Lecture 20

$\varphi_k = \varphi_k^{(1)} - \varphi_k^{(2)} \quad \forall k$

$\varphi_k(x) \to f(x) \quad \forall x$ as $k \to \infty$

To show: $\{ |\varphi_k| \}$ is an increasing sequence.

& $|\varphi_k| = \varphi_k^{(1)} + \varphi_k^{(2)}$.

$|\varphi_k| = \varphi_k^{+} + \varphi_k^{-}$

& $\varphi_k = \varphi_k^{+} - \varphi_k^{-} = \varphi_k^{(1)} - \varphi_k^{(2)}$, where $\varphi_k^{(1)}, \varphi_k^{(2)}$ are non-negative functions.

Enough to show: $\varphi_k^{+} = \varphi_k^{(1)}$ &
$\varphi_k^{-} = \varphi_k^{(2)}$. (Try it!)

<u>Theorem</u>:- Suppose f is a measurable function on $\mathbb{R}^d$.
Then there exists a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$ that converges pointwise to f almost every where.

ie, $\{ x \mid \psi_k(x) \not\to f(x) \}$ has measure zero.

Littlewood's three principles.

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① Every measurable set is nearly a finite union of intervals.

(2) Every measurable function is nearly continuous.

(3) Every convergent sequence is nearly uniformly convergent.

<u>Def</u>:- A sequence of functions $\{f_n\}$, $f_n : E \to \mathbb{R}$, $E \subseteq \mathbb{R}^d$, is said to be <u>uniformly convergent</u> on E to $f : E \to \mathbb{R}$, if for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ (independent of x) such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N \;\&\; \forall\, x \in E.$$

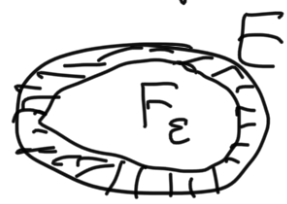
<u>Theorem (3rd principle)</u> (Egorov)

Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$ & assume that $f_k \to f$ pointwise almost everywhere on E. Given $\varepsilon > 0$, there exists a closed set $A_\varepsilon \subseteq E$ such that $m(E \setminus A_\varepsilon) \leq \varepsilon$ & $f_k \to f$ uniformly on A_ε.

Theorem (Lusin) (2nd principle)

Suppose f is measurable & finite valued on E i.e., $f: E \to \mathbb{R}$ measurable & $m(E) < \infty$. Then for every $\varepsilon > 0$ there exists a closed set $F_\varepsilon \subseteq E$ such that $\boxed{m(E \setminus F_\varepsilon) \leq \varepsilon}$ & $f\big|_{F_\varepsilon}$, the restriction map is continuous.

$\boxed{f: E \to \mathbb{R} \cup \{\pm\infty\}}$



$$f\big|_{F_e} : F_e \to \mathbb{R}, \qquad f\big|_{F_e}(x) = f(x) \quad \forall\, x \in F_e \subseteq E.$$

$f: E \subseteq \mathbb{R}$

$m(E) = +\infty$

$\Rightarrow$ $\boxed{\text{i.e. } \{x \in E \mid f \text{ is not contin at } x\} \subseteq E \setminus F_e.}$

1st principle already stated.

Given $\varepsilon > 0$, $E \subseteq \mathbb{R}^d$ measurable set. & $m(E) < \infty$. Then there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that

$$m(E \,\Delta\, F) \leq \varepsilon$$

The Lebesgue integral.

$f(x)$

$\int_a^b f(x)\,dx$

Riemann integration

The general notion of the Lebesgue integral on $\mathbb{R}^d$ will be defined in a step-by-step fashion

1. Simple functions.
2. Bounded funtions suppored on a set of finite measure.
3. Non-negative funtions.
4. Integrable funtions.

Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ be a simple funtion

where E_k are measurable sets of finite measure & a_k are constants.

The Canonical form of φ is the unique decomposition $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$, where a_k are distinct & the sets E_k are disjoint.

Remark:- Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ be a simple funtion.

$$\text{range}(\varphi) = \{ y \in \mathbb{R} \,/\, y = \varphi(x) \text{ for some } x \in \mathbb{R}^d \}.$$

$$= \{ c_1, c_2, \ldots, c_M \}$$

where $c_j = a_{k_j}$ distinct.