* Inner Product Space :-

$(X, \|\cdot\|)$

$\|\cdot\| : X \to \mathbb{R}^2$ 	norm

1. $\|x\| \geq 0$, $\|x\| = 0$ iff $x = 0$

2. $\|\alpha x\| = |\alpha| \|x\|$

3. $\|x+y\| \leq \|x\| + \|y\|$

An inner product on a vector space X is a functional s.t.,

$\langle , \rangle : X \times X \to \mathbb{K}$ 	$\langle x, x \rangle = 0$ iff $x = 0$

(i) $\langle x, x \rangle \geq 0 \quad \forall x \in X$

(ii) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(iii) $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$

(iv) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

$\langle \alpha x + \beta y, z \rangle = \langle \alpha x, z \rangle + \langle \beta y, z \rangle$

$= \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

$\langle x, \alpha y + \beta z \rangle = \overline{\langle \alpha y + \beta z, x \rangle}$

$= \overline{\alpha \langle y, x \rangle + \beta \langle z, x \rangle}$

$= \bar{\alpha} \overline{\langle y, x \rangle} + \bar{\beta} \overline{\langle z, x \rangle}$

$= \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$

A vector space X where an inner product is defined is called inner product space.

Ex: $X = \mathbb{C}^n$ & $\langle x, y \rangle = \sum^n x_i \bar{y_i}$

$\langle x, x \rangle = \sum |x_i|^2$

Ex: $X = \mathbb{R}^n$

$\langle x, y \rangle = \sum x_i y_i$

$\longrightarrow$ Inner product is the generalisation of dot product.

$\rightarrow$ If X is an inner product space,

let. $\|x\| = \sqrt{\langle x, x \rangle}$

$\Rightarrow \|x\|^2 = \langle x, x \rangle$

$\rightarrow$ ~~Suppose x~~ Show that in an IPS :-

$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$

$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y \rangle + \langle x-y, x-y \rangle$

$= \langle x, x+y \rangle + \langle y, x+y \rangle + \langle x, x-y \rangle - \langle y, x-y \rangle$

$\boxed{\text{Parallelogram Law}}$

$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$
$+ \langle x, x \rangle - \langle x, y \rangle$
$+ \langle y, y \rangle - \langle y, x \rangle$

$= 2\langle x, x \rangle + 2\langle y, y \rangle = 2\left(\|x\|^2 + \|y\|^2\right)$

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→ Every IPS satisfies the parallelogram law.

If a normed space does not satisfy ~~inner product~~ parallelogram law, then it does'nt form an inner product space.

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Lemma: Let $\langle .,\rangle$ be an IPS on a vector space X. $[\|x\|^2 = \langle x,x\rangle]$

a) Polarization Identity : For all $x, y \in X$

$$4\langle x,y\rangle = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\left[\langle x+iy, x+iy\rangle - \langle x-iy, x-iy\rangle\right]$$

$$= \|x+y\|^2 - \|x-y\|^2 + i\left(\|x+iy\| - \|x-iy\|\right)$$

$$\Rightarrow \langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\left(\|x+iy\| - \|x-iy\|\right)\right)$$

b) Let $x \in X$, then $\langle x,y\rangle = 0\ \forall y$, iff $x = 0$

c) (Schwarz Inequality) For $x, y \in X$, $|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$

$$= \|x\|^2\|y\|^2$$

i.e $|\langle x,y\rangle| \leq \|x\|\|y\|$

~~d) $\langle x+y, x+y\rangle$~~

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle - \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle$

$+ i\big(\langle x, x\rangle + \bar{i}\langle y, x\rangle + i\langle x, y\rangle - \langle y, y\rangle$
$- \langle x, x\rangle + \langle y, y\rangle + \bar{i}\langle y, x\rangle + i\langle x, y\rangle\big)$

$= \langle x, y\rangle + \langle y, x\rangle + \langle x, y\rangle + \langle y, x\rangle$

b) Let $x = 0$.

$\langle 0, y\rangle = \langle 0 + 0, y\rangle$
$= \langle 0, y\rangle + \langle 0, y\rangle$

$\Rightarrow \langle 0, y\rangle = 0 \quad \forall y.$

& $\langle x, y\rangle = 0 \quad \forall\, y \in X$

Let $y = x$
$\Rightarrow \langle x, x\rangle = 0$
$\Rightarrow \|x\|^2 = 0 \Rightarrow x = 0$

c) Let $z = \langle y, y\rangle x - \langle x, y\rangle y$ for $x, y \in X$.

$0 \leq \langle z, z\rangle = \langle \langle y, y\rangle x - \langle x, y\rangle y, \langle y, y\rangle x - \langle x, y\rangle y\rangle$

$= \langle\langle y, y\rangle x, \langle y, y\rangle x\rangle - \langle\langle y, y\rangle x, \langle x, y\rangle y\rangle$
$- \langle\langle x, y\rangle y, \langle y, y\rangle x\rangle + \langle\langle x, y\rangle y, \langle x, y\rangle y\rangle$

$= \langle y, y\rangle^2 \langle x, x\rangle - \langle y \cdot y\rangle |\langle x, y\rangle|^2$

$= \langle y, y\rangle \left(\langle x, x\rangle \langle y, y\rangle - |\langle x, y\rangle|^2 \right)$

$\Rightarrow \langle y, y\rangle \left[\langle y, y\rangle \langle x, x\rangle - |\langle x, y\rangle|^2 \right] \geq 0$

For $\langle y, y\rangle \neq 0$

$\Rightarrow$ i.e $\|y\|^2 \|x\|^2 \geq |\langle x, y\rangle|^2$

& if $\langle y, y\rangle = 0 \Rightarrow y = 0$

$\Rightarrow \langle x, y\rangle = 0$

$\therefore |\langle x, y\rangle| \leq \|x\| \, \|y\|$

$0 = 0$

~~Equality hold~~

Equality holds

$|\langle x, y\rangle| = \|x\| \, \|y\|$

iff x, y are linearly independent

If $|\langle x, y\rangle| = \|x\| \, \|y\|$

i.e. $|\langle x, y\rangle|^2 = \|x\|^2 \|y\|^2$

then $z = 0$

$\langle y, y\rangle x - \langle x, y\rangle y = 0$

Conversely suppose,

the set $\{x, y\}$ are L.D

$\Rightarrow x = \alpha y$

$\Rightarrow y = \beta x$

$\Rightarrow |\langle x, y \rangle|^2 = |\langle x, \beta x \rangle|^2 = |\beta|^2 |\langle x, x \rangle|^2$

$= |\beta|^2 \|y\|^2 \|x\|^2$

Theorem :-

* A norm on an IPS X satisfies the parallelogram law. ~~of vectors~~

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

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l^p space : l^p is not an IPS for $p \neq 2$

l^2 is Hilbert space, l^p is banach for $1 \leq p \leq \infty$

$x = (1, 1, 0 \ldots) \in l^p$, $y = (1, -1, 0 \ldots) \in l^p$

$x+y = (2, 0, 0 \ldots)$ $\|x+y\| = 2$, $\|x-y\| = 2$

$\|x\| = (1^p + 1^p + 0^p + \ldots)^{1/p} = (\Sigma |x_i|^p)^{1/p} = 2^{1/p}$

$\|y\| = 2^{1/p}$

$$H = 2 \cdot 2^{2/p}$$

$$\Rightarrow \quad p = 2$$

$\therefore$ If $p \neq 2$, parallelogram law does not hold.

$$C[a,b] = \{ x(t) \mid x : [a,b] \to \mathbb{R}/\mathbb{C} \}$$

$x \in C[a,b]$ $\quad \|x\| = \max |x(t)|$, $t \in [a,b]$

$$\langle x, y \rangle = \int_a^b x(t)\, \overline{y(t)}\, dt$$

$$\|x\| = \left(\int_a^b |x(t)|^2 \right)^{1/2}$$

Let $x(t) = 1$, $y(t) = \dfrac{t-a}{b-a}$

$\|x\| = \sup x(t) = 1$

$\|y\| = \sup y(t) = 1$

$\|x+y\| = 2$

$$\|x-y\| = \sup_{t \in [a,b]} \left(1 - \frac{t-a}{b-a} \right) = 1$$

$\therefore$ $\|x+y\|^2 + \|x-y\|^2 = 5$

$5 \neq 4$

Parallelogram law not satisfied

Not IPS

$\|x\|^2 = \langle x, x \rangle$

$\|x\| = \sqrt{\langle x, x \rangle}$

$\|x+y\| \leq \|x\| + \|y\|$

Theorem: Let $\|\cdot\|$ be a norm on a vector space X. Then $\exists$ an ~~IPS~~ $\langle \cdot \rangle$ on x such that $\langle x, x \rangle = \|x\|^2 \; \forall x \in X$ iff the norm satisfies the parallelogram law

i.e. $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \; \forall \, x, y$

In this case the inner product is unique and given by

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

Proof: Suppose $\exists$ an $\langle \cdot \rangle$ on X such that

$$\langle x, x \rangle = \|x\|^2$$

Then,

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y \rangle + \langle x-y, x-y \rangle$$
$$= 2(\|x\|^2 + \|y\|^2)$$

And from Polarization property :-

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) \quad - *$$

Converse, Suppose $*$ holds true & Parallelogram law holds.

to show that $\langle , \rangle$ is an Inner Product

a) $\langle x, x\rangle \geq 0$ — From *

b) $\langle x, x\rangle = 0 \Rightarrow x = 0$ — From *

c) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ — From *

- $\langle x, x\rangle = \frac{1}{4}\left[\,\|2x\|^2 - \|0\|^2 + i\|x\|\,|1+i| - i\|x\|\,|$

$$= \|x\|^2 \geq 0$$

$$\langle x, x\rangle = 0 \Rightarrow x = 0$$

Similarly it follows that

$$\langle -x, y\rangle = -\langle x, y\rangle$$
$$\langle ix, y\rangle = i\langle x, y\rangle$$

To show,

$$\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$$

$$\rightarrow 4\langle x+z, y\rangle = \|x+z+y\|^2 - \|x+z-y\|^2 + i\|x+z+iy\|^2 - i\|x+z-iy\|^2$$

$$= \|(x+y/2)+(z+y/2)\|^2 - \|(x-y/2)+(z-y/2)\|^2 + i(\quad)$$

$$= 2(\|x+y/2\|^2 + \|z+y/2\|^2) - \|x-z\|^2 - 2(\|x-y/2\|^2 + \|z-y/2\|^2) + \|x-z\|^2 + 2i(\quad)$$

$$= 2(\|x+y/2\|^2 - \|x-y/2\|^2 + \|z+y/2\|^2 - \|z-y/2\|^2) + 2i(\quad)$$

(Replacing y by iy)

$$= 2(\|x+y/2\|^2 - \|x-y/2\|^2 + \|z+y/2\|^2 - \|z-y/2\|^2) + 2i(\quad)$$

$$= 2(4\langle x, y/2\rangle + 4\langle z, y/2\rangle)$$

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$\Rightarrow \langle x+z, y\rangle = 2\langle x, y/2\rangle + 2\langle z, y/2\rangle$

By taking $z=0$

$\langle x,y\rangle = 2\langle x, \frac{y}{2}\rangle$

$\Rightarrow \langle x+z, y\rangle = \langle x,y\rangle + \langle z,y\rangle$

By taking $x=z$

$\langle 2x, y\rangle = \langle x,y\rangle + \langle x,y\rangle$

$= 2\langle x,y\rangle$

$\Rightarrow \langle nx, y\rangle = n\langle x,y\rangle \quad n\in\mathbb{N}$

Take $z=-x \rightarrow \quad 0 = \langle x,y\rangle + \langle -x,y\rangle$

$\Rightarrow \langle -x, y\rangle = -\langle x,y\rangle$

$n<0$

$\langle -nx, y\rangle = -n\langle x,y\rangle$

$\therefore \langle nx, y\rangle = n\langle x,y\rangle \quad \forall \quad n\in\mathbb{Z}$

If $p,q\in\mathbb{Z}$, $q\neq 0$

$p\langle x,y\rangle = \langle px, y\rangle = \langle p\frac{qx}{q}, y\rangle = q\langle \frac{px}{q}, y\rangle$

$\Rightarrow p/q \langle x,y\rangle = \langle \frac{p}{q}x, y\rangle$

$\therefore \forall\ r\in\mathbb{Q},\ \langle rx,y\rangle = r\langle x,y\rangle$

Now, if $s\in\mathbb{R}$, $\exists\ (r_n) \in \mathbb{Q}$ $\Rightarrow r_n \to s$

$r_n \langle x, y \rangle \longrightarrow s \langle x, y \rangle$

$\langle r_n x, y \rangle \longrightarrow s \langle x, y \rangle$

$r_n \langle x, y \rangle - s \langle x, y \rangle$

$|(r_n - s) \langle x, y \rangle| \leq |r_n - s| \, |\langle x, y \rangle|$

$\leq |r_n - s| \, \|x\| \, \|y\|$

$\Rightarrow \quad r_n \to s \quad \Rightarrow \quad |r_n - s| \to 0$

$\langle r_n x, y \rangle \longrightarrow s \langle x, y \rangle$

$\|r_n x + y\| \longrightarrow \|s x + y\|$

$\big| \|x\| - \|y\| \big| \leq \|x - y\|$

$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$

$\Rightarrow \|x\| - \|y\| \leq \|x - y\| \Rightarrow \big| \|x\| - \|y\| \big| \leq \|x - y\|$

$\& \; -\|x\| + \|y\| \leq \|x - y\|$

$\big| \|r_n x + y\| - \|s x + y\| \big| \leq \|r_n x + y - s x - y\|$

$= \|(r_n - s) x\|$

$= |r_n - s| \, \|x\|$

$\Rightarrow \quad r_n x \to s x$

$\therefore \quad r_n \langle x, y \rangle \to \langle s x, y \rangle$

$(= \langle r_n x, y \rangle)$

$\& \quad r_n \langle x, y \rangle \longrightarrow s \langle x, y \rangle$

$\Rightarrow \quad \langle s x, y \rangle = s \langle x, y \rangle$

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Since $\langle ix, y\rangle = i\langle x, y\rangle$

We take $\langle kx, y\rangle = k\langle x, y\rangle$

i.e. $\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$

7) Let (x_n) be a sequence in the metric space (X, d) and let $x \in X$. Show that if every subsequence of (x_n) converge to x then $x_n \to x$.

Proof: Suppose that x_n does not converge to x. Then $\exists\ \epsilon_0 > 0$ s.t for ~~every~~ each $n \in \mathbb{N}$, there is some $m > n$ s.t $d(x_m, x) \geq \epsilon$. Now choose $n_1 > 1$ s.t $d(x_{n_1}, x) > \epsilon_0$. Then choose $n_2 > n_1$ with $d(x_{n_2}, x_0) > \epsilon_0$ and so on proceeding this way we get $n_1 < n_2 < n_3 \dots$

s.t $d(x_{n_k}, x) > \epsilon_0 \ \forall\ k \in \mathbb{N}$

Now, (x_{n_k}) is a subsequence of (x_n) & no subsequence of (x_{n_k}) converges to x.

18) Let $f \neq 0$ be a linear functional on vector space X. Show that $R(f)$ is a scalar field of X.

Proof: $R(f)$ is subspace of $\mathbb{K}$ (Property of Linear Transform)

If $\mathbb{K}$ is a vector space, $\dim \mathbb{K} = 1$. Since $f \neq 0$,

$R(f) \neq \{0\}$

$\Rightarrow \exists\ x \in X$ s.t $f(x) \neq 0$

$R(f)$ is a subspace of $\mathbb{K}$, $R(f) \neq \{0\} \Rightarrow R(f) = \mathbb{K}$

8 a) Let $\{x_n\}$ be a Cauchy sequence in X (normed space). Show that $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ s.t the series $\sum_{k=1}^{\infty} \|(x_{n_k} - x_{n_{k+1}})\| < \infty$

Soln: Let $\{x_n\}$ be a Cauchy sequence in a normed space X. For $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ s.t $n_{k+1} > n_k$ & $\|x_m - x_n\| < 1/2^k$ for $m, n \geq n_k$

$$\text{Now, } \sum_{k=1}^{\infty} 1/2^k < \infty$$

$$\| x_{n_k} - x_{n_{k+1}} \| < 1/2^k$$

$$\therefore \sum \| x_{n_k} - x_{n_{k+1}} \| < \sum 1/2^k < \infty$$

b) Show that a normed space X is complete iff every absolutely convergent series is convergent.

Solution: Let $\sum x_n$ be an absolutely convergent series.
i.e $\sum_{n=1}^{\infty} \|x_n\| < \infty$

Let $S_n = x_1 + x_2 + \dots + x_n$
& $t_n = \|x_1\| + \|x_2\| + \dots + \|x_n\|$

Since $\{t_n\}$ is convergent, $\{t_n\}$ is a Cauchy sequence.

For any $m, n \in \mathbb{N}$ with $n > m$

$$\begin{aligned} \|S_n - S_m\| &= \| x_{m+1} + \dots + x_n \| \\ &\leq \| x_{m+1} \| + \dots + \|x_n\| \\ &= |t_n - t_m| \end{aligned}$$

For $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ s.t $|t_n - t_m| < \varepsilon$, $\forall$ $m, n \geq n_0$. Then $\|S_n - S_m\| < \epsilon$, $m, n \geq n_0$.

Thus $\{S_n\}$ is a Cauchy sequence in the complete metric space X.

So $\{S_n\}$ is convergent.

$\lim_{n \to \infty} \Sigma x_n$ exist. $\sum_{n=1}^{\infty} x_n < \infty$

Conversely,

If X not complete.

Then $\exists$ a Cauchy sequence $\{x_n\}$ in X that is not convergent. Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\sum_{k=1}^{\infty} \| x_{n_{k+1}} - x_{n_k} \| < \infty$$

$y_1 = x_{n_1}$

$y_2 = x_{n_2} - x_{n_1}$

$y_3 = x_{n_3} - x_{n_2}$

$\vdots$

$y_{k+1} = x_{n_{k+1}} - x_{n_k}$

$\Rightarrow$ $y_1 + y_2 + \dots + y_{k+1} = x_{n_{k+1}}$

& $\|y_1\| + \dots \|y_{k+1}\|$

$= \sum_{j=1}^{k} \| x_{n_j} - x_{n_{j+1}} \|$

Since $\{x_n\}$ is a Cauchy sequence that is not convergent, no subsequence of $\{x_n\}$ is convergent.

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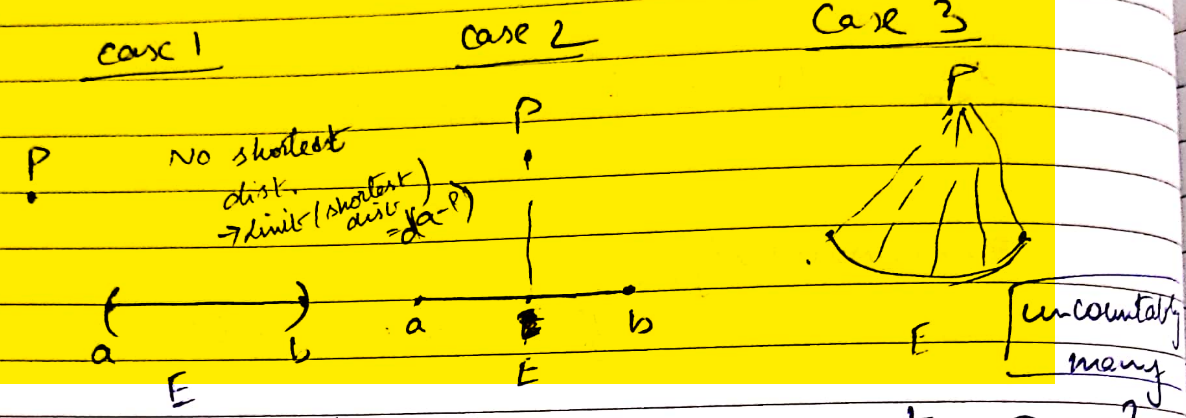
Date 25/9/19

* Dist. from point to set :-

$M \subset X \quad : \quad \delta = \inf_{\tilde{y} \in M} d(x, \tilde{y})$



$\boxed{\text{No min as open set}}$ $\quad \boxed{\text{exist}}$ $\quad P \in \mathbb{R}^2, \; E \subset \mathbb{R}^2$

→ Under what conditions a minimum dist. $\tilde{y} \in E$ (point) exist ?

Ans.

When M (E) is a convex & complete subset of X. (X is Hilbert) [X is a normed space, $\delta = \|x - y\|$]

Completeness → for openess prevent case 1

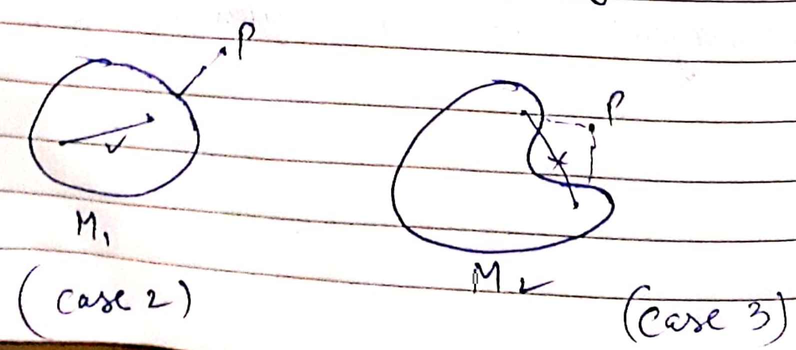
convex → to prevent ~~multiple~~ case 3

A set is a convex set if the segment joining any 2 points of the set, belongs to the set.

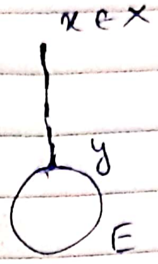
Let $x, y \in X$, segment of x, y is

$\{ z \mid z = \alpha x + (1-\alpha) y, \; 0 \le \alpha \le 1 \}$

Eg:



→ $y \in E$ is said to be the best approximation of x into $E \subset X$ if



$$\|x-y\| \leq \|x-z\| \quad \forall\, z \in E$$

$\therefore$ To find y : $\min\limits_{z \in E} \|x-z\|$

→ Existence of best approximation :-

Theorem : Let H be a ~~Hilbert~~ Inner Product space (Hilbert similar to $\mathbb{R}^n$)

Hilbert $\subset$ Inner Product

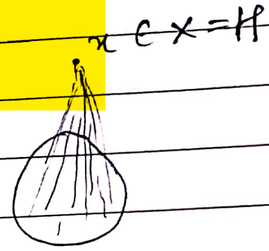
→ If ~~norm can be obtained from~~ inner product in complete normed space

Let E is a convex complete subset of X * (w.r.t metric induced from inner product)

$E \neq \phi$

Then for every $x \in X$ $\exists$ a unique $y \in E$ s.t

$$\delta = \inf_{\tilde{y} \in E} \|x-\tilde{y}\| = \|x-y\|$$



Proof : Consider $\|x-y_1\|$

$\|x-y_2\|$

$\vdots$ $\quad y_n \in E$

$\|x-y_n\|$

As $\delta = \inf \|x-\tilde{y}\|$, $\exists\, y_n \in E$, $\tilde{y} \in E$

s.t $\delta_n = \|x-y_n\| \to \delta$ as $n \to \infty$

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→ Next, to show (y_n) is a cauchy sequence in E

let $y_n - x = v_n$ $\quad \|v_n\| = \delta_n$

$y_m - x = v_m$

$\Rightarrow \|v_n + v_m\| = \|y_n - x + y_m - x\| = 2\|\frac{1}{2}(y_n+y_m) - x\|$

$\left[\frac{1}{2}(y_n+y_m) \in E \text{ as } E \text{ is convex } \& \ y_n, y_m \in E\right]$

$\Rightarrow 2\|\frac{1}{2}(y_n+y_m) - x\| \geq 2\delta$ $\quad$ ('δ' is $\inf \|\tilde{y} - x\|$)

$\Rightarrow \|v_n + v_m\|^2 \geq 4\delta^2 \Rightarrow -\|v_n + v_m\|^2 \leq -4\delta^2$

For $y_n - y_m$:-

$\rightarrow y_n - y_m = v_n - v_m$

$\Rightarrow \|y_n - y_m\|^2 = \|v_n - v_m\|^2 = -\|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|^2)$

$\leq -4\delta^2 + 2(\delta_n^2 + \delta_m^2)$

$\leq -4\delta^2 + 2(2\delta^2)$ $\quad$ $\left[\begin{array}{l}\delta_n \rightarrow \delta \\ \delta_m \rightarrow \delta\end{array}\right]$

$\longrightarrow 0$

as $n \rightarrow \infty$

As, $\|y_n - y_m\|^2 \rightarrow 0$

$\Rightarrow \|y_n - y_m\| \rightarrow 0 \Rightarrow (y_n)$ is a cauchy sequence

As E is complete $\lim\limits_{n\to\infty} y_n = y \in E$

$\therefore \delta = \inf\limits_{\tilde{y} \in E} \|x - \tilde{y}\| = \|x - y\|$

→ Next to show uniqueness of y –

Suppose $\exists\ y_1 \in E$ s.t. $\delta = \|x - y_1\|$

(To show $y = y_1 \to \|y - y_1\| = 0$)

$$\|y - y_1\|^2 = \|(y - x) - (y_1 - x)\|^2$$
$$= 2\|y - x\|^2 + 2\|y_1 - x\|^2 - \|(y - x) + (y_1 - x)\|^2$$
$$= 2\delta^2 + 2\delta^2 - 2^2\left\|\tfrac{1}{2}(y + y_1) - x\right\|^2$$

($\frac{1}{2}(y + y_1) \in E$ as E is convex
$\therefore \left\|\frac{1}{2}(y + y_1) - x\right\|^2 \geq \delta^2$)

$$\|y - y_1\|^2 \leq 2\delta^2 + 2\delta^2 + 4(-\delta^2)$$

$$\|y - y_1\|^2 \leq 0$$

$$\Rightarrow \|y - y_1\| = 0$$

Date 2/7/17

* Finding the best approximation when it exists :

Theorem :-

Lemma 1: Orthogonality : 2 vectors are orthogonal if their inner product is 0.

x orthogonal $(\perp)$ to $y \Rightarrow \langle x, y \rangle = 0$

$x \perp M \Rightarrow \langle x, y \rangle = 0 \ \forall \ y \in M \Rightarrow x \in M^{\perp}$

$A \perp B \Rightarrow \langle a, b \rangle = 0 \ \forall \ a \in A, \ b \in B$

$M^{\perp} = \{ z \in X \mid \langle z, m \rangle = 0 \ \forall \ m \in M \}$
(Orthogonal complement)

Considering the theorem of existence of best approximation as theorem 1 :-

Th: In theorem 1, let M be a complete subspace Y & $x \in X$. Then $z = x - y$ is orthogonal to Y.

Proof: As Y is convex & complete by theorem 1,
$\forall \ x \in X, \ \exists \ y \in M = Y$ s.t $\delta = \|x - y\|$

To show ~~that~~ $\langle z, y^* \rangle = 0 \ \forall \ y^* \in Y$:-

Suppose $\exists \ y_1 \in Y$ s.t $\langle z, y_1 \rangle \neq 0$

Then $y_1 \neq 0$ as $\langle z, 0 \rangle = 0$.

& $\|z - \alpha y_1\|^2 = \langle z - \alpha y_1, z - \alpha y_1 \rangle$
$= \langle z, z \rangle - \bar{\alpha} \langle z, y_1 \rangle - \alpha \langle y_1, z \rangle + \alpha \bar{\alpha} \langle y_1, y_1 \rangle$

Let $\beta = \langle z, y_1 \rangle \neq 0$

$\therefore \quad \|z - \alpha y_1\|^2 = \|z\|^2 - \bar{\alpha}\beta - \alpha\bar{\beta} + |\alpha|^2 \|y_1\|^2$

$\alpha \in \mathbb{F} \Rightarrow$ set $\alpha = \bar{\beta}/\|y_1\|^2$ As result holds for all 'α'.

$$= \|z\|^2 - 2\frac{|\beta|^2}{\|y_1\|^2} + \frac{|\beta|^2}{\|y_1\|^2}$$

$$= \|z\|^2 - \underbrace{\frac{|\beta|^2}{\|y_1\|^2}}_{>0} < \|z\|^2 = \delta^2 \quad - \text{(A)}$$

Now, $z - \alpha y_1 = x - y - \alpha y_1$

$= x - (y + \alpha y_1) = x - y_2 \quad \left(\begin{array}{c} Y \text{ is subspace} \\ y_2 \in Y \end{array} \right)$

But By definition of δ :-

$\|z - y_2\| > \delta$ (infimum property)

$\Rightarrow \|z - \alpha y_1\| > \delta > 0$

But from (A) $\|z - \alpha y_1\|^2 < \delta^2$

$\boxed{\text{Contradiction.}}$ $\Rightarrow \langle z, y^* \rangle = 0 \;\; \forall \, y \in Y$

where $z = x - y$,

$\|x - y\| = \inf_{\tilde{y} \in Y} \|x - \tilde{y}\|$

Lemma 2: A necessary & sufficient condition that $y \in M$ (M is a (complete) subspace of X) be the unique best approximation to x from y is that $(x-y) \perp M$.

$A \Rightarrow B$: Proved earlier

$A \Leftarrow B$: ~~Proof~~ :- $\forall x \in X, y \in M$, if $x-y \perp M$, then y is the best approximation

i.e $\|x-y\| \leq \|x-\tilde{y}\| \quad \forall \tilde{y} \in M$

Proof :-

Given $x-y \perp M \Rightarrow$ for any $z \in M$

$\langle x-y, y-z \rangle = 0 \quad$ [$y-z \in M$ as M is vector space]

$\Rightarrow \|(x-y)+(y-z)\|^2 = \|x-y\|^2 + \|y-z\|^2$

$\Rightarrow \|x-z\|^2 = \|x-y\|^2 + \|y-z\|^2$

$\Rightarrow \|x-y\|^2 \leq \|x-z\|^2$, if $y \neq x$

$\Rightarrow \|x-y\| \leq \|x-z\|$ if, $y \neq x$ $(\forall z \in Y)$

Hence Proved

$x \perp y \Rightarrow \langle x,y \rangle = 0$

$\|x+y\|^2 = \langle x+y, x+y \rangle$

$= \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle$

$= \|x\|^2 + \|y\|^2$

Date 27/9/19

Hilbert space as direct sum :-

Direct sum: Let Y, Z be 2 vector subspaces of X & $Y \cap Z = \{0\}$.

$X = Y \oplus Z$

iff every $x \in X \Rightarrow x = y + z,\ y \in Y, z \in Z$

y, z are unique

Then,

$$H = Y \oplus Y^{\perp}$$

; H is a Hilbert space

Y is a closed ~~sub~~ space of H.

Proof :-

$$Y^{\perp} = \{ z \in H \mid z \perp Y \}$$

Since Y is closed subspace of Hilbert space H, it is complete. By "Existence of best approximation" theorem & "$x - y \perp Y$" lemma

$\forall x \in H$, $\exists y \in Y$, st $x - y \perp Y$

Let $x - y = z \Rightarrow x = y + z$, $y \in Y$, $z \in Y^{\perp}$

Uniqueness :-

Suppose $\exists\ y_1 \in Y$, $z_1 \in Z = Y^{\perp}$ st $x = y_1 + z_1$

As $z, z_1 \in Y \Rightarrow z - z_1 \in Y$

$x = y + z = y_1 + z_1$

$\Rightarrow y - y_1 = z_1 - z$

$\Rightarrow$ As $y - y_1 \in Y$, $z_1 - z \in Y^{\perp}$

$\Rightarrow y - y_1, z_1 - z \in Y \cap Y^{\perp} = \{0\}$

$\Rightarrow y = y_1$, $z_1 = z$

Proved

Eg: $L^2[a,b] = \{ f : \int_a^b |f|^2 < \infty \}$

L^2 is Hilbert
L^p is not Hilbert if $p \neq 2$

Y = space of even functions
$Y^\perp$ - space of odd function

$L^2[a,b] = Y \oplus Y^\perp$

* **Projection Function :-**

$P : X \longrightarrow X$ [(or) $P : X \to Y$, onto function]
$x \longmapsto y = Px$ $\quad$ ($x = y + z$, $y \in Y$, $z \in Y^\perp$)

→ The projection function is linear :-

? $P(x+x_1) = Px + Px_1$?

~~$P(\alpha x)$~~

$x + x_1 = y + z + y_1 + z_1 = (y+y_1) + (z+z_1)$
$\Rightarrow P(x+x_1) = y + y_1$
$= Px + Px_1$

? $P(\alpha x) = \alpha Px$?

$\alpha x = \alpha y + \alpha z$
$P(\alpha x) = \alpha y = \alpha Px$

→ $P^2 = P \circ P = P$:- [P is idempotent]

$P^2 x = P(Px) = P(y)$
Now $y \in X$, $y = y + 0$, $y \in Y$, $0 \in Y^\perp$
$\therefore P^2 x = P(y) = y = Px$

→ An Hilbert space can be written as the direct sum of the range space & null space of projection map.

$$H = \text{ran } P + \text{null } P = R(P) + N(P)$$

Proof :-

$\text{Range}(P) = Y$

$\text{Null}(P) = Y^{\perp}$

$\therefore \quad H = Y \oplus Y^{\perp} = R(P) + N(P)$

$R(P) \subset Y \rightarrow$ Prove $Y \subset R(P)$

$N(P) \subset Y^{\perp} \rightarrow$ Prove $Y^{\perp} \subset N(P)$

Theorem :-

Let X be a Hilbert space.

(a) If $P : X \to X$ is a projection, then $X = \text{ran } P \oplus \text{null } P$).

(b) If $X = M \oplus N \ \exists \ P : X \to X$ with $\text{ran } P = N$, $\text{null } P = M$.

M, N are vector subspaces of X

Proof :-

a) We first show that $x \in \text{range}(P)$ iff $x = Px$

If $x = Px$, then $x \in R(P)$. If $x \in R(P)$ then $x = Py$ for some $y \in X$ & since $P^2 = P$

$Px = \cancel{P^2x} = P(Py) = Py = x$

→ To show $R(P) \cap N(P) = \{0\}$

Let $x \in$ ran P $\cap$ null P

$\Rightarrow$ $x = Px$ & $Px = 0$

$\Rightarrow$ $x = 0$

$\therefore$ $R(P) \cap N(P) = \{0\}$

→ All elements ~~have unique~~ have unique representation & ~~N(P)~~)

$x = Px + (x - Px)$

$\in R(P)$ ~~$\in N(P)$~~

$P(x - Px) = Px - P^2x$

$= Px - Px = 0$

$\therefore$ $x - Px \in N(P)$

b) If $X = M \oplus N$, then $x \in X$ has a unique decomposition

$x = y + z$ with $y \in M$, $z \in N$

$Px = y$ defines the required projection map.

$x \mapsto y = Px$

$M = \text{ran}(P)$, $N = N(P)$

(Construction)

→ In case of Hilbert space :-

$H = M \oplus M^{\perp}$

If $x = y + z$, $x_1 = y_1 + z_1$, $y, y_1 \in M$, $z, z_1 \in M^{\perp}$

$\langle Px, x_1 \rangle = \langle y, y_1 + z_1 \rangle = \langle y, y_1 \rangle + \langle y, z_1 \rangle$

$= \langle y, y_1 \rangle$ $[\because y \perp z_1]$

$= \langle y, y_1 \rangle + \langle z, y_1 \rangle$

$[\because z \perp y_1]$

$= \langle z + y, y_1 \rangle$

$= \langle x, Px_1 \rangle$

Any projection operator P is a self-adjoint operator

$\therefore \langle Px, y \rangle = \langle x, Py \rangle$

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Def : An orthogonal projection on a Hilbert space H is a linear map $P: H \to H$ satisfying $P^2 = P$, $\langle Px, y\rangle = \langle x, Py\rangle$ for $x, y \in H$

Proposition : If P is a non-zero orthogonal projection then $\|P\| = 1$

Proof : If $x \in H$ & $Px \neq 0$:-

$$\|Px\| = \frac{\langle Px, Px\rangle}{\|Px\|}$$

self-adjoint operator P.

$$= \frac{\langle x, P^2x\rangle}{\|Px\|} = \frac{\langle x, Px\rangle}{\|Px\|}$$

$$\leq \frac{\|x\| \ \|Px\|}{\|Px\|} = \|x\|$$

$\therefore \|Px\| \leq \|x\| \Rightarrow \|P\| \leq 1$ — (1)

If $P \neq 0$, $\exists\ x \in H$, s.t $Px \neq 0$

$$\|P(Px)\| = \|Px\|$$

$$P(x) = y$$

$$\|P(y)\| = \|y\|$$

$\Rightarrow \|y\| \leq \|P\| \cdot \|y\|$

$\Rightarrow \|P\| \geq 1$ — (2)

$\therefore$ From (1), (2) $\|P\| = 1$

Date 10/3/19

The → **Theorem**: Let H be a Hilbert space
(a) If P is an orthogonal projection on H, then $ran(P)$ is closed and $H = ran\, P \oplus Null\, P$

(b) If M is closed subspace of H, then $\exists$ an orthogonal Projection P on H with $ran\, P = M$ & $null\, P = M^{\perp}$

$f : H \to \mathbb{R}$
$f(x) = \langle x, y \rangle \quad y \in H.$
$\|f\| = \|y\|$

10/3/19

* Orthonormal :-

→ When M is orthogonal, $\forall x \in M$, $\|x\| = 1$
Then the set M is said to be orthonormal

→ If $\{e_1, e_2, \dots, e_n\}$ is an orthonormal vector basis Then any $x \in X$ can be expressed as a linear combination of e_i's.

$$x = \sum \alpha_i e_i$$

& $\langle x, e_i \rangle = \langle \sum \alpha_j e_j, e_i \rangle = 0 + \langle \alpha_i e_i, e_i \rangle + 0$

$= \alpha_i \|e_i\|^2 = \alpha_i \quad$ ($\langle e_j, e_i \rangle = 0$ if $j \neq i$ orthonormal)

$$\therefore x = \sum \langle x, e_i \rangle e_i$$

→ Fourier coefficients

Date 10/3/19

→ For infinite dimensional set :-

basis = $\{ x_\alpha , \alpha \in I \}$ I = index set

→ Let M be a closed subspace of a Hilbert space

Then, $M^{\perp\perp} = (M^{\perp})^{\perp} = M$

Proof :-

For any subset :-

($\Rightarrow$) ~~$M \subset M^{\perp\perp}$~~

$x \in M \Rightarrow x \perp M^{\perp} \Rightarrow x \in M^{\perp\perp}$

$\Rightarrow M \subset M^{\perp\perp}$

($\Leftarrow$) Let $y \in M^{\perp\perp}$ $M \oplus M^{\perp}$

Any Hilbert space $H =$ ~~$M^{\perp} \oplus (M^{\perp})^{\perp}$~~

Then $x = y + z$, where $y \in M \subset M^{\perp\perp}$

Since $M^{\perp\perp}$ is a vector space $\Rightarrow$ $M^{\perp\perp}$ is a vector space

& $x \in M^{\perp\perp}$ $\Rightarrow z = x - y \in M^{\perp\perp}$,

$\therefore z \perp M^{\perp}$, $z \in M^{\perp}$

$\Rightarrow \langle z, z \rangle = 0$

$\Rightarrow z = 0$

$\in M$

$z \in M^{\perp}$

→ Pythagoras theorem :-

Let $\{x_1, x_2, \dots, x_n\}$ be an orthogonal set in an IPS

Then,

$$\left\|\sum_1^n x_i\right\|^2 = \sum_1^n \|x_i\|^2$$

Proof :-

$$\left\|\sum x_i\right\|^2 = \left< \sum x_i , \sum x_i \right>$$

$$= \sum_{j=1}^n \left< \sum_1^n x_i , x_j \right> = \sum_{j=1}^n \sum_{i=1}^n \left< x_i , x_j \right>$$

$$\left[\left< x_i , x_j \right> = 0 \quad \text{if} \quad i \neq j \right]$$

$$= \sum_{j=1}^n \left< x_j , x_j \right> = \sum_{j=1}^n \|x_j\|^2$$

$$= \sum_1^n \|x_i\|^2$$

→ If E is an orthonormal subset of an IPS X, then E is linearly independent.

Proof :-

Let $F = \{e_1, \dots, e_n\}$ be a finite subset of E which is an orthogonal set in X. To show it is linearly independent.

To show: $\sum \alpha_i e_i = 0 \Rightarrow \alpha_i = 0 \ \forall i$

$$\left< \sum \alpha_j e_j , e_i \right> = \left< \alpha_i e_i , e_i \right> = \alpha_i \|e_i\|^2$$

If $\sum \alpha_j e_j = 0 \Rightarrow \left< \sum \alpha_j e_j , e_i \right> = 0$

$\Rightarrow \alpha_i \|e_i\|^2 = 0 \Rightarrow \alpha_i = 0$ For any i

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Gram-Schmidt Orthonormalization Process :-

Let $\{x_1, \ldots, x_n\}$ $\{x_n : n \in \mathbb{N}\}$ $n \to \infty$ be a linearly independent set in an IPS.

Define $y_1 = x_1$, $u_1 = y_1/\|y_1\|$

& for $n = 2, 3, \ldots$

$$y_n = x_n - \sum_{j=1}^{n-1} \langle x_n, y_j \rangle u_j$$

Then $\{u_n : n \in \mathbb{N}\}$ is an orthonormal set ~~and span~~ in X and ~~span~~

$\text{span}\{x_1, \ldots, x_n\} = \text{span}\{u_1, \ldots u_n\}$

Proved using method of induction

Let $X = \ell^2$ & for $n = 1, 2, \ldots$

set $x_n = (\underbrace{1, 1, 1, \ldots 1}_{n \text{ terms}}, 0, 0, 0, 0 \ldots \ldots)$

i.e '1' occurs in first n entries & '0' in remaining

$x_n \in \ell^2$ $\because$ $\sum \|x_n\|^2 = 1^2 + 1^2 + \ldots + 0^2 + 0^2 + \ldots < \infty$

Let $x_1 = (1, 0, \ldots)$, $x_2 = (1, 1, 0, \ldots 0, 0 \ldots)$, ..., $x_n = (\underbrace{1, \ldots, 1}_{n}, 0, \ldots)$ 'ous

Let $\sum \alpha_i x_i = 0$

$\Rightarrow \alpha_i = 0 \rightarrow L.I$

G.S.ON :-

$y_1 = x_1$, $u_1 = x_1/\|x_1\| = x_1$

$y_2 = x_2 - \langle x_2, u_1 \rangle u_1 = (1, 1, 0, \ldots) - [(1 \times 1 + 1 \times 0 + 0)](1, 0, \ldots)$

$= (1, 1, 0, \ldots) - 1(1, 0 \ldots)$

$= (0, 1, 0, \ldots)$

$u_2 = y_2/\|y_2\| = y_2$

& $u_n = (0, 0, \ldots, \underset{n^{th} \text{ bit}}{1}, 0 \ldots)$

→ Bessel's Inequality (Theorem)

Let $\{u_1, u_2, \dots, u_n\}$ be an orthonormal set in X (ips). Then for every $x \in X$

$$\sum_1^\infty |\langle u_n, x\rangle|^2 \leq \|x\|^2,$$

where equality holds iff $x \in \text{span}\{u_1, u_2, \dots, u_n\}$

i.e. $\sum_1^\infty |\langle u_n, x\rangle|^2 = \|x\|^2$ iff $x = \sum \alpha_i u_i$

Proof :-

Let $x \in X$ & define $x_m = \sum_{n=1}^{m} \langle x, u_n\rangle u_n$

Now by orthonormality of $\{u_1, u_2, \dots, u_n\}$, we have

$$\langle x, x_m\rangle = \langle x_m, x_m\rangle = \sum_1^m |\langle x, u_n\rangle|^2$$

$$\because \langle x, x_m\rangle = \langle x, \sum_1^m \langle x, u_m\rangle u_m\rangle$$

$$= \sum_1^m \langle x, \langle x, u_n\rangle u_n\rangle$$

$$= \sum_1^m \overline{\langle x, u_n\rangle}\langle x, u_n\rangle$$

$$= \sum_1^m |\langle x, u_n\rangle|^2$$

$$\therefore \langle x_m, x_m\rangle = \overline{\langle x_m, x_m\rangle} = \langle x, x_m\rangle \text{ Real no.}$$

$$\& \quad \langle x_m, x_m\rangle = \|x_m\|^2 \cancel{= \left(\sum_1^m \|\langle x, u_n\rangle u_n\|\right)^2}$$

$$= \sum |\langle x, u_n\rangle|^2$$

Now,

$$0 \leq \|x - x_m\|^2 = \langle x - x_m, x - x_m \rangle$$

$$= \langle x, x \rangle - \cancel{\langle x_m, x \rangle} - \cancel{\langle x, x_m \rangle} + \cancel{\langle x_m, x_m \rangle}$$

$$= \langle x, x \rangle - \langle x, x_m \rangle$$

$$= \|x\|^2 - \langle x, x_m \rangle$$

$$\Rightarrow \quad \langle x, x_m \rangle \leq \|x\|^2$$

$$\Rightarrow \quad \sum_{1}^{m} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

As R.H.S is independent of 'm', taking $m \to \infty$

$$\sum_{1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

Equality holds iff $\|x - x_m\| = 0$ i.e $x \in \text{span}\{u_1, \ldots, u_m\}$

X : norm space $\qquad$ X : vector space

X' : dual space $\qquad$ X^* : Algebraic Dual

X'' : $\qquad$ X^{**}

Ex: $(\ell^p)' = \ell^q$, p, q are conjugates

Ex: H is a Hilbert space

$$f \in H'$$
$$\Rightarrow f : H \longrightarrow \mathbb{K}$$

f is linear & bounded

* Riesz's Representation Theorem (Functionals on Hilbert space)

Every bounded linear functional f on H (Hilbert space) can be represented in terms of inner product, namely $f(x) = \langle x, z \rangle$ where z depends on f, is uniquely determined by f and has norm $\|z\| = \|f\|$

~~Bounded linear~~

Proof :-

Uniqueness: $\langle x, z \rangle = \langle x, z_1 \rangle$, to show $z = z_1$

$$\Rightarrow \langle x, z \rangle - \langle x, z_1 \rangle = 0$$
$$\Rightarrow \langle x, z - z_1 \rangle = 0 \quad \forall x \in H$$

Let $x = z - z_1$

$$\Rightarrow \langle z - z_1, z - z_1 \rangle = 0$$
$$\Rightarrow \|z - z_1\|^2 = 0$$
$$\Rightarrow z = z_1$$

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Norm: To show $\|z\| = \|f\|$

If $f = 0 \Rightarrow \langle x, z\rangle = 0 \; \forall x$
$\Rightarrow z = 0$
$\Rightarrow \|f\| = \|z\|$

If $f \neq 0$, $z \neq 0$
consider $f(z) = \langle z, z\rangle = \|z\|^2$
$\Rightarrow \|z\|^2 = f(z) \leq \|f\| \|z\|$
$\Rightarrow \|z\| \leq \|f\|$ — ①
To show $\|f\| \leq \|z\|$:-

$|f(x)| = |\langle x, z\rangle| \leq \|x\| \|z\|$
$\Rightarrow \|f\| = \sup_{(\|x\|=1)} |\langle x, z\rangle| \leq \|z\|$

$\Rightarrow$ ie $\|f\| \leq \|z\|$ —②

①, ② $\Rightarrow \|f\| = \|z\|$

Existence: If $f = 0$, $f(x) = \langle x, 0\rangle = 0$ $(z=0)$, the theorem follows

Assume $f \neq 0$, $\exists\, z_0 \in H$ st $f(z_0) \neq 0$
$F = f^{-1}\{0\} = \{x \in H : f(x) = 0\}$
$\rightarrow F$ is a subspace of H
$x, y \in F \Rightarrow \alpha x + \beta y \in F \Rightarrow f(\alpha x + \beta y) = 0$
$\Rightarrow$
$\rightarrow F$ is a closed subspace of H
$\Rightarrow H = F \oplus F^{\perp}$
But $f \neq 0 \Rightarrow F \neq H$. Hence $F^{\perp}$ contains atleast one non-zero element & hence an element $\|y\| = 1$ ($\because$ let $y = \frac{u}{\|u\|} \in F^{\perp}$)
(say u)
$\forall x \in H$, $f(x)y - f(y)x \in F$
$\because f(\underbrace{f(x)}_{\text{scalar}} y - f(y)x) = f(x)f(y) - f(y)f(x) = 0$
~~$\Rightarrow y \in F^{\perp}$~~

~~And~~ $\Rightarrow \langle f(x)\,y - f(y)\,x \,,\, y \rangle = 0$

$\Rightarrow f(x)\,\|y\|^2 - f(y)\langle x, y\rangle = 0$

$\|y\|^2 = 1$

$\Rightarrow f(x) = \langle x, \overline{f(y)}\,y \rangle$

$= \langle x, z\rangle, \quad z = \overline{f(y)} \cdot y$

which is the representation of f.

→ Riesz Representation Theorem :-

Bounded Linear Map $f : H \to \mathbb{K}$

$\exists$ a unique $z \in H$,

s.t. $f(x) = \langle x, z\rangle$, $z \in H$

$\|f\| = \|z\|$

→ Heine-Banach Extension Theorem :-

(in Hilbert space, ~~can~~ ~~be~~ also applies to metric space)

Let X be a Hilbert space & Y be its subspace.

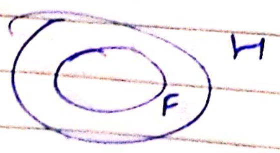
Let $f \in Y'$ (Bounded Linear maps of Y) then f can be extended to X, ~~$\exists g \in X'$~~ i.e.

$\exists\, g \in X'$ s.t. $g|_Y = f$, $\|g\| = \|f\|$

(that is unique)

Let H be a Hilbert space, F a subspace of H & g a bounded linear ~~map~~ ~~of~~ functional on F. Then there is a unique $f \in H'$ st $f|_F = g$ & $\|f\| = \|g\|$

$g : F \to \mathbb{K}$



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Axiom of Choice : Arbitrary product of non empty sets is non empty.

Proof :-

Since g is a bounded linear functional on F, then g admits a norm preserving linear extension to the closure of F in H i.e.

$$\tilde{g} : \bar{F} \to H \quad \& \quad \|\tilde{g}\| = \|g\|$$

Without loss of generality, F is a closed subspace of H. Since H is a Hilbert space and F is a closed subspace of the Hilbert space, F is a Hilbert space.

Using Riesz Representation ~~of~~ on F,

$$\Rightarrow g(x) = \langle x, z \rangle, \ x \in F$$

Also $\|g\| = \|z\|$

Defining $f : H \to K$

$f(x) = \langle x, y \rangle$, $x \in H$; $\|f\| = \|g\| = \|y\|$

$f \in Y'$, $\exists \underset{\text{unique}}{g} \in X'$ s.t $g|_Y = f$

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Uniqueness : Let $h \in H'$ with $h|_F = g$ & $\|h\| = \|g\|$. Let $z \in H$ be the representation of H so that $\|h\| = \|z\|$.

Now, $\frac{1}{2}(f+h)|_F = g$ & $\frac{1}{2}(y+z)$ is the representation of $\frac{1}{2}(f+h)$. Hence

$\|y\| = \|g\| \leq \|\frac{1}{2}(f+h)\| \cancel{\leq \|\frac{1}{2}(g+g)\|} \leq \frac{1}{2}(\|y\| + \|z\|) = \|y\|$

$\Rightarrow \frac{1}{2}(y+z) = \|y\| = \|z\|$

$$\|y+z\|^2 + \|y-z\|^2 = 2\left(\|y\|^2 + \|z\|^2\right)$$

$$\Rightarrow \|y-z\|^2 = 2\left(\|y\|^2 + \|z\|^2\right) - \|y+z\|^2$$

$$= 2\left(\|y\|^2 + \|z\|^2\right) - 4\|z\|^2$$

$$= 2\left(\|y\|^2 - \|z\|^2\right)$$

$$= 0$$

~~Now~~

$$\Rightarrow \|y-z\| = 0$$

$$\Rightarrow y = z \rightarrow \boxed{f = h}$$

General Heine Banach Theorem :-

X - vector space

Y - subspace

$f \in Y^*$

$g \in X^*$ s.t $g|_Y = f$

Then $|f| \leq \alpha$

$\Rightarrow |g| \leq \alpha$

A Hyperplane in a vector space X is a maximal proper linear variety. i.e Here if H is a hyperspace with $H \neq X$ & V is any hyperspace containing H, then $V = X$ (or) $V = H$

Let H be a subspace of X. $x_0 \in X$, then a linear variety is written as $H + x_0$. A maximal proper subspace of a vector space is called a <u>hyper space</u>. Shifting a hyperspace is called a hyperplane.

A proper subspace Z of X is said to be maximal iff $X = \text{span}\{Z \cup \{a\}\}$ for each $a \notin Z$.

$\dim(X/Z) = 1$ (or) $\text{codim } Z = 1$

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(also a hyperplane)

If Z is a hyper space (Proper subspace), $Z + x$, $x \in X$ is a hyperplane (may not be a subspace)

$$\text{span of } (Z \cup \{x_0\}) = X$$

$$x_0 \notin Z$$

$$\dim(X/Z) = 1$$

~~Proof:~~

Let H be a hyper plane in a vector space X. Then there is a linear functional f on X and a constant 'c' s.t $H = \{x \mid f(x) = c\}$. Conversely, if f is a non-zero linear functional on X, Then the set $\{x, f(x) = c\}$ is a hyperplane

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Proof :-

Let H be a hyper~~space~~plane ~~of~~ in X. Then $H = x_0 + M$, where M is hyperspace in X, $x_0 \notin M$.

$$X = \text{span}\{M \cup \{x_0\}\}, \text{ then}$$

$$x = \alpha x_0 + m, \text{ uniquely with } \alpha \in \mathbb{R} \text{ \& } m \in M$$

Define $f(x) = \alpha$, then $H = \{x \mid f(x) = 1\}$.

If $x_0 \in M$ take $x_1 \notin M$, & $X = \text{span}\{M \cup \{x_1\}\}$

$H = M$, & define $x = \alpha x_1 + m$, $f(x) = \alpha$, $H = \{x : f(x) = 0$

$(\Leftarrow)$

$f \neq 0$, i.e. $\exists\, x^* \in X$ such that $f(x^*) = 0$

Let $M = \{x \in X \mid f(x) = 0\}$ M is a subspace of X, called nullspace. Let $x_0 \in X$, $f(x_0) = 1$. Then for $x \in X$

$$z = x - f(x).x_0, \quad f(z) = f(x - f(x)x_0) = f(x) - f(x).f(x_0) = 0$$

Hence,

$$x - f(x)x_0 \in M \Rightarrow x - f(x)x_0 = m$$

$$\Rightarrow x = m + f(x)x_0$$

$$X = \text{span}\{M \cup \{x_0\}\}$$

i.e. M is a maximal proper subspace. For any c let 'x_1' be any element for which $f(x_1) = c$

$$\{x : f(x) = c\} = \{x : f(x - x_1) = 0\}$$

$$= M + x_1, \text{ which is a hyper plane.}$$

Scanned by CamScanner

1) A hyperplane H in a normed space X is either closed or dense in X because as H is maximal proper subspace of X.

$$H = \bar{H} \quad (\text{or}) \quad \bar{H} = X$$

Let f be a non-zero functional on X (normed space). Then the hyperplane $H = \{x : f(x) = c\}$ is closed for every c iff f is bounded.

If $f \neq 0$, linear functional on X, we associate with the hyperplane $H = \{x \mid f(x) = c\}$ 4 sets $\{x : f(x) \leq c\}$, $\{x : f(x) < c\}$, $\{x : f(x) \geq c\}$, $\{x : f(x) > c\}$ called half spaces determined by H. The 1st two are called negative half spaces, last two are positive half spaces.

Definition (- Category :-

A subset M of X is said to be

a) rare (nowhere dense) in X if its closure $\bar{M}$ has no interior pts. i.e. $(\bar{M}^{\circ}) = \phi$

Ex: Every finite set is nowhere dense in $\mathbb{R}$.

b) meager (or the first category) in X if M is the ~~countable~~ union of countably many sets each of which is rare in X.

Ex: $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$

c) Non-meager (or 2nd category) in X if M is not ~~of~~ 1st category or (rare)

Baire's Category Theorem (Complete Metric space)
If a metric space $X \neq \phi$ is complete, it is of 2^{nd} category (or) non-meager.

Note: If $X \neq \phi$ is complete and $X = \bigcup_{k=1}^{\infty} A_k$, ($A_k$ is closed) Then atleast one A_k contains a non-empty open-subset. ($\exists$ $B(x;r) \subset A_k$ for some k)

$f_i : X \to \mathbb{R}$
collection of f^n's $\{f_i\}_{i \in I}$
where f_i are pointwise bounded.
i.e. $f_i(x) \leq C_x$. $\forall f_i$

A collection of functions $A = \{f_i : X \to \mathbb{R}\}$ on X is said to be uniformly bounded if $\exists$ $M \in \mathbb{R}$ s.t $|f_i(x)| \leq M$ $\forall f_i \in A$, $\forall x \in X$
$\Rightarrow$ $\|f_i(x)\| \leq M$ $\forall i$

It is ~~said~~ pointwise bounded if $\exists$ $M_x \in \mathbb{R}$ s.t $|f_i(x)| \leq M_x$ $\forall f_i \in A$

Ex: $A = \{f_1 = x, f_2 = 2x, \ldots f_n = nx\}$
$A \subset C[0,1]$, then each function is pointwise bounded, but not uniformly bounded because,

For M is any real number, however large, $\exists$ $n_0 \in \mathbb{N}$ with $n_0 > M$ & $f_{n_0}(x_0) = n_0 > M$, $x_0 = 1$.

Ex 2: $A = \{ f_1 = \sin x_1, f_2 = \sin x_2, \dots, f_n = \sin x_n \}$

$A \subset C(\mathbb{R})$ are uniformly bounded.

$|f_i(x)| = |\sin x_i| \leq 1$

$\forall i, x$

$\boxed{M = 1}$

Uniform Boundedness Principle (or) Theorem :-

Let $\{T_n\}$ be a sequence of bounded linear operators. $T_n : X \to Y$ from a Banach space X into a normed space Y such that $\|T_n x\|$ is bounded for every $x \in X$, (pointwise bounded)

say,

$$\|T_n x\| \leq C_x \quad n = 1, 2, 3, \dots, \quad — (2)$$

where C_x is a real number. Then the sequence of norms $\|T_n\|$ is bounded, i.e. $\exists\, C$ s.t $\|T_n\| \leq C$, $n = 1, 2, \dots,$

$— (3)$

23/10/19

Proof :-

For any $k \in \mathbb{N}$ $A_k = \{ x \in X \mid \|T_n(x)\| \leq k \}$ $n = 1, 2, \dots$

To show that A_k is closed, Let $x \in \bar{A}_k$, $\exists\, (x_n) \in A_k$ s.t $x_j \to x$ i.e for every fixed 'n', we have $\|T_n(x_j)\| \leq k$.

As T_n is continuous $T_n(x_j) \to T_n x$. Again $\|\cdot\|$ is also continuous

$\|T_n(x_j)\| \to \|T_n x\|$ $\quad \|T_n(x)\| = \|\lim_{j \to \infty} T_n(x_j)\|$

$\Rightarrow \quad \|T_n(x)\| \leq k$

$\Rightarrow \quad x \in A_k \quad \Rightarrow \quad A_k$ is closed in X.

As X is complete & non-empty
$X = \bigcup_{k=1}^{\infty} A_k$ with atleast one $\overline{A_k}^{\circ} \neq \phi$
(Baire Category Theorem)

i.e some A_k contains an ~~fat~~ open ball say
$B_0 = B(x_0 ; r) \subset A_k$

Let x be an arbitrary, non-zero element of X.
$z = x_0 + \gamma x$, $\gamma = \frac{r}{2\|x\|}$

$$\|z - x_0\| = \|\gamma x\| = \frac{r}{2\|x\|} \cdot \|x\| = \frac{r}{2} < r$$
$\Rightarrow z \in B_0$

i.e. $z \in A_{k_0}$, $\|T_n z\| \leq K_0 \ \forall n$
Also, $\|T_n x_0\| \leq K_0$

$$\therefore \|T_n x\| = \left\|T_n\left(\frac{1}{\gamma}(z - x_0)\right)\right\| = \frac{1}{\gamma}\|T_n z - T x_0\|$$

$$\leq \frac{1}{\gamma}\left(\|T_n z\| + \|T x_0\|\right)$$

$\Rightarrow \|T_n x\| \leq \frac{4K_0\|x\|}{r}$ $= \frac{2K_0}{\gamma}$

Taking $\|x\| = 1$ $= \frac{4K_0\|x\|}{r}$

$$\Rightarrow \|T_n\| = \sup_{\|x\|=1} \|T_n x\| \leq \frac{4K_0}{r} = C$$
(Let)

$\therefore \|T_n\| \leq C$

Geometrical Representation of Uniform Boundedness Principle:-

It says that either T_n maps a given bounded subset of a Banach space X into a fixed ball in the normed space Y, or else there is some $x \in X$ s.t no ball in Y contains all $T_n(x)$.

The normed space X of all polynomials with norm defined by $\|x\| = \max_j |a_j|$

$$x(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$$
$$= \sum_{j=0}^{\infty} a_j t^j \, , \quad a_j = 0 \quad \forall \; j > N_x$$

'x' is a polynomial of degree N_x.

Let $T_n = f_n$, sequence of functionals on X.

$$T_n(0) = f_n(0) = 0$$
$$T_n x = f_n(x) = a_0 + a_1 + \dots + a_{n-1}$$

Weakly Convergent :- 24/10/19

→ $\{x_n\}$ is a sequence in n.l.s X (normed linear space)

$x_n \to x$

$\lim_{n\to\infty} \|x_n - x\| = 0$

(x_n) converges to x strongly

And we say $x_n \xrightarrow{w} x$ (weakly) if $f(x_n) \to f(x) \;\; \forall \; f \in X'$ dual space

→ X: Space of all polynomials

$$x \in X \Rightarrow x = \sum_{i=0}^{n} \alpha_i t^i \, , \quad \alpha_n \neq 0 \, , \text{ for any } n \in \mathbb{N}$$

$$\|x\| := \max_{i \in \{0, \dots, n\}} (|\alpha_i|)$$

$(X, \|\cdot\|)$ is normed linear space.

$(X, \|\cdot\|)$ is not a Banach space.
(complete)

Proof:-

Let $T_n = f_n$ be a sequence of functionals defined,
$T_n(0) = f_n(0) = 0$ (Linear map)

Now,

$T_n : X \to \mathbb{K}$, sequence of operators
$T_n x = f_n(x) = \sum_{0}^{n-1} \alpha_i$

(f_n) is a series of linear functionals

$$\boxed{\begin{array}{l} f_n : X \to \mathbb{R} \\ f_n(x+y) = f_n(x) + f_n(y) \\ f_n(\alpha x) = \alpha f_n(x) \end{array}}$$

Proof

$|f_n(x)| = |\sum_{0}^{n-1} \alpha_i|$
$\leq \sum_{0}^{n-1} |\alpha_i|$
$\leq n\|x\|$

~~Taking supremum for fixed 'x'~~

Any polynomial x of degree N_x has $N_x + 1$ coefficients.

So $|f_n(x)| \leq (N_x+1) \max|\alpha_j| = (N_x+1)\|x\|$
$\Rightarrow f_n(x) \leq C_x = (N_x+1)\|x\|$

If X is banach then $\|f_n\|$ must be uniformly bounded.

Eg: $x_n(t) = 1 + t + \cdots + t^{n-1}$
$\|x_n\| = 1$
$T x_n = f_n x_n = n$

$\|f_n\| = n$
$(\|f_n\|)$ is unbounded
X is not a Banach space.

classmate

Date

Page

(Exam point of view)

Theorem : Let X be normed linear space

a) $x_n \to x$ (strong) implies $x_n \rightharpoonup x$ (weakly)

b) $x_n \rightharpoonup x$ does not imply $x_n \to x$

c) If $\dim X < \infty$, then $x_n \rightharpoonup x \Rightarrow x_n \to x$

Proof : $x_n \to x$ i.e $\|x_n - x\| \to 0$ as $n \to \infty$

To show that $x_n \rightharpoonup x$

$\forall f \in X'$, $f(x_n) \to f(x)$ (To show)

$$|f(x_n) - f(x)| = \|f(x_n - x)\| \leq \|f\| \cdot \underbrace{\|x_n - x\|}_{\downarrow 0} \quad \text{as } n \to \infty$$

$\Rightarrow \lim_{n\to\infty} f(x_n) = f(x)$ (or)

$f(x_n) \to f(x)$ as $n \to \infty$

Let X be a Hilbert space & (e_n) be a sequence of orthonormal set $\|e_n\| = 1$

$\langle e_n, e_m \rangle = 0$, $n \neq m$

Let $f \in H'$, $f(x) = \langle x, z \rangle$ Riesz's Representation Theorem :

$f(e_n) = \langle e_n, z \rangle$

(e_n) $f(e_n)$

$$\sum_{1}^{\infty} |\langle e_n, x \rangle|^2 \leq \|x\|^2$$

$$\sum_{1}^{\infty} |\langle e_n, z \rangle|^2 \leq \|z\|^2$$

$\langle e_n, z \rangle \to 0$

$f(e_n) \to 0$ $\rightarrow$ $e_n \xrightarrow{w} 0$

or $e_n \rightharpoonup 0$

But for $n \neq m$ $\|e_n - e_m\|^2 = 2$

$\therefore e_n \not\to 0$ Not strongly convergent