Lecture 4

Correction:- (a,b) is ~~perfect~~

(a,b) is not a perfect set. ✓

$A \subseteq \mathbb{R}$

Theorem:- (i) $m^*(A) \geq 0$.

(ii) $m^*(\phi) = 0$

(Monotone property) $\leftarrow$ (iii) if $A \subseteq B \subseteq \mathbb{R}$, then $m^*(A) \leq m^*(B)$.

(iv) $m^*(\{x\}) = 0$, for any $x \in \mathbb{R}$.

proof:-

$$m^*(A) = \inf\left\{ \sum_{n=1}^{\infty} l(I_n) \;\middle|\; \begin{array}{l} I_n = [a_n, b_n) \text{ intervals} \\ \& \ A \subseteq \bigcup_{n=1}^{\infty} I_n \end{array} \right\}$$

≥ 0.

(ii) $\phi = [a,a)$ $\quad m^*(\phi) = m^*([a,a)) = a - a = 0$.

(iii) Let $A \subseteq B \subseteq \mathbb{R}$.

To show: $m^*(A) \leq m^*(B)$.

Let $\{I_n\}$ be a collection of intervals such that

$B \subseteq \bigcup_{n=1}^{\infty} I_n$

Then $A \subseteq \bigcup_{n=1}^{\infty} I_n$

$$m^*(B) = \inf_{B \subseteq \cup I_n} \left(\sum_n l(I_n) \right)$$

FACT.

$S_1 \subseteq S_2 \subseteq \mathbb{R}$

$\inf(S_1) \geq \inf(S_2)$

$$\geq \inf_{A \subseteq \cup I_n} \left(\sum_n l(I_n) \right)$$

Let

$S_1 = \left\{ \sum_n l(I_n) \;/\; I_n\text{'s intervals}, \; B \subseteq \bigcup I_n \right\}$

$S_2 = \left\{ \sum_n l(I_n) \;/\; I_n\text{'s intervals}, \; A \subseteq \bigcup I_n \right\}$

$S_1 \subseteq S_2$.

$$= m^*(A).$$

Thus $m^*(A) \leq m^*(B)$.

(iv) $\{x\} \subseteq J_n = [x, x+\frac{1}{n}) \quad \forall n$.

$\therefore$ By (iii), we have $m^*(\{x\}) \leq m^*(J_n) = l(\;\;)$
$= x+\frac{1}{n} - x$
$= \frac{1}{n}$.

$\therefore m^*(\{x\}) \leq \frac{1}{n} \quad \forall n \geq 1$.

$\Rightarrow m^*(\{x\}) \leq 0$.

But $m^*(\{x\}) \geq 0$, therefore $m^*(\{x\}) = 0$.

<u>Proposition</u>:- The outer measure is translation invariant.

i.e., For $A \subseteq \mathbb{R}$, $m^*(A) = m^*(A+x)$ for any $x \in \mathbb{R}$.

$A + x := \{ a+x \,/\, a \in A \} \subseteq \mathbb{R}$.

<u>proof</u>:- Let $\varepsilon > 0$.

Then by using the inf. property, there

exists a collection of intervals $\{I_n\}$ such that $A \subseteq \bigcup_n I_n$ & $m^*(A) + \varepsilon \geq \sum_n l(I_n)$.

Now $A + x \subseteq \bigcup_n (I_n + x)$ — intervals

$$\Rightarrow m^*(A+x) = \inf \left\{ \sum_n l(J_n) \,\middle|\, \begin{array}{l} \{J_n\} \text{ intervals s.t} \\ (A+x) \subseteq \bigcup J_n \end{array} \right\}.$$

$$\leq \sum_{n=1}^{\infty} l(I_n + x)$$

But $\displaystyle\sum_{n=1}^{\infty} l(I_n + x) = \sum_{n=1}^{\infty} l(I_n)$ $\left(\because\ l(I_n + x) = l(I_n) \right)$

$$\therefore m^*(A+x) \leq \sum_{n=1}^{\infty} l(I_n)$$

$$\leq m^*(A) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we get

$$m^*(A+x) \leq m^*(A).$$

Also $A = \underline{(A+x)} - x$.

By above argument, $m^*(A) = m^*(\underbrace{(A+x)}_{A'} - x)$

$$\leq m^*((A+x)).$$

$$= A'$$

$$\therefore m^*(A+x) = m^*(A).$$

Theorem: The outer measure of any interval is equal to its length.

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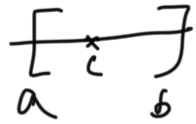

proof:

Case 1: Let $I = [a,b]$

To show: $m^*(I) = b-a$.

Let $\varepsilon > 0$.

We have $[a,b] \subseteq [a, b+\varepsilon)$

$$\Rightarrow m^*([a,b]) \le m^*([a,b+\varepsilon)) = b+\varepsilon-a = b-a+\varepsilon$$

True for any $\varepsilon > 0$.

$\therefore \; m^*([a,b]) \le b-a$.

$[a,b) \subseteq [a,b]$

Now $[a,b) \subseteq [a,b]$

$$\Rightarrow \underbrace{m^*([a,b))}_{= b-a} \le m^*([a,b]).$$

$\therefore \; m^*([a,b]) = b-a$.

Case 2: Suppose $I = (a,b]$ $\quad a > -\infty$.

If $a=b$, then $I = \emptyset$, then its length is zero.

Suppose $a < b$.

Let $\varepsilon > 0$ such that $\varepsilon < b-a$.

Let $I' = [a+\varepsilon, b]$, $I' \subseteq I$.

$$\Rightarrow m^*(I') \leq m^*(I)$$
$$m^*(I') = b-a-\varepsilon \quad (\text{by case}(1))$$

$\therefore \quad m^*(I) \geq l(I) - \varepsilon.$

for sufficiently small $\varepsilon > 0$, this implies that
$$m^*(I) \geq l(I).$$

Consider $I'' = [a, b+\varepsilon)$

Then $I \subseteq I''$.

$$\Rightarrow m^*(I) \leq m^*(I'') = b-a+\varepsilon = l(I)+\varepsilon$$

$\therefore \quad m^*(I) \leq l(I).$

Hence $m^*(I) = l(I)$.

Case 3: Suppose $I = (-\infty, a]$

For any $M > 0$, there exists k such that the finite interval $I_M = [k, k+M)$ is contained in I.

i.e., $\quad I_M \subseteq I$

$$m^*(I_M) \leq m^*(I).$$
$$m^*(I_M) = l(I_M) = k+M-k = M.$$

$\therefore \quad m^*(I) \geq M.$

But $M \gg 0$ (sufficiently bigger)
we get $m^*(I) \geq \infty$
i.e, $m^*(I) = \infty = l(I)$

EXERCISE: prove for (a, ∞), $[a, \infty)$.

Theorem:- Outer measure is countably subadditive.

i.e, For any sequence of subsets $\{E_i\}$ of $\mathbb{R}$ we have $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$

Remarks: $m^*(\mathbb{Q}) = ?$

Say $\mathbb{Q} = \{x_1, x_2, \ldots\}$

Let $E_i = \{x_i\}$ $\forall i \geq 1$

Then $m^*(\mathbb{Q}) = m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$

$$= \sum_{i=1}^{\infty} m^*(\{x_i\})$$

$$= \sum_{i=1}^{\infty} 0 = 0.$$

$\therefore m^*(\mathbb{Q}) = 0.$