INDIAN INSTITUTE OF TECHNOLOGY

DATE | PROBLEM SHEET 1 | SHEET NO.

FUNCTIONAL ANALYSIS

AKASH MANDAL

Roll. No.: 16 MA 20007

1. Metric Spaces

1. $d(x,y)$ is a metric on $\mathbb{R}$ if $\forall x,y,z \in \mathbb{R}$

i) $d(x,y) \geq 0$ ~~$\forall x,y \in \mathbb{R}$~~

ii) $d(x,y) = 0 \Leftrightarrow x = y$

iii) $d(x,y) = d(y,x)$

iv) $d(x,y) \leq d(x,z) + d(z,y)$

a) $d(x,y) = (x-y)^2$

i), ii), iii) are satisfied trivially

$$d(x,z) + d(y,z) = (x-z)^2 + (x-y)^2$$
$$= x^2 + z^2 - 2xz + z^2 + y^2 - 2yz$$
$$= x^2 + y^2 - 2xy + 2z^2 - 2xz - 2yz + 2xy$$
$$= (x-y)^2 + 2\left(z(z-x) - y(z-x)\right)$$
$$= (x-y)^2 + 2(x-z)(y-z)$$

which may not be true.

Let $x = 3$ $y = 6$ $z = 4$

$(x-y)^2 = 9$ $\quad (x-z)^2 + (y-z)^2 = 1 + 4 = 5 < 9$

NOT A METRIC

P.R.E.

(b) $d(x, y) = \sqrt{|x-y|}$

(i), (ii), (iii) True trivially

$d(x, z) + d(z, y) = \sqrt{|x-y|} + \sqrt{|y-z|}$

Squaring

$$\left(d(x, z) + d(y, z)\right)^2 = |x - z| + |y - z| + 2\sqrt{|(x-z)(y-z)|}$$

$$\cancel{\geq |x - y + y - z| + 2\sqrt{|(x-y)(y-z)|}}$$

$$\cancel{= |x - z|}$$

$$= |x - z| + |z - y| + 2\sqrt{|(x-z)(y-z)|}$$

$$\geq |x - z + z - y| + 2\sqrt{|(x-z)(y-z)|}$$

$$\geq |x - y|$$

$$= [d(x, y)]^2$$

$\therefore$ Since $d(x,y) \geq 0 \;\forall\, x, y \in \mathbb{R}$

$d(x, y) < d(x, z) + d(z, y)$

$\therefore$ d is a metric on $\mathbb{R}$

2. Metric on ℓ^∞: $x = (x_1, x_2, \dots)$ $y = (y_1, y_2, \dots)$

$$d(x,y) = \sup_{i \in \mathbb{N}} |x_i - y_i|$$

Now if the subspace is restricted to binary sequences

$$(x_i - y_i) = \begin{cases} 0 & x_i = 1, y_i = 1 \text{ or } x_i = 0\ y_i = 0 \\ 1 & x_i = 1\ y_i = 0 \\ -1 & x_i = 0\ y_i = 1 \end{cases}$$

$$\therefore |x_i - y_i| = \begin{cases} 0 & x_i = y_i \\ 1 & x_i \neq y_i \end{cases}$$

Now if $x \neq y$ $\exists$ an index j such that $x_j \neq y_j$ and $|x_j - y_j| = 1$

and since $|x_i - y_i|$ can only be $0, 1$

$$\therefore d(x,y) = \sup_{i \in \mathbb{N}} |x_i - y_i| = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Which is the discrete metric.

3. ~~$X = \{(x, y, z$~~

$X = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \{0, 1\}\}$

$\therefore$ Since each of x_1, x_2, x_3 has 2 choices

X has $2^3 = 8$ elements.

$d(x, y)$ = No. of places where x, y differ

(i) $d(x, y) \geq 0$ as d is defined as a count

(ii) $d(x,y) = 0 \Leftrightarrow x, y$ differ at 0 positions

$\Leftrightarrow x = y$

~~and $x = y \Rightarrow$~~

(iii) $d(x, y) = d(y, x)$ (by definition

(iv) Consider $x = (x_1, x_2, x_3)$

$y = (y_1, y_2, y_3)$

$z = (z_1, z_2, z_3)$

Let $\exists\ i \in \{1, 2, 3\}$

such that $x_i \neq y_i$

This contributes 1 to $d(x, y)$

(If No such i exists, $d(x,y) = 0$

Triangle inequality is trivial).

Now $x_i = z_i$ and $y_i = z_i$ cannot be simultaneously true, otherwise $x_i = y_i$

$\therefore$ Either $x_i \neq z_i$ or $y_i \neq z_i$

$\therefore$ 1 is contributed to $d(x, z) + d(z, y)$

$\therefore$ For each mismatch in x, y, 1 is contributed to both $d(x, y)$ and $d(x, z) + d(z, y)$

$\therefore d(x, y) \leq d(x, z) + d(z, y)$

INDIAN INSTITUTE OF TECHNOLOGY

DATE SHEET NO.

4 (a) Consider

$$x = (x_n) = \Big(1, \underbrace{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}}_{4 \text{ terms}}, \underbrace{\tfrac{1}{3}, \tfrac{1}{3}, \ldots, \tfrac{1}{3}}_{27 \text{ term}}, \ldots, \underbrace{\tfrac{1}{n}, \tfrac{1}{n}, \ldots, \tfrac{1}{n}}_{n^n \text{ terms}}, \ldots\Big)$$

The sequence converges to 0

but for any p $1 \leq p < \infty$

$$\sum_1^\infty |x_n|^p = 1 + \frac{2^2}{2^p} + \frac{3^3}{3^p} + \cdots + \frac{n^n}{n^p} + \cdots$$

But $\forall n > p$ $\quad \frac{n^n}{n^p} > 1$ $\quad$ Given $1 \leq p < \infty$

$$\therefore \sum_1^\infty |x_n|^p = \infty$$

$\therefore$ x does not belong to l^p for any $1 \leq p < \infty$

b) Let $p > 1$ Consider $x = \left(\frac{1}{n}\right)$

We know $\displaystyle\sum_{n=1}^\infty \left|\frac{1}{n}\right| = \infty$ ies $x \notin l^1$

but $\displaystyle\sum_{n=1}^\infty \left|\frac{1}{n^p}\right| < \infty$ ie $x \in l^p$ $p > 1$

Use integral test $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}$, $p > 1$

P.R.E.

5. Given metric space (X, d)

$$D(A, B) = \inf_{a\in A, b\in B} d(a, b) \quad \forall\ A, B \subseteq X$$

To show D is not a metric.

Let $A \neq B$ and they have a common elt. a^*

Since $a \in A$, $a \in B$ and $d(a^*, a^*) = 0$

$$D(A,B) = \inf_{a\in A, b\in B} d(a,b) \leqslant d(a^*, a^*) = 0$$

But by assumption $A \neq B$

$\therefore$ d is not a metric.

6) Consider $f : (-1, 1) \to \mathbb{R}$

where $f(x) = x^2$

Image of $(-1, 1)$ is $[0, 1)$

But $(-1, 1)$ is open in $\mathbb{R}$

whereas $[0, 1)$ is not

7. A metric space (X, d) is said to be separable if it has a countable subset Y which is dense in X. (ie $\bar{Y} = X$ and Y is countable)

(i) $\Rightarrow$ Given (X, d) is separable

To show $\exists$ countable subset Y of X s.t $\forall \epsilon > 0$, $\forall x \in X$ $\exists y \in Y$ s.t $d(x, y) < \epsilon$

Since X is separable $\exists$ a dense subset say Y st ~~Now~~ $\bar{Y} = X$. Now $\bar{Y} = Y \cup Y'$

Choose any $\epsilon > 0$

$\therefore$ Every element of $X \in Y$ or $\in Y'$

Let $x \in X$. Then if

$x \in Y$ choose $y = x \in Y$

Then $d(x, y) = 0 < \epsilon$

otherwise

$x \in Y'$

$\Rightarrow x$ is a limit pt of Y $\therefore \exists$ an elt of Y (other than x) in every nhbd of x. Consider the $N_\epsilon(x)$ for ϵ chosen before. $\exists y \in Y$ s.t $y \in N_\epsilon(x)$ $\therefore$ $d(x, y) < \epsilon$

(ii) $\Leftarrow$

Given Y is a countable subset of X s.t

$\forall \varepsilon > 0$ and $\forall x \in X$ $\exists y \in Y$ s.t. $d(x, y) < \epsilon$

To show X is separable. ie X has a countable subset that is dense in X.

We claim given Y is dense in X.

Choose $x \in X$ and any $\epsilon > 0$.

Now $\exists y \in Y$ s.t. $d(x, y) < \epsilon$

Now if $y = x$ then $x \in Y$

otherwise $y \in N_\epsilon(x)$ and $y \neq x$.

$\Rightarrow x \in N_\epsilon(y)$ and $y \neq x$.

$x \in Y'$

$\therefore x \in Y$ or $x \in Y'$ $\therefore x \in Y \cup Y'$

$\Rightarrow x \in \overline{Y}$

Since x is an arbitrary element

$X \subseteq \overline{Y}$

But X is the whole metric space

$\therefore X = \overline{Y}$

$\therefore Y$ is dense in X

$\Rightarrow X$ is separable

Hence proved.

8. a) Given (X, d) and an uncountable subset Z such that topology on Z generated by the induced metric is the discrete topology.

Consider a ~~countable~~ dense subset of X, say D
(If it does not exist, X is not separable)

Claim: D is uncountable.

For each $x \in Z$, consider $B_x = B_{1/2}(x)$
ie an open ball of radius $\frac{1}{2}$ with centre x.

Since D is dense in X $\quad B_x \cap D \neq \phi$ ($\because x \in X$ also) for each $x \in Z$

~~Consider~~ $\therefore \exists \; a_x \in B_x \cap D$

Now $x \neq y \Rightarrow B_x \cap B_y = \phi$ ($\because d(x, y) = 1$ in discrete metric)

$\therefore (B_x \cap D) \cap (B_y \cap D) = \phi$

$\Rightarrow \quad a_x \neq a_y$

$\therefore$ The mapping $x \longmapsto a_x$ is injective

$\Rightarrow$ D is also uncountable

$\therefore$ X cannot be separable

Hence proved.

(b) $B[a,b]$ is the space of bounded functions on $[a,b]$

Consider an element $t \in [a,b]$

Let $f_t(x) = \delta_{xt} = \begin{cases} 0 & x \neq t \\ 1 & x = t \end{cases} \quad \forall x \in [a,b]$

This is obviously bdd. ie $f_t \in B[a,b]$

Consider $\{ f_t \mid t \in [a,b] \} = F$

F is obviously uncountable subset of $B[a,b]$

Also

$$d(f_x, f_y) = \sup_{t \in [a,b]} |f_x(t) - f_y(t)|$$

$$= \sup_{t \in [a,b]} |\delta_{x,t} - \delta_{yt}|$$

$$= \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

[If $x \neq y$, $|\delta_{xx} - \delta_{yx}| = |0 - 1| = 1$ which is the maximum value that can be attained by $|\delta_{xt} - \delta_{yt}|$]

$\therefore$ F is an uncountable subset of $B[a,b]$ and d induces a discrete metric on it

$\therefore$ From part (a), $B[a,b]$ is not separable

INDIAN INSTITUTE OF TECHNOLOGY

DATE SHEET NO.

9. ~~For some~~ ϵ

Assume (x_n) does not converge to x

For some $\epsilon > 0$ and for every N $\exists n \geq N$

such that $|x_n - x| \geq \epsilon$

Thus there must be a subsequence (x_{k_n})

of (x_n) such that $|x - x_{k_n}| \geq \epsilon$

$\therefore$ no subsequence of (x_{k_n}) converges to x

which is a contradiction.

$\therefore$ (x_n) converges to x.

10) Consider a subsequence (x_{n_k}) of (x_n) s.t. $x_{n_k} \to x$ in X.

~~Consider (x_{n_k})~~

~~$\exists$ p such that $|x - x_p| < \frac{\epsilon}{2}$~~

Choose $\epsilon > 0$

$\exists$ N such that

$$\forall n, m > N$$

$$|x_m - x_n| < \frac{\epsilon}{2}$$

Also $\exists$ p, $x_p \in (x_{n_k})$ such that

$$|x - x_n| < \frac{\epsilon}{2} \quad \forall n > p$$

Choose $M = \max(N, p)$ $\quad M \geq N$

Then $|x_M - x_n| < \frac{\epsilon}{2}$ for $n >$

and $|x - x_M| < \frac{\epsilon}{2}$ for $M \geq p$

$$d(x, x_n) \leq d(x - x_M) + d(x_M - x_n)$$

$$\therefore = |x - x_M| + |x_M - x_n|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore x_n \to x$

Alt Another way is to note say (x_{n_k}) converges to L. Complete X to $\bar{X}$ so that the (x_n) converges in $\bar{X}$. Say (x_n) converges to M in $\bar{X}$. But every subsequence must converge to M $\Rightarrow$ M = L. Thus (x_n) converges to L

INDIAN INSTITUTE OF TECHNOLOGY

SHEET NO.

Q 11. $d_1, d_2 \to$ metrics on X, $a, b > 0$ $\forall x, y \in X$

$$a\, d_1(x,y) \leq d_2(x,y) \leq b\, d_1(x,y)$$

Consider A is an open subset of (X, d_1)

Choose $x \in A$

Choose r such that $B_{d_1}(x, \frac{r}{a})$ is contained in A

$B_{d_2}(x,r) = \{y \mid d_2(x,y) < r, y \in X\}$

$\therefore \forall y \in B_{d_2}(x,r)$

$a\, d_1(x,y) < d_2(x,y) < r$

$\therefore B_{d_2}(x,r) \subseteq \{y \mid a\, d_1(x,y) < r\}$

$= \{y \mid d_1(x,y) < \frac{r}{a}\}$

$= B_{d_1}(x, \frac{r}{a}) \subset A$

$\therefore \exists r$ such that

$B_{d_2}(x,r) \subset A$

$\therefore A$ is open in (X, d_1)

P.R.E.

Now let A be open in (X, d_2)

Similarly choose ϵ such that $B_{d_2}(x, b\epsilon)$ contained in A

$$B_{d_1}(x, \epsilon) = \{y \mid d_1(x,y) < \epsilon, y \in X\}$$

Now, $\forall y \in B_{d_1}(x, \epsilon)$

$$d_2(x, y) \leq b\,d_1(x, y) < b\epsilon$$

$$\therefore B_{d_1}(x, \epsilon) \subseteq \{y \mid d_2(x, y) < b\epsilon\}$$
$$= B_{d_2}(x, b\epsilon)$$
$$\subset A$$

Thus $B_{d_1}(x, \epsilon)$ is an open ball in A

$\therefore$ A is open in (X, d_1)

Hence proved

(b) Let (x_n) be a cauchy sequence in (X, d_1)
$\forall \epsilon > 0\ \exists N_1$, such that $d_1(x_m, x_n) < \frac{\epsilon}{b}$ $\forall m, n > N_1$

But $d_2(x_m, x_n) \leq b\,d_1(x_m, x_n) < \epsilon$ $\forall m, n > N_1$

$\therefore (x_n)$ is cauchy in (X, d_1)

Let (y_n) be Cauchy sequence in (X, d_2)
$\forall \epsilon > 0\ \exists N_2$ s.t. $d_2(x_m, x_n) < a\epsilon$ $\forall m, n > N_2$

But $a\,d_1(x_m, x_m) \leq d_2(x_m, x_n) < a\epsilon$ $\forall m, n > N_2$

$\Rightarrow d_1(x_m, x_n) < \epsilon$

$\therefore (x_n)$ is cauch in (X, d_2)

Hence proved.

12. $(\mathbb{N}, d)$ is complete or not

(a) Consider a Cauchy sequence (x_n) in $(\mathbb{N}, d)$

~~st~~. $\forall \epsilon > 0 \ \exists N$ st.

$d(x_m, x_n) < \epsilon \quad \forall m, n > N$

ies $|x_m - x_n| < \epsilon \quad \forall m, n > N$

Choose $\epsilon = \frac{1}{2}$

if $x_m, x_n \in \mathbb{N}$ $|x_m - x_n| < \frac{1}{2}$

iff $x_m = x_n = x$ (say)

$\therefore$ The elements of the series are all x except for finitely many elements.

$x_m = x \quad \forall m > N$ (chosen above)

$\therefore$

$\forall \epsilon > 0 \quad |x_m - x| = 0 < \epsilon \quad \forall m > N$

$\therefore$ The series converges to x.

$\therefore$ Every cauchy sequence converges

$(\mathbb{N}, d)$ is complete.

(b) Let $(x_n) = (n)$

Now $\forall \epsilon > 0 \quad \exists N$ s.t.

$$d(x_m, x_n) = \left|\frac{1}{m} - \frac{1}{n}\right| < \epsilon \quad \forall m, n > N$$

$\left(\because \left(\frac{1}{n}\right)\right.$ is convergent hence cauchy$\Big)$

$\therefore (x_n) = (n) = (1, 2, 3, \ldots)$ is a Cauchy sequence

However it is obviously unbounded hence not convergent.

$\therefore (\mathbb{N}, d)$ is not complete.

Q 13.

Consider $\varphi : [a,b] \to [0,1]$

$$\varphi(t) = \frac{t-a}{b-a}$$

Obviously $\varphi(t)$ is a homeomorphism

Define $T : C[0,1] \longrightarrow C[a,b]$

$$T(x) = x \circ \varphi$$

~~[a,b]~~
$$\begin{tikzcd} {[a,b]} \arrow[r, "\varphi"] \arrow[rr, bend right, "T"'] & {[0,1]} \arrow[r, "x"] & \mathbb{R} \end{tikzcd}$$

$$d(Tx, Ty) = \sup_{t \in (a,b)} |Tx(t) - Ty(t)|$$

$$= \sup_{t \in (a,b)} \left| x\left(\frac{t-a}{b-a}\right) - y\left(\frac{t-a}{b-a}\right) \right|$$

$$= \sup_{t \in (0,1)} |x(t) - y(t)|$$

$$= d(x,y)$$

$\therefore$ T preserves distance

$h \in C[a,b] \Rightarrow h \circ \varphi^{-1} \in C[0,1]$ $\therefore T(h \circ \varphi) = h$

$\therefore$ T is surjective

Since $d(Tx, Ty) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$.

$\Leftrightarrow Tx = Ty$

$\therefore$ T is injective $\therefore$ T is bijective $\Rightarrow$ $C[a,b]$ and $C[0,1]$ are isometric

Q14

$\bar{d} = \frac{d}{1+d} \leq d$

$\Rightarrow$ Let (x_n) be a Cauchy sequence in (X, d)

$\forall \epsilon > 0 \ \exists N$ s.t.

$d(x_m, x_n) < \epsilon \quad \forall m, n > N$

Since $\bar{d} \leq d \ \forall x \in X$

$\bar{d}(x_m, x_n) < \epsilon \quad \forall m, n > N$

$\therefore (x_n)$ is Cauchy sequence in $(X, \bar{d})$

$\therefore (X, d)$ is Complete ~~$\Rightarrow (X, d)$~~.

$\Rightarrow (x_n)$ convergent (say to x)

$\Rightarrow (x_n)$ convergent to x in $(X, \bar{d})$

$\Rightarrow (x_n)$ convergent in $(X, \bar{d})$

$\therefore (x_n)$ is a Cauchy sequence in $(X, \bar{d})$

~~and~~ $\Rightarrow (x_n)$ is convergent $\Rightarrow (X, \bar{d})$ complete

$\Leftarrow$ Let (x_n) be Cauchy sequence in $(X, \bar{d})$ which is complete

If $\bar{d}(x_m, x_n) < \epsilon < \frac{1}{2}$

$$d(x_m, x_n) = \frac{\bar{d}(x_m, x_n)}{1 - \bar{d}(x_m, x_n)} < 2\bar{d}(x_m, x_n)$$

$\therefore (x_n)$ is Cauchy in (X, d) with the same limit and convergent.

$\therefore (X, d)$ is complete.

Scanned by CamScanner

Q15.

(a) $\tilde{X} = \{(x_n) \mid (x_n)$ is a Cauchy sequence in $X, d\}$

$(x_n), (y_n) \in \tilde{X}$

Choose $\epsilon > 0$ $\exists$ N_1, N_2

$$d(x_m, x_n) < \frac{\epsilon}{2} \qquad m, n > N_1$$

$$d(y_m, y_n) < \frac{\epsilon}{2} \qquad m, n > N_2$$

Define (a_n) $a_n = d(x_n, y_n)$

$\therefore$ (a_n) is a sequence in $\mathbb{R}$

Since $\mathbb{R}$ is complete, it is enough to show (a_n) is Cauchy.

$$a_m = d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m)$$
$$\leq d(x_m, x_n) + a_n + d(y_m, y_n)$$

$$\Rightarrow a_m - a_n \leq d(x_m, x_n) + d(y_m, y_n)$$

Slly, reversing role of m, n

$$a_n - a_m \leq d(x_m, x_n) + d(y_m, y_n)$$

$$|a_m - a_n| \leq d(x_m, x_n) + d(y_m, y_n)$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad (\forall\, m, n > \max(N_1, N_2))$$
$$< \epsilon$$

Since ϵ was arbitrary (a_n) is Cauchy in $\mathbb{R}$

$\Rightarrow$ (a_n) converges in $\mathbb{R}$

~~Hence~~ Proved

(b) Given $(x_n), (y_n) \in \tilde{X}$ and $(d(x_n, y_n)) \to 0$

$\Rightarrow$ Let (y_n) converge to z

Choose $\epsilon > 0$, $\exists\ N_1, N_2$ s.t

~~$|y_n - z|$~~

$d(y_n, z) < \frac{\epsilon}{2} \quad \forall n > N_1$

~~$d(x_n, z) \leq d(x_n, y_n) + d(y_n, z$~~

$|d(x_n, y_n) - 0| < \frac{\epsilon}{2} \quad \forall n > N_2$

$\therefore d(x_n, z) \leq d(x_n, y_n) + d(y_n, z)$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall n > \max(N_1, N_2)$

$= \epsilon$ Since ϵ arbitrary

$\therefore x_n \to z$

Slly, reversing role of x_n, y_n (WLOG)

if $x_n \to z$

$y_n \to z$

Hence Proved.

(c) (i) $(x_n) \sim (x_n)$ is true as $(d(x_n, x_n))$ is zero sequence

(ii) $(x_n) \sim (y_n) \Leftrightarrow (d(x_n, y_n))$ converges to $0 \Leftrightarrow (y_n) \sim (x_n)$

$\Leftrightarrow (d(y_n, x_n))$ converges to 0

(iii) Let $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$

$d(x_n, y_n) \longrightarrow 0 \qquad d(y_n, z_n) \to 0$

Choose ϵ $\exists$ N_1, N_2 such that

$|d(x_n, y_n) - 0| < \frac{\epsilon}{2} \quad \forall n > N_1$

$|d(y_n, z_n) - 0| < \frac{\epsilon}{2} \quad \forall n > N_2$

$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ (\forall n > \max(N_1, N_2))$

$\therefore |d(x_n, y_n) - 0| < \epsilon$. Since ϵ arbitrary

$(d(x_n, y_n)) \longrightarrow 0 \therefore (x_n) \sim (z_n)$

Proved.

Scanned by CamScanner

2. Normed Spaces

1. (a) Let $(X, \|\cdot\|)$ be a normed space.

If $X \neq \{0\}$ to show $\{\|x\| : x \in X\} = [0, \infty)$

Since X is a vector space

and $X \neq \{0\}$

$\exists\ v \in X \quad \|v\| \neq 0$

$\therefore \quad \alpha v \in X \quad \forall\ \alpha \in \mathbb{R}$

Consider the subset
$S = \{\|\alpha v\| : \alpha \in \mathbb{R}\}$ of $\{\|x\| : x \in X\}$

$S = \{|\alpha|\ \|v\| \mid \alpha \in \mathbb{R}\}$

By definition of norm $|\alpha|\,\|v\| \in [0, \infty)$

$\Rightarrow S \subseteq [0, \infty)$

Choose any no $M \in [0, \infty)$

Let $\alpha = \dfrac{M}{\|v\|} \Rightarrow |\alpha|\,\|v\| = M \in S$

$\therefore [0, \infty) \subseteq S$

$\therefore S = [0, \infty)$

But $S \subseteq \{\|x\| : x \in X\} \subseteq [0, \infty)$

$\therefore \{\|x\| : x \in X\} = [0, \infty)$

(b) d is a metric on vector Space $X \neq \{0\}$

$$\bar{d}(x,y) = \begin{cases} 0 & x = y \\ d(x,y)+1 & \text{if } x \neq y \end{cases} \quad \forall x,y \in X$$

(i) $\bar{d}(x,y) \geqslant 0$

(ii) $\bar{d}(x,y) = 0 \Leftrightarrow x = y$

(iii) $\bar{d}(x,y) = 1 + d(x,y) = 1 + d(y,x) = \bar{d}(y,x)$

(iv) $\bar{d}(x,y) = 1 + d(x,y) \leq 2 + d(x,z) + d(y,z)$

$= 1 + d(x,z) + 1 + d(y,z)$

$= \bar{d}(x,z) + \bar{d}(y,z)$

$\therefore$ $\bar{d}$ is a metric on X

Assume $\bar{d}$ is induced by a norm $\|\cdot\|$ $\bar{d}(x,y) = \|x-y\|$

Let $y = (1-\epsilon)x$ for some $0 \leq \epsilon \leq 1$

$\bar{d}(x,y) = \|x-(1-\epsilon)x\| = \|\epsilon x\| = |\epsilon|\,\|x\|$

Let $\epsilon = \frac{1}{2\|x\|} \Rightarrow \bar{d}(x,y) = \frac{1}{2}$

but $\bar{d}(x,y) = \begin{cases} 0 & x = y \\ 1 + d(x,y) & x \neq y \end{cases}$

$= 1 + d(x,y) = \frac{1}{2}$

But $d(x,y) \geqslant 0$

$\therefore$ This is a contradiction

$\therefore$ $\bar{d}$ cannot be induced by a norm

Q2. A set A is said to be bounded if

diameter (A) $< \infty$

$\sup_{x,y \in A} d(x,y) < \infty$

($\Rightarrow$) Assume $\exists\ m > 0$ s.t $\|x\| \leq m$

$\text{diam}(A) = \sup_{x,y \in A} \|x - y\|$

$\leq \sup_{x,y \in A} (\|x\| + \|y\|)$ [Triangle inequality]

$\leq 2m < \infty$

$\therefore$ $\text{diam}(A) < \infty \Rightarrow$ A is bounded.

($\Leftarrow$) Assume A is bounded.

$\sup_{x,y \in A} \|x - y\| < \infty$

Let $y = 0$ $\|x - 0\| < \sup_{x,y \in A} \|x - y\| < \infty$

$\therefore \|x\| < \infty$

$\Rightarrow \exists\ m$ such that $\|x\| < m\ \forall x$

Q3

(a) X is a normed space, $Y \neq \{0\}$ is a subspace.

$\therefore \quad \exists\, y \in Y,\ y \neq 0 \Rightarrow \|y\| \neq 0$

Let $\operatorname{diam}(Y) = \delta(Y) = \sup\limits_{y_1, y_2 \in Y} \|y_1 - y_2\|$

For $y_2 = 0$, $y_1 = \alpha y \in Y$ $\alpha \in \mathbb{R}$ is arbitrary

$\cancel{\|y - 0\| \leftarrow \sup}$

$\|\alpha y - 0\| \leqslant \sup\limits_{y_1, y_2 \in Y} \|y_1 - y_2\|$ [Def^n of sup]

$|\alpha| \|y\| \leq \delta(Y)$

Since α is arbitrary and $\alpha \in \mathbb{R}$ and $\|y\| \neq 0$

$\Rightarrow |\alpha| \|y\|$ is not bounded.

[If it was bounded by M, choose $\alpha > \frac{M}{\|y\|}$ to contradict]

$\delta(Y) \geq |\alpha| \|y\|$

$\therefore$ cannot be bounded

Y is unbounded.

(b) Let us assume $\|\cdot\|$ induces discrete metric d on X

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

~~First, rule out $X = \{0\}$~~
~~if $X = \{0\}$ $\|x\|$ 0 but $d(x,y)$~~

Assume $X \neq \{0\}$

$\exists\, x \in X$ such that $x \neq 0 \Rightarrow \|x\| \neq 0$ ~~$0 < \epsilon < 1$~~

Now choose $y = (1-\epsilon)x$

$d(x,y) = 1$ (since $\epsilon \neq 1$)

But $d(x,y) = \|x - y\|$
$= \|\epsilon x\|$
$= |\epsilon|\,\|x\|$

Choose $\epsilon = \frac{1}{2\|x\|}$

$\therefore d(x,y) = |\epsilon|\,\|x\| = \frac{1}{2}$

But $d(x,y) = 1$ for discrete metric $\Rightarrow \Leftarrow$

$\therefore$ Contradiction.

Such $\|\cdot\|$ cannot exist.

Q4.

Given Y subspace of X.

Assume $(a+Y) \cap (b+Y) \neq \phi$

~~To show a+Y and b+Y are disjoint~~

Assume they have a common element y.

Let $y = a + y_1 = b + y_2 \Rightarrow a - b = y_1 - y_2 \in Y$

Choose any element $a + y_3$ in $a + Y$, $y_3 \in Y$

$$a + y_3 = a + b + y_3 - b$$

$$= b + (a-b) + y_3$$

But $a - b \in Y$ and $y_3 \in Y$

$\therefore a + y_3 = b + y_3'$, $y_3' \in Y$

$\therefore a + y_3 \in b + Y$

$\Rightarrow (a+Y) \subseteq (b+Y)$

Slly, choose any elt. in $b + y_4$ in $b + Y$, $y_4 \in Y$

$$b + y_4 = a + ((b-a) + y_4)$$

Again $b - a \in Y$ and $y_4 \in Y$

$\therefore b + y_4 = a + y_4'$, $y_4' \in Y$

$\Rightarrow b + y_4 \in a + Y$

$\Rightarrow b + Y \subseteq a + Y$

$\therefore a + Y = b + Y$

If $a+Y$ and $b+Y$ have a common elt.
$a + Y = b + Y$. Otherwise they are disjoint

Hence Proved.

Scanned by CamScanner

INDIAN INSTITUTE OF TECHNOLOGY

DATE SHEET NO.

Q5. $X/Y = \{x+Y \mid x \in X\}$ Y subspace of X.

Operations $(a+Y)+(b+Y) = (a+b)+Y$

$\alpha(a+Y) = \alpha a + Y$

Closure properties are obvious by definition

(i) $(a+Y)+(b+Y) = \underline{(a+b)} + Y$

$= \underline{(b+a)} + Y$

$= (b+Y) + (a+Y)$

(commutative)

(ii) $(a+Y)+((b+Y)+(c+Y)) = (a+Y)+((b+c)+Y)$

$= \underline{(a+(b+c))} + Y = ((a+b)+c) + Y$

$= ((a+b)+Y) + (c+Y) = ((a+Y)+(b+Y)) + (c+Y)$

(associative)

(iii) $\bar{0}$ in $X/Y = 0+Y = Y$ (add. identity)

$\therefore a+Y + \underline{0+Y} = (a+0)+Y = a+Y$

(iv) $(a+Y)+\underline{(-a+Y)} = (a+(-a))+Y = 0+Y = Y$

add. inverse

(v) $\alpha(\beta(a+Y)) = \alpha(\beta a+Y) = \alpha(\beta a)+Y$

$= (\alpha\beta)a + Y$

(vi) $1.(a+Y) = 1.a+Y = a+Y$

P.R.E.

Scanned by CamScanner

(VII) $\alpha((a+Y)+(b+Y)) = \alpha((a+b)+Y)$

$= \alpha(a+b)+Y = (\alpha a+\alpha b)+Y$

$= (\alpha a+Y)+(\alpha b+Y)$

$= \alpha(a+Y)+\alpha(b+Y)$

(VIII) $(\alpha+\beta)(a+Y) = (\alpha+\beta)a+Y = (\alpha a+\beta a)+Y$

$= (\alpha a+Y)+(\beta a+Y) = \alpha(a+Y)+\beta(a+Y)$

$\therefore X/Y$ is a vectorspace.

INDIAN INSTITUTE OF TECHNOLOGY

DATE SHEET NO.

Q6. c = Set of all convergent seq. of real no's.
c_0 = set of all seq. of real no's converging to 0
c_{00} = set of all seq. with finitely many non zero terms.

a) zero sequence is convergent
$(0, 0, \dots) \in c$
also if $c_1, c_2 \in c$ $\quad c_1 + c_2 \in c$
as sum of convergent sequences converge.

To show c is closed.
$$c = \{ (x^{(j)})_{j \in \mathbb{N}} \in l^\infty \mid \exists \lim_{j \to \infty} x^{(j)} \in \mathbb{R} \}$$

~~Let $(x_n) \in c$ and $x_n \to x$~~

~~Since l^∞ is complete~~

Consider $(x_n)_{n \in \mathbb{N}}$ is a sequence of elements of c and it converges to x. Since l^∞ is complete $x \in l^\infty$ $\quad \therefore \|x_n - x\|_\infty \to 0$ as $n \to \infty$

Now, Let $x = (x^{(j)})$ To show $\exists\, x^* \in \mathbb{R}$ which is the limit of $(x^{(j)})$ $\lim_{j \to \infty} x^{(j)} = x^*$, ~~because then~~ ~~$x \in c$ and c will be closed.~~

P.R.E.

Consider the sequence (x_n^*) where x_n^* is the limit of $(x_n^{(j)})_{j \in \mathbb{N}}$

Since $x_n \in C$ $\therefore$ x_n^* exists in $\mathbb{R}$ for all $n \in \mathbb{N}$

$$|x_n^* - x_m^*| \leq |x_n^* - x_n^{(j)}| + |x_n^{(j)} - x_m^{(j)}| + |x_m^{(j)} - x_m^*|$$

$$\leq |x_n^* - x_n^{(j)}| + \|x_n - x_m\|_\infty + |x_m^{(j)} - x_m^*|$$

(Prop. of sup.

~~Now as $m, n \to \infty$ RHS $\to 0$~~

Choose $\epsilon > 0$ $\exists$ N_1, N_2, N_3

s.t. $|x_n^{(j)} - x_n^*| < \epsilon/3$ $\forall j > N_1$

$\|x_n - x_m\|_\infty < \epsilon/3$ $\forall n, m > N_2$

$|x_m^{(j)} - x_m^*| < \epsilon/3$ $\forall j > N_3$

Choose $j, n, m > \max(N_1, N_2, N_3)$

$\therefore |x_n^* - x_m^*| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$

Since $\mathbb{R}$ is complete (x_n^*) converges to x^* in $\mathbb{R}$

To show x^* is also the lim of $(x^{(j)})$

$$|x^{(j)} - x^*| \leq \|x - x_n\|_\infty + |x_n^{(j)} - x_n^*| + |x_n^* - x^*|$$

$\to 0$ as $j \to \infty$ and $n \to \infty$

INDIAN INSTITUTE OF TECHNOLOGY

DATE | SHEET NO.

~~Q6) a)~~

(b) Let $x = (x_j) \in \overline{c_0}$

$\therefore \exists\, x_n = (x_j^n) \in c_0$ s.t. $x_n \to x$ in l^∞

Given $\epsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t $\forall n \geq N$ & j

$$|x_j^n - x_j| \leq \|x_n - x\| < \frac{\epsilon}{2}$$

in particular for $n = N$, all j

$x_N \in c_0$ x_j^N is convergent seq.

limit 0 $\exists\, N_1 \in \mathbb{N}$ s.t $\forall j \geq N_1$

$$|x_j^N| < \frac{\epsilon}{2}$$

$$|x_j| \leq |x_j - x_j^N| + |x_j^N|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad j \geq N_1$$

$\therefore x = (x_j)$ is convergent, limit 0

$x \in c_0$ $\therefore$ c_0 is closed in C.

(c) C_{00} is not closed in c, c_0, ℓ^p for any $1 \leq p \leq \infty$

Consider $x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots\right)$

$x_n \in C_{00}$

To show (x_n) does not converge in C_{00}

if x_n converged to say $x \in C_{00}$

Then $x = (x^{(i)})_{i \in \mathbb{N}}$ would have finitely many non zero terms

$\therefore \exists\, I \in \mathbb{N}$ such that $x^{(j)} = 0 \ \forall j \geq I$

But consider

$x_I \in (x_n)$

$x_I = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{I}, 0, 0, \dots\right)$

Thus $x_I^{(I)} \neq 0$

and $x_n^{(I)} = \frac{1}{I} \quad \forall\, n \geq I$

Then $x^{(I)}$ cannot be 0 $(\Rightarrow \Leftarrow)$

$\therefore x_n$ converges to $\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) = x$

but $x \notin C_{00}$. $\therefore C_{00}$ is not closed in c, c_0 or ℓ^p

INDIAN INSTITUTE OF TECHNOLOGY

DATE

SHEET NO.

7. Let X be a normed space (e_i) a Schauder basis

Also choose $\|e_i\| = 1$ by scaling.

Consider $Q = \{ \sum_{i=0}^{n} q_i e_i \mid n \in \mathbb{N},\ q_i \in \mathbb{Q} \}$

is countable.

Let $x \in X$

$\therefore\ \exists (\alpha_i)_{i \geq 0}$ s.t. $x = \sum_{i=0}^{\infty} \alpha_i e_i$

Also $\exists$ N such that

$$\|x - \sum_{i=0}^{N} \alpha_i e_i\| < \epsilon/2 \ \forall n \geq N.$$

and $|q_i - \alpha_i| \leq \frac{\epsilon}{2(N+1)}$ Since $\mathbb{Q}$ is dense in $\mathbb{R}$

Consider $y = \sum_{i=0}^{N} q_i e_i \in Q$

$$\|x - y\| \leq \|x - \sum_{i=0}^{N} \alpha_i e_i\| + \|\sum_{i=0}^{N} \alpha_i e_i - \sum_{i=0}^{N} q_i e_i\|$$

$$\cancel{\leq \frac{\epsilon}{2} + |\alpha_i - q_i| \sum_{i=0}^{N} \|e_i}$$

$$< \frac{\epsilon}{2} + \sum_{i=0}^{N} |\alpha_i - q_i| \|e_i\|$$

$$< \frac{\epsilon}{2} + \sum_{i=0}^{N} \frac{\epsilon}{2(N+1)}$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\therefore$ Q is dense in X

$\therefore$ X is separable.

P.R.

Q 8) a)

Choose (x_n) cauchy in X

Given $k \in \mathbb{N}$ $\exists N_k \in \mathbb{N}$ s.t $\forall m, n \geq N_k$

$$\|x_m - x_n\| < \frac{1}{2^k}$$

Let N_k be increasing by construction

Let $y_k = x_{N_{k+1}} - x_{N_k}$

Then $$\sum_{k=1}^{\infty} \|y_k\| = \sum_{k=1}^{\infty} \|x_{N_{k+1}} - x_{N_k}\|$$

$$< \sum_{1}^{\infty} \frac{1}{2^k} < \infty$$

(x_{N_k}) is the required subsequence

(b) Let X be complete.

$\sum x_n$ be the absolutely convergent series

To show $\sum x_n$ converges.

Let $S_n = \sum_{i=1}^{n} x_i$

We have $\sum \|x\| < \infty$

$\therefore \forall \epsilon > 0 \ \exists N$ s.t. $\sum_{i=N}^{\infty} \|x_i\| < \epsilon$

$\therefore \|S_n - S_m\| = \|\sum_{i=m+1}^{n} x_i\| \leq \sum_{i=m+1}^{n} \|x_i\|$

Choose $m, n > N$

We have $\|S_n - S_m\| \leq \sum_{m+1}^{n} \|x_i\| \leq \sum_{i=N}^{\infty} \|x_i\| < \epsilon$

$\therefore (S_n)$ is a cauchy sequence

$\Rightarrow S_n$ converges

Let X be a normed vector space where very absolutely convergent series converges.

Let (x_n) be Cauchy sequence.

To show (x_n) converges.

For each $k \in \mathbb{N}$

Choose N_k such that $\|(x_m - x_n)\| < 2^{-k}$ $m, n \geq n_k$

in particular $\|x_{n_{k+1}} - x_{n_k}\| < 2^{-k}$

Let $y_1 = x_{n_1}$ and $y_{k+1} = x_{n_{k+1}} - x_{n_k}$ for $k \geq 1$

$\Sigma \|y_n\| \leq \|x_{n_1}\| + 1$ is absolutely convergent and hence convergent

$\therefore x_{n_k} = x_{n_1} + (x_{n_2} - x_{n_1}) + (x_{n_3} - x_{n_2}) + \dots + (x_{n_k} - x_{n_{k-1}})$

~~is a convergent~~

$\therefore (x_{n_k})$ is a convergent ~~series~~ sequence

$\therefore (x_n)$ has a convergent subsequence

X is complete.

INDIAN INSTITUTE OF TECHNOLOGY

DATE

SHEET NO.

Q9 Y subspace of $(X, \|\cdot\|)$

$\|x+Y\|_0 = \inf\{\|x+y\| \mid y \in Y\} \quad \forall x \in X$

(a) To show $\|x+Y\|_0 = D(x, Y)$

$$= \inf_{y \in Y} d(x, y)$$

$$= \inf_{y \in Y} \|x-y\|$$

$$= \inf \{\|x-y\| \mid y \in Y\}$$

for each $y \in Y$, $-y \in Y$

$$= \inf\{\|x+y\| \mid y \in Y\}$$

$\therefore \{\|x+y\| \mid y \in Y\} = \{\|x-y\| \mid y \in Y\}$

$\therefore \inf \{\|x+y\| \mid y \in Y\} = \inf \{\|x-y\| \mid y \in Y\}$

P.R.E.

(b) Let Y be closed.

$$\|x+Y\|_0 = d(x,Y)$$
$$= \inf\{\|x-y\| \mid y \in Y\}$$

$\|x+Y\|_0 = 0$ iff $\forall \epsilon > 0 \ \exists \ y_\epsilon \in Y$

s.t. $\|x - y_\epsilon\| < \epsilon$

This is the same as saying $x \in \bar{Y}$

Since $\exists$ an elt. of y in every ϵ-Nhbd of x.

Also

$$\therefore \{x \in X \mid \|x+Y\|_0 = 0\} = \{x \in X \mid D(x,Y) = 0\}$$
$$= \bar{Y} = Y$$
$$= \{x \in X \mid x+Y = 0_{X/Y}\}$$

$\therefore \ \|x+Y\|_0 = 0 \Leftrightarrow x+Y = 0_{X/Y}$

$$\|\alpha(x+Y)\|_0 = \|\alpha x + Y\|_0$$
$$= \inf\{\|\alpha(x - \alpha^{-1}y)\| \mid y \in Y\}$$
$$= |\alpha| \inf\{\|x - \alpha^{-1}y\| \mid y \in Y\}$$
$$= |\alpha| \inf\{\|x - y'\| \mid y' \in Y\}$$
$$= |\alpha| \, \|x+Y\|_0$$

Let $x_1, x_2 \in X$. $\exists \ y_1, y_2 \in Y$ Choose $\epsilon > 0$ By inf defn. $\|x_1 - y_1\| < \|x_1 + Y\|_0 + \frac{\epsilon}{2}$ and $\|x_2 - y_2\| < \|x_2 + Y\|_0 + \frac{\epsilon}{2}$

$$\therefore \|x_1 + Y + x_2 + Y\|_0 = \inf\{\|x_1 + x_2 - y\| \mid y \in Y\}$$
$$\leq \|(x_1 - y_1) + (x_2 - y_2)\|$$
$$\leq \|x_1 + Y\|_0 + \frac{\epsilon}{2} + \|x_2 + Y\|_0 + \frac{\epsilon}{2}$$
$$= \|x_1 + Y\|_0 + \|x_2 + Y\|_0 + \epsilon$$

$\therefore$ Taking inf. Triangle inequality holds.

Q10 $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent

$$\alpha \|x\|_a \leq \|x\|_b \leq \beta \|x\|_a$$

Show open sets in $(X, \|\cdot\|_a)$ and $(X, \|\cdot\|_b)$ are the same. Identity map:

$$I : (X, \|\cdot\|_b) \longrightarrow (X, \|\cdot\|_a)$$

Choose $x_0 \in X$ Given any $\epsilon > 0$ choose $\delta = \alpha\epsilon > 0$ $\|x - x_0\|_b < \delta$

$$\Rightarrow \|x - x_0\|_a \leq \frac{1}{\alpha}\|x - x_0\|_b < \frac{\alpha\epsilon}{\alpha} = \epsilon$$

Since $x_0 \in X$ is arbitrary I is continuous over X. Thus $M \subset X$ open in $(X, \|\cdot\|_a)$ has preimage M is open in $(X, \|\cdot\|_b)$

Slly, $\bar{I} : (X, \|\cdot\|_a) \rightarrow (X, \|\cdot\|_b)$

Choose $x_0 \in X$. Given $\epsilon > 0$ Choose $\delta = \frac{\epsilon}{\beta}$

For all x s.t. $\|x - x_0\|_a < \delta$

$$\|x - x_0\|_b \leq \beta \|x - x_0\|_a < \beta \frac{\epsilon}{\beta} = \epsilon$$

$\therefore \bar{I}$ is continuous For any open set in $(X, \|\cdot\|_b)$ its preimage is also open in $(X, \|\cdot\|_a)$

Q 11. Suppose $\| \|_a$ on X and $\| \|_b$ on X are equivalent

$\exists \alpha, \beta > 0$ $\quad \alpha \|x\|_a \leq \|x\|_b \leq \beta \|x\|_a$. ~~a,b>0~~ $\forall x \in X$

$(x_n) \to$ Cauchy sequence in $(X, \| \|_b)$

Given $\epsilon > 0$ $\exists$ $N_1 \in \mathbb{N}$ s.t.

$$\|x_m - x_n\|_b < \alpha \epsilon \qquad \forall m,n > N_1$$

which implies

$$\|x_m - x_n\|_a \leq \frac{1}{\alpha} \|x_m - x_n\|_b < \frac{\alpha \epsilon}{\alpha} = \epsilon$$

$\therefore$ (x_n) is a cauchy seq. in $(X, \| \|_a)$

Slly, We can prove the converse

$(y_n) \to$ Cauchy in $(X, \| \|_a)$

Given $\epsilon > 0$ $\exists$ $N_2 \in \mathbb{N}$ s.t.

$$\|y_m - y_n\|_a < \frac{\epsilon}{\beta} \qquad \forall m,n > N_2$$

$$\Rightarrow \|y_m - y_n\|_b \leq \beta \frac{\epsilon}{\beta} = \epsilon$$

$\therefore$ $(y_n) \to$ Cauchy in $(X, \| \|_b)$

$\therefore$ Cauchy sequences are same in $(X, \| \|_a)$ and $(X, \| \|_b)$.

Thus if (x_n) Cauchy sequence converges in $(X, \| \|_b)$ in $(X, \| \|_a)$ it also converges and vice versa.

$(X, \|\|_a)$ complete $\iff$ $(X, \| \|_b)$ complete.

Q12. a) Given $x_1, x_2, \dots x_n$ are lin. ind.

Let $x_{n+1} \in X$

$\Rightarrow$ Assume $x_1, \dots, x_{n+1}$ is lin. ind.

Then $x_{n+1} \neq \sum_{i=1}^{n} \alpha_i x_i$ for any $(\alpha_i)_{i=1}^{n} \in \mathbb{R}$

Let $Y = \text{span}\{x_1, x_2, \dots, x_n\}$

$D(x_{n+1}, Y) = \inf_{y \in Y} \| x_{n+1} - y \|$

$= \inf_{\beta \in \mathbb{R}^n} \| x_{n+1} - \sum_{i=1}^{n} \beta_i x_i \|$

if $D(x_{n+1}, Y) = 0$ Given $\epsilon > 0$ $\exists$ $\beta_\epsilon \in \mathbb{R}^n$ s.t. ~~$\epsilon > 0$~~

s.t. $\| x_{n+1} - \sum_{i=1}^{n} \beta_{\epsilon i} x_i \| < \epsilon$

Letting $\epsilon \to 0$

$\sum_{i=1}^{n} \beta_{\epsilon i} x_i \to x_{n+1}$

$\therefore$ $x_{n+1} = \sum_{i=1}^{n} \beta_{\epsilon i} x_i$

Which is a contradiction to lin ind.

($\Leftarrow$) Let $D(x_{n+1}, Y) = \rho > 0$

Assume $x_{n+1} = \sum_{i=1}^{n} \alpha_i x_i$ (to show contradiction)

choose $y_\alpha = \sum \alpha_i x_i = x_{n+1}$

$$\|x_{n+1} - y_\alpha\| \geq \inf_{y \in Y} \|x_{n+1} - y\| = \rho$$

$$\|0\| \geq \rho$$

$$0 \geq \rho > 0$$

$$\Rightarrow \Leftarrow$$

$\therefore \{x_i\}_{i=1}^{n+1}$ are lin. ind.

(b) ~~Let $x^* \in X \setminus Y$ since Y is proper subspace.~~

~~Let $Y = \text{span}\{y_1, y_2, \dots y_n\}$ and $\|y_i\| = 1$ WLOG.~~

~~and $\dim Y = n$. i.e. $\{y_1, \dots y_n\}$ is a basis.~~

Let $a = D(x, Y)$ $\exists (y_n)$ in Y such that

$\|x - y_n\| \le a + \frac{1}{n}$ .. so (y_n) is bounded

and $\|y_n\| \le M \quad \forall n$

$\therefore A = \{y \in Y \mid \|y\| \le M\}$ is closed

and bounded subset of F.D.V.S and

A is compact

$\therefore y_n$ has convergent subsequence (y_{n_k})

$y_{n_k} \to y_0$ $\cancel{\Rightarrow \|x - y_0\| \le a + \frac{1}{n_k} + \|y_{n_k} - y_0\|}$

$\|x - y_0\| > d(x, Y) = a$

$\|x - y_0\| < \|x - y_{n_k}\| + \|y_{n_k} - y_0\| \le a + \frac{1}{n_k} + \|y_{n_k} - y_0\|$

$\therefore \quad a \le \|x - y_0\| \le a$

$\therefore \|x - y_0\| = a = \text{dist}(x, Y)$

Since x was arbitrary

$\forall x \in X \ \exists \ y_0$ s.t. $\|x - y_0\| = d(x, Y)$

Let $v \in X \setminus Y$ $\quad b = d(v, Y)$ $\quad b > 0$

as Y (FDVS) is closed.

$\therefore \exists\, y_0 \in Y$ s.t. $b = \| v - y_0 \|$

Let $x = c(v - y_0)$ where $c = \dfrac{1}{\|v - y_0\|} = \dfrac{1}{b}$

$\therefore \|x\| = 1$ and

$$\|x - y\| = \| c(v - y_0) - y \|$$
$$= c\| v - y_1 \|$$

$\therefore \|x - y\| \geq c\, d(v, Y) = cb = 1$

$\therefore D(x, Y) \geq 1$

Since $0 \in Y$ $\quad D(x, Y) \leq 1$

$$D(x, Y) = 1 = \|x\|$$

(c) Choose $x_1 \in B$ such that $\|x_1\| = 1$ [where $B = \{x \mid \|x\| \leq 1\}$]

Let $A_1 = \text{span}\{x_1\}$

We can now choose x_2 such that $\|x_2\| = 1$

and $d(x_2, A_1) = \|x_2 - x_1\| \geq \frac{1}{2}$ by Riesz Lemma.

Let $A_2 = \text{span}\{x_1, x_2\}$

Choose x_3 s.t. $\|x_3\| = 1$

$d(x_3, A_2) \geq \frac{1}{2}$. ies $\|x_3 - x_2\| \geq \frac{1}{2}$

Continuing thus

we get a seq. of elts. x_n

s.t $\|x_m - x_n\| \geq \frac{1}{2}$ $m \neq n$.

x_n cannot have a convergent subsequence.

$\therefore$ ~~B~~ X cannot be compact

INDIAN INSTITUTE OF TECHNOLOGY

DATE

SHEET NO.

Q13 $T : X \to Y$ linear

(a) Let M_X be subspace of X

~~Let $m_{x_1}, m_{x_2} \in M_x$~~

~~$\Rightarrow \alpha m_{x_1} + \beta m_{x_2} \in M_x$~~

~~$T(m_{x_1})$~~ Let $M_Y = T(M_x)$

Choose m_{y_1}, m_{y_2} in M_Y

$m_{y_1} = T(m_{x_1})$ some $m_{x_1} \in M_x$

$m_{y_2} = T(m_{x_2})$ some $m_{x_2} \in M_x$

$$\alpha m_{y_1} + \beta m_{y_2} = \alpha T(m_{x_1}) + \beta T(m_{x_2})$$
$$= T(\alpha m_{x_1}) + T(\beta m_{x_2})$$
$$= T(\underbrace{\alpha m_{x_1} + \beta m_{x_2}}_{\in M_x})$$

$\in M_Y$

~~Also~~

~~$T(0) = 0 \in M_Y$~~

$0_X \in M_X$

$T(0_X) = 0_Y \in M_Y$

M_Y is a subspace.

P.R.E.

if M_Y is a subspace to show $T^{-1}(M_Y) = M_X$ is a subspace. We know T^{-1} is linear

~~Choose m_{x_1}, m_{x_2} in $T^{-1}(M_Y)$~~

~~$T(m_{x_1}), T(m_{x_2}) \in M_Y$~~

~~$\therefore \alpha T(m_{x_1}) + \beta T(m_{x_2}) \in M_Y$~~

Choose $m_{y_1}, m_{y_2} \in M_Y$

$$\alpha T^{-1}(m_{y_1}) + \beta T^{-1}(m_{y_2})$$

$$= T^{-1}(\alpha m_{y_1}) + T^{-1}(\beta m_{y_2})$$

$$= T^{-1}(\underbrace{\alpha m_{y_1} + \beta m_{y_2}}_{\in M_Y})$$

M_X is a subspace

INDIAN INSTITUTE OF TECHNOLOGY

DATE

SHEET NO.

(b): Given $Tx_1, Tx_2, Tx_3 \cdots Tx_n$ are L.I

$\therefore \sum_{i=1}^{n} c_i T(x_i) = 0_Y \Rightarrow c_i = 0 \quad \forall i = \{1, \dots, n\}$

$\therefore \sum_{i=1}^{n} T(c_i x_i) = 0_Y \Rightarrow c_i = 0$

$\therefore T(\sum_1^n c_i x_i) = 0_Y \Rightarrow c_i = 0$

$\sum c_i x_i = 0_X \Rightarrow c_i = 0$

$\therefore x_i$ are L.I. $i = \{1, \dots n\}$

(c). x_i are L.I.

$\sum_{i=1}^{n} c_i x_i = 0_X \Rightarrow c_i = 0$

if T is injective $x_1 = x_2 \Rightarrow T(x_1) = T(x_2)$

$\therefore T(\sum_1^n c_i x_i) = T(0_X) \Rightarrow c_i = 0$

$\therefore \sum_1^n T(c_i x_i) = 0_Y \Rightarrow c_i = 0$

$\Rightarrow \sum_1^n c_i T x_i = 0_Y \Rightarrow c_i = 0$

Tx_i are L.I.

P.R.E.

(14) Let T map bdd sets in X to bdd sets in Y

WLOG choose bdd sets centred 0.

Given $R > 0$ $\exists$ $M_R > 0$ $\|x\| \leq R \Rightarrow \|Tx\| \leq M_R$

Choose $y \neq 0 \in X$ set

$$x = R \frac{y}{\|y\|} \Rightarrow \|x\| = R$$

Thus $\frac{R}{\|y\|}\|Ty\| = \left\|T\left(\frac{R}{\|y\|}y\right)\right\| = \|Tx\| \leq M_R$

$$\Rightarrow \|Ty\| \leq \frac{M_R}{R}\|y\|$$

Taking sup. over y with norm 1

Shows T is bdd.

Let T be bdd.

Let $A = \{ y \mid \|x - y\| \leq C_x \}$ be bdd.

$\|Tx + Ty\| \leq \|Tx\| + \|Ty\| \leq C_1\|x\| + C_2\|y\|$

$\therefore$ Image of A is bdd.

INDIAN INSTITUTE OF TECHNOLOGY

DATE SHEET NO.

Q15.

$\|Tx\| < \|T\|\|x\| < \|T\|$ if $\|x\| < 1$

Q16

(a) $T(\alpha x + \beta y) = \left(\alpha \frac{x_j}{j} + \beta \frac{y_j}{j}\right)$

$= \alpha\left(\frac{x_j}{j}\right) + \beta\left(\frac{y_j}{j}\right)$

$= \alpha Tx + \beta Ty$

$\left|\frac{x_j}{j}\right| \leq |x_j| \leq \sup |x_j| = \|x\|$

Taking sup. over j

$\|Tx\| \leq \|x\|$

$\therefore T$ is bdd.

$T(x) = T(y) \Rightarrow \left(\frac{x_j}{j}\right) = \left(\frac{y_j}{j}\right)$

For term wise equating

$x_j = y_j$

$\therefore (x_j) = (y_j) \Rightarrow x = y.$

P.R.E.

(b) Let (x_n) be sequence in l^∞ $x_n = (x_j^n)$

$$x_j^n = \begin{cases} \sqrt{j} & j \le n \\ 0 & j > n \end{cases}$$

$T : l^\infty \to l^\infty$ as above

$$Tx_n = (y_j^n)$$

$$y_j^n = \frac{x_j^n}{j} = \begin{cases} \frac{1}{\sqrt{j}} & j \le n \\ 0 & j > n \end{cases}$$

(y_n) converges to y in l^∞

with $y = (y_j)$ $y_j = \frac{1}{\sqrt{j}}$

$$\|y_n - y\|_\infty = \sup_{j \in \mathbb{N}} |y_j^n - y_j| = \frac{1}{\sqrt{n+1}} \to 0 \text{ as } n \to \infty$$

But $y \notin R(T)$ $\therefore$ $R(T)$ is not closed in l^∞

(c) $x = T^{-1}y$ ~~defined~~ $= (x_j)$ $x_j = j\,y_j$ $y = (y_j)$

To show T not bdd.

$$y_n = (\delta_{jn})_{j=1}^{\infty} \quad \|y_n\| = 1$$

$$\|T^{-1}y_n\| = \|(j\,\delta_{jn})\| = n.$$

$$\Rightarrow \frac{\|T^{-1}y_n\|}{\|y_n\|} = n$$

n is arbit. $\exists$ T^{-1} is unbounded.

Q17 If $x_1 \neq x_2$ $\|x_1 - x_2\| > 0$

$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \geq b\|x_1 - x_2\| > 0$

$Tx_1 \neq Tx_2$

$\therefore$ T is injective.

T^{-1} exists (surj. is by def^n)

$\|T(T^{-1}y)\| \geq b\|T^{-1}y\|$

$\Rightarrow \|T^{-1}y\| \leq \frac{1}{b}\|y\|$

$C = \frac{1}{b} > 0$

Q 18

Let $f \neq 0$.

$f(y) \neq 0 \quad \forall y \in Y$

$f(y_0) = \alpha \quad$ for some $\alpha \in K$

~~Is subspace~~

$\beta y_0 \in Y \quad \forall \beta \in K$

$f(\beta y_0) = \beta f(y_0) = \beta \alpha \in R(f)$

As β is arbitrary $\quad R(f) = K$

Q19 $\quad f$ is linear functional $f \neq 0$.

Let $x_0 \in X \setminus N$

Claim for each $x \in X \; \exists \; \alpha$ s.t

$x = \alpha x_0 + y \quad y \in N$

$f(x) = f(\alpha x_0 + y)$

$\Rightarrow f(y) = f(x) - \alpha f(x_0)$

Let $\alpha = \dfrac{f(x)}{f(x_0)} \quad f(y) = f(x) - f(x) = 0$

$y \in N$

To show uniqueness

Let $x = \alpha_1 x_0 + y_1 = \alpha_2 x_0 + y_2$

$(\alpha_1 - \alpha_2) x_0 = y_2 - y_1$

$f((\alpha_1 - \alpha_2) x_0) = f(y_2 - y_1)$

$(\alpha_1 - \alpha_2) f(x_0) = f(y_2) - f(y_1) = 0$

$f(x_0) \neq 0 \Rightarrow \alpha_1 - \alpha_2 = 0 = y_1 - y_2$

Scanned by CamScanner

INDIAN INSTITUTE OF TECHNOLOGY

DATE

SHEET NO.

Q9

$x, x' \in X \quad z = x f_1(x') - x' f_1(x)$

$f_1(x) = 0 \Rightarrow x \in N(f_1) = N(f_2)$

$0 = f_2(x) = f_2(x) f_1(x') - f_2(x') f_1(x)$

$f_1 \neq 0 \quad \exists \; x' \in X \setminus N(f_1)$ s.t. $f_1(x') \neq 0$

$f_2(x') \neq 0$ as $N(f_1) = N(f_2)$

$$f_2(x) = \frac{f_2(x')}{f_1(x')} f_1(x)$$

$\therefore f_1, f_2$ are lin. dependent

P.R.E.

$L = \{ x \in X \mid f(x) = 1 \}$

20) $|f(x)| \leq \|f\| \|x\|$

$1 \leq \|f\| \|x\| \quad \forall x \in L$

$\frac{1}{\|f\|} \leq \|x\| \quad \forall x \in L$

$\frac{1}{\|f\|} \leq \inf_{x \in L} \|x\| = \inf_{x \in L} (d(0, x)) = D(0, L)$

$\therefore \inf \{ \|x\| \mid f(x) = 1 \} \geq \frac{1}{\|f\|}$

$\forall \epsilon > 0$ ~~$\|x\|$~~ we find x, $\|x\| = 1$

$|f(x)| \geq \frac{\|f\|}{1+\epsilon}$

$y = \frac{x}{f(x)} \qquad f(y) = 1$

$\|y\| \geq D(0, L)$

$1 = \|x\| \geq |f(x)| D(0, L) \geq \frac{\|f\|}{1+\epsilon} D(0, L)$

$1 \geq \|f\| D(0, L)$

$\inf \{ \|x\| \mid f(x) = 1 \} \leq \frac{1}{\|f\|}$

$\therefore \frac{1}{\|f\|} = \inf \{ \|x\| \mid f(x) = 1 \}$