MA 51002: Measure Theory and Integration
Assignment - 1, (Spring 2021)
Riemann Integration
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1. A function f is defined on $[0,1]$ by $f(0)=0$, and
$$f(x)=\frac{1}{2^n},\quad \text{when } \frac{1}{2^{n+1}}<x\le\frac{1}{2^n}\quad (n=0,1,2,\cdots).$$
Prove that (i) f is Riemann integrable on $[0,1]$. (ii) Find $\int_0^1 f$.

2. Suppose that $f:[a,b]\to\mathbb{R}$ is bounded and Riemann integrable on $[a,r]$ for every $a<r<b$. Then f is Riemann integrable on $[a,b]$ and
$$\int_a^b f=\lim_{r\to b^-}\int_a^r f.$$
Using this result or otherwise, show that $f:[0,1]\to\mathbb{R}$ is Riemann integrable, where
$$f(x)=\begin{cases}\sin(1/x) & \text{if } 0<x\le 1,\\ 0 & \text{if } x=0.\end{cases}$$
Show that the assumption in the result that f is bounded is essential.

3. Determine whether the following functions are Riemann integrable or not?

   (i) $f:[0,2\pi]\to\mathbb{R}$ defined by
$$f(x)=\begin{cases}\sin(1/\sin x) & \text{if } x\ne 0,\pi,2\pi,\\ 0 & \text{if } x=0,\pi,2\pi\end{cases}$$

   (ii) $f:[0,1/\pi]\to\mathbb{R}$ defined by
$$f(x)=\begin{cases}\mathrm{sgn}[\sin(1/\mathrm{x})] & \text{if } x\ne 1/n\pi,\ n\in\mathbb{N},\\ 0 & \text{if } x=0 \text{ or } 1/n\pi.\end{cases}$$
   Where sgn is the sign function,
$$\mathrm{sgn(x)}=\begin{cases}1 & \text{if } x>0,\\ 0 & \text{if } x=0,\\ -1 & \text{if } x<0.\end{cases}$$

   (iii) $f:[0,1]\to\mathbb{R}$ defined by
$$f(x)=\begin{cases}x & \text{if } x\in[0,1]\cap\mathbb{Q},\\ x^2 & \text{if } x\in[0,1]\cap\mathbb{Q}^c.\end{cases}$$

4. If $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and continuous at $c\in(a,b)$. Then prove that
$$\lim_{h\to 0^+}\frac{1}{2h}\int_{c-h}^{c+h} f=f(c).$$

5. Using the first fundamental theorem of the integral calculus find $\int_0^1 f(x)\,dx$, where
$$f(x)=\begin{cases}-\cos(1/x)+2x\sin(1/x) & \text{if } 0<x\le 1,\\ 0 & \text{if } x=0.\end{cases}$$

6. A function f is continuous for all $x\ge 0$ and $f(x)\ne 0$ for all $x>0$. If $[f(x)]^2=2\int_0^x f(t)\,dt$, prove that $f(x)=x$, for all $x\ge 0$.

7. A function f is defined on $[0,3]$ by
$$f(x)=\begin{cases}x, & 0\le x\le 1\\ 1, & 1<x\le 2\\ x-1, & 2<x\le 3.\end{cases}$$

Show that f is Riemann integrable on $[0,3]$. Let $F(x) = \int_0^x f(t)\,dt$, $x \in [0,3]$, then find F. Verify that $F'(x) = f(x)$ on $[0,3]$.

8. Let $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be both continuous on $[a,b]$ and $\int_a^b f = \int_a^b g$. Prove that there exists a point $c \in [a,b]$ such that $f(c) = g(c)$.

9. A function $f : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and $\int_0^x f(t)\,dt = \int_x^1 f(t)\,dt$, for all $x \in [0,1]$. Prove that $f(x) = 0$, for all $x \in [0,1]$.

10. Suppose $\{f_n\}$ be a sequence of Riemann integrable functions in $[a,b]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a,b]$. Is it true that f is also Riemann integrable on $[a,b]$? Suppose f is also Riemann integrable on $[a,b]$, Does it necessarily hold that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f \text{ ?}$$