Group Action

Lecture 14

<u>Cayley's Thm</u>: Every finite gp G is isomorphic to a subgp of a permutation gp. If G has order n then it is isomorphic to a subgp of S_n.

<u>Pf</u>: Consider the the gp action

$$G \times G \longrightarrow G$$
$$(g, x) \longmapsto gx \quad \text{(left multipl)}$$

↳ operation of the gp.

Using permutation representation of a gp action we define a gp homo

$$\psi : G \longrightarrow S_{|G|}$$
$$g \longmapsto \sigma_g : G \to G$$
$$\sigma_g(x) = gx.$$

ψ is a gp homo. (check!)

$$\ker \psi = \{ g \in G \mid \sigma_g = Id \}.$$

$$= \{ g \in G \mid \sigma_g(x) = x \ \ \forall x \in G \}$$

$$= \{ g \in G \mid gx = x \ \ \forall x \in G \}$$

$$= \{ 1 \}.$$

$\therefore$ G is isomorphic to its image in $S_{|G|}$.

i.e if $|G| = n$ then G is isomorphic to a subgp of order n of S_n.

Group action by conjugation :

Let G be a gp and consider $A = G$.
Then we define the gp action by conjugation as

$$G \times G \longrightarrow G$$

$$(g, x) \longmapsto gxg^{-1}$$ (The operation is the gp operation).

$$\text{stb}(x) = \{ g \in G \mid gxg^{-1} = x \}.$$

$$= \{ g \in G \mid gx = xg \}.$$

:= centralizer of x and is denoted by $C(x)$

$$O(x) = \{ gxg^{-1} \mid g \in G \}$$

:= conjugacy class of x $= \mathcal{C}_x$.

Since the conjugacy classes are orbits they partition G.

Note that an elt $x \in G$ have conjugacy class of size 1. then it implies that $gxg^{-1} = x \quad \forall g \in G$.

$$\Rightarrow \quad gx = xg \quad \forall g \in G.$$

$$\Rightarrow \quad x \in Z(G).$$

i.e An elt $x \in G$ have conjugacy class of size 1 iff $x \in Z(G)$.

Let G be a gp of finite order and $Z(G) = \{1, z_1, \ldots, z_m\}$ and let $C_{g_1}, \ldots, C_{g_r}$ be be the conjugacy classes of G not contained in the centre.

Since G is a partition of all conjugacy classes we have

$$G = C_1 \cup C_{z_1} \cup \cdots \cup C_{z_m} \cup C_{g_1} \cdots \cup C_{g_r}.$$

$$|G| = 1 + 1 + \cdots + 1 + \sum_{i=1}^{r} |C_{g_i}|$$

$$|G| = |Z(G)| + \sum_{i=1}^{r} |C_{g_i}|$$

This is known as the class eqn..

Example. Let $G = S_3$. Write down the class eqn. of S_3.

Note that $Z(S_3) = (1)$.

Let $g_1 = (12)$. $C_{g_1} = \{ \sigma (12) \sigma^{-1} / \sigma \in S_3 \}$

$= \{(12), (23), (13)\}$.

$g_2 = (23)$ $\quad C_{g_2} = C_{g_1}$

$g_3 = (123)$ $\quad C_{g_3} = \{\sigma(123)\sigma^{-1} \mid \sigma \in S_3\}$

$g_4 = (132)$ $\quad = \{(123), (132)\}$

$= C_{g_4}$

The class eqn of S_3 is

$$6 = 1 + 3 + 2.$$

Ex. Describe the conjugacy of S_4.

Some Application:

Defn A gp of order p^n, p is a prime no. and $n \in \mathbb{N}$. is called a p-group.

Propn Let G be a p-gp. Then $|Z(G)| \geq p$.

Pf: Suppose that $Z(G) = \{1\}$.

Then $|G| = |Z(G)| + \sum_{i=1}^{r} |C_{g_i}|$

$$\Rightarrow \quad |G| = 1 + \sum_{i=1}^{r} |C_{g_i}| \quad \text{——} \quad (1)$$

$$|C_{g_i}| = [G : C(g_i)] = \frac{|G| = p^m}{|C(g_i)|}$$

Since $p \mid |G|$

and $p \mid |C_{g_i}|$

we get a contradiction from eq (1).

$\therefore \quad |Z(G)| \geq p.$

Cor. Any gp of order p^2 is abelian.

Pf: By previous propn we have $|Z(G)| \geq p$.

If $|Z(G)| = p^2$ then $Z(G) = G$.

Thus G is abelian.

Let $|Z(G)| = p$. Let $x \in G \setminus Z(G)$.

Note that $Z(G) \subset C(x)$.

and also $x \in C(x) \setminus Z(G)$.

$\therefore$ $C(x)$ is strictly bigger than $Z(G)$

$\therefore$ $|C(x)| = p^2$. which is a contradict

$\therefore$ $Z(G) = G$. $\quad C(x) = G$.

$\Rightarrow x \in Z(G)$.

Hence G is abelian.

Cor. Any gp of order p^2 is isomorphic to either $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Pf: Let $x \neq 1 \in G$.

then $|x| = p$ or p^2.

If $|x| = p^2$ then $G \cong \mathbb{Z}/p^2\mathbb{Z}$.

Let there doesn't exist any elt of order p^2 and $1 \neq x \in G$. s.t $|x| = p$. Consider $H = \langle x \rangle$.

and $1 \neq y \notin H$ then $|y| = p$.

Let $K = \langle y \rangle$. Then $H \cap K = \{1\}$.

Since G is of order p^2 then G is abelian.

H and K both are normal subgps.

Then HK is a subgp of G which is strictly bigger than H.

$\therefore$ $|HK| = p^2$. Then $G = HK$.

$\therefore$ $G \cong H \times K \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

Q7. $GL_n(\mathbb{R}) \xrightarrow{f} \{1,-1\}$.

$$f(A) = \frac{\det A}{|\det A|}$$

$\ker f = \{$ collection all matrices have $\det > 0 \}$.

$$GL_n(\mathbb{R}) / \ker f \cong \{1,-1\}.$$

Q9. $\varphi : G \longrightarrow G/M \times G/N$

$g \longmapsto (gM, gN)$.

(xM, yN)

Given that $G = MN$. $\underline{(xM, yN)}$

let $x = mn$ and $y = m'n'$

then $xM = mnM = n n^{-1} mn M$

$= nM$.

and $yN = m'n'N = m'N$.

$\phi(m'n) = (m'nM, m'nN)$

$= (n(n^{-1}m'n)M, m'N)$

$= (nM, m'N)$

$= (xM, yN)$.

Q10. $Z(G \times H) = Z(G) \times Z(H)$.

Let $(x, y) \in Z(G \times H)$

$$(x, y) \cdot (a, b) = (a, b)(x, y)$$

$$\Rightarrow (xa, yb) = (ax, by)$$

$$\Rightarrow xa = ax \quad \& \quad yb = by$$

$$\Rightarrow x \in Z(G), \quad y \in Z(H).$$

$\therefore Z(G \times H) \subset Z(G) \times Z(H)$.

Q $[H : H \cap K] \leq [G : K]$.

$H : H \cap K = HK : K$

$$\boxed{H / H \cap K \cong HK / K}$$

$K \trianglelefteq G$ and $H \subset G$ then $H \cap K \trianglelefteq H$

$K.$

$$\left[H : H\cap K\right] \leq \left[G : K\right].$$

$h\, H\cap K.$ where $h \in H.$

$$\begin{tikzcd}
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & K \arrow[dl, dash] \\
 & H\cap K &
\end{tikzcd}$$

If $h \in K.$ $\boxed{h\, H\cap K = H\cap K}$

h

1

$$f : H/H\cap K \longrightarrow G/K.$$

$$f(h(H\cap K)) = hK.$$

Let $f(h_1(H\cap K)) = f(h_2(H\cap K))$

$\Rightarrow \quad h_1 K = h_2 K$

$\Rightarrow \quad h_2^{-1} h_1 \in K.$

$\Rightarrow \quad h_2^{-1} h_1 \in H \cap K.$ $\rightarrow h_1(H\cap K) = h_2(H\cap K).$

$\therefore f$ is inj.

Prove that:

$$|HK| = \frac{|H|\,|K|}{|H \cap K|}.$$

Consider the gp $G = H \times K$.

and set $A = HK = \{hk \mid h \in H, k \in K\}$

$$G \times A \xrightarrow{\varphi} A$$

$$((h,k), x) \longmapsto h x k^{-1}$$

check this is a gp action.

There is only one orbit.

let $1 \in HK$. and $h_1 k_1 = \varphi((h_1, k_1^{-1}), 1)$

i.e there is only one orbit.

$$|HK| = |O(1)| = [H \times K : stb(1)]$$

$$= \frac{|H \times K|}{|stb(1)|}$$

$$stb(1) = \{(g,h) \in H \times K \mid (g,h)1 = 1\}$$

$$= \{(g,h) \in H \times K \mid g h^{-1} = 1\}$$

$$= \{(g,h) \in H \times K \mid g = h\}$$

$$= \{(g,g) \in H \times K \mid g \in H \cap K\}.$$

$$\therefore \quad |HK| = \frac{|H \times K|}{|H \cap K|}.$$