Measure theory & Integration – PART-II

Borel-Cantelli Lemma :-

$\{E_k\}$ be countable family of measurable subsets of $\mathbb{R}^n$ & let $\sum_{n=1}^{\infty} m(E_n) < \infty$. Let

$$E = \limsup_{n\to\infty} (E_n)$$

$$= \left\{ x \in \mathbb{R}^n \;\middle|\; x \in E_n \text{ for infinitely many } n \right\}$$

Then,

(a) E is measurable

(b) $m(E) = 0$

Proof. a) $E = \bigcap_{N=1}^{\infty} \bigcup_{k \geq N} E_k$

Since each E_k is measurable, thus E is also measurable.

b) $m(E)$

$$= \lim_{N \to \infty} m\left(\bigcup_{k=N}^{\infty} E_k\right) \quad \left[\text{Continuity of measure}\right]$$

$$\leq \lim_{N \to \infty} \underbrace{\sum_{k=N}^{\infty} m(E_k)}$$

$$= 0$$ because tail of a convergent series must go to zero.

We know
$m(E) \geq 0$

we conclude $m(E) = 0$. ✓

Every Borel set is measurable.

$$\mathcal{B}_{\mathbb{R}^n} \subsetneq \checkmark \ \mathcal{L}_{\mathbb{R}^n}$$

$\mathcal{B}_{\mathbb{R}^n}$ ↑ Borel σ-algebra

$\mathcal{L}_{\mathbb{R}^n}$ ↑ σ-algebra of all measurable sets.

$$|\mathbb{R}| = |(0,1)| \qquad \underline{\underline{(\Omega, \mathcal{F}, P)}}$$

$C \to$ Cantor set

$|C| = |(0,1)|$, because there is a bijection between C & $(0,1)$.

$$|2^C| = |2^{(0,1)}|$$

$$2^C \subseteq \mathcal{L}_{\mathbb{R}}.$$

$\underline{A \in 2^C}$ $m(C) = 0$

$\Rightarrow A \subseteq C$

$0 \le m(A) \le m(C) = 0$

$\Rightarrow m(A) = 0$

$\Rightarrow A$ is meas. $\Rightarrow A \in \mathcal{L}_{\mathbb{R}}$

$$\Rightarrow |2^C| \le |\mathcal{L}_{\mathbb{R}}|$$

$$|(0,1)| \lneq |2^{(0,1)}| = |2^C| \le |\mathcal{L}_{\mathbb{R}}| \quad \text{—— } (*)$$

Now one can show that –

$$|\mathcal{B}_{\mathbb{R}}| \le |(0,1)| \quad \text{—— } (**)$$

$(*)$ & $(**)$ $\Rightarrow$

$$|\mathcal{B}_{\mathbb{R}}| < |\mathcal{L}_{\mathbb{R}}|$$

$$\Rightarrow \quad \mathcal{B}_{\mathbb{R}} \subsetneq \mathcal{L}_{\mathbb{R}}.$$

(strict inclusion) proper subset.

Lebesgue Integration

$\{\phi_n\} \longrightarrow f$ a.e. x.

1) If $\lim_{n\to\infty} \int \phi_n$ exists, define

$$\int f = \lim_{n\to\infty} \int \phi_n.$$

2) If $f = 0$ a.e., then

$$\lim_{n\to\infty} \int \phi_n = 0.$$

✓

done ✓

Bounded Convergence theorem (BCT)

MCT

DCT

Statement :- Suppose $\{f_n\}$ is a sequence of measurable functions s.t. $|f_n| \leq M \;\; \forall x \in E$, $m(E) < \infty$.

Assume $f_n(x) \longrightarrow f(x)$ a.e. x as $n \to \infty$

Then, f is measurable, f is bounded.

(if we assume that f_n's are supported on E, then f is also supported on E)

Most importantly,

$$\int |f_n - f| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

$$\left(\left| \int (f_n - f) \right| \leq \int |f_n - f| \longrightarrow 0 \quad \text{as } n \to \infty \right.$$

$$\Rightarrow \left| \int (f_n - f) \right| \longrightarrow 0 \quad \text{''}$$

$$\Rightarrow \int f_n - \int f \longrightarrow 0 \quad \text{''}$$

$$\Rightarrow \int f_n \longrightarrow \int f \quad \text{''}$$

$$\Rightarrow \lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = \int f.$$

<u>Proof</u> :- $|f_n - f| < N_0$;

Let $\varepsilon > 0$, applying Egorov's thm, we can find $A_\varepsilon \subseteq E$ s.t. $m(E \setminus A_\varepsilon) \le \varepsilon$ & $f_n \to f$ converges uniformly on A_ε.

<u>i.e.</u>

$$|f_n(x) - f(x)| \le \varepsilon \quad \forall\, x \in A_\varepsilon$$

$$\int |f_n(x) - f(x)|$$

$$\le \int_{A_\varepsilon} |f_n(x) - f(x)| + \int_{E\setminus A_\varepsilon} |f_n(x) - f(x)|$$

$$\le \varepsilon\, m(A_\varepsilon) + 2M\, m(E \setminus A_\varepsilon)$$

Since ε is arbitrary, $\varepsilon \to 0$,

$$\int |f_n(x) - f(x)| \longrightarrow 0$$

$\square$

$$|f_n| \le M$$

$$f_n \longrightarrow f$$

$$\begin{aligned} |f| &= |f - f_n + f_n| \\ &\le |f - f_n| + |f_n| \\ &\le M + M \\ &= 2M. \end{aligned}$$

Corollary