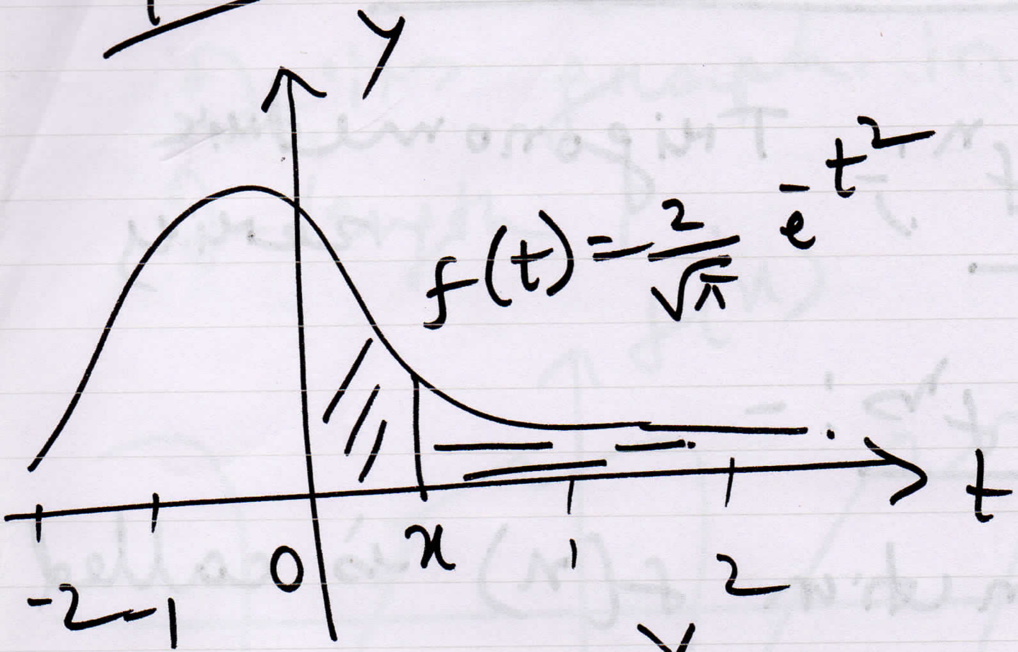
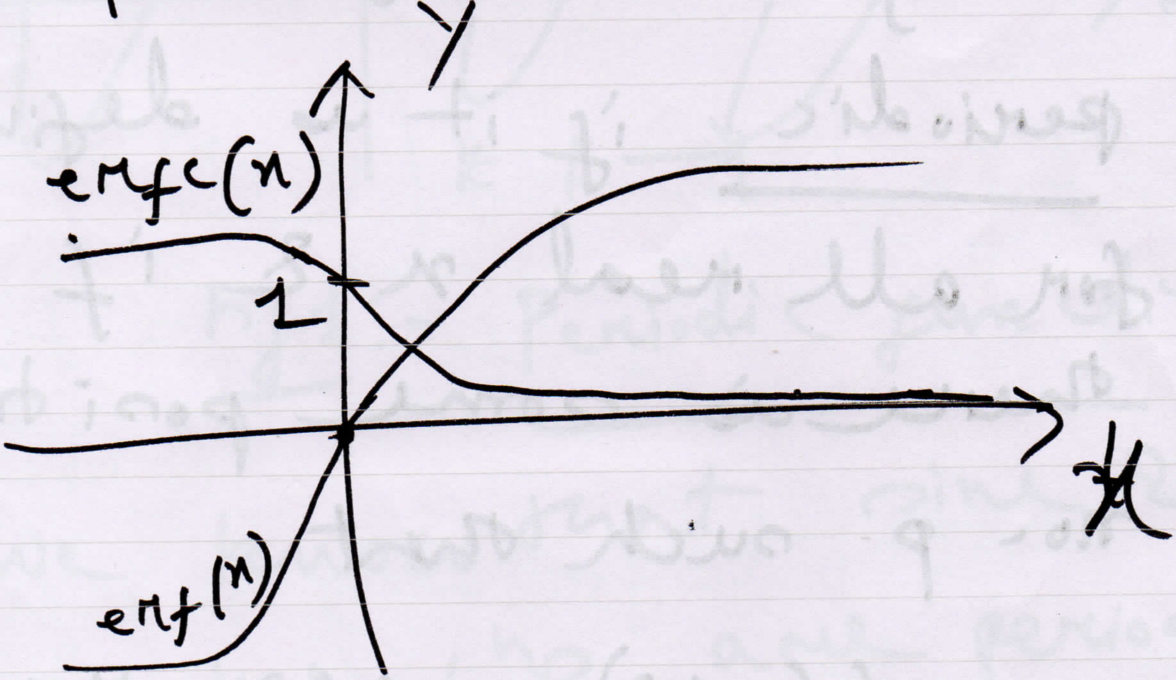
Done 04/09/2017

Lecture 14





$\mathrm{erf}(0) = 0$

& $\mathrm{erfc}(0) = 1$.

Ex Draw the graph

$\ln x = \int_1^x dt/t$, $t > 0$

$\longrightarrow$ periodic f^{ns}:

Fourier Series

Periodic f^n; Trigonometric series

Periodic f^{ns}:-

A function $f(x)$ is called **periodic** if it is defined for all real x & if there is some positive no. p such that

$$f(x+p) = f(x), \quad \forall x \qquad \longrightarrow (1)$$

This number p is called a **period** of $f(x)$. The graph of such a function is obtained by periodic repetition of its graph in any interval of length p

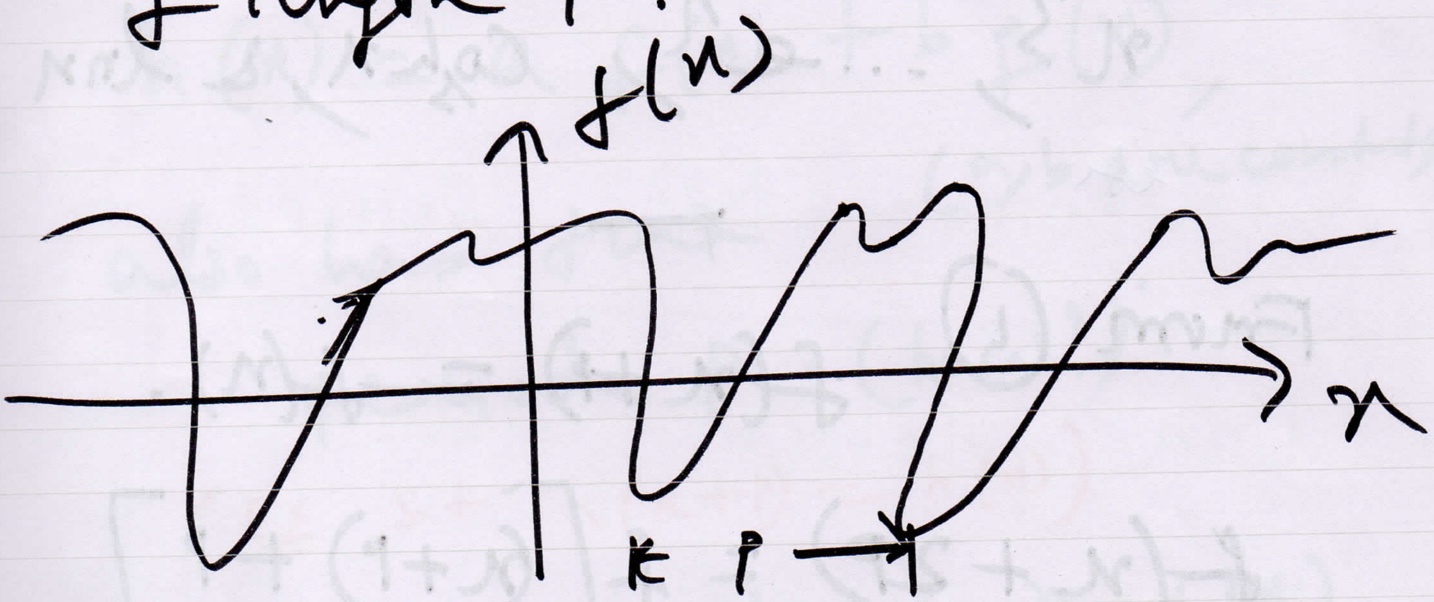


Fig 1:- Periodic function

We know that sine & cosine f^{ns} are periodic f^{ns} of period 2π.

$f(x) = c$, $f(x+p) = c$

$f = $ constant (why?) is also a periodic f^n in the sense of the given defn (1)

Examples of f^{ns} that are not periodic are x, x^2, x^3, ... e^x, $\cosh x$, & $\ln x$

etc.

From ① $f(x+p) = f(x)$.

$$f(x+2p) = f[(x+p)+p]$$
$$= f(x+p) = f(x)$$

(how?)

etc. & for any integer n,

$$f(x+np) = f(x), \forall x. \rightarrow ②$$

Hence, $2p$, $3p$, $4p$, ... are also periods of $f(x)$.

gain, if $f(x)$ & $g(x)$ have period p, then the f^n

$$h(x) = a f(x) + b g(x),$$

also has the (a, b are constants) period p. (how?).

(EX s.t $h(x+p) = h(x)$)

If a periodic f^n $f(x)$ has a smallest period p $(p>0)$, this is often called the <u>fundamental period</u> of $f(x)$. eg, for $\cos x$ & $\sin x$ the fundamental period is 2π, for $\cos 2x$ & $\sin 2x$, it is π, & so on.

A function having ~~no~~ no fundamental period is a constant f^n.

Trigonometric Series :-

Our problem will be the representation of various $f^{n}s$ of period $p = 2\pi$ in terms of the simple $f^{n}s$

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots \longrightarrow (3)$$

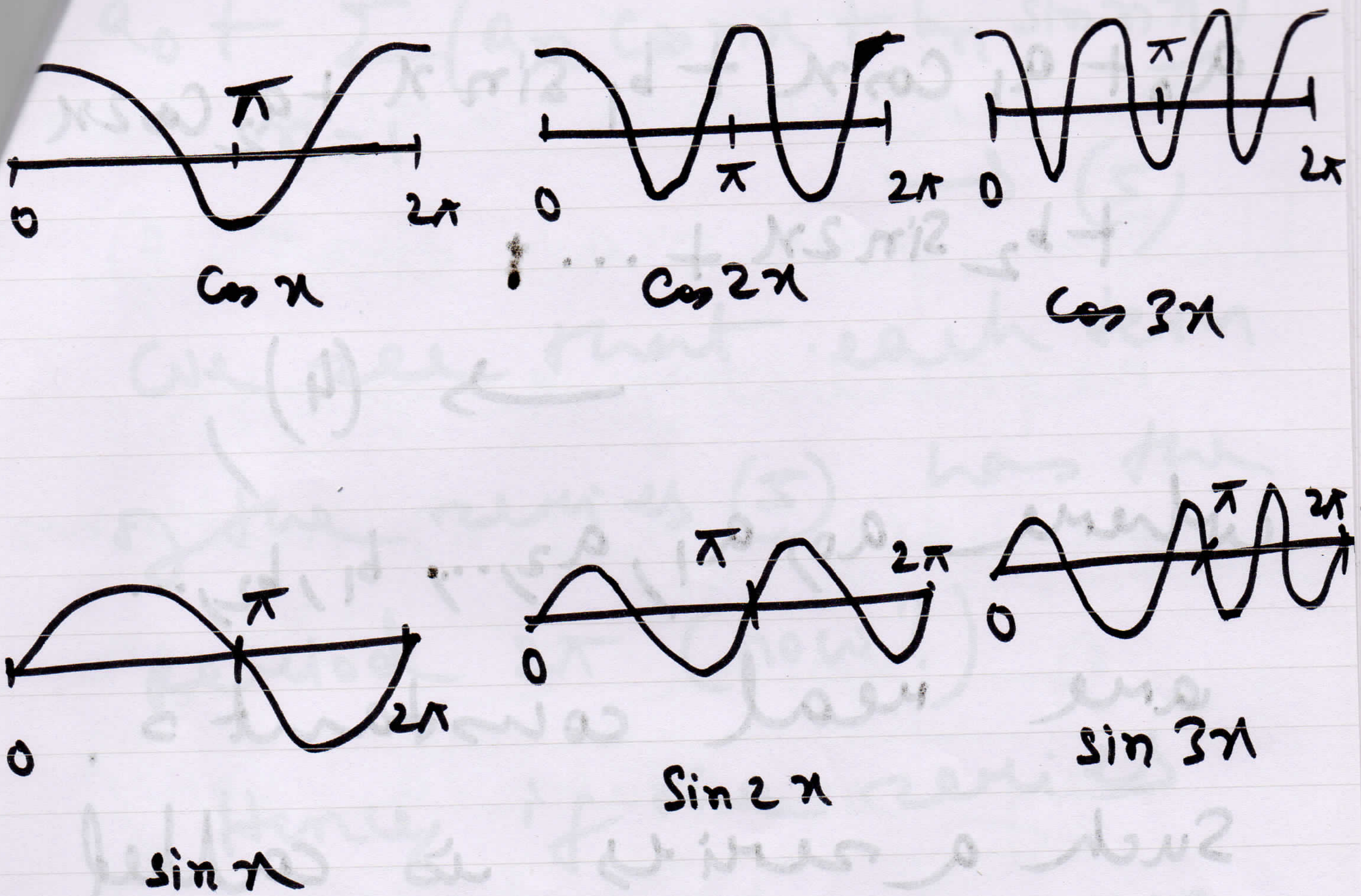


Fig 2:- Cosine & sine f^{ns} having the period 2π.

These f^{ns} have the period 2π in Fig 2

The series that will arise in this connection will be of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots ;$$

$$\longrightarrow (4),$$

where $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ are real constants.

Such a series is called a trigonometric series & the a_n & b_n are called the co-efficients of the series.

Using the summation sign, we may write this series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \rightarrow (5)$$

We see that each term of the series (5) has the period 2π (how?)

Hence, if the series (5) converges, its sum will be a function of period 2π.

Fourier Series

Fourier series arise from the practical task of representing a given periodic f^n $f(x)$ in terms of cosine & sine f^{ns}. They constitute a very important tool in solving problems that involve O.D. eq^n & P.D eq^{ns}.

Euler formulas for the Fourier Co-efficients

Let us assume that $f(x)$ is a periodic f^n of period 2π & is integrable over a period.

Let us further assume that $f(x)$ can be represented by a trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

ie, we assume that this series converges & has $f(x)$ as its sum.

Given such a function $f(x)$, we want to determine the co-efficients a_n & b_n of the corresponding series (1)

(I) <u>Determination of the constant term a_0</u>

Integrating both sides of (i) from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)\right]dx$$

If term-by-term integration of the series is allowed, we get

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(nx)\,dx + b_n \int_{-\pi}^{\pi} \sin(nx)\,dx \right)$$

$$= 2\pi a_0 + 0 + 0$$

(how?)

The first term on the R.H.S equal $2\pi a_0$. All the other integrals on the R.H.S are zero,

Hence, our first result is

$$\boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx} \rightarrow \text{(ii)}$$

Determination of the co-efficients a_n of the cosine terms.

Slly, we multiply (i) by $\cos mx$, where m is any fixed positive integer & integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$$

$$= \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)\right]\cdot \cos mx\,dx. \longrightarrow \text{(iii)}$$

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The first integral is zero (how?)

& $\int_{-\pi}^{\pi} \cos nx \cos mx \, dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos (n+m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos (n-m)x \, dx$$

$= 0$. (check!) $(n \neq m)$

& $\int_{-\pi}^{\pi} \sin nx \cos mx \, dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin (n+m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin (n-m)x \, dx$$

$= 0$.

Integration shows that the four terms on the R.H.S are zero, except for the last term in the first line, which equals π when $\underline{\underline{n=m}}$

Since in (iii), this term is multiplied by a_m, the R.H.S in (iii) equals $a_m \pi$.

$$\therefore a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx \, dx,$$

$m = 1, 2, \ldots$

$\longrightarrow$ (4)

II) Determination of the co-efficients b_m of the sine terms.

Here, we multiply (I) by $\sin(mx)$, where m is any fixed positive integer, & then integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \right] \sin mx \, dx$$

$\rightarrow$ (V)

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9. $\int_{-\pi}^{\pi} \sin mx \, dx = 0$ (how?)

$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0$ (how?)

$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx$

$= \frac{1}{2} \int_{-\pi}^{\pi} \cos (n-m)x \, dx$

$- \frac{1}{2} \int_{-\pi}^{\pi} \cos (n+m)x \, dx$

$= 0$, except when $\underline{\underline{n=m}}$

& it is π when $n=m$.

Since, in (5) this term is multiplied by b_m.

$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx$, $m=1, \dots$

Summary

from (2), (4) & (6) we see ~~that~~, we have the so-called Euler formulas

(a) $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

(b) $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n = 1, 2 \ldots$$

(c) $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots$$

$\longrightarrow$ (7)

These nos. given by eqn (vii) are called the Fourier co-efficients of $f(x)$

The trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(8)

with co-efficients given in (~~VII~~ 7) is called the Fourier series of $f(x)$.

(regardless of convergence)

Ex 1/ Rectangular wave

Q) Find the Fourier coefficients of the periodic f^n $f(x)$ given by

$$f(x) = \begin{cases} -k, & \text{if } -\pi < x < 0 \\ k, & 0 < x < \pi . \end{cases}$$

Function of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits etc. ——— . . (Find out more about them)

The value of $f(x)$ at a single point does not affect the integral hence (why? study Riemann integral) we can leave $f(x)$ undefined at $x=0$, & $x = \pm\pi$.

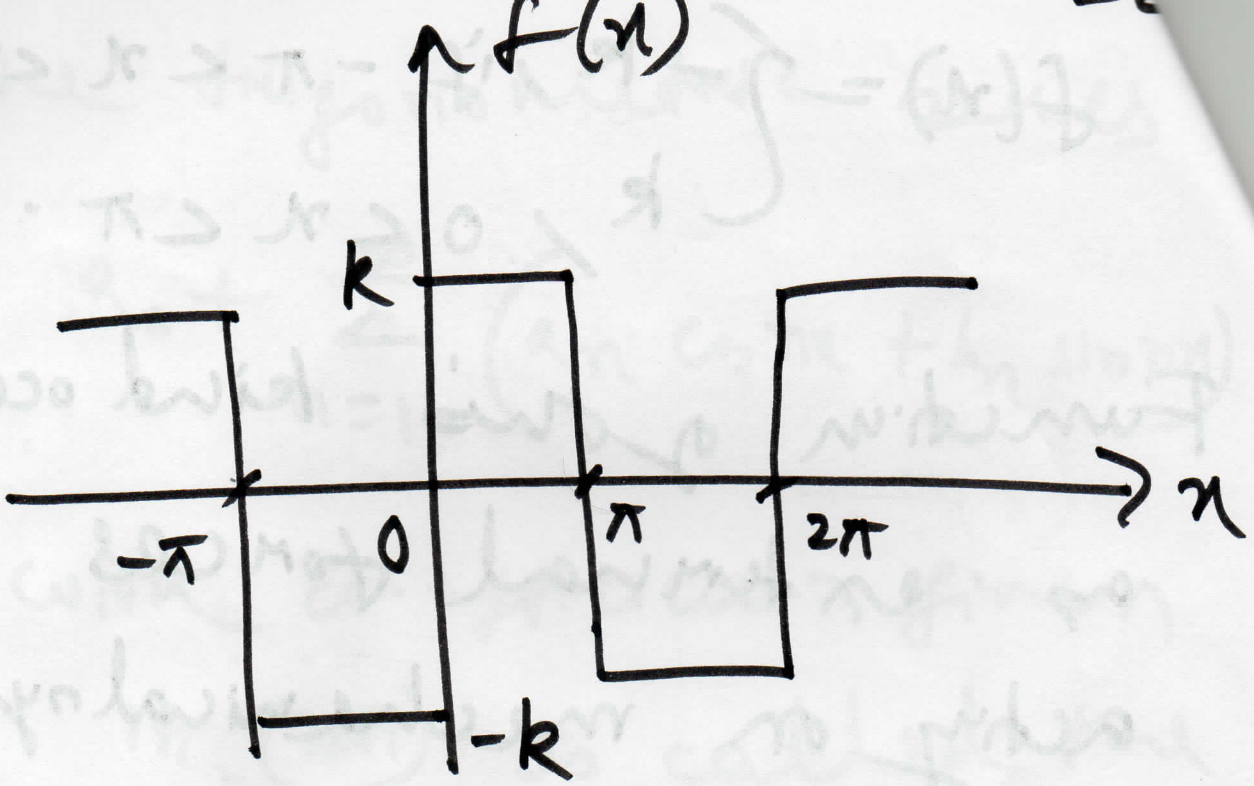

Fig 1 :- The given f^n $f(x)$

(periodic square wave)

From ~~[illegible]~~ the eq^n 7) a)

we obtain $a_0 = 0$ (how?)

This can also be seen

without integration, since

the area under the curve

of $f(x)$ bet^n $-\pi$ & π is zero

From (7b), we get

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[-k\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[k\frac{\sin nx}{n}\right]_{0}^{\pi}\right]$$

$= 0$. / because $\sin nx = 0$ at $-\pi$, 0 & π for all $n = 1, 2, \ldots$

Slly, from eq^n 7)c), we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\ dx.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[k\frac{\cos nx}{n}\right]_{-\pi}^{0} - \left[k\frac{\cos nx}{n}\right]_{0}^{\pi}\right\}$$

$$\Rightarrow b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right]$$

$$= \frac{2k}{n\pi}\left[1 - \cos n\pi\right]$$

Now, $\cos n\pi = \begin{cases} -1, & \text{for odd } n \\ 1, & \text{for even } n \end{cases}$