(1.1)

(Q) $u_t + u u_x = \nu u_{xx}$

(i) $u(x,0) = \sin \pi x, \quad 0 < x < 1$
$u(0,t) = u(1,t) = 0, \ t > 0,$ } H.T.

(Lab $\nu = 1$)

Also,
(ii) $u(x,0) = 4x(1-x) \quad 0 < x < 1$
$u(0,t) = u(1,t) = 0 \quad t > 0$ } H.T. & Lab

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ADI Scheme

$\frac{\partial u}{\partial t} = \nu \nabla^2 u$, in $\mathbb{R}$.

$u(x,y,0) = f(x,y) \rightarrow$ I.C.

$u(x,y,t)$ is prescribed at the boundary

$t_n \rightarrow t_{n+1}$. $\quad u_{ij}^{n} \rightarrow u_{ij}^{n+1}$

Two Step Method

Step-I: $t_n \rightarrow t_{n+1/2}$, x-derivatives implicitly (or y-derivatives) and y-derivatives (x-derivatives) explicitly

$$\frac{u_{ij}^{n+1/2} - u_{ij}^{n}}{\delta t} = \nu \left[\frac{u_{i+1j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i-1j}^{n+1/2}}{\delta x^2} + \frac{u_{ij+1}^{n} - 2u_{ij}^{n} + u_{ij-1}^{n}}{\delta y^2} \right]$$

At a fixed j $(=1,2,\dots, M-1)$

Solve for $i = 1, 2, \dots, N-1$, leading to a tri-diag. syst.

Step II. $t_{n+1/2} \rightarrow t_{n+1}$ y-derivatives (x-derivatives) implicit & x-derivatives (y-deriv) explicit

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+1/2}}{\delta t} = \frac{\nu}{2} \left[\frac{u_{i+1j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i-1j}^{n+1/2}}{\delta x^2} + \frac{u_{ij+1}^{n+1} - 2u_{ij}^{n+1} + u_{ij-1}^{n+1}}{\delta y^2} \right]$$

At a fixed i $(=1,2,\dots,N-1)$

solve for $j = 1,2,\dots, M-1$, $\rightarrow$ tri-diagonal system forms

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H.T Expand by Taylor Series show that ADI scheme is $O(\delta t^2, \delta x^2, \delta y^2)$ and is consistent Von-Newman analysis reveals that the ADI scheme is unconditionaly stable.

H.T Q $u_t = \nabla^2 u$, $0 \le x, y \le 1$, $t > 0$.

$u(x, y, 0) = \sin \pi x \cdot \sin \pi y$

$u = 0$, on the boundary

Derive the ensuing tridiagonal system at each step.

Q $u_t = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$, $u(x, y, z, t)$.

$\nabla^2 Q = 0$ Laplace Equation

$\dfrac{\partial Q}{\partial t} + \nabla^2 Q = 0$, $\underset{t=0}{0} \longrightarrow \underset{t=T}{0}$

Laplace or Poisson Equation

$\left.\begin{array}{l} \nabla^2 Q = 0, \rightarrow \text{Laplace Eq}^n \\ \nabla^2 Q = f(x, y) \rightarrow \text{Poisson Eq}^n \end{array}\right\}$ Elliptic PDE

$b^2 - ac < 0$

$\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0$, $b^2 - ac > 0 \rightarrow$ hyperbolic

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Elliptic PDE. $\nabla^2 Q = 0 = f(x, y)$.

all conditions on Q is prescribed on the boundary for this it also refers as "boundary value problem."

$\nabla^2 Q = f(x, y)$, $x, y \in \mathbb{R}$.

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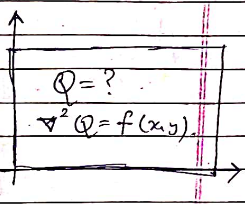
Q is prescribed on the boundary $\partial \mathbb{R}$ of $\mathbb{R}$

$\mathbb{R}$: $0 < x < a$, $0 < y < b$

$\partial \mathbb{R} = x = 0, a$; $y = 0, b$.

$\therefore \dfrac{\partial^2 Q}{\partial x^2} + \dfrac{\partial^2 Q}{\partial y^2} = f(x, y)$.



$x_i = i\delta x$, $i = 0, 1, \ldots, N$

$y_j = j\delta y$, $j = 0, 1, \ldots, M$.

$\therefore Q(x_i, y_j) = Q_{ij}$ $i = 1, 2, \rightarrow N-1$

$j = 1, 2, \ldots, M-1$.

Discretize the equation through central difference scheme.

$$\frac{Q_{i+1,j} - 2Q_{ij} + Q_{i-1,j}}{\delta x^2} + \frac{Q_{i,j+1} - 2Q_{ij} + Q_{i,j-1}}{\delta y^2} = f_{ij} \quad (*)$$

$i = 1, 2, \ldots, N-1$, $j = 1, 2, \ldots, M-1$

which are $(N-1) \times (M-1)$ equations involving $(N-1) \times (M-1)$ variables Q_{ij}.

$\boxed{\begin{array}{ll} Q_{0j}, Q_{Nj} & j = 0, 1, \rightarrow M \\ Q_{i0}, Q_{iM}, & i = 0, 1, \ldots, N \end{array}} \rightarrow$ Known

Eq. (*) forms a closed system

Q $\nabla^2 u = -10(x^2 + y^2 + 10)$, $0 < x < 3$, $0 < y < 3$.

$u = 0$ on the boundary $x = 0, 3$

$y = 0, 3$

$\delta x = \delta y = 1$

u_{ij} ; $2 \times 2 = 4$ points

$(u_{i-1,j} - 2u_{ij} + u_{i+1,j}) + (u_{i,j+1} - 2u_{ij} + u_{i,j-1}) = -10(x_i^2 + y_j^2 + 10)$

$i = 1, 2$; $j = 1, 2$. $\rightarrow$ H.T

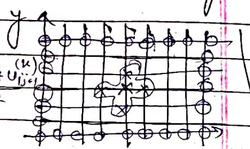
$i = 1, 2, j = 1, 2$

$\delta x, \delta y \sim O(0.01)$, $M = \frac{3}{0.01} = 300$

(299×299) equation

Gauss – Seiddel method to solve the system of equations

$\nabla^2 u = f(x, y)$



$$\frac{u_{i-1\,j}^{(k+1)} - 2u_{ij}^{k+1} + u_{i+1\,j}^{k}}{\delta x^2} + \frac{u_{ij-1}^{k+1} - 2u_{ij}^{k+1} + u_{ij+1}^{(k)}}{\delta y^2} = f_{ij}$$

At the $(k+1)^{th}$ iteration, $k \geq 0$,

$u_{ij}^{(0)} \rightarrow$ known

u_{ij} is the unknown at the ij^{th} equation

$u_{ij}^{2}\left(\frac{1}{\delta x^2} + \frac{1}{\delta y^2}\right)$

Other variables are treated as known.

$$u_{ij}^{(k+1)} = \frac{u_{i+1\,j}^{(k)} + u_{ij+1}^{(k)} + \beta^2\left(u_{i-1\,j}^{(k+1)} + u_{ij-1}^{(k+1)}\right)}{2(1+\beta^2)} - \frac{f_{ij}}{2}$$

$\beta = \frac{\delta x}{\delta y}$ $\quad i = 1, 2, \ldots, N-1$

$j = 1, 2, \ldots, M-1$ for $k \geq 0$

Start the iteration with $u_{ij}^{(0)}, \forall i, j$

$$-2u_{ij}\left(\frac{1}{\delta x^2} + \frac{1}{\delta y^2}\right) = f_{ij} - \frac{u_{i+1\,j}^{(k-1)} + u_{i-1\,j}^{(k-1)}}{\delta x^2} - \frac{u_{ij+1}^{(k)} + u_{ij-1}^{(k+1)}}{\delta y^2}$$

$$-2u_{ij}\frac{(\beta^2+1)}{\delta x^2} = -\frac{f_{ij}\,\delta x^2}{2(1+\beta^2)} + \frac{u_{i+1\,j}^{(k)} + u_{i-1\,j}^{(k-1)}}{2(1+\beta^2)} + \frac{\beta^2\left(u_{ij}^{(k)}\right)}{2(1+\beta^2)}$$

$$u_{ij} = \frac{u_{i+1\,j}^{(k)} + u_{i-1\,j}^{(k+1)} + \beta^2\left(u_{ij+1}^{(k)} - u_{ij-1}^{(k+1)}\right)}{2(1+\beta^2)} - \frac{f_{ij}\,\delta x^2}{2(1+\beta^2)}$$

$i = 1, 2, \ldots$

To start the iteration,

given $u_{ij}^{(0)}$ at all grid points,

This procedure is repeated till

$$\max_{i,j}\left|u_{ij}^{(k+1)} - u_{ij}^{(k)}\right| < \epsilon$$

$\epsilon = 0.5 \times 10^{-5}$ convergence criteria.

$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{M} |a_{ij}|, \forall i, \rightarrow$ diagonally dominant

which is the sufficient condition for convergence.

G.S-iteration converges at a slow rate.

$u_{ij}^{(1)} \quad u_{ij}^{(2)} \quad u_{ij}^{(k)} \quad u_{ij}^{(k+1)} \quad \bar{u}_{ij}^{(k+1)}$

Successive over relaxation.

$$u_{ij}^{(k+1)} = \bar{u}_{ij}^{(k)} + \omega\left(u_{ij}^{(k+1)} - \bar{u}_{ij}^{(k)}\right).$$

$\omega > 1$

$\bar{u}_{ij}^{(k+1)}$ is the modified value at $(k+1)^{th}$ iteration

ω is the relaxation parameter $1 \leq \omega \leq 2$

$\omega = 1$ is the Gauss-Seidel iteration.

$u_{ij}^{(k)} \quad * \quad u_{ij}^{(k+1)}$

$0 < \omega < 1 \rightarrow$ interpolated value

↳ successive-under-relaxation

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Lab

(Q) $\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad 0<x<1, \; t>0.$

$u=0.1, \quad \alpha = 0.01$

$T(0,t)=0, \; T(1,t)=100, \; t>0$

$T(x,0) = 100x$

HT

(Q) $\frac{\partial T}{\partial t} + T\frac{\partial T}{\partial x} = \gamma \frac{\partial^2 T}{\partial x^2}, \quad 0<x<1, \; t>0,$

$T(x,0)=f(x), \quad T(0,t)=T(1,t)=0$

Discretize by the Crank-Nicolson scheme and use Newton's linearization technique to ~~generate~~ determine the ensuing tri-diagonal system which needs to be solved

Hint: $t_n \to t_{n+1}, \quad (T_i^{(n+1)})^{(k+1)} = (T_i^n)^{(k)} + \Delta T_i$

$\delta x = 0.25, \; T=1$

(Q) $\nabla^2 u - 2\frac{\partial u}{\partial x} = -2, \text{ in } R, \quad u=0 \text{ on } \partial R$

H.T

$R: 0<x<1, \; 0<y<1, \quad h=1/3$

(Q) $-\nabla^2 u + 0.1u = 1, \quad 0<x<1$

HT & (Lab)

$u=0, \text{ on } x=0, \; y=0.$

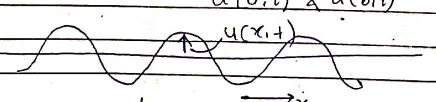
$\frac{\partial u}{\partial n} = 0, \text{ on } x=1, y=1, \quad n \to \text{unit normal.}$

$\delta x = \delta y = 0.5$

Hyperbolic PDE :

$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \; \frac{\partial u(x,0)}{\partial t} = g(x)$

$u(0,t)$ & $u(b,t)$ are given



$t_n \to t_{n+1}$

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$\nu = c\frac{\delta t}{\delta x}$

$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\delta t^2} = c^2\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\delta x^2}\right)$

Explicit Scheme

$u_j^{n+1} = \nu^2\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1-\nu^2)u_j^n - u_j^{n-1}$

$j = 1, 2, \ldots, N-1$

$n=0, \quad \frac{\partial u}{\partial t}\Big|_j^0 = g_j, \quad \frac{u_j^1 - u_j^{-1}}{2\delta t} = g_j, \quad u_j^{-1} = u_j^1 - 2\delta t g_j$

$u_j^1 = \nu^2\left(u_{j+1}^0 + u_{j-1}^0\right) + 2(1-\nu^2)u_j^0 - \{u_j^1 - 2\delta t g_j\}$

$u_j^1 = \frac{\nu^2}{2}\left[(f_{j+1} + f_{j-1}) + (1-\nu^2)f_j\right] + \delta t g_j$

$\nu < 1$, stability

H.T Lab Task

$C=1, \quad \delta x = 1/5, \quad \nu = 0.5$

$u(0,t) = u(1,t) = 0$

$u(x,0) = \sin \pi x, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \le x \le 1$

Find for $n = 1, 2, 3$. Leap frog scheme

$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \quad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$

$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = z, \quad \left.\frac{\partial z}{\partial t} - c\frac{\partial z}{\partial x} = 0\right\}$ Hyperbolic PDE

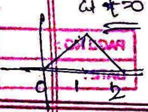
wave eqn

$\boxed{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0} \to$ 1st order hyperbolic PDE

$u(x,0) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x\le 2 \\ 0 & \text{otherwise} \end{cases}$

Gas dynamics
Euler eqn for inviscid flow
Transport governed by convection. no diffusion

at $t=0$



$$\therefore \frac{dx}{c} = \frac{dt}{1} = \frac{du}{0}$$

$$u = c_1, \quad x - ct = c_2$$

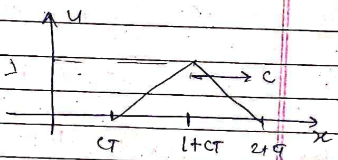
$$f(c_1, c_2) = 0$$

So, $u = f(x - ct)$. , f is any arbitrary funcn.

$$u(x,0) = f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x; & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

$$f(x-ct) = \begin{cases} x-ct, & 0 \le x-ct \le 1 \\ 2-x+ct & 1 \le x-ct \le 2 \end{cases}$$

$u(x,T) = f(x - cT)$



velocity $= c$

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0 \Rightarrow u = g(x+ct)$$

$u(x,t) = g(x+ct)$ $\qquad$ $u(x,T) = g(x+cT)$

$= u(x+cT, 0)$

§. $u_t + cu_x = 0$, c is either $c(x,t)$ or a constant.

$u(x,0) = u_0(x)$, $\to$ I.C. ; BC over x is prescribed

FT-CS $\qquad t_n \to t_{n+1} \quad n \geq 0$

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x} = 0$$

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$

Stability :- $u_j^n \to A^n e^{i\theta j}$, $\xi = \frac{A^{n+1}}{A^n}$

$\xi = 1 - i\nu \sin\theta$ $\qquad$ $|\xi| \le 1$

↳ for stability

$|\xi|^2 = 1 + \nu^2 \sin^2\theta \geq 1$

↳ unconditionally unstable

FTBS (Euler scheme)

(i) $$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\left(\frac{u_j^n - u_{j-1}^n}{\delta x}\right) = 0$$

$$u_j^{n+1} = (1-\nu)\,u_j^n + \nu\, u_{j-1}^n$$

$$\xi = \frac{A^{n+1}}{A^n} = 1 - \nu + \nu(\cos\theta - i\sin\theta)$$

$$|\xi|^2 = 1 - 2\nu(1-\nu)(1-\cos\theta)$$

stable $\qquad 0 < \nu \le 1$, for stability

$\nu = c\frac{\delta t}{\delta x} > 0$, if $c > 0$

< 0 if $c < 0$

(ii) if $c<0$, FTFS

Q Show that $u_t + cu_x = 0$, stable for

(a) FTBS when $c > 0$

(b) FTBS when $c < 0$

Find the value of ν.

check for consistency.