) g_i(t)\, dt \quad — ②$$

Let us assume,

$$c_i = \int_a^b u(t) g_i(t)\, dt \quad \forall\ i = 1, \ldots, n \quad — ③$$

$$\Rightarrow u(x) = F(x) + \lambda \sum_{i=1}^{n} f_i(x) c_i \quad — ④$$

to determine c_i's multiply ④ by $g_1(x), \ldots, g_n(x)$ one by one respectively.

$$\left.\begin{aligned} \int_a^b u(x) g_1(x)\, dx &= \int_a^b F(x) g_1(x)\, dx + \lambda \sum_{i=1}^{n} c_i \int_a^b f_i(x) g_1(x)\, dx \\ &\vdots \\ \int_a^b u(x) g_n(x)\, dx &= \int_a^b F(x) g_n(x)\, dx + \lambda \sum_{i=1}^{n} c_i \int_a^b f_i(x) g_n(x)\, dx \end{aligned}\right\}$$

— ⑤

Let us define

$$\alpha_{ij} = \int_a^b g_i(x) f_j(x)\, dx \quad \forall\ i,j = 1, 2, \ldots, n$$

$$\beta_j = \int_a^b F(x) g_j(x)\, dx \quad \forall\ j = 1, 2, \ldots, n$$

By eqⁿ (v.)

$$c_1 = \beta_1 + \lambda \sum_{i=1}^{m} c_i \alpha_{i1}$$

$$c_2 = \beta_2 + \lambda \sum_{i=1}^{m} c_i \alpha_{i2}$$

$$\vdots$$

$$c_n = \beta_m + \lambda \sum_{i=1}^{n} c_i \alpha_{in}$$

$$\Rightarrow \left.\begin{aligned} (1-\lambda\alpha_{11})c_1 - \lambda c_2 \alpha_{11} - \cdots - \lambda c_n \alpha_{1n} &= \beta_1 \\ -\lambda c_1 \alpha_{12} + (1-\lambda\alpha_{22})c_2 \cdots - \lambda c_n \alpha_{n2} &= \beta_2 \\ &\vdots \\ -\lambda c_1 \alpha_{1m} - \lambda c_2 \alpha_{2n} \cdots + (1-\lambda\alpha_{mn})c_m &= \beta_n \end{aligned}\right\} \text{— (6)}$$

For the existence of solⁿ (unique / infinitely many / no) we must calculate Determinant of (6)

$$D(\lambda) = \begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{21} & \cdots & -\lambda\alpha_{m1} \\ -\lambda\alpha_{12} & (1-\lambda\alpha_{22}) & \cdots & -\lambda\alpha_{n2} \\ \vdots & & & \\ -\lambda\alpha_{1m} & -\lambda\alpha_{2n} & \cdots & +(1-\lambda\alpha_{mm}) \end{vmatrix} = 0$$

$\Rightarrow D(0) = 1$, i.e. $D(0) \neq 0 \Rightarrow \lambda = 0$ is not an eigenvalue.

several cases may arise.

Case 1: when $F(x) = 0$, then $\beta_j = 0 \;\forall\, j = 1, 2, \ldots, n$. ∴ if (a) $D(\lambda) \neq 0$ then will imply all $c_i = 0$, $i = 1, 2, \ldots, n$ & gives $u(x) \equiv 0$.

(b) $D(\lambda) = 0$, then any non-zero solⁿ of (5) will give a solⁿ of the IE (1)

Case 2: when $F(x) \neq 0$

(a) $F(x)$ & $g_j(x)$'s are orthogonal i.e. $\int_a^b F(x) g_j(x)\,dx = 0 \quad \forall j = 1, 2, \ldots, n$

If $D(\lambda) \neq 0$, then unique solⁿ of (5) is $c_i = 0 \;\forall\, i = 1, \ldots, n$. This gives $u(x) = F(x)$ as a solⁿ.

If $D(\lambda) = 0$, then the system ⑤ provides infinite non-zero solⁿ.

$\therefore$ eqⁿ ① has infinite non-zero solⁿ, together $\bar{c}$ $F(x)$.

Case 3: Atleast one of the $\beta_j \neq 0$ $\forall j = 1, \ldots, n$. Then if $D(\lambda) \neq 0$, we get non-zero unique solⁿ of the system ⑥. This gives a non-zero solⁿ of the eqⁿ ①.

If $D(\lambda) = 0$, No solⁿ at all. Then doesn't exist any solⁿ to ①.

Ex-1 Solve $g(s) = f(s) + \lambda \int_0^1 st\, g(t)\, dt$ — ①

Solⁿ ~~Let us take~~

Here $K(s, t) = st = \phi(s)\psi(t)$, $(\phi(s) = s,\ \psi(t) = t)$

Let us take $c = \int_0^1 t g(t) dt$, then the solⁿ

$$g(s) = f(s) + \lambda s c \quad \text{— ②}$$

$$g(t) = f(t) + \lambda t c \quad \text{— ③}$$

Multiply ③ by t & integrate,

$$\int_0^t t g(t) dt = \int_0^1 t f(t) dt + \lambda c \int_0^1 t^2 dt$$

$$\Rightarrow c = \int_0^1 t f(t) dt + \frac{\lambda c}{3}$$

$$\Rightarrow c - \frac{\lambda c}{3} = \int_0^1 t f(t) dt$$

$$\Rightarrow c = \frac{3}{3-\lambda} \int_0^1 t f(t) dt \quad \text{for } \lambda \neq 3$$

$\therefore$ the eigenvalue of ① are $\lambda \in \mathbb{R}$ s.t. $\lambda \neq 3$ & the solⁿ is given by

$$g(s) = f(s) + \frac{3\lambda s}{\lambda - 3} \int_0^1 t f(t) dt.$$

Ex Solve $\phi(x) = \cos x + \lambda \int_0^{\pi} \sin x\, \phi(t)\, dt$.

Solⁿ $K(x, t) = \sin x \cdot 1 \Rightarrow K(t, x) = \sin t \cdot 1$

$\lambda = 1/2$, $c = \int_0^{\pi} \phi(t) dt \Rightarrow c(1 - 2\lambda) = 0$

$\lambda = 1/2 \Rightarrow c \neq 0 \Rightarrow$ a unique solⁿ.

$\lambda \neq 1/2 \Rightarrow c = 0 \Rightarrow \phi(x) = \cos x$

□ Volterra IE $\bar{c}$ Symmetric Kernel :

Def^n: A Kernel $K(x,t)$ of an integral eq^n is said to be sym. (also for complex ~~f~~^{n} Hermit^n) if

$$K(x,t) = \overline{K(t,x)}$$

For real kernels, $K(x,t) = \overline{K(t,x)} = K(t,x)$, $\forall x,t \in D \subset \mathbb{R}^2$

Thm: If a kernel is symmetric, then all the iterated kernels are also symmetric.

Pf: (Iterated Kernels) For Fredholm IEs of 2nd kind, we define the iterated kernel $K_n(x,\xi)$ for $n=1,2,\ldots$, as below.

$$K_1(x,\xi) = K(x,\xi)$$
$$K_n(x,\xi) = \int_a^b K(x,\xi)\, K_{n-1}(z,\xi)\, dz, \quad n=2,3,\ldots$$

Slly, for Volterra's IEs of 2nd, we define the iterated kernel $K_n(x,\xi)$ as

$$K_1(x,\xi) = K(x,\xi)$$
$$K_n(x,\xi) = \int_\xi^x K(x,z)\, K_{n-1}(z,\xi)\, dz, \quad n=2,3,\ldots$$

Pf

Let $K(x,t)$ be a sym kernel, $K(x,t) = \overline{K(x,t)}$, by definit^n

$$K_1(x,t) = K(x,t) \qquad \text{—— } 1(a)$$
$$K_n(x,t) = \int_a^b K(x,z)\, K_{n-1}(z,t)\, dz, \text{ where } n \in \mathbb{Z}^+ \setminus 1 \qquad \text{—— } (1b)$$

Furthermore, $\cancel{K_n(z,t) = \int K_{n-1}(x,z)\, K(z,x)\, dx}$ —— (1c) (substitute $x \to z$ & $z \to x$)

$$K_n(x,t) = \int_a^b K_{n-1}(x,z)\, K(z,t)\, dz \longrightarrow \text{Proof ??}$$

Use Mathematical Induct^n, For $n=2$ we have

$$K_2(x,t) = \int_a^b K(x,z)\, K_1(z,t)\, dz \text{ from (1b)}$$
$$= \int_a^b \bar{K}(z,x)\, \bar{K}_1(t,z)\, dz = \bar{K}_2(t,x)$$

Also $\because \bar{K}(z,x) = K(x,z)$, $\overline{K_1}(t,z) = K_1(z,t)$

Proceeding in the same way, we assume that $K_n(x,t)$ is symmetric, $K_{n+1} = \int_a^b K(x,z)\, K_n(z,t)\, dz$

We can be able to show that K_{n+1} is symmetric.

Regularity condit^n:

For the kernel is $\int_a^b \int_c^d K(x,t)\,dx\,dt < \infty$

Regular w.r.t. x for every fixed t $\forall\, t \in [c,d]$

$$\int_a^b K(x,t)\,dx < \infty$$

12/02/2020

Wednesday

FIE of 2nd kind via successive approximat^n :-

Consider a 2nd kind IE

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\,\phi(t)\,dt, \text{ where } \quad \text{—(1)}$$

we assume $f \in C[a,b]$, $K(x,t)$ is a complex-valued cont^n f^n on $[a,b] \times [a,b] = Q[a,b]$, $\lambda \in \mathbb{C}$ is the eigenvalue. If this eq^n has a sol^n $\phi(x)$, then the eq^n itself provides a representat^n of it.

From (1),

$$\phi(t) = f(t) + \lambda \int_a^b K(t,s)\,\phi(s)\,ds \quad \text{—(2)}$$

By ① & ②.

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\left[f(t) + \lambda \int_a^b K(x, s)\,\phi(s)\,ds\right] dt$$

$$= f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t)\,K(\lambda, s)\,\phi(s)\,ds\,dt$$

$$\Rightarrow \phi(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b K_2(x,t)\,\phi(t)\,dt$$

where $K_2(x,t) = \int_a^b K(x, s)\,K(s,t)\,ds$

Proceeding in the same way,

$$\phi(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x,t) f(t)\,dt + \lambda^{n+1} \int_a^b K_{n+1}(x,t)\,\phi(t)\,dt$$

Initially we set $K_1(x,t) = K(x,t)$ & then

$$K_m(x,t) = \int_a^b K_{m-1}(x, s)\,K(s,t)\,ds. \quad \forall\ m = 1, 2, \ldots, n$$

↑ iterated kernel

$K_m(x,t)$ is cont on $Q[a,b]$, $\because$ all the previous kernels are continuous. $\therefore$ if $K(x,t) \le M$, then $|K_m(x,t)| \le M^m (b-a)^{m-1}$, $\forall (x,t) \in Q$

Now let

$$\sigma_n(x) = \sum_{m=1}^{n} \lambda^{m-1} \int_a^b K_m(x,t) f(t)\,dt$$

$$g_n(x) = \lambda^{n+1} \int_a^b K_{n+1}(x,t)\,\phi\,dt$$

s.t. that

$$\phi(x) = f(x) + \lambda\sigma_n(x) + g_n(x)$$

So, $\sigma_n(x)$ & $g_n(x)$ are cont. on $[a,b]$

$\Rightarrow$ $\sigma_n(x)$ is also uniformly cont. on $[a,b]$ (verify) & it will converge to some $\sigma(x)$ on $[a,b]$

Now, we will obtain,

$$\sigma(x) = \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^b K_m(x,t) f(t)\,dt$$

We see that $\{\sigma_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence (show..)

& that $g_n(x) \longrightarrow 0$ uniformly (show..)

13/02/2020
Thursday

We see that $\{\sigma_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence & $\rho_n(x) \to 0$ uniformly.

To show that $\{\sigma_n(x)\}_{n=1}^{\infty}$ is uniformly convergent to a continuous f^n $\sigma(x)$. To prove this we show $\{\sigma_n(x)\}_{n=1}^{\infty}$ is Cauchy.

$$|\sigma_n(x) - \sigma_p(x)| = \left|\sum_{m=1}^{n} \lambda^{m-1} \int_a^b K_m(x,t) f(t)\,dt - \sum_{m=1}^{p} \lambda^{m-1} \int_a^b K_m f\,dt\right|, \text{ where } n > p$$

$$\leq \sum_{m=p+1}^{n} |\lambda|^{m-1} \int_a^b |K_m(x,t)|\,|f(t)|\,dt$$

$$\leq [|\lambda| M (b-a)]^p \frac{M \|f\|_{C[a,b]}}{1 - |\lambda| M (b-a)}$$

$< \epsilon$, only when

p is large & $|\lambda| M (b-a) < 1$

$\Rightarrow \{\sigma_n(x)\}_{n=1}^{\infty}$ is Cauchy on $[a,b]$

$\Rightarrow \sigma_n(x)$ is uniformly conv on $[a,b]$

Now $|\sigma_n(x) - \sigma(x)| = \left|\sum_{m=1}^{n} \lambda^{m-1} \int_a^b K_m(x,t) f(t)\,dt - \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^b K_m(x,t) f(t)\,dt\right|$

$\Rightarrow \sigma_n(x)$ is uniformly conv to $\sigma(x)$

Similarly,

$$|\rho_n(x)| = \left|\lambda^{n+1} \int_a^b K_{n+1}(x,t)\,\phi(t)\,dt\right|$$

$$\leq |\lambda|^{n+1} M^{n+1} (b-a)^n \int_a^b |\phi(t)|\,dt$$

$$= |\lambda| M \left[M^n |\lambda|^n (b-a)^n\right] \int_a^b |\phi(t)|\,dt$$

$= |\lambda| M [M|\lambda|(b-a)]^n \int_a^b |f(t)|\,dt$

$\to 0$ as $n \to \infty$

$\Rightarrow |S_n(x)| \to 0$ as $n \to \infty$ $\Rightarrow S_n(x) \to 0$ as $n \to \infty$ uniformly & absolutely

$\therefore$ the $\phi(x) = f(x) + \lambda \sigma_n(x) + S_n(x)$ of the solⁿ eqⁿ (1) will converge to,

$$\phi(x) = f(x) + \lambda \sigma(x) \quad \text{where}$$

$$\sigma(x) = \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^b K_m(x,t) f(t)\,dt$$

$$= \int_a^b \left(\sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \right) f(t)\,dt$$

$$= \int_a^b R(x,t;\lambda) f(t)\,dt$$, where $R(x,t;\lambda)$ is called the <u>resolvent</u> of the IE & it is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

~~Find~~ Final step verify $\phi(x) = f(x) + \lambda\sigma(x)$ is a solⁿ of (1) or not

We show that,

$$f(x) + \lambda\sigma(x) = f(x) + \lambda \int_a^b K(x,t)\,(f(t) + \lambda\sigma(t))\,dt$$

$$\Rightarrow \sigma(x) = \int_a^b K(x,t)\,(f(t) + \lambda\sigma(t))\,dt$$

$$\therefore \lambda \int_a^b K(x,t)\,[f(t) + \lambda\sigma(t)]\,dt$$

$$= \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b K(x,t)\sigma(t)\,dt$$

$$= \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_{t=a}^b \left(K(x,t) \int_{s=a}^b \sum_{m=1}^{\infty} \lambda^{m-1} K_m(t,s)\,ds \right) dt$$

$$= \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_{s=a}^b \sum_{m=1}^{\infty} \lambda^{m-1} \left(\int_{t=a}^b K_m(x,t)\, K(t,s)\,dt \right) f(s)\,ds$$

$$= \lambda \int_a^b K(x,t) f(t)\,dt + \lambda \int_a^b \sum_{m=1}^{\infty} \lambda^m K_{m+1}(x,s) f(s)\,ds$$

$$= \lambda\sigma(x)$$

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