Lecture 1

Measure Theory & Integration

Book: ① Measure Theory & Integration
by G. de Barra.

② Real Analysis: Measure Theory,
Integration & Hilbert Space.
by E. M. Stein & Rami Shakarchi.

$X \neq \phi$ set.

$A, B \subseteq X$.

$A \setminus B := \{ x \in A \,/\, x \notin B \}$
$= A \cap B^c$.

$A \Delta B := (A \setminus B) \cup (B \setminus A)$
the symmetric difference of A & B.

Properties.

① $A \Delta B = B \Delta A$.

② $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

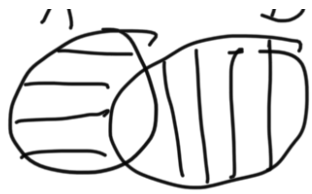
③ $(A \Delta B) \Delta (C \Delta D) = (A \Delta C) \Delta (B \Delta D)$

④ $\left(\bigcup_{i=1}^{n} E_i\right) \Delta \left(\bigcup_{i=1}^{n} F_i\right) = \bigcup_{i=1}^{n} (E_i \Delta F_i)$.

A B

proof:-

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$
$$= (A \cap B^c) \cup (B \cap A^c).$$



$A^c = X \setminus A.$

② Consider

$$(A \Delta B)^c = \left((A \cap B^c) \cup (B \cap A^c) \right)^c$$
$$= (A^c \cup B) \cap (B^c \cup A)$$
$$= (A^c \cap B^c) \cup (A \cap B).$$

$$(A \Delta B) \Delta C = \left((A \Delta B) \cap C^c \right) \cup \left((A \Delta B)^c \cap C \right)$$

(by def.)

$$= \left(\left((A \cap B^c) \cup (B \cap A^c) \right) \cap C^c \right) \cup$$
$$\left(\left((A^c \cap B^c) \cup (A \cap B) \right) \cap C \right).$$

$$= \underline{(A \cap B^c \cap C^c)} \cup \underline{(A^c \cap B \cap C^c)} \cup \underline{(A^c \cap B^c \cap C)}$$
$$\cup \underline{(A \cap B \cap C)}.$$

$\therefore$ By Symmetry this is equal to $A \Delta (B \Delta C)$.

$$\overset{\shortparallel}{(B \Delta C) \Delta A.}$$

$$\therefore (A \Delta B) \Delta C = A \Delta (B \Delta C).$$

③ $(A \Delta B) \Delta (C \Delta D) = ((A \Delta B) \Delta C) \Delta D$

$$= (A \Delta (B \Delta C)) \Delta D$$

(by ②)

$$= ((B \Delta C) \Delta A) \Delta D.$$

$$= (B \Delta C) \Delta (A \Delta D)$$

$$= (A \Delta D) \Delta (B \Delta C).$$

④ EXERCISE.

$B \subset A$

$$\boxed{\begin{aligned} A \Delta B &= (A \setminus B) \cup \underbrace{(B \setminus A)}_{= \phi} \\ &= \underline{A \setminus B} \end{aligned}}$$

Recall:- Let $E_1 \supseteq E_2 \supseteq \cdots\cdots$ Then

$$\bigcup_{i=1}^{\infty} (E_1 \setminus E_i) = E_1 \setminus \bigcap_{i=1}^{\infty} E_i$$

Pf:- EXERCISE.

Defn An equivalence relation R on a set E is a subset of $E \times E$ with the following properties.

(i) $(x, x) \in R$ for any $x \in E$. (reflexive)

(ii) $(x,y) \in R \Rightarrow (y,x) \in R$ (Symmetric).

(iii) if $(x,y), (y,z) \in R$, then $(x,z) \in R$.
(transitive).

We also write $x \sim y$ if $(x,y) \in R$.

Then R partitions E into disjoint equivalence classes such that x & y are in the same class if and only if $x \sim y$.

$$[x] = \{ y \in E \mid y \sim x \}$$

$$\bigcup [x] = E.$$

Axiom of choice :-

If $\{E_\alpha\}_{\alpha \in A}$ is a non-empty Collection of non-empty disjoint subsets of a set X, then there exists a set $V \subseteq X$ Containing just one element from each E_α.

Recall metric spaces.

Let X be a non-empty set.

A map $d: X \times X \to \mathbb{R}$ such that

- $d(x,y) \geq 0 \quad \forall x,y \in X$
- $d(x,y) = 0 \iff x = y.$
- $d(x,y) = d(y,x) \quad \forall x,y \in X.$
- $d(x,z) \leq d(x,y) + d(y,z)$

$\forall x,y,z \in X.$

d is called a metric on X.

& (X, d) is called a metric space.

(also a topological space).

Ex ① $(\mathbb{R}, d)$, $d(x,y) = |x-y| \quad \forall x,y \in \mathbb{R}.$

metric space.

② $(\mathbb{R}^n, d)$, $d(\underline{x}, \underline{y}) = \sqrt{|x_1-y_1|^2 + \cdots + |x_n-y_n|^2}$

$\forall \underline{x}, \underline{y} \in \mathbb{R}^n.$

metric space.

③ $X \neq \phi$ set. Define $d: X \times X \to \mathbb{R}$

or $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y. \end{cases}$

$\forall x, y \in X.$

(X, d) is a metric space called "discrete space".

Let (X, d) be a metric space.

Define a ball in X with centre at $x \in X$ & radius $r > 0$

is $B(x, r) := \{ y \in X \mid d(x,y) < r \}$

Also called an open ball in X.

closed ball $\overline{B(x,r)} = \{ y \in X \mid d(x,y) \leq r \}.$

Examples ① $(\mathbb{R}, d)$ $d(x,y) = |x-y|$. usual metric.

$$B(x, r) = \{ y \in \mathbb{R} \mid |x-y| < r \}$$

$$= (x-r, x+r) \text{ open interval.}$$

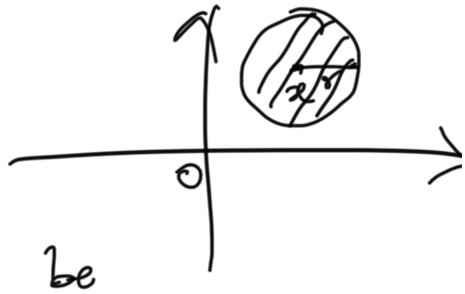
② $(\mathbb{R}^2, d)$, $d(\underline{x}, \underline{y}) = \sqrt{|x_1-y_1|^2 + |x_2-y_2|^2}$

$\underline{x} = (x_1, x_2)$

$\underline{y} = (y_1, y_2).$

$$B(\underline{x}, r) = \left\{ \underline{y} \in \mathbb{R}^2 \,\middle|\, \sqrt{|x_1-y_1|^2 + |x_2-y_2|^2} < r \right\}$$

$$=$$



Def:- A subset $A \subseteq X$ is said to be an <u>open set</u> if given any $x \in A$, there exists $\varepsilon > 0$ such that the open ball around x

$$B(x, \varepsilon) \subseteq A.$$

Def:- A subset $A \subseteq X$ is called a <u>closed set</u> if its complent is an open set

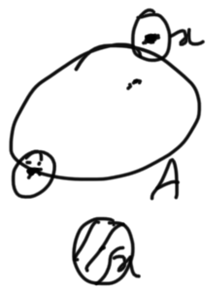
i.e, A^c is an open set.

Def:- The <u>closure</u> of a set $A \subseteq X$ is defined as $\overline{A} = \bigcap$ all closed sets containing A.

$$= \bigcap_{\substack{V \supseteq A \\ V \subseteq X \\ \text{closed set}}} V$$

Def:- A point $x \in X$ is called a <u>limit point</u> of a subset A of X, if given $\varepsilon > 0$, there exists $y \in A$, $y \neq x$ such that
$$d(x,y) < \varepsilon.$$

i.e., $$\boxed{\left(B(x,\varepsilon) \setminus \{x\}\right) \cap A \neq \phi.}$$



Def:- A subset $A \subseteq X$ is said to be <u>dense</u> if $\overline{A} = X$.

Def:- A subset $A \subseteq X$ is called <u>nowhere dense</u> if $\overline{A}$ contains no non-empty open set.

Def:- A subset A is said to be a <u>perfect</u> set if $\{x \in X \,/\, x \text{ is a limit pt. of } A\} = A$.

Eg:- Every openset is a perfect set.
$[a,b] \subset \mathbb{R}$ is perfect.

Result:- ① $\overline{A} = A \cup \{\text{the set of all limit points of } A\}$.

Eg:- $[a, b] = [a, b]$

(2) Arbitrary union of open sets is open.
i.e., if $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets in X, then $\bigcup_{\alpha \in I} V_\alpha$ is also an open set.

(2) Arbitrary intersection of closed sets is also closed. i.e., $\{V_\alpha\}_{\alpha \in I}$ is a collection of closed sets in X, then $\bigcap_{\alpha \in I} V_\alpha$ is also closed.

(3) Finite intersection of open sets is open.

(4) Finite union of closed sets is closed.

Examples:- (1) $\underbrace{\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)}_{\text{open sets}} = \underbrace{\{0\}}_{\text{closed set}}$

(2) $\underbrace{\bigcup_{n=1}^{\infty} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]}_{\text{closed}} = \underbrace{(-1, 1)}_{\text{open}}$.