Product Form of Inverse(PFI)of a Basis Matrix and Revised Simplex Method (RSM)

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We wish to compute the inverse of a basis matrix, B_c, that is differ by one column from the basis matrix, B, whose inverse is known. The product form of the inverse allows us to determine this new inverse in an efficient manner. We want to find B_c^{-1}.

First, let us consider the following definitions:

B is the original basis matrix of size $m \times m$.

B_c is the new basis matrix, which is identical to B *except* for the column r.

c is the rth column of matrix B_c, the only column different from those in B.

$$\boldsymbol{e} = B^{-1}c = (e_1, e_2, \ldots, e_m)^T \tag{1}$$

$$\eta = \left(-\frac{e_1}{e_r}, \ldots, -\frac{e_{r-1}}{e_r}, \frac{1}{e_r}, -\frac{e_{r+1}}{e_r}, \ldots, -\frac{e_m}{e_r}\right)^T, e_r \neq 0 \tag{2}$$

where e_r is the r-th component of $\boldsymbol{e}$ as computed in (1) and m is the total number of elements of the column vector $\boldsymbol{e}$. Thus,

$$B_c^{-1} = E_r B^{-1} \tag{3}$$

where B_c^{-1}= inverse of B_c
B^{-1} = inverse of the previous matrix
E_r= an identity matrix with its r-th column replaced by η.

We now use (3) to illustrate the computation of the inverse of a basis matrix that differs by only a single column from another basis matrix, whose inverse is known.

Example 1:

Consider the two matrices shown below. Both are non-singular and differ by only one column, the first. The inverse of B is given and we wish to find the inverse of B_c.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_c = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \quad B_c^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

We first compute $\boldsymbol{e}$ from (1), where

$$c_1 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(i.e., the first column in B_c)

$$\boldsymbol{e} = B^{-1}c_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

Next, from (2) we establish η:

$$\eta = \begin{pmatrix} \frac{1}{2} \\ -1 \\ -2 \end{pmatrix}$$

Thus,

$$E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$B_c^{-1} = E_1 B^{-1}$$

$$B_c^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Example 2:

Consider the two matrices shown below. Both are non singular and differ by only one column, the second. The inverse of B is given and we wish to find the inverse of B_c.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B_c = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix} \qquad B_c^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

We first compute $\boldsymbol{e}$ from (1), where

$$c_2 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(i.e., the second column in B_c)

$$\boldsymbol{e} = B^{-1}c_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

Next, from (2) we establish η:

$$\eta = \begin{pmatrix} -1 \\ \frac{1}{2} \\ -2 \end{pmatrix}$$

Thus,

$$E_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$B_c^{-1} = E_2 B^{-1}$$

$$B_c^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

Example 3:

Consider the two matrices shown below. Both are non singular and differ by only one column, the third. The inverse of B is given and we wish to find the inverse of B_c.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B_c = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \qquad B_c^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

We first compute $\boldsymbol{e}$ from (1), where

$$c_3 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(i.e., the third column in B_c)

$$\boldsymbol{e} = B^{-1}c_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

Next, from (2) we establish η:

$$\eta = \begin{pmatrix} -1/2 \\ -1/2 \\ 1/4 \end{pmatrix}$$

Thus,

$$E_3 = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$B_c^{-1} = E_3 B^{-1}$$

$$B_c^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Let B be a basis matrix of size $m \times m$.
Let $B = I_{m\times m}$ (an identity matrix of size $m \times m$)
Then $B = B^{-1} = I_{m\times m}$.
Let $B_1, B_2, \ldots, B_m$ are m non-singular matrices of size $m \times m$.
B and B_1 are differ by first column.
B_1 and B_2 are differ by second column.
B_2 and B_3 are differ by third column.
B_3 and B_4 are differ by fourth column.
$\vdots$
B_{m-1} and B_m are differ by m-th column.
Now $B_1^{-1} = E_1 B^{-1} = E_1 I_{m\times m} = E_1$
Then $B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$
$B_3^{-1} = E_3 B_2^{-1} = E_3 E_2 E_1$
$B_4^{-1} = E_4 B_3^{-1} = E_4 E_3 E_2 E_1$
$B_m^{-1} = E_m B_{m-1}^{-1} = E_m E_{m-1} \ldots E_1$
where $E_r, (r = 1, 2, ..., m)$ is defined in equation (3).

Example 4:

Consider four different matrices shown below. All are non-singular matrices and differ by only one column. The inverse of the matrices are computed as follows:

$$B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 6 & 1 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 6 & 5 \end{pmatrix} = B_{new}$$

We first compute $\boldsymbol{e}$ from (1), where

$$c_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

From B and B_1 we find c_1.

(i.e., the first column in B_1)

$$\boldsymbol{e} = B^{-1}c_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Next, from (2) we establish η:

$$\eta = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$$

Thus,

$$E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_1^{-1} = E_1 B^{-1}$$

$$B_1^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then we compute c_2.

$$c_2 = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

From B_1 and B_2 we find c_2.

(i.e., the second column in B_2)

$$\boldsymbol{e} = B^{-1}c_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$$

Next, from (2) we establish η:

$$\eta = \begin{pmatrix} -1 \\ 1/2 \\ -3 \end{pmatrix}$$

Thus,

$$E_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$$

$$B_2^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

Then we compute c_3.

$$c_3 = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

(i.e., the third column in B_3)

$$\boldsymbol{e} = B_2^{-1} c_3 = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

Next, from (2) we establish η:

$$\eta = \begin{pmatrix} 0 \\ 0 \\ 1/5 \end{pmatrix}$$

Thus,

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}$$

$$B_3^{-1} = E_3 B_2^{-1} = E_3 E_2 E_1 = B_{new}^{-1}.$$

$$B_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3/5 & 1/5 \end{pmatrix}$$

Hence $B_{new}^{-1} = B_3^{-1} = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3/5 & 1/5 \end{pmatrix} = E_3 E_2 E_1$

Revised Simplex Method

Original simplex method calculates the stores all numbers in the simplex Tableau. Many are not needed.
Revised Simplex Method (more efficient for computing):
It is used in all commercially packages (e.g. IBM MPSX, CDC APEX III).

$$LPP \quad \max: \quad Z = c^T x$$

Subject to

$$\begin{aligned} Ax &\leq b, \quad b \geq 0 \\ x &\geq 0. \end{aligned}$$

Initially constraints becomes (standard form):

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ x_s \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

x_s = slack variables
Basis matrix: Column relating to basic variables.

$$B = \begin{pmatrix} B_{11} & \dots & \dots & \dots & B_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ B_{m1} & \dots & \dots & \dots & B_{mm} \end{pmatrix}_{m \times m}$$

Initially $B = I_{m \times m}, B^{-1} = I_{m \times m}$.

Basic variable values: $X_B = \begin{pmatrix} X_{B1} \\ \dots \\ \dots \\ \dots \\ X_{Bm} \end{pmatrix}$

At any iteration all the non-basic variables are zero.

$$BX_B = b$$

Therefore $X_B = B^{-1}b$ where B^{-1}, inverse basis matrix.
At any iteration, given the original b vector and the inverse matrix B^{-1}, X_B can be calculated.
$Z = c_B^T x_B$, where c_B =objective coefficients of basic variables.

Steps in the Revised Simplex Method

Step 1. Determine the entering variable, x_j, with associated vector P_j.
–compute $Y = c_B^T B^{-1}$
–compute $z_j - c_j = YP_j - c_j$ for all non-basic variables.
Select the largest negative value (For Max type LPP) among all $z_j - c_j$.
Break the ties arbitrarily. If all the $z_j - c_j \geq 0$, optimal solution is reached.
$X_B = B^{-1} b$
$Z = c_B^T X_B$
Otherwise go to Step 2.
Step 2. Determine leaving variable, x_r, with associated vector P_r.
–compute the current basic variable $X_B = B^{-1} b$
– compute constraint coefficients of entering variables for P_j:

$$\alpha^j = B^{-1} P_j$$

Leaving variable x_r must be associated with

$$\theta = \min_k \left\{ \frac{(B^{-1}b)_k}{\alpha_k^j}, \alpha_k^j > 0 \right\}.$$

using minimum ratio rule.
If $\alpha_k^j \leq 0,\ \forall k$, then the problem is unbounded.
Step 3. Determination of the next basis matrix and B_{next}^{-1}
For the given B^{-1} the B_{next}^{-1} is computed by
$B_{next}^{-1} = E_r\ B^{-1}$, where r is the column number of the entering vector
Set $B^{-1} = B_{next}^{-1}$
Go to step 1. Note E is computed using equation (3).

(See the next slide for the numerical example) Note: If you detect any typo please inform me.