$f : \mathbb{R} \to \mathbb{R}$ Real valued function

$|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}$

$|x|$

General space $X = C[a,b]$.

$T : X \longrightarrow X$ Infinite dimensional vector space

$f : X \longrightarrow \mathbb{F}$

↳ functional

generalize this modulus function.

1.) Metric spaces
2.) Normed linear space

$f : V \longrightarrow \mathbb{R}$

$f(x) = \langle x, y \rangle \longrightarrow \exists\, y$ in V such that the inner product enforces a functional.

ℓ^∞ = space of bounded sequence $\{x_i\}_{i=1}^{\infty}$

$X \neq \{0\}$ ↓ every non zero vector space has a basis

$\ell^\infty = \{x \mid x = (x_i)_i^\infty, |x_i| \leq C_x\}$

$$d(x, y) = \sup_{i \in \mathbb{N}} |x_i - y_i|$$

space s : space of all sequences (Bounded or unbounded)

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i (1+|x_i - y_i|)}$$

$x = (x_1, x_2 \ldots \ldots x_n \ldots) = (x_i)$

$y = (y_1, y_2 \ldots \ldots y_n \ldots) = (y_i)$

1) $d(x,y) \geq 0$
2) $d(x,y) = 0 \iff x = y$
3) $d(x,y) = d(y,x)$
4) $d(x,y) + d(y,z) \geq d(x,z)$.

$$f(t) = \frac{t}{1+t}$$

$f'(t) > 0 \rightarrow f(t)$ is monotonically increasing

$|a+b| \leq |a| + |b|$

$\Rightarrow f(|a+b|) \leq f(|a|+|b|)$

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|} = \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|}$$

$$\leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

$$\Rightarrow \frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

$d(x,z) + d(z,y) \geq d(x,y)$

$x = (x_i) \quad y = (y_i) \quad z = (z_i)$

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$a = z_i - x_i \quad b = y_i - z_i$

$a + b = y_i - x_i$

$$\Rightarrow \frac{|y_i - x_i|}{1 + |y_i - x_i|} \leq \frac{|z_i - x_i|}{1 + |z_i - x_i|} + \frac{|y_i - z_i|}{1 + |y_i - z_i|}$$

Multiply by $\frac{1}{2^i}$

Take summation wrt $i = 1, \infty$

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|y_i - x_i|}{1 + |y_i - x_i|} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|z_i - x_i|}{1 + |z_i - x_i|} + \frac{1}{2^i} \frac{|y_i -}{1 + |y_i -}$$

$$\Rightarrow \quad d(x, y) \leq d(x, z) + d(y, z)$$

p is fixed Real number.

$$\ell^p := \{ x = (x_i) \mid \sum_{i=1}^{\infty} |x_i|^p \leq \infty \}$$

$x = (x_i) \quad y = (y_i)$

$$d(x, y) = \left(\sum_{i=1}^{\infty} |y_i - x_i|^p \right)^{1/p} \quad p > 1$$

when $p = 2$

Hilbert space

$$\ell^1 = \{ x : \sum_{i=1}^{\infty} |x_i| < \infty \}$$

$$d(x, y) = \left(\sum_{i=1}^{\infty} |y_i - x_i| \right)$$

In general ℓ^p is a Banach space.

$p > 1$, find $q \ni$

$$\frac{1}{p} + \frac{1}{q} = 1 \quad (p, q \text{ are conjugate})$$

$$\frac{p + q}{pq} = 1 \quad \Rightarrow (1 - p)(1 - q) = 1$$

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~~Holder's Inequality~~ Minkowski's Inequality

For $x = (x_i)$, $y = (y_i)$, $p > 1$

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}.$$

Holder's Inequality

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1/q}$$

$\boxed{(p, q \text{ are conjugate})}$

18th July

$(X, d) \longrightarrow$ Metric space

$d : X \times X \longrightarrow \mathbb{R}^+$

1) $d(x, y) \geq 0$
2) $d(x, y) = 0 \iff x = y$
3) $d(x, y) = d(y, x)$
4) $d(x, y) + d(y, z) \geq d(x, z)$ $\quad \forall\, x, y, z \in X$.

Ex 2 : Discrete Metric

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

$$d(x, y) \leq d(x, z) + d(z, y)$$

Case 1

Let $x = y$

$$0 \leq 2d(x, z)$$

(which is true)

Case 2

Let $x \neq y$

$$\Rightarrow 1 \leq d(x, z) + d(y, z)$$

Let $y = z \Rightarrow z \neq x$

$$\Rightarrow 1 \leq d(\cancel{x, z}) + 0 \quad \text{(which is true)}$$

Let $y \neq z$ and $x \neq z$

$$1 \leq 2 \quad \text{(which is true)}$$

$\blacksquare$ (X, d) is a discrete metric space.

$x = (x_i),\ y = (y_i) \in \ell^p,\ p > 1$

$$\sum_{i=1}^{\infty} |x_i y_i| \le \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1/q}$$

or $\sum_{i=1}^{\infty} |x_i y_i| \le \|x\|_p \|y\|_q$

where $1/p + 1/q = 1$

$$\boxed{\begin{array}{l} x \in \ell^p \\ y \in \ell^q \end{array} \Rightarrow xy \in \ell^1}$$

$$f \in L^p[a,b] = \left\{ f : \int_a^b |f(x)|^p dx < \infty \right\}$$

$$g \in L^q[a,b] = \left\{ g : \int_a^b |g(x)|^q dx < \infty \right\}$$

Eg. $x = (1/n)$

$= (1, 1/2, 1/3 \ldots)$

$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$(1/n) \notin \ell^1$

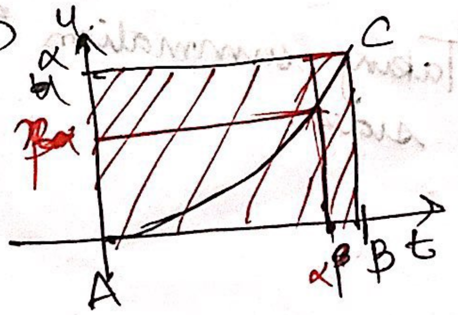
For $p > 1$

$x = \left(\frac{1}{n}\right) \in \ell^p$

because $\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$ (Proof Ex)

$$\int_a^b |fg|\,dx \le \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |f(x)|^q dx\right)^{1/q}$$

$u = t^{p-1}$



$\alpha\beta$: Area of Rectangle ABCD

$$\alpha\beta \leq \int_0^{\alpha} t^{p-1}\,dt + \int_0^{\beta} u^{q-1}\,du$$

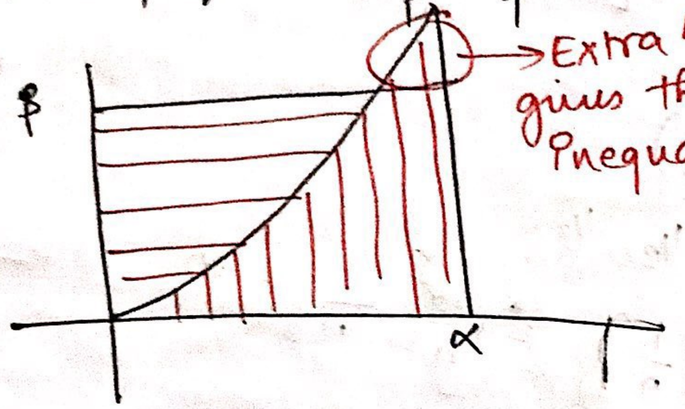
(Because $u^{q-1} = t^{(p-1)(q-1)} = t$

$$\boxed{\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}}$$

Young's Inequality.

For any 2 +ve numbers α, β.

for fixed $p > 1$ $\exists$ q. s.t. $\frac{1}{p} + \frac{1}{q} = 1$



Define.

$$\hat{x} = (\hat{x}_i) \quad , \quad \hat{y} = (\hat{y}_i)$$

$$\sum_{i=1}^{\infty} |\hat{x}_i|^p = 1 \qquad \sum_{i=1}^{\infty} |\hat{y}_i|^q = 1$$

$$|\hat{x}_i \hat{y}_i| \leq \frac{|\hat{x}_i|^p}{p} + \frac{|\hat{y}_i|^q}{q}.$$

Taking summation $i = 1$ to ∞ both sides.

$$\therefore \sum_{i=1}^{\infty} |\hat{x_i}\hat{y_i}| \le \frac{1}{p}\sum_{i=1}^{\infty} |\hat{x_i}|^p + \frac{1}{q}\sum_{i=1}^{\infty}|\hat{y_i}|^q$$

$$\sum_{q=1}^{\infty} |\hat{x_i}\hat{y_i}| \le \frac{1}{p} + \frac{1}{q} = 1$$

Take any non zero $x = (x_i) \in l^p$

and $y = (y_i) \in l^q$.

$$\hat{x_i} = \frac{x_i}{\left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}}, \quad \hat{y_i} = \frac{y_i}{\left(\sum_{j=1}^{\infty} |y_j|^q\right)^{1/q}}.$$

$$\sum_{i=1}^{\infty} \left| \frac{y_i}{\left(\sum_{j=1}^{\infty} |y_j|^q\right)^{1/q}} \cdot \frac{x_i}{\left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}} \right| \le 1$$

$$\sum_{i=1}^{\infty} (x_i y_i) \le \left(\sum_{j=1}^{\infty} |y_j|^q\right)^{1/q} \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}.$$

19th July

$$\|xy\|_1 \le \|x\|_p \|y\|_q$$

$$p^{-1} + q^{-1} = 1$$

Set $p = 2 \Rightarrow q = 2$

$$\|xy\|_1 \le \|x\|_2 \|y\|_2$$

$$\sum_{i=1}^{\infty} |x_i y_i| \le \sqrt{\sum_{j=1} x_j^2}\sqrt{\sum_{j=1} y_j^2}$$

(Cauchy schwartz Inequality)

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}$$

$p \geq 1$

$x = (x_i) \quad x \in \ell^p$

$y = (y_i) \quad y \in \ell^p$

Proof: Case 1, $p = 1$

$$\sum_{i=1}^{\infty} |x_i + y_i| \leq \sum_{i=1}^{\infty} |x_i| + \sum_{i=1}^{\infty} |y_i|$$

(From Δ inequality).

Case 2, $p > 1$

$z_k = x_k + y_k \quad k = 1, \dots \infty.$

$$|z_i|^p = |z_i| \, |z_i|^{p-1}$$

$$= |x_i + y_i| \, |z_i|^{p-1}$$

$\hookrightarrow$ (Δ inequality)

$$\leq (|x_i| + |y_i|) \, |z_i|^{p-1}$$

$$\leq (|x_i| |z_i|^{p-1} + |y_i| |z_i|^{p-1})$$

$$\sum_{i=1}^{n} |z_i|^p \leq \sum_{i=1}^{n} |x_i| |z_i|^{p-1} + \sum_{i=1}^{n} |y_i| |z_i|^{p-1}$$

(We don't know if it is convergent, that's why finite sum)

Use Holder's Inequality on $\sum_{i=1}^{n} |x_i| |z_i|^{p-1}$

$$\sum_{i=1}^{n} |x_i| |z_i|^{p-1} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |z_i|^{(p-1)q}\right)^{1/q}$$

$$(p-1) q = p$$

$$\Rightarrow \sum_{i=1}^{n} |x_i| |z_i|^{p-1} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/q} \quad \text{— (A)}$$

Similarly

$$\sum_{i=1}^{n} |\cancel{x_i}\, y_i|\,|z_i|^{p-1} \le \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/q}$$

Hence

$$\sum_{i=1}^{n} |z_i|^p \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/q} + \left(\sum_{i=1}^{n} |z_j|^p\right)^{1/q} \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$$

$$\Rightarrow \sum_{i=1}^{n} |z_i|^p \le \left(\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}\right) \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/q}$$

Divided by $\left(\sum_{i=1}^{n} |z_i|^p\right)^{1/q}$

$$\Rightarrow \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

Taking $n \longrightarrow \infty$

$$\left(\sum_{i=1}^{\infty} |z_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}.$$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right) < \infty$$

Hence the following proof.

l^p is a vector space ? (HOW THO)

① $x \in l^p,\ y \in l^p$

$\Rightarrow (x+y) \in l^p$ by Minkowski's inequality.

Now show that

$\boxed{l^p \text{ is a metric space}}$

$$d(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p}$$

$x = (x_i)$

$y = (y_i)$

① ② ③ $\longrightarrow$ Trivial

$$d(x,y) \leq d(x,z) + d(z,y) \qquad z = (z_i)$$

$$d(x,y) = \left(\sum_{i=1}^{\infty} |(x_i - y_i)|^p\right)^{1/p}$$

$$= \left(\sum_{i=1}^{\infty} |(x_i - z_i) + (z_i - y_i)|^p\right)^{1/p}$$

(By minkowski's ineq.)

$$\Rightarrow d(x,y) \leq \left(\sum_{i=1}^{\infty} |x_i - z_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |z_i - y_i|^p\right)^{1/p}$$

$$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$$

Ex

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{|x_i - y_i|}{1 + |x_i - y_i|}\right)$$

We can replace $\frac{1}{2^i}$ by μ_i

S.T.

$$\sum_{i=1}^{\infty} \mu_i < \infty.$$

$$d(x,y) = \sum_{i=1}^{\infty} \mu_i \left(\frac{|x_i - y_i|}{1 + |x_i - y_i|}\right)$$

Kreyszig Pg 16 Pb 1

(2) For $\alpha, \beta > 0$

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^p}{p}.$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

Set $\alpha^* = \sqrt{\alpha^*}$
$\beta^* = \sqrt{\beta^*}$. $p = 2$

$$\Rightarrow \sqrt{\alpha^*\beta^*} \le \frac{\beta^* + \alpha^*}{2}$$

(3) Show that Cauchy Schwartz inequality

$$\sum_{i=1}^{\infty} |x_i y_i| \le \sqrt{\sum_{i=1}^{\infty} |x_i|^2} \sqrt{\sum_{i=1}^{\infty} |y_i|^2}$$

implies that

$$\boxed{\left(|x_1| + |x_2| + \dots + |x_n|\right)^2 \le n\left(|x_1|^2 + \dots + |x_n|^2\right)}$$

~~Replace~~
Take finite sum till n.

$$\sum_{i=1}^{n} |x_i y_i| \le \sqrt{\sum_{i=1}^{n} |x_i|^2} \sqrt{\sum_{i=1}^{n} (y_i)^2}$$

Replace $y_i = \frac{1}{n}$

$$\sum_{i=1}^{n} \frac{|x_i|}{n} \le \sqrt{\sum_{i=1}^{n} |x_i|^2} \times \sqrt{1/n}$$

$\Rightarrow$ squaring both sides

$$\boxed{\Rightarrow \left(\sum_{i=1}^{n} |x_i|\right)^2 \le n\left(\sum_{i=1}^{n} |x_i|^2\right)}$$

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④ (l^p space) find a sequence which converges to 0, but is not in any space l^p, where $1 \leq p < +\infty$.

Construct a sequence

$$(x_n) = \frac{1}{n^\alpha}$$

Chose α S.T ~~such that~~ $\alpha p \leq 1$

Q5 Find a sequence x which is in l^p for $p > 1$ but $x \notin l^1$

$$x_n = \left(\frac{1}{n}\right), \quad \frac{1}{n} \notin l^1$$

$$\text{As } \sum_{n=1}^{\infty} \frac{1}{n} \not< \infty$$

$$\text{But } \left(\frac{1}{n^p}\right) \in l^p \text{ As } \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$$

⑥ The diameter $\delta(A)$ of $A \neq \phi$ in a metric space X is defined by

$$\delta(A) = \sup_{y, x \in A} (d(x, y))$$

A is bounded if $\delta(A) < \infty$

$$A \subset B \implies \delta(A) \leq \delta(B)$$

diameter is a monotone concept

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(7) Show that $\delta(A) = 0$ iff A consists of a single pt.

(8) <u>Distance Between 2 sets</u>

Let $D(A, B)$ denote the distance b/w two nonempty sets A and B.

$$D(A, B) = \inf_{\substack{a \in A \\ b \in B}} (d(a, b))$$

Ex (Not a metric)

Take $A = \{1, 2, 3\}$
$B = \{2, 4, 6\}$

Similarly dist- b/w pt and a set
$D(a, A)$.

24th July:

$x \longmapsto d(x, A)$ is a continuous map

$$|d(x, A) - d(y, A)| < d(x, y)$$

Proof: $d(x, z) \leq d(x, y) + d(y, z)$.

$$d(x, A) = \inf_{z \in A} (d(x, z)) \leq \inf_{z \in A} \{d(x, y) + d(y, z)\}$$

$$= d(x, y) + \inf_{z \in A} d(y, A).$$

$$d(x, A) \leq d(x, y) + d(y, A)$$

i.e. $\Rightarrow d(x, A) - d(y, A) \leq d(x, y)$

Similarly, $d(y, A) - d(x, A) \leq d(x, y)$

$$f(x, y) - f(x_1, y) \leq \alpha |x - x_1|$$

$$0 < \alpha < L$$

L - Lipschitz continuous

(X_1, d_1) (X_2, d_2)

$$X_1 \times X_2 = \left\{ (x_1, x_2) \,\middle|\, \begin{matrix} x_1 \in X_1 \\ x_2 \in X_2 \end{matrix} \right\}$$

How to construct a metric on $X_1 \times X_2$?

Let $x \in X_1 \times X_2$

$y \in X_1 \times X_2$

$x = (x_1, x_2)$

$y = (y_1, y_2)$

Let d be defined on $X_1 \times X_2$.

(13) $$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

Trivially, it is a metric on $X_1 \times X_2$. (Exercise)

(14) $$\bar{d}(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)}$$

is also a metric on $(X_1 \times X_2)$.

(15) $$\tilde{d}(x, y) = \max\{d(x_1, y_1), d(x_2, y_2)\}$$

<u>Open Ball</u>:

$B(x_0, \cancel{x}\, r) = \{x \in X \mid d(x, x_0) < r\}$

$r > 0$

$x_0 \in X$.

$B(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\}$

$S(x_0, r) = \{x \in X \mid d(x, x_0) = r\}$

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Limit Point:

$\epsilon > 0$, $B(x_0, \epsilon)$ is also called the nbhd of x_0.

$= \{x \in X, d(x, x_0) < \epsilon\}$.

$A \subset X$.

Let $x \in X$, x is called a limit pt. if every nbhd of A; (x is not necessarily an element of $\hat{A}$) intersects A in at least one point other than x.

Eg None of the points of $\mathbb{Z}$ is a limit point.

$\overline{A} = A \cup A'$

- In euclidean space
  $\overline{B}(x_0, r) = \overline{B(x_0, r)}$.

$r \neq 0$

(X, d) d is discrete metric

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

$B(x_0, r) = \{$

in discrete metric d.

$\{x_0\}$ is open in discrete metric d

Also $\{x_0\}$ is closed in discrete metric d

$\mathcal{T}$ be a collection of subsets of X

$(X, \mathcal{T})$ is called a Topology on X if

① $X, \phi \in \mathcal{T}$

② $G_1, G_2, \ldots G_i \quad i \in I \Rightarrow \bigcup_{i \in I} G_i \in \mathcal{T}$

③ $F_1, F_2, \ldots F_n \in \mathcal{T} \Rightarrow \bigcap_{j=1}^{n} F_j \in \mathcal{T}$

(discrete Topology $\longleftrightarrow$ Discrete Metric ?)

25th July

$r > 0$

$S(x_0, r) = \{y \mid d(y, x_0) = r\}$

Take $r \neq 1$

$S(x_0, r) = \phi$

$B(x_0, r) = \{y \in X \mid d(y, x_0) < r\}$.

$\overline{B}(x_0, r) = \{y \in X \mid d(y, x_0) \leq r\}$

$S(x_0, r) = \{y \in X \mid d(y, x_0) = r\}$

$\overline{B} - B = S$ (Euclidean space)

$\overline{B(x_0, r)} = \overline{B}(x_0, r)$

True in euclidean metric but not in general true for arbitrary metric.

X is a ~~X be~~ (X, d) be a metric space which contains more than one element (where metric is discrete) Let $x_0 \in X$ $\{x_0, x_1\}$

$$B(x_0, 1/2) = \{x \in X \mid d(x, x_0) < 1/2\}$$
$$= \{x_0\}$$

$$B(x_0, 1) = \{x_0\}$$
$$\bar{B}(x_0, 1) = \{x \in X \mid d(x, x_0) \leq 1\}$$
$$= X$$
$$\overline{B(x_0, 1)} = \{x_0\}$$ (Because $\{x_0\}$ is closed and open)

$A \subset X$ open $\forall\, x \in A$ $\epsilon > 0$
s.t. $B(x, \epsilon) \subset A$

$X = \{a, b, c\}$

$\mathcal{J} = \{\phi, \{a\}, \{a, b\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

$\mathcal{J}_1 = \{\phi, X\}$ $\quad \mathcal{J}_2 = \{\phi, \{a\}, \{a, b, c\}$

$\mathcal{J}_3 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}\}$.

$\mathcal{J}_1 \subset \mathcal{J}$.

$T : X \longrightarrow Y$

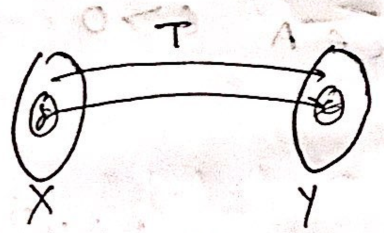
(X, d) $(Y, \tilde{d})$ $T(x) = Tx$

Let $x_0 \in X$

T is said to be continuous at $x = x_0$ if $\epsilon > 0$ if $\exists\, \delta(\epsilon)$ $\tilde{d}(Tx, Tx_0) < \epsilon$ whenever $d(x, x_0) < \delta$.

$\Rightarrow$ $\forall$ $Tx_0 \in N$, $\exists$ an open $N_0 \ni x_0$ s.t. $T(N_0) \subseteq N$.

26th July



It maps at δ-nbhd of x_0 to an ϵ nbhd of Tx_0

For each ϵ nbhd C of Tx_0, $\exists$ a δ nbhd C_0 s.t. $T(C_0) \subseteq C$

$B \subset X$, Image of B.

$T(B) = \{Tx : x \in B\}$

Thm: A mapping $T: X \longrightarrow Y$ (X, Y are metric spaces) is continuous if and only if the inverse image of any open subset of Y is an open subset of X

i.e. $\forall$ C open in Y, $T^{-1}(C)$ is open in X

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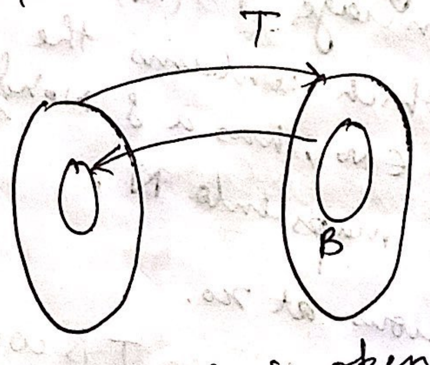
Proof: suppose T is continuous. Let C be open in Y and $C_0 = T^{-1}(C)$

If $C_0 = \phi$, as ϕ is open the result follows

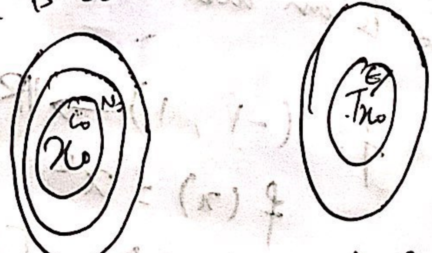
i.e. it takes open set into open set.

Suppose $C_0 \neq \phi$. Let $x_0 \in C_0$ be an arbit elt.



$Tx_0 \in C$. But C is open. Let N be the ϵ nbhd of Tx_0 which is contained in C.

$Tx_0 \in N \subset C$



As T is continuous at x_0, there is a δ-nbhd N_0 of x_0 which maps to N.

$d(Tx, Tx_0) < \epsilon$

~~As $N \subset C$~~ $T^{-1}(N) \subset T^{-1}(C) = C_0$

Since $N \subset C$, $T^{-1}(N) \subset T^{-1}(C) = C_0$

$\Rightarrow N_0 \subset T^{-1}(N) \subset C_0$

$\Rightarrow x_0 \in N_0 \subset C_0$.

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Converse

Suppose that $\forall$ C is open in Y $T^{-1}(C)$ is open in X, To show that T is continuous.

For $x_0 \in X$, any ϵ-nbhd N of Tx_0, the inverse image $T^{-1}(N) = N_0$ is open in X which contains the $C N_0$ pt x_0 i.e. $x_0 \in N_0$, has a δ-Nbhd, which maps into N. Hence N_0 maps into N $\therefore$

So T is continuous at x_0.

As x_0 is an arbitrary pt, so T is continuous on X.

$$f: (-1, 1) \rightarrow \mathbb{R}$$

$$f(x) = x^2$$

$$f(-1, 1) = \underset{\text{Not open}}{[0, 1)}$$

Open map: A map $T: X \longrightarrow Y$ is said to be open if $\forall$ open set A in X $T(A)$ is open in Y.

Every open map is continuous

Every open map is continuous

Ex1: Show that $T: X \longrightarrow Y$ is continuous iff $\forall$ closed set F of Y, $T^{-1}(F)$ is closed in X.

$f^{-1}(B) = \{x \in X \mid Tx \in B\}$

Proof: As F is closed $F^c = Y - F$ is open.

$T^{-1}(Y - F)$ is open in X.

$T^{-1}(Y - F)$ is open in X

$= T^{-1}(Y - F)$

$= T^{-1}(Y) - T^{-1}(F)$ $= X - T^{-1}(F)$ is open

$= T^{-1}(F)$ is closed.

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$T : X \longrightarrow Y$ One-to-one and onto
is bijective

If T and T^{-1} are both continuous then T is said to be a homeomorphism.

$T^{-1} : Y \longrightarrow X \supset A$

$(T^{-1})^{-1}(A)$ is open in Y.

$T(A)$ is open in Y.

i.e. T is an open map.

$\Rightarrow$ Homeomorphism = Continuous inverse + open map.

Let $A \subset X$, A is said to be dense in X if $\bar{A} = X$.

$\mathbb{Q}$ = set of rational numbers

$\bar{\mathbb{Q}} = \mathbb{R}$, $r \in \mathbb{Q}$ $x \in \mathbb{R}$

$\forall$ $N(x, r) \cap \mathbb{Q} \neq \phi$

Countable dense set

Separable metric space: If (X, d) is a metric space then it is said to be separable if it has a countable dense set.

EX $\mathbb{R}, \mathbb{R}^2$ are separable.

$\bar{\mathbb{Q}} = \mathbb{R}$. $\mathbb{Q}$ is countable. $\Rightarrow \mathbb{R}$ is separable

l^{∞} space is non separable

$x = (x_1, x_2, \ldots)$ $y = (y_1, y_2, \ldots)$

$|x_i| < M$

$d(x, y) = \sup |x_i - y_i|$

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$x_i = 0$ or 1 $\quad (1,0,0,1,0,1,0 \ldots)$

$y_i = 0$ or 1 $\quad (1,1,1,1,0,1,0 \ldots)$

$d(x,y) = 1$

M space of sequence which contains only 0, 1.

$M \subset l^\infty$.

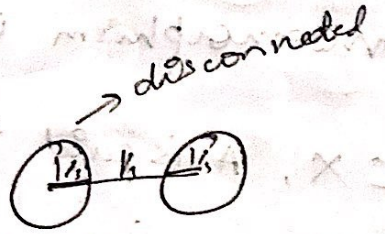
M is uncountable.

$x \neq y$

$d(x,y) = 1$

$B(x; 1/3)$

$B(y; 1/3)$.



l^∞ is not separable

Proof: Let M be the set of sequences of 0 and 1 i.e. $x = (1,0,0,1,0,\ldots)$

$y = (0,1,0,1,1,0,0\ldots)$.

$x, y \in M$. We know M is uncountable (cantor argument)

$M \subset l^\infty$, $d(x,y) = \sup |x_i - y_i| = 1$

Now $B(x, 1/3)$ and $B(y, 1/3)$ do not intersect and we have uncountable many of them.

If N is any dense set in l^∞, then each of these non intersecting balls must ~~be~~ contain an element of N. So N is not countable. Since N is arbitrary dense set, this ~~be~~ shows that l^∞ can't be separable.

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$x, y \in l^p$ $\quad x = (x_1, x_2 \ldots)$
$\quad y = (y_1, y_2, \ldots)$

$$\Rightarrow \sum_{i=1}^{\infty} |x_i|^p < \infty$$

$$\sum_{j=1}^{n} |x_j|^p + \sum_{j=n+1}^{\infty} |x_j|^p < \infty$$

Proof: Let M be the set of all sequences y of the form

$y = (y_1, \ldots y_n, \ldots)$ $\quad n$ is +ve ~~int~~ integer $\quad y_i \in Q$

Hence M is countable.

l^p space $1 \le p < \infty$ is separable

$M = \{(\boxed{x_1, x_2, \ldots x_n}, 0, 0, 0.) \mid x_i \in Q\}$.

$M \simeq Q_1 \times Q_2 \times \ldots Q_n$

M is countable

M is dense in l^p.

$$\sum_{n=0}^{\infty} a_n < \infty \Rightarrow \lim_{n \to \infty} a_n = 0$$

i.e. To show that $\overline{M} = l^p$

let $x = (x_1, x_2, \ldots) \in l^p$

$$\sum_{i=1}^{\infty} |x_i|^p < \infty.$$

$$\sum_{i=1}^{n} |x_i|^p + \sum_{i=n+1}^{\infty} |x_i|^p < \infty$$

$$\Rightarrow \lim_{i \to \infty} |x_i|^p = 0$$

Let x_n be an arbit elt.

since $n > 0$ $\quad \exists\, n_0(\epsilon)$ s.t.

$$\sum_{j=n+1}^{\infty} |x_i|^p < \frac{\epsilon^p}{2} \quad \text{— (A)}$$

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$\overline{\mathbb{Q}} = \mathbb{R}$ $\quad \forall \epsilon > 0$.

$B(x^*, \epsilon) \cap \mathbb{Q} \neq \phi$

(because its remainder ⊛ is a 'convergent series)

Now $\forall x_j \in \mathbb{R}$, $\exists y_j \in \mathbb{Q}$ close to it

or For all $\epsilon > 0$ for given $x \in \mathbb{R}$ $\exists y \in \mathbb{Q}$ s.t. $|x - y| < \epsilon$.

Then $\exists y = (y_1, y_2, y_3 \ldots y_n, 0, 0 \ldots) \in M$

satisfying

$$\sum_{i=1}^{n_0} |x_i - y_i|^p < \frac{\epsilon^p}{2} \quad — \text{B}$$

$$d(x,y) = \left[\sum_{i=1}^{\infty} |x_i - y_i|^p\right]^{1/p}.$$

$$d(x,y)^p = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right) = \sum_{i=1}^{n_0} |x_i - y_i|^p + \sum_{i=n_0+1}^{\infty} |x_i - y_i|^p$$

$$= \sum_{i=1}^{n_0} |x_i - y_i|^p + \sum_{n_0+1}^{\infty} |x_i|^p$$

$$\leq \frac{\epsilon^p}{2} + \frac{\epsilon^p}{2} = \epsilon^p$$

$\Rightarrow \boxed{d(x,y) \leq \epsilon}$ and $x \in l^p$

We proved that $\forall \epsilon > 0$, $\exists y \in M \cong \mathbb{Q}^n \subseteq l^p$ such that $d(x,y) \leq \epsilon$ which is a countable dense set

Hence M is dense in l^p.

1.

$f : \mathbb{N} \longrightarrow X$

(X, d)

$f(1) = x_1$, $f(2) = x_2$ ... $f(n) = x_n$

$\in X.$

$\boxed{x_i = f(i)}$

$x_1, x_2, \ldots x_n$

(x_n) sequence in X.

(x_{n_k}) is a subsequence of (x_n).

$n_1 < n_2 < n_3 < \ldots$

$(x_2, x_6, x_9, x_{12}, \ldots)$

$(x_4, x_8, x_{12}, x_{16} \ldots)$

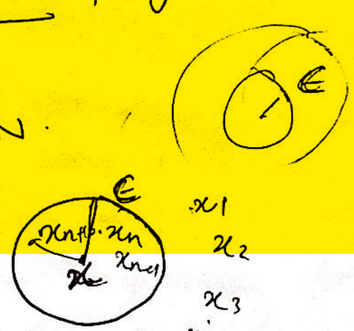
$d_n = d(x_n, x)$
$n = 1, 2, \ldots$

(d_n) is a sequence of Real numbers.

$(d_1, d_2, \ldots d_n)$ is a sequence in $\mathbb{R}$. $|d_n| < \epsilon$

$\epsilon > 0$ $\exists$ N s.t. $|d_n - d_0| < \epsilon$, $n \geq N$

A sequence (x_n) of X converges to $x \in X$ if for given $\epsilon > 0$
$\exists$ N s.t.
$d(x_n, x) < \epsilon$ whenever $n \geq N$.



x_1
x_2
x_3
x_{n-1}

or $\boxed{\lim_{n \to \infty} x_n = x}$

$\boxed{\lim x_n \longrightarrow x}$

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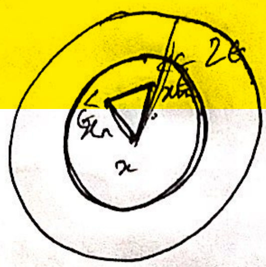
Cauchy sequence / Fundamental sequence

A sequence (x_n) in X is said to be a cauchy sequence in X if $\forall \epsilon > 0 \; \exists N$ s.t.

$$d(x_n, x_m) < \epsilon \quad x_n, m > N$$



$d(x_n, x) < \epsilon/2$

$d(x_m, x) < \epsilon/2$

$\boxed{d(x_n, x_m) < 2\epsilon}$

Show that
Every convergent sequence is a cauchy sequence

A metric space is said to be a complete metric space is every cauchy sequence converges.

$\mathbb{Q}$ is not ~~concrete~~ complete but $\mathbb{R}$ is complete

$$d(f, g) = \sup_{x \in [a, b]} [f(x) - g(x)]$$

$$d(f, g) = \int_a^b |f(x) - g(x)| \, dx$$

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1^{st} August

Thm: Let $\{x_n\}$ be a convergent sequence in a metric space (X, d). Then $(x_n) < M$ for some $M < \infty$ and the limit of $\{x_n\}$ is unique

Proof:

Given $x_n \longrightarrow l$ where l is the limit of the sequence.

i.e. x_n converges to l

Given $x_n > 0$ $\exists$ N s.t $d(x_n, l) < \epsilon$ $\forall n \geq N$.

self Note: $\boxed{\text{To find some } M > 0 \text{ such that } d(x_N, l) < M \ \forall n \in \mathbb{N}.}$

Take $\epsilon = 1$

$d(x_n, l) < 1$ for $n \geq N$.

$\alpha = \max\{d(l, x_1), d(l, x_2), \dots d(l, x_n)\}$

(Because $n < \infty$ the max exists)

$\Rightarrow d(x_n, l) < \alpha + 1$ $\forall n \in \mathbb{N}$.

$S = \{x_1, x_2, \dots x_n \dots\}$

diam $S < \infty$

$\Downarrow$

bounded.

Thm Limit of a sequence is unique

~~On the~~ Proof On the contrary, let's assume there exists two limits namely $z \neq x$. & x

$\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$d(x_n, x) < \epsilon/2$

$d(x_n, z) < \epsilon/2$

Then $x_n \longrightarrow x$ $z \neq x$

$x_n \longrightarrow z$

$d(z, x) \leq d(x_n, z) + d(x, x_n)$

$\Rightarrow d(z, x) \leq \epsilon/2 + \epsilon/2 = \epsilon$

$\Rightarrow d(z, x) \leq \epsilon$ $\forall \epsilon > 0$ and $n \geq N(\epsilon)$

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$\Rightarrow d(z, x) = 0$

$\Rightarrow z = x$

But we assumed the contrary that $x \neq z$

But we assumed the contrary has been contradicted.

Thus the assumption has been contradicted.

$\Rightarrow$ The limit is unique

If $x_n \longrightarrow x$ and $y_n \longrightarrow y$ then $d(x_n, y_n) \longrightarrow d(x, y)$

Proof : $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y_n, y)$

$\Rightarrow d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y)$ ⊛

Also $d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$

self note: since $x_n \longrightarrow x$, $y_n \longrightarrow y$

$\Rightarrow d(x, y) - d(x_n, y_n) \leq d(x, x_n) + d(y, y_n)$ ⊕

From ⊕ & ⊛

$\Rightarrow |d(x, y) - d(x_n, y_n)| \leq d(x, x_n) + d(y, y_n)$

Since given $\epsilon > 0$ $\exists$ $N(\epsilon) \in \mathbb{N}$ such that

$d(x_n, x) < \epsilon/2$

$d(y_n, y) < \epsilon/2$ $\forall$ $n > N(\epsilon)$

$\Rightarrow |d(x, y) - d(x_n, y_n)| < \epsilon$

$\forall$ $\epsilon > 0$ for an $N(\epsilon) \in \mathbb{N}$.

$\Rightarrow d(x, y) \longrightarrow d(x_n, y_n)$

i.e. $\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$

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Thm Every convergent sequence in a metric space (X,d) is fundamental / cauchy sequence

Proof: Given $x_n \longrightarrow x$

$\Rightarrow$ Given $\epsilon > 0$ $\exists$ $N(\epsilon) \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon/2$ $\forall n > N(\epsilon)$

$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \epsilon$ $\forall n, m > N(\epsilon)$

$\Rightarrow d(x_n, x_m) < \epsilon$

which implies every ~~cauchy sequence~~ convergent seq is ~~con~~ cauchy.

But the converse is not true (in general)

(∵ In $\mathbb{R}^n, \mathbb{R}, \mathbb{C}$ Cauchy sequence $\Longleftrightarrow$ Convergent Sequence)

$\mathbb{R}$ and $\mathbb{C}$ are complete metric spaces

To prove: In $\mathbb{R}$ or $\mathbb{C}$ every cauchy sequence is convergent.

OR

A sequence (x_n) in $\mathbb{R}$ or $\mathbb{C}$ is convergent iff (x_n) is a cauchy sequence

We only need to prove the reverse statement since the forward proposition is already proved. (see at top of the page)

Theorem: Let M be a nonempty subset of (X, d) and $\overline{M}$ be its closure. Then

(a) $x \in \overline{M}$ iff $\exists$ a sequence $\{x_n\}$ s.t. $x_n \to x$

(b) M is closed iff closed iff $\exists$ a sequence $\{x_n\}$ in M, $x_n \to x$ then $x \in M$ (Basically $M = \overline{M}$)

Proof:

Let $x \in \overline{M}$. Then $x \in M$ or $x \notin M$.

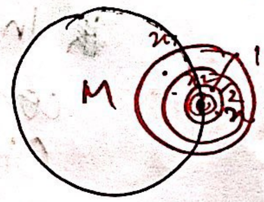
If $x \in M$ then the sequence is of the type $\{x, x, \dots\}$

$x_n = (x)$

$x_n \to x$.

If $x \notin M$, then it is a lt. point.

Hence every nbhd $B(x, \epsilon)$, $\epsilon > 0$ contains a point other than x in M.



Let $\epsilon = 1/n \quad n = 1, 2, 3 \dots$

$B(x, 1) \supseteq B(x, 1/2) \supseteq \dots$

$\Rightarrow B(x, 1)$ contains $x_1 \in M$.

We have a sequence $(x_1, x_2, \dots)$ which is contained in $(B(x, 1), B(x, 1/2), B(x, 1/3) \dots)$

$\frac{1}{n} \to 0$ as $n \to \infty$

$\Rightarrow$ As $\frac{1}{n} \to 0 \quad x_n \to x$.

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$\Leftarrow$ For $x_n \in M$ with $x_n \longrightarrow x$ to show that $x \in \overline{M}$.

$x_n \longrightarrow x \Rightarrow$ either $x \in M$ or $x \notin M$.

if $x \in M$ then its very very very trivial
($M \subseteq \overline{M} \Rightarrow x \in \overline{M}$)

But if $x_n \notin M$, then every nbhd of x contains $x_n \neq x$. so $\Rightarrow$ x is a limit point

~~$M = \overline{M}$~~

~~(b)~~ (b) It follows from (a) as M is closed iff $\boxed{\overline{M} = M}$

Thm: A subspace M of a complete metric space X is itself complete iff M is closed.

Proof: $\Rightarrow$ Let M be a ~~subset of X~~ complete metric space

Show by the previous theorem every $x \in \overline{M}$. $\exists (x_n)$ in M s.t. $x_n \longrightarrow x$

As M is complete, (x_n) is a convergent sequence in M. since (x_n) is a ~~con~~ cauchy sequence
(Every convergent seq. is cauchy seq)

(x_n) converges. I.E. $x \in M$

$\Rightarrow$ ~~x~~ is M is closed

$\Leftarrow$ ~~let~~ ~~let~~ M is closed in X.
$\Rightarrow \overline{M} = M$

Let (x_n) be a cauchy sequence in M. which implies

(Because X is complete) $x_n \longrightarrow x$ in X,
$x \in \overline{M} = M$ as M is closed
$\Rightarrow x_n \longrightarrow x$ in M
As (x_n) is abit cauchy seq. in M.
$\Rightarrow M$ is complete.

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Thm A mapping $T: X \longrightarrow Y$ is continuous at a point $x = x_0 \in X$ iff $x_n \longrightarrow x_0 \Rightarrow Tx_n \longrightarrow Tx_0$

Proof: Assume T is continuous at $x = x_0$

Given $\epsilon > 0$ $\exists$ $\delta(\epsilon)$ s.t.

$\hat{d}(Tx, Tx_0) < \epsilon$

whenever

$d(x, x_0) < \delta$. — (A)

To show that $x_n \longrightarrow x_0 \Rightarrow Tx_n \longrightarrow Tx_0$

$x_n \longrightarrow x_0$

Then $\forall$ given $\delta > 0$, $\exists$ an $N \in \mathbb{N}$

s.t. $\forall n > N$ $d(x_n, x_0) < \delta(\epsilon)$

By (A) $d(Tx_n, Tx_0) < \epsilon$ for $n > N$.

i.e. $\lim_{n \to \infty} Tx_n = Tx_0$

$\Rightarrow Tx_n \longrightarrow Tx_0$

$\Leftarrow$ $x_n \longrightarrow x_0$ and $Tx_n \longrightarrow Tx_0$

$\Rightarrow T$ is continuous

We can instead assume contrapositive

That ~~T is not continuous~~ ~~if~~ x_n

suppose T is not continuous

$\Downarrow$

Then $\exists$ $\epsilon > 0$, $\forall$ $\delta(\epsilon) > 0$ There is $x \neq x_0$

s.t. $\hat{d}(Tx_0, Tx) \geq \epsilon$

if $d(x_0, x) < \delta$

Choose

$\delta = \frac{1}{n}$, $n = 1, 2, \ldots$

Then there is an x_n satisfying

$d(x_n, x_0) < \delta = 1/n$ but $d(Tx_n, Tx_0) \geq \epsilon$.

As $n \to \infty$ $1/n \to 0$

$\Rightarrow$ so $x_n \to x$

$\Rightarrow Tx_n \not\to Tx_0$

But this contradicts the initial assumption

$\Rightarrow$ This means T is continuous.

(x_n) sequence in $\mathbb{R}$ for given $\epsilon > 0$, $\exists$ N S.T.

$|x_n - x_m| < \epsilon$; $n, m > N$

(cauchy criterion)

Limit Point (limit of a sequence)

A point "A" is called a lt pt of a sequence in $\mathbb{R}$ or $\mathbb{C}$ if for every $\epsilon > 0$ we have $|x_n - a| < \epsilon$ for infinitely many n.

The Bolzano Weierstrass Thm

Every boundary infinite sequence has a limit pt.

(x_n) converges in $\mathbb{R}$

given $\epsilon > 0$ choose $n = k > N$ S.T.

$|x_m - x_k| < \epsilon$ whenever $k, m > N$

$B(x_k; \epsilon)$ contains all pts except the pts

$\{x_1, \ldots x_{N-1}\} \notin B(x, \epsilon)$

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Let D be the set containing the set

$B(x_K, \epsilon)$ and $x_1, \ldots \ldots x_{N-1}$.

$\Rightarrow$ D is bounded and infinite

$\Rightarrow$ D has a limit point 'a'.

since ① holds for all $\varepsilon > 0$, an $\epsilon > 0$ being given there is an N^* s.t.

$$|x_n - x_m| < \epsilon/2 \quad m, n > N^*.$$

Choose a fixed $n > N^*$

$$\Rightarrow |x_n - a| < \epsilon/2 \quad \text{for } n > N^*$$

$$\Rightarrow |x_m - a| = |x_m - x_n + x_n - a|$$

$$\leq |x_m - x_n| + |x_n - a|$$

$$< \epsilon$$

$$\Rightarrow |x_m - a| < \epsilon \quad m > N^*$$

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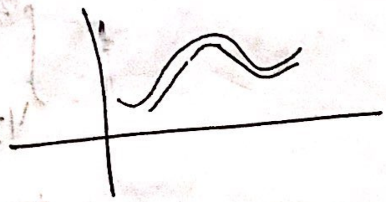
7th August

$C[a,b]$

$$d(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|$$



$x_m(t)$

$d(x_m, x_n) < \epsilon$; $m,n > N$

$$\boxed{\lim_{n \to \infty} x_n(t) = x(t)}$$

Thm $C[a,b]$ is complete metric space with the metric

$$d(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|$$

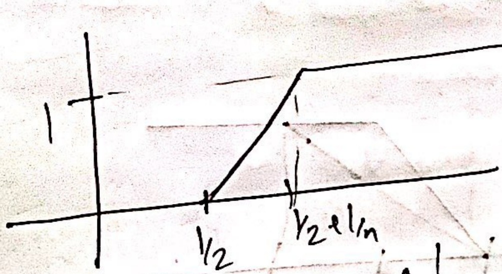
set $a = 0$, $b = 1$

$$\hat{d}(x,y) = \int_0^1 |x(t) - y(t)|\, dt$$

$\hat{d}$ is a metric on $C[0,1]$

$C[0,1]$ is not complete wrt $\hat{d}$

Let
$$x_n(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1/2 \\ n(t - 1/2) & \text{if } 1/2 < t \le 1/2 + 1/n \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} < t \le 1 \end{cases}$$



Now
$$d(x_m, x) = \int_0^1 |x_m(t) - x(t)|\, dt$$

$$= \int_0^{1/2} |x(t)|\, dt + \int_{1/2}^{1/2+1/m} |x(t) - x_m(t)|\, dt + \int_{1/2+1/m}^{1} |x(t) - 1|\, dt$$

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$$d(x_n, x_m) = \int_0^1 |x_n(t) - x_m(t)|\, dt$$

$$= \int_{1/2}^{1/2+1/n} (n-m)(t-1/2)\, dt$$

$$+ \int_{1/2+1/n}^{1/2+1/m} (1 - m(t-1/2))\, dt.$$

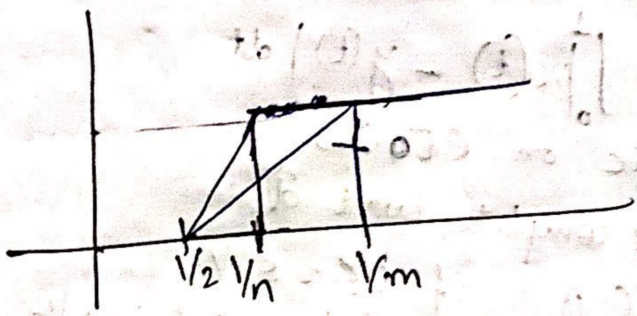
because between $1/2 + 1/n$ and $1/2 + 1/m$.

$x_n(t) = 1$

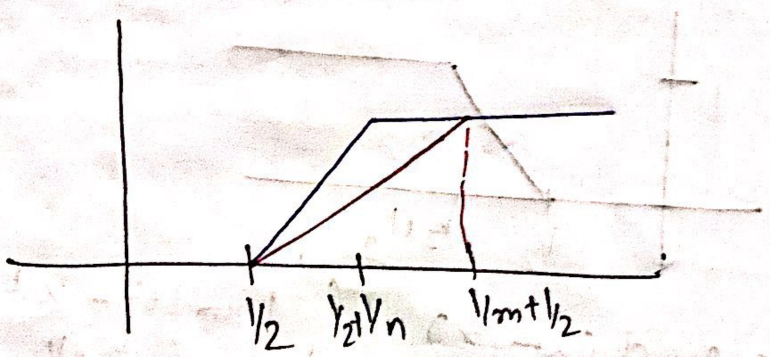
whereas $x_m(t) = m(t - 1/2)$ $n > m$.

$1/n < \frac{1}{m}$

$$= \frac{1}{2m} - \frac{1}{2n}$$



$$\left(\frac{1}{m} - \frac{1}{2}\right) \times \frac{1}{2} \; \cancel{1/2} \; \left(1/n - 1/2\right) \times 1/2 = \frac{1}{2m} - \frac{1}{2n}$$



$$= \frac{1}{\alpha\, n m} \left(\frac{n-m}{mn} \right)$$

If $d(x_n, x) \longrightarrow 0$

$x(t) = 0$ if $t \in [0, 1/2]$

$x(t) = 1$ if $t \in [1/2, 1]$

We have 2 metric spaces (X, d) and $(\hat{X}, \hat{d})$

Two metric spaces (X, d) & $(\hat{X}, \hat{d})$ are said to be **ISOMETRIC** if $\exists$ a map

$T : X \longrightarrow \hat{X}$

which preserves the ~~diffe~~ distance

i.e $\hat{d}(Tx, Ty) = d(x, y)$

$\forall\ x, y \in X$.

Isometric onto map $\Longrightarrow$ bijective map

$T : X \longrightarrow Y$

Two metric spaces X and Y are said to be homeomorphic with each other if there exists a homeomorphism T between them

1. T is bijective
2. T is continuous
3. T^{-1} is continuous

Not in book

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Each isometric map is a homeomorphism

$(-1, 1)$ is not complete

Pg 46 Q5

Show that ~~nowdense~~ if X and Y are isometric they are homeomorphic

Show by an example that an incomplete and complete metric space may be homeomorphic

$T : \mathbb{R} \longrightarrow (-1, 1)$ not complete

$Tx = \frac{2}{\pi} \tan^{-1}(x)$

8th August

Let $Y \subset X$ be a subspace of X

$(X/Y) = \{x + Y : x \in X\}$

$(x+Y) + (z+Y) = (x+z) + Y$

$\alpha(x+Y) = \alpha x + Y$

$(X \backslash Y)$ is a vector space which is called its quotient space.

Dimension of $(X|Y)$ is called the codimension of Y

$\dim (X/Y) = \operatorname{codim}(Y)$

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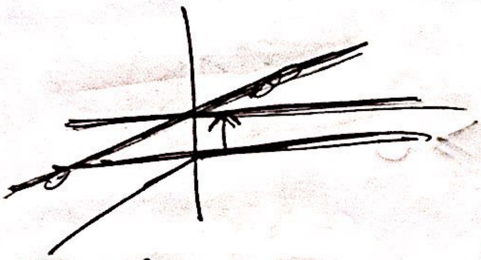
$$Y = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$

If $X = \mathbb{R}^3$ $\quad Y = \{(x, 0, 0) \mid x \in \mathbb{R}\}$

X/Y, the set of all lines parallel to x axis



$X/X = \{0\}$

$X/\{0\} = X$

A norm is a mapping

$\|x\| : X \longrightarrow \mathbb{R}$

with the following prop.

1. $\|x\| \geq 0$
2. $\|x\| = 0 \iff x = 0$
3. $\|\alpha x\| = |\alpha| \, \|x\|$
4. $\|x+y\| \leq \|x\| + \|y\|$

$(X, \|\cdot\|) \longrightarrow$ Normed space

$\downarrow$

Norm or normed linear space

$d(x, y) = \|x - y\|$

(Normed space is always a metric space)

$\Rightarrow$ (X, d) is a metric space

① $d(x, y) = 0$

$\Rightarrow \|x - y\| = 0$

$\Leftrightarrow x - y = 0 \Leftrightarrow x = y$

② $d(x, y) \geq 0$

because $\|x - y\| \geq 0$

③ $d(x, y) = d(y, x)$

(Trivially)

④ $d(x, y) = \|x - y\|$

$= \|x - z + z - y\|$

$\leq \|x - z\| + \|z - y\|$

$= d(x, z) + d(z, y)$

$\Rightarrow d(x, y) \leq d(x, z) + d(z, y)$

$\Rightarrow$ Every normed linear space is a metric space. The metric is $\boxed{\text{induced}}$ by the norm

$\boxed{\text{Metric} \supseteq \text{Norm}}$

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$x = \mathbb{R}^n$, $\|x\| = \sqrt{\sum_{i=1}^{n} (x_i)^2}$

$x = (x_1, \dots x_n)$

$d(x,y) = \|x-y\|$
$= \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$

$1 \le p < \infty$

<u>l^p space</u> l^p is a normed space

$x \in l^p$

$x = (x_1, x_2, \dots)$

$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$

$d(x,y) = \|x-y\|$

<u>Banach space</u>.

A complete normed space is called a banach space (here the metric ~~indu~~ induced from the norm)

All finite dimensional vector spaces are Banach spaces.

l^p, l^∞, $C[a,b]$ are examples of Banach spaces.

Is Norm continuous? Let's see

Consider $(X, \|\cdot\|)$ be a normed linear space

$x \longmapsto \|x\|$

$\|\cdot\| : X \longrightarrow \mathbb{R}$

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$$\|x\| = \|x-y+y\|$$
$$\leq \|x-y\| + \|y\|$$
$$\Rightarrow \|x\| - \|y\| \leq \|x-y\|$$

Similarly
$$\|y\| - \|x\| \leq \|x-y\|$$
$$\Rightarrow \big| \|x\| - \|y\| \big| \leq \|x-y\| \quad \text{— } (A)$$

$\delta = \epsilon$

$\forall$ All $\epsilon > 0$, $\|x-y\| < \epsilon$
$$\Rightarrow \big| \|x\| - \|y\| \big| < \epsilon$$

δ

<u>**Normed space**</u> $\mathbb{R}^n$, $\mathbb{C}^n$

$x \in \mathbb{C}^n$ $\qquad$ $x \in \mathbb{R}^n$

$\|x\| = \sqrt{\sum_{i=1} |x_i|^2}$ $\qquad$ $\|x\| = \sqrt{\sum_{i=1} x_i^2}$

l^p if $x \in l^p$

then $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$

$d(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p}$

* $C[a,b] = \{ f : [a,b] \longrightarrow \mathbb{R},\ f \text{ is continuous} \}$

$\|x\| = \max \{x(t)\}$, $t \in (a,b)$

$d(x,y) = \max |x(t) - y(t)|$ $\quad t \in (a,b)$

* $C[a,b]$ is a banach space wrt. the metric induced from sup norm

$\|x\| = \int_a^b |x(t)|\,dt$

$d(x,y) = \int_a^b |x(t) - y(t)|\,dt$

. $(C[a,b]$ is not norm$) \longrightarrow$ is not complete space

• $\mathbb{R} \longrightarrow$ complete

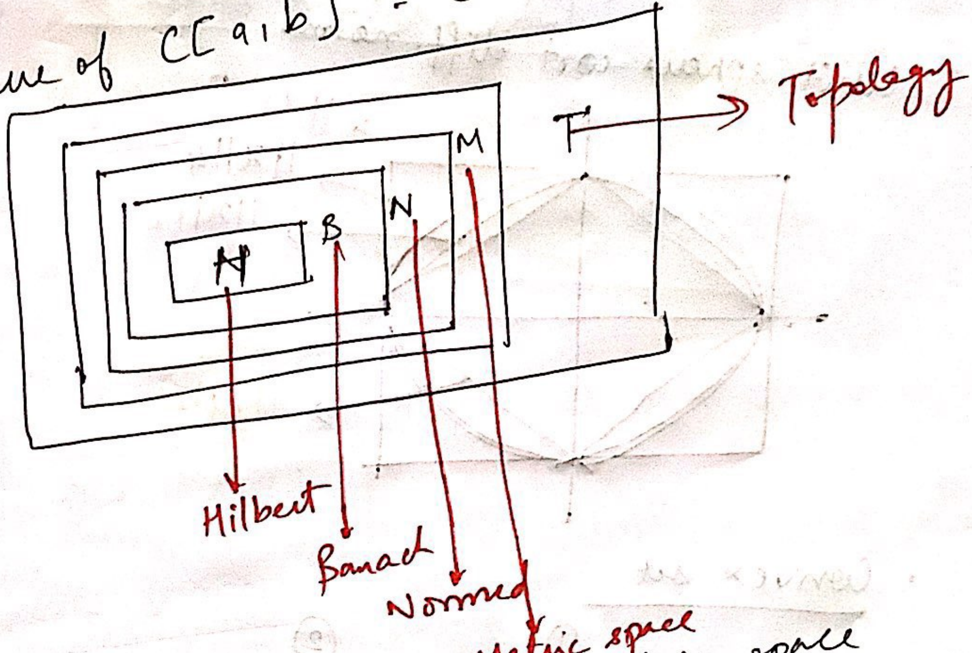
$\mathbb{Q} \longrightarrow$ Incomplete.

$L^2[a,b] = \{ f : \int_a^b |f|^2 < \infty \} \longrightarrow$ complete metric space

$f = g$ a.e.

$f(x) = \begin{cases} x & x \neq 1 \\ 2 & x = 1 \end{cases}$ $\qquad g(x) = \begin{cases} x & x \neq 1 \\ 10 & x = 1 \end{cases}$

closure of $C[a,b] = \overline{C[a,b]} = L^2[a,b]$



i.e. Every normed space is a metric space

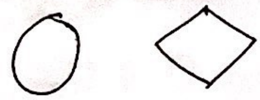
Every metric space is a topological space

l^p is Hilbert iff $p = 2$

$L^p[a,b] = \{ |f| = \int_a^b |f|^p < \infty \}$

L^p is Hilbert iff $p = 2$

$S(x;1) = \{ x \in X : \|x\| = 1 \}$



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Q Let X be the vector space of all ordered pairs of Real no's.
$x = (x_1, x_2)$, $y = (y_1, y_2)$ ...
Show that norms on X defined by

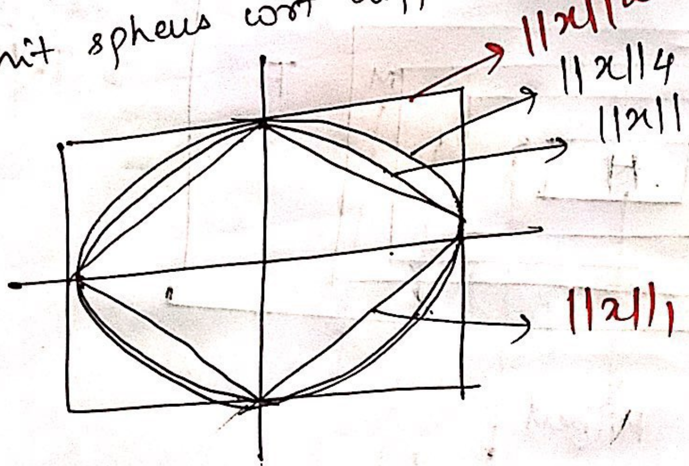
$$\|x\|_1 = |x_1| + |x_2|$$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$

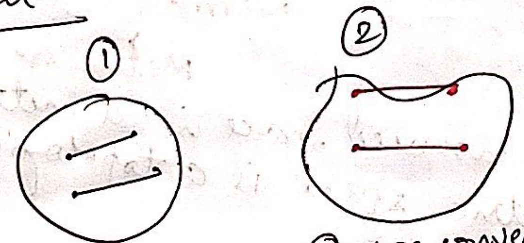
$$\|x\|_\infty = \max\{|x_1|, |x_2|\}$$

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$$

Unit spheres wrt diff. norms



· Convex set



① Convex set

② Non convex set

A subset A of a vector space X is called to be convex iff $x, y \in A$ implies
$M = \{z \in X \mid z = \alpha x + (1-\alpha) y,\ 0 \le \alpha \le 1\} \subset A$

$A \subset X$.

(12) Show that $\phi(x) = (\sqrt{|x_1|} + \sqrt{|x_2|})^2$ $x = (x_1, x_2)$

$= |x_1| + |x_2| + 2\sqrt{x_1 x_2}$

is not a normed space

$\phi(x_1, x_2) \neq \phi(x_2, x_1)$

$\phi(\alpha x) = |\alpha| \phi(x)$

① $\phi(x) \geq 0$

② $\phi(x) = 0$ iff $x = 0$

③ $\phi(\alpha x) = |\alpha| \phi(x)$

$\phi(x+y) \leq \phi(x) + \phi(y)$

* $Y \subset X \rightarrow$ Banach space

A complete normed space is said to be a Banach space

(Metric derived from the norm)

THM A subspace Y of a Banach space is complete iff Y is closed in X.

Ans: same proof of that of Metric space

X is a metric space

E is complete in $X \Longrightarrow E$ is closed in X.

If X is complete, E is closed subset of X

$\Rightarrow E$ is complete in X

Sequence in Normed Space X

A sequence $(x_n) = (x_1, x_2, \ldots)$ in a normed space X is convergent if X contains an x s.t.

$$\lim_{n \to \infty} \|x_n - x\| = 0$$

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$x_n \longrightarrow x$ in X

iff $\lim_{n \to \infty} \|x_n - x\| = 0$

A sequence (x_n) in X is said to be a cauchy sequence if for given $\epsilon > 0$, $\exists N$ s.t. $\|x_n - x_m\| < \epsilon$, whenever $n, m > N$

Infinite series in a normed space X.

Let $\{x_n\}$ be a sequence in X.

Defined $S_n = x_1 + x_2 + \dots + x_n$

for $n = 1, 2, \dots$

$S_1 = x_1$, $S_2 = x_1 + x_2$,

$S_3 = x_1 + x_2 + x_3$, $S_4 = x_1 + x_2 + x_3 + x_4, \dots$

sequence of Partial sum or partial sum of sequence is given by $S_1, S_2, S_3, S_4 \dots$

If (S_n) converges from $S_n \longrightarrow S$

i.e. $\|S_n - S\| \longrightarrow 0$ as $n \longrightarrow \infty$

Then the infinite series

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + x_4 \dots \quad \text{--- (1)}$$

is said to be convergent.

S is called the sum of the series and we write

$$S = \sum_{n=1}^{\infty} x_n$$

If $\|x_1\| + \|x_2\| + \|x_3\|$ ①

$= \sum_{i=1}^{\infty} \|x_i\| < \infty$, then the series is said to be absolutely convergent.

* For Real sequence (a_n)

$$\sum_{a=1}^{\infty} |a_n| < \infty \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

but $\sum_{n=1}^{\infty} a_n < \infty \not\Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

Example :

$$a_n = \frac{(-1)^n}{n}$$

$\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}_{\text{convergent}} < \infty$ but $\underbrace{\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right|}_{\text{not convergent}} = \sum_{n=1}^{\infty} \frac{1}{n} \not< \infty$

But in a normed space X, absolute convergence implies convergence iff X is complete.

$B \subset X$
B is L. I.
if span $= X$, then B is called Hamel Basis

If a normed space X contains a sequence $\{e_n\}$ with the property that for every $x \in X$ there is a unique property of scalars $\{\alpha_n\}$ such that

$$\left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| \longrightarrow 0 \quad \text{for } n \longrightarrow \infty$$

Here (e_n) is called a ~~Schauder~~ Schauder Basis for X.

$$\boxed{x = \sum_{i=1}^{\infty} \alpha_i x_i}$$

$x, y \in S$

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^n} \frac{|x_i - y_i|}{1+|x_i - y_i|}$$ ★

The metric defined on S is not obtained from a norm

$x = (x_i)$
$y = (y_i)$

$d(x,y) = \|x-y\|$ (Metric induced by the norm)

1) $d(x+z, y+z) = d(x,y)$
2) $d(\alpha x, \alpha y) = |\alpha| d(x,y)$

Translation invariance A metric induced by a norm on a normed space X satisfies

i) $d(x+a, y+a) = d(x,y)$ ii) $d(\alpha x, \alpha y) = |\alpha| d(x,y)$

Proof: $d(x+a, y+a) = \|x+a-(y+a)\| = \|x-y\| = d(x,y)$

$d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = |\alpha|\,\|x-y\| = d(x,y)$

★ suppose it is obtained from a norm $\|\cdot\|$

$d(x,y) = \|x-y\|$

So d must satisfies the translation invariant prop.

i.e. $d(\alpha x, \alpha y) = |\alpha| d(x,y)$
& $d(x+\alpha, y+\alpha) = d(x,y)$

$$d(\alpha x, \alpha y) = \sum_{i=1}^{\infty} \frac{1}{2^i}\left(\frac{|\alpha x_i - \alpha y_i|}{1+|\alpha x_i - \alpha y_i|}\right)$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i}\left(\frac{|\alpha|\,|x_i - y_i|}{1+|\alpha|\,|x_i - y_i|}\right)$$

$$\neq |\alpha| d(x,y) = \sum_{i=1}^{\infty} \frac{|\alpha|\,|(x_i - y_i)|}{1+|x_i - y_i|}$$

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Thm (completion Theorem) Let $X = (X, \|\cdot\|)$ be a normed space then there is a Banach space $\tilde{X}$ and an isometry T from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$. The space is unique, except for isometries.

$$\mathbb{Q} \longrightarrow \mathbb{R} \qquad \hat{X} \quad \boxed{\overline{W} = \hat{X}}$$

$$e_n = (0, 0, \ldots \overset{n}{1}, 0, 0, \ldots 0 \ldots)$$

$$e_1 = (1, 0, \ldots - 0, \ldots 0)$$

$$e_2 = (0, 1, \ldots - \quad , 0 \ldots)$$

$$e_3 = (0, 0, 1, \ldots - \quad 0, 0 \ldots)$$

$\vdots$

$\|e_n\| = 1$ $\forall n$

$$S = \{e_n : n = 1, 2, \ldots\}$$

S is closed as it doesn't have a lt. point.

take $B(e_n; 1) = e_n$.

S is bounded.

But it is not compact.

$\{x_1, x_2, \dots x_n\}$ is a basis for X.

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n \qquad c > 0$$

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n\| \geq c\,(|\alpha_1| + |\alpha_2| + \dots + |\alpha_n|)$$

α_j, lesser value

Rough statement of the lemma

In case of Linearly independent vectors we can't find a linear combination that involves large scalars but represents a small vector.

Lemma: (Linear combination)

Let $\{x_1, \dots x_n\}$ be a l.I set of vectors. In a normed space X (of any dimension). Then $\exists$ a no. $c > 0$ s.t. $\forall$ choice of scalars $\alpha_1, \dots \alpha_n$ we have

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n\| \geq c\,(|\alpha_1| + |\alpha_2| + \dots |\alpha_n|)$$

1st Application of this lemma

Every finite dimensional subspace Y of X (normed space) is complete. In particular every finite dimensional vector space is complete.

Ex: $X = C[0,1]$

$Y = \text{span}\{x_0(t), x_1(t), \dots, x_n(t)\}$

$x_j(t) = t^j$

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Completeness of Metric Spaces

Friday, 16 August 2019 9:06 AM

Theorem (Completeness) Every finite dimensional subspace V of X (normed space) in particular every finite dimensional vector space is complete

Proof: Let (y_n) be an arbitrary cauchy sequence in V and show that it converges in V. i.e. its limit say y belongs to V.

Let $\dim(V) = n$ and $\{e_1, e_2 \ldots e_n\}$ be any basis for V. So each y_n has a unique representation of the form.

$$y_m = \alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + \ldots + \alpha_n^{(m)} e_n$$
$$y_1 = \alpha_1^{(1)} e_1 + \alpha_2^{(1)} e_2 + \ldots + \alpha_n^{(1)} e_n$$
$$y_2 = \alpha_1^{(2)} e_1 + \alpha_2^{(1)} e_2 + \ldots + \alpha_n^{(2)} e_n$$

Since (y_m) is a cauchy sequence for every $\epsilon > 0$ there is an N S.T. $\|y_m - y_r\| < \epsilon$, $m, r > N$.

$$\epsilon > \|y_m - y_r\| = \|\alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + \ldots \alpha_n^{(m)} e_m (-)\{\alpha_1^{(r)} e_1 + \alpha_2^{(r)} e_2 + \ldots + \alpha_n^{(m)} e_m\}\|$$
$$= \left\|\sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(r)}) e_i\right\| \geq c \sum_{i=1}^{n} \|\alpha_i^{(m)} - \alpha_i^{(r)}\|$$

(Using Boundedness of linear combination)

$$\Rightarrow |\alpha_j^{(m)} - \alpha_j^{(r)}| \leq \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(r)}| < \epsilon/c$$
$$\Rightarrow |\alpha_j^{(m)} - \alpha_j^{(r)}| < \epsilon/c$$

i.e. n sequences α_j, $j = 1, \ldots n$

This shows that each of n sequences $(\alpha_j^{(m)}) = (\alpha_j^{(1)}, \alpha_j^{(2)} \ldots)$ is a cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$. Hence it converges. Let α_j denote the limit

$$\alpha_j^{m} \longrightarrow \alpha_j \quad j = 1, \ldots n$$

Define $y = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n$

As $\{e_1, \ldots e_n\}$ is a basis for V, $y \in V$.

$$\|y_m - y\| = \left\|\sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i) e_i\right\|$$

Using Δ inequality.

$$\leq \sum_{i=1}^{n} \|\alpha_i^{(m)} - \alpha_i\| \|e_i\|$$
$$\alpha_j^{m} \longrightarrow \alpha_j$$
$$\Rightarrow |\alpha_i^{(m)} - \alpha_i| \longrightarrow 0$$
$$\Rightarrow \|y_m - y\| \longrightarrow 0 \text{ as } m \longrightarrow \infty$$

Theorem (Closedness) Every finite dimensional space V of X (normed) is closed in X.

Banach

Proof: We know if X is a metric space $E \subset X$, E is complete $\Longrightarrow$ E is closed.

If X is complete and E is closed subset of X, then E is complete. (See Real Analysis)

As X is a metric space, the result follows.

But if $\dim(Y) = \infty$, Y need not be closed.

For $X = C[0,1]$, $Y = \text{span}\{x_0, x_1, \ldots\}$ (infinite no. of elts)

where $x_j(t) = t^j$

$$\therefore y_n = 1 + \frac{t}{1} + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!}$$

$$\lim_{n\to\infty} y_n = e^t \notin Y.$$

Y is not closed. ((Q) is Y dense in X?)

In finite dimensional vector space X, any 2 norms (or every) are equivalent $\Rightarrow$ The two norms induce the same topology.

i.e. the open subsets of X are same regardless of the particular choice of the norm on X.

Def (Equivalent Norm) A norm $\|\cdot\|$ on a vector space X is said to be equivalent to a norm $\|\cdot\|_0$ on X if there are +ve numbers a and b such that for all $x \in X$

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0$$

Similarly $\exists\, \alpha, \beta > 0 \ni$

$$\alpha\|x\| \leq \|x\|_0 \leq \beta\|x\|$$

Thm: (Equivalence of norms) In finite dimensional vector spaces X, any norm $\|x\|$ is equivalent to any other norm $\|\cdot\|_0$

Proof Let $\dim(X) = n$ and $\{e_1, e_2, \ldots e_n\}$ any basis for X.

Then every $x \in X$ has a unique representation.

$$x = \alpha_1 e_1 + \alpha_2 e_2 \ldots + \alpha_n e_n$$

$$\|x\| = \|\alpha_1 e_1 + \ldots + \alpha_n e_n\| \geq c\left(\sum_{i=1}^{n} |\alpha_i|\right)$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha_i| \leq \frac{\|x\|}{c} \quad \text{—— (1)}$$

On the other hand the Δ inequality gives

$$\|x\|_0 \leq \sum_{i=1}^{n} |\alpha_i| \|e_i\|_0 \leq \sum_{i=1}^{n} k|\alpha_i| \qquad \underbrace{k}_{\max \|e_i\|_0}$$

$$\Rightarrow \|x_0\| \leq \sum_{i=1}^{n} |\alpha_i| \|e_i\|_0 \leq k \sum_{i=1}^{n} |\alpha_i|$$
$$\leq k \frac{\|x\|}{c} \quad \text{(using 1)}$$
$$\leq \frac{\|x\|}{c/k}$$

Let $a = c/k$

$$\Rightarrow a\|x_0\| \leq \|x\| \longrightarrow \text{(I)}$$

On the other hand

$$\|x\|_0 \geq c(|\alpha_1| + \ldots + |\alpha_n|) \Longrightarrow \frac{\|x_0\|}{c} \geq \sum_{i=1}^{n} |\alpha_i|$$

(Δ ineq)
$$\|x\| \leq \sum_{i=1}^{n} \|\alpha_i e_i\| = \sum_{i=1}^{n} |\alpha_i| \|e_i\|$$
$$\leq \underbrace{k}_{\max \|e_i\|} \sum_{i=1}^{n} |\alpha_i|$$

$$\Rightarrow \|x\| \leq b\|x\|_0 \quad \text{where } b = k/c$$

(II)

Using (I) & (II)

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0$$

Compactness and finite dimension

A metric space X is said to be compact if every sequence has a convergent subsequence. (sequential compactness)

A subset M of X is said to be compact if M is compact considered as a subspace of X, that is, if every sequence in M has a convergent subsequence whose limit is an element of M.

Lemma: (compactness) A compact subset M of X (metric space) is closed and bounded.

In finite dimensions $\longrightarrow$ iff

Not finite $\not\longrightarrow$ iff

PROOF For every $x \in \overline{M}$ $\exists$ a sequence (x_j) in $M \ni x_n \longrightarrow x$.

Since M is compact, $x \in M$

i.e. M is closed.

Suppose M is not bounded, then $\exists$ an unbounded sequence (y_n) $\ni$ (S.T.) $d(y_n, a) > n$ where a is a fixed element in X. Now this sequence can not have a converging subsequence since a convergent subsequence must be bounded.

But the converse need not be true.

Let $X = \ell^2$.

$$Y = \{e_n \in \ell^2 : e_n = (0, 0 \ldots, \overset{n}{1} \ldots 0 \ldots)$$
$$\|e_n\| = 1 \quad \forall n$$

$\Rightarrow$ Y is not compact.

X and

Thm (compactness) In a finite dimensional normed space any subset $M \subset X$ is compact iff M is closed and bounded.

Proof Compactness $\Longrightarrow$ Closed and Bounded

Now to prove the converse. Let M be closed and bounded subset of X.

Let $\dim X = n$ and $\{e_1, \ldots e_n\}$ be a basis for X. We consider and sequence (x_m) in M

Now each (x_m) has a representation

$$x_m = \alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + \ldots + \alpha_n^{(m)} e_n$$

Since M is bounded so is (x_m) for $\|x_m\|$

$$k \geq \|x_m\| = \|\alpha_1^{m} e_1 + \ldots + \alpha_n^{m} e_n\|$$
$$\geq c\sum_{i=1}^{n} |\alpha_i^{m}|$$

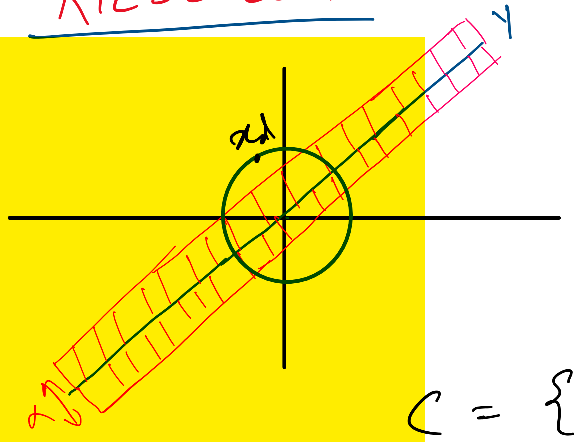
Hence the sequence $|\alpha_i^{(m)}|$ (fixed i) is bounded and by Bolzano Weierstrass Thm it has a limit pt. α_i $1 \leq i \leq n$

So (x_m) has a subsequence (z_m) which converges to

$$z = \sum_{i=1}^{n} \alpha_i e_i$$

Wednesday, 21 August 2019 12:04 PM

RIESZ'S LEMMA



$0<\alpha<1$

$X=\mathbb{R}^2$ $\quad Y\neq X$

$C=\{x\in\mathbb{R}^2 \mid \|x\|=1\}$

Y is proper closed subspace of X

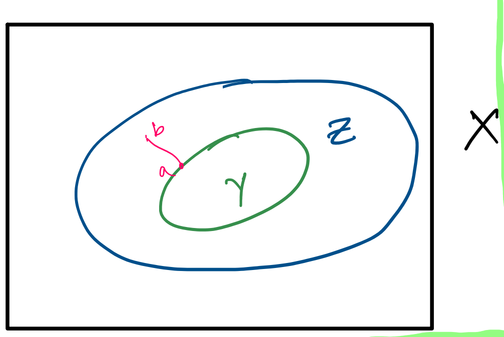
$0<\alpha<1$

$\exists\ x_\alpha\in C$ s.t. $d(x_\alpha, Y)>\alpha$

Let $d(x_\alpha, Y)\le 1$

RIESZ'S LEMMA: Let Y and Z be subspaces of X (normed space) of any dimension and suppose Y is closed and proper subset of Z. Then for every real $\theta\in(0,1)$ there is a $z\in Z$ such that $\|z\|=1$ $\quad \|z-y\|\ge\theta\ \forall\ y\in Y$.

Construct a pt. $b\in Z/Y$.



Proof

Consider point $v\in Z-Y$ and denote its distance from Y is 'a'.

$a=\inf_{y\in Y}\|v-y\|$
$\Downarrow$
dist(v,Y)

Claim: $a>0$

Suppose $a=0$, then $v\in cl(Y)$

$\Rightarrow v\in Y$ $\Rightarrow\Leftarrow$

Hence $a>0$.

using the property of infimum, $\exists\ y_0\in Y$.

$d(v,y_0)\ge a$
$\quad\|$
$\|v-y_0\|$

$\Rightarrow\quad a\le\|v-y_0\|\le\frac{a}{\theta}$ $\quad$ (as $\frac{a}{\theta}>a$)

$\theta\in(0,1)$ assumption

Let $z=c(v-y_0)$, where $c=\frac{1}{\|v-y_0\|}$

$\Rightarrow \|z\|=1$

Next to show that $\|z-y\|\ge\theta$ for every $y\in Y$.

$\|z-y\|=\|c(v-y_0)-y\|=c\|v-y_0-y/c\|$

Let $y_1=y_0+y/c$

Let $y_0\in Y$, $y/c\in Y$

$\Rightarrow y_1=y_0+y/c\in Y$.

As $y_1\in Y$, $\|v-y_1\|\ge a$

by the def^n of inf.

$\|z-y\|=c\|v-y_1\|\ge ca$

$\Rightarrow \|z-y\|\ge ca=\frac{a}{\|v-y_0\|}$

since $\|y_0-v\|\le\frac{a}{\theta}$

$\Rightarrow \|z-y\|\ge ca=\frac{a}{\|v-y_0\|}\ge\frac{a}{a/\theta}=\theta$

$\Rightarrow \|z-y\|\ge\theta$ ■

Theorem: (finite dimensional) If a normed space X has the property that the closed unit ball $M=\{x\in X \mid \|x\|\le 1\}$ is compact, then X is finite dimensional. (Hint: contradiction)

Proof: Suppose $\dim X=\infty$. Now choose any $x_1\in X$ with $\|x_1\|=1$, $X_1=\text{span}\{x_1\}$, X_1 is one dimensional subspace of X. Hence it is closed, so by Riesz's lemma $\exists\ x_2\in X$ s.t. $\|x_2-x_1\|\ge\frac{1}{2}$

Let $X_2=\text{span}\{x_1,x_2\}$, $\quad X_2\neq X$

Something something

Thursday, 22 August 2019 11:14 AM

$T: X \xrightarrow{\text{continuous}} Y$

suppose M is a compact subset of X.

T_M is the image of M in Y.

T_M is a compact subset of Y.

$\Rightarrow T_M$ is a compact

(Basically a continuous map sends a compact set to a compact set)

compact in $X \longmapsto$ compact in Y

Thm Let X and Y be metric spaces and $T: X \to Y$ is a continuous map. Then the image of a compact set M under T is compact.

Proof: To show T_M is compact.

Let (y_n) be a sequence in T_M.

To show that (y_n) has a convergent subsequence in T_M.

As $y_n \in T_M$. $\exists\ x_n \in M$ s.t. $T(x_n) = y_n$.

Now we have a sequence (x_n) in M.

Now for (x_n) in M $\exists\ (x_{n_k})$ in M which converges in M. (compactness of M)

$$x_{n_k} \longrightarrow x^* \Rightarrow \underset{y_{n_k}}{\underset{\parallel}{T(x_{n_k})}} \longrightarrow T(x^*)$$

We have a subsequence (y_{n_k}) in $T(M)$ which converges in $T(M)$.

Thm (Max & Min) A continuous mapping T of a compact subset M of X (metric space) into $\mathbb{R}$ assumes a maximum and minimum at some points of M.

$f: [a,b] \longrightarrow \mathbb{R}$ (Intuition)

$m \le f(x) \le M$

$\exists$ $x^* \in [a,b]$ s.t. $f(x^*) = M$

$x_* \in [a,b]$ s.t. $f(x_*) = M$

Proof: As T is continuous and M is compact $T(M) \subset \mathbb{R}$ is compact. Hence $T(M)$ is closed and bounded.

$\Rightarrow \inf T(M) \in T(M)$, $\sup T(M) \in T(M)$

$\exists\ x^*, x_*$ in M s.t.

$T(x_*) = \inf(T(M)) = \min$

$T(x^*) = \sup(T(M)) = \max$

$E_1 \subset X$ (normed space) — open

$E_2 \subset X$ (E_2 can be any subset)

$$\underbrace{E_1 + E_2}_{\text{open}} = \{e_1 + e_2 \mid e_1 \in E_1, e_2 \in E_2\} = \bigcup_{e_2 \in E_2} (E_1 + e_2)$$

Thm: If $E_1 \subset X$ is open and $E_2 \subset X$ then $E_1 + E_2$ is open.

Proof: As E_1 is open for $x_1 \in E_1$ $\exists\ r > 0$ s.t. $B(x_1; r) \subset E_1$.

Let $x_2 \in E_2$,

$B(x_1 + x_2; r) = B(x_1; r) + x_2 \subset E_1 + x_2$ Hence open

$E_1 + E_2 = \bigcup_{x_2 \in E_2} (E_1 + x_2)$ (Arbit union of open sets)

As arbitrary union of open sets is open.

$\Rightarrow E_1 + E_2$ is open.

If Y is a subspace of X, $Y \neq X$ iff $Y^\circ = \phi$.

$Y = X$ iff $Y^\circ \neq \phi$

$Y = X \Rightarrow Y^\circ = X^\circ = X \neq \phi$.

$Y^\circ \neq \phi$, $Y = X$ (To prove) $Y \subset X$, $X \subset Y$

Let $x \in Y^\circ$, $\exists\ r > 0$ s.t. $B(x; r) \subset Y^\circ \subset Y$.

$\Rightarrow B(x; r) \subset Y$.

Now for any $x \neq 0$, $a + r\dfrac{x}{2\|x\|} \in B(a, r)$

As $\left\| a + \dfrac{r}{2}\left(\dfrac{x}{\|x\|}\right) - a \right\| = r/2 < r$

$\Rightarrow a + \dfrac{r x}{2\|x\|} \in Y$

and $-a \in Y$

$\Rightarrow x \in Y$.

Friday, 23 August 2019 9:12 AM

$T: X \longrightarrow Y$
any map b/w 2 vector spaces is called an operator.

$x, y \in X$
$x + y \in X$
$T(x+y) = Tx + Ty$
$T(\alpha x) = \alpha Tx$
$\forall \alpha \in \mathbb{K}, x, y \in X.$

$T: X \longrightarrow Y$ is said to be a linear operator

$D(T)$: Domain of T is a vector space
$R(T)$: Range of T is a vector space

1) $T(x+y) = Tx + Ty$
2) $T(\alpha x) = \alpha T(x)$

$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$

$T(0) = 0$

Ex 1 $Tx : X \longrightarrow X$
$I_x(x) = x$
i.e $Ix = x \quad \forall x \in X.$

Ex 2 Zero map $0(x) = 0 \ \forall x \in X.$

Ex 3: $P(x) = \{ p(x) \mid p(x)$ is a polynomial in $[0,1] \}$
$T: P(x) \longrightarrow P(x)$
$T(x(t)) = x'(t).$

Ex 4: $T: C[0,1] \longrightarrow C[0,1]$
$T(x(t)) = \int_0^t x(t')\, dt'$

Ex 5 (Used in Quantum Mech).
$T: C[0,1] \longrightarrow C[0,1]$
$T(x(t)) = t(x(t))$
$T(x+y)t = t((x+y)(t))$
$= t\,x(t) + t\,y(t)$
$= Tx(t) + Ty(t)$

Ex 6: $T: \mathbb{R}^n \longrightarrow \mathbb{R}$
$Tx = a \cdot x \qquad a \in \mathbb{R}^n \longrightarrow$ fixed.
$= \sum_{i=1}^{n} a_i x_i \qquad x \in \mathbb{R}^n$

Reisz Representation Thm Any linear operator from $\mathbb{R}^n$ to $\mathbb{R}$ can be written in form of an inner product.
$T(x) = \langle a, x \rangle.$

Ex 7 $A = (\alpha_{ij})_{m \times n}.$
$\underset{m \times n}{A} \ \underset{n \times 1}{x}$
$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m.$
$(y_i) = \sum_{i=1}^{n} \alpha_{ij} x_i$
$Tx = Ax = y$

$N(T)$ = null space
$= \{x \in X \mid Tx = 0\}$
$T: D(T) \longrightarrow R(T)$
onto.
$T: D(T) \longrightarrow Y$
into.
$T: X \longrightarrow Y.$

Thm: Let T be a linear operator. (No restriction on the dimension of the spaces)
(a) Range of T $(R(T))$ is a vector space
(b) If $\dim(D(T)) = n$ then $\dim(R(T)) \leq n$
(c) The null space $N(T)$ is a vector space

We have a linear operator $T: X \longrightarrow Y$. T is bdd if $\exists\, c \in \mathbb{R}$ if $\|Tx\| \leq c\|x\|$

A linear operator $T: D(T) \subset X \longrightarrow Y$ T is bounded if $\exists\, c \in \mathbb{R}$ s.t. $\|Tx\| \leq c\|x\| \ \forall x \in D(T)$
(bounded linear operator is completely diff from bdd map).

When $x = 0$, $Tx = 0$
$\|Tx\| = 0 \qquad \|x\| = 0$
$0 \leq c \cdot 0$

Suppose $\|x\| \leq \alpha \qquad \|x\| > 0$
$x \neq 0$

$\frac{\|Tx\|}{\|x\|} \leq c \Rightarrow c \geq \frac{\|Tx\|}{\|x\|}$

$\Rightarrow c \geq \sup_{x \in D(T)} \frac{\|Tx\|}{\|x\|}$

$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$ — (2)
(Norm of a linear operator)

if T is bounded.
$\Rightarrow \|Tx\| \leq \|T\| \|x\|$
$\|T\| = \sup_{\|x\|=1} \|Tx\|$ — (3)

$\|T\|$ is called the norm of .
Now if $D(T) = \{0\}$, define $\|T\| = 0$
$T = 0$
$Tx = 0 \ \forall x$
$T0 = 0$

$BL(X, Y) \longrightarrow$ collection of all Bounded linear map.

An alternative formula for the norm T is
$\|T\| = \sup_{\substack{x \in D(T) \\ \|x\|=1}} \|Tx\|$

Proof: Let $\alpha = \|x\|$, $x \neq 0$ and set $y = \left(\frac{1}{\alpha}\right)x$

$\|y\| = \left\|\left(\frac{1}{\alpha}x\right)\right\| = \frac{1}{\alpha}\|x\| = 1$

As T is linear
(since linear operator)

Eq (2) $\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{1}{\alpha}\|T(x)\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \left\|T\left(\frac{x}{\alpha}\right)\right\| = \sup_{\substack{y \in D(T) \\ \|y\|=1}} \|T(y)\|$

since $\frac{1}{\alpha}x \in D(T)$

Identity operator
$I: X \longrightarrow X$
$Ix = x$
$\sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \neq 0 \\ x \in D(T)}} \frac{\|Tx\|}{\|x\|} = 1$

Zero Operator
$0: X \longrightarrow X$
$\|0\| = 0$

Ex3 Let X be the normed space of all polynomial on $J = [0,1]$ its norm
$\|x\| = \max_{t \in [0,1]} |x(t)|$
$T: X \to X \qquad T(x(t)) = x'(t)$

Ex 4: Integral operator
We can define an integral operator $T: C[0,1] \longrightarrow C[0,1]$
$T(x(t)) = \int_0^1 k(t, \tau)\, x(\tau)\, d\tau$

$k: [0,1] \times [0,1] \longrightarrow \mathbb{R}$ known continuous f^n.

Whether it is a bounded operator?
As k is continuous k has maximum value on $[0,1] \times [0,1]$ as well as min.

$y = Tx(t) \qquad \hat{J} = [0,1]$
$\|y\| = \|Tx\| = \max_{t \in J} \left| \int_0^1 k(t, \tau)\, x(\tau)\, d\tau \right| \quad G = J \times J$
$\leq \max_{t \in J} \int_0^1 |k(t, \tau)| |x(\tau)|\, d\tau \leq \underline{\underline{k_0 \|x\|}}$

Let $k_0 = \max_{[0,1] \times [0,1]} k(t, \tau)$

$M = \{ A_{n \times n} : \cdot \} \longrightarrow$ Vector space

$\|A\| = (\alpha_{ij})$

$\|A\|_1 =$
$\|A\|_2 = \sqrt{\sum\sum (a_{ij})^2} = \sqrt{Tr(A^T A)}$
$\|A\|_\infty =$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \begin{matrix} 6 \\ 4 \\ 3 \end{matrix}$$
$\quad 2 \quad 3 \quad 8$

$1 + 4 + 9 + 1 + 9 + 1 + 4 = 29$

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$Tx = y$
$y = Ax$
$y_i = \sum_{j=1}^{n} \alpha_{ij} x_j$
$\|Tx\|^2 = \sum_{i=1}^{n} y_i^2 = \left(\sum_{j=1}^{m}\left(\sum_{i=1}^{n} \alpha_{ij} x_j\right)\right)^2$

(Cauchy Schwartz Ineq.)
$\leq \left[\sum_{i=1}^{m}\left(\sum_{j=1}^{n} \alpha_{ij}^2\right)^{1/2} \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}\right]^2$
$= \|x\|^2 \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}^2$

$\|T\| = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}^2\right)^{1/2}$

28th – 29th August

Wednesday, 28 August 2019 12:04 PM

Let X be a finite dimensional normed space. Then any linear transform on X is a bounded linear map.

$T: X \longrightarrow Y \qquad \|T\| \le m \Rightarrow \|Tx\| \le c\|x\|$

Proof: Let $\{e_1, \dots, e_n\}$ be a basis for X, i.e. $\dim X = n$

Now any $x \in X$, can be expressed uniquely as

$x = \sum_{i=1}^{n} \alpha_i e_i, \ \alpha_i \in \mathbb{K}$.

$$Tx = T\left(\sum_{i=1}^{n} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i T(e_i)$$

In "sxr"

1)

$$\|Tx\| = \left\|\sum_{i=1}^{n} \alpha_i T(e_i)\right\| \le \sum_{i=1}^{n} |\alpha_i| \|T(e_i)\| \le M \sum_{i=1}^{n} |\alpha_i| \quad — (A)$$

$M = \max_{i=1,2,\dots n} \|T(e_i)\|$

2) $\|\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n\| \ge c(|\alpha_1| + |\alpha_2| + \dots |\alpha_n|)$

$\|\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n\| \ge c \sum_{i=1}^{n} |\alpha_i|$

$\Rightarrow \sum_{i=1}^{n} |\alpha_i| \le \frac{1}{c}\|\alpha_1 x_1 + \dots + \alpha_n x_n\| = \frac{\|x\|}{c}$

Using (A)

$\Rightarrow \|Tx\| \le \frac{M}{c}\|x\|$ Let $\gamma = \frac{M}{c}$

$\Rightarrow \|Tx\| \le \gamma\|x\|$

Eg $\left|\int_a^b x(t)\,dt\right| \le \int_a^b |x(t)|\,dt \le \int_a^b \|x\|\,dt$

$\|x\| = \sup_{t \in \Omega} |x(t)|$

$\Rightarrow |x(t)| \le \|x\| \quad \forall t \in \Omega$

functional

Eg: $f: C[a,b] \longrightarrow \mathbb{R}$

$f(x) = \int_a^b x(t)\,dt$

$|f(x)| \le (b-a)\|x\|$

$T: D(T) \longrightarrow Y$ is linear.

T is said to be bounded

$\|Tx\| \le c\|x\|$

$\|Tx\| \le \|T\|\,\|x\|$

(Formalism starts here, last few sentences bullshit)

$T: D(T) \subset X \longrightarrow Y$ is an operator.

Let $x_0 \in D(T)$. T is said to be continuous at $x \longrightarrow x_0$ if given $\epsilon > 0 \ \exists \ \delta(\epsilon) > 0$ s.t. $\|Tx - Tx_0\| \le \epsilon$ whenever $\|x - x_0\| < \delta$.

T is continuous on $D(T)$ if it is continuous at every pt of $D(T)$.

Theorem (Continuity and boundedness)

Let $T: D(T) \longrightarrow Y$ be a linear operator. where $D(T) \subset X$, X, Y are normed spaces.

Then:

(a) T is continuous $\boxed{\text{iff}}$ T is bounded.

(b) T is continuous at a single pt, T is continuous.

Suppose T is bounded $\|T\| = 0 \Rightarrow T$ is continuous

Proof: i) If $T = 0 \Rightarrow$ nothing to prove.

Assume $T \ne 0$, $\|T\| \ne 0$ $\qquad \|T\| = \sup_{x \in D(T)} \frac{\|Tx\|}{\|x\|}$

Let $x_0 \in D(T)$. TP: Tx is cont.

Let $\epsilon > 0$ be arbitrary $\qquad (B(X,Y), \|\cdot\|)$

$\|Tx - Tx_0\| = \|T(x - x_0)\| \le \|T\|\,\|x - x_0\|$

norms are not same

Take $\delta = \epsilon / \|T\|$ since T is bdd. as $\|x - x_0\| < \delta$

$\Rightarrow \|Tx - Tx_0\| \le \|T\|\,\|x - x_0\| \le \|T\| \frac{\epsilon}{\|T\|}$

$\Rightarrow \|Tx - Tx_0\| \le \epsilon$.

Hence by def^n of continuity T is continuous

Suppose T is continuous to show that T is bdd.

TP: $\|Tx\| \le c\|x\| \ \forall x \in X$.

Given $\epsilon > 0 \ \exists \ \delta(\epsilon)$ s.t.

$\Rightarrow \|Tx - Tx_0\| < \epsilon \qquad \|x - x_0\| < \delta$.

Let $x = x_0 + \frac{\delta}{2\|y\|} y \Rightarrow \|x - x_0\| = \left\|\frac{\delta}{2\|y\|} y\right\| = \delta/2$

$\Rightarrow \left\|T\left(x_0 + \frac{\delta}{2\|y\|} y\right) - Tx_0\right\| \le \epsilon$.

$\Rightarrow \left\|T\left(\frac{\delta y}{2\|y\|}\right)\right\| \le \epsilon$

$\Rightarrow \frac{\delta}{2\|y\|}\|Ty\| \le \epsilon$. $\Rightarrow \|Ty\| \le \frac{2\epsilon}{\delta}\|y\| \qquad \frac{2\epsilon}{\delta} = m$

$\Rightarrow \|Ty\| \le m\|y\|$

Corollary (continuity, null space)

Let T be a bounded linear operator. Then:

(a) $x_n \longrightarrow x$ [where $x_n, x \in D(T)$] implies $Tx_n \longrightarrow Tx$

(b) $N(T)$: The Null space is closed

Proof: (a) $x_n \longrightarrow x$

(A) $\|Tx_n - Tx\| = \|T(x_n - x)\| \le \|T\|\,\|x_n - x\| \le \|T\|\epsilon \longrightarrow 0$ as $n \longrightarrow \infty$

$x_n \longrightarrow x$ means give a $\epsilon > 0 \ \exists N, \forall n > N$.

$\|x_n - x\| < \epsilon$.

(A) becomes $\|T\|\epsilon = \epsilon$, arbitrary.

(b) Let $x \in \overline{N(T)}$. $\bar{A} = A \cup A'$.

so $\exists \ (x_n) \in N(T)$

Such that $x_n \longrightarrow x$.

$N(T) = \{x \in D(T) \mid Tx = 0\}$

As $x_n \in N(T)$

$\Rightarrow Tx_n = 0$

$\Rightarrow$ As $x_n \longrightarrow x$, $Tx_n \longrightarrow Tx$ (By (a))

$Tx = \lim_{n \to \infty} Tx_n = 0$

$\Rightarrow Tx = 0$

$\Rightarrow x \in N(T)$

Q) What about $R(T)$?

Not closed.

(look for counterexample) (Construction)

Hint: $T: \ell^\infty \longrightarrow \ell^\infty$

$T(x) = y$

$x = (x_j) \in \ell^\infty$

$y = (y_j) \in \ell^\infty$.

$T(x_j) = \frac{x_j}{j} \quad \forall j = 1, 2, \dots$

$x = (x_1, x_2, \dots)$

$y = (x_1/1, x_2/2, x_3/3 \dots)$

$R(T) = \{Tx : x \in \ell^\infty\}$.

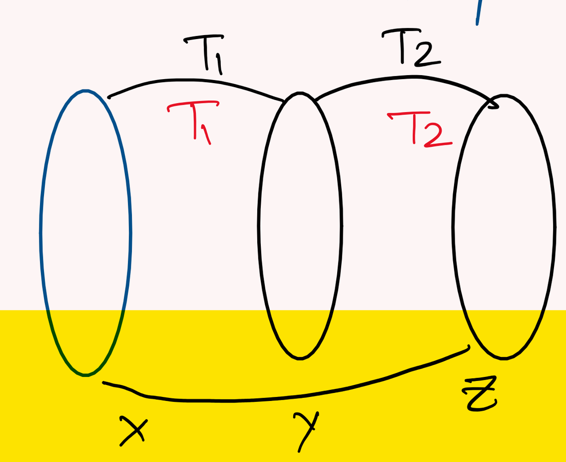
$T_1: X \longrightarrow Y$

$T_2: Y \longrightarrow Z$

$T: X \longrightarrow X$.

X, Y, Z are normed spaces

T, T_1, T_2 are bounded linear maps



$\|T_2 T_1\| \le \|T_1\|\,\|T_2\|$

$\|T\|^n = \|T^n\|$

T_1, T_2

$T_1 = T_2$

$\Updownarrow$

$D(T_1) = D(T_2)$

& $T_1 x = T_2 x \ \forall x \in D(T_1) = D(T_2)$

The restriction of an operator (T)

$T: D(T) \longrightarrow Y$

Vector Space $\qquad$ Normed space.

$B \subset D(T)$

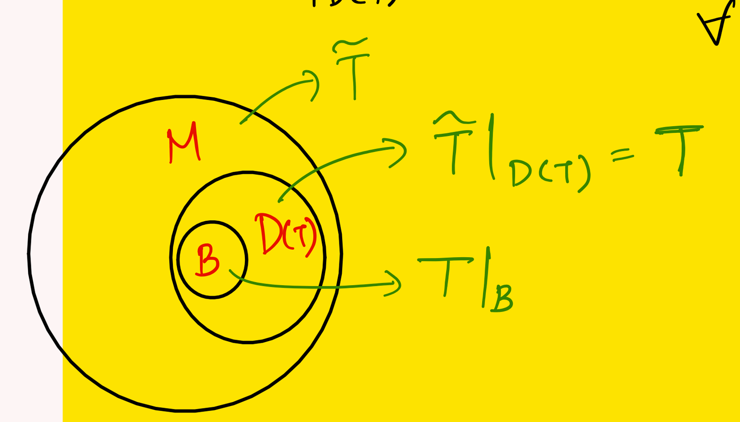
The restriction of T on B is denoted by $T|_B x = Tx \quad \forall x \in B$

Extension of a linear operator

$T: D(T) \longrightarrow Y$. to a set $M \supset D(T)$ is an operator

$\tilde{T}: M \longrightarrow Y$

ST $\tilde{T}|_{D(T)} = T$ i.e. $\tilde{T}x = Tx \quad \forall x \in D(T)$



30th August

Friday, 30 August 2019 9:15 AM

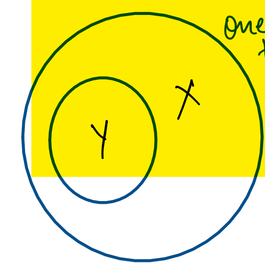
Theorem (Bounded linear extension)

Let T be a bounded linear operator ($T: D(T) \longrightarrow Y$) where $D(T)$ lies in X (Normed space) and Y is a Banach Space.

Then T has an extension $\tilde{T}: \overline{D(T)} \longrightarrow Y$, where $\tilde{T}$ is bounded linear operator of norm

$$\|\tilde{T}\| = \|T\|$$

(Four fundamental Thm of functional analysis) One of them: 1) Hahn Banach Extension Thm



Proof: Let $x \in \overline{D(T)}$, $\exists$ (x_n) in $D(T)$ s.t. $x_n \longrightarrow x$.

Given T is a bounded linear map on $D(T)$

$T(x_n) \longrightarrow Tx$ as $n \longrightarrow \infty$

$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\| \|x_n - x_m\|$ $n, m > N$

i.e. (Tx_n) is a cauchy sequence in Y.

As Y is complete, so $\lim_{n\to\infty} Tx_n = y$

$T: \overline{D(T)} \longrightarrow Y$

$\tilde{T}(x) = y = Tx$ $\forall x \in D(T)$

As limits are unique, $\tilde{T}$ is well defined.

$\tilde{T}(\alpha x + \beta z) = \alpha \tilde{T}(x) + \beta \tilde{T}(z)$

$\alpha x_n \longrightarrow \alpha x$, $\beta z_n \longrightarrow \beta z$

$$\tilde{T}(\alpha x + \beta z) = \lim_{n\to\infty} [T(\alpha x_n + \beta z_n)]$$

$$= \lim_{n\to\infty} [\alpha T(x_n) + \beta T(z_n)]$$

$$= \alpha \lim_{n\to\infty} T(x_n) + \beta \lim_{n\to\infty} T(z_n)$$

$$= \alpha \tilde{T}(x) + \beta \tilde{T}(z)$$

$\tilde{T}$ is bdd

T is bdd

$\|Tx_n\| \leq \|T\| \|x_n\|$ $\quad x \xrightarrow{\text{continuous}} \|x\|$

Taking $n \longrightarrow \infty$ as $\|\cdot\|$ is continuous

$\lim_{n\to\infty} \|Tx_n\| \leq \|T\| \lim_{n\to\infty} \|x_n\|$

$\|\lim_{n\to\infty} Tx_n\| \leq \|T\| \|\lim_{n\to\infty} x_n\|$

$\|\tilde{T}x\| \leq \|T\| \|x\|$

$\Rightarrow$ Taking supremum both sides.

$\|\tilde{T}\| \leq \|T\|$

$\tilde{T}$ is an extension of T from the definition of extension.

BOUNDED LINEAR FUNCTIONALS

$f: D(f) \longrightarrow \mathbb{K}$

$D(f)$ is a vector space. $\mathbb{K} \Rightarrow \mathbb{R}$ or $\mathbb{C}$

$D(f) \subset X$, X is a vector space.

$f(x+y) = f(x) + f(y)$

$f(\alpha x) = \alpha f(x)$

$x, y \in D(f)$

$\alpha \in \mathbb{K}$

Convention: Small letters $\rightarrow$ functionals; Capital letters $\rightarrow$ Operators

f is bounded if $\exists$ α s.t.

$|f(x)| \leq \alpha \|x\|$

Suppose f is bounded linear map

$\|f\| = \sup_{x \in D(x)} \frac{|f(x)|}{\|x\|}$ $\rightarrow$ Due to linear functional.

$\|f\| = \alpha$

$\boxed{\|f\| \leq \alpha \quad \|f\| \geq \alpha}$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$f(x) = a \cdot x = \sum_{i=1}^{3} a_i x_i$

$a = (a_1, a_2, a_3)$

$x = (x_1, x_2, x_3)$

Find $\|f\|$

$|f(x)| = |a \cdot x| = |\sum_{i=1}^{3} a_i x_i|$

$\leq \|a\| \|x\|$

$\Rightarrow \sup_{\substack{x \in D(x) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \leq \|a\|$ i.e.

① $\boxed{\|f\| \leq \|a\|}$

$\|f\| = \sup_{\|x\| = 1} |f(x)|$

$|f(x)| \leq \|f\| \|x\|$

$\Rightarrow \|f\| \geq \frac{|f(x)|}{\|x\|}$ $\forall x \in D(x)$

Also,

$f(a) = \|a\|^2$

$\Rightarrow \|f\| \geq \frac{|f(a)|}{\|a\|} = \frac{\|a\|^2}{\|a\|} = \|a\|$

$\Rightarrow \|f\| \geq \|a\|$ — ②

from ① & ②

$\boxed{\|f\| = \|a\|}$

(on ℓ^2)

$f: \ell^2 \longrightarrow \mathbb{R}$

$f(x) = \langle a, x \rangle = \sum_{i=1}^{\infty} a_i x_i$

$a = (a_i) \in \ell^2$

$x = (x_i) \in \ell^2$

$|f(x)| = |\langle a, x \rangle| = |\sum_{i=1}^{\infty} a_i x_i|$

$\leq \left(\sum_{i=1}^{\infty} a_i^2\right)^{1/2} \left(\sum_{i=1}^{\infty} x_i^2\right)^{1/2}$

$= \|a\| \|x\|$

$\Rightarrow \frac{|f(x)|}{\|x\|} \leq \|a\|$

$\Rightarrow \|f\| \leq \|a\|$

$f: C[a,b] \longrightarrow \mathbb{R}$

$f(x) = \int_a^b x(t)\,dt$, $x \in C[a,b]$

What is $\|f\|$?

$|f(x)| = \left|\int_a^b x(t)\,dt\right| \leq \int_a^b |x(t)|\,dt$

$\leq \int_a^b \|x\|\,dt$

$\leq \|x\| (b-a)$

$\Rightarrow |f(x)| \leq \|x\| (b-a)$

$\Rightarrow \frac{|f(x)|}{\|x\|} \leq b-a$

$\Rightarrow \sup_{\substack{x \in D(x) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \leq b-a$

$\Rightarrow \|f\| \leq b-a$ — Ⓐ

② We know that

$\|f\| \geq \frac{|f(x)|}{\|x\|}$

Take $x = 1$ as $x = 1$ is also continuous in $[a,b]$; $\|x\| = 1$

$\Rightarrow \|f\| \geq \frac{|\int_a^b dt|}{1}$

$\Rightarrow \|f\| \geq |(b-a)|$

$\Rightarrow \|f\| \geq (b-a)$ — Ⓑ

From Ⓐ & Ⓑ

$\|f\| = b-a$

Ex 1: $f_1: C[a,b] \longrightarrow \mathbb{R}$

$f_1(x) = x(t_0)$, t_0 is a that value in $[a,b]$?

$\|f_1\| = ?$

$|f_1(x)| \leq |x(t_0)| \leq \|x\|$ (sup norm)

$\|f_1\| \leq 1$ — Ⓐ

Let $x_0 = 1$

$\|f_1\| \geq \frac{|f_1(x_0)|}{\|x_0\|}$ $\longrightarrow 1$, $\longrightarrow 1$

$\Rightarrow \|f_1\| \geq 1$. — Ⓑ

$\Rightarrow \|f_1\| = 1$

X, Y be vector spaces.

$L(X, Y) \longrightarrow$ Space of linear functionals from $X \longrightarrow Y$

$T_1, T_2 \in L(X, Y)$

$(T_1 + T_2)(x) = T_1 x + T_2 x$

$(\alpha T) x = \alpha T x$

$L(X, \mathbb{R}) = X^* \longrightarrow$ Algebraic Dual.

X^* is a vector space.

$(X^*)^* \longrightarrow$ Algebraic Double Dual.

$= X^{**} =$ Space of linear functional of X^*.

$X \longrightarrow X^{**}$.

Space	General element	Value at a point
X	x	—
X^*	f	$f(x)$
X^{**}	g	$g(f)$.

$g: X^* \longrightarrow \mathbb{K}$

$g(f)$

$f: X \longrightarrow \mathbb{K}$

$g: X^* \longrightarrow \mathbb{K}$

$g(f) = f(x)$

for $x \in X$

$g(f) = g_x(f) = f(x)$ ($x \in X$ and $f \in X^*$ is variable)

g is linear in X^*.

$g(\alpha f + \beta h) = (\alpha f + \beta h)(x)$

$= \alpha f(x) + \beta h(x)$

$= \alpha g(f) + \beta g(h)$

Natural one-one map.

$C: X \longrightarrow X^{**}$

$x \longrightarrow g_x$.

4th September

$$X, X^*, X^{**}$$

Space	element	Value at a point
X	x	—
X^*	f	$f(x)$
X^{**}	g	$g(f)$

$$C : X \longrightarrow X^{**}$$
$$x \longmapsto g_x$$

C is injective

C is linear

$$g_x(f) = f(x)$$

$$g_x(\alpha f + \beta h) = (\alpha f + \beta h)(x)$$
$$= \alpha f(x) + \beta h(x)$$
$$= \alpha g(f) + \beta g(h)$$

X^* = algebraic dual space

X^{**} = algebraic double dual space

$$\boxed{X = X^{**}}$$

$B(X, Y)$

$= \{$ bounded linear operator from X into $Y \}$

X, Y are vector spaces over field $\mathbb{K}$.

$T_1, T_2 : X \longrightarrow Y$

$(T_1 + T_2)(x) = T_1 x + T_2 x$

$(\alpha T)(x) = \alpha T x.$

$B(X, Y)$ becomes a vector space

It is also a normed space.

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{\|x\|=1 \\ x \in X}} \|Tx\|$$

$$\{ B(X,Y), \|T\|, T \in B(X,Y) \}$$

If Y is complete i.e. Y is Banach then $B(X,Y)$ is Banach space

$Y = \mathbb{K}$

$B(X, \mathbb{K}) = X'.$

X' is Banach. whether X is Banach or not.

Thm: $B(X,Y)$ is a Banach space.

Proof: Let (T_n) be a cauchy sequence in $B(X,Y)$.

To show that $T_n \longrightarrow T$, $T \in B(X,Y)$.

$\epsilon > 0 \ \exists N$ s.t. $\|T_n - T_m\| < \epsilon$ for $n, m > N$

$\forall x \in X.$

$$\|T_n x - T_m x\| = \|(T_n - T_m) x\| \leq \|T_n - T_m\| \|x\|$$

$$\leq \epsilon \|x\|$$

For fixed X, any arbitrary $\tilde{\epsilon}$,

Choose $\epsilon = \epsilon_x$ s.t. $\epsilon \|x\| < \tilde{\epsilon}.$

Hence $\|T_n x - T_m x\| \leq \tilde{\epsilon}.$ — ②

i.e. $(T_n x)$ is a cauchy sequence in Y.

As Y is complete, $T_n x$ converge in Y.

Let $T_n x \longrightarrow y$

Define $y = Tx = \lim_{n\to\infty} T_n x$.

Show that T is linear.

$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$

$\Rightarrow T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y) = \lim_{n\to\infty} \alpha T_n(x) + \beta T_n(y)$

$= \alpha \cdot T(x) + \beta Ty$

$\Rightarrow T$ is linear.

From (2)

let $m \longrightarrow \infty$

$\lim_{m\to\infty} T_m x = Tx.$

$\Rightarrow \lim_{m\to\infty} \|T_n x - T_m x\| = \|T_n x - \lim_{m\to\infty} T_m x\|$

$= \|T_n x - Tx\| = \lim_{m\to\infty} \|T_n x - T_m x\| \leq \epsilon \|x\|$

As norm is a continuous

$\Rightarrow \|(T_n - T)x\| \leq \epsilon \|x\|$ bound.

$\Rightarrow T_n - T$ is bounded.

$T_n - (T_n - T)$ is an element of $B(X, Y)$

$\Rightarrow T \in B(X, Y)$

T is a bounded linear map

$T : X \longrightarrow Y.$

$\|T_n x - Tx\| = \lim_{m\to\infty} \|T_n x - T_m x\| \leq \epsilon \|x\|$ — (2)

Take supremum in both sides $\|x\| = 1$

$\Rightarrow \sup_{\|x\|=1} \|T_n x - Tx\| \leq \epsilon.$

$\Rightarrow \|T_n - T\| \leq \epsilon. \Rightarrow T_n \longrightarrow T. \ \forall \epsilon > 0$

i.e. $T_n \longrightarrow T$
$T \in B(X, Y)$

"MOST POWERFUL RESULT IN FUNCSIONAL ANALYSEES"
— LORD NAHAK

Corollary $X' = B(X, \mathbb{K})$ is a Banach Space.

Proof: Set $Y = \mathbb{K}$, which is Banach, Hence the thm follows.

$$X \longleftrightarrow \tilde{X}.$$

If 2 spaces are isomorphic, their "behaviour" in terms of operations is same.

$$\ell^{1'} = \ell^{\infty}, \quad \ell^{p'} = \ell^{q}$$

$$\frac{1}{p} + \frac{1}{q} = 1 \quad p > 1, q > 1$$

5th September

C.S. → Cauchy Schwartz

$X \quad \tilde{X}$

$T : X \longrightarrow \hat{X}$

linear bijective

and $\|Tx\| = \|x\|$

We say T is an Isomorphism

X and $\hat{X}$ are Isomorphic to each other.

$\Rightarrow X = \tilde{X}$

Dual of $\mathbb{R}^n = \mathbb{R}^{n'} = ?$

we show that $\mathbb{R}^{n'} \cong \mathbb{R}^n$

Isomorphism

There is an isomorphism from $\mathbb{R}^{n'} \longrightarrow \mathbb{R}^n$.

$\mathbb{R}^{n*} =$ Space of linear functionals on $\mathbb{R}^n$.

$\parallel$

$\mathbb{R}^{n'}$

Proof: We know any linear map on a finite dimensional vector space is bounded.

$\dim(\mathbb{R}^n) = n$

Hence $\mathbb{R}^{n*} = \mathbb{R}^{n'}$ → BL (Bounded linear)

Take Linear

Any $f \in \mathbb{R}^{n*}$

$$f(x) = \sum_{i=1}^{n} a_i x_i \qquad a = (\alpha_i) \in \mathbb{R}^n \quad x = (x_i) \in \mathbb{R}^n$$

$$= \langle a, x \rangle.$$

(Basically taking any elt from the dual space

Let $x \in \mathbb{R}^n$ and $\{e_1, e_2, \ldots\ldots e_n\}$ be a basis for $\mathbb{R}^n$.

For any element of $x \in \mathbb{R}^n$ can be written as

$$x = \sum_{i=1}^{n} x_i e_i$$

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i)$$

Let $\gamma_j = f(e_j)$

$$\Rightarrow f(x) = \sum_{i=1}^{n} x_i \gamma_i$$

$$\Rightarrow |f(x)| = \left|\sum_{i=1}^{n} x_i \gamma_i\right| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} \gamma_i^2\right)^{1/2}$$

(Cauchy schwartz inequality)

$$\Rightarrow |f(x)| \leq \|x\|_2 \left(\sum_{i=1}^{n} \gamma_i^2\right)^{1/2}$$

Taking sup $\|x\| = 1$ both sides

$$\Rightarrow \|f\| \leq \left(\sum_{i=1}^{n} \gamma_i^2\right)^{1/2} \quad \text{— (A)}$$

$\gamma = (\gamma_j) \in \mathbb{R}^n$

$\gamma_j = f(e_j)$

$$\boxed{\begin{array}{l}\text{Taking } x = (\gamma_j) \\ \text{C.S. Inequality becomes equality}\end{array}}$$

i.e. $$\|f\| = \left(\sum_{j=1}^{n} \gamma_j^2\right)^{1/2}$$

Define T (linear map)

$T : \mathbb{R}^{n'} \longrightarrow \mathbb{R}^n$

$f \longmapsto c = (\gamma_j)$

$\gamma_j = f(e_j)$

$\Rightarrow T(f) = (\gamma_j)$

To show T is linear

TP $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$.

$$T(\alpha f + \beta g) = (\alpha \gamma_j + \beta J_j) = \alpha(\gamma_j) + \beta(J_j)$$
$$= \alpha T(f) + \beta T(g)$$

$\gamma_j = f(e_j)$

$J_j = g(e_j)$

Null of T

$Z(T) = \{ f \in \mathbb{R}^{n'} \mid T(f) = 0 \}$

$\Rightarrow (\gamma_j) = 0$

i.e. $\forall j \; \gamma_j = 0$

$\Rightarrow f = 0$

T is also onto.

2-10.6 The dual space of ℓ^1 is ℓ^∞.

$$\ell^{1'} \cong \ell^\infty.$$

Take $f \in \ell^{1'}$

Then $f \longmapsto c = (\gamma_j)$

$(\gamma_j) \in \ell^\infty.$

(Proving that the isomorphism preserves distance)

$$\|f\| = \|c\|_\infty$$

if $x \in \ell^1_\infty$ $x = \sum_{i=1} x_i e_i$ $e_i = \delta_{ki}$

Proof: Let $f \in \ell^{1'}$ and let (e_k) be a Schauder Basis.

$e_1 = (1, 0, 0 \dots)$
$e_2 = (0, 1, 0 \dots)$
$e_3 = (0, 0, 1 \dots)$

$\|e_k\| = 1 \quad \forall \; k = 1, 2, \dots$

$$f(x) = f\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{\infty} x_i f(e_i)$$

Take $f(e_i) = \gamma_i$ $\Rightarrow |f(x)| = \left|\sum_{i=1}^{\infty} x_i \gamma_i\right|$

$$\Rightarrow |f(x)| = \left|\sum_{i=1}^{\infty} x_i \gamma_i\right|$$

$$|\gamma_j| = |f(e_j)| \leq \|f\| \, \|e_j\|$$

As $\|e_j\| = 1$

$$\Rightarrow |f(e_j)| \leq \|f\|$$

$$\gamma \in \ell^\infty.$$