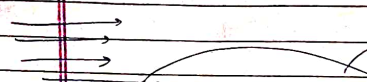
wind or fluid movemer (H.T) (Lab Test)



Q $f''' + ff'' + 1 - (f')^2 = 0$

$f(0) = f'(0) = 0$

$f'(10) = 1$

get the reduced block-tridiagonal system which needs to be solved at every iteration.

$F = f'$

$F'' + fF' + 1 - F^2 = 0$

$X_i = \begin{pmatrix} f_i \\ F_i \end{pmatrix}$

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$f''' + ff'' + 1 - (f')^2 = 0$, $f(0) = 0$, $f'(0) = 0$, $f'(10) = 1$

$f' = F$, $F'' + fF' + 1 - F^2 = 0$

$f(0) = 0$, $F(0) = 0$, $F(10) = 1$

$f_i - f_{i-1} - \frac{h}{2}(F_i + F_{i-1}) = 0$ — (a)

$\frac{F_{i+1} - 2F_i + F_{i-1}}{h^2} + f_i \frac{F_{i+1} - F_{i-1}}{2h} - F_i^2 = -1$ — (b)

At any iteration $(k+1)$

$f_i^{(k+1)} = f_i^{(k)} + \Delta f_i$

$F_i^{(k+1)} = F_i^{(k)} + \Delta F_i$

$A_{11} \Delta f_{i-1} + A_{12} \Delta F_{i-1} + B_{11} \Delta f_i + B_{12} \Delta F_i + C_1 \Delta f_{i+1} + C_2 \Delta F_{i+1} = D_1$

$A_{21} \Delta f_{i-1} + A_{22} \Delta F_{i-1} + B_{21} \Delta f_i + B_{22} \Delta F_i + C_{21} \Delta f_{i+1} + C_{22} \Delta F_{i+2} = D_2$

$i = 1, 2, \dots, n-1$, $\Delta f_0 = \Delta F_0 = 0$

$\Delta f_n = \Delta F_n = 0$

$A_{11} = -1$, $A_{12} = -\frac{h}{2}$, $B_{11} = 1$, $B_{12} = -h/2$, $C_1 = C_2 = 0$.

$D_1 = -\left(f_i^{(k)} - f_{i-1}^{(k)}\right) + \frac{h}{2}\left(F_i^{(k)} + F_{i-1}^{(k)}\right)$

$A_{21} = 0$, $A_{22} = \frac{1}{h^2} - \frac{f_i^{(k)}}{2h}$, $B_{21} = \frac{(F_{i+1}^{(k)} - F_{i-1}^{(k)})}{2h}$

$B_{22} = \frac{-2}{h^2} - 2F_i^{(k)}$, $C_{21} = 0$, $C_{22} = \frac{1}{h^2} + \frac{f_i^{(k)}}{2h}$

$D_2 = -1 - \frac{1}{h^2}\left(F_{i+1}^{(k)} - 2F_i^{(k)} + F_{i-1}^{(k)}\right) - \frac{1}{2h} f_i^{(k)}\left(F_{i+1}^{(k)} - F_{i-1}^{(k)}\right)$

$+ \left(F_i^{(k)}\right)^2$. $i = 1, 2, \dots, n-1$, $k \geq 0$

$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i$

Lab Test

$X_i = \begin{pmatrix} \Delta f_i \\ \Delta F_i \end{pmatrix}$; $A_i = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $D_i = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$

$AX = D$, A = Block Tridiagonal matrix

iteration starts with $f_i^{(0)}$, $F_i^{(0)}$ are prescribed for $i = 0, 1, \dots$

$F^{(0)} = \eta/10$, $f^{(0)} = \frac{\eta^2}{20}$, → initial approximation to start the iteration

(2) (H.T) $F''' + (2F + 4)F' = 0$, $F(0) = 0$, $F''(0) = -K$, $F'(\omega) = 0$.

$K = 0.1$, $\omega = 0.087$.

Ans $F' = g$, $g'' + (2F + 4)g = 0$ $0 < \eta < \omega$

$F(0) = 0$, $g'(0) = -K$, $g(\omega) = 0$

$g_0' = (-3g_0 + 4g_1 - g_2)/2h = -K$

$F_i - F_{i-1} - \frac{h}{2}(g_i + g_{i-1}) = 0$ | $\left(\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2}\right) + (2F_i + 4) g_i = 0$

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} + (2F_i+4)g_i = 0, \quad i=1,2..$$

For $i=1$, Substitute $\frac{(-3g_0 + 4g_1 - g_2)}{2h} = g_0' = -k$

$$\Rightarrow -3g_0 + 4g_1 - g_2 = -2hk$$

$$\Rightarrow g_0 = \frac{(4g_1 - g_2 + 2hk)}{3}$$

$$F_1 - F_0 - \frac{h}{2}\left[g_1 + \left(\frac{4g_1 - g_2 + 2hk}{3}\right)\right] = 0 \quad \text{—(I)}$$

$$\left(\frac{g_2 - 2g_1 + \left(\frac{4g_1 - g_2 + 2hk}{3}\right)}{h^2}\right) + (2F_1+4)g_1 = 0. \quad \text{—(II)}$$

$$F_2 - F_1 - \frac{h}{2}[g_2 + g_1] = 0 \quad \text{—(III)}$$

$$\left(\frac{g_3 - g_2 + g_1}{h^2}\right) + (2F_2+4)g_2 = 0 \quad \text{—(IV)}$$

H.T

$$F_i^{(K+1)} = F_i^{(K)} + \Delta F_i, \quad g_i^{(K+1)} = g_i^{(K)} + \Delta g_i$$

$i = 1, 2, \ldots, n-1$

Quasilinearization Technique

Consider the BVP

$$F(x, y, y', y'') = 0, \quad a < x < b$$

$$y(a) = y_a, \quad y(b) = y_b$$

we treat F as a continuously differentiable function of the functions y, y', y'' at every point x in $a < x < b$. Let $y^{(0)}(x), y'^{(0)}(x), y''^{(0)}(x)$ be an approximate form of the unknown variables $y(x), y'(x), y''(x)$ respectively.

we expand $F(x, y, y', y'') = 0$ about $y^{(0)}, y'^{(0)}, y''^{(0)}$ by Taylor's series expansion when $a < x < b$

$$F(x, y, y', y'') = F(x, y^{(0)}, y'^{(0)}, y''^{(0)}) + (y - y^{(0)})\frac{\partial F}{\partial y}\Big|^{(0)} + (y' - y'^{(0)})\frac{\partial F}{\partial y'}\Big|^{(0)} + (y'' - y''^{(0)})\frac{\partial F}{\partial y''}\Big|^{(0)} + (\ldots) = 0$$

If we drop the square & higher orders, we get the eq^n as

$$F(x, y^{(0)}, y'^{(0)}, y''^{(0)}) + (y^{(1)} - y^{(0)})\frac{\partial F}{\partial y}\Big|^{(0)} + (y'^{(1)} - y'^{(0)})\frac{\partial F}{\partial y'}\Big|^{(0)} + (y''^{(1)} - y''^{(0)})\frac{\partial F}{\partial y''}\Big|^{(0)} = 0$$

The variables y, y', y'' are appearing linearly in this eq^n, and solutions $y^{(1)}, y^{(1)'}, y^{(1)''}$ are the approx. solution for the unknowns y, y', y''.

with b.c.s $y^{(1)}(a) = y_a$, $y^{(1)}(b) = y_b$

same b.c as satisfied by y.

In general, at any iteration $(K+1)$ we can express the linear BVP as

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$$F(x, y^{(k)}, y'^{(k)}, y''^{(k)}) + \left(y^{(k+1)} - y^{(k)}\right)\frac{\partial F}{\partial y}\Big|^{(k)}$$

$$+ \left(y'^{(k+1)} - y'^{(k)}\right)\frac{\partial F}{\partial y'}\Big|^{(k)} + \left(y''^{(k+1)} - y''^{(k)}\right)\frac{\partial F}{\partial y''}\Big|^{(k)} = 0$$

where the BCs

$$y^{(k+1)}(a) = y_a,\ y^{(k+1)}(b) = y_b, \quad k \geq 0$$

iteration starts with $y^{(0)}, y'^{(0)}, y''^{(0)}$

repeat the process till.

$$\| y^{(k+1)} - y^{(k)} \| < \epsilon, \ a < x < b$$

$$\max_{a \leq x \leq b} \left| y^{(k+1)}(x) - y^{(k)}(x) \right| < \epsilon$$

$k = 1, 2 \rightarrow$

At every iteration, the non linear BVP is reduced to a linear BVP.

Q) $3yy'' + (y')^2 = 0,\ y(0) = 0,\ y(1) = 1$.

Reduced Quasilinear form of the non linear BVP

$F(x, y, y', y'') = 3yy'' + (y')^2 = 0$

At any iteration $(k+1)$

$$F(x, y^{(k+1)}, y'^{(k+1)}, y''^{(k+1)}) = F(x, y^{(k)}, y'^{(k)}, y''^{(k)})$$
$$+ (y^{(k+1)} - y^{(k)})\, 3y''^{(k)} + (y'^{(k+1)} - y'^{(k)})\, 2y'^{(k)}$$

$$+ \left(y''^{(k+1)} - y''^{(k)}\right) 3y^{(k)} = 0$$

$$\Rightarrow 3y^{(k)} \frac{d^2 y^{(k+1)}}{dx^2} + 2y'^{(k)} \frac{dy^{(k+1)}}{dx} + 3y''^{(k)} y^{(k+1)} = 0$$

$$= \cancel{-3y^{(k)}} y''^{(k)} - \left(y'^{(k)}\right)^2 + \cancel{3y^{(k)} y''^{(k)}} + \cancel{2}(y'^{(k)})^2$$
$$+ 3y^{(k)} y''^{(k)}$$

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HW $= 3y^{(k)} y''^{(k)} + (y'^{(k)})^2$

$y^{(k+1)}(0) = 0,\quad y^{(k+1)}(1) = 1$

which is a linear BVP, $k \to k+1$.

Q $y'' - (y')^2 + y + 1 = 0,\ y(0) = 0.5,\ y(\pi) = -0.5$

Q $f''' + ff'' + 1 - (f')^2 = 0,\ f(0) = 0,\ f'(0) = 0,\ f'(10) = 10$

Recap Quasilinearization Technique 22/02/2024.

$F(y, y', y'', x) = 0$, $y^{(0)}, y'^{(0)}, y''^{(0)} \rightarrow$ Known fns

F is treated as a function of functions y, y', y''.

Expand F about the known fns of $y^{(0)}, y'^{(0)}$, $y''^{(0)}$ and retain only upto linear orders

In general, at the $(k+1)^{th}$ iteration, the quasilinear form of the non linear BVP is

$$\frac{d^2 y^{(k+1)}}{dx^2}\frac{\partial F}{\partial y''}\Big|^{(k)} + \frac{dy^{(k+1)}}{dx}\frac{\partial F}{\partial y'}\Big|^{(k)} + y^{(k+1)}\frac{\partial F}{\partial y}\Big|^{(k)}$$

$$= -F(y^{(k)}, y'^{(k)}, y''^{(k)}, x), \quad a < x < b.$$

with $y^{(k+1)}(a) = y_a,\ y^{(k+1)}(b) = y_b$. reduced linear BVP

$$\max_{a < x < b} \left| y^{(k+1)}(x) - y^{(k)}(x) \right| < \epsilon, \quad \text{STOP.}$$

next else, we repeat,

(P)

Q $f''' + ff'' + 1 - (f')^2 = 0,\quad f(0) = f'(0) = 0,\ f'(10) = 1.$

$F(f''', f'', f', f) = 0 \qquad 0 < \eta < 10$

A. At the $(k+1)^{th}$ iteration

$$F(x, y^{(k+1)}, y'^{(k+1)}, y''^{(k+1)}) = F(x, y^{(k)}, y'^{(k)}, y''^{(k)})$$
$$+ \left(y^{(k+1)} - y^{(k)}\right).$$

$M_1 + 3$

$+4y_1$

$F(x, \cancel{f^{(K+1)}, f'^{(K+1)}, f''^{(K+1)}}) = \cancel{F(x, f}$

$$\frac{d^3 f^{(K+1)}}{d\eta^3} \cdot \left.\frac{\partial F}{\partial f'''}\right|^{(K)} + \frac{d^2 f^{(K+1)}}{d\eta^2} \cdot \left.\frac{\partial F}{\partial f''}\right|^{(K)} +$$

$$\frac{df^{(K+1)}}{d\eta} \cdot \left.\frac{\partial F}{\partial f'}\right|^{(K)} + f^{(K+1)} \cdot \left.\frac{\partial F}{\partial f}\right|^{(K)}$$

$$= -F\left(f'''^{(K)}, f''^{(K)}, f'^{(K)}, f^{(K)}\right)$$

$$+ f'''^{(K)} \cdot \left.\frac{\partial F}{\partial f'''}\right|^{(K)} + f''^{(K)} \left.\frac{\partial F}{\partial f''}\right|^{(K)} + f'^{(K)} \left.\frac{\partial F}{\partial f'}\right|^{(K)} + f^{(K)} \left.\frac{\partial F}{\partial f}\right|^{(K)}$$

B.C. $f^{(K+1)} = 0 = f'^{(K+1)}$, $f'^{(K+1)}(10) = 1$, ∞.

$f^{(0)}(\eta)$ needs to be guessed

$f'''^{(K+1)}$ $+ f''^{(K+1)} \cdot f^{(K)}$

$$F\left(y^{(K)}, y'^{(K)}, y''^{(K)} (x)\right) + \left(y''^{(K+1)} - y'^{(K)}\right) \left.\frac{\partial F}{\partial y''}\right|^{(K)}$$

$$+ \left(y'^{(K+1)} - y'^{(K)}\right) \left.\frac{\partial F}{\partial y'}\right|^{(K)} + \left(y^{(K+1)} - y^{(K)}\right) \left.\frac{\partial F}{\partial y}\right|^{(K)} = 0$$

$$\Rightarrow \frac{d^2 y^{(K+1)}}{dx^2} \cdot \left.\frac{\partial F}{\partial y''}\right|^{(K)} + \frac{\partial y^{(K+1)}}{\partial x} \left.\frac{\partial F}{\partial y'}\right|^{(K)} + y^{(K+1)} \cdot \left.\frac{\partial F}{\partial y}\right|^{(K)}$$

$$= -F\left(y^{(K)}, y'^{(K)}, y''^{(K)}\right) + y''^{(K)} \left.\frac{\partial F}{\partial y''}\right|^{(K)}$$

$$+ y'^{(K)} \left.\frac{\partial F}{\partial y'}\right|^{(K)} + y^{(K)} \left.\frac{\partial F}{\partial y}\right|^{(K)}$$

Thus $\dfrac{d^3 f^{(K+1)}}{d\eta^3} + \dfrac{d^2 f^{(K+1)}}{d\eta^2} f^{(K)} + \dfrac{df^{(K+1)}}{d\eta}\left(-2f'^{(K)}\right)$

$$+ f^{(K+1)}\left(f''^{(K)}\right)$$

$$= -\left(\cancel{f'''^{(K)}} + f^{(K)} f''^{(K)} + 1 - \cancel{\left(f'^{(K)}\right)^2}\right)$$

$$+ \cancel{f'''^{(K)}} + f''^{(K)} \cdot f^{(K)} + f'^{(K)} \cdot \left(-2f'^{(K)}\right)$$

$$+ f''^{(K)} f^{(K)} \cdot -1$$

$$\Rightarrow f'''^{(K+1)} + f^{(K)} f''^{(K+1)} - 2f'^{(K)} f'^{(K+1)} + f^{(K+1)} f''^{(K)}$$

$$= f^{(K)} f''^{(K)} - \left(f'^{(K)}\right)^2 - 1$$

with b.c.s. $f^{(K+1)}(0) = f'^{(K+1)}(0) = 0$

& $f'^{(K+1)}(10) = 1$, $K \geq 0$

let $F^{(K+1)} = f'^{(K+1)}$

Lab Task

$$F''^{(K+1)} + f^{(K)} F'^{(K+1)} - 2F^{(K)} F^{(K+1)} + f^{(K+1)} F'^{(K)}$$

$$= -1 + F^{(K)} f'^{(K)} - \left(F^{(K)}\right)^2$$

H.T

$f^{(K+1)}(0) = 0 = F^{(K+1)}(0)$, $F^{(K+1)}(10) = 1$.

→ derive the Block tridiagonal System which needs to be solved at every iter.

(2) $f''' + f f'' + \beta\left(1 - (f')^2\right) = 0 \quad | \quad f(0) = f'(0) = 0, \ f'(10) = 1$

$\beta \geq 0 \Rightarrow$ Known parameter

in general ~~≤ 0~~ $\alpha\beta < 2$

→ derive the ensuing block tri diagonal system

Reduce the nonlinear BVP

Apply the quasilinearization method to cope with the nonlinearity & derive the block tri diagonal system ~~then~~ for the discretized quasilinear eq^n

Partial Differential Equations :-

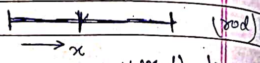
$u = u(x, t)$.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad a<x<b, \quad t \geq 0$$

temperature dist. ← $u(x,0) = F(x), \quad a < x < b)$

$u(a,t) = U_a$ & $u(b,t) = U_b$.

$t \geq 0$



$u(x,t), \; t \geq 0$

$a < x < b$

PDE: $$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0<x<l, \quad t>0$$

(initial Condition) I.C. : $u(x,0) = F(x)$

Boundry Condition B.C : $u(0,t) = U_0, \quad u(l,t) = U_l, \quad t \geq 0$

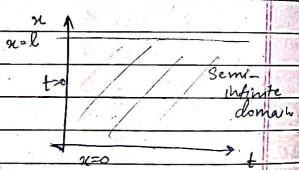
Find $u(x,t) = ??$

$u(x,t) = g(x,t)$

analytic for $e^{2t} \sin x$

$0 < x < l$

$t > 0$



$$A u_{xx} + 2B u_{xy} + C u_{yy} + G(u, u_x, u_y, x, y) = 0$$

$B^2 - AC = 0$, Parabolic PDE

$B^2 - AC < 0$, Elliptic PDE

$B^2 - AC > 0$; Hyperbolic PDE

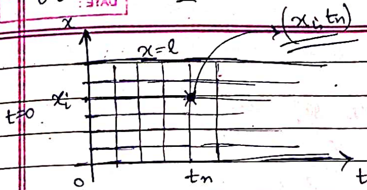
Q# $u_t = \nu u_{xx} + f(x,t), \quad 0<x<l, \quad t>0$

I.C. : $u(x,0) = F(x)$

B.C : $u(0,t) = U_0; \quad u(l,t) = U_l, \quad t>0$

Define the grid points at which we obtain solution :

$$\frac{\partial u}{\partial t} = \frac{u(x, t+\delta t) - u(x, t)}{\delta t} = \frac{u_i^{n+1} - u_i^n}{h} \qquad \left. \frac{\partial^2 u}{\partial x^2} \right|_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$$



$x_i = i\delta x, \quad i = 0,1,2,\dots,N$.

$x_0 = 0, \quad x_N = l, \quad t_n = n\delta t$

$n = 0, 1, \dots$

Denote $u(x_i, t_n) = u_i^n$

task is to find u_i^n for $n = 1, 2, \dots$

& $i = 1, 2, \dots, N-1$

$u_i^0 = F_i$ is known for $i = 1, 2, \dots, N-1$

$u_0^n = U_0, \quad u_N^n = U_l$.

Find u_i^n, for $i = 1, 2, \dots, N-1$, when $n = 1, 2, \dots$

Forward marching in time, let u_i^n be known, find $u_i^{n+1}, \forall i, n \geq 0$

Satisfy the PDE at $(x_i; t_n)$.

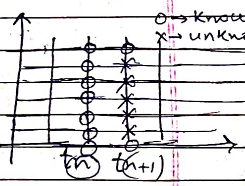
$$\left. \frac{\partial u}{\partial t} \right|_i^n = \nu \left. \frac{\partial^2 u}{\partial x^2} \right|_i^n + f(x_i, t_n)$$

u_i^n is known for $i = 1, 2, \dots, N-1$.

$u_0^n = U_0, \; u_N^n = U_l$ are known from B.Cs.

Task is to find $u_i^{n+1}, \; i = 1, 2, \dots, N-1$?

Apply forward diff app. to the PDE at (x_i, t_n).



$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\delta x^2} + f(x_i, t_n), \quad i = 1, 2, \dots, N-1$$

variables with superscript n are known & $n+1$ are unknown

$\delta t \times 4 \le 1/2$ Explicit forward time & central space discretization

$\delta t \le \frac{1}{8}$

$\gamma = \nu \frac{\delta t}{(\delta x)^2} \le 1/2$

$u_i^{n+1} = a_i u_{i-1}^n + b_i u_i^n + c_i u_{i+1}^n + d_i f_i^n$, $i = 1, 2, .., N-1$

Explicit method

$u_0^{n+1} = U_0$, $u_N^{n+1} = U_L$

So, start from $n=0$, Known at $n=0$ And $n=1$ & so on.

$\delta t, \delta x \rightarrow$ to consider

Q $u_t = u_{xx}$, $u(x,0) = \sin(\pi x)$, $0 < x < 1$

$\gamma = \nu\delta t/(\delta x)^2 \le 1/2$ $u(0,t) = u(1,t) = 0$, $t > 0$.

↳ for stability

HT

Hab

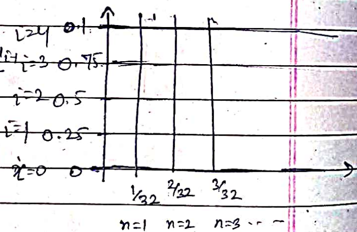
Tush

$\delta x = 1/4$, $\delta t(16) \le 1/2$

$\Rightarrow \delta t \le \frac{1}{32}$

if we take $\gamma = 1/2$, $\delta t = 1/32$

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\delta x^2}$$



$$\Rightarrow u_i^{n+1} = \frac{\delta t}{(\delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + u_i^n$$

$\frac{1}{32} \times 16 = \frac{1}{2}$

$$\Rightarrow u_i^{n+1} = \frac{\delta t}{(\delta x)^2} u_{i-1}^n + \left(1 - \frac{2\delta t}{(\delta x)^2}\right) u_i^n + \frac{\delta t}{(\delta x)^2} u_{i+1}^n$$

$1 - \frac{1}{2} \times 2$

$u_0^{n+1} = u_4^{n+1} = 0$ $i = 1, 2, 3$ $N = 4$

$$u_i^{n+1} = \frac{u_{i-1}^n}{2} + \frac{u_{i+1}^n}{2}$$

$i \downarrow$ / $n \rightarrow$	0	1	2	3	4
0	0	0	0	0	0
1	0.707	0.5	$1/2\sqrt{2}$	0.25	$1/4\sqrt{2}$
2	1	$1/\sqrt{2}$	0.5	$1/2\sqrt{2}$	0.25
3	$1/\sqrt{2}$	0.5	$1/2\sqrt{2}$	0.25	$1/4\sqrt{2}$
4	0	0	0	0	0

- - - →