6/1/2020

Stochastic Process.

CT-1 8 marks

CT-2 8 marks

Attendance 4 marks

Probability Review:

Events $A_1, A_2, \ldots$

$\bigcup_{i=1}^{n} A_i \rightarrow$ atleast one event occur.

$\bigcap_{i=1}^{n} A_i \rightarrow$ event that all occurs.

$P(\Omega) = 1$, $P(\phi) = 0$, $0 \le P(A) \le 1$

$A \cap B = \phi \Rightarrow P(A \cap B) = 0$

$\rightarrow P(A \cup B) = P(A) + P(B)$.

Law of total prob.:-

$A_i \cap A_j = \phi$; $i, j = 1, 2, \ldots$ $i \neq j$

$\Omega = A_1 \cup A_2 \cup \ldots$

$$P(B) = \sum_{i=1}^{\infty} P(A_i \cap B)$$

Event A and B independent,

$P(A \cap B) = P(A) \cdot P(B)$.

$\rightarrow A_1, A_2, \ldots$ are independent,

$P(A_{i_1} \cap A_{i_2} \ldots \cap A_{i_n}) = P(A_{i_1}) \cdot P(A_{i_2}) \ldots \ldots P(A_{i_n})$

Scanned by CamScanner

$P(X > a) = 1 - F(a) = \bar{F}(a)$

$P(a < X \le b) = F(b) - F(a)$

$P(X = x) = F(x) - F(x-)$

X discrete if finite or denumerable set of $x_1, x_2, \ldots$ st

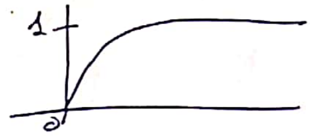
$p_i = P(X = x_i) \ldots, \quad i = 1, 2, \ldots$

$\sum_{i=1}^{\infty} p_i = 1$

$p_i = F(x_i) - F(x_i-)$

$\rightarrow F(x) = \sum_{x_i \le x} P(x_i) \rightarrow$ step f^n.

$\Rightarrow$ If $P(X = x) = 0$ for every value of x then X is unit and its CDF $F(x)$ is continous f^n.



CDF $\rightarrow F(x) = \int_{-\infty}^{x} f(\xi)\, d\xi$

$f(x) \rightarrow$ PDF

$$E(X^m) = \begin{cases} \sum_i x_i^m \, P(X = x_i) \\ \int_{-\infty}^{\infty} x^m f(x)\, dx \end{cases}, \quad \text{provided } E(|X^m|) < \infty.$$

$\Rightarrow$ X random variable, g function, $Y = g(X)$, $X \sim f_X^{(x)}$ pdf.

$$E(Y) = \begin{cases} \sum_i g(x_i) \cdot P(X = x_i) & \text{if } X \text{ discrete.} \\ \int g(x) \cdot f_X^{(x)}\, dx, & \text{if } X \text{ continous.} \end{cases}$$

<u>Joint CDF</u> :-

$F(x, y) = P(X \le x, Y \le y)$

$\nearrow \sum_{x_i \le x} \sum_{y_i \le y} P(x_i, y_i)$

$\searrow \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, du\, dv.$

$F_X(x) = \lim_{y \to \infty} F(x, y)$

$F_Y(y) = \lim_{x \to \infty} F(x, y)$.

Scanned by CamScanner

$E(X+Y) = E(X) + E(Y)$

X, Y are independent if $F(x,y) = F_X(x) \cdot F_Y(y)$

Joint pdf $\rightarrow f(x,y) \rightarrow f(x,y) = f_X(x) \cdot f_Y(y)$.

$\text{Cov}(X,Y) = E\left[(X-\mu_X)\cdot(Y-\mu_Y)\right] = E(XY) - \mu_X \mu_Y$.

convolution.

$\rightarrow$ $X, Y \rightarrow$ independent

$Z = X+Y$

$F_Z(z) = P(X+Y \le z) = E\left(P(X+Y \le z \mid Y)\right)$

$= \int P(X+Y \le z \mid Y=y) \, f_Y(y) \, dy.$

$= \int P(X \le z-y) \, f_Y(y) \, dy.$

$= \int_{-\infty}^{\infty} F_X(z-y) \, f_Y(y) \, dy.$

$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) \, f_Y(y) \, dy.$

$\Rightarrow$ $X \sim f_X$ (pdf) ; g st $\uparrow$, diff. function, $Y = g(X)$

$\{Y \le y\} \equiv \{X \le g^{-1}(y)\}$ $\quad y = g(x)$.

$F_Y(y) = F_X(g^{-1}(y))$

$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = f_X(x) \frac{dx}{dy} = f_X(x) \cdot \frac{1}{g'(x)}$

$\Rightarrow$ <u>conditional probabilities</u>:-

A, B events.

$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ if $P(B) > 0$

$P(A \cap B) = P(A \mid B) \cdot P(B)$

Sim, $P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_2 \cap A_1) \cdot P(A_2 \mid A_1) \cdot P(A_1)$

Scanned by CamScanner

Law of total prob., $\Omega = A_1 \cup A_2 \cup \dots$; $A_i \cap A_j = \phi$

$$P(B) = \sum_{i=1}^{\infty} P(A_i \cap B) = \sum_{i=1}^{\infty} P(B \mid A_i) \cdot P(A_i)$$

⇒ **Bernoulli distribution:-**

$$X \sim Ber(p)$$

$p(1) = P(X=1) = p$; $p(0) = P(X=0) = q = 1-p$, $0 \le p \le 1$.

⇒ **Binomial distribution:-**

$$Y \sim Bin(n, p)$$

$P(Y=k) = {}^{n}C_{k} \cdot p^{k}(1-p)^{n-k}$; $k = 0, 1, \dots, n$

$E(Y) = np$; $var(Y) = np(1-p)$.

⇒ **Geometric distribution:-**

$Z \sim Geo(p)$; $p_Z(k) = p(1-p)^{k}$; $k = 0, 1, 2, \dots$

Z counts # of failures prior to the first success.

$E(Z) = \frac{1-p}{p}$; $V(Z) = \frac{1-p}{p^2}$.

→ $Z' = 1 + Z$ # of trails untill first success.

$p_{Z'}(k) = p \cdot (1-p)^{k-1}$; $k = 1, 2, \dots$

$E(Z') = \frac{1}{p}$; $V(Z') = \frac{1-p}{p^2}$.

⇒ **Poission distribution:-**

$X \sim P(k) = \frac{e^{-\lambda} \lambda^{k}}{k!}$; $k = 0, 1, 2, \dots$
pmf

$E(X) = \lambda = V(X)$

$\lim_{\substack{n \to \infty \\ np = \lambda}} B(n, p) \equiv Poiss(\lambda)$.

⇒ **Normal distribution:-** $\boxed{X \sim N(\mu, \sigma^2)}$

$Z \sim N(0,1)$; $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

CDF
Φ

⇒ Central Limit Theorem :-

$$S_n = \xi_1 + \cdots + \xi_n$$

$\xi_1, \xi_2, \ldots$ IID ; $\mu = E(\xi_k)$, $\sigma^2 = V(\xi_k)$

$k = 1, 2, \ldots$

$$\lim_{n \to \infty} P\left(\frac{S_n - \mu \cdot n}{\sigma \sqrt{n}} \le x \right) = \Phi(x) \quad , \forall x .$$

∴ In practical terms we expect the normal distribution to arrive whenever the numerical outcome of experiment results from numerous small additive effects, all operating independently, and when no single or small group of effect is dominant.

⇒ Exponential distribution :-

$T \sim exp(\lambda)$

pdf, $f_T(t) = \begin{cases} \lambda e^{-\lambda t} , & t \geq 0, \lambda \geq 0 \\ 0 , & t < 0 \end{cases}$

CDF, $F_T(t) = \begin{cases} 1 - e^{-\lambda t} , & t \geq 0 \\ 0 , & t < 0 \end{cases}$

$$\int_t^{\infty} f_T(u)\, du = \bar{F}_T(t) = 1 - F_T(t) .$$

⇒ Gamma distribution :- $X \sim Gamma(\alpha, \lambda)$

$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma \alpha} \cdot e^{-\lambda x} \cdot x^{\alpha - 1} , & x > 0, \lambda > 0, \alpha > 0 \\ 0 , & \text{otherwise.} \end{cases}$

If $Y_1, \ldots Y_n \sim$ independent $exp(\lambda)$

then $Y_1 + Y_2 + \cdots + Y_n \sim Gamma(n, \lambda)$.

$$M_Y(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$M_Z(t) = E(e^{tZ})$$

$$= \prod_{i=1}^{n} M_{Y_i}(t) = \left(1 - \frac{t}{\lambda}\right)^{-n}$$

Scanned by CamScanner

$\Rightarrow$ **Exponential distribution :-**

independent $\leftarrow$ $X_0, X_1 \sim \exp$ with parameters λ_0, λ_1.

$$X_i \sim \exp(\lambda_i), \quad i = 0, 1, ..$$

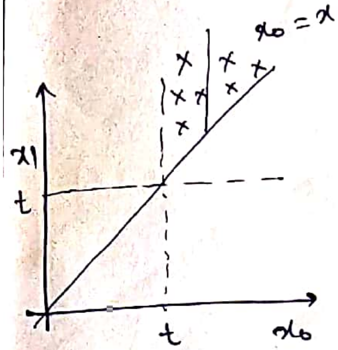
$$N = \begin{cases} 0 & \text{if } X_0 \le X_1 \\ 1 & \text{if } X_1 \le X_0 \end{cases}$$

$$U = \min(X_0, X_1) = X_N, \quad M = 1 - N$$

$$V = \max(X_0, X_1) = X_M$$

$$W = V - U = |X_0 - X_1|$$

a.) $P(N=0, U>t) = P(t < X_0 \le X_1)$



$$= \iint_{t < x_0 \le x_1} \lambda_0 \cdot e^{-\lambda_0 x_0} \lambda_1 e^{-\lambda_1 x_1} dx_1 \cdot dx_0.$$

$$= \int_t^\infty \left(\int_{x_0}^\infty \lambda_1 \cdot e^{-\lambda_1 x_1} dx_1 \right) \lambda_0 e^{-\lambda_0 x_0} dx_0$$

$$= \int_t^\infty e^{-\lambda_1 x_0} \lambda_0 e^{-\lambda_0 x_0} dx_0 = \frac{\lambda_0}{\lambda_0 + \lambda_1} \int_t^\infty (\lambda_0 + \lambda_1) \cdot e^{-(\lambda_0 + \lambda_1) x_0} dx_0.$$

$$= \frac{\lambda_0}{\lambda_0 + \lambda_1} \cdot e^{-(\lambda_0 + \lambda_1) \cdot t}$$

$F(x) = 1 - e^{-\lambda x}$

$\bar{F}(x) = 1 - F(x)$

$= e^{-\cancel{\lambda(x)} \lambda x}$

b.) $P(N=0) = P(N=0, U>0)$.

$\downarrow$

put, $t=0$ in above result.

$$\Rightarrow \frac{\lambda_0}{\lambda_0 + \lambda_1}$$

$$P(N=1) = \frac{\lambda_1}{\lambda_0 + \lambda_1}$$

c.) $P(U>t) = P(\min(X_0, X_1) > t) = P(X_0 > t, X_1 > t)$

$= P(X_0 > t) \cdot P(X_1 > t) = e^{-\lambda_0 t} \cdot e^{-\lambda_1 t} = e^{-(\lambda_0 + \lambda_1) t}$

$$P(N=0, U>t) = \frac{\lambda_0}{\lambda_0 + \lambda_1} \cdot e^{-(\lambda_0 + \lambda_1) t}$$

$$P(N=1, U>t) = \frac{\lambda_1}{\lambda_0 + \lambda_1} \cdot e^{-(\lambda_0 + \lambda_1) \cdot t}$$

Scanned by CamScanner

$$P(U>t) = P(N=0, U>t) + P(N=1, U>t)$$
$$= e^{-(\lambda_0+\lambda_1)\cdot t}.$$

$\rightarrow$ $P(N=0, U>t) = P(N=0)\cdot P(U>t)$

$P(N=1, U>t) = P(N=1)\cdot P(U>t)$

So, N & U are independent.

$$\rightarrow P(W>t, N=0) = P(|X_0 - X_1|>t, \; X_0 \le X_1)$$
$$= P(X_1 - X_0 > t)$$
$$= \iint_{x_1 - x_0 > t} \lambda_0 \cdot e^{-\lambda_0 x_0} \lambda_1 e^{-\lambda_1 x_1} \, dx_1 dx_0.$$
$$= \int_0^\infty \left(\int_{x_0+t}^\infty \lambda_1 \cdot e^{-\lambda_1 x_1} dx_1 \right) \lambda_0 \cdot e^{-\lambda_0 x_0} dx_0.$$
$$= \int_0^\infty e^{-\lambda_1(x_0+t)} \lambda_0 \cdot e^{-\lambda_0 x_0} dx_0.$$
$$= \frac{\lambda_0 \cdot e^{-\lambda_1 t}}{\lambda_0+\lambda_1} \int_0^\infty (\lambda_0+\lambda_1)\cdot e^{-(\lambda_0+\lambda_1)x_0} dx_0.$$
$$= \frac{\lambda_0}{\lambda_0+\lambda_1} \cdot e^{-\lambda_1 t} = P(N=0)\cdot e^{-\lambda_1 t}.$$

$$P(W>t \mid N=0) = e^{-\lambda_1 \cdot t}$$
$$P(W>t \mid N=1) = e^{-\lambda_0 t}$$

$$P(W>t) = \frac{\lambda_0}{\lambda_1+\lambda_0} \cdot e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_0+\lambda_1} \cdot e^{-\lambda_0 t}, \quad t \ge 0$$

To show

$\rightarrow$ U, $W = V - U$ independent, $\quad U = \min(X_0, X_1)$

$V = \max(X_0, X_1)$

$$P(U \ge u, W > \omega) = P(U > u)\cdot P(W > \omega)$$
$$= P(u < X_0 < X_1 - \omega)$$
$$= \iint_{u < x_0 < x_1 - \omega} \lambda_0 \cdot e^{-\lambda_0 x_0} \lambda_1 e^{-\lambda_1 x_1} \, dx_1 dx_0.$$

Scanned by CamScanner

$$= \int_u^\infty \left(\int_{x_0+w}^\infty \lambda_1 \cdot e^{-\lambda_1 x_1} \cdot dx_1 \right) \lambda_0 e^{-\lambda_0 x_0} dx_0$$

$$= \int_u^\infty e^{-\lambda_1(x_0+w)} \cdot \lambda_0 e^{-\lambda_0 x_0} dx_0$$

$$= \frac{\lambda_0}{\lambda_0+\lambda_1} \cdot e^{-\lambda_1 w} \cdot \int_u^\infty (\lambda_0+\lambda_1) e^{-(\lambda_0+\lambda_1)x_0} dx_0.$$

$$= \frac{\lambda_0}{\lambda_0+\lambda_1} e^{-\lambda_1 w} \cdot e^{-(\lambda_0+\lambda_1)u} \quad \text{—— (1)}$$

$\rightarrow P(N=1, U>u, W>w) = \frac{\lambda_1}{\lambda_0+\lambda_1} \cdot e^{-\lambda_0 w} \cdot e^{-(\lambda_0+\lambda_1)u}$ —— (2)

$$P(U>u, W>w) = \left[\frac{\lambda_0}{\lambda_0+\lambda_1} \cdot e^{-\lambda_1 w} + \frac{\lambda_1}{\lambda_0+\lambda_1} \cdot e^{-\lambda_0 w} \right] e^{-(\lambda_0+\lambda_1)u}$$

$$= P(W>w) \cdot P(U>u)$$

$\therefore$ W and U are independant random variables.

$\Rightarrow$ Let, $X \sim Bin(\cancel{p,N}\ N, p)$, where $N \sim Poiss(\lambda)$.

i.e., $P_{X|N}(x|n) \sim B(n,p)$.

$$P(X=k) = \sum_{n=0}^\infty P_{X|N}(x|n) . P_N(n)$$

$$= \sum_{n=0}^\infty \frac{n!}{k!(n-k)!} . p^k . (1-p)^{n-k} \times \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= \frac{\lambda^k . e^{-\lambda} . p^k}{k!} . \sum_{n=k}^\infty \frac{[\lambda(1-p)]^{n-k}}{(n-k)!} = \frac{(\lambda p)^k . e^{-\lambda}}{k!} . e^{\lambda(1-p)}$$

Scanned by CamScanner

6/1/20.

$$E(g(x) \mid Y=y) = \sum_x g(x) \cdot \underbrace{P_{X|Y}(x|y)}_{\substack{\text{pmf} \\ \text{(mass f}^n\text{)}}}, \quad \text{if } P_Y(y) > 0$$

$$E(g(X)) = \sum_x E(g(x) \mid Y=y) \cdot P_Y(y)$$

$$= E(E(g(X) \mid Y))$$

$$\therefore E(X) = E(\underbrace{E(X|Y)}_{\psi(Y)})$$

$$= \sum_y E(X|Y=y) P_Y(y)$$

$$= \sum_y \sum_x x \cdot P_{X|Y}(x|y) P_Y(y)$$

$$= \sum_y \sum_x x \, P_{X,Y}(x,y)$$

$$= \sum_x x \sum_y P_{X,Y}(x,y)$$

$$= \sum_x x \, P_X(x) = E(X);$$

1.) $E(c_1 g_1(X_1) + c_2 g_2(X_2) \mid Y=y)$

$\Rightarrow c_1 \cdot E(g_1(X_1) \mid Y=y) + c_2 E(g_2(X_2) \mid Y=y)$

2.) $g \geq 0$; $E(g(X) \mid Y=y) \geq 0$

3.) $E(\nu(X,Y) \mid Y=y) = E(\nu(X,y) \mid Y=y)$

4.) $E(g(X) \mid Y=y) = E(g(X))$ if X and Y are independent.

5.) $E(g(X)h(Y) \mid Y=y) = h(y)\, E(g(X) \mid Y=y)$

6.) $E(g(X) \cdot h(Y)) = E(h(Y) E(g(X)) \mid Y)$

$\searrow E(g(X) \cdot h(Y) \mid Y)$.

$E(c \mid Y=y) = c$.

$E(h(Y) \mid Y=y) = h(y)$

★ Partial Sums:-

$X = \xi_1 + \cdots + \xi_N$, where N is random,

$\xi_1, \xi_2, \ldots$ IID random variables. N discrete type r.v independent of $\xi_1, \xi_2, \ldots$, and

$P_N(n) = P(N=n)$, $n = 0, 1, 2, \ldots$

$$X = \begin{cases} 0 & \text{if } N=0 \\ \xi_1 + \xi_2 + \cdots + \xi_N & \text{if } N>0. \end{cases}$$

Scanned by CamScanner

Ex:- 1.) Queuing: , N = # of customers

ξ_i = ~~sense~~ service time for i^{th} customer.

$X = \xi_1 + \cdots + \xi_N$ total ~~sence~~ service time

2.) Risk Theory:- N claims arrive at insurance company in a given week.

ξ_i amount of i^{th} claim.

$X = \xi_1 + \cdots + \xi_N$ → total liability of company.

3.) Population model:- Amount of seeds distributed.

4.) Biometrics:

★ Conditional distribution mixed case:

X, N $\quad N \sim$ discrete , $n = 0, 1, 2, \cdots$

$$F_{X|N}(x|n) = \frac{P(X \le x, N = n)}{P(N=n)} \; ; \text{ if } P(N=n) > 0$$

if $X \to$ continous random variable,

$$f_{X|N}(x|n) = \frac{d}{dx} F_{X|N}(x|n).$$

$$\Rightarrow P(a \le X \le b, N = n) = \int_a^b f_{X|N}(x|n) \cdot p_N(n)\, dx.$$

$$f_X(x) = \sum_{n=0}^{\infty} f_{X|N}(x|n) \cdot p_N(n).$$

$$E(g(X)) = E(E(g(X)|N)) = \sum_{n=0}^{\infty} E(g(X)|N=n)\, p_N(n)$$

⇒ Moments of random sum:- $N \to$ discrete.

Assume ξ_k, N finite (moments),

$E(\xi_k) = \mu, \quad V(\xi_k) = \sigma^2$

$E(N) = \nu, \quad V(N) = \tau^2$

$X = \xi_1 + \cdots + \xi_N$

Scanned by CamScanner

$$E(X) = E(\xi_1 + \dots + \xi_N) = E\left(E\left(\sum_{i=1}^{N} \xi_i \mid N\right)\right)$$

$$= \sum_{n=0}^{\infty} E(\xi_1 + \dots + \xi_N \mid N = n) \cdot P_N(n)$$

$$= \sum_{n=0}^{\infty} E(\xi_1 + \dots + \xi_N) \cdot P_N(n).$$

$$= \sum_{n=0}^{\infty} n \cdot \mu \cdot P_N(n) = \mu \underbrace{\sum_{n=0}^{\infty} n P_N(n)}_{E(N)} = \mu\nu.$$

variance

$$\rightarrow V(X) = E(X - \mu\nu)^2 = E(X - N\mu + N\mu - \nu\mu)^2.$$

$$= \underbrace{E(X - N\mu)^2}_{(\sigma^2\nu)} + \underbrace{E(\mu(N-\nu))^2}_{(\mu^2\tau^2)} + 2E[\underbrace{(X - N\mu)(N\mu - \nu\mu)}_{0}].$$

$$\rightarrow E(X - N\mu)^2 = \sum_{n=0}^{\infty} E((X - N\mu)^2 \mid N = n) P_N(n)$$

$$E(E((X - N\mu)^2 \mid N))$$

$$= \sum_{n=0}^{\infty} \cancel{E(X - n\mu)^2}$$

$$= \sum_{n=0}^{\infty} E(\xi_1 + \dots + \xi_n - n\mu)^2 P_N(n).$$

$$= \sum_{n=0}^{\infty} n\sigma^2 P_N(n) = \sigma^2 \underbrace{\sum_n n P_N(n)}_{E(N)}$$

$$= \sigma^2\nu \quad \{\nu = E(N)\}.$$

$$\rightarrow \mu^2 E(N - \nu)^2 = \mu^2\tau^2$$

$$\rightarrow \mu \cdot E((X - N\mu)(N - \nu)) = \mu \cdot \sum_n E((X - N\mu)(N - \nu) \mid N = n) P_N(n).$$

$$= \mu \cdot \sum_n (n - \nu) E(X - N\mu \mid N = n) P_N(n).$$

$$= \mu \cdot \sum_n (n - \nu) \cdot 0 \cdot P_N(n)$$

$$= 0.$$

$\because E(h(X)g(Y) \mid Y = y) = g(y) E(h(X) \mid Y = y)$

$V(X) = \sigma^2 \nu + \mu^2 \tau^2.$

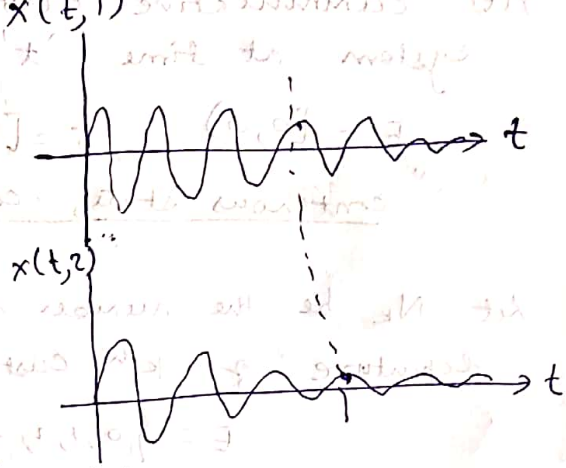
$E(X) = \mu \nu.$

7/1/20.

Stochastic Process (SP)

A S.P is a family of random variables $\{X(t), t \in T\}$ defined on a given probability space, indexed by the parameter t, where $t \in T \rightarrow$ index set.

e.g.) consider the experiment of randomly choosen a resistor s from a set S of thermally agitated resistors and meaning the noise voltage $X(t,s)$ across resistance at time 't'. $s \in S = \{1, 2, \ldots\}$



$\rightarrow X(t) \equiv X(t,s)$, $t \in T$ $s \in S \rightarrow$ sample space

The values of r.v's $X(t)$ are called states and set of all possible values from state space (E).

→ discrete state, discrete parameter S.P

→ — " — , continous — " — SP

→ const state, — " — SP

→ cont state, discrete " S P

★ $\boxed{\text{Discrete state SP also known as 'chain'}}$

Ex:- Consider a queueing system with jobs arriving at random point in time, queueing for service and departing from the system after service complition.

→ Nextpage.

a.) $X(t)$ # of jobs in the system at time 't'.

$\{X(t), t \in T\}$

$E = \{0, 1, 2, \ldots\}$, $T = \{t : 0 \leq t < \infty\}$

Above is "discrete state, continous parameter/time S.P"

b.) W_K is the time that the K^{th} customer has to wait in the system before recieving ~~it~~ service.

$\{W_K : K = 1, 2, \ldots\}$

$E = \{x \mid 0 \leq x < \infty\}$; $T = \{1, 2, 3, \ldots\}$

"continous state, discrete parameter S.P."

c.) $Y(t)$ cummulative service requirement of all jobs in the system at time 't'.

$\therefore$ $E = [0, \infty)$, $T = [0, \infty)$

"continous state, cont. parameter SP".

d.) Let N_K be the number of jobs in the system at time of departure of K^{th} customer (after service complition).

$E = \{0, 1, 2, \ldots\}$; $T = \{1, 2, 3, \ldots\}$

"discrete state, discrete parameter SP".

Counting Process:-

$X(t) =$ # of events in $(0, t]$

i) $X(0) = 0$.

ii) $s < t \Rightarrow X(s) \leq X(t)$.

iii) $X(t) - X(s) \longrightarrow$ # of events in $(s, t]$

Independent increaments:

$\{X(t), t \in T\}$

$X(b_1) - X(a_1)$, $X(b_2) - X(a_2)$, ... are independent

Events occuring in disjoint time ~~[illegible]~~ interval are independant.

Scanned by CamScanner

Stationary Increments:

$$X(2t) - X(t) \overset{d}{=} X(t)$$

$$X(t+h) - X(s+h) \overset{d}{=} X(t) - X(s)$$
$$\overset{d}{=} X(t-s)$$

★ **Martingales:**

SP are charaterized by the dependence relationships that exists among their variables. The martingale proprety is one such relationship that captures a notion of a game being fare.

⇒ A S.P $\{X_n : n = 0, 1, 2, \dots\}$ is martingale if for $n = 0, 1, 2, \dots$

a) $E|X_n| < \infty$

b) $E(X_{n+1} | X_0, \dots, X_n) = X_n$

- $E\left(E(X_{n+1} | X_0, \dots, X_n)\right) = E(X_n)$

$E(X_{n+1}) = E(X_n)$, $\forall n$

⇒ martingale has constant mean.

→ ⓑ extends to future times in the form $E(X_m | X_0, \dots, X_n) = X_n$ for $m \geq n$.

e.g., fairness in gambling

consider X_n → players fortune after n^{th} play of game

fair game if players' fortune neither increase nor decrease at each play.

13/1/20.

13/1/20

Ex:- Let $S_0 = 0$, and for $n \geq 1$, let $S_n = \epsilon_1 + \dots + \epsilon_n$ be the sum of 'n' independent random variables each exponential with mean $E(\epsilon) = 1$.

Show that, $X_n = 2^n \cdot e^{-S_n}$, $n \geq 0$ defines a martingale.

Solution:- $S_n = \epsilon_1 + \epsilon_2 + \dots + \epsilon_n$.

$$X_{n+1} = 2^{n+1} \cdot e^{-S_{n+1}} = 2^{n+1} \cdot e^{-\sum_{i=1}^{n+1} \epsilon_i} = 2 \cdot 2^n \cdot e^{-\sum_{i=1}^{n} \epsilon_i} \cdot e^{-\epsilon_{n+1}}$$

$$= 2 X_n \cdot e^{-\epsilon_{n+1}}.$$

$$E(X_{n+1} | X_1, \dots, X_n) = E(2X_n e^{-\epsilon_{n+1}} | X_1, X_2, \dots, X_n)$$

$$= 2 \cdot X_n \cdot E(e^{-\epsilon_{n+1}} | X_1, \dots, X_n)$$

$$= 2 \cdot X_n \cdot E(e^{-\epsilon_{n+1}} | \epsilon_1, \epsilon_2, \dots, \epsilon_n).$$

$$= 2 \cdot X_n \cdot E(e^{-\epsilon_{n+1}}).$$

$$= 2 \cdot X_n \cdot \int_0^\infty e^{-x} \cdot f_{\epsilon_{n+1}}(x)\, dx.$$

$$= 2 X_n \int_0^\infty e^{-x} \cdot e^{-x}\, dx$$

$$= X_n$$

$\epsilon \sim \exp(\lambda)$; $\lambda = 1$

$f_\epsilon(x) = \lambda e^{-\lambda x}$

$E(X) = 1/\lambda = 1$

Ex:- Let $\xi_1, \xi_2, \dots$ be independent Bernoulli random variable with parameter p, $0<p<1$. Show that $X_0 = 1$ and

$$X_n = p^{-n} \cdot \xi_1 \cdots \xi_n \quad ; \quad n = 1, 2, \dots$$

define a non-negative martingale. What is the limit of X_n as $n \to \infty$.

Solution:-

$$X_{n+1} = p^{-(n+1)} \xi_1 \cdot \xi_2 \cdots \xi_{n+1}$$

$$X_{n+1} = \frac{1}{p} \cdot p^{-n} \cdot \xi_1 \cdots \xi_n \cdot \xi_{n+1}$$

$$X_{n+1} = \frac{1}{p} \cdot X_n \cdot \xi_{n+1}$$

Scanned by CamScanner

$$E(X_{n+1} \mid X_1, X_2, \ldots X_n)$$

$$= E\left(\frac{1}{p} \cdot X_n \cdot \xi_{n+1} \mid X_1, X_2, \ldots X_n\right)$$

$$= E\left(\frac{1}{p} \cdot X_n \cdot \xi_{n+1} \mid \xi_1, \ldots, \xi_n\right)$$

$$= \frac{1}{p} X_n \, E(\xi_{n+1} \mid \xi_1, \ldots, \xi_n)$$

$$= \frac{1}{\cancel{p}} \cdot X_n \cdot \cancel{p}$$

$$= X_n.$$

mean of bernoulli $= p$.

$$\xi_i = \begin{cases} 1 & \text{with } p \\ 0 & \text{with } q = 1-p. \end{cases}$$

$$X_n = p^{-n} \cdot \prod_{i=1}^{n} \xi_i = \begin{cases} \frac{1}{p^n} & \text{with } p^n \\ 0 & \text{with } 1-p^n. \end{cases}$$

$$\lim_{n\to\infty} X_n = \begin{cases} 0 & \text{with } 1 \end{cases}$$

★★ Let ξ be a random variable with mean μ and stand. deviation σ. Let $X = (\xi - \mu)^2$. Apply Markov's inequality to X to deduce Chebyshev's inequality.

→ $$P(X \geq \lambda) < \frac{E(X)}{\lambda}.$$

$$P((\xi-\mu)^2 \geq \epsilon^2) \leq \frac{E(\xi-\mu)^2}{\epsilon^2}, \quad \lambda = \epsilon^2$$

$$\Rightarrow P(|\xi - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \quad \text{for any } \epsilon > 0.$$

Ex:- Use the law of total prob. for conditional expectation.

$$E(E\{X \mid Y, Z\} \mid Z) = E(X \mid Z)$$

to show,

$$E(X_{n+2} \mid X_0, \ldots, X_n) = E[E\{X_{n+2} \mid X_0, \ldots, X_{n+1}\} \mid X_0, \ldots, X_n]$$

conclude that when X_n is a martingale.

$$E(X_{n+2} \mid X_0, \ldots, X_n) = X_n.$$

Scanned by CamScanner

Exg: S.P. If $x_n = 0$ then $x_{n+1} = 0$, whereas if $x_n > 0$, then

$$x_{n+1} := \begin{cases} x_n + 1 & , \text{with } 1/2 \\ x_n - 1 & , \text{with } 1/2 \end{cases}$$

a) Show that $|x_n|$ is martingale.

[Above prob. is know as "Gamblers Problem"]

~~Solution:~~ b.) ~~Show~~ Suppose that $x_0 = i > 0$. Use the maximal probability to bound $P(X_n \geq N$ for some $n \geq 0 \mid X_0 = i)$.

13/1/20 · **Markov inequality:**

Let $X \geq 0$, for any +ve constant λ.

$$E(X) \geq \lambda \cdot P(X \geq \lambda).$$

Proof:-

$$E(X) = E\left(X 1_{(-\infty,\lambda)}(x)\right) + E\left[X 1_{[\lambda,\infty)}(x)\right] \quad \longrightarrow \text{using law of total prob.}$$

where, $1_{(a,b)}(u) = \begin{cases} 1 & , a < u \leq b \\ 0 & , \text{otherwise} \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} x f(x) \cdot dx = \int_{-\infty}^{\lambda} x f(x)\, dx + \int_{\lambda}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x f(x)\, 1_{(-\infty,\lambda)}(x)\, dx + \int_{-\infty}^{\infty} x f(x)\, 1_{[\lambda,\infty)}\, dx$$

$$\geq E[X 1_{[\lambda,\infty)}(x)]$$

$$\geq E[\lambda 1_{[\lambda,\infty)}(x)]$$

$$= \lambda E[1_{[\lambda,\infty)}(x)]$$

$$= \lambda \cdot P(X \geq \lambda)$$

$$1_{[\lambda,\infty)}(x) = \begin{cases} 1 & , x \geq \lambda \\ 0 & , x < \lambda \end{cases}$$

★ **Maximal inequality for non-negative martingale**

martingale has constant mean, then using markov inequality

$$P(X_n \geq \lambda) \leq \frac{E(X_0)}{\lambda}, \quad \lambda > 0$$

$E(X_{n+1} \mid X_0 \ldots X_n) = X_n$

$E(E(X_{n+1} \mid X_0 \ldots X_n)) = E$

$E(X_{n+1}) = E(X_n)$

Theorem: Let $X_0, X_1, \ldots$ be a martingale with non-negative values.
i.e, $P(X_n \geq 0) = 1$ for $n = 0, 1, 2, \ldots$ For any $\lambda > 0$

$$P\left(\max_{0 \leq n \leq m} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda}, \text{ for } 0 \leq n \leq m$$

m can tend to ∞.

Sol: $\{X_0, \ldots, X_m\}$ sequence rises above λ for first time at some index n or else it remains always below λ.

Using law of total prob:-

$$E(X_m) = \sum_{n=0}^{m} E(X_m 1\{X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda\}) + E(X_m 1(X_0 < \lambda, \ldots, X_m < \lambda))$$

$$\geq \sum_{n=0}^{m} E(X_m 1\{X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda\})$$

We know, $E(g(x).h(y)) = E(h(y). E(g(x) | Y))$

$Y = (X_0, \ldots, X_n)$

Let, $X = X_m$, $Y = (X_0, \ldots, X_n)$

$$= \sum_{n=0}^{m} E(1(X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda) E(X_m | X_0, \ldots X_n))$$

$$= \sum_{n=0}^{m} E(X_n 1(X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda))$$

since, $\{X_n\}$ is martingale.

$$\geq \sum_{n=0}^{m} \lambda . E(1(X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda))$$

$$= \lambda \sum_{n=0}^{m} P(X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda)$$

$$= \lambda P\left(\bigcup_{n=0}^{m} (X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda)\right)$$

$$= \lambda . P\left(\max_{0 \leq n \leq m} X_n \geq \lambda\right)$$

$$\Rightarrow P\left(\max_{0 \leq n \leq m} X_n \geq \lambda\right) \leq \frac{E(X_m)}{\lambda} = \frac{E(X_0)}{\lambda}.$$

Scanned by CamScanner

Eg 1.) A gambler begins with a unit amount of money and faces a series of independent fair games.

$X_0 = 1$.

$0 < p < 1$

gambler wages pX_n,

$$X_{n+1} = \begin{cases} X_n + pX_n & \text{with } \frac{1}{2} \\ X_n - pX_n & \text{with } \frac{1}{2} \end{cases} \qquad X_1 = \begin{cases} 1 + pX_0 & \text{with } \frac{1}{2} \\ 1 - pX_0 & \text{with } \frac{1}{2} \end{cases}$$

proof: To prove if it is martingale.

$$E(X_{n+1} \mid X_0, \ldots, X_n) = X_n$$

$$E(X_{n+1} \mid X_0 = x_0, \ldots, X_n = x_n) = (1+p)x_n \times \tfrac{1}{2} + (1-p)x_n \times \tfrac{1}{2}$$

$$= x_n$$

$$E(X_{n+1} \mid X_0, X_1, \ldots X_n) = X_n$$

Thus, $\{X_n\}$ is martingale

Let $\lambda = 2$, $E(X_0) = 1$

Eg. 2) Let $X_1, X_2, \ldots$ be independent r.v's with mean $= 0$. Let

$Z_n = \sum_{i=1}^{n} X_i$ $\{Z_n, n \geq 1\}$ is martingale?

Sol:-

$$E(Z_{n+1} \mid Z_1, \ldots, Z_n) = E(Z_n + X_{n+1} \mid Z_1, \ldots, Z_n)$$

$$= E(Z_n \mid Z_1, \ldots, Z_n) + E(X_{n+1} \mid Z_1, \ldots, Z_n)$$

$$= Z_n + E(X_{n+1} \mid X_1, \ldots, X_n)$$

$$= Z_n + \underbrace{E(X_{n+1})}_{\to\, 0 \text{ mean}=0} \qquad \{X_1, X_2 \text{ are independent}\}$$

$$= Z_n.$$

14/1/20

Ex:- Consider a two-state M.C $\{X_n\}$, having state space $E = \{0, 1\}$ with TPM.

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 1 & \tfrac{1}{3} & \tfrac{2}{3} \end{array}$$

whether $Z_n = (X_{n-1}, X_n)$ is a M.C. If so, determine state space and TPM.

Sol: Clearly $\{Z_n\}$ is a M.C state space.

$$E_1 = \{(0,0), (0,1), (1,0), (1,1)\}$$

$$Z_{n+1} = \{i, j\}.$$

TPM

$$P = \begin{array}{c} X_{n-1}X_n \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} X_nX_{n+1}\ 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 \\ 0 & 0 & \tfrac{1}{3} & \tfrac{2}{3} \\ \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 \\ 0 & 0 & \tfrac{1}{3} & \tfrac{2}{3} \end{pmatrix} \end{array}$$

$$P(Z_{n+1} = (i,j) \mid Z_n = (i', j')) = P_{((i',j')(i,j))} = P(X_{n+1} = j, X_n = i \mid X_n = j', X_{n-1} = i')$$

$$P(AB \mid BC) = \frac{P(ABC)}{P(BC)} = P(A \mid BC)$$

$$P(Z_{n+1} = (0,1) \mid Z_n = (0,0)) = P(X_{n+1} = 1, X_n = 0 \mid X_n = 0, X_{n-1} = 0)$$

$$= P(X_{n+1} = 1 \mid X_n = 0, X_{n-1} = 0) = P(X_{n+1} = 1 \mid X_n = 0)$$

$$= P_{01} = \tfrac{1}{2}.$$

★ <u>n-step transition probability</u>:

$$P_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i) = P(X_n = j \mid X_0 = i)$$

state space $E = \{0, 1, 2, \ldots\}$

n-step TPM

$$P^{(n)} = \begin{array}{c} \\ X_0 = i \end{array} \begin{array}{c|cccc} & & & X_n = j & \\ & 0 & 1 & 2 & \cdots \\ \hline 0 & P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & \cdots \\ 1 & & & & \\ 2 & & & & \\ \vdots & & & & \end{array}$$

$$\sum_{j \in E} P_{ij}^{(n)} = 1$$

Scanned by CamScanner

★ Chapman Kolmogrov equation :- $\boxed{X_n \to M.C}$

$$P_{ij}^{(m+n)} = \sum_{k\in E} P_{ik}^{(m)} \cdot P_{kj}^{(n)} = \sum_{k\in E} P_{ik}^{(n)} P_{kj}^{(m)} .$$

Sol :-

$$P_{ij}^{(m+n)} = P(X_{m+n}=j \mid X_0=i)$$

$$= \sum_{k\in E} P(X_{m+n}=j, X_n=k \mid X_0=i)$$

$$P(AB|C) = P(A|BC) \cdot P(B|C)$$

$$\Rightarrow \sum_{k\in E} P(X_{m+n}=j \mid X_n=k, \cancel{X_0=i})\, P(X_n=k \mid X_0=i)$$

$$\Rightarrow \sum_{k\in E} P(X_{m+n}=j \mid X_n=k)\, P(X_n=k \mid X_0=i) \quad \left\{ \because \{X_n\} \text{ is a M.C.} \right.$$

$$\Rightarrow \sum_{k\in E} P_{kj}^{(m)} P_{ik}^{(n)}$$

$$\begin{array}{c|c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} P_{00}^{(m+n)} & P_{01}^{(m+n)} & \cdots \\ P_{10}^{(m+n)} & P_{10}^{(m+n)} & \cdots \end{pmatrix} \end{array} = \begin{pmatrix} P_{00}^{(m)} & P_{01}^{(m)} & \cdots \\ P_{10}^{(m)} & P_{11}^{(m)} & \cdots \end{pmatrix} \begin{pmatrix} P_{00}^{(n)} & P_{01}^{(n)} & \cdots \\ P_{10}^{(n)} & P_{11}^{(n)} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

$$P^{(m+n)} = P^{(m)} \cdot P^{(n)}$$

$$\to P^{(2)} = P^{(1)} \cdot P^{(1)} = P \cdot P = P^2$$

$$P^{(n)} = P^n .$$

$$P(X_n=i) = P_i^{(n)} \quad , \quad i \in E = \{0,1,2,\ldots\}$$

proof of x_n is:-

$$P^{(n)} = (P_0^{(n)}, P_1^{(n)}, P_2^{(n)}, \ldots)$$

~~proof of x_0 is~~:-

$$P^{(0)} = (P_0^{(0)}, P_1^{(0)}, P_2^{(0)}, \ldots)$$

$$P^{(n)} = P^{(0)} \cdot P^n \qquad \underline{P \to TPM}$$

Scanned by CamScanner

To prove above we need to show :-

$$p^{(n)} = p^{(n-1)} \cdot p$$

$$(p_0^{(n)}, p_1^{(n)}, \ldots, p_i^{(n)}, \ldots) = (p_0^{(n-1)}, p_1^{(n-1)}, \ldots) \begin{pmatrix} p_{00} & p_{01} & \ldots p_{0i} & \ldots \\ p_{10} & p_{11} & p_{1i} & \ldots \\ - & - & & \\ - & & & \end{pmatrix}$$

$$\boxed{p_i^{(n)} = \sum_{k \in E} p_k^{(n-1)} \cdot p_{ki}}$$

Now we have to prove this.

Sol^n.
$$p_i^{(n)} = P(X_n = i)$$
$$= \sum_{k \in E} P(X_n = i, X_{n-1} = k)$$
$$= \sum_{k} P(X_n = i \mid X_{n-1} = k) \cdot P(X_{n-1} = k)$$
$$= \sum_{k} p_{ki}\, p_k^{(n-1)}$$

Ex. Suppose that the chance of rain tomorrow depends on previous weather condition only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with prob α, and if it does not rain today, then it will rain tomorrow with prob. β.
(a) TPM ?
(b) $\alpha = 0.7$, $\beta = 0.3$. Calculate the prob. that it will rain two days from today given that it is raining today.

Sol:- X_n, weather condition on today $E = \{0, 1\}$.

a.)
$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix} \end{array}$$
0 → rain
1 → not rain

b.)
$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$
$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

$$P^{(2)} = P^2 = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$

$p_{00}^{(2)}$
$\Downarrow$
0.7×0.3
$+ 0.3 \times 0.7$

$p_{00}^{(4)} = ?$

$$p_{00}^{(2)} = \sum_K p_{0K}^{(1)} p_{K0}^{(1)} = P_{00} P_{00} + P_{01} P_{10}$$

$$= 0.7 \times 0.7 + 0.3 \times 0.3$$

$$P^{(4)} = P^{(2)} \cdot P^{(2)}$$

$$P_{01}^{(2)} = 0.7 \times 0.3 + 0.3 \times 0.7$$

20/1/20

Ex:- TPM of Markov chain with states $E = \{1, 2, 3\}$ is

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.1 & 0.5 & 0.4 \\ 2 & 0.6 & 0.2 & 0.2 \\ 3 & 0.3 & 0.4 & 0.3 \end{array}$$

if the initial state probability distribution is $(0.7, 0.2, 0.1)$

Find;

a) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

b) $P(X_3 = 2, X_0 = 2, X_1 = 3)$

c) $P(X_2 = 3)$

d) $P(X_1 = 3, X_2 = 3 | X_0 = 2)$

Sol:- $p^{(0)} = (0.7, 0.2, 0.1)$

$P(X_0=1)$ $P(X_0=2)$ $P(X_0=3)$.

a) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

$\Rightarrow P(X_3 = 2 | X_2 = 3, X_1 = 3, X_0 = 2) \cdot P(X_2 = 3 | X_1 = 3, X_0 = 2) \cdot P(X_1 = 3 | X_0 = 2) \cdot P(X_0 = 2)$

$\Rightarrow P(X_3 = 2 | X_2 = 3) \cdot P(X_2 = 3 | X_1 = 3) \cdot P(X_1 = 3 | X_0 = 2) \cdot P(X_0 = 2)$

$\Rightarrow P_{32} \cdot P_{33} \cdot P_{23} \cdot P(X_0 = 2) = 0.4 \times 0.3 \times 0.2 \times 0.2 = \underline{0.0048}$.

b) $P(X_3 = 2 | X_1 = 3, X_0 = 2) \cdot P(X_1 = 3 | X_0 = 2) \cdot P(X_0 = 2)$.

$\Rightarrow P(X_3 = 2 | X_1 = 3) \cdot P(X_1 = 3 | X_0 = 2) \cdot P(X_0 = 2)$.

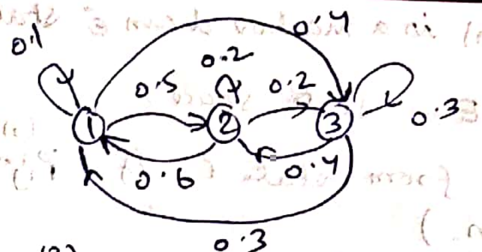
$\Rightarrow P_{32}^{(2)} \, P_{23} \cdot P(X_0 = 2)$

$$P^{(2)} = P^2$$

$$P_{32}^{(2)} = \sum_{K=1}^{3} P_{3K}^{(1)} \cdot P_{K2}^{(1)} = P_{31} \cdot P_{12} + P_{32} \cdot P_{22} + P_{33} \cdot P_{32}$$

$$= 0.3 \times 0.5 + 0.4 \times 0.2 + 0.3 \times 0.4 = 0.35$$

Scanned by CamScanner



$\Rightarrow P_{32}^{(2)} \; P_{23} \cdot P(X_0=2)$

$\Rightarrow 0.35 \times 0.2 \times 0.2$

d) $P(X_1=3, X_2=3 \mid X_0=2)$

<u>Apply</u> :- $\boxed{P(AB \mid C) = P(A \mid BC) \cdot P(B \mid C)}$

$\rightarrow P(X_2=3 \mid X_1=3, X_0=2) \cdot P(X_1=3 \mid X_0=2)$ $\{\{X_n\} \mapsto$ markov chain$\}$.

$\rightarrow P(X_2=3 \mid X_1=3) \cdot P(X_1=3 \mid X_0=2)$

$\rightarrow P_{33} \cdot P_{23} = 0.3 \times 0.2 = \underline{0.06}$.

c) $P(X_2=3) = P_3^{(2)}$ $\quad \{ p^{(2)} = (p_1^{(2)}, p_2^{(2)}, p_3^{(2)}) \}$

$p_1^{(2)} \to P(X_2=1)$, $p_2^{(2)} \to P(X_2=2)$, $p_3^{(2)} \to P(X_2=3)$

$p^{(2)} = p^{(1)} \cdot p$

$p^{(1)} = p^{(0)} \cdot p$

$p^{(2)} = p^{(0)} \cdot p^2$

$$p^{(1)} = p^{(0)} \cdot p = (0.7 \quad 0.2 \quad 0.1) \cdot \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$= \{0.7 \times 0.1 + 0.2 \times 0.6 + 0.1 \times 0.3, \underbrace{\quad}_{0.43}, \underbrace{\quad}_{0.35}\}$$

$$= (0.22, 0.43, 0.35)$$

$$p^{(2)} = p^{(1)} \cdot p = (0.22, 0.43, 0.35) \cdot \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$p_3^{(2)} = 0.22 \times 0.4 + 0.43 \times 0.2 + 0.35 \times 0.3$$

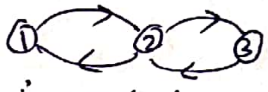
$$\boxed{p_3^{(2)} = 0.279}$$

Scanned by CamScanner

★ Classification of states : $\{X_n\}$ is a markov chain with state space E.

$i, j \in E \rightarrow$ state space.

1.) $i \rightarrow j$ (state j is accessible from state i, if $P_{ij}^{(n)} > 0$ for some n.)

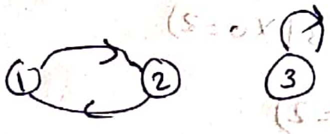


2.) $i \leftrightarrow j$ (state i and j communicate) if $i \rightarrow j$ and $j \rightarrow i$.

3.) Markov chain is irreducible if all states communicate with each other.

Otherwise M.C is said to be reducible.

Ex :-



$\Rightarrow$ If $i \leftrightarrow j$, $j \leftrightarrow k$, then $i \leftrightarrow k$

Sol: Since, $i \rightarrow j$, $j \rightarrow k$

$\exists\ m, n$ s.t $P_{ij}^{(n)} > 0$, $P_{jk}^{(m)} > 0$

$$P_{ik}^{(n+m)} = \sum_{l \in E} P_{il}^{(n)} \cdot P_{lk}^{(m)}$$

$$\geq P_{ij}^{(n)} \cdot P_{jk}^{(m)}$$

$$> 0$$

$\Rightarrow$ so, $i \rightarrow k$

similarly, $k \rightarrow i$

$\therefore$ $i \leftrightarrow k$.

4.) Period of state i : $d(i)$:

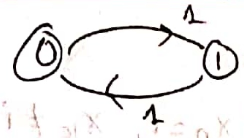
$d(i)$ = gcd of $I^+ = \{1, 2, 3, \ldots\}$, '$n$' s.t $P_{ii}^{(n)} > 0$

(If $P_{ii}^{(n)} = 0$ $\forall$ $n \geq 1$, define $d(i) = 0$)

$$P_{ii}^{(n)} = P_{ii}^{(n+d)} \quad \text{for } n \geq N.$$

Scanned by CamScanner

Example:- (1) $E = \{0,1\}$



$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

$$P^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P^3 = P^2 \cdot P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = P = P^3$$

$$P^4 = P^2$$

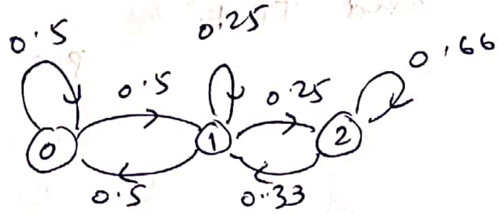
$$d(0) = \gcd \{\underbrace{2, 4, 6, \ldots}_{\text{no: of steps to reach '0' from '0'.}}\} = 2 = d(1).$$

→ $\boxed{d(i) \rightarrow \text{period of state } i.}$

Here both states are communicating ē each other $(i \leftrightarrow j)$ so it is irreducible markov chain.

(2) TPM.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.33 & 0.66 \end{pmatrix} \end{array}$$



so, $0 \leftrightarrow 1 \leftrightarrow 2$

→ Hence, it is an irreducible markov chain.

equivalence class $\{0, 1, 2\}$.

→ $d(0) = \gcd\{n\}$ s.t $P_{ii}^{(n)} > 0$.

$$d(0) = \gcd\{\underbrace{1, 2, 3, 4, \ldots}_{\text{can be directly written from figure.}}\} = 1. = d(1) = d(2).$$

$P_{00}^{(n)} > 0$; $\boxed{\text{state } 0 \text{ has period } 1}$, i.e., a periodic state.

→ For state i ;

$(f_{ii}^{(n)}$ or$)$ $f_i^{(n)} = P(X_n = i,\ X_k \neq i\ ;\ k = 1, 2, \ldots, n-1 \mid X_0 = i)$

↳ prob. of first visit of state i in 'n' steps / transitions starting from state i.

{recurrence time probability}.

$(f_{ii}$ or$)$ $f_i = f_i^{(1)} + f_i^{(2)} + \ldots\ldots$

{∴ $f_i^{(0)} = 1$, reaching to itself is certain}

↓ prob. of first ever visit to state i.

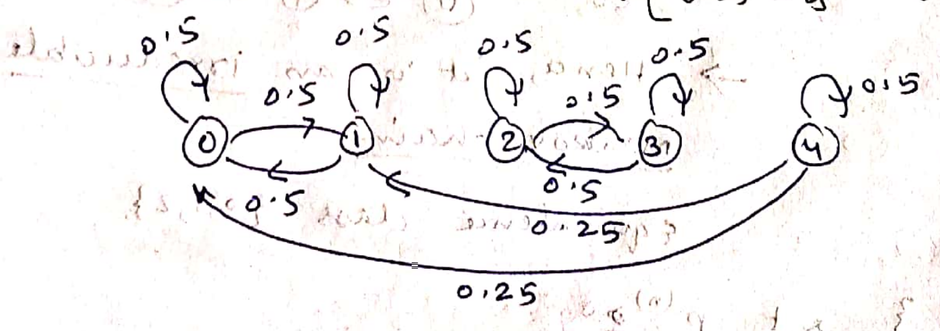
→ state 'i' is recurrent if $f_i = 1$, i.e., return to state 'i' is certain

→ if $f_i < 1$, return to state i is uncertain.

state 'i' is transient.}

Ex:- consider a markov chain with states $E = \{0, 1, 2, 3, 4\}$ and TPM:

$$P = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0.5 \end{bmatrix} \end{array}$$



$0 \leftrightarrow 1$ $\quad 2 \leftrightarrow 3 \quad 4$

equivalence class $\{0,1\}$ $\{2,3\}$ $\{4\}$

→ Hence, this is reducible markov chain.

→ Aperiodic states.

→ states 0, 1, 2, 3 are recurrent states and 4 is transient state.

$\rightarrow f_0 = f_0^{(1)} + f_0^{(2)} + f_0^{(3)} + \cdots$

$= \frac{1}{2} + \left(\frac{1}{2}\times\frac{1}{2}\right) + \left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right) + \cdots$

$= \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{4} + \cdots\right]$

$= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1$

$f_0 = 1$ $\{\because$ state '0' is recurrent$\}$

$f_4 = f_4^{(1)} + f_4^{(2)} + f_4^{(3)} + \cdots$

$= \frac{1}{2} + 0 + 0 + \cdots = \frac{1}{2} < 1$

$\therefore$ Hence, (4) is transient $\longrightarrow$ {we cannot stay there for very longer time}.

★ X_n is markov chain and E is state space.

Let $I_n = \begin{cases} 1, & \text{if } X_n = i \\ 0, & \text{if } X_n \neq i \end{cases}$

starting ē state i

$\rightarrow \sum_{n=1}^{\infty} I_n \rightarrow$ # of time periods the process is in state i.

$E\left(\sum_{n=1}^{\infty} I_n \mid X_0 = i\right) = \sum_{n=1}^{\infty} E(I_n \mid X_0 = i)$

$= \sum_{n=1}^{\infty} P(X_n = i \mid X_0 = i)$

$= \sum_{n=1}^{\infty} P_{ii}^{(n)} \longrightarrow$ [if this diverges ($=\infty$) then also it is recurrent].

[if 'i' is recurrent $\Leftrightarrow f_i = 1$] $\Leftrightarrow \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$

★ Let i is the transient state. $\{f_i < 1\}$

$N \rightarrow$ # of time periods, so that the process is in state i, starting from state i.

$P(N = n) = f_i \cdot f_i \cdots f_i\,(1 - f_i)$

$= f_i^{n-1} \cdot (1 - f_i)$, $n = 1, 2, \cdots$

Hence, $\boxed{N \sim \text{Geo.}(1 - f_i)}$

$E(N) = \frac{1}{1 - f_i}$

Scanned by CamScanner

$\rightarrow$ i is transient $\Leftrightarrow$ $f_i < 1$ $\Leftrightarrow$ $\sum P_{ii}^{(n)} < \infty$ {not divergence}

★ Properties:

1.) $i \rightarrow$ recurrent , $i \leftrightarrow j$, then j is recurrent.

2.) $i \rightarrow$ transient , $i \leftrightarrow j$, then j is transient.

3.) In a finite state markov chain, all states cannot be transient.

4.) In a finite state, irreducible M.C, all states are recurrent.

5.) Markov chain irreducible, all states are either recurrent or transient.

Sol:- 1.) Given $\sum P_{ii}^{(n)} = \infty$, $i \leftrightarrow j$

To show: $\sum P_{jj}^{(n)} = \infty$

$i \leftrightarrow j$, $\exists$ m, n s.t $P_{ij}^{(m)} > 0$, $P_{ji}^{(n)} > 0$

$$P_{jj}^{(m+n+v)} \geq P_{ji}^{(n)} \cdot P_{ii}^{(v)} \cdot P_{ij}^{(m)}$$ {using C-K eq^n $\downarrow$ chaplan}

$$\sum P_{jj}^{(n+m+v)} \geq P_{ji}^{(n)} \underbrace{\left(\sum P_{ii}^{(v)} \right)}_{\infty \text{ (diverges)}} P_{ij}^{(m)}$$ {Hence LHS will also diverge}

$$\sum P_{jj}^{(n)} = \infty \Leftrightarrow j \text{ recurrent}$$

2.) Given, $i \rightarrow$ transient , $i \leftrightarrow j$

To show, $j \rightarrow$ transient.

On contrary suppose $j \rightarrow$ recurrent.

given, $i \leftrightarrow j$, using 1.) $\underline{i \text{ is recurrent}}$, a contradiction

Hence, $\underline{j \text{ is transient}}$.

Scanned by CamScanner

3.) on a contrary, let us assume that all states are transient.

$E = \{1, 2, \ldots, M\}$.

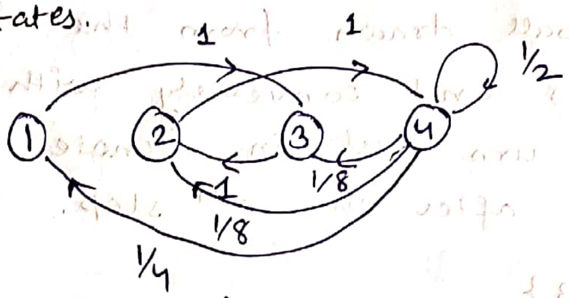
4.) Using 1.) and 3.) we can prove it.

5.) Here states can be infinite.

Ex:- M.C having TPM as:-

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \end{array}$$

→ classify the states.



Equivalence class: $\{1, 2, 3, 4\}$ $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$

finite state irreducible markov chain.

therefore all states are recurrent.

→ If state i is recurrent,

$$m_i = \sum_{n=1}^{\infty} n \cdot f_i^{(n)} \longrightarrow \text{mean recurrence time.}$$

$$f_{ii}^{(n)} = f_i^{(n)} = P(X_n = i, X_k \neq i, k = 1, 2, \ldots, n-1 \mid X_0 = i)$$

If $m_i = \infty$, then state i is called recurrent null.

If $m_i < \infty$, then state i is non-null recurrent or positive recurrent.

Ex: (above)

$f_4^{(1)} = \frac{1}{2}$, $f_4^{(2)} = \frac{1}{8} \times 1 = \frac{1}{8}$, $f_4^{(3)} = \frac{1}{8} \times 1 \times 1 = \frac{1}{8}$

$f_4^{(4)} = \frac{1}{4}$, $f_4^{5} = 0$ at $m \geq 5$

$$f_4 = \sum_{n=1}^{\infty} f_4^{(n)} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + 0 + 0 + \dots = 1$$

so, (4) $\rightarrow$ recurrent.

$$\rightarrow m_4 = \sum_{n=1}^{\infty} n \cdot f_4^{(n)} = 1\times\frac{1}{2} + 2\times\frac{1}{8} + 3\times\frac{1}{8} + 4\times\frac{1}{4} + 0 + \dots < \infty$$

Hence, (4) is positive recurrent.

Ex:- 3 W, 3B balls are distributed in two urns in such a way that each contain 3 balls. We say that the system is in state i; $i = 0, 1, 2, 3$. if the first urn contain i W balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system after the n^{th} step. TPM.

$\rightarrow$ $E = \{0, 1, 2, 3\}$.

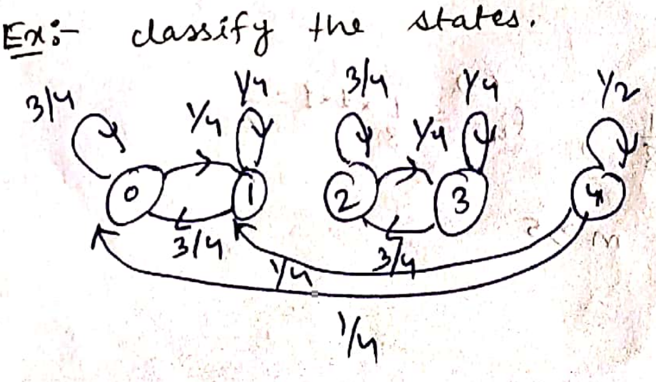
$$X_{n+1} = j$$

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3}\times\frac{1}{3} & 4/9 & \frac{2}{3}\times\frac{2}{3} & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$X_n \rightarrow$ state of system at n^{th} step.

$$P_{ij} = P(X_{n+1} = j \mid X_n = i)$$

Ex:- classify the states.



$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{vmatrix} 3/4 & 1/4 & 0 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \end{vmatrix} \end{array}$$

Scanned by CamScanner

$E = \{2,3\} \quad \{0,1\} \quad \{4\}$

recurrent $\qquad$ transient

Scanned by CamScanner

$f_4 = f_4^{(1)} + f_4^{(2)} + f_4^{(3)} + \cdots$

$= \frac{1}{2} + 0 + 0 \cdots = \frac{1}{2} < 1$; state 4 is transient.

$f_0 = f_0^{(1)} + f_0^{(2)} + f_0^{(3)} + \cdots$

$= \frac{3}{4} + \frac{1}{4} \times \frac{3}{4} + \left(\frac{1}{4}\right)^2 \times \frac{3}{4} + \cdots$

$= \frac{3}{4} \cdot \left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right]$

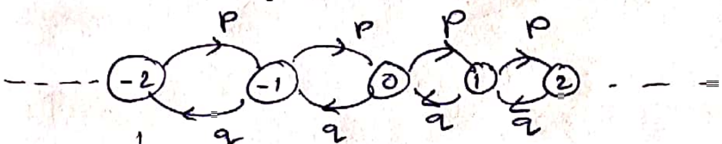
$= \frac{3}{4} \times \frac{1}{1 - 1/4} = 1$

Hence, state '0' is recurrent.

Ex:- One-dimensional random walk. $\{E = \{\ldots, -2, -1, 0, 1, 2, \ldots\}\}$

$P_{i,i+1} = p$; $P_{i,i-1} = 1-p = q$

$P_{ij} = 0$ if $j \neq i-1, i+1$, $0 < p < 1$



$X_n \rightarrow$ position of particle at step 'n'.

irreducible Markov chain.

$\rightarrow$ Here, if particular state is recurrent (or transient) then all are recurrent (or transient).

$\sum P_{ii}^{(n)} = \infty \rightarrow$ state i recurrent

$< \infty \rightarrow$ state i transient.

$P_{i,i}^{(n)} = 0$, $n = 2k+1$, $k = 1, 2, 3, \ldots$; ~~$n = 2k+1$, $k=1,2,3,\ldots$~~

$\underbrace{{}^{2k}C_k \, p^k \cdot (1-p)^k}_{a_k}$; $n = 2k$

$\rightarrow \frac{a_{k+1}}{a_k} = \frac{\binom{2k+2}{k+1} \cdot p^{k+1} \cdot (1-p)^{k+1}}{\binom{2k}{k} \cdot p^k \cdot (1-p)^k}$

$$\sum_n P_{i,i}^{(n)} = \sum_{k=1}^{\infty} a_k$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \begin{cases} <1, & \sum a_k \text{ converges} \\ >1, & \sum a_k \text{ diverges} \end{cases}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 4p(1-p)$$

$$= \begin{cases} 1, & p = \frac{1}{2} \\ <1, & p \neq \frac{1}{2} \end{cases} \qquad \underline{\sum a_k < \infty}.$$

$p \neq \frac{1}{2}$; $i \rightarrow$ transient

$\therefore$ $p = \frac{1}{2}$: Hint : use sterling's approx.

$$n! \sim n^{n+\frac{1}{2}} \cdot e^{-n} \cdot \sqrt{2\pi}$$

and, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{converges if } p>1 \\ \text{diverges if } p \leq 1 \end{cases}$

21/1/20. **Gambler's ruin game:**

Rs i Aim Rs N

$i = 0, 1, 2, \dots, N$

Z_i i^{th} bet

s.t $P(Z_i = 1) = p$; $P(Z_i = -1) = q = 1-p$.

X_n = for time of gambler after n steps / bets.

$= Z_1 + Z_2 + \dots + Z_n + i$.

$\rightarrow$ $\{X_n\}$ is markov chain.

$\rightarrow$ $E = \{0, 1, 2, \dots, N\}$.

TPM

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & \dots & N \\ \hline 0 & 1 & 0 & 0 & \text{---} & 0 \\ 1 & q & 0 & p & \dots & 0 \\ 2 & 0 & q & 0 \;\; p & \dots & 0 \\ \vdots & & & & & \\ N & \text{---} & & & & 1 \end{array}$$

Scanned by CamScanner

classes $\{0\}$ $\{1, 2, \cdots, N-1\}$ $\{N\}$.

$\{0\}$ absorbing, $\{1, 2, \cdots, N-1\}$ transient, $\{N\}$ absorbing

T_0 = time he broke $= \inf\{n : X_n = 0\}$

T_N = time he gets Rs N $= \inf\{n : X_n = N\}$

$$\rightarrow P(T_N < T_0) = P(T_N < T_0 \mid Z_1 = -1) \cdot P(Z_1 = -1) + P(T_N < T_0 \mid Z_1 = +1) \cdot P(Z_1 = +1)$$

{probability he achieved before he broke}. {using law of total prob.}

$P_{i-1} \cdot q$; $P_{i+1} \cdot p$

$$P_i = P(T_N < T_0) = E[P(T_N < T_0 \mid Z_1)]$$

{ $X_n = Z_1 + Z_2 + \cdots + Z_n + i$; i, N

$Z_n \rightarrow$ amount gain or loss in n^{th} step. }

$$P_i = P_{i-1} \cdot q + P_{i+1} \cdot p$$

$$p \cdot P_i + q \cdot P_i = P_{i-1} \cdot q + P_{i+1} \cdot p$$

$$(P_{i+1} - P_i)\, p = q \cdot (P_i - P_{i-1})$$

$$\Rightarrow P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}) \qquad \left\{ \begin{array}{l} i = 1, 2, \cdots \\ P_0 = 0 ;\ P_N = 1 \end{array} \right.$$

$$\rightarrow P_2 - P_1 = \frac{q}{p} \cdot P_1$$

$$\rightarrow P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 \cdot P_1$$

$$\rightarrow P_4 - P_3 = \frac{q}{p}(P_3 - P_2) = \left(\frac{q}{p}\right)^3 \cdot P_1$$

$\vdots$

$$\rightarrow P_i - P_{i-1} = \left(\frac{q}{p}\right)^{i-1} \cdot P_1$$

Now add all the above equations.

$$P_i - P_1 = P_1 \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1} \right]$$

Scanned by CamScanner

$\rightarrow P_i = P_1 \left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1} \right]$

$$\rightarrow P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-q/p} \cdot P_1 & ; \quad \frac{q}{p} \neq 1 \\ i P_1 & ; \quad \frac{q}{p} = 1 \end{cases}$$

we know, $P_N = 1$.

$$1 = P_N \Rightarrow P_1 = \begin{cases} \dfrac{1-q/p}{1-\left(\frac{q}{p}\right)^N} & ; \text{ if } \frac{q}{p} \neq 1 \\ \dfrac{1}{N} & ; \text{ if } \frac{q}{p} = 1 \end{cases} ; P_0 = 0, P_N = 1$$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & ; \text{ if } \frac{q}{p} \neq 1 \Leftrightarrow p \neq 1/2 \\ \dfrac{i}{N} & ; \frac{q}{p} = 1 \Leftrightarrow p = 1/2 \end{cases}$$

$\Rightarrow$ Let us take N is large $(N \to \infty)$.

$$P_i = \begin{cases} 0 & ; p = 1/2 \Leftrightarrow \frac{q}{p} = 1 \\ 1-\left(\frac{q}{p}\right)^i & ; \frac{q}{p} < 1 \Leftrightarrow p \neq 1/2 \\ 0 & ; \frac{q}{p} > 1 \end{cases}$$

$$P_i = \begin{cases} 1-\left(\frac{q}{p}\right)^i & , \frac{q}{p} < 1 \Leftrightarrow p > 1/2 \\ 0 & , \frac{q}{p} \geq 1 \Leftrightarrow p \leq 1/2 \end{cases}$$

Scanned by CamScanner

** consider two state M.C having TPM.

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \end{array}, \quad 0 \le a, b \le 1$$

s.t, $|1-a-b| < 1$

violated if $a=0, b=0$ or $a=1, b=1$.

$$(p_{ij}^{(n)}) = P^n = \begin{bmatrix} \frac{b+a(1-a-b)^n}{a+b} & \frac{a-a(1-b-a)^n}{a+b} \\ \frac{b-b(1-a-b)^n}{a+b} & \frac{a+b(1-a-b)^n}{a+b} \end{bmatrix}$$

$P_{00} = 1-a, \quad P_{01} = a, \quad P_{10} = b, \quad P_{11} = 1-b$

Let us determine, $p_{00}^{(n)} = p_{00}^{(n-1)} \cdot P_{00} + P_{01}^{(n-1)} \cdot P_{10}$

$\left\{ P_{00}^{(n+m)} = \sum_K P_{0K}^{(n)} P_{K0}^{(m)} \right.$

$$P^{(n)} = \begin{pmatrix} p_{00}^{(n)} & p_{01}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} \end{pmatrix}$$

$$p_{00}^{(n)} = p_{00}^{(n-1)} \cdot (1-a) + (1 - p_{00}^{(n-1)}) \cdot b$$

$$p_{00}^{(n)} = b + (1-a-b)\, p_{00}^{(n-1)}.$$

$$= b + b(1-a-b) + b(1-a-b)^2 + \dots + b(1-a-b)^{n-2} + P_{00}(1-a-b)^{n-1}$$

$P_{00} \to (1-a)$

$$p_{00}^{(n)} = b\left[\sum_{K=0}^{n-2} (1-a-b)^K\right] + (1-a)\cdot(1-a-b)^{n-1}$$

$$p_{00}^{(n)} = b\cdot\left[\frac{1-(1-a-b)^{n-1}}{1-(1-a-b)}\right] + (1-a)\cdot(1-a-b)^{n-1}.$$

$$p_{00}^{(n)} = \frac{b}{a+b} + \left[\frac{-b}{a+b} + 1-a\right]\cdot(1-a-b)^{n-1}.$$

$$((p_{ij}^{(n)}))\ P^{(n)} = \begin{bmatrix} \frac{b+a(1-a-b)^n}{a+b} & \frac{a-a(1-a-b)^n}{a+b} \\ \frac{b-b(1-a-b)^n}{a+b} & \frac{a+b(1-a-b)^n}{a+b} \end{bmatrix}$$

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$$

Scanned by CamScanner

Note that, $|1-a-b| < 1$, $0 < a, b < 1$

as $n \to \infty$

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

X_0 $\quad P^{(0)} = \left(\frac{1}{3}, \frac{2}{3}\right) = (P(X_0 = 0), P(X_0 = 1))$

X_n $\quad P^{(n)} = P^{(0)} \cdot P^n .$

$$\lim_{n\to\infty} P^{(n)} = \lim_{n\to\infty} P^{(0)} \cdot P^n = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)$$

Scanned by CamScanner

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

$X_0 \qquad P^{(0)} = \left(\frac{1}{3}, \frac{2}{3}\right) = \left(P(X_0=0), P(X_0=1)\right)$

$X_n \qquad P^{(n)} = P^{(0)} \cdot P^n.$

$$\lim_{n\to\infty} P^{(n)} = \lim_{n\to\infty} P^{(0)} \cdot P^n = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)$$

27/1/20.

Ex: A rat is put into the linear maze.

shock	1	2 rat is here	3	4	5 food

At each step the rat moves to right with ($p = 3/4$) and to the left $\bar{c}$ prob $= 1/4$. What is the prob. that the rat finds the food before getting shocked?

Sol: $p \neq 1/2$

$i = 2, \quad N = 5$ Gambler's ruin problem.

$$P(\text{food before shock}) = \frac{1-(q/p)^i}{1-(q/p)^N} = \frac{1-(1/3)^2}{1-(1/3)^5} = 0.892.$$

Ex: An organisation has N-employees where N is large. Each employee has one of the three possible job classification and changes classification (independently) according to M.C with TPM.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

What % of employees are in each classification?

Scanned by CamScanner

$\boxed{P \rightarrow \text{regular TPM}}$

$\boxed{\underset{\sim}{\pi} = \underset{\sim}{\pi} P}$ ** Imp eqⁿ.

∴ A P is said to be regular TPM if p^k has all the elements positive.

$p_{ij} > 0$

$$\underset{\sim}{\pi} = (\pi_1, \pi_2, \pi_3)$$

$$\sum_i \pi_i = 1.$$

$$\rightarrow \pi_1 + \pi_2 + \pi_3 = 1$$

$$\Rightarrow \underset{\sim}{\pi} = \underset{\sim}{\pi} P$$

$$\pi_1 = 0.7\pi_1 + 0.2\pi_2 + 0.1\pi_3 \quad \text{——} (1)$$

$$\pi_2 = 0.2\pi_1 + 0.6\pi_2 + 0.4\pi_3 \quad \text{——} (2)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \quad \text{——} (3)$$

solve above 3 eqⁿ:-

$$\pi_1 = \frac{6}{17}, \quad \pi_2 = \frac{7}{17}, \quad \pi_3 = \frac{4}{17}$$

$\rightarrow \left(\frac{6}{17} \times 100\right)\%$ in classification I.

$\rightarrow \left(\frac{7}{17} \times 100\right)\%$ in —"— 2 -

$\rightarrow \left(\frac{4}{17} \times 100\right)\%$ in —"— 3 .

Ex: Every time that the team wins a game, it was its next game with $p = 0.8$; every time it loses a game it wins its next game $\bar{c}$ $p = 0.3$. If the team wins a game then it has dinner together $\bar{c}$ $p = 0.7$, whereas if the team loses then it has dinner together $\bar{c}$ $p = 0.2$. what proportion of games results in a team dinner?

X_n: team performance in n^{th} game; $E = \{W, L\}$

Scanned by CamScanner

$$
\begin{array}{c|cc}
 & W & L \\
\hline
W & 0.8 & 0.2 \\
L & 0.3 & 0.7
\end{array}
$$

$x_n = i$

$i, j \in \{W, L\}$

$P_{ij} = P(X_{n+1} = j \mid X_n = i)$

Let us take $W = 0$; $L = 1$.

$$(\pi_0, \pi_1) = (\pi_0, \pi_1)\begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

$\pi_0 + \pi_1 = 1$ — ①

$\pi_0 = 0.8\pi_0 + 0.3\pi_1$ — ②

$\hookrightarrow 0.2\pi_0 - 0.3\pi_1 = 0$

$\rightarrow 0.5\pi_0 = 0.3$

$\pi_0 = \frac{3}{5}$

$\pi_1 = 2/5$

In long run team wins $\bar{c}$ $p = 3/5$ & loses $\bar{c}$ $p = 2/5$

$$\text{Dinner} = \begin{cases} 0.7 & \bar{c} \quad p = 3/5 \\ 0.2 & \bar{c} \quad p = 2/5 \end{cases}$$

$E(\text{dinner together}) = 0.7 \times \frac{3}{5} + 0.2 \times \frac{2}{5}$

$= \frac{25}{50} = \frac{1}{2} = 50\%$

Ex: The random variables $\xi_1, \xi_2, \ldots$ are independent and $\bar{c}$ the common pmf.

k	0	1	2	3
$P(\xi = k)$	0.1	0.3	0.2	0.4

Set $X_0 = 0$ and let $X_n = \max\{\xi_1, \ldots, \xi_n\}$ be the largest ξ observed to date. Determine TPM of MC $\{X_n\}$.

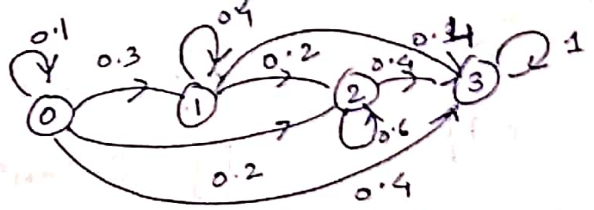
Sol: $E = \{0, 1, 2, 3\}$

$P_{ij} = P(X_{n+1} = j \mid X_n = i)$
$i, j \in E$

$X_{n+1} = j$

$$
X_n = i \quad
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0.1 & 0.3 & 0.2 & 0.4 \\
1 & 0 & 0.4 & 0.2 & 0.4 \\
2 & 0 & 0 & 0.6 & 0.4 \\
3 & 0 & 0 & 0 & 1
\end{array}
$$

Scanned by CamScanner



Hence, it is reducible M.C.

class = $\underbrace{\{0\}, \{1\}, \{2\}}_{\text{transient}}, \underbrace{\{3\}}_{\text{recurrent}}$.

Ex: suppose A and B play a game to flip a Rs 1 coin. Both A and B have only coins of ₹1 type. Assume B is the better player, has a prob. 0.6 of winning in each flip.

If B starts with 5 coins and A with 10, what is the prob. that B will wipe out A?

Sol: $p = 0.6$
$q = 0.4$

$i = 5$
$N = 15$

$$\Rightarrow \frac{1 - \left(\frac{0.4}{0.6}\right)^5}{1 - \left(\frac{0.4}{0.6}\right)^{15}} = 0.876$$

{using Gamblers ruin problem}.

Ex:

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & 1/2 & 1/3 & 1/6 \\ 1 & 0 & 1/3 & 2/3 \\ 2 & 1/2 & 0 & 1/2 \end{array} \leftarrow \text{TPM}$$

$\{X_n\}$ is M.C.

If $P(X_0 = 0) = P(X_0 = 1) = 1/4$; Find $E(X_3)$.

Sol: Hint, $p^{(0)} = \left(\frac{1}{4}, \frac{1}{4}, \frac{2}{4}\right)$

$\underbrace{p^{(0)}, p^{(1)}, p^{(2)}, p^{(3)}}$ these are prob. distribution.

$p^{(1)} = p^{(0)} \cdot P$

$p^{(2)} = p^{(1)} P$

$p^{(3)} = p^{(2)} \cdot P = (a, b, c)$

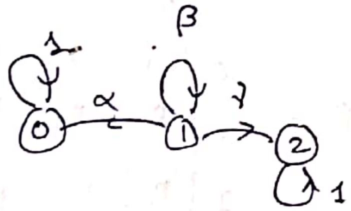
$$\underline{E(X_3) = 0 \times a + 1 \times b + 2 \times c.}$$

Scanned by CamScanner

28/1/20

Simple first step analysis:

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$



1 → transient

0, 2 → absorbing.

$T = \min\{n : X_n = 0 \text{ or } X_n = 2\}$

↓

time of absorbtion

(min. time required to absorb in 0 or 2 steps).

$P(X_T = 0 \mid X_0 = 0) = 1$; $P(X_T = 0 \mid X_0 = 1) = u$; $P(X_T = 0 \mid X_0 = 2) = 0$

mean no. of steps req. for absorbtion ν.

$\nu = E(T \mid X_0 = 1)$

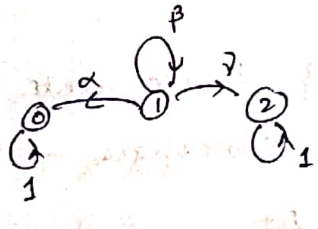
$u = P(X_T = 0 \mid X_0 = 1) = \sum_{j=0}^{2} P(X_T = 0 \mid X_1 = j, X_0 = 1)\, P(X_1 = j \mid X_0 = 1)$ [it is a M.C]

$= \sum_{j=0}^{2} P(X_T = 0 \mid X_1 = j)\, P_{ij}$

$= \alpha \times 1 + \beta \times u + \gamma \times 0$

$u = \alpha + \beta u.$

$$\boxed{u = \frac{\alpha}{1-\beta}}$$ → prob. of ultimate absorbtion starting from state 1.



→ $\nu = E(T \mid X_0 = 1)$

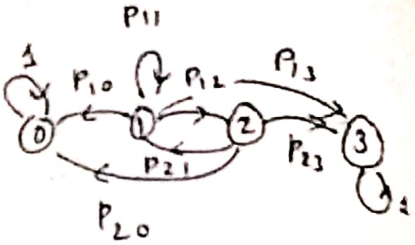
$\nu = 1 + \underbrace{\alpha \times 0 + 1 \times 0}_{X_1 = 0 \text{ or } 2} + \underbrace{\beta \times \nu}_{X_1 = 1}$

$\nu = 1 + \beta\nu$

$$\boxed{\nu = \frac{1}{1-\beta}}$$

Scanned by CamScanner

Ex: consider 4 state M.C.



$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} ;$$

$\left.\begin{array}{l} 1, 2 \rightarrow \text{transient} \\ 0, 3 \rightarrow \text{absorbing} \end{array}\right\}$ states

$T = \min\{n : X_n = 0 \text{ or } X_n = 3\}$
↳ time to absorbtion.

$u_1 = P(X_T = 0 \mid X_0 = 1)$

$u_2 = P(X_T = 0 \mid X_0 = 2)$

$\nu_1 = E(T \mid X_0 = 1)$

$\nu_2 = E(T \mid X_0 = 2)$

$u_1 = P(X_T = 0 \mid X_0 = 1)$

$= \sum_{j=0}^{3} P(X_T = 0 \mid X_1 = j, X_0 = 1)\, P(X_1 = j \mid X_0 = 1)$

$u_1 = P_{10} + P_{11} \cdot u_1 + P_{12} u_2 + 0$

$u_2 = P_{20} + P_{21} u_1 + P_{22} u_2 + 0$

solve above two eq^n:

$\nu_1 = 1 + P_{11} \cdot \nu_1 + P_{12} \cdot \nu_2$

$\nu_2 = 1 + P_{21} \cdot \nu_1 + P_{22} \cdot \nu_2$

solve above to get ν_1 & ν_2.

★ $\{X_n\}$ finite state M.C.

statespace $\rightarrow \{0, 1, \ldots, N\}$

let $0, 1, \ldots, r-1 \rightarrow$ transient

$r, r+1, \ldots, N \rightarrow$ absorbing state $\rightarrow (P_{ii} = 1$ for $r \le i \le N)$

$P_{ij}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$; $0 \le i, j < r$

Scanned by CamScanner

$$P = \begin{pmatrix} Q & R \\ O & I \end{pmatrix}$$

$$Q_{r \times r} \qquad R_{r \times (N-r+1)}$$

$$O_{(N-r+1)\times r} \qquad I_{(N-r+1)\times(N-r+1)}$$

$u_{ik} = u_i = P(\text{absorbtion in state } k \mid X_0 = i)$

$$= \sum_{j=0}^{N} P(\text{absorbtion in } k \mid X_1 = j, \cancel{X_0 = i})\, P_{ij}$$

$$u_i = P_{ik} + \sum_{\substack{j=r \\ j \neq k}}^{N} P_{ij} \times 0 + \sum_{j=0}^{r-1} P_{ij}\, u_i$$

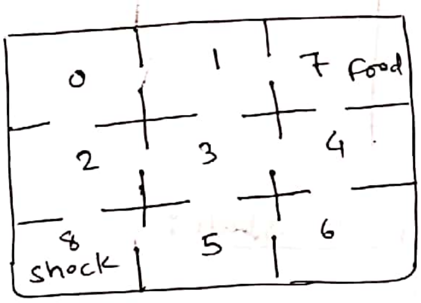
$$\boxed{u_i = P_{ik} + \sum_{j=0}^{r-1} P_{ij} \cdot u_j}$$

Previous results generalised.

$\nu_i = E(T \mid X_0 = i)$

$$\boxed{\nu_i = 1 + \sum_{j=0}^{r-1} P_{ij}\, \nu_j} \qquad \text{for } i = 0, 1, \dots, r-1$$

Ex:-



Find the prob. that rat finds food first then shock given that it is dropped initially in compartment i. $i = 0, 1, 2, \dots, 6$

$u_i = u_i(7) \to$ prob. of absorbtion in food compartment 7, given that the rat is dropped initially in compartment i, $i = 0, 1, 2, \dots, 6$.

TPM.

$$P = \begin{array}{c|ccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 \\ 2 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 1/3 \\ 3 & 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\ 5 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 1/3 \\ 6 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Scanned by CamScanner

$0, 1, \ldots, 6 \longrightarrow$ transient

$7, 8 \longrightarrow$ absorbing.

Equations:

$$u_0 = \frac{1}{2} \cdot u_1 + \frac{1}{2} u_2$$

$$u_1 = \frac{1}{3} u_0 + \frac{1}{3} u_3 + \frac{1}{3}$$

$$u_2 = \frac{1}{3} u_0 + \frac{1}{3} u_3 + 0$$

$$u_3 = \frac{u_1}{4} + \frac{u_2}{4} + \frac{u_4}{4} + \frac{u_5}{4}$$

$$u_4 = \frac{u_3}{3} + \frac{u_6}{3} + \frac{1}{3}$$

$$u_5 = \frac{u_3}{3} + \frac{u_6}{3}$$

$$u_6 = \frac{u_4}{2} + \frac{u_5}{2}$$

By symmetry $\begin{cases} u_0 = u_6 \\ u_1 = u_4 \\ u_2 = u_5 \\ u_3 = \frac{1}{2} \end{cases}$

$u_0 = \frac{1}{2}$; $u_1 = \frac{2}{3}$; $u_2 = \frac{1}{3}$.

3/2/20 Mean time spent in transient state :-

Scanned by CamScanner

Ex:- consider Gambler's ruin problem with $p = 0.4$ & $N = 4$. starting with 2 units, determine.

a.) expected amount of time the gambler has 3 units.

b.) " " ", ______ 1 units.

Sol:-

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0.6 & 0 & 0.4 & 0 & 0 \\ 2 & 0 & 0.6 & 0 & 0.4 & 0 \\ 3 & 0 & 0 & 0.6 & 0 & 0.4 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$C_1 = \{0\}$ absorbing, $C_2 = \{1, 2, 3\}$ transient, $C_3 = \{4\}$ absorbing

$$S = (I - P_T)^{-1}$$

$$P_T = \begin{bmatrix} 0 & 0.4 & 0 \\ 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 \end{bmatrix}$$

Scanned by CamScanner

$$(I-P_T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.4 & 0 \\ 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -0.4 & 0 \\ -0.6 & 0 & -0.4 \\ 0 & -0.6 & 1 \end{bmatrix}$$

$$S = (I-P_T)^{-1} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1.46 & 0.76 & 0.31 \\ 1.15 & 1.92 & 0.76 \\ 0.69 & 1.15 & 1.46 \end{bmatrix} \end{array}$$

$S_{23} = 0.76$

$S_{21} = 1.15$

$i, j \in T$

f_{ij} = prob. that M.C ever makes a transition into state j given that it starts in state i.

s_{ij} = E(time in j | start in i)

= E(time in j | start in i, ever transient to j) $\cdot f_{ij}$

= E(time in j | start in i, never transient to j) $\cdot (1-f_{ij})$.

+ E(time in j | start in i,

$\{ E(X) = E(E(X|Y)) = \Sigma E(X|Y=i) \cdot P(Y=i) \}$

$= (\delta_{ij} + S_{jj}) \cdot f_{ij} + \delta_{ij}(1-f_{ij})$

$= \cancel{\delta_{ij} \cdot f_{ij}} + S_{jj} \cdot f_{ij} + \delta_{ij} - \cancel{\delta_{ij} f_{ij}}$

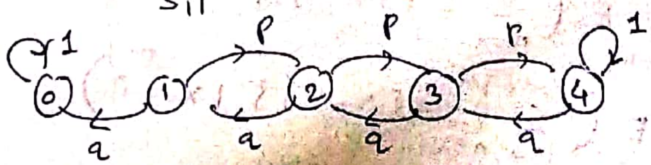
$s_{ij} = \cancel{\delta_{ij} \cdot f_{ij}} + S_{jj} \cdot f_{ij} + \delta_{ij} - \cancel{\delta_{ij} f_{ij}}$

$$f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}$$

Q- cont.

(c) What is the prob. that the gambler ever has a fortune of 1 ?

$$f_{21} = \frac{S_{21} - \delta_{21}}{S_{11}} = \frac{1.15 - 0}{1.46} = 0.78$$



$$f_{21} = q + pq^2 + p^2q^3 + \cdots$$
$$= q + pq(q + pq^2 + \cdots) = q + pq \cdot f_{21}$$

$$\rightarrow f_{21} = \frac{q}{1-pq} = \frac{0.6}{1-0.4\times 0.6} = 0.78$$

$f_{21} \longrightarrow$ start in 2, visit 1 before 4.

$\longrightarrow$ start in 1, visit 0, before 3.

$p = 0.4,\ q = 0.6$
$i = 1,\ N = 3$

$$f_{21} = \frac{1-(q/p)^i}{1-(q/p)^N} = \frac{1-\left(\frac{0.6}{0.4}\right)^1}{1-\left(\frac{0.6}{0.4}\right)^3} = 0.78.$$

★ Particular case:

TPM, $P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$ $\quad 0, 1, \ldots, r-1$ transient; $r, \ldots, N$ absorbing.

$P_T = Q$

$S = (I-Q)^{-1}$

$$S_{ij} = \delta_{ij} + \sum_{l=0}^{r-1} P_{il} \cdot S_{lj} \quad ; \quad i, j = 0, 1, 2, \ldots, r-1.$$

$U_i = E(T \mid X_0 = i)$

$T = \min\{n ;\ r \le X_n \le N\}$.

$$S_{ij} = E\left(\sum_{n=0}^{T-1} \{X_n = j\} \mid X_0 = i\right), \quad 0 < i, j < r.$$

$$\sum_{j=0}^{r-1} \sum_{n=0}^{T-1} 1\{X_n = j\} = \sum_{n=0}^{T-1} \left(\sum_{j=0}^{r-1} 1\{X_n = j\}\right) = \sum_{n=0}^{T-1} 1 = T.$$

$$\sum_{j=0}^{r-1} s_{ij} = \sum_{j=0}^{r-1} E\left(\sum_{n=0}^{T-1} 1\{X_n = j\} \mid X_0 = i\right)$$
$$= E\left(\sum_{j=0}^{r-1}\sum_{n=0}^{T-1} 1\{X_n = j\} \mid X_0 = i\right) = E(T \mid X = i)$$
$$= \nu_i$$

Scanned by CamScanner

$\{ U = U_{ij}$ prob of absorbtion $\}$

$U = S \cdot R$

$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$

$u_{ij} = u_{i'} = P(\text{absorption in } j \mid X_0 = i)$

i = transient

j = absorbing states.

$S = (I - Q)^{-1}$

$$u_{ik} = \sum_{j=0}^{r-1} S_{ij} R_{jk} \quad ; \; 0 \le i \le r, \; r \le k \le N$$

we know,

$$S_{ij} = \delta_{ij} + \sum_{l=0}^{r-1} p_{il} \cdot S_{lj} \quad , \; 0 \le j, i < r$$

$$\sum_{j=0}^{r-1} S_{ij} R_{jk} = \sum_{j=0}^{r-1} \delta_{ij} R_{jk} + \sum_{j=0}^{r-1}\sum_{l=0}^{r-1} P_{il} \cdot S_{lj} R_{jk}$$

$$U_{ik} = R_{ik} + \sum_{l=0}^{r-1} P_{il} \underbrace{\sum_{j=0}^{r-1} s_{lj} \cdot R_{jk}}_{\to u_{lk}}$$ ← if chain holds

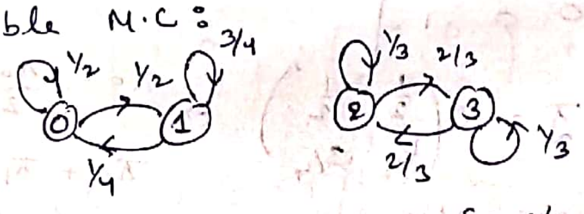
$$u_{ik} = P_{ik} + \sum_{j=0}^{r-1} P_{ij} u_{jk}$$

Reducible M.C:

①

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1/2 & 1/2 & 0 & 0 \\ 1 & 1/4 & 3/4 & 0 & 0 \\ 2 & 0 & 0 & 1/3 & 2/3 \\ 3 & 0 & 0 & 2/3 & 1/3 \end{array}$$

Reducible M.C:



$C_1 = \{0, 1\}$ $\quad C_2 = \{2, 3\}$

recurrent, aperiodic.

$$\lim_{n\to\infty} P^n = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \pi_0 & \pi_1 & 0 & 0 \\ 1 & \pi_0 & \pi_1 & 0 & 0 \\ 2 & 0 & 0 & \pi_2 & \pi_3 \\ 3 & 0 & 0 & \pi_2 & \pi_3 \end{array}$$

Scanned by CamScanner

C_1 | $\underset{\sim}{\pi} = (\pi_0, \pi_1)$ $\qquad \underset{\sim}{\pi} = \underset{\sim}{\pi}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 3/4 \end{pmatrix} \Rightarrow \pi_0 = \frac{\pi_0}{2} + \frac{\pi_1}{4}$

$\pi_0 + \pi_1 = 1$

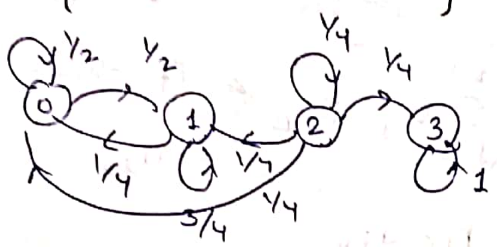
$\Rightarrow \pi_0 = \frac{1}{3}, \ \pi_1 = 2/3$

C_2 = doubly stochastic submatrix

$(\pi_2, \pi_3) = (1/2, 1/2)$

(2)

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1/2 & 1/2 & 0 & 0 \\ 1 & 1/4 & 3/4 & 0 & 0 \\ 2 & 1/4 & 1/4 & 1/4 & 1/4 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$



$C_1 = \{0, 1\}$; $C_2 = \{2\}$; $C_3 = \{3\}$

recurrent, absorbing — transient — absorbing.

$$\lim_{n\to\infty} P^n = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1/3 & 2/3 & 0 & 0 \\ 1 & 1/3 & 2/3 & 0 & 0 \\ 2 & 2/9 & 4/9 & 0 & 1/3 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$C_1 = [\ \pi = (\pi_0, \pi_1) \quad \underset{\sim}{\pi} = \pi\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1/4 & 3/4 \end{pmatrix} \to \pi_0 = \frac{\pi_0}{2} + \frac{\pi_1}{4}$, $\ \pi_0 + \pi_1 = 1$ $\quad \pi_0 = 1/3$, $\pi_1 = 2/3$

Let U_2 = prob. of ultimate absorb. prob in class 1 as opposed to C_3 start with 2.

$$U_2 = \frac{1}{4}\times 1 + \frac{1}{4}\times 1 + \frac{1}{4}\cdot U_2 + \frac{1}{4}\times 0$$

$$U_2 = 2/3.$$

3/2/20.

Ex:

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0.7 & 0 & 0.3 & 0 \\ 2 & 0 & 0.3 & 0 & 0.7 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$u_{20} = 0.3 \times 0.7 + 0.3 \times 0.3 \times$ ...

$1 - 0.09$

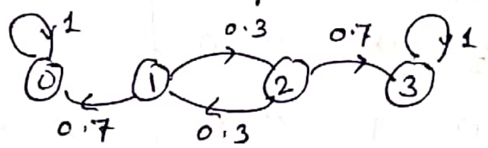
0.91

Determine,

(i) Prob. of absorbtion into state 0, starting from state 1.

(ii) the mean time spent in each of the states 1 and 2 prior to absorption.

(iii) mean time to absorption.



$C_1 = \{0\}$, $C_2 = \{1, 2\}$, $C_3 = \{3\}$

absorbing   transient   absorbing.

(i) $u_{10} = 0.7 \times 1 + 0.3 \times u_{20}$

$u_{20} = 0.3 \times u_{10}$

$\Rightarrow$ $u_{10} = \frac{70}{91} = 0.769$

$u_{20} = 0.3 \times 0.769$

(iii) $v_1 = 1 + 0 \times v_1 + 0.3 v_2$

$v_2 = 1 + 0.3 v_1 + 0 \times v_2$

$\Rightarrow$ $v_1 = 10/7$

$v_2 = 10/7$

(ii) $S = (I - Q)^{-1}$

$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$; $Q = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 0.3 \\ 2 & 0.3 & 0 \end{array}$; $R = \begin{array}{c|cc} & 0 & 3 \\ \hline 1 & 0.7 & 0 \\ 2 & 0 & 0.7 \end{array}$

$I - Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0.3 \\ 0.3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -0.3 \\ -0.3 & 1 \end{pmatrix}$

$S = (I - Q)^{-1} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1.03 & 0.33 \\ 2 & 0.33 & 1.03 \end{array} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$

$\rightarrow v_i = \sum_{j=0}^{r-1} s_{ij}$; $v_1 = s_{11} + s_{12} = 1.03 + 0.33 = 10/7$

$v_2 = s_{21} + s_{22} = 10/7$

$\therefore r =$ no. of transient states.

$\Rightarrow s_{ij} = \delta_{ij} + \sum_{l=0}^{r-1} P_{il} \cdot s_{lj}$; $U = SR$

prob. of absorption.

Scanned by CamScanner

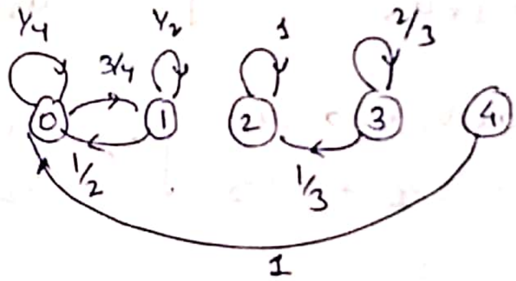
$$\rightarrow \quad U = SR = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 100/91 & 30/91 \\ 2 & 30/91 & 100/91 \end{array} \cdot \begin{array}{c|cc} & 0 & 3 \\ \hline 1 & 0.7 & 0 \\ 2 & 0 & 0.7 \end{array}$$

$$= \begin{array}{c|cc} & 0 & 3 \\ \hline 1 & 70/91 & 21/91 \\ 2 & 21/91 & 70/91 \end{array}$$

Ex: 4.5.1

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1/4 & 3/4 & 0 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1/3 & 2/3 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \end{array}$$

$\lim_{n\to\infty} P_{ij}^{(n)} = ?$



$C_1 = \{0,1\}$, $C_2 = \{2\}$, $C_3 = \{3\}$, $C_4 = \{4\}$.

recurrent, aperiodic — absorbing — transient — transient.

$$\lim_{n\to\infty} P_{ij}^{(n)} = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 2/5 & 3/5 & 0 & 0 & 0 \\ 1 & 2/5 & 3/5 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 4 & 2/5\times 1 & 3/5\times 1 & 0 & 0 & 0 \end{array}$$

$\underset{\sim}{\pi} = \underset{\sim}{\pi} P^* = (\pi_0, \pi_1)$

$\pi_0 = \frac{\pi_0}{4} + \frac{\pi_1}{2}$

$\pi_0 + \pi_1 = 1$

$\Rightarrow \pi_0 = 2/5, \quad \pi_1 = 3/5$

Scanned by CamScanner

1

$u_4 C_1 =$
$\downarrow$
class C_1.

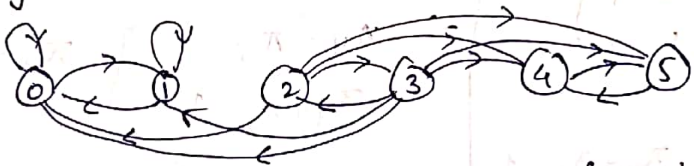
For state 2.

$$u_3 = \frac{1}{3}\times 1 + \frac{2}{3}\times u_3 \Rightarrow u_3 = 1$$

Ex:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 0 & 1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\lim_{n\to\infty} P_{ij}^{(n)} = ?$$



$C_1 = \{0, 1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4, 5\}$
$\downarrow$ recurrent, aperiodic
$\downarrow$ transeint
$\downarrow$ recurrent, periodic. $(d(5) = 2)$.

$\downarrow$

$$\lim_{n\to\infty} P_{ij}^{(n)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 2/5 & 3/5 & 0 & 0 & 0 & 0 \\ 2/5 & 3/5 & 0 & 0 & 0 & 0 \\ \frac{8}{17}\times\frac{2}{5} & \frac{8}{17}\times\frac{3}{5} & 0 & 0 & \times & \times \\ \frac{7}{17}\times\frac{2}{5} & \frac{7}{17}\times\frac{3}{5} & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix} \end{array}$$

$\times \rightarrow$ Does not exist

For $\overset{\text{class}}{\uparrow}{C_1}$, $\underset{\sim}{\Pi} = (\Pi_0, \Pi_1)$, solve, $\underset{\sim}{\pi} = \underset{\sim}{\pi}\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$
$\pi_0 + \pi_1 = 1$

$\Rightarrow \left.\begin{array}{l} \pi_0 = \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 \\ \pi_0 + \pi_1 = 1 \end{array}\right\} \begin{array}{l} \pi_0 = 2/5 \\ \pi_1 = \frac{3}{5} \end{array}$

For C_2, u_i = prob. of absorbtion with class C_1 starting from i, $i = 2, 3, \ldots$

$$u_2 = \frac{1}{3}\times 1 + 0\times 1 + 0\times u_2 + \frac{1}{3}\times u_3 + \frac{1}{6}\times 0 + \frac{1}{6}\times 0.$$

Scanned by CamScanner

$$u_3 = \frac{1}{6}\times 1 + \frac{1}{6}\times 1 + \frac{1}{6}\cdot u_2 + 0\cdot u_3 + \frac{1}{3}\times 0 + \frac{1}{6}\times 0 \quad \Big] \Rightarrow \quad u_2 = 8/17, \quad u_3 = 7/17$$

$\rightarrow$ $3u_2 = 1 + u_3$

$\rightarrow$ $6u_2 = 2 + u_2$

* <u>Time average</u>:

$$\lim_{n\to\infty} \frac{1}{n}\cdot\sum_{m=0}^{n-1} P_{ij}^{(m)} = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 2/5 & 3/5 & 0 & 0 & 0 & 0 \\ 1 & 2/5 & 3/5 & 0 & 0 & 0 & 0 \\ 2 & \frac{8}{17}\times\frac{2}{5} & \frac{8}{17}\times\frac{3}{5} & 0 & 0 & \frac{9}{17}\times\frac{1}{2} & \frac{9}{17}\times\frac{1}{2} \\ 3 & \frac{7}{17}\times\frac{2}{5} & \frac{7}{17}\times\frac{3}{5} & 0 & 0 & \frac{10}{17}\times\frac{1}{2} & \frac{10}{17}\times\frac{1}{2} \\ 4 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 5 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{array}$$

<u>For C_3</u>:

$$\tilde{\pi} = \tilde{\pi}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\tilde{\pi} = (\pi_4, \pi_5)$$

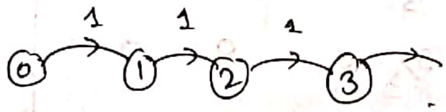
$$\pi_4 + \pi_5 = 1$$

$\Rightarrow$ $\pi_4 = \pi_5$, $\pi_4 + \pi_5 = 1$ $\Big]$ $\pi_4 = 1/2 = \pi_5$

$u_i' =$ prob. of absorption into class C_3, starting from i, $i = 2, 3, ..$

$$u_2' = 1 - u_2 = \frac{9}{17} \; ; \; u_3' = 1 - u_3 = \frac{10}{17}$$

<u>Example</u>:-



$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \; \cdots \\ \hline 0 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 1 & \\ 2 & 0 & 0 & 0 & - \\ 3 & & & & \\ \vdots & & & & \end{array}$$

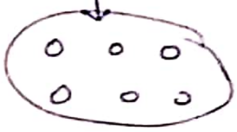
Here, all the states are transient.

Scanned by CamScanner

* ## Branching Process:

Each individual

$\downarrow$ j new offsprings with prob. P_j
s.t $p_0 > 0$



$X_0 \rightarrow$ size of zeroth generation

$X_1 \rightarrow$ size of first generation i.e., all offsprings of zeroth generation.

$\vdots$

$X_n \longrightarrow$ size of n^{th} generation.

state space, $E = \{ \underset{\text{zero member}}{0}, 1, \underset{\text{2 members}}{2}, \dots \}$

$$P_{00}^{(n)} = 1 = P(X_n = 0 \mid X_0 = 0)$$

$$\cancel{f_{ii}^{(n)} = P(X_n = i, \underset{j = 1,2,\dots,n-1}{X_j \neq i} \mid X_0 = i)}$$

$$\cancel{f_{ii} = f_{ii}^{(1)} + f_{ii}^{(2)} + \dots}$$

$\sum_{n=1}^{\infty} P_{00}^{(n)} = \infty$, state 0 is recurrent

$$P_{0i}^{(n)} = 0 \qquad i = 1, 2, \dots$$

$$P(X_n = 0 \mid X_0 = i) = P_{i0}^{(n)} \geq P_0^{i}$$

$$P(X_{n+1} = 0 \mid X_0 = i) = \sum P(X_{n+k} = 0, X_n = j \mid X_0 = i)$$

population will either die out or its size will convergence to ∞. mean no. of offsprings of single individual

$$\mu = \sum_{j=0}^{\infty} j \cdot P_j$$

variance of ——— $= \sigma^2 = \sum_j (j - \mu)^2 P_j$

let us suppose $X_0 = 1$ $\qquad (n-1)$ st X_{n-1}

$$X_n = Z_1 + Z_2 + \dots + Z_{X_{n-1}} \qquad (n) \rightarrow (X_n)$$

where, Z_i = # of offsprings of i^{th} individual of $(n-1)^{st}$ generation s.t,

$$E(Z_i) = \mu$$
$$V(Z_i) = \sigma^2.$$

Scanned by CamScanner