Lecture 2

(4) $\left(\bigcup_{i=1}^{n} E_i\right) \Delta \left(\bigcup_{i=1}^{n} F_i\right) \subseteq \bigcup_{i=1}^{n} (E_i \Delta F_i)$. (EXERCISE)

Def:- A subset $A \subseteq X$ is called a G_δ-set, if $A = \bigcap_{i=1}^{\infty} G_i$, for G_i open sets.

Def:- A subset $A \subseteq X$ is called an F_σ-set, if $A = \bigcup_{i=1}^{\infty} F_i$, for F_i closed sets.

Notation: $A \subseteq \mathbb{R}$.

- supremum of A = the least upper bound of A & denote by $\sup(A)$.
- infimum of A = the greatest lower bound of A & denote by $\inf(A)$.

Properties: (1) Let $\varepsilon > 0$. Consider

$$\sup(A) - \varepsilon < \sup(A).$$

there exists $x \in A$ such that

$$\sup(A) - \varepsilon < x.$$

② Let $\varepsilon > 0$, $\inf(A) + \varepsilon > \inf(A)$.

Then there exists $y \in A$ such that

$$\inf(A) + \varepsilon > y.$$

Theorem (Heine-Borel Thm).

Let $A \subset \mathbb{R}$ be a closed & bound set. Suppose $A \subseteq \bigcup_{\alpha \in I} G_\alpha$, where the sets G_α are open & I is some index set. Then there exists a finite subcollection of the sets G_α, say $\{G_1, \ldots, G_n\}$ such that

$$A \subseteq \bigcup_{i=1}^{n} G_i.$$

~~$A = [0,1]$~~

Let $\{x_n\}$ be a sequence of real numbers.

Def:- The upper limit of $\{x_n\}$ is defined as

$$\limsup_{n \to \infty} x_n := \inf \left\{ \sup_{m \geq n} \{x_m\} \;\middle|\; n \in \mathbb{N} \right\}$$

$$= \inf \left\{ \sup\{x_1, x_2, \ldots\}, \sup\{x_2, x_3, \ldots\}, \ldots \right.$$

$$\left. \ldots \right\}$$

Def:-

D: $\liminf_{n\to\infty} x_n := \sup\left\{ \inf_{m\geq n}\{x_m\} \;\middle|\; n \in \mathbb{N} \right\}$

lower limit of $\{x_n\}$.

Simply we write $\limsup x_n$ & $\liminf x_n$.

If $\limsup(x_n) = \liminf(x_n)$, then we write the common value as $\lim x_n$.

Remark:- $\limsup(x_n) = -\liminf(-x_n)$.

Let $f: \mathbb{R} \to \mathbb{R}$ be a function.
Then the upper limit of f at α_0 is defined as

$$\limsup_{\alpha\to\alpha_0} (f(\alpha)) := \inf\left\{ \sup_{0<|\alpha-\alpha_0|<h} (f(\alpha)) \;\middle|\; h>0 \right\}$$

IIly

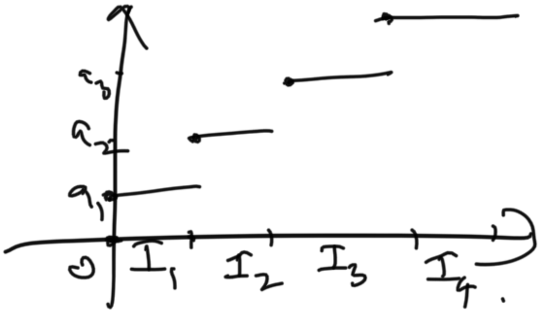
$$\liminf_{\alpha\to\alpha_0} (f(\alpha)) := \sup\left\{ \inf_{0<|\alpha-\alpha_0|<h} (f(\alpha)) \;\middle|\; h>0 \right\}.$$

Let (X, d) be a metric space.

- $A \subseteq X$. Then the characteristic function of A.
χ_A is defined as $\chi_A : X \to \mathbb{R}$.

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A. \end{cases}$$

Eg:-



Step function.

Then the step function, $\sum_{i=1}^{n} a_i \chi_{I_i}$.

Theorem (Lindelöf Theorem)

Let $\mathcal{G} = \{ I_\alpha \mid \alpha \in A \}$ be a collection of open intervals in $\mathbb{R}$. Then there exists a subcollection say $\{I_1, I_2, \dots\}$ of $\mathcal{G}$ at most countable in number (i.e, $\{I_1, I_2, \dots\} \subseteq \mathcal{G}$) such that

$$\bigcup_{i=1}^{\infty} I_i = \bigcup_{\alpha \in A} I_\alpha .$$

Theorem:- Let G be a non-empty open set in $\mathbb{R}$. Then G is equal to union of disjoint open intervals, at most countable in number.

i.e. there exists open intervals $I_1, I_2, \dots$, disjoint,

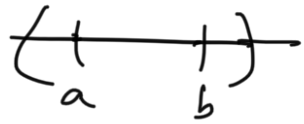
such that $G = \bigcup_{i=1}^{\infty} I_i$.

<u>proof:-</u> Given that $G \subseteq \mathbb{R}$ open.

Define a relation $\sim$ on G as follows.

$a \sim b$ if the closed interval $[a,b]$ or $[b,a]$ lies in G. for $a, b \in G$. if $b<a$

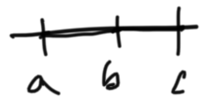
<u>claim:-</u> $\sim$ is an equivalence relation.



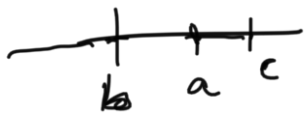
$a \sim a$, since $[a,a] = \{a\}$ closed interval in G.

$a \sim b \Rightarrow b \sim a$.

Ily $a \sim b$ & $b \sim c \Rightarrow a \sim c$.



$\therefore$ G is the union of disjoint equivalence classes.



$b < c$.

$\boxed{a < c}$

Let $C(a)$ = the equivalence class containing a.

Then $C(a)$ is an interval.

(if not then there exists two pts $b, c \in C(a)$ such that $[b,c]$ is not contained in G.

$\Rightarrow b \not\sim c \Rightarrow\Leftarrow$

$\because a \sim b$ & $a \sim c$.
$\Rightarrow b \sim c$.)

Further we show that $C(a)$ is open.

for if $k \in C(a)$, to show,

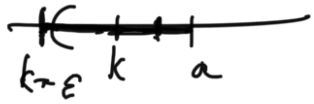
$$(k-\varepsilon, k+\varepsilon) \subseteq C(a).$$

Pf:- Let $k \in C(a) \subseteq G$
open.

There exists an $\varepsilon > 0$ such that

$$(k-\varepsilon, k+\varepsilon) \subseteq G.$$

interval



& $k \in C(a)$.

$\Rightarrow (k-\varepsilon, k+\varepsilon) \subseteq C(a)$ $(\because k \sim a)$

as required.

$[a,k]$ or $[k,a]$
$(k-\varepsilon) \sim a$

$\therefore G = \bigcup C(a)$
open interval

$= $ Countable disjoint union of $C(a)$

$= \bigcup_{i=1}^{\infty} C(a_i)$ (by Lindelöf's thm)

$\mathcal{Y} = \{C(a)\}_{a \in \Lambda}$ disjoint family.

$\exists \; \{C(a_1), \ldots, C(a_n), \ldots\} \subseteq \mathcal{Y}$

still disjoint.