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# CALCULUS OF VARIATIONS:

References

① Problems & exercises in the calculus of variations. by M. L. Krasnov, G. I. Marageno, etc.

② Calculus of variat$^n$s $\bar{c}$ applicat$^n$s by A. S. Gupta

③ Calculus of Variat$^n$s by Gelfand.

□ **Extremum of a f$^n$ of several variables:**

$f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, where $\Omega$ is an open subset of $\mathbb{R}^n$

Let $x_0 \in \Omega$ & we say $x_0$ is local minimum of $f(x)$ if $\exists$ nbd $U$ of $x_0$ s.t $f(x) \geq f(x_0)$ $\forall x \in U \subset \Omega$.
'$x_0$' is a local maximum of $f(x)$ if $\exists$ a nbd $U$ of $x_0$ s.t $f(x_0) \geq f(x)$ $\forall x \in U$. The points of maximum & minimum of $f(x)$ are called extreme points of $f(x)$

□ **Necessary Condit$^n$ for a f$^n$ $f(x)$ to have extreme points**

$f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$.

Let $x_0 \in \Omega$, if $x_0$ is local extremum of $f(x)$ then $\frac{\partial f}{\partial x_i}(x_0) = 0$ for $i = 1, 2, \cdots, n$ (~~$x_0$ must be an interior pt.~~)

→ We say $x_0 \in \Omega$ is a critical pt. of $f(x)$ if $\frac{\partial f}{\partial x_i}(x_0) = 0$ for $i = 1, 2, \ldots, n$ or $\frac{\partial f}{\partial x_i}$ does not exist at $x_0$.

**Q** Is Every critical pt a local extremum for $f(x)$?

**Ans** No.

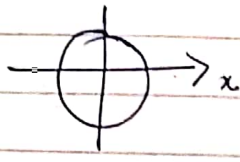
**Ex** $f(x,y) = x^2 - y^2$ , $x, y \in \mathbb{R}$

$\frac{\partial f}{\partial x} = 2x = 0 \Rightarrow x = 0$

$\frac{\partial f}{\partial y} = -2y = 0 \Rightarrow y = 0$

$(0,0)$ is a critical pt. of $f(x,y)$

$B_\epsilon(x_0, y_0) = \{(x,y) : \sqrt{(x-x_0)^2 + (y-y_0)^2} < \epsilon$




$\because$ $f(0,y) = -y^2 < 0 \quad \forall y \ (y \neq 0)$

$f(x,0) = x^2 > 0 \quad \forall x \ (x \neq 0)$

& $f(0,0) = 0$.

$\Rightarrow$ $(0,0)$ is neither local maximum nor local minimum.

- The quadratic form $A(x) = A(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$

  $a_{ij} = a_{ji}$ for $i, j = 1, 2, \ldots, n$

  is +ve definite if $A(x) > 0 \quad \forall x \in \mathbb{R}^n - \{0\}$ &

  $A(x) = 0$ only when $x = 0$

Similarly, $A(x)$ is -ve definite if $A(x) < 0 \quad \forall x \in \mathbb{R}^n - \{0\}$ & $A(x) = 0$ only when $x = 0$.

**Sylvester's Criteria:**

Necessary & sufficient condⁿ

$A(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$

for $(n=2)$ $= \sum_{i=1}^{n} (a_{i1} x_i x_1 + a_{i2} x_i x_2)$ . . . ~~$+ a_{in} x_i x_n$~~)

$$= a_{11} x_1^2 + a_{12} x_1 x_2 + a_{21} x_2 x_1 + a_{22} x_2^2$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

→ $A(x) = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} x_i x_j$ is +ve definite if

$$a_{11} > 0 , \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$$

→ $A(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ is +ve definite if

$$a_{11} > 0 , \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 , \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0 \ldots$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0$$

Similarly $A(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ is said to be -ve definite if

$$a_{11} < 0 , \quad \begin{vmatrix} a_{11} , & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 , \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0 \ldots$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} (-1)^n > 0$$

- A quadratic form c is either +ve or -ve definite is called a '<u>definite</u> <u>quadratic</u> <u>form</u>'

A quadratic form c is not definite is called 'indefinite quad~~form~~ratic form'

## Sufficient Condⁿ for a local extremum

Let $f(x)$ be fⁿ defined on $\Omega \subseteq \mathbb{R}^n$ and have continuous $2^{nd}$ order partial derivatives in a nbd of the pt $x_0$. & the pt. $x_0$ is a critical ~~local~~ pt. ~~extremum~~ of $f(x)$, if the quadratic form

$$A(dx_1, dx_2, \dots, dx_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)\, dx_i\, dx_j \quad \text{i.e.}$$

Assume $f_{xy} = f_{yx}$

$2^{nd}$ order differential of $f(x)$ at the pt $x_0$ is a +ve definite (-ve definite) quadratic form, then the pt $x_0$ is strict minimum (maximum)

We apply this idea for a fⁿ of two variable

$$= \frac{\partial^2 f}{\partial x^2}(x_0)\, dx^2 + \frac{\partial^2 f}{\partial x \partial y}(x_0)\, dx\, dy + \frac{\partial^2 f}{\partial y^2}(x_0)\, dy^2 \quad \text{--- (1)}$$

(1) is +ve definite if $f_{xx}(x_0) > 0$, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$
then $x_0$ is strict minimum

(1) is -ve definite if $f_{xx} < 0$ ($f_{yy} < 0$),
$$\begin{vmatrix} f_{xx}(x_0) & f_{xy}(x_0) \\ f_{yx}(x_0) & f_{yy}(x_0) \end{vmatrix} > 0$$

then in this case $x_0$ is strict maximum of $f(x,y)$

→ If $f_{xx}(x_0) f_{yy}(x_0) - f_{xy}^2(x_0)$ ~~$f_{yx}f_{xx}$~~ $< 0$ then we say $x_0$ is neither local maximum or local minimum

$\rightarrow$ If $f_{xx}(x_0)\, f_{yy}(x_0) - f^2_{xy}(x_0) = 0$, then no conclusion

In this case, $x_0$ may or may not be extremum of $f$

Ex $f(x,y) = x^4 + y^4$

$(0,0)$ is a critical pt.

$f(x,y) > f(0,0)$ $\forall (x,y)$ in a nbd of $(0,0)$

$(0,0)$ is strict minimum of $f(x,y)$ but

$f_{xx}(0,0) = 0$, $f_{yy}(0,0) = 0$, $f_{xy}(0,0) = 0$

$\Rightarrow f_{xx} f_{yy} - f^2_{xy} = 0$

Ex $f(x,y) = -x^4 - y^4$

$(0,0)$ is a critical pt.

$f_{xx}(0,0) = f_{xy}(0,0) = f_{yy}(0,0) = 0$

$\Rightarrow f_{xx}(0,0)\, f_{yy}(0,0) - f^2_{xy}(0,0) = 0$

$f(x,y) = -x^4 - y^4 < 0 = f(0,0)$, $f(x,y)$ is a nbd of $f(0,0)$. $\Rightarrow (0,0)$ is strict maximum of $f(x,y)$

Ex $f(x,y) = x^4 - y^4$

$(0,0)$ is a critical pt.

$f_{xx}(0,0) = 0$, $f_{xy}(0,0) = 0$, $f_{yy}(0,0) = 0$

$\Rightarrow f_{xx}(0,0)\, f_{yy}(0,0) - f^2_{xy}(0,0) = 0$

$f(x,0) = x^4 > 0 = f(0,0)$

$f(0,y) = -y^4 < 0 = f(0,0)$

So $(0,0)$ is neither maximum nor minimum

$\Rightarrow (0,0)$ is saddle pt.

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→ $f(x)$ defined on $\Omega \subseteq \mathbb{R}^n$, $x \in \Omega$
In additⁿ to $f(x)$, we have $m$ accessary conditⁿs

$$\left.\begin{aligned} \phi_1(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ \phi_m(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}\right\} - (1)$$

(1) are called '__equatⁿs of constraint__'

» __Conditional Maximum / Minimum__ :

We say the point $x_0 \in \Omega$ is conditional maximum of $f(x)$ if $f(x) \le f(x_0)$ for all $x$ in some nbd of $x_0$ & also $x$ & $x_0$ satisfy the eqⁿs of constraint (1)

__Example__: Let $f(x,y) = x^2 + y^2$
$\phi(x,y) = x + y - 1 = 0$ (constraint)

__Solⁿ__

$y = 1 - x$

$f(x,y) = f(x, 1-x) = x^2 + (1-x)^2 = F(x)$

$F'(x) = 0 \Rightarrow 2x + 2(1-x)(-1) = 0 \Rightarrow x = 1/2$

$F''(x) = 2 + 2 = 4 > 0$

At $x = 1/2$, $F(x)$ has minimum
when $x = 1/2$, $y = 1/2$

$\left(\frac{1}{2}, \frac{1}{2}\right)$ lies on $x + y - 1 = 0$
$f(x,y)$ has conditional minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$

In general, Max / Min $f(x,y)$
subject to $\phi(x,y) = 0$

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if $\frac{\partial\phi}{\partial y} \neq 0 \Rightarrow y = \psi(x) \Rightarrow f(x,y) = f(x, \psi(x)) = F(x)$

## Lagrangian method of multipliers

(1) $f(x)$ & $\phi_i(x)$, $x \in \Omega \subseteq \mathbb{R}^n$, have $1^{st}$ order continuous derivatives i.e. $\frac{\partial f}{\partial x_i}$ & $\frac{\partial \phi_j}{\partial x_i}$ exists & those are continuous $\forall\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$

(2) $m < n$ & the rank of the matrix

$$\begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial\phi_m}{\partial x_1} & \frac{\partial\phi_m}{\partial x_2} & \cdots & \frac{\partial\phi_m}{\partial x_n} \end{bmatrix}$$

is $m$ $\forall$ the points defined on the domain $\Omega$.

Let us define Lagrange $f^n$

$$\psi(x) = f(x) + \sum_{i=1}^{m} \lambda_i \phi_i \quad \text{—— (2)}$$

where $\lambda_i$'s are unknowns & are to be determined.

For the existence of maximum of $\psi(x)$, we have

$\frac{\partial\psi}{\partial x_i} = 0$ & $\phi_j(x) = 0$ for $i = 1, 2, \ldots, n$ ; $j = 1, 2, \ldots, m$

$$\begin{cases} \frac{\partial\psi}{\partial x_1} = 0, \frac{\partial\psi}{\partial x_2} = 0, \ldots, \frac{\partial\psi}{\partial x_n} = 0 \\ \phi_1(x) = 0, \phi_2(x) = 0, \ldots, \phi_m(x) = 0 \end{cases}$$

We compute $\lambda_j$ ; $j = 1, 2, \ldots, m$ & the possible pts $x$ & given extremum of $\psi(x)$.

If the points ~~xy~~ $x_n$ is conditional extremum of $f(x)$ then $x_n$ is a stationary pt of Lagrange $f^n$ of $\psi(x)$

In order to find conditional extremum of $f(x)$ we do further investigat$^n$

We write the quadratic form.

$$B(dx_1, dx_2, \dots, dx_{n-m}) = \sum_{i=1}^{n-m}\sum_{j=1}^{n-m} b_{ij}\, dx_i\, dx_j \quad , \qquad \text{— } (3)$$

i.e $2^{nd}$ differential of the $f^n$ $\varphi(x)$ evaluated at the stationary pt(s) & ī allowance condit$^n$s.

$$d\phi_1 = \frac{\partial \phi_1}{\partial x_1} dx_1 + \frac{\partial \phi_1}{\partial x_2} dx_2 + \dots + \frac{\partial \phi_1}{\partial x_n} dx_n = 0$$

$$\vdots$$

$$d\phi_m = \frac{\partial \phi_m}{\partial x_1} dx_1 + \dots + \frac{\partial \phi_m}{\partial x_m} dx_m = 0$$

If (3) is +ve definite at $x_0$ then $x_0$ is conditional minimum of $f(x)$

If (3) is −ve definite at $x_0$ " " " " maximum of $f(x)$

If (3) is indefinite at $x_0$ then $x_0$ is not the conditional extremum of $f(x)$.

Remark 1: If $x_0$ is an absolute extremum of Lagrange $f^n$ $\varphi(x)$ then $x_0$ is conditional extremum of $f(x)$ subject to $\phi_j(x) = 0$, $j = 1, 2, \dots, m$

Remark 2: The absence of absolute extremum of Lagrange $f^n$ $\varphi(x)$ does not signify the absence of conditional extremum of $f(x)$ subject to $\phi_j(x) = 0$ ; $j = 1, 2, \dots, m$ means,

suppose $\varphi(x)$ has no absolute extremum but $f(x)$ may or may not ~~be~~ have conditional extremum on prescribed domain.

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Ex $f(x,y) = xy$ subject to $\underbrace{y - x = 0}_{\phi}$

$\psi(x,y) = f(x,y) + \lambda\phi(x,y) = xy + \lambda(y-x) = 0$

$\frac{\partial\psi}{\partial x} = y - \lambda = 0$ , $y - x = 0$

$\frac{\partial\psi}{\partial y} = x + \lambda = 0$

$\left.\begin{array}{l} x + y = 0 \\ -x + y = 0 \end{array}\right\} \Rightarrow x = 0,\ y = 0 \quad ; \quad \lambda = y$

$\Rightarrow \boxed{\lambda = 0}$ , $x = 0, y = 0$

$(0,0)$ is stationary pt. of $\psi(x,y)$

$\Rightarrow \psi(x,y) = xy$

$\psi_{xx} = 0, \quad \psi_{yy} = 0, \quad \psi_{xy} = 1$

$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$

quadratic form is indefinite, so $\psi(x,y)$ has no absolute extremum at $(0,0)$.

$f(x,y) = xy$ , $\phi(x,y) = y - x = 0 \Rightarrow y = x$.

$f(x,y) = f(x,x) = F(x) = x^2$

$\frac{\partial F}{\partial x} = 0 \Rightarrow x = 0$ is a stationary pt. of $F(x)$

$\frac{d^2F}{dx^2} = 2 > 0$

$F(x)$ has minimum at 0

$(0,0)$ is conditional minimum of $f(x,y) = xy$ s.t. $y - x = 0$

Q. Find the conditional extremum of the $f^n$

$$f(x,y,z) = xyz$$

s.t. $x+y-z-3 = 0$

$x-y-z-8 = 0$

Sol$^n$ Lagrange $f^n$ $\phi(x,y,z) = xyz + \lambda_1(x+y-z-3) + \lambda_2(x-y-z-8)$

$$\frac{\partial \phi}{\partial x} = yz + \lambda_1 + \lambda_2 = 0 \qquad x+y-z-3 = 0$$

$$x-y-z-8 = 0.$$

$$\frac{\partial \phi}{\partial y} = xz + \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial \phi}{\partial z} = xy - \lambda_1 - \lambda_2 = 0$$

$\lambda_1 = \frac{11}{32}$, $\lambda_2 = \frac{-231}{32}$, $x = \frac{-11}{4}$

$y = \frac{-5}{2}$, $z = \frac{-11}{4}$ (check)

$\left(\frac{-11}{4}, \frac{-5}{2}, \frac{-11}{4}\right)$ is a critical pt. of $\phi$

$$\Rightarrow d^2\phi = \frac{\partial^2 \phi}{\partial x^2}dx^2 + \frac{\partial^2 \phi}{\partial y^2}dy^2 + \frac{\partial^2 \phi}{\partial z^2}dz^2 + 2\frac{\partial^2 \phi}{\partial x \partial y}dx\,dy$$

$$+ 2\frac{\partial^2 \phi}{\partial y \partial z}dy\,dz + \frac{2\partial^2 \phi}{\partial x \partial z}dx\,dz$$

$$\Rightarrow d^2\phi = 2z\,dx\,dy + 2y\,dx\,dz + 2x\,dy\,dz$$

$$d\phi_1 = dx + dy - dz = 0$$

$$d\phi_2 = dx - dy - dz = 0.$$

$$\Rightarrow dx = dz, \quad dy = 0.$$

$$\Rightarrow d^2\phi = 2y\,dx^2$$

$$d^2\phi\big|_{\text{critical pt.}} = -5\,dx^2$$

quadratic form is $-ve$ definite, so $f(x,y,z)$ has a conditional maximal at obtained critical pt $f_{max} = \frac{605}{32}$ (Verify)

## Linear Functional :

$T: V \rightarrow F$ is a linear functional if
($V$ is a vector space over the field $F$.)

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall x, y \in V \quad \forall \alpha, \beta \in F$$

## Functional :

Let $M$ be certain class of $f^n$s say $y(x)$

$$J : M \longrightarrow \mathbb{R}.$$
$$y(x) \longmapsto \underbrace{J[y(x)]}_{\text{Real no.}}$$

$J$ is a functional

eg (1) $M = C[0,1]$ : space of all continuous $f^n$s defined on $[0,1]$

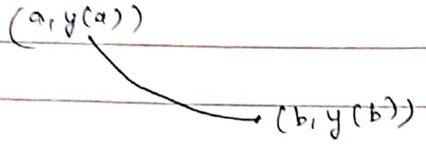
$M = \{ y(x) , y : [0,1] \rightarrow \mathbb{R} \rightarrow y(x)$ is continuous on $[0,1] \}$
$J : M \rightarrow \mathbb{R}$ is defined, as $J[y(x)] = \int_0^1 y(x)\,dx$
If $y(x) = x^2$ , $J[x^2] = \int_0^1 x^2 dx = \frac{1}{3}$.

eg (2) $M = C'[0,1]$ : space of all continuous differentiable $f^n$s defined on $[0,1]$

$M : \{ y , y : [0,1] \rightarrow \mathbb{R} \rightarrow y'(x)$ exists $\forall x \in [0,1]$ & $y'(x)$ is continuous on $[0,1] \}$

$J : C^1[0,1] \to \mathbb{R} \to J[y(x)] = \int_0^1 \sqrt{1+y'(x)^2}\, dx$

Construct a curve $y(x)$ which passes through $(a, y(a))$ & $(b, y(b))$ & the curve has smallest length




The variation or increment of the argument $y(x)$ of the functional $J[y(x)]$ is defined as
$\delta y = y(x) - y_1(x)$ where $y, y_1 \in M$

If $y(x)$ & $y_1(x)$ are continuous on $[a,b]$ & we say those are proximity of the zeroth order if $|y(x) - y_1(x)|$ is small on $[a,b]$

We say the curves $y(x)$ & $y_1(x)$ are close in the sense of proximity $k^{th}$ order if

$|y(x) - y_1(x)|$, $|y'(x) - y_1'(x)|$, $|y''(x) - y_1''(x)|$ ...

$|y^k(x) - y_1^k(x)|$ are small on $[a,b]$.

Ex $y(x) = \frac{\sin n^2 x}{n}$, $y_1(x) = 0$, $x \in [0,\pi]$ & for large $n$
$|y(x) - y_1(x)| = \left|\frac{\sin n^2 x}{n}\right| \leq \frac{1}{n}$

As $n$ is large, $|y(x) - y_1(x)|$ is small $y'(x) = \frac{\cos n^2 x \, n^2}{n}$
$= n\cos n^2 x$, $y_1'(x) = 0$

$|y'(x) - y_1'(x)| = |n\cos n^2 x - 0|$

In particular, $x = \frac{2\pi}{n^2}$

$|y'(x) - y_1'(x)| = n$

As $n \to \infty$, $|y'(x) - y_1'(x)| \to \infty$

$y(x) = \frac{\sin n^2 x}{n}$, $y_1(x) = 0$ are close in the sense of proximity of zeroth order.

Ex $y(x) = \frac{\sin nx}{n^2}$, $y_1(x) = 0$, $x \in [0, \pi]$ & $n$ is large

$|y(x) - y_1(x)| = \left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$

$|y(x) - y_1(x)|$ is small for large $n$

$|y'(x) - y_1'(x)| = \left|\frac{\cos nx}{n}\right| \leq \frac{1}{n} = 1$ is small. those are close in the proximity of first order

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# Variational - Methods

- Distance b/w the curves:

Let $f(x)$ and $g(x)$ be two continuing curves on $[a,b]$.
The distance b/w $f(x)$ and $g(x)$ is defined as:
zeroth order

$$S_0 = S_0(f(x), g(x)) = \max_{a \le x \le b} |f(x) - g(x)|$$

Ex. Find the zeroth order distance b/w $f(x) = x$ & $g(x) = x^2$ on $[0,1]$

Soln

$$S_0 = \max_{0 \le x \le 1} |x - x^2|$$

Let $h(x) = x - x^2$, $h'(x) = 1 - 2x = 0 \Rightarrow x = 1/2$

$x = 1/2$ is a critical point.

$h''(x) = -2 \le 0$, $h(x)$ has maximum at $x = 1/2$

$$h(1/2) = 1/2 - 1/4 = -1/4$$

$$\boxed{S_0 = 1/4}$$

The nth order distance b/w $f(x)$ & $g(x)$ is defined as:

$$S_n = S_n(f(x), g(x)) = \max_{0 \le k \le n} \max_{a \le x \le b} \left| f^{(k)}(x) - g^{(k)}(x) \right|$$

Ex. Find the $1^{st}$ order distance b/w $x^2$ & $x^3$ on $[0,1]$.

Soln

$$S_1 = \max\left\{ \max_{0 \le x \le 1} |x^2 - x^3|, \max_{0 \le x \le 1} |2x - 3x^2| \right\}$$

$h(x) = x^2 - x^3$, $h'(x) = 2x - 3x^2 = 0 \Rightarrow x(2 - 3x) = 0 \Rightarrow x = 0, \frac{2}{3}$

$h''(x) = 2 - 6x$, $h''(0) = 2 > 0$ (minimum at $x = 0$)

$h''(2/3) = 2 - 4 = -2 < 0$ max occur at $x = 2/3$

$h(2/3) = 4/4 - 8/27 = 4/27$, $\max_{0 \le x \le 1} h(x) = 4/27$

$h(0) = 0$, $h(1) = 1$

$K(x) = 2x - 3x^2 \Rightarrow K'(x) = 2 - 6x = 0 \Rightarrow x = 1/3$

$K''(x) = -6 < 0$

$K(1/3) = 2/3 - 1/3 = 1/3$

$K(0) = 0, \quad K(1) = -1$

$$\max_{0 \le x \le 1} |2x - 3x^2| = \max\left\{0, 1, \tfrac{1}{3}\right\} = 1 \text{ on } [0,1]$$

$$S_1 = \max\{1, 4/27\} = 1$$

- $\epsilon$ nbd of a curve $f(x)$ :

Let $f(x)$ be continuous on $[a,b]$. The zeroth order $\epsilon$ nbd of $f(x)$ on $[a,b]$ is defined as $\{g(x) : \max_{a \le x \le b} |g(x) - f(x)| < \epsilon\}$

In other words, $\{g(x) : \rho_0(f(x), g(x)) < \epsilon\}$

Strong nbd of $f(x)$ on $[a,b]$

- $n^{th}$ order nbd of $f(x)$ on $[a,b]$ is defined as set of all admissible curves $g(x)$ defined on $[a,b]$ s.t $\rho_n(f(x), g(x)) < \epsilon$. In other words,

$$\left\{g(x) : \max_{0 \le k \le n} \max_{a \le x \le b} |f^{(k)}(x) - g^{(k)}(x)| < \epsilon\right\}$$

$\Rightarrow$ First order nbd of $f(x)$ on $[a,b]$ is called '<u>weak nbd</u>' of $f(x)$

- Continuity of the functional $J[y(x)]$

We say $J[y(x)]$ ? is defined on the class of functions $y(x)$ is continuous at $y_0(x)$ in the sense of $n^{th}$ order proximity if for any given $\epsilon > 0$ we can find a +ve real number $\delta$ s.t

$$\rho_n(y(x), y_0(x)) < \delta \Rightarrow |J[y(x)] - J[y_0(x)]| < \epsilon ;$$

$$y(x) \in M$$

( $f(x)$ is continuous at $x_0$ if for given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$ )

set $y^{(k)}(x) = y_0^{(k)}(x) + \alpha w^{(k)}$; $\alpha \in \mathbb{R}$, $w(x) \in M$
for $k = 0, 1, 2, \dots, n$

$\lim_{\alpha \to 0} y^{(k)}(x) = y_0^{(k)}(x)$, then

$$\lim_{\alpha \to 0} J[y_0^{(k)} + \alpha w^{(k)}(x)] = J[y_0^{(k)}(x)] \; ; \text{ for } k = 0, 1, 2, \dots, n$$

Ex: $J[y(x)] = \int_0^1 [y(x) + 2y'(x)]\,dx$, Prove that $J$ is continuous on $C^1[0,1]$

Soln: Choose $\varepsilon > 0$, ~~set~~ arbitrary function $y_0(x) \in C^1[0,1]$
we want to $J[y(x)]$ is continuous at $y_0(x)$.

$$|J[y(x)] - J[y_0(x)]| = \left| \int_0^1 (y(x) + 2y')\,dx - \int_0^1 (y_0 + 2y_0')\,dx \right|$$

$$= \left| \int_0^1 (y(x) - y_0(x))\,dx + 2\int_0^1 (y'(x) - y_0'(x))\,dx \right|$$

$$\leq \int_0^1 |y(x) - y_0(x)|\,dx + 2\int_0^1 |y'(x) - y_0'(x)|\,dx$$

$$\leq \delta \int_0^1 dx + 2\delta \int_0^1 dx = \delta + 2\delta = 3\delta < \varepsilon$$

~~Choose $\delta = \varepsilon/3$~~

We choose $\delta < \varepsilon/3$

we say $J[y(x)]$ is discontinuous in the sense of $n^{th}$ order proximity if there exist $\varepsilon$ such that no matter what $\delta > 0$, there is always exists a function $y(x)$ such that

$$\rho_n(y(x), y_0(x)) < \delta \text{ follows } |J[y(x)] - J[y_0(x)]| \geq \varepsilon$$

Ex: $J[y(x)] = y'(x_0)$, where $y(x) \in C^1[a,b]$ & $x_0 \in [a,b]$.
s.t. the above functional is discontinuous of zeroth order proximity but is continuous of first order proximity on $C^1[a,b]$

$\underline{Sol^n}$ choose $\epsilon < 1$, for any $\delta > 0$

let us choose $\phi(x) \ni |\phi(x)| < \delta$ on $[a,b]$ & $\phi'(x_0) = 1$ for some $x_0 \in [a,b]$

let us ~~continuous~~ choose $y_0(x)$ arbitrary in $C'[a,b]$

set $y(x) = y_0(x) + \phi(x)$

$$\rho_0(y(x), y_0(x)) < \delta \quad \left( \because \rho_0(y(x), y_0(x)) = \max_{a \le x \le b} |y(x) - y_0(x)| = \max_{a \le x \le b} |\phi(x)| < \delta \right)$$

Now, $|J[y(x)] - J[y_0(x)]| = |y'(x_0) - y_0'(x_0)| = |\phi'(x_0)| = 1 > \epsilon$

$J[y(x)]$ is discontinuous on $C'[a,b]$ of zeroth order proximity.

We want to check for first order proximity if $\rho_1(y(x), y_0(x)) < \epsilon$

$$\Rightarrow \max\left\{ \max_{a \le x \le b} |y(x) - y_0(x)|,\ \max_{a \le x \le b} |y'(x) - y_0'(x)| \right\} < \delta$$

We have $|J[y(x)] - J[y_0(x)]| < \epsilon = |y'(x_0) - y_0'(x_0)| < \delta$

take $\delta = \epsilon$

( Using $f^n$ $J[y(x)]$ continuous at $y_0(x)$ of $n^{th}$ order proximity then $J(y(x))$ not necessarily be continuous at $y_0(x)$ of lower order proximity )

Let $M$ be normed linear space of $f^n$'s $y(x)$

$$L : M \longrightarrow \mathbb{R} \ni L[y(x)]$$
$$y(x) \longmapsto L[y(x)]$$

We say '<u>L is linear functional on</u>' M if

$$L[c_1 y_1(x) + c_2 y_2(x)] = c_1 L[y_1(x)] + c_2 L[y_2(x)]$$

where $y_1(x), y_2(x) \in M$ & $c_1, c_2 \in \mathbb{R}$ on $C'[a,b]$

- Increment or variat$^n$ of a functional :

$\Delta J = J[y(x) + \delta y(x)] - J[y(x)]$, where

$\delta y(x)$ is an increment in $y(x)$

or

$\Delta J = J[y_1(x)] - J[y_2(x)]$, where $y_1, y_2 \in M$, $\delta y = y_2(x) - y_1(x)$

eg $J[y(x)] = \int_0^1 y(x)\, y'(x)\, dx$, $y(x) \in C'[0,1]$

for $y_1(x) = x$ & $y_2(x) = x^2$ on $[0,1]$

Sol$^n$ $\Delta J = J[x^2] - J[x] = \int_0^1 2x\, x^2 dx - \int_0^1 x\, dx = 0$ ( verify )

- Differentiability of a functional :

If the increment of the functional $J[y(x)]$

$\Delta J = J[y(x) + \delta y(x)] - J[y(x)]$ can be written as

$\Delta J = L[y(x), \delta y] + \beta[y(x), \delta y]\, \|\delta y\|$, where

'L' is linear w.r.t $\delta y$ for a fixed y. & $\beta[y(x), \delta y] \to 0$ as $\|\delta y\| \to 0$, then $L[y(x), \delta y]$ is called 'variational of the functional' & it is denoted by $\delta J$.

In this case, we say the functional $J[y(x)]$ is differentiable at $y(x)$.

$$J[y(x)] = \int_a^b y^2(x)\, dx, \quad y(x) \in C'[a,b]$$

$$\Delta J = J[y+\delta y] - J[y] = \int_a^b (y+\delta y)^2 dy - \int_a^b y^2 dx$$

$$= \underbrace{\int_a^b 2y\, \delta y\, dy}_{\text{Linear wrt } \delta y} - \int_a^b \delta y^2 dx$$

~~$\beta[y, dy]$~~

$$\int_a^b \delta y^2 dx \le \|\delta y\|^2 \int_a^b dx = (b-a)\|\delta y\|^2 \quad \cancel{(b-a)(\|\delta y\|)(}$$

$$= ((b-a)\|\delta y\|)\, (\|\delta y\|)$$

According to def.

$\beta[y, \delta y] = (b-a)\|\delta y\| \to 0$ as $\|\delta y\| \to 0$ $\quad \left( \|\delta y\| = \max\limits_{a \le x \le b} \delta y \right)$

$J[y(x)]$ is differentiable on $C'[a,b]$

$$\delta J = 2\int_a^b y\,\delta y\,dx$$

$J[y(x)] = \int_a^b f(x, y, y', \ldots, y^{(n)})\,dx$; $y(x) \in C^m[a,b]$, then

$$\delta J = \int_a^b \left( \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + \cdots + \frac{\partial f}{\partial y^{(n)}}\delta y^{(n)} \right) dx$$

24/01/2020
Friday

- **Second definit^n of variat^n of a functional:**

Suppose we consider $\phi(\alpha) = J[y + \alpha\,\delta y]$

$$\delta J = \left.\frac{d\phi}{d\alpha}\right|_{\alpha=0}$$

Ex Find the variat^n of a functional

$$J[y(x)] = \int_a^b y^2(x)\,dx, \quad \text{where } y(x) \in C[a,b]$$

Sol^n $\phi(\alpha) = J[y + \alpha\,\delta y] = \int_a^b (y + \alpha\,\delta y)^2\,dx$

$$\frac{d\phi}{d\alpha} = \frac{d}{d\alpha}\int_a^b (y + \alpha\,\delta y)^2\,dx = \int_a^b 2(y + \alpha\,\delta y)\,\delta y\,dx$$

$$\left.\frac{d\phi}{d\alpha}\right|_{\alpha=0} = 2\int_a^b y\,\delta y\,dx$$

~~$J[x,y]$ is said to be a bilinear~~ functional

A functional $J[x,y]$ dependent on 2 elts $x$ & $y$ is called a '**bilinear functional**' if following holds:

i) $J[\alpha x_1 + \beta x_2, y] = \alpha J[x_1, y] + \beta J[x_2, y]$

ii) $J[x, \alpha y_1 + \beta y_2] = \alpha J[x, y_1] + \beta J[x, y_2]$, $\alpha, \beta \in \mathbb{R}$

→ set $y = x$ then $J[x, x]$ is called 'quadratic functional'
→ A quadratic functional $J[x, x]$ is said to be '+ve definite' if $J[x, x] > 0$ for every non zero elt $x$ & '−ve definite' if $J[x, x] < 0$ $\forall$ non-zero elt $x$.

Eg $J[x, y] = \int_a^b A(t)x(t)y(t)\,dt$, for fixed $A(t)$, $J[x, y]$ is bilinear functional.

set $y = x$ & fix $A(t)$, then $J = \int_a^b A(t)x(t)^2\,dt$ is a quadratic functional. Now we restrict quadratic functional for $A(t) > 0$.

$$\Rightarrow \quad J = \int_a^b A(t)x(t)^2\,dt > 0, \text{ c is +ve definite.}$$

▫ Second variatⁿ of a functional :

Consider the functional $J[y, x]$.

Now, consider $\Delta J = J[y + \delta y] - J[y]$. c can be written as

$$\Delta J = L_1[\delta y] + \frac{1}{2}L_2[\delta y] + \beta\,\|\delta y\|^2$$

where $L_1[\delta y]$ is a linear functional
$L_2[\delta y]$ " " quadratic " & $\beta \to 0$ as $\|\delta y\| \to 0$

The quadratic functional $L_2[\delta y]$ is called 'second variation of functional $J[y(x)]$' & it is denoted by $\delta^2 J$

Eg. Find the $2^{nd}$ variatⁿ of the functional

$$J[y(x)] = \int_0^1 (xy^2 + y'^3)\,dx \quad \text{for } y \in C'[0,1]$$

Solⁿ

$$\Delta J = J[y + \delta y] - J[y]$$

$$= \int_0^1 \left(x(y+\delta y)^2 + (y' + \delta y')^3\right)dx - \int_0^1 (xy^2 + y'^3)\,dx$$

$$= \int_0^1 (\cancel{xy^2} + x\delta y^2 + 2xy\delta y + \cancel{y'^3} + \delta y'^3 + 3y'^2\delta y' + 3y'\delta y'^2)\,dx - \int_0^1 (\cancel{xy^2} + \cancel{y'^3})\,dx$$

$$= \int_0^1 (2xy\delta y + 3y'^2\delta y')\,dx + \int_0^1 (x\delta y^2 + 3y'\delta y'^2)\,dx + \int_0^1 \delta y'^3\,dx$$

$$\because \int_0^1 \delta y'^3\,dx \leq \left(\max_{0 \le x \le 1} \delta y'\right)^2 \int_0^1 \delta y'\,dx \leq \|\delta y\|^2 \int_0^1 \delta y'\,dx ;$$

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where $\|\delta y\| = \max\left\{\max_{0\le x\le 1} \delta y, \max_{0\le x\le 1} \delta y'\right\}$

$$\left|\int_0^1 \delta y'^3 dx\right| \le \|\delta y\|^2 \int_0^1 |\delta y'| dx.$$

$$\beta \le \int_0^1 |\delta y'| dx \quad \text{as } \|\delta y\| \to 0, \ \beta \to 0$$

$\Rightarrow L_2[\delta y] = 2\int_0^1 (x\,\delta y^2 + 3y'\delta y'^2)\,dx$ is the second variatⁿ of the functional.

## Extremum of a functional :

$\rightarrow$ $f(x)$ has a relative maximum at $x_0$, if $f(x) \le f(x_0)$ $\forall$ $x$ near $x_0$

We say that a functional $J[y(x)]$ attains a maximum on a curve $y = y_0(x)$ if the values of the functⁿal $J[y(x)]$ on any curve close to $y = y_0(x)$ do not exceed $J[y_0(x)]$ that is

$$\Delta J = J[y(x)] - J[y_0(x)] \le 0$$

If $\Delta J = 0$ only when $y = y_0(x)$ then we say $J[y]$ has a strict maximum at $y_0$.

We say that a functional $J[y(x)]$ attains a minimum on a curve $y = y_0(x)$ if for the values of the functional $J[y(x)]$ on any curve close to $y = y_0(x)$, we have $\Delta J$ ~~≥0~~ $= J[y(x)] - J[y_0(x)] \ge 0$

If $\Delta J = 0$ only when $y = y_0(x)$ we say $y_0(x)$ is a strict minimum.

## Strong Maximum :

We say that a functional $J[y(x)]$ attains a strong maximum on a curve $y = y_0(x)$ if for all admissible curves $y = y(x)$ located on $\epsilon$-nbd of zeroth order proximity of the curve $y = y_0(x)$, we have $J[y(x)] \le J[y_0(x)]$

Slly, we can define strong minimum

$$J[y(x)] \ge J[y_0(x)] \quad \forall\, y \ni \rho_0(y(x), y(x)) < \epsilon$$

## Weak Maximum :

We say that a functional $J[y(x)]$ attains a weak minimum on a curve $y = y_0(x)$ if $\forall$ admissible curves $y = y(x)$

located on $\epsilon$- nbd of first order of a curve $y = y_0(x)$, we have

$$J[y(x)] \geq J[y_0(x)]$$

In other words.

$$J[y(x)] \leq J[y_0(x)] \quad \forall \; y(x) \text{ s.t. } \rho_1(y(x), y_0(x)) < \epsilon$$

where $\rho_1 = \max\left\{ \max_{x \in [a,b]} |y(x) - y_0(x)|, \max_{x \in [a,b]} |y' - y_0'| \right\}$

Q Weak maximum of a functional J implies strong maximum of a functional.

Sol$^n$ $J[y(x)] \leq J[y_0(x)]$ if $\rho_0(y, y_0) < \epsilon$ $\quad (\because \rho_1 > \rho_0)$

$J[y(x)] \leq J[y_0(x)]$ if $\rho_1(y, y_0) < \epsilon$

~~$\because \rho_1 > \rho_0$~~ so, TRUE .

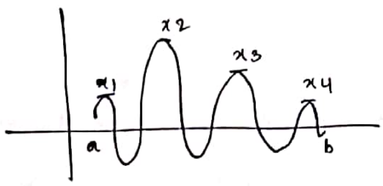
But converse need not be true.

Sly, we define strong minimum & weak minimum.

strong minimum: $J[y(x)] \geq J[y_0(x)]$ whenever $\rho_0(y(x), y_0(x)) < \epsilon$

weak minimum: $J[y(x)] \geq J[y_0(x)]$ whenever $\rho_1(y(x), y_0(x)) < \epsilon$

□ $f : \Omega \subseteq R^2 \to R$, $(x_0, y_0)$ is absolute maximum if




⊕ here '$x_2$' is absolute max.,

$\because$ $x_1, x_2, x_3, x_4$ : local maximum values of $f$, but $x_2$ (max. of local max.) is absolute maximum.

□ An extremum of a functional $J[y(x)]$ on the entire collect$^n$ of elements on $\subseteq$ it (J) is defined is called an 'absolute extremum'

Ex Consider the functional

$$J[y(x)] = \int_0^{\pi} y^2(1 - y'^2)\, dx, \quad y(x) \in C^1[0, \pi]$$

Sol$^n$ $y_0 = 0$, $y(x) = \dfrac{\sin nx}{\sqrt{n}}$

$$|y(x) - y_0(x)| = \left| \frac{\sin nx}{\sqrt{n}} \right| = \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty$$

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$$J[y(x)] - J[y_0(x)] = \int_0^{\pi} \frac{\sin^2 nx}{n}(1 - n\cos^2 nx)\,dx = \int_0^{\pi}\left(\frac{\sin^2 nx}{n} - \sin^2 nx\cos^2 nx\right)dx$$

(as $y'(x) = \sqrt{n}\cos nx$ )

$$= \int_0^{\pi}\left(\frac{\sin^2 nx}{n} - \frac{1}{4}\sin^2 2nx\right)dx$$

$$= \int_0^{\pi}\left(\left(\frac{1-\cos 2nx}{2n}\right) - \frac{1}{4}\left(\frac{1-\cos 4nx}{2}\right)\right)dx = \frac{\pi}{2n} - \frac{\pi}{8}$$

$$\Delta J = J\left[\frac{\sin nx}{\sqrt{n}}\right] - J[0] = \frac{\pi}{2n} - \frac{\pi}{8} < 0$$

Given functional $J[y(x)]$ has no strong minimum on the ~~curve~~ curve $y \equiv 0$, but it has weak minimum on $y \equiv 0$ for $c < 1$

$$\rho_1 = \max\{|y(x)|, |y'(x)|\}$$

& $\max\limits_{x \in [0,\pi]} |y'(x)| < 1$

$$J[y(x)] - J[0] = \int_0^{\pi} y^2(1-y'^2)\,dx \geq 0$$

$\because |y'| < 1 \Rightarrow (1 - y'^2) > 0$, so $J[y(x)] - J[0] \geq 0$

31/01/2020
Friday

Lemma: If $\alpha(x)$ is cont. on $[a,b]$ & if $\int_a^b \alpha(x)h(x)\,dx = 0$ for every $f^n$ $h(x) \in C[a,b]$ s.t. $h(a) = h(b) = 0$, then $\alpha(x) = 0$ $\forall x$ in $[a,b]$.

Proof: suppose $\alpha(x) \neq 0$ for some $x_0 \in [a,b]$. WLOG, we assume that $\alpha(x_0) > 0$.

$\because$ $\alpha(x)$ is continous at $x_0$, we can find a nbd say $(x_1, x_2) \subset [a,b]$ s.t. in $\subseteq$ $\alpha(x)$ is +ve i.e. $\alpha(x) > 0$ $\forall x \in (x_1, x_2)$.

Now, construct $h(x) = \begin{cases} (x-x_1)(x_2-x), & \text{if } x \in (x_1,x_2) \\ 0, & \text{o/w} \end{cases}$

It is clear that $h(x)$ is cont.

$h(a) = 0$, $h(b) = 0$ as $a,b \notin (x_1, x_2)$

so, $h(x)$ satisfies the condⁿ of Lemma.

$$\int_a^b \alpha(x) h(x) dx = \int_{x_1}^{x_2} \alpha(x)(x-x_1)(x_2-x) dx \quad \text{--- (1)}$$

integrand of R.H.S. of (1) is +ve, so $\int_{x_1}^{x_2} \alpha(x)(x-x_1)(x_2-x) dx$

but given $\int_a^b \alpha(x)(x-x_1)(x_2-x) dx = 0 \quad (\Rightarrow\Leftarrow)$

Hence, our assumptⁿ is wrong.

so, $\alpha(x) = 0 \quad \forall x \in [a,b]$

<u>Lemma</u>: If $\alpha(x)$ is cont. on $[a,b]$ & if $\int_a^b \alpha(x) h'(x) dx = 0 \quad \forall$ fⁿ $h(x) \in C^1[a,b] \to h(a) = h(b) = 0$, then $\alpha(x) = c \quad \forall x \in [a,b]$ where $c$ is a constant.

<u>Proof</u>: Let $c$ be a const. defined by the condⁿ

$$\int_a^b (\alpha(x) - c) dx = 0$$

Now construct $h(x) \to h(x) = \int_a^x (\alpha(A) - c) d(A)$

$$\to h'(x) = \alpha(x) - c \Rightarrow h(x) \in C^1[a,b]$$

$$h(a) = \int_a^a (\quad) dx = 0, \quad h(b) = \int_a^b (\alpha(A) - c) dA = 0$$

$$\to \int_a^b (\alpha(x) - c) h'(x) dx = \int_a^b \alpha(x) h'(x) dx - c\int_a^b h'(x) dx$$

$$= 0 - c[h(b) - h(a)] = 0$$

$$\Rightarrow \int_a^b (\alpha(x) - c) h'(x) dx = 0 \quad \text{--- (2)}$$

Now, the other way, $\int_a^b (\alpha(x) - c) h'(x) dx = \int_a^b (\alpha(x) - c)^2 dx$

from (2) we have $\int_a^b (\alpha(x) - c)^2 dx = 0$,

this is possible only when $(\alpha(x) - c) = 0$

$$\Rightarrow \alpha(x) = c \quad \forall x \in [a,b]$$

<u>Lemma</u>: If $\alpha(x)$ is cont. on $[a,b]$ & if $\int_a^b \alpha(x) h''(x) dx = 0$ for every fⁿ $h(x) \in C^2[a,b]$ s.t. $h(a) = h(b) = 0$ & $h'(a) = h'(b) = 0$ then $\alpha(x) = c_0 + c_1 x \quad \forall x \in [a,b]$ where $c_0, c_1$ are constants.

<u>Proof</u>: we choose $c_0, c_1$ those satisfy the following condⁿ.

$$\int_a^b (\alpha(x) - c_0 - c_1 x) dx = 0 \quad \&$$

$$\int_a^b dx \int_a^x (\alpha(t) - c_0 - c_1 t)\,dt = 0$$

Now, construct $h(x)$ as

$$h(x) = \int_a^x d\xi \int_a^\xi (\alpha(t) - c_0 - c_1 t)\,dt$$

$$\Rightarrow \quad h'(x) = \int_a^x (\alpha(\xi) - c_0 - c_1\xi)\,d\xi$$

$$\Rightarrow \quad h''(x) = \alpha(x) - c_0 - c_1 x, \quad \text{so } h(x) \in C^2[a,b]$$

$$h(a) = 0 \quad \& \quad h(b) = \int_a^b d\xi \int_a^\xi (\alpha(t) - c_0 - c_1 t)\,dt = 0$$

$$h'(a) = \int_a^a (\quad)\,d\xi = 0$$

$$h'(b) = \int_a^b (\alpha(\xi) - c_0 - c_1\xi)\,d\xi = 0$$

$$\rightarrow \int_a^b (\alpha(x) - c_0 - c_1 x)\,h''(x)\,dx = \int_a^b \alpha(x) h''(x)\,dx - c_0\int_a^b h''(x)\,dx - c_1\int_a^b x\,h''(x)\,dx = 0$$

$$= 0 - c_0[h'(b) - h'(a)] - c_1[bh'(b) - ah'(a)] - [h(b) - h(a)]$$

$$= 0$$

$$\Rightarrow \int_a^b (\alpha(x) - c_0 - c_1 x)\,h''(x)\,dx = 0 \quad \text{— (3)}$$

$$\int_a^b (\alpha(x) - c_0 - c_1 x)\,h''(x)\,dx = \int_a^b (\alpha(x) - c_0 - c_1 x)^2\,dx$$

from (3), we have $\int_a^b (\alpha(x) - c_0 - c_1 x)^2\,dx = 0$ only when $\alpha(x) - c_0 - c_1 x = 0$

$$\Rightarrow \quad \alpha(x) = c_0 + c_1 x \quad \forall x \in [a,b]$$

<u>Lemma</u>: If $\alpha(x)$ & $\beta(x)$ are cont. on $[a,b]$ & if

$$\int_a^b (\alpha(x) h(x) + \beta(x) h'(x))\,dx = 0 \quad \text{— (4)} \quad \forall \text{ f}^n\ h(x) \in C^1[a,b]$$

s.t $h(a) = h(b) = 0$ then $\beta(x)$ is differentiable & $\beta'(x) = \alpha(x)\ \forall x \in [a,b]$

<u>Proof</u>: Let $A(x) = \int_a^x \alpha(\xi)\,d\xi$ —(5)

$$\int_a^b \alpha(x) h(x)\,dx = \left[h(x)\int_a^x \alpha(\xi)\,d\xi\right]_a^b - \int_a^b h'(x) A(x)\,dx$$

$$= h(a)A(a) - h(b)A(b) - \int_a^b h'(x) A(x)\,dx$$

$$\int_a^b \alpha(x)h(x)\,dx = -\int_a^b A(x)h'(x)\,dx \quad \text{— (6)}$$

$$\int_a^b \left(\alpha(x)h(x) + \beta(x)h'(x)\right)dx = 0 \quad (\text{from (6)})$$

$$\int_a^b \left(-A(x)h'(x) + \beta(x)h'(x)\right)dx = 0$$

$$\Rightarrow \int_a^b \left(\beta(x) - A(x)\right)h'(x)\,dx = 0$$

from lemma (2), we have

$$\beta(x) - A(x) = c \quad \forall x \in [a,b], \text{ where } c \text{ is a constant}$$

$\Rightarrow \beta(x) = c + A(x)$

$\Rightarrow \beta(x)$ is differentiable & $\beta'(x) = A'(x) = \alpha(x)$, where $A(x) = \int_a^x \alpha(s)\,ds$

Thm: A necessary condⁿ for the differentiable functional $J[y]$ to have an extremum on the curve $y = y_0(x)$ is that first variatⁿ vanishes for $y = y_0(x)$ i.e. $\delta J = 0$.

Proof: Let us suppose $J[y]$ has a minimum on the curve $y = y_0(x)$. Accord. to the definitⁿ, we have,

$$\Delta J = J[y_0 + h] - J[y_0] = \delta J[h] + \beta\|h\| \quad \text{— (1)}$$

where '$h$' is an increment in $y_0(x)$ & $\beta \to 0$ as $\|h\| \to 0$

The sign of $\Delta J$ is same as the sign of $\delta J$. Suppose for some $h_0(x)$, we have $\delta J[h_0] \neq 0$.

For any $\alpha > 0$, however small $\delta J[-\alpha h_0(x)] = -\delta J[\alpha h_0(x)]$

The RHS of (1) takes both the signs, but we have

$$\Delta J = J[y_0 + h] - J[y_0] \geq 0 \quad (\Rightarrow\Leftarrow)$$

$\Rightarrow \delta J = 0$

Thm: Let $J[y]$ be a functional of the form

$$\int_a^b F(x, y, y')\,dx$$

defined on the set of functions those have cont. derivatives on $[a,b]$ and satisfy the boundary conditions $y(a) = A$ & $y(b) = B$ then a necessary condⁿ on $J[y]$ to have an extremum for a given fⁿ $y(x)$ is that $y(x)$ satisfies the Euler eqⁿ

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Proof: let $h(x)$ be an increment in $y(x)$ s.t. $h(a) = h(b) = 0$

$$\Delta J = J[y+h] - J[y] = \int_a^b F(x, y+h, y'+h')\,dx - \int_a^b F(x, y, y')\,dx$$

$$= \int_a^b \left(F(x, y+h, y'+h') - F(x, y, y')\right)dx$$

$$= \int_a^b \left(\frac{\partial F}{\partial y}(x, y, y')h + \frac{\partial F}{\partial y'}(x, y, y')h' + \dots\right)dx$$

$$\delta J = \int_a^b \left(F_y(x, y, y')h + F_{y'}(x, y, y')h'\right)dx$$

Necessary cond$^n$ for the functional $J$ to have extremum is $\delta J = 0$

$$\Rightarrow \int_a^b \left(F_y(x, y, y')h + F_{y'}(x, y, y')h'\right)dx = 0$$

by using lemma 4, we have $\frac{d}{dx}(F_{y'}) = F_y$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\boxed{\frac{\partial F}{\partial y} - F_{y'x} - F_{y'y}y' - F_{y'y'}y'' = 0} \quad 2$$

Second order ODE $\bar{c}$ unknown as $y(x)$, $y(a) = A$, $y(b) = B$

eg $\int_1^2 (y'^2 - 2xy)\,dx$, $y(1) = 0$, $y(2) = -1$

Sol$^n$ $F(x, y, y') = y'^2 - 2xy$

$F_y - \frac{d}{dx}(F_{y'}) = -2x - 2y'' = 0 \Rightarrow \boxed{y'' + x = 0}$

$y'' = -x \Rightarrow y' = \frac{-x^2}{2} + c_1 \Rightarrow \boxed{y = -\frac{x^3}{6} + c_1 x + c_2}$

$c_1 + c_2 = \frac{1}{6}$

$2c_1 + c_2 = -1 + \frac{8}{6} = \frac{2}{6} \Rightarrow c_1 = \frac{1}{6}, \; c_2 = 0$

$y(x) = -\frac{x^3}{6} + \frac{x}{6}.$

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$J[y] = \int_a^b F(x, y, y')\,dx$ , $y(a) = A$ , $y(b) = B$

$\delta J = 0$ , $F_y - \frac{d}{dx}(F_{y'}) = 0$

eg $J[y] = \int_1^3 (3xy - y^2)\,dx$ , $y(1) = 1$, $y(3) = 9/2$ (No extremal)

Sol$^n$ Associated Euler equat$^n$

$F_y - \frac{d}{dx}(F_{y'}) = 3x - 2y - \frac{d}{dx}(0) = 0 \Rightarrow 3x - 2y = 0$

$\Rightarrow y = \frac{3x}{2}$

$\Rightarrow y(1) = 3/2$ (but $y(1) = 1$)

~~$y(1) = 3/2$~~

So, the variational problem does not have solution.

eg Find the extremal of the f$^n$: (Infinite extremal)

$J[y] = \int_0^{2\pi} (y'^2 - y^2)\,dx$ s.t. $y(0) = 1$ , $y(2\pi) = 1$

Sol$^n$ $F_y - \frac{d}{dx}(F_{y'}) = (-2y) - \frac{d}{dx}(2y') = 0$

$\Rightarrow -2y - 2y'' = 0$

$\Rightarrow y + y'' = 0$

$\Rightarrow y = A\cos x + B\sin x$ (general sol$^n$)

$\because y(0) = A = 1$ , $y(2\pi) = A = 1$

So, finally our extremal is $y(x) = \cos x + B\sin x$, B: arbitrary constant.

□ Existence & Uniqueness Theorem for extremal of the functional :

(Bernstein)

Thm: Suppose we have the functional $J[y] = \int_a^b F(y, x, y')\,dx$ & the continuous $y(a) = A$ & $y(b) = B$. The corresponding Euler eq$^n$ of the functional can be written as

$$y'' = G(x, y, y')$$

If the functions $G$, $G_y$ & $G_{y'}$ are cont. at every finite pt $(x, y)$ for any finite $y'$ & if there exists a constant $k > 0$, functions $\alpha(x, y) \geq 0$, $\beta(x, y) \geq 0$ bounded on every finite partit$^n$ of the plane s.t.

$G_y(x, y, y') \geq k$ & $|G(x, y, y')| \leq \alpha y'^2 + \beta$

then one & only one extremal of Euler eq^n passing through any 2 points $(a, A)$ & $(b, B)$ in the plane that have different abscissas $(a \neq b)$.

In other words, Given $\int_a^b F(x, y, y')dx \ni y(a) = A$, $y(b) = B$.

Euler eq^n $F_y - \frac{d}{dx}(F_{y'}) = y'' - G(x, y, y') = 0$. If $G, G_y, G_{y'}$ are cont. for finite values of $x, y, y'$. If $\exists\ k > 0$, bounded f^ns $\alpha(x, y) \geq 0$, $\beta(x, y) \geq 0 \ni G_y \geq k$ & $|G| \leq \alpha y'^2 + \beta$ then the extremal of Euler eq^n is unique.

eg $\int e^{-2y^2}(y'^2 - 1)\,dx$, Prove that any two pts on the plane $\bar{c}$ different abscissas there passes one & only extremal.

Sol^n $F(x, y, y') = e^{-2y^2}(y'^2 - 1)$

$\Rightarrow F_y - \frac{d}{dx}(F_{y'}) = e^{-2y^2}(-4y)(y'^2 - 1) - \frac{d}{dx}(e^{-2y^2} 2y') = 0$

$\Rightarrow -4yy'^2 + 4y + 8yy'^2 - 2y'' = 0$

$\Rightarrow 2yy'^2 + 2y - y'' = 0$

$\Rightarrow y'' = 2y(1 + y'^2) = G(x, y, y')$

$\Rightarrow G_y = 2(1 + y'^2) \geq 2 \Rightarrow k = 2$

$|G(x, y, y')| = |2y(1 + y'^2)| = 2|y|(1 + y'^2) = \alpha y'^2 + \beta$, where

$\alpha - 2|y| = \beta$

By Bernstein theorem, $\exists$ only one extremal $\ni$ passes through any 2 pts in the plane $\bar{c}$ different abscissas.

$\rightarrow$ Suppose we have

$J[y] = \int_a^b F(x, y, y')dx$ $\bar{c}$ $y(a) = A$, $y(b) = B$

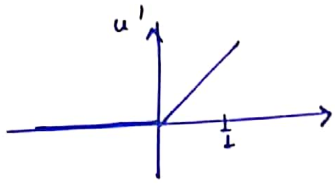
Associated Euler eq^n is a 2^nd order ODE.

For some functional $\ni$ attains an extremum value on some curve, say $y(x)$. But $y(x)$ is not twice differentiable.

Ex: $\int_{-1}^{1} y^2(2x - y')^2 dx$, $y(-1) = 0$, $y(1) = 1$

J takes minimum value 0 on the curve, $u(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ x^2, & 0 < x \leq 1 \end{cases}$

$$u'(x) = \begin{cases} 0, & \text{if } -1 \le x \le 0 \\ 2x, & \text{if } 0 < x \le 1 \end{cases}$$




but $u''(0)$ does not exist.

$$F_y - \frac{d}{dx}(F_{y'}) = 2y(2x-y')^2 - \frac{d}{dx}\left(-2y^2(2x-y')\right)$$

$y = u(x)$, $F_y = 0$ when $y = u(x)$

$$\frac{d}{dx}(F_{y'}) = \frac{d}{dx}\left(-2y^2(2x-y')\right) = \frac{d}{dx}(0) = 0.$$

So, $u(x)$ satisfies Euler equat$^n$ but it is not twice differentiable.

Thm: Suppose $y = y(x)$ has a first order cont. derivatives & satisfies Euler eq$^n$

$F_y - \frac{d}{dx}(F_{y'}) = 0$. If the f$^n$ $F(x, y, y')$ has $1^{st}$ & $2^{nd}$ order cont. partial derivatives w.r.t. its arguments, then $y(x)$ has a cont. $2^{nd}$ order derivatives at all pts $(x, y)$ for $F_{y'y'}(x, y, y') \neq 0$.

In earlier example $F = y^2(2x - y')^2$

$$F_{y'} = -2y^2(2x - y')$$

$$F_{y'y'} = 2y^2 \neq 0$$

Let $2y^2 = 0 \Rightarrow y = 0$

In order to solve Euler eq$^n$. We consider some special cases, where $J = \int_a^b F(x, y, y')\,dx$

- Case-I: F does not depend on y i.e.

$$J = \int_a^b F(x, y')\,dx$$

Euler eq$^n$ is $\frac{d}{dx}(F_{y'}) = 0 \Rightarrow F_{y'} = c$, c: arbitrary constant

$y' = f(x, c)$, & can be integrated.

Ex $\int_1^2 y'(1 + x^2 y')\,dx$, $y(1) = 3$, $y(2) = 5$

Sol$^n$ $F_{y'} = 1 + 2x^2 y' = c \Rightarrow y' = \frac{c-1}{2x^2} \Rightarrow y = \frac{A}{x} + B$, $A = \frac{1-c}{2}$

$y(x) = \frac{A}{x} + B$, $y(1) = 3 \Rightarrow A + B = 3$ $\Rightarrow A = -4$, $B = 7$

$y(2) = 5 \Rightarrow \frac{A}{2} + B = 5$

$y(x) = 7 - \frac{4}{x}$

$J = \int_a^b F(x, y, y')\,dx$

- **Case - II** : F does not depend on x, i.e. $J = \int_a^b F(y, y')\,dx$. Euler eq$^n$ is $F_y - \frac{d}{dx}(F_{y'}) = 0$

$$F_y - F_{y'y}\,y' - F_{y'y'}\,y'' = 0$$

$$\Rightarrow \quad y'F_y - \cancel{F_{y'y}}\; y'^2 F_{y'y} - y'y''F_{y'y'} = 0$$

$$\frac{d}{dx}(F - y'F_{y'}) = 0$$

$$\left(\text{as } F_y y' + \cancel{F_{y'}y''} - \cancel{y''}F_{y'} - y'F_{y'y}y' - y'F_{y'y'}y'' = 0\right)$$

$F - y'F_{y'} = c$ where c is an arbitrary constant it represent first order ODE & can be solved easily.

Ex : $\int_0^1 yy'^3\,dx$ $\bar{c}$ $y(0) = 0$, $y(1) = 2$

Sol$^n$ $F_{y'} = 3yy'^2$

$F - y'F_{y'} = c \Rightarrow yy'^3 - y'.3yy'^2 = c$

$\Rightarrow yy'^3 - 3yy'^3 = c$

$\Rightarrow -2yy'^3 = c \Rightarrow yy'^3 = c_1 \Rightarrow y'^3 = \frac{c_1}{y}$, where $c_1 = \frac{c(-1)}{2}$

$\Rightarrow y' = \frac{c_1}{\sqrt[3]{y}} \Rightarrow y = (c_1 x + c_2)^{3/4}$

$y(0) = (c_2)^{3/4} = 0 \Rightarrow c_2 = 0$

$y(x) = (c_1 x)^{3/4}$

$y(1) = c_1^{3/4} = 2 \Rightarrow c_1 = 2^{4/3}$

$\Rightarrow \boxed{y(x) = 2x^{3/4}}$

- **Case - III** : F does not depend on $y'$, i.e. $J = \int_a^b F(x, y)\,dx$.

Euler eq$^n$ is ~~Fy~~ $F_y(x, y) = 0$

$\underline{\underline{Ex}}$ : $\int_0^{\pi/2} (2xy - y^2)\, dx$ , $y(0) = 0$, $y(\pi/2) = 1$

$\underline{\underline{Sol^n}}$ Euler $eq^n$ is $2x - 2y = 0 \Rightarrow x = y$

So, $y(0) = 0$ & $y(\pi/2) = \pi/2 \neq 1$ $(\Rightarrow\Leftarrow)$

$\Rightarrow$ this functional has no $sol^n$.

$\underline{\underline{Ex}}$ if in above $y(\pi/2) = \pi/2$

$\underline{\underline{Sol^n}} \Rightarrow$ this functional attains an extremum.

$\underline{\underline{Ex}}$ $\int_a^b f(x,y)\sqrt{1+y'^2}\, dx$ , Compute Euler $eq^n$, where $f$: arbitrary $f^n$.

e

$\frac{\pi}{2}(-1)$