Lecture 12

Date
28/08/2017

Q1) Solve the integral eq^n

$$y(t) - \int_0^t \underbrace{y(\tau)\sin(t-\tau)}\,d\tau = t$$

Sol^n :- $\Rightarrow y(t) = t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$

1st step :- Eq^n in terms of convolution. We see that the given eq^n can be written as

$$y(t) = t + y(t) * \sin(t)$$

2nd step :- Application of the convolution theorem we write

$$Y(s) = \mathcal{L}\{y(t)\}$$

By the Convolution Theorem,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{y * \sin t\}$$

$$\Rightarrow Y(s) = 1/s^2 + Y(s) \cdot \frac{1}{(s^2+1)}$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

3rd step:- Taking the inverse transform

This gives the sol^n

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s^2}\} + \mathcal{L}^{-1}\{\frac{1}{s^4}\}$$

$$\boxed{y(t) = t + \frac{t^3}{6}}$$

is the reqd. solⁿ,

Check this by substitution!

Q) **Solve**:-

$$y(t) - \int_0^t (1+\tau)\, y(t-\tau)\, d\tau = 1 - \sinh t$$

Solⁿ:-

$$y - (1+t) * y(t) = 1 - \sinh t$$

Investigate about them

$$Y(s) = \mathcal{L}\{y(t)\}.$$

$$\mathcal{L}\{y\} - \mathcal{L}\{(1+t) * y(t)\} = \mathcal{L}\{1 - \sinh t\}$$

$$\Rightarrow Y(s) - \left(\tfrac{1}{s} + \tfrac{1}{s^2}\right) \cdot Y(s) = \tfrac{1}{s} - \frac{1}{(s^2-1)}$$

$$\Rightarrow Y(s) = \frac{s}{(s^2-1)} \Rightarrow \boxed{y(t) = \cosh t}$$

Q) Find the following inverse L.Ts

(i) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

Sol^n: —

$\mathcal{L}\{\cos t\}\cdot\mathcal{L}\{\sin t\}$

$= \frac{s}{(s^2+1)^2}$.

& so using the convolution theorem,

$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \cos t * \sin t$

$= \frac{1}{2} t \sin t$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}t^2$

& $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

thus $\mathcal{L}\{t^2\}\,\mathcal{L}\{\sin t\} = 2/s^3(s^2+1)$.

$= \boxed{\frac{1}{2} t^2 * \sin t}$

$= \frac{1}{2}(t^2 + 2\cos t - 2)$.

Partial fractions

D·eqns

case 1 :- (Unrepeated factors $(s-a)$)

Solve the IVP

$$y''(t) + y'(t) - 6y(t) = 1,$$

$$y(0) = 0, \; y'(0) = 1.$$

$$\Rightarrow Y(s) = \frac{(s+1)}{s(s-2)(s+3)}$$

$$= \frac{A_1}{s} + \frac{A_2}{s-2} + \frac{A_3}{s+3}$$

$$\Rightarrow s+1 = A_1(s-2)(s+3) + A_2 s(s+3)$$

$$+ A_3 s(s-2)$$

Taking $s = 0, 2, -3$ successively

$1 = (-2)\cdot 3 A_1$, $3 = 2\cdot 5 A_2$, $-2 = (-3)(-5) A_3$ we get

$$\Rightarrow A_1 = -1/6, \; A_2 = 3/10, \; A_3 = -2/15.$$

$\therefore$ $y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{6} + \frac{3}{10}e^{2t}$

$-\frac{2}{15}e^{-3t}$

Case 2 :- **Repeated factor**

$(s-a)^m$

$(s-a)^2$, $(s-a)^3$ etc.

P.F

$\frac{A_1}{(s-a)} + \frac{A_2}{(s-a)^2}$, $\frac{A_1}{(s-a)} + \frac{A_2}{(s-a)^2} + \frac{A_3}{(s-a)^3}$

eg; $y''(t) - 3y'(t) + 2y(t) = 4t$

$y(0) = 1, y'(0) = -1.$

$\therefore$ G.S is $y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2t + 3 - e^{2t} - e^{t}$.

Case 3 :- Unrepeated Complex factors

$$(s-a)(s-\bar{a})$$

such factors occur, for instance, in connection, with vibrations.

$a = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, $i = \sqrt{-1}$.

$\bar{a} = \alpha - i\beta$

$$\therefore (s-a)(s-\bar{a}) = (s-\alpha+i\beta)(s-\alpha-i\beta)$$

$$= (s-\alpha)^2 + \beta^2.$$

These corresponds to the partial fraction

$$\frac{As+B}{(s-a)(s-\bar{a})} \text{ or, } \frac{As+B}{(s-\alpha)^2+\beta^2}.$$

~~x x x x~~

eg, $y''(t) + 2y'(t) + 2y(t)$

$= r(t) = 10\sin 2t$, if $0 < t < \pi$

$= 0$ if $t > \pi$

$y(0) = 1$, $y'(0) = -5$.

$\therefore$ G.S is

$$y(t) = e^{-t}\left[(3+2e^{\pi})\cos t + 4e^{\pi}\sin t\right]$$

Case 4 :- Repeated Complex factors

$$\left[(s-a)(s-\bar{a})\right]^2$$

In this case the partial fraction are the form

$$\frac{As+B}{\left[(s-a)(s-\bar{a})\right]^2} + \frac{Ms+N}{(s-a)(s-\bar{a})}$$

$\rightarrow$ this case is important in connection with resonance

e.g., $y''(t) + \omega_0^2 y(t) = k \sin \omega_0 t$

where $\omega_0^2 = k/m$, k is the spring constant & m is the mass of the body attached to the spring,

$y(0) = 0$, $y'(0) = 0$.

$\therefore$ A.S is $y(t) = \dfrac{k}{2\omega_0^2}\left(-\omega_0 t \cos(\omega_0 t) + \sin(\omega_0 t)\right)$.

Systems of D. eq^{n}s

The Laplace transform method may also be used for solving systems of d. eq^{n}s.

For a first order linear system

$$y_1' = a_{11} y_1 + a_{12} y_2 + g_1(t)$$
$$y_2' = a_{21} y_1 + a_{22} y_2 + g_2(t)$$
$$\rightarrow (1)$$

Writing $Y_1 = \mathcal{L}\{y_1\}$, $Y_2 = \mathcal{L}\{y_2\}$

$G_1 = \mathcal{L}\{g_1\}$, $G_2 = \mathcal{L}\{g_2\}$,

we obtain from eq^n(1), the subsidiary eq^n

$$sY_1 - y_1(0) = a_{11} Y_1 + a_{12} Y_2 + G_1(s)$$
$$sY_2 - y_2(0) = a_{21} Y_1 + a_{22} Y_2 + G_2(s).$$

collecting the Y_1 & Y_2 - terms

$$\left.\begin{aligned}(a_{11}-s)Y_1 + a_{12}Y_2 &= -y_1(0) - A_1(s)\\ a_{21}Y_1 + (a_{22}-s)Y_2 &= -y_2(0)\\ &\quad - A_2(s)\end{aligned}\right\} \to (2)$$

This must be solved algebraically for $Y_1(s)$ & $Y_2(s)$.

The sol^n of the given system is then obtained if we take the inverse

$$y_1 = \mathcal{L}^{-1}(Y_1),$$

$$y_2 = \mathcal{L}^{-1}\{Y_2\}.$$

Simultaneous Ordinary d.e_{q^n}

$\times\times\times\times$

Solve:- $\begin{cases} \dfrac{dx}{dt} = 2x - 3y \\ \dfrac{dy}{dt} = y - 2x \end{cases}$

subject to $x(0) = 8$, $y(0) = 3$.

Soln:- Taking the Laplace transform we get

$$\mathcal{L}\{x\} = X, \quad \mathcal{L}\{y\} = Y$$

$$\mathcal{L}\{x'\} = \mathcal{L}\{2x\} - \mathcal{L}\{3y\}$$

$$\mathcal{L}\{y'\} = \mathcal{L}\{y\} - \mathcal{L}\{2x\}$$

$$\Rightarrow \quad sX - \underset{(=8)}{x(0)} = 2X - 3Y$$

$$sY - \underset{(=3)}{y(0)} = Y - 2X.$$

$$\Rightarrow (s-2)X + 3Y = 8 \quad \rightarrow (1)$$

$$2X + (s-1)Y = 3 \quad \rightarrow (2).$$

Solving 1 & 2 simultaneously (By Cramer's Rule)

$$X = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8s-17}{(s+1)(s-4)}$$

(Using Partial Fractions)

$$Y = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{5}{s+1} + \frac{3}{(s-4)}$$

$$= \frac{3s-22}{s^2-3s-4} = \frac{5}{s+1} - \frac{2}{s-4}$$

Then

$$x = \mathcal{L}^{-1}\{X\} = 5e^{-t} + 3e^{4t}$$

$$y = \mathcal{L}^{-1}\{Y\} = 5e^{-t} - 2e^{4t}$$

C.W

Q) Solve

$$\begin{cases} x'' + y' + 3x = 15e^{-t} \\ y'' - 4x' + 3y = 15\sin 2t \end{cases}$$

subject to $x(0) = 35$, $x'(0) = -48$

$y(0) = 27$, $y'(0) = -55$.

$\therefore$ G.S is

$$x = \mathcal{L}^{-1}\{X\} = 30\cos t - 15\sin 3t + 3e^{-t} + 2\cos 2t$$

$$y = \mathcal{L}^{-1}\{Y\} = 30\cos 3t - 60\sin t - 3e^{-t} + \sin 2t.$$

1) $\left[\frac{dy}{dx}\right]^3 + y = \sin x \longrightarrow$ first order d.eqn

2) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + \ln x = 0 \longrightarrow$ 2nd order d.eqn

3) $\left(\frac{d^3y}{dx^3}\right)^4 + \left(\frac{dy}{dx}\right)^7 + y^8 = 0$ — 3rd order d.eqn

Such eqns cannot be solved by L.T (why?)

we shall be restricted to solving first & 2nd order linear o.d.eqns.

- time now ... later.

(1) Solve the first order diff eq^n

$$\frac{dx}{dt} + 3x = 0, \quad x(0) = 1.$$

$$\Rightarrow \mathcal{L}\{x'(t)\} + 3\mathcal{L}\{x\} = 0$$

$$\boxed{\mathcal{L}\{x\} = \bar{x}(s)}$$

$$\Rightarrow s\bar{x}(s) - x(0) + 3\bar{x}(s) = 0.$$

$$\Rightarrow \bar{x}(s) = \frac{1}{(s+3)}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\{\bar{x}(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= e^{-3t},$$

check that this is indeed the sol^n.

(2) $\frac{dx}{dt} + 3x = 0$, $x(1) = 1$

$\Rightarrow x(t) = \mathcal{L}^{-1}\left\{\frac{x(0)}{s+3}\right\}$

$\Rightarrow x(t) = x(0)\, e^{-3t}$.

$\Rightarrow x(1) = x(0)\, e^{-3} = 1$.

$\Rightarrow x(0) = e^{3}$

$\therefore$ sol^n is $x(t) = e^{3(1-t)}$.

(3) $\frac{dx}{dt} + 3x = \cos 3t$, $x(0) = 0$.

Taking L.T, we obtain

$s\bar{x}(s) - x_{t=0}(0) + 3\bar{x}(s) = s/s^2+9$

$\Rightarrow \bar{x}(s) = \frac{s}{(s+3)(s^2+9)}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t.$$

$$\therefore x(t) = \mathcal{L}^{-1}\left\{\bar{x}(s)\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+3}\cdot\frac{s}{s^2+9}\right\}$$

$$= e^{-3t} * \cos(3t)$$

$$= \int_0^t e^{-3(t-\tau)} \cos(3\tau)\, d\tau$$

$$= \int_0^t e^{-3\tau} \cos 3(t-\tau)\, d\tau$$

$$\therefore x(t) = e^{-3t} \int_0^t e^{3\tau} \cos(3\tau)\, d\tau$$

$$x(t) = \frac{1}{6}(\cos 3t + \sin 3t) - \frac{1}{6}e^{-3t}$$

$$\int_0^t e^{3\tau}\cos(3\tau)\,d\tau = \frac{1}{6}\left[e^{3t}\cos 3t + e^{3t}\sin 3t - 1\right]$$

Ex/ Find the G.S to the d.eq

$$\frac{dx}{dt} + 3x = f(t), \quad x(0) = 0$$

& $f(t)$ is of exponential order.

$$\therefore x(t) = \mathcal{L}^{-1}\left\{\frac{\bar{F}(s)}{s+3}\right\}$$

$$= \int_0^t e^{-3(t-\tau)} f(\tau)\,d\tau$$

$$= e^{-3t}\int_0^t e^{3\tau} f(\tau)\,d\tau.$$

H.W

EX/ Use L.T to solve the eq

$$\boxed{\frac{d^2x}{dt^2} + x = 0}, \text{ with}$$

$x(0) = 1, \; x'(0) = 0.$

S.H.M problem.

Taking L.T / $s^2\bar{x}(s) - s\underset{(=1)}{x(0)} - \underset{(=0)}{x'(0)} + \bar{x}(s) = 0 \Rightarrow \bar{x}(s) = s/s^2+1$

$$\Rightarrow x(t) = L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$$

This is the soln to this S.H.M problem.

N-H Second order O.D eqn

H.W

$$\frac{d^2x}{dt^2} + x = t, \text{ with}$$

$x(0) = 1, \; x'(0) = 0.$

$\bar{x}(s) = s/s^2+1 + \frac{1}{s^2(s^2+1)}$

$$\therefore x(t) = \underbrace{\cos t - \sin t}_{C.F} + \underbrace{t}_{P.I}$$

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

$(t \geq 0)$

where a, b, c are constants.

<u>In Mechanics</u>, a is the mass,

b is the damping constant,
(diagrammatically, represented by a dashpot)

c is the spring constants (or, stiffness)

x is the displacement of the mass.

<u>In electrical circuits</u>, a is the inductance

b is the resistance,

c is the reciprocal of capacitance / reactance

& x (replaced by q) is the charge, the rate of change of which correct time is the more familiar electric current

The R.H.S is called the forcing or excitation.

In terms of systems engineering

$f(t)$ is the system input

$x(t)$ is the system output.

Since a, b, c are constants the system described by the eqn is termed linear & time invariant.

Taking L.T

$$a\left(s^2\bar{x}(s) - s\,x(0) - x'(0)\right)$$
$$+ b\left(s\bar{x}(s) - x(0)\right)$$
$$+ c\,\bar{x}(s) = \bar{f}(s)$$

$$\Rightarrow \bar{x}(s) = \frac{\bar{f}(s) + (as+b)x(0) + a x'(0)}{(as^2+bs+c)}$$

In particular

$$\Rightarrow \bar{x}(s) = \frac{1}{(as^2+bs+c)}\,\bar{f}(s)$$ [when $x(0) = x'(0) = 0$]

$\Rightarrow$ response = transfer f^n × input.

This is why L.Ts are highly regarded by engineers.

Q) Use L.T. Techniques, to solve

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = \underbrace{2e^{-t}} \quad (t \geq 0)$$

subject to the condn,

$x = 1$ & $x' = 0$ at $t = 0$.

Explain the physical significance of the soln.

Hint:- $x(t) = \underbrace{e^{-t}}_{\cancel{P.I}\ P.S} + \underbrace{e^{-2t} - e^{-3t}}_{C.F} \quad (t \geq 0)$

Note:-

① If $\bar{f}(s) \not\to 0$ for large $|s|$, generalized fn are to be expected.

If $\bar{f}(s)$ has a reg. root or a more elaborate multi-valued, error f^ns, Bessel f^n & the likes are expected!

Ex/ Use L-T to solve the

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = \sin t$$

$$x(0) = 0 = x'(0). \quad (t \geq 0).$$

Explain the physical significance of the sol^n.

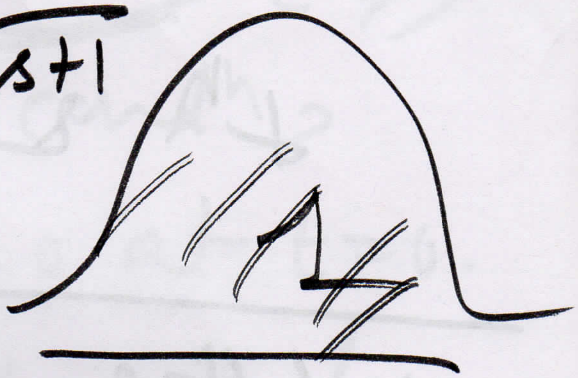
sol^n: $x(t) = \dfrac{e^{-3t}}{50}(5t+3) - \frac{3}{50}\cos t + \frac{2}{25}\sin t$

Def^n:—

The Error function :—

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

$$\mathcal{L}\{erf\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}$$



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