Nichomachus Theorem.

$1^3 = 1$

$2^3 = 3+5$

$3^3 = 7+9+11$

$4^3 = 13+15+17+19.$

$\vdots$

$n^3 = ?$

Prove that $\Rightarrow 1^3+2^3+3^3+\cdots+n^3 = (1+2+\cdots+n)^2$

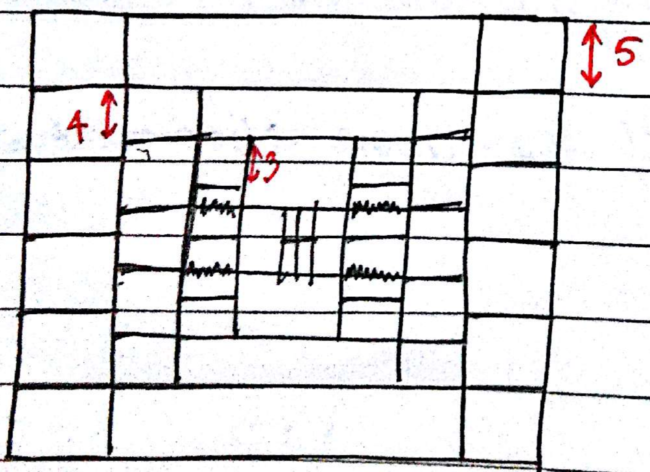
$$1+(3+5)+\cdots+(\cdots+\cdots+(n^2+n-1)) = \left\{\frac{n(n+1)}{2}\right\}^2$$

using induction $\rightarrow$ also can be proved.
$\hookrightarrow$ fails to find the formula.

Geometrically. $\rightarrow$ consider a square of length of side $n(n+1)$

eg:- $n=5$

a square having side length 5×6.



$n(n+1)$
$\downarrow$
$(n-1)n$
$\downarrow$
$(n-2)(n-1)$
$\vdots$
$1\times 2.$

Area $= \{2(1+2+3+4+5)\}^2$

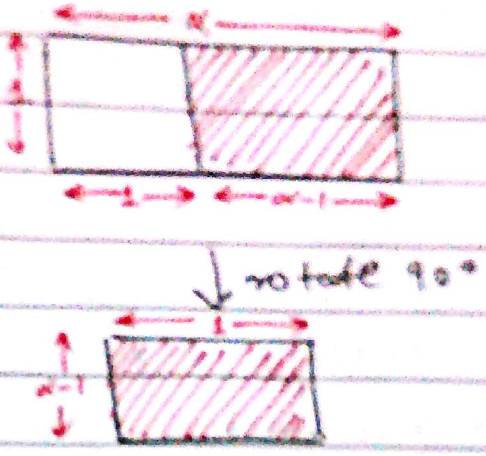
Area $= 4\times 1^2 + 8\times 2^2 + \cdots + 20\times 5^2$

$= 4\times 1^3 + 4\times 2^3 + \cdots + 4\times 5^3 = 4(1^3+2^3+\cdots+5^3)$

Example

$$\alpha = \frac{1+\sqrt{5}}{2} = 1.61803\ldots$$

(Golden Ratio)



$\alpha \times 1$

↓

take out 1×1 square

↓

remaining $1 \times (\alpha - 1)$ rectangle

↓

rotate 90° has the same ratio as that of the original one

$$\frac{\alpha}{1} = \frac{1}{\alpha - 1}$$

$$\Rightarrow \boxed{\alpha^2 - \alpha = 1}$$

Binet formula

$$\boxed{F(n) = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)}$$

$$\alpha = \frac{1+\sqrt{5}}{2} = 1.61803\ldots$$

$$\beta = \frac{1-\sqrt{5}}{2} = -0.61803$$

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1. \\ F(n-1) + F(n-2) & \text{, if } n \geq 2. \end{cases}$$

$f_0 = 0$, $f_5 = 5$
$f_1 = 1$, $f_6 = 8$
$f_2 = 1$, $f_7 = 13$
$f_3 = 2$, $f_8 = 21$
$f_4 = 3$, $f_9 = 34$.

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Proof that:

$$\boxed{\alpha^{n-2} < F(n) \leq \alpha^{n-1}}$$

proof using Induction.

Let $P(n)$ be the statement that

$F(n) > \alpha^{n-2}$, for $n \geq 3$.

Basis step:

$F(3) = 2 > \alpha = \alpha^{3-2}$

$P(3)$ is True

Inductive step.

Let $P(3), P(4), \ldots, P(k)$ be true.

claim →. $P(k+1)$ is true.

$P(k+1) = P(k) + P(k-1) > \alpha^{k-2} + \alpha^{k-3}$

$> \alpha^{k-3}(\alpha+1) > \alpha^{k-3}(\alpha^2) = \alpha^{k-1} = \alpha^{(k+1)-2}$

Hence proved.

Lame's Theorem.

a, b be two integers with $a \geq b$. Then the # of divisors used by the Euclidean algorithm to find gcd (a, b) is less than or equal to $5(\log_{10} b + 1)$.

Proof. → Let $a = r_0$, $b = r_1$

Divison algorithm ⇒

'n' steps

$$\begin{cases} r_0 = r_1 q_1 + r_2, & 0 \leq r_2 < r_1 \\ r_1 = r_2 q_2 + r_3, & 0 \leq r_3 < r_2 \\ \vdots \\ r_{n-2} = r_{n-1} \cdot q_{n-1} + r_n, & 0 \leq r_n < r_{n-1} \\ r_{n-1} = r_n q_n + r_{n+1}, & 0 \leq r_{n+1} < r_n \end{cases}$$

2, ..., n.

$$\begin{array}{c|c|c} \overset{r_1}{\overset{\|}{b}} & \overset{r_0}{\overset{\|}{a}} & q_1 \end{array}$$

$\vdots$

$r_2 \mid r_1 \mid q_2$

$\vdots$

$r_3 \mid r_2 \mid q_3$

$\vdots$

Thus, we have, $r_n \geq 1 = f_2$

$r_{n-1} \geq r_n \cdot q_n \geq 2 \cdot r_n \geq 2 f_2 = f_3.$

$r_{n-2} \geq r_{n-1} + r_n \geq f_2 + f_3 = f_4.$

$\vdots$

$r_2 \geq r_3 + r_4 \geq f_{n-1} + f_{n-2} = f_n.$

$b = r_1 \geq r_2 + r_3 \geq f_n + f_{n-1} = f_{n+1}.$

$\Rightarrow \log_{10}(b) > (n-1) \log_{10} \alpha$ ← some approximation

$\sim 0.208 > \frac{1}{5} = 0.2.$

$\Rightarrow \log_{10}(b) > \frac{n-1}{5}$

$\therefore n - 1 < 5 \log_{10}(b)$

'b' has suppose 'k' digits.

$\Rightarrow b < 10^k$

$\Rightarrow \log_{10}(b) < k.$

$\Rightarrow k = \lfloor \log_{10}(b) + 1 \rfloor$

$\leq (\log_{10}(b) + 1)$

$n < 5k = 5(\log_{10}(b) + 1).$

NEXT CLASS → Huffman Codes (Greedy Algorithm), use mathematical induction to prove how greedy approach of Huffman codes give optimal solution.

EEK-11
EDAY
2017

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- no. of distinct rooted unlabeled binary tree with n nodes = catalan no. $= \frac{1}{n+1}\binom{2n}{n}$

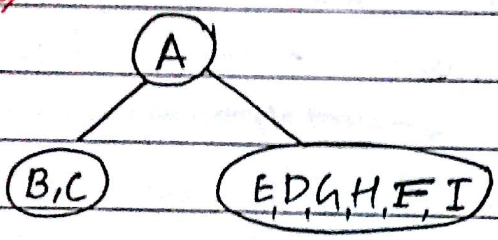
- no. of different ways to multiply n-matrices

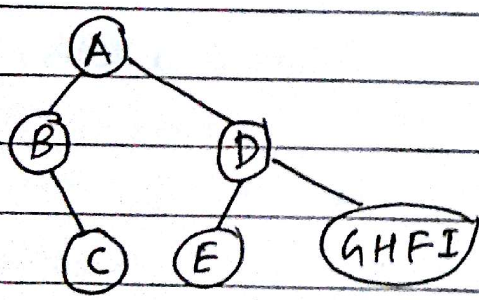
- given a preorder permutation of a n-node binary tree → no. of distinct inorder permutations.

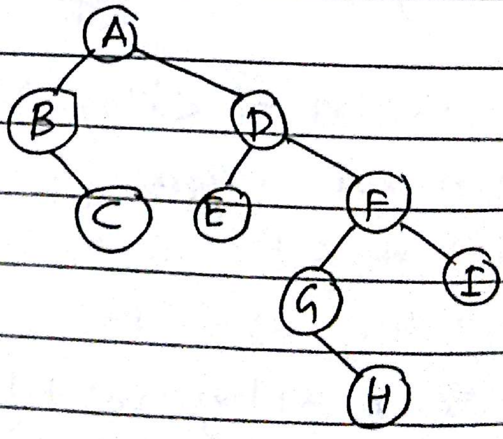
Example:→

preorder → ABCDEFGHI (VLR)
inorder → BCAFDGHFI (LVR)

ACT → The pair (preorder, inorder) sequence $\xrightarrow[\text{defines}]{\text{uniquely}}$ a binary tree.
proof (Exercise)







Generalized Induction.
↳ Extension of Mathematical induction to other sets besides N where well-ordering principle hold.

eg.- N×N

It can be extended to N×N×N×- - - as well.

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Lexicographic Ordering:

$\times N$ (x_1, y_1) less than (x_2, y_2)
if either $(x_1 < x_2)$ or $(x_1 = x_2$ and $y_1 < y_2)$

$z_1^{x_1} \cdot z_2^{y_1} + z_1^{x_2} \cdot z_2^{y_2} \rightarrow$ multivariate polynomial.

Example:

$m, n \in N \times N$

$a_{m,n}$ defined as:

$$a_{0,0} = 0$$

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1 & \text{if } n=0 \text{ and } m>0 \\ a_{m,n-1} + n & \text{if } n>0. \end{cases}$$

Show that $a_{m,n} = m + \frac{n(n+1)}{2} \quad \forall (m,n) \in N \times N.$

Solution $\rightarrow$ (Using variation of Mathematical Induction).

Base step $\rightarrow$ $a_{0,0} = 0.$

$m + \frac{n(n+1)}{2} = 0$ at $(m,n) = (0,0)$

$= a_{0,0}$

Inductive step $\rightarrow$ Let $a_{m',n'} = m' + \frac{n'(n'+1)}{2}$

where (m', n') is less than (m,n).

Case 1 $\rightarrow$ $n = 0$

$a_{m,n} = a_{m-1,n} + 1.$; $(m-1,n)$ less than (m,n)

$\hookrightarrow$ induction hypothesis holds.

so we can go on proofing.

Case 2. $\rightarrow$ $n > 0$

$a_{m,n} = a_{m,n-1} + n$; $(m, n-1)$ less than (m,n)

$\hookrightarrow$ induction hypothesis holds.
so we can go on proofing.

Recurrance Relations.

Example. → $\{a_r\}$ ← any sequence

Sequence Natural nos. to Real nos.

(discrete) Numeric function

$f: N \rightarrow R.$

these Numeric functions are solution of recurrence relations

$a_r = 3^r \quad , r \geq 0$

- ordinary generating function

$$A(z) = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots$$
$$= 1 + 3z + 3^2 z^2 + 3^3 z^3 + \cdots$$
$$= \frac{1}{1-3z}$$

for $\boxed{a_r = 3^r} \longrightarrow \boxed{a_r = 3a_{r-1}}$

Always it is not easy to get the closed form of the recurrence relation. eg:- fibonnaci series. (not that obvious, closed form exists for Fibbonnaci series)

Example →

$\boxed{\begin{array}{l} F_{n+1} = F_n + F_{n-1} \\ F_0 = 0 \\ F_1 = 1 \end{array}}$ ← 1, 1, 2, 3, 5, 8, 13, 21, 34, - - - -

- $a_r = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ {general expression}.
- $A(z) = \frac{1}{1-z-z^2}$ {closed form of generating function}.

→ how to solve recurrance relation with constant coefficients.

There are several Discrete computational Problems that involves Recurrance Relations.

<u>Example</u> → { Legitimate code words }.

$\{0, 1, 2, 3\}$.

- codewords from the alphabet $\{0,1,2,3\}$ are to be recognized as legitimate iff they have an even no. of zeros.
- how many legitimate codewords are there of length k?

<u>Solution</u> → let a_k = no. of legitimate codewords of length k.

consider $(k+1)$ length codewords.

- ↳ starts with 1 or 2 or 3 → $3a_k$.
- ↳ starts with 0. → $4^k - a_k$.

$$\cancel{a_{k+1} = \begin{cases} 3\cdot a_k + (4^k - a_k), & \text{if start with zero} \\ 3a_k, & \text{if starts with 1 or 2 or 3} \end{cases}}$$

$\therefore \; a_{k+1} = 3\cdot a_k + (4^k - a_k)$.

$\boxed{a_{k+1} = 2a_k + 4^k}$; $a_1 = 1$.

↳ not a homogeneous recurrance relation.

<u>NEXT CLASS</u> → 2 methods of solving Recurrance Relation.

WEEK-12

MONDAY

20.3.2017

Example:→ $\Sigma = \{0, 1, 2, 3\}$.

$a_{k+1} = 2 \cdot a_k + 4^k \quad , a_1 = 3$.

a_k = # code words of length 'k' with even # of zeros.

Solving of the Recurrence Relation using Generating functions

Ordinary Generating function for $\{a_k\}$

$$G(x) = \sum_{k=0}^{\infty} a_k \cdot x^k.$$

$$= a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$$

Need to define 'a_0' so that (1) holds

$k=0 \rightarrow a_1 = 2a_0 + 4^0$

$\Rightarrow 3 = 2a_0 + 1 \Rightarrow a_0 = 1.$

Then (1) holds for all $k \geq 0$.

$$\sum_{k=0}^{\infty} a_{k+1} x^k = \sum_{k=0}^{\infty} 2 \cdot a_k \cdot x^k + \sum_{k=0}^{\infty} 4^k \cdot x^k$$

$\downarrow$

$\Rightarrow a_1 + a_2 x + a_3 \cdot x^2 + \cdots$

$\Rightarrow \frac{1}{x}(G(x) - a_0)$

$$\Rightarrow \frac{1}{x}[G(x) - a_0] = 2G(x) + \frac{1}{1-4x}.$$

$$\Rightarrow G(x) = \frac{x}{(1-2x)\cdot(1-4x)} + \frac{1}{1-2x}.$$

$$= \frac{1/2}{1-4x} + \frac{1/2}{1-2x}$$

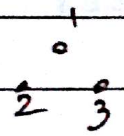
$$\boxed{a_k = \frac{1}{2}\cdot 4^k + \frac{1}{2}\cdot 2^k.} \quad \forall \; k \geq 0.$$

Example :→ (The number of Labelled graphs)
↳ with 'n' vertices.

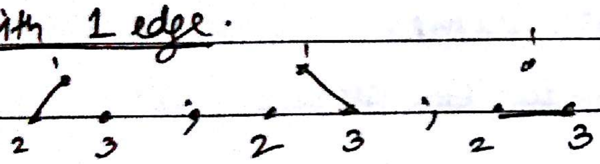
'n' vertices
with 'e' edges $\quad 0 \leq e \leq \binom{n}{2} = r$

$L(n,e)$ = # of labelled graphs with 'n' vertices and 'e' edges. $= \binom{r}{e}$

n=3
no edge



with 1 edge.



and so on....

For a fixed n, we let
$a_k = L(n,k)$; $k = 0,1,2,\ldots, r = \binom{n}{2}$

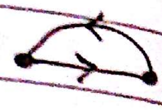
Consider the ordinary generating function for $\{a_k\}$

$$G_n(x) = \sum_{k=0}^{r} a_k \cdot x^k.$$

$$= \sum_{k=0}^{r} \binom{r}{k} \cdot x^k = (1+x)^r$$

$L(n)$ = # of labelled graphs with 'n' vertices

$$= \binom{r}{0} + \binom{r}{1} + \cdots + \binom{r}{r} = 2^r = 2^{\binom{n}{2}}$$

$M(n)$ = # of labelled digraphs with 'n' vertices 

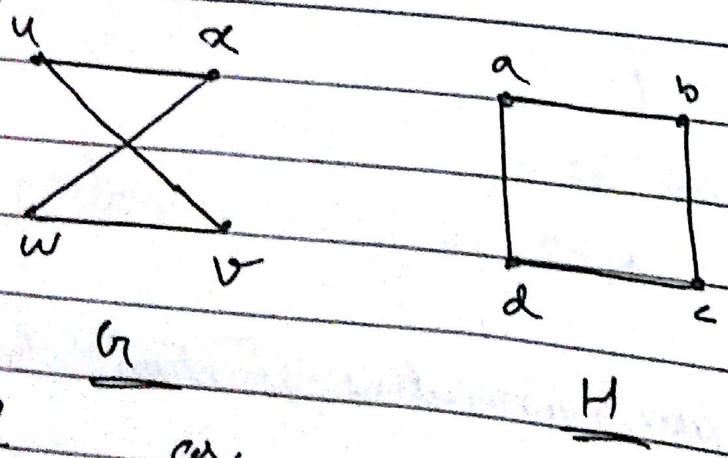
Isomorphism.

- Two unlabelled graphs (digraphs) G & H, each having 'n' vertices are considered the same if the vertices of both can be labelled with integers $1, 2, \ldots, n$ so that the edge sets (arc sets) consist of the same unordered (ordered) pairs.

  i.e. if two graphs (digraphs) can be given a labelling that shows them to be the same.

  G, H are then said to be isomorphic.

Example.



Label cos.

u, a → 1

v, b → 2

w, c → 3

x, d → 4

Isomorphism Problem :→ Given two unlabelled graphs, decide whether they are isomorphic or not.

computationally hard.

The exponential generating function for $\{a_k\}$.

$$H(x) = a_0 \cdot \frac{x^0}{0!} + a_1 \cdot \frac{x^1}{1!} + a_2 \cdot \frac{x^2}{2!} + \cdots + a_k \cdot \frac{x^k}{k!} + \cdots$$

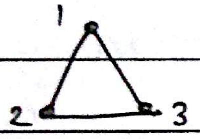
Example :→ { Eulerian graphs }.

↳ every vertex having even degree.

* can traverse every edge only once without taking out the pen and return to the same point.

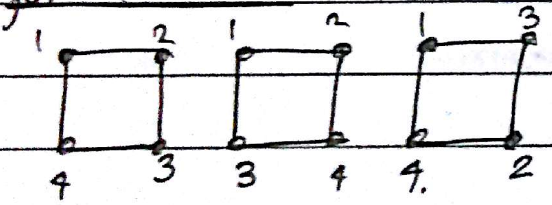
u_n = # of labelled Eulerian graphs of 'n' vertices.

for 3 vertices.



only 1 Eulerian graph)

for 4 vertices.



Eulerian graph : → connected graph with degree of each vertex even (without any isolated vertex).

The exponential generating function for $\{u_k\}$.

38 → for n=5 ve

3 → for n=4 vert

$$u(x) = x + \frac{x^3}{3!} + \frac{3 \cdot x^4}{4!} + \frac{38 \cdot x^5}{5!} + \cdots$$

(Using exponential generating function, we can get this result example :→ (Counting permutations)

suppose we have 'p' types of objects, with n_i indistinguishable objects of type i, $i = 1, 2, \cdots, p$.

The # of distinguishable permutations of length k with up to n_i objects of type i is the co-efficient of $\frac{x^k}{k!}$ in the exponential generating function.

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n_1}}{n_1!}\right)\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_2}}{n_2!}\right) \cdots \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_p}}{n_p!}\right)$$

$$= \boxed{\frac{n!}{n_1! \cdot n_2! \cdots n_p!}}$$

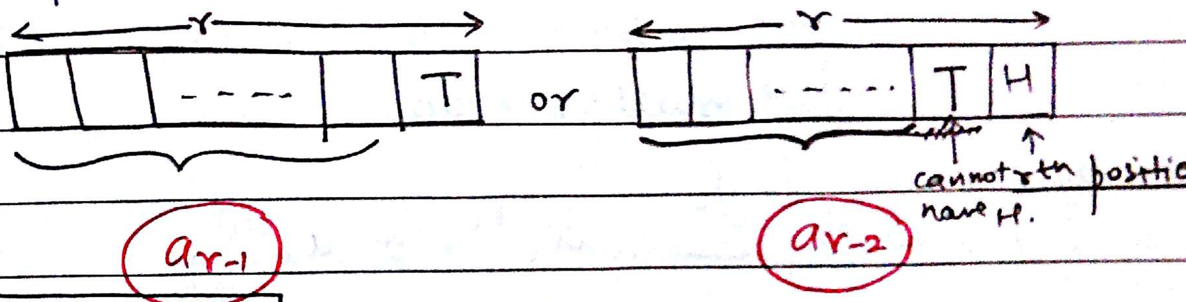
with ordinary generating function we cannot manage this, but using exponential generating function we can do this.

NEXT CLASS → HATCHECK problem.

TUESDAY 21.3.2017

Example → suppose we toss a coin 'r' times. These are 2^r possible sequences of outcomes. Find # of sequences of outcomes in which heads never appears on successive tosses. by forming a recursive relative.

a_r = required no.



$$\Rightarrow \boxed{a_r = a_{r-1} + a_{r-2}} \quad \text{—①} \quad ; \quad a_1 = 2, \quad a_2 = 3$$

similar to fibbonacci sequence but initial conditions are different.

Linear recurrence relation with constant coefficients (Homogenous).

Linear ↓ no product terms: $a_{r-1} \cdot a_{r-2} \cdots$

Let $a_r = \alpha^r$ and substitute in the relation.

$\Rightarrow \alpha^r = \alpha^{r-1} + \alpha^{r-2}$

$\Rightarrow \alpha^2 - \alpha - 1 = 0$ → characteristic equation of ①

$\{a_n = c_0 a_k + c_1 a_{k+1} + \cdots + c_n \cdot a_{n-1}.\}$

characteristic Root Method.

$$\alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

$$a_r = \lambda_1 \left(\frac{1+\sqrt{5}}{2}\right)^r + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right)^r \quad ; \lambda_1, \lambda_2 \text{ constant.}$$

linear combination is also a solution.

solving for λ_1 & λ_2 using the initial conditions.

$c_0 \cdot a_r + c_1 \cdot a_{r-1} + c_2 \cdot a_{r-2} + \cdots + c_k \cdot a_{r-k} = f(r)$

$a_r = \underset{\text{homogenous}}{a_r^{(h)}} + \underset{\text{particular}}{a_r^{(p)}}$ $f(r) = 0 \leftarrow$ Homogenous equation.

To find $\rightarrow a_r^{(h)}$ (making $f(r) = 0$)

put $a_r = \alpha^r$ and get the charasteristic equation.

- $\rightarrow$ distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_k$
- $\rightarrow$ multiple roots

Case of distinct roots $\Rightarrow$ $\boxed{\sum_{i=1}^{k} \lambda_i \cdot \alpha_i^{r_i} = a_r^{(h)}}$

Case of Multiple roots

$a_r^{(h)}$ $\alpha_1, \alpha_2, \ldots, \alpha_l$ $\leftarrow$ roots of the ch. equation.

$\downarrow \quad \downarrow \quad \ldots \quad \downarrow$ with

$m_1 \quad m_2 \quad \ldots \quad m_l$ multiplicity

$\alpha_i^r, r \cdot \alpha_i^r, r^2 \cdot \alpha_i^r, \ldots, r^{m_i - 1} \cdot \alpha_i^r$; $1 \leq i \leq l$.

$b_1, b_2, \ldots, b_k \rightarrow$ all basic solutions.

$$a_r^{(h)} = \sum_{i=1}^{k} \lambda_i \cdot b_i^r$$

For finding Particular solutions

ar on the form $\rightarrow f(r) = F_1 r^t + F_2 \cdot r^{t-1} + F_3 \cdot r^{t-2} + \cdots + F_t \cdot r + F_{t+1}$; polynomial form

$F_1, F_2, \ldots, F_{t+1}$ are constants.

$a_r^{(p)} = \cancel{p_1 r^t + p_2} \; p_1 \cdot r^t + p_2 \cdot r^{t-1} + \cdots + p_t \cdot r + p_{t+1}$;

assume the $a_r^{(p)}$.

$p_1, p_2, p_3, \ldots, p_{t+1}$ are constants

Example → $a_r + 5a_{r-1} + 6a_{r-2} = 3 \cdot r^2$ ① (2nd order linear difference equation with constant coefficients)

② → $a_r^{(P)} = P_0 + P_1 \cdot r + P_2 \cdot r^2$

substitute ② in ①

$$(P_0 + P_1 \cdot r + P_2 \cdot r^2) + 5[P_0 + P_1(r-1) + P_2(r-1)^2] + 6[P_0 + P_1(r-2) + P_2(r-2)^2] = 3r^2.$$

Equating the coefficients

$$\left.\begin{aligned} 12P_1 &= 1. \\ 12P_2 - 34P_1 &= 0. \\ 12P_3 - 17P_2 + 29P_1 &= 0. \end{aligned}\right\} \quad \begin{aligned} P_1 &= 1/4. \\ P_2 &= 17/24. \\ P_3 &= 115/288 \end{aligned}$$

$$a_r^{(P)} = \frac{1}{4} + \frac{17}{24} \cdot r + \frac{115}{288} \cdot r^2$$

→ particular solutions may not satisfy the boundary conditions so both particular as well as general solutions are needed.

~~XXXX~~

$$\Rightarrow f(r) = (F_1 \cdot r^t + F_2 \cdot r^{t-1} + \cdots + F_t \cdot r + F_{t+1}) \cdot \beta^r.$$

↓ corresponding particular solution has the form.

$$a_r^{(P)} = (P_1 \cdot r^t + P_2 \cdot r^{t-1} + \cdots + P_t \cdot r + P_{t+1})\, \beta^r.$$

provided β is not a root of the ch. equation.

In case β is a root of ch. equation with multiplicity $(m-1)$,

then $\boxed{a_r^{(P)} = (p_1 \cdot r^t + \cdots + p_{t+1}) \cdot r^{m-1} \cdot \beta^r}$

Example.

$$a_r - 5a_{r-1} + 6\cdot a_{r-2} = \underbrace{2^r}_{P_2\cdot r\cdot 2^r} + \underbrace{r}_{P_0 + P_1\cdot r}$$

* sum of two particular solution.

Ch. equation.

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x - 3x - 2x + 6 = 0$

$\Rightarrow (x-3)(x-2) = 0$

$x = 2, 3$

multiplicity 1

$a_r^{(p)} = P_0 + P_1\cdot r + P_2\cdot r\cdot 2^r$

$a_r^{(p)} = \frac{7}{4} + \frac{r}{2} - r\cdot 2^{r+1} - 5\cdot 2^r + \frac{17}{4}\cdot 3^r$ (check)

Example.

$a_r = a_{r-1} + 7$

'7' can be considered as 7 (constant) or $7\cdot 1^r$

The Hatcheck Problem. (Derangement)

Imagine that 'n' gentlemen attend a party and check their hats.

The checker returns their hats at random.

What is the probability that no gentleman recieves their own hat.

$\boxed{\text{DERRANGEMENT}}$

A permutation or arrangement in which i-th object is not placed in the i-th place for any i.

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eg:- $n = 3$.

231 Yes.

213 No.

D_n = # of derrangements of 'n' objects

Using Recurrence Relation for D_n.

1	2	3	-----	K	-----	n	n+1
1st	2nd	3rd	----	Kth	----	nth	(n+1)th
K	2	3	-----	1	------	n	← 1st and Kth interchanged.

n choices. Derranged.

D_{n+1} → Kth position has 1 → D_{n-1} (2 are derranged)

→ Kth position does not hold 1 → D_n (only 1 is derranged)

$$\boxed{D_{n+1} = \underset{\substack{\uparrow \\ \text{for each of the} \\ n \text{ elements.}}}{n} (D_n + D_{n-1})} \quad -① \; ; \quad D_1 = 0, \quad D_2 = 1$$

solving using generating function.

need to defined D_0 so that ① holds.

$n=1$

$D_2 = D_1 + D_0 \Rightarrow D_0 = D_2 - D_1 \Rightarrow D_0 = 1 - 0 = 1$.

Algebraic Manipulation.

$$\Rightarrow D_{n+1} = (n+1) D_n + (-1)^{n+1}, \quad n \geq 0 \quad ②$$

Exercise.

$$G(x) = \sum_{n=0}^{\infty} D_n \cdot x^n \qquad \sum_{n=0}^{\infty} D_{n+1} \cdot x^n = \sum_{n=0}^{\infty} (n+1) \cdot D_n \cdot x^n + \sum_{n=0}^{\infty} (-1)^{n+1} \cdot x^n$$

$D_1 \cdot x^0 + D_2 \cdot x^1 + \cdots$

$$\Rightarrow \frac{1}{x}(G(x)-D_0) = \sum_{n=0}^{\infty} n \cdot D_n \cdot x^n + \underbrace{\sum_{n=0}^{\infty} D_n \cdot x^n}_{\sum_{n=0}(-1)^n x^n}$$

$$= x \cdot \underbrace{\sum n \cdot D_n \cdot x^{n-1}}_{G'(x)} + G(x)$$

$$\Rightarrow \boxed{G'(x) + \left(\frac{1}{x} - \frac{1}{x^2}\right) \cdot G(x) = \frac{1}{x+x^2} - \frac{1}{x^2}}$$

* Difficult to solve this differential equation.

So solving by exponential Generating Function.

$$H(x) = \sum_{n=0}^{\infty} D_n \cdot \frac{x^n}{n!}$$

$$H(x) = \frac{e^{-x}}{1-x} = \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right]\left[1 + x + x^2 + \cdots\right]$$

$$D_n = \text{coefficient of } \frac{x^n}{n!} = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\right]$$

Probability Generating Function.

an experiment performed.

$p_k \rightarrow$ probability that the k-th event occurs, $k = 0, 1, 2, 3, \ldots$

$$G(x) = \sum_{k=0}^{\infty} p_k \cdot x^k. \qquad p_1 + p_2 + \cdots + p_k + \cdots = 1$$

Example $\rightarrow$ coin Toss $\quad P(H) = \frac{1}{2}, \; P(T) = \frac{1}{2}$

$$G(x) = \frac{1}{2} + \frac{1}{2} \cdot x.$$

Bernoulli's trials.

Example. → Bernoulli's Trials. Series give Binomial distribution.
'n' trials

each trial.
→ success
→ fail

$b(k, n, p) = \binom{n}{k} p^k \cdot q^{n-k}$ $\boxed{0 \le k \le n}$

probability that out of 'n' trials, there will be k-success.

probability Generating function $G(x)$

$$G(x) = \sum_{k=0}^{n} \binom{n}{k} \cdot p^k \cdot q^{n-k} \cdot x^k \cdot = (px+q)^n.$$

$G(1) = 1$

System. $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix}$

$$G(x) = \sum_{k=1}^{n} p_k \cdot x^k.$$

$$G'(x) = \sum k \cdot p_k \cdot x^{k-1}, \quad G''(x) = \sum k(k-1)\, p_k \cdot x^{k-2}$$

Expectation. $= G'(1) = \sum k \cdot p_k$

Variance. $= V = G''(1) + G'(1) - [G'(1)]^2$ {Exercise}.

for permutation counting.
n_i many i-th type object.
1, 2, 3, ----, p types of objects.

NEXT CLASS → using generating function to calculate formula for permutation counting.

Example :- A code can use three letters a, b or c. A sequence of five or fewer letters gives a codeword. The codeword can use at most one b, at most one c, and upto three a's.
How many possible codewords are there of length k with $k \le 5$?!

$$\{ \underset{\le 3}{a}, \underset{\le 1}{b}, \underset{\le 1}{c} \}$$ codewords of length ≤ 5

ordinary generating function.

$(1 + ax + a^2x^2 + a^3x^3)(1+bx)(1+cx)$.

$= 1 + (a+b+c)x + (bc + a^2 + ab + ac)x^2 + (a^3 + abc + a^2b + a^2c)x^3$
$+ (a^2bc + a^3b + a^3c)x^4 + a^3bcx^5$

coefficient of x^k → ways of getting k letters.

eg:- 3 letters.

3 a's → $\frac{3!}{3!} = 1$ permutation aaa

1a, 1b, 1c → 3! permutation.

2a, 1b → $\frac{3!}{2!\,1!}$

2a, 1c → $\frac{3!}{2!\,1!}$

actual information

ways to obtain codewords of length 3.

$= \frac{3!}{3!}a^3 + \frac{3!}{1!1!1!}abc + \frac{3!}{2!1!}a^2b + \frac{3!}{2!1!}a^2c$.

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Setting $a=b=c$. (Actual count)

we consider $\frac{(ax)^p}{p!}$ instead of $a^p x^p$ to derive the generating function.

$$\left(1+\frac{ax}{1!}+\frac{a^2x^2}{2!}+\frac{a^3x^3}{3!}\right)\left(1+\frac{bx}{1!}\right)\left(1+\frac{cx}{1!}\right)$$

$$=1+\left(\frac{a}{1!}+\frac{b}{1!}+\frac{c}{1!}\right)\frac{x}{1!}+\left(\frac{bc}{1!1!}+\frac{a^2}{2!}+\frac{ab}{1!1!}+\frac{ac}{1!1!}\right)\frac{x^2}{2!}\times 2!$$

$$+4!\left(a^2bc+a^3b+a^3c\right)\frac{x^4}{4!}$$

$$+3!\left(\frac{a^3}{3!}+\frac{abc}{1!1!1!}+\frac{a^2b}{2!1!}+\frac{a^2c}{2!1!}\right)\frac{x^3}{3!}+5!\frac{a^3bc}{3!1!1!}\frac{x^5}{5!}$$

setting $a=b=c=1$. {if we are only interested in counting}.

$$\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\left(1+\frac{x}{1!}\right)\left(1+\frac{x}{1!}\right) \leftarrow \text{coefficient of } \frac{x^3}{3!}$$

'p' types of objects.

'n_i' objects, i-th type

→ # of distinguishable permutations of length k with upto 'n_i' objects of type 'i'.

= coefficient of $\frac{x^k}{k!}$ in the exponential generating function.

$$\prod_{i=1}^{p}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^{n_i}}{n_i!}\right)$$

Simultaneous Equation for Generating function.

Example :→ alphabet $\{0,1,2,3\}$.

$k \geq 1$ $\quad a_k$ = # of codewords of length k with an even # of zeros and an even no. of 3's.

we can solve directly for a_k also.

Let $a_k \to$ even 0 and even 3.

$b_k \to$ even 0 and odd 3.

$c_k \to$ odd 0 and even 3

$d_k \to$ odd 0 and odd 3.

$a_k + b_k + c_k + d_k = 4^k$.

$a_{k+1} = 2 \cdot a_k + b_k + c_k$

$b_{k+1} = 2b_k + a_k + d_k$.

$c_{k+1} = 2c_k + a_k + d_k$.

~~$d_{k+1} = 2d_k + b_k + c_k$.~~

4 equations are there.

$a_{k+1} = 2a_k + b_k + c_k$

$b_{k+1} = b_k - a_k + 4^k$.

$c_{k+1} = c_k - b_k + 4^k$

eliminate d_k.

$a_1 = 2,\ b_1 = 1,\ c_1 = 1$

define a_0, b_0, c_0

$a_0 = 1$

$b_0 = c_0 = 0$

→ such that the relation holds.

multiply by x^k and take summation.

$$\frac{1}{x}\left[A(x) - a_0\right] = 2A(x) + B(x) + C(x). \quad — ①$$

$$\frac{1}{x}\left[B(x) - b_0\right] = B(x) - C(x) + \frac{1}{1-4x}. \quad — ②$$

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$A(x) = \sum_{k=0}^{\infty} a_k . x^k$ $\qquad C(x) = \sum_{k=0}^{\infty} c_k . x^k$

$B(x) = \sum_{k=0}^{\infty} b_k . x^k$

$$\frac{1}{x}\left[C(x) - c_0\right] = C(x) - B(x) + \frac{1}{1-4x} \quad - (3)$$

Solve (1), (2) and (3) to get

$$B(x) = C(x) = \frac{x}{1-4x}.$$

$$A(x) = \frac{2x^2 - 4x + 1}{(1-2x)(1-4x)}$$

$\downarrow$

partial fraction.

$$A(x) = 1 + \frac{x}{1-4x} + \frac{x}{1-2x}.$$

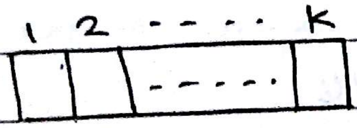
$$a_k = 4^{k-1} + 2^{k-1}, \quad k>0 \; ; \; a_0 = 1.$$

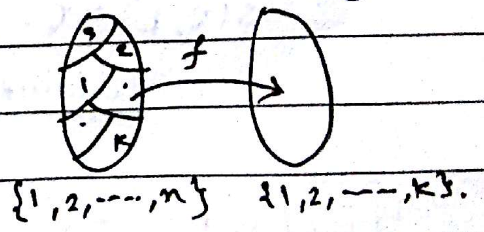
WEEK-13
MONDAY
27.3.2017

Occupancy Problem.

Distribution of Distinguishable Balls into Indistinguishable cells

n balls into k cells. with no cell empty.





$S(n,k)$ = The stirling no. of 2nd kind.

let $T(n,k)$ = # of ways of distributing n distinguishable balls into k distinguishable cells with no cell empty.

$$\boxed{k! \times S(n,k) = T(n,k)}.$$

$T(n,k) \rightarrow$ find a distribution of n distinguishable balls into k indistinguishable cells with no cell empty and then labelling (ordering) the cells.

$T(n, k) = ?$

$c(i)$ ~~denote~~ ; let i-th balls goes to cell $c(i)$

$c(1), c(2), \cdots, c(n) \rightarrow$ an n-permutation of the set $\{1, 2, \cdots, k\}$, with each label j in k-set $\{1, 2, \cdots, k\}$ used at least once.

Using Exponential Generating function:

$\boxed{\frac{x^n}{n!} \text{ is}}$

$$H(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^k.$$

$$= (e^x - 1)^k.$$

$T(n, k)$ is the coefficient of $\frac{x^n}{n!}$ in.

$$= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \times (e^x)^{k-i}$$

$$= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \times \sum_{n=0}^{\infty} \frac{(x(k-i))^n}{n!}$$

$$\boxed{T(n, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n}$$

$s(0, 0) = 1.$

$s(n, 0) = 0$

$s(0, k) = 0.$

n distinguishable balls.

$\boxed{\text{EXERCISE}} \rightarrow$ $s(m, n) = s(m-1, n-1) + n \cdot s(m-1, n)$; $m, n > 0$

prove the recurrance relation.

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The Principle of Inclusion and Exclusion.

Theorem → If N is the no. of ~~digits~~ objects in a set A, the no. of objects in A having none of the properties $a_1, a_2, \dots, a_r$ is

$$N(a_1' a_2' \dots a_r') = N - \sum_i N(a_i) + \sum_{i \neq j} N(a_i a_j) - \sum_{\substack{i,j,k \\ \text{different}}} N(a_i a_j a_k) + \dots + (-1)^r N(a_1 a_2 \dots a_r)$$

Example → How many integers between 1 to 1000

i) not divisible by 2? $N(a_1') = 500$

ii) not divisible by 2 or 5? $N(a_2' a_1') = 400$

iii) not divisible by 2, 5 or 11? $N(a_1' a_2' a_3') = 364.$

Let a_1 = the property of being divisible by 2

a_2 = - - - - - - - - - - - - 5

a_3 = - - - - - - - - - - - - 11.

$N = 1000$

$N(a_1') = N - \sum N(a_1)$

$= 1000 - \lfloor 1000/2 \rfloor = 500.$

$N(a_1' a_2') = N - [N(a_1) + N(a_2)] + N(a_1 a_2)$

$= 1000 - \left[500 + \left\lfloor \frac{1000}{5} \right\rfloor\right] + \left\lfloor \frac{1000}{2 \times 5} \right\rfloor$

PROOF {Combinatorial Proof}.

an object having 'p' properties; $a_{i1}, a_{i2}, \dots, a_{ip}$ is counted once the RHS.

In the LHS, it counted once.

$$N - \sum_i N(a_i) + \sum_{i \neq j} N(a_i a_j) - \dots$$

$$\binom{p}{0} - \binom{p}{1} + \binom{p}{2} - \dots + (-1)^r \binom{p}{r}$$

Book

Applied combinatorics.

Fred S. Roberts. Ch. 4, 5, 6.

$$\binom{p}{0} - \binom{p}{1} + \cdots + (-1)^p \binom{p}{p} = (1-1)^p = 0.$$

LHS = # of objects none of the properties $a_1, a_2, \ldots, a_r$

RHS = every object having none of the properties $a_1, a_2, \ldots, a_r$ counted once

+

every object having at least one of the properties is counted exactly zero times.

proof for when the objects is have some properties; LHS = 0, RHS = 0)

Derrangements. (Using principle of inclusion & Exclusion)

D_n = # of derrangements of 'n' objects.

a_i = the property that i-th object is placed at the i-th place.

Then,

$$D_n = N(a_1' a_2' \cdots a_n')$$

$$N = n!$$

$$N(a_i) = (n-1)!$$

$$N(a_i a_j) = (n-2)!$$

$\vdots$

$$\sum' N(a_{i_1} a_{i_2} \cdots a_{i_t}) = \binom{n}{t}(n-t)!$$

$i_1, i_2, \ldots, i_t$ all present

putting in the formula of Inclusion & Exclusion principle

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

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WEEK-13
TUESDAY
28.3.2017

Classification of Occupancy problems. (placing balls into cells)

• The stirling no. of 2nd kind $= S(n,k)$
$$= \sum_{i=0}^{k} \binom{k}{i} (-1)^i (k-i)^n \times \frac{1}{k!}$$

	Distinguishable Balls?	Distinguishable cells?	can cells be empty?	# ways to place 'n' balls into 'k' cells.
case 1:	YES	YES	YES	k^n.
	YES	YES	NO	$T(n,k) = k!\, S(n,k)$.
case 2:	NO	YES	YES	$C(k+n-1, n)$ = ${}^nC_r$
	NO	YES	NO	$C(k-1, n-1)$
case 3:	YES	NO	YES	$S(n,1) + S(n,2) + \cdots + S(n,k)$
	YES	NO	NO	$S(n,k)$
case 4:	NO	NO	YES	# of partitions of n into k or fewer parts.
	NO	NO	NO	# of partitions of n into exactly k parts. (partition function).

• $$C(n,i) = \binom{n}{i} = \begin{cases} \dfrac{n!}{i!(n-i)!} & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

• $\underbrace{(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots}_{\text{'k' terms.}}$

coefficient of x^n in $(1+x+x^2+\cdots)^k = (1-x)^{-k}$.
$$= \sum_{i=0}^{\infty} \binom{-k}{i} (-x)^i$$

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(Negative Binomial)

coefficient of $x^n = \binom{-k}{n}(-1)^n = \frac{(-k)(-k-1)(-k-2)\cdots(-k-n+1)}{n!}$

$= \binom{k+n-1}{n}$

- $T(n,k)$ = # of ways of placing 'n' distinguishable balls in 'k' distinguishable cells with no cell empty.

$$\underbrace{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(1+x+\frac{x^2}{2!}+\cdots\right)\cdots}_{k\text{ terms}}$$

$$T(n,k) = \sum_{i=0}^{k}\binom{k}{i}(-1)^i(k-i)^n.$$

Principle of Inclusion and Exclusion.

Total of N objects

$a_1, a_2, a_3, \ldots, a_r \rightarrow$ 'r' properties that a set of objects in A have.

$$N(a_1', a_2', \ldots, a_r') = N - \sum_i N(a_i) + \sum_{\substack{i,j\\ i\neq j}} N(a_i a_j) - \sum_{\substack{i,j,k\\ \text{diff.}}} N(a_i a_j a_k) + \cdots + (-1)^r N(a_1, a_2, \ldots, a_r)$$

$N(a_1', a_2', \ldots, a_r')$ → # of objects in A having none of the properties $a_1, a_2, \ldots, a_r$

case 1 :- an object having none of the properties $a_1, a_2, a_3, \ldots, a_r$

case 2 :- an object having atleast one of the properties $a_1, a_2, \ldots, a_r$.

MOD principle of inclusion exclusion

e_m = # of objects in A having exactly 'm' properties out of 'r' properties.
(general case)

$a_1, a_2, a_3, \ldots, a_r$

Book → Hernsteine.
(for Groups, Rings, Fields).

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An alternative proof of $T(n,k)$ using the principle of Inclusion and Exclusion Principle.

$T(n,k) \rightarrow$ n distinguishable balls. k distinguishable cells. with no cell empty

s = set of distribution of balls to cells.

a_i = the property that i-th cell is empty.

To find $N(a_1' a_2' \cdots a_r') = N - \sum_i N(a_i) + \sum_{\substack{i,j \\ i \neq j}} N(a_i a_j) - \cdots$

$$k^n - (k-1)^n \binom{k}{1} + (k-2)^n \binom{k}{2} - \cdots$$

$$N(a_1' a_2' \cdots a_r') = k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \cdots + (-1)^k \binom{k}{k}(k-k)^n$$

$$= \sum_{i=0}^{k} \binom{k}{i} (-1)^i (k-i)^n.$$

The number of objects having exactly 'm' properties if there are 'r' properties $a_1, a_2, \ldots, a_r$ and $m \leq r$ is given by

$$e_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^p \binom{m+p}{p} S_{m+p} + \cdots + (-1)^{r-m} \binom{m-(r-m)}{r-m} S_r$$

where $S_t = \sum_{i_1, i_2, i_3, \ldots, i_t} N(a_{i_1}, a_{i_2}, a_{i_3}, \ldots, a_{i_t})$; $t \geq 1$.

$i_1, i_2, i_3, \ldots, i_t$ are different

Special case.

$m = 0$ considering $S_0 = N$

0 ← case 1 : an object having less than m properties.

1 ← case 2 : an object having exactly equal to m properties.

0 ← case 3 : an object having greater than m properties

an object having $(m+j)$ properties.

→ it is counted $\binom{m+j}{m}$ times in S_m.

→ it is counted $\binom{m+j}{m+1}$ times in S_{m+1}.

→ it is counted $\binom{m+j}{m+p}$ times in S_{m+p}.

claim! → $\binom{m+j}{m+p}\binom{m+p}{p} = \binom{m+j}{m}\binom{j}{p}$.

RHS : This object with $(m+j)$ properties is counted

$$\binom{m+j}{m} - \binom{m+1}{1}\binom{m+j}{m+1} + \binom{m+2}{2}\binom{m+j}{m+2} - \cdots\cdots + \binom{m+j}{j}\binom{m+j}{m+j}$$

$= 0$

Using Derrangements.

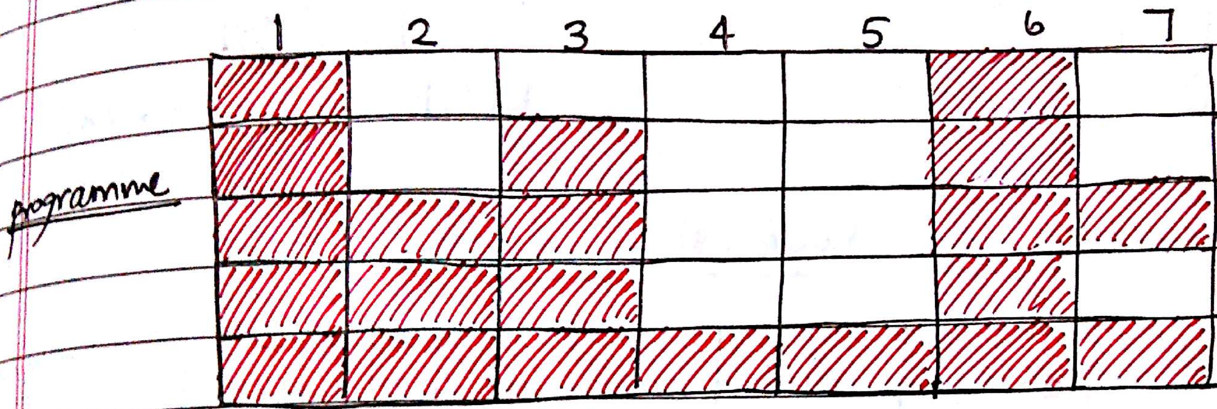
$e_1 = {}^nC_1 \times D_{n-1}$

Using the formula

calculate e_1.

Rook Polynomial.

consider a Board $B_{n \times m}$



B

To determine # of ways to assign each program to a storage location with sufficient storage capacity, at most one programme per location.

(i,j)th position darkened.

→ storage location j has sufficient storage capacity for program i.

Find $r_k(B)$ = # of ways in which k non-taking rooks can be placed in acceptable (darkened) squares of B.

↓

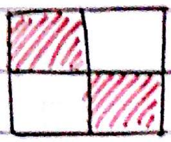
generating function for $r(B)$ is the Rook polynomial.

so we have to find $r_5(B)$?

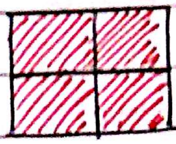
Rook polymial.

$$R(x, B) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots$$

Example.



B_1

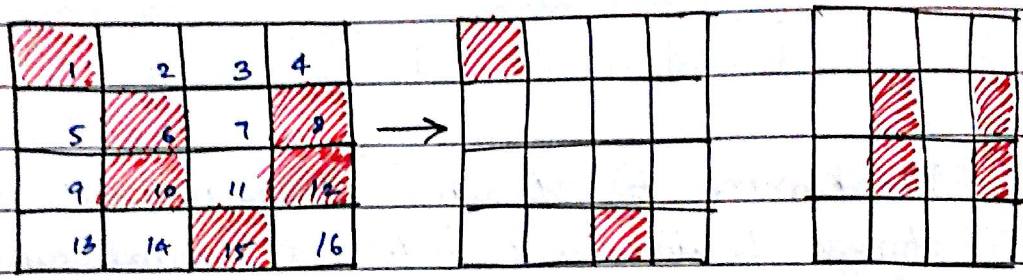


B_2

$R(x, B_1) = 1 + 2x + x^2$

$R(x, B_2) = 1 + 4x + 2x^2$

Reduction for Rook Polynomials.



B B_I B_J.

$I = \{1, 15\}$ $J = \{6, 10, 8, 12\}$.

…f we can place k rooks in B_I and $(n-k)$ rooks in B_J and the ~ take the convolution of the two.

* Breaking into 2 disjoint sets (no squares in I lies in the same row or column of any square J. and then taking the convolution of the two.

$R(x, B) = R(x, B_1) \cdot R(x, B_2)$

$r_k(B_I) * r_k(B_J) = r_k(B) = r_0(B_I) \cdot r_k(B_J) + r_1(B_I) \cdot r_{k-1}(B_J) + \dots + r_k(B_I) \cdot r_0(B_J)$