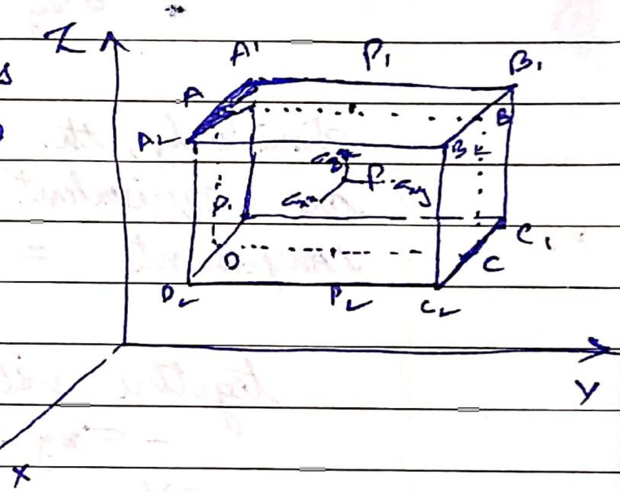
Required moment at B

$$= a\rho K U \cos\alpha + 2\pi\rho U^2 \left[2bc \sin(\alpha+\beta)\cos(\alpha+\eta) + a^2 \sin 2\alpha \right]$$

$$= 4a\pi b U^2 \rho \cos\alpha + 2\pi\rho U^2 \left[2bc \sin(\alpha+\beta) \cos(\alpha+\eta) + a^2 \sin 2\alpha \right]$$

Navier-Stokes Equation



With $P(x,y,z)$ as center and edges of length δx, δy, δz parallel to coordinates axes, we construct an elementary rectangular parallelopiped. Let us consider the fluid motion is viscous. We assume that the fluid element is moving & its mass is $\rho\,\delta x\,\delta y\,\delta z$. Let the coordinate of P_2 & P_1 be $(x+\frac{\delta x}{2}, y, z)$ & $(x-\frac{\delta x}{2}, y, z)$ respectively. At P the force components // to OX, OY, OZ on the rectangular surface ABCD of area $\delta y\,\delta z$ through P & having $\hat{i}$ as the unit normal are $= (\sigma_{xx}\,\delta y\,\delta z,\ \sigma_{xy}\,\delta y\,\delta z,\ \sigma_{xz}\,\delta y\,\delta z)$ — ①

At the point $P_2(x+\delta x/2, y, z)$ the components of the force are $A_2B_2C_2D_2$ is

$$= \left[\left(\sigma_{xx} + \frac{\delta x}{2}\frac{\partial \sigma_{xx}}{\partial x}\right)\delta y\,\delta z,\ \left(\sigma_{xy} + \frac{\delta x}{2}\frac{\partial \sigma_{xy}}{\partial x}\right)\delta y\,\delta z,\ \left(\sigma_{zx} + \frac{\delta x}{2}\frac{\partial \sigma_{zx}}{\partial x}\right)\delta y\,\delta z \right] \quad -②$$

Slly force on P_1 in $A_1B_1C_1D_1$,

$$= \left[-\left(\sigma_{xx} - \frac{\delta x}{2}\frac{\partial \sigma_{xx}}{\partial x}\right)\delta y\,\delta z,\ -\left(\sigma_{xy} - \frac{\delta x}{2}\frac{\partial \sigma_{xy}}{\partial x}\right)\delta y\,\delta z,\ -\left(\sigma_{xz} - \frac{\delta x}{2}\frac{\partial \sigma_{xz}}{\partial x}\right)\delta y\,\delta z \right] \quad -③$$

Hence the forces on // planes $A_2B_2C_2D_2$ & $A_1B_1C_1D_1$ passing through P_2 & P_1 are equivalent to a single force at P with components

$$= \left[\frac{\partial \sigma_{xx}}{\partial x} \delta x \delta y \delta z, \frac{\partial \sigma_{xy}}{\partial x} \delta x \delta y \delta z, \frac{\partial \sigma_{zx}}{\partial x} \delta x \delta y \delta z \right] \quad -(4)$$

together with couple whose moment are

$$\left. \begin{array}{l} -\sigma_{xz}\, \delta x \delta y \delta z \text{ about } OY \\ \sigma_{xy}\, \delta x \delta y \delta z \text{ about } OZ \end{array} \right\} \quad -(5)$$

Similarly, the forces parallel on ~~to the planes~~ planes ~~each other and~~ $\perp^r$ to z-axis are equivalent to a single force at P with components $= \left[\frac{\partial \sigma_{zx}}{\partial z}, \frac{\partial \sigma_{zy}}{\partial z}, \frac{\partial \sigma_{zz}}{\partial z} \right] \delta x \delta y \delta z$

together with couple of moment,
$-\sigma_{zy}\, \delta x \delta y \delta z$ about OX & $\sigma_{zx}\, \delta x \delta y \delta z$ about OY.

And forces on parallel planes $\perp^r$ to y-axis are equivalent to single force at P

$$= \left[\frac{\partial \sigma_{yx}}{\partial y}, \frac{\partial \sigma_{yy}}{\partial y}, \frac{\partial \sigma_{zy}}{\partial y} \right] \delta x \delta y \delta z$$

with couple of moment,
$-\sigma_{yx}\, \delta x \delta y \delta z$ about OZ & $\sigma_{yz}\, \delta x \delta y \delta z$ about OX

Thus the surface forces on all 6 faces of rectangular //piped are equivalent to a single force at P having compon

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}, \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}, \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right.$$

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with couple

$$= \left[(\sigma_{yz}-\sigma_{zy})\,\delta x\,\delta y\,\delta z,\ (\sigma_{zx}-\sigma_{xz})\,\delta x\,\delta y\,\delta z,\ (\sigma_{xy}-\sigma_{yx})\,\delta x\,\delta y\,\delta z \right]$$

Let $\vec{q} = U\hat{i} + V\hat{j} + W\hat{k}$ be the velocity at P &
$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ be the body force P. (per unit mass) The total force is given by

$$= \text{Body force} + \text{surface force}$$

$$= \vec{F} + \vec{S}$$

Total force along i-direction

$$= \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \right) \delta x\,\delta y\,\delta z$$

$$+ \rho F_x\,\delta x\,\delta y\,\delta z$$

By Newton's 2nd Law :-

$$\rho\,\delta x\,\delta y\,\delta z\,\frac{DU}{Dt} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho F_x \right) \delta x\,\delta y\,\delta z$$

$$\Rightarrow \rho\frac{DU}{Dt} = \rho F_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}$$

Slly, $$\rho\frac{DV}{Dt} = \rho F_y + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}$$

$$\rho\frac{DW}{Dt} = \rho F_z + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

From constitutive law of Newtonian fluid, by stoke's law of fluid motion

$$\sigma_{xx} = 2\mu \,\partial u/\partial x - 2\mu/3\, \vec{\nabla}\cdot\vec{q} - p$$

$$\sigma_{yy} = 2\mu \,\partial v/\partial y - 2\mu/3\, \vec{\nabla}\cdot\vec{q} - p$$

$$\sigma_{zz} = 2\mu \,\partial w/\partial z - 2\mu/3\, \vec{\nabla}\cdot\vec{q} - p$$

$$\sigma_{xy} = \sigma_{yx} = \mu\left(\partial u/\partial y + \partial v/\partial x\right)$$

$$\sigma_{yz} = \sigma_{zy} = \mu\left(\partial v/\partial z + \partial w/\partial y\right)$$

$$\sigma_{zx} = \sigma_{xz} = \mu\left(\partial w/\partial x + \partial u/\partial z\right)$$

$$\rho\frac{Du}{Dt} = \rho F_x + 2\mu\left(\partial^2 u/\partial x^2\right) + 2\mu/3\,\frac{\partial}{\partial x}\left(\vec{\nabla}\cdot\vec{q}\right) - \frac{\partial p}{\partial x}\Big\} + \mu\left(\partial^2 u/\partial y^2 + \partial^2 v/\partial y\partial x\right) + \mu\left(\partial^2 w/\partial z\partial x + \partial^2 u/\partial z^2\right)$$

$$\rho\frac{Dq_i}{Dt} = \rho F_{x_i} - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left[\mu\left\{2\,\partial q_i/\partial x_i - 4/3\,\vec{\nabla}\cdot\vec{q}\right\}\right]$$

$$+ \partial/\partial x_j\left[\mu\left(\cancel{\partial u/\partial y + \partial v/\partial x}\right)\right] + \frac{\partial}{\partial x_k}\left[\mu\left(\cancel{\partial u/\partial z + \partial w/\partial x}\right)\right]$$

$$\partial q_i/\partial x_j + \frac{\partial q_j}{\partial x_i} \qquad \partial q_k/\partial x_i + \partial q_i/\partial x_k$$

$$\cancel{\rho\frac{Dv}{Dt} = \rho F_y - \partial p/\partial y}$$

$$\rho\frac{D\vec{q}}{Dt} =$$

DISSIPATION OF ENERGY

It is that energy which is dissipated in a viscous fluid in motion on account of internal friction

To determine the rate of dissipation of energy of a fluid due to dissipation :-

Suppose we follow a particle of a viscous ~~incomp~~ incompressible fluid of density ρ & volume δV, such that its mass is $\rho \delta V$. It moves with velocity $\vec{q}$ at any time 't'. Then KE $T = \frac{1}{2}\rho \delta V \vec{q}^2$. Hence the rate of change of energy as the particle moves with time is given by,

$$\frac{D}{Dt}\left(\frac{1}{2}\rho \delta V \vec{q}^2\right) = \cancel{\frac{1}{2} \frac{D}{Dt}(\rho \delta V)\vec{q}^2}^{\,0} + \frac{1}{2}\rho \delta V \frac{D}{Dt}(\vec{q}^2)$$

$$= \rho \delta V \, \vec{q} \, \frac{D\vec{q}}{Dt} \quad -(1)$$

Let total volume of the fluid be V & S be the total surface area.

$$\frac{DT}{Dt} = \frac{d}{dt}\int_V \frac{1}{2}\rho \vec{q}^2 \, dV$$

$$= \rho \int_V \vec{q} \, \frac{d\vec{q}}{dt} dV \quad -(2)$$

From Navier-Stokes Equation for incompressible fluid.

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{\nabla p}{\rho} + \nu \vec{\nabla}^2 \vec{q} \quad -(3), \quad \nu = \frac{\mu}{\rho} \text{ Kinematics of viscosity}$$

From ②, ③ :-

$$\frac{dT}{dt} = \rho \int_V \vec{q}\left[\vec{F} - \frac{\nabla p}{\rho} + \nu \nabla^2 \vec{q}\right] dV$$

$$= \int_V \vec{q}(\rho \vec{F})\, dV - \cancel{\int_V \vec{q}\cdot\nabla p\, dV} + \int_V \vec{q}\rho\nu \nabla^2 \vec{q}\, dV - \int_S p\, \vec{q}\cdot\vec{n}\, dS \quad -④$$

$$dV = -\left[-\int_V \overset{0}{\nabla\cdot\vec{q}}\, p\, dV + \int_S p\,\vec{q}\cdot\vec{n}\, dS\right]$$

The ~~dissipation~~ first term on RHS of ④ represents rate at which the external force $\vec{F}$ is doing work throughout the mass of the fluid while the second term represents the rate at which pressure is doing work at the boundary. For ideal fluid, the work done by the force $\vec{F}$ in the volume V and work by the pressure at the boundary are same.

$$\frac{dT}{dt} = D = \rho \int_V \frac{\mu}{\rho}\vec{q}\cdot\nabla^2\vec{q}\, dV = \int_V \mu\, \vec{q}\cdot\nabla^2\vec{q}\, dV$$

where D is the dissipation of energy. Now if the flow is rotational s.t $\vec{\Omega}$ represents the vorticity. Then $\vec{\nabla}\times\vec{q} = \vec{\Omega}$

We know $\vec{\nabla}\times(\vec{\nabla}\times\vec{q}) = \nabla(\underbrace{\vec{\nabla}\cdot\vec{q}}_{0}) - \nabla^2\vec{q} = -\nabla^2\vec{q}$

$$\Rightarrow \vec{\nabla}\times\vec{\Omega} = -\nabla^2\vec{q}$$

$$\Rightarrow \vec{q}\cdot(\nabla^2\vec{q}) = -\vec{q}\cdot(\vec{\nabla}\times\vec{\Omega})$$

Again,

$$\vec{\nabla}\cdot(\vec{q}\times\vec{\Omega}) = \vec{\Omega}\cdot(\vec{\nabla}\times\vec{q}) - \vec{q}\cdot(\vec{\nabla}\times\vec{\Omega})$$

$$= \vec{\Omega}\cdot\vec{\Omega} - \vec{q}(\vec{\nabla}\times\vec{q})$$

$$\Rightarrow \vec{\nabla}(\vec{q}\times\vec{\Omega}) = |\vec{\Omega}|^2 - \vec{q}(\vec{\nabla}\times\vec{\Omega})$$

$$\Rightarrow -\vec{q}(\vec{\nabla}\times\vec{\Omega}) = \vec{\nabla}(\vec{q}\times\vec{\Omega}) - |\vec{\Omega}|^2$$

If

This given,

$$\vec{q}(\nabla^2\vec{q}) = \vec{\nabla}(\vec{q}\times\vec{\Omega}) - |\Omega|^2$$

This gives,

$$D = \int_V \mu[\vec{\nabla}\cdot(\vec{q}\times\vec{\Omega}) - |\vec{\Omega}|^2]\,dV$$

$$\cancel{= \int_V \vec{\nabla}(\vec{q}\cdot\vec{\Omega})}$$

$$= \mu\int_S (\vec{q}\times\vec{\Omega})\cdot\hat{n}\,dS - \mu\int_V |\vec{\Omega}|^2\,dV$$

(no flux at boundary)

If We assume no-slip condition, i.e. $\vec{q}\cdot\vec{n} = 0$, then first term will vanish,

$$D = -\int_V \mu|\vec{\Omega}|^2\,dV$$

$$\Omega = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \xi\hat{i} + \eta\hat{j} + \zeta\hat{k}$$

$$|\Omega|^2 = \xi^2 + \eta^2 + \zeta^2$$

$$|D| = \mu\int_V ((\xi^2 + \eta^2 + \zeta^2)\,dV$$