Tutorial 7 Discussion

Defn Let V be a subset of $\mathbb{C}^n$.

V is said to be a variety if

$$V = Z(f_1, \dots, f_r)$$

where $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_n]$.

and $Z(f_1, \dots, f_r)$ denotes the set of common zeros of the polys $f_1, \dots, f_r$.

$$\{(x, y) \mid y = e^x, x \in \mathbb{C}\}$$

$$\subseteq \mathbb{C}^2.$$

This is not a variety.

$f(x,y) = x^2 + y^2 - 9$. $Z(f)$ is a variety in $\mathbb{C}^2$.

$f_1 = x^2 + y^2 - 1$, $f_2 = x^2 - y + 1$.

$Z(f_1, f_2)$ is a variety.

↳ set of common zeros of f_1 & f_2.

Let $f_1, f_2, \ldots, f_r$

$V = Z(f_1, \ldots, f_r)$

<u>Q1</u>. Is V is non-empty?

<u>Q2</u>. If V is non-empty then is it finite?

Grobner bases technique is useful to study the soln.

$\mathbb{C}[x,y]$.

$I = (x^2+y^2-1)$.

$V = Z(\underline{\underline{x^2+y^2-1}})$.

$\mathbb{C}[x,y]/I$.

$(x-a_1, y-a_2)/I$.

s.t $(x-a_1, y-a_2) \supseteq I$.

$\underline{\underline{(x-1, y)}}/I \rightsquigarrow \underline{\underline{(1,0)}}$

$Z(f_1, \ldots, f_n) = \phi$

means there is no maximal ideal containing $I = (f_1, \ldots, f_n)$

Therefore I is the whole ring

$\underline{I} = (1)$.

$(f_1, \ldots, f_n) = (1)$.

$\underline{1} = g_1 f_1 + \cdots + g_n f_n$.

Tut 7

$\boxed{\underline{\underline{Q\,15}}}$

$\mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z}\}$.

$$\mathbb{Z}[i] \cong \frac{\mathbb{Z}[x]}{(x^2+1)}.$$

$$\mathbb{Z}[i] \Big/ (3) \cong \frac{\mathbb{Z}[x]/(x^2+1)}{3\ \mathbb{Z}[x]/(x^2+1)}$$

$$\boxed{\mathbb{Z}[i] \cong \frac{\mathbb{Z}[x]}{(x^2+1)}} \cong \frac{\mathbb{Z}[x]/(x^2+1)}{(3, x^2+1)/(x^2+1)}$$

$$\cong \frac{\mathbb{Z}[x]}{(3, x^2+1)}$$

$$\boxed{\mathbb{Z}[x]\Big/ p \cong \mathbb{Z}/p\mathbb{Z}[x]} \cong \frac{\mathbb{Z}[x]/(3)}{(3, x^2+1)/(3)}$$

$\downarrow$

<u>Ex</u> Prove it

$$\cong \frac{\mathbb{Z}/3\mathbb{Z}[x]}{(x^2+\bar{1})}. \text{ field.}$$

Now $x^2+\bar{1}$ is irreducible over $\mathbb{Z}/3\mathbb{Z}[x]$ hence the quotient is a field.

Thus (3) is a maximal ideal

Ex. Show that $\mathbb{Z}[i]/(1+3i) \cong \mathbb{Z}/10\mathbb{Z}$.

An

$$\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}[i]/(1+3i) = \overline{R}$$

$$\varphi(1) = 1 + (1+3i).$$

$$\varphi(n) = n + (1+3i).$$

In $\overline{R}$,

$$1+3i = 0$$
$$\Rightarrow 3i = -1.$$
$$\Rightarrow 3i^2 = -i.$$
$$\Rightarrow i = 3.$$

$$i^2 = -1$$
$$\Rightarrow 3^2 = -1$$
$$\Rightarrow 10 = 0$$

$$a+ib+(1+3i) = a+3b+(1+3i),$$

$\therefore$ φ is surjective

Let $n \in \ker \varphi$.

then $n \in (1+3i)$

$\Rightarrow \quad n = (1+3i)(a+ib)$,

$\Rightarrow \quad n = (a-3b) + (3a+b)i$

$\Rightarrow \quad 3a+b = 0$

$\Rightarrow \quad b = -3a$.

$\therefore \quad n = a(1+3i)(1-3i) = 10a$.

$n \in 10\mathbb{Z}$.

$\therefore \quad \ker \varphi \subseteq 10\mathbb{Z}$.

$10\mathbb{Z} \subseteq \ker \varphi$

$\therefore \quad \ker \varphi = 10\mathbb{Z}$.

Hence $\quad \mathbb{Z}[i]/(1+3i) \cong \mathbb{Z}/10\mathbb{Z}$

Q9. $\mathbb{Z}[i] \big/ (3+i) \cong \mathbb{Z}/10\mathbb{Z}$.

$3+i=0$.

$\Rightarrow i=-3$

$\Rightarrow 10=0$.

Q10. $\varphi: R \longrightarrow \mathbb{R}$

$\varphi(f) = f(0)$.

$\ker\varphi = I \therefore R/\ker\varphi \cong \mathbb{R}$

field.

$\therefore$ I is maximal ideal.

Q4. $R = \dfrac{\mathbb{F}_5[x]}{(x^2+x+1)}$.

Q7. $I = (y^2+x^3-17) \subset \mathbb{C}[x,y]$.

$$R/I \rightsquigarrow m_a/I,$$

$a \in Z(y^2+x^3-17)$.

$$m_{\underline{a}} = (x-a_1,\ y-a_2)/I.$$

s.t $\quad a_2^2+a_1^3-17=0.$

Q3. $\left\{ \frac{f}{g} \;\middle|\; f, g \in R[[x]] \text{ and } g \neq 0 \right\}$.

Ex. If R is a field. can you give more explicit description of the quotient field.