Riemann Integration

Recall "uniform convergence is very powerful assumption"

$\{f_n\} : [a,b] \to \mathbb{R}$ of Riemann integrable function & $f_n \xrightarrow{\text{unif.}} f$ on $[a,b]$ as $n \to \infty$. Then $f : [a,b] \to \mathbb{R}$ is Riemann integrable &

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x) = \int_a^b f(x)\,dx$$

Q $f_n \to f$ pt. wise

then what can you say about

$$f_n' \xrightarrow{??} f'$$

Thm :- Let $f_n : (a,b) \to \mathbb{R}$ be sequence of diff. functions. Assume f_n' is integrable on (a,b). Assume $f_n \to f$ pt. wise

Also assume, $f_n' \to g$ uniformly on $[a,b)$, where g is continuous. Then, f is continuously differentiable &

$f' = g.$

Proof Let $a < c < b$. Since f_n' is integrable, apply F.T.I.C.

$$f_n(x) = f_n(c) + \int_c^x f_n'(x)\,dx \quad \text{—} (*)$$

$$a < x < b$$

$$\left\{ \begin{array}{lll} f_n \longrightarrow f & \text{pt. wise} & \\ f_n' \longrightarrow g & \underline{\text{uniformly}} & [c, x] \end{array} \right.$$

Then we get from $(*)$,

$$f(x) = f(c) + \int_c^x g(x)\,dx$$

Since, g is continuous, applying again F.T.I.C. (another version), we get f is diff'l

$$\underline{f'(x) = g(x)}.$$

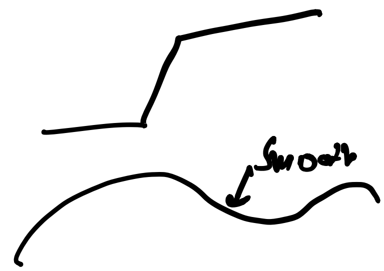
Example :-

$$f_n(x) = \begin{cases} n, & 0 < x < \frac{1}{n} \\ 0 & x = 0, \text{ or } \frac{1}{n} \le x \le 1 \end{cases}$$

$f_n \longrightarrow 0$ as $n \to \infty$ pt. wise

$$\int_0^1 f_n(x)\,dx = n \cdot \frac{1}{n} = 1 \quad \forall\, n \in \mathbb{N}$$

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \neq 0 = \int_0^1 f(x)\,dx \quad ✓$$



Example

$f_n(x) = ne^{-nx}, \ x \in [0,1]$

each f_n is Riemann integrable

$f_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$.

$$\int_0^1 f_n(x)\,dx = \int_0^1 ne^{-nx}\,dx$$

$$= -e^{-n} + 1$$

$\downarrow$ as $n \to \infty$

$$1 \neq 0 = \int_0^1 f(x)\,dx$$

Important.

$$f_n(x) = \begin{cases} n^2, & 0 < x < \frac{1}{n} \\ 0, & x = 0, \ \frac{1}{n} \leq x \leq 1 \end{cases}$$

$f_n \to 0$ as $n \to \infty$

$$\int f_n(x)\,dx = n^2 \cdot \frac{1}{n} = n \to \infty$$

Example $\{q_k\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$

& $f_n : [0,1] \longrightarrow \mathbb{R}$ by,

$$f_n(x) = \begin{cases} 1, & \text{if } x = q_k \text{ for } k \leq n \\ 0, & \text{otherwise} \end{cases}$$

Each f_n is Riemann integrable.

$$f_n \xrightarrow{\text{ptwise}} f.$$

$f(x)$ is Dirichlet function $= \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$

Is f Riemann integrable?

No

1) $\int_a^\infty$ 2) $\int_{-\infty}^b$ 3) $\int_{\infty}^{\infty}$ 4) $\int_a^b f(x)$, $f(c) = \infty$, $a < c < b$

Improper Riemann integral.

$$\lim_{b \to \infty} \boxed{\int_a^b f(x)\,dx}$$

Lebesgue criteria for Riemann Integrability

Let f be a bounded function on $[a,b]$. Let D denote the set of discontinuities of f. Then f is Riemann integrable on $[a,b]$ iff $m(D) = 0$

bounded function may not be Riemann integrable (Dirichlet function)

Riemann integrable
$\Downarrow$
bounded.

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}$$

Discontinuity $\mathbb{R}$

$x_0 \in \mathbb{Q}$, $y_0 \in \mathbb{I}$

$f(x_0) = 1$

$\mathbb{I} \ni x_n \longrightarrow x_0$

$f(x_n) \not\to f(x_0) = 1$

$f(x_n) = 0 \longrightarrow 0$

$\mathbb{Q} \ni y_n \longrightarrow y_0$

$f(y_n)$ $\quad$ $f(y_0) = 0$

$f(y_n) = 1 \longrightarrow 1 \neq 0$

Def^n 1. f is bdd on $[a,b]$. For any interval J, the oscillation of f on J is defined by

$$\omega_f(J) = \sup\{f(x) : x \in J \cap [a,b]\} - \inf\{ \quad " \quad \}$$

Def^n 2

For $x_0 \in [a,b]$, the oscillation of f at x_0 is defined by

$$\omega_f(x_0) = \inf\{\omega_f(I) : I \text{ is an open interval containing } x_0\}$$

Lemma: Let f be bdd on $[a,b]$. Then f is continuous at $x_0 \in [a,b]$ iff $\omega_f(x_0) = 0$