Recap

- The set of strings accepted by some automaton M (not necessarily a DFA) is called a *language*, or the language of M, and is denoted by $L(M)$.

- A language accepted by a DFA is called a *regular set* or *regular language*.

Some closure properties of regular sets

Let A and B be two regular sets. Then the following sets are also regular:

- $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ (union)
- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ (intersection)
- $\sim A = \{x \in \Sigma^* \mid x \notin A\}$ (complement)
- $AB = \{xy \mid x \in A \text{ and } y \in B\}$ (concatenation)
- $A^* = \{x_1 x_2 \ldots x_n \mid n \geq 0 \text{ and } x_i \in A,\ 1 \leq i \leq n\} = A^0 \cup A^1 \cup A^2 \cup A^3 \cup \ldots$ (asterate or Kleene closure)

Product Construction

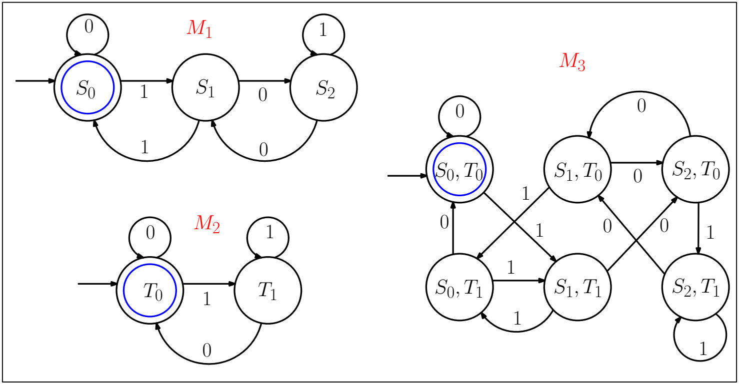
Suppose that A and B are regular. Then there are automata $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ with $L(M_1) = A$ and $L(M_2) = B$.

Let $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$, such that

- $Q_3 = Q_1 \times Q_2 = \{(p, q) \mid p \in Q_1 \ and \ q \in Q_2\}$,
- $F_3 = F_1 \times F_2 = \{(p, q) \mid p \in F_1 \ and \ q \in F_2\}$,
- $s_3 = (s_1, s_2)$, and
- $\delta_3 : Q_3 \times \Sigma \to Q_3$ is defined by $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$.

Product Construction

M_1 and M_2 accept binary strings divisible by 3 and 2 respectively. M_3 has been constructed from them using the product construction method.



Product Construction

Now we prove that for M_3 constructed earlier,
$L(M_3) = L(M_1) \cap L(M_3)$.

We define

- $\hat{\delta}_3((p, q)\epsilon) = (p, q)$
- $\hat{\delta}_3((p, q), xa) = \delta_3(\hat{\delta}_3((p, q), x), a)$

Product Construction

Lemma

For all $x \in \Sigma^*$, $\hat{\delta}_3((p,q),x) = (\hat{\delta}_1(p,x), \hat{\delta}_2(q,x))$

Proof.

By Induction on $|x|$. Base case:

$$\hat{\delta}_3((p,q),\epsilon) = (p,q) = (\hat{\delta}_1(p,\epsilon), \hat{\delta}_2(q,\epsilon))$$

Induction step:
$\hat{\delta}_3((p,q),xa) = \delta_3(\hat{\delta}_3((p,q),x),a) = \delta_3((\hat{\delta}_1(p,x),\hat{\delta}_2(q,x)),a) =$
$(\delta_1(\hat{\delta}_1(p,x),a),(\delta_2(\hat{\delta}_2(q,x),a)) = (\hat{\delta}_1(p,xa),\hat{\delta}_2(q,xa))$ $\square$

Exercise: Justify the steps involved above.

Product Construction

Theorem

$L(M_3) = L(M_1) \cap L(M_2)$

Proof.

For all $x \in \Sigma^*$ the following holds:

$$
\begin{aligned}
& x \in L(M_3) \iff \hat{\delta}_3(s_3, x) \in F_3 \iff \hat{\delta}_3((s_1, s_2), x) \in F_3 \\
& \iff \hat{\delta}_3((s_1, s_2), x) \in F_1 \times F_2 \iff (\hat{\delta}_1(s_1, x), \hat{\delta}_2(s_2, x)) \in F_1 \times F_2 \\
& \iff \hat{\delta}_1(s_1, x) \in F_1 \textit{ and } \hat{\delta}_2(s_2, x) \in F_2 \\
& \iff x \in L(M_1) \textit{ and } x \in L(M_2) \iff x \in L(M_1) \cap L(M_2) \qquad \square
\end{aligned}
$$

Exercise: Justify the steps involved above.

Product Construction

Exercise: Use the product construction method to prove that $A \cup B$ and $\sim A$ are regular.