Solution of Test 4

Test 4

Q1. Let $R = \mathbb{Z}[i]$ & $I = (5)$ be an ideal.

Then R/I is not an integral domain as 5 is not a prime elt.

$5 = (2+i)(2-i)$, which implies 5 is not irreducible.

$$R/I = \mathbb{Z}[i]/5 \cong \frac{\mathbb{Z}[x]/(x^2+1)}{(5, x^2+1)/(x^2+1)}$$

$$= \frac{\mathbb{Z}/5\mathbb{Z}[x]}{(x^2+\bar{1})}.$$

Any elt of R/I is of the form

$\bar{a}x + \bar{b}$ where $\bar{a}, \bar{b} \in \mathbb{Z}/5\mathbb{Z}$.

$\therefore$ R/I has $5 \times 5 = 25$ elts.

More generally, if $\gcd(a,b)=1$, $\mathbb{Z}[i]/a+ib \cong \mathbb{Z}/(a^2+b^2)\mathbb{Z}$

Q2. Let $f(x) \in \mathbb{Z}/p\mathbb{Z}[x]$ be an irreducible poly of deg n. where p is a prime no.

Then $(f(x))$ is a prime ideal as $f(x)$ is irreducible poly over $\mathbb{Z}/p\mathbb{Z}[x]$.

Hence $\mathbb{Z}/p\mathbb{Z}[x]/(f(x))$ is an int domain.

Since every non-zero prime ideal is a maximal ideal in $\mathbb{Z}/p\mathbb{Z}[x]$ therefore $\mathbb{Z}/p\mathbb{Z}[x]/(f(x))$ is a field.

Since deg $f(x)$ is n, every elt in $\mathbb{Z}/p\mathbb{Z}[x]/(f(x))$ can be written as

$$a_{n-1}x^{n-1} + \cdots + a_1x + a_0 + \langle f(x) \rangle.$$

Since each $a_i \in \mathbb{Z}/p\mathbb{Z}$ there are p

options for each coeff. Hence there will be total p^n elts in $\dfrac{\mathbb{Z}/p\mathbb{Z}[x]}{(f(x))}$.

Q3. $100 = 5^2 \times 2^2$

Group of order 100 which doesn't contain an elt of order 4.

$\mathbb{Z}_{5^2} \times \mathbb{Z}_{2^2}$

$\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ ✓

$\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{2^2}$

$\mathbb{Z}_{5^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ ✓

Q4 Choose from the list which groups are not simple.

Ans. Any gp of order 97 is a cyclic gp as 97 is prim no. It has no proper divisor hence it will not have any non-trivial proper subgp. Hence it is simple.

$1365 = 5 \times 3 \times 91.$

$n_5 \mid 273$ & $n_5 \equiv 1 \pmod 5$.

$\Rightarrow n_5 = 1, 91$

$n_3 \mid 455$ & $n_3 \equiv 1 \pmod 3$

$\Rightarrow n_3 = 1, 91.$

$n_{91} \mid 15$ & $n_{91} \equiv 1 \pmod{91}$

$\Rightarrow n_{91} = 1$

As $n_{91} = 1$ i.e $\exists$ only one Sylow 91-subgp hence it is normal. So it is not simple.

$6545 = 5 \times 7 \times 11 \times 17$

Note that $n_5 \mid 7 \cdot 11 \cdot 17$ & $n_5 \equiv 1 \pmod 5$

$\Rightarrow n_5 = 1$ or 11

Similarly $n_7 = 1$ or 85

$n_{11} = 1$ or 595

$n_{17} = 1$ or 35

Suppose $n_5 = 11$, $n_7 = 85$, $n_{11} = 595$ and $n_{17} = 35$ then we have total $11 \cdot (5-1) + 85(7-1) + 595 \cdot (11-1)$

$+ 35(17-1) = 44 + 510 + 5950 + 560$

which exceeds the number of elts of the gp. Thus G has at least a single Sylow sub-gp. Hence it will be normal. Thus it is not simple.

<u>Q5</u>. Class eqn of a gp of order 15.

Since any gp of order 15 is abelian. Thus the class eqn is $1+1+\cdots+1$ (15 times).

As from Sylow's Thm it follows that any gp of 15 is $\mathbb{Z}_{15}$.

Q6. Let $R = \mathbb{Z}/6\mathbb{Z}[x]$. Then R is not an integral domain as $\mathbb{Z}/6\mathbb{Z} \subset \mathbb{Z}/6\mathbb{Z}[x]$ and $\mathbb{Z}/6\mathbb{Z}$ is not an int domain.

Because $\bar{2} \cdot \bar{3} = \bar{0}$ but none of them are zero.

Q7. Let $R = \mathbb{C}[x,y]/I$ where $I = (x^2+y-19)$.

Maximal ideals of R is of the form m/I where m is a maximal ideal of $\mathbb{C}[x,y]$ i.e $m = (x-a, y-b)$ and $a^2+b-19=0$.

$a=1$, $b=18$ satisfies the eqn.

$a=0$, $b=19$ satisfies the eqn.

$a=-3$, $b=10$ satisfies the eqn.

Thus $(x-1, y-18)/I$

$(x, y-19)/I$

$(x+3, y-10)/I$

are maximal ideals whereas

$a=5$, $b=10$ doesn't satisfy the

eqn. So $(x-5, y-10)/I$ is

not a maximal ideal.

Test 3

Q1(c). Show that $\mathbb{Z}[x]/\mathbb{Z} \cong \mathbb{Z}[x]$.

Ans. Consider the map

$$\phi : \mathbb{Z}[x] \longrightarrow \mathbb{Z}[x]$$

$$\phi(f(x)) = f(x) - f(0).$$

Then ϕ is a gp homo & surjective also

as $\phi(f(x)+f(0)) = f(x).$

and $\ker \phi = \{ f \in \mathbb{Z}[x] \mid \phi(f) = 0 \}$

$$= \{ f(x) \in \mathbb{Z}[x] \mid f(x) - f(0) = 0 \}$$

$$= \mathbb{Z} .$$

Thus $\mathbb{Z}[x] / \mathbb{Z} \cong \mathbb{Z}[x]$.