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13MA20041

MATHEMATICAL METHODS

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Reference Books:

1. Mathematical methods in the physical sciences. Mary L. Baas (Wiley)
2. Mathematical methods. Jacob Korevaar. (Dover)
3. Mathematical methods for physicists. Arfken & Weber. (Elsvie
4. Theory and problems of vector analysis and an introductio to Tensor Analysis. Murray R Spiegel. (Schaum's series)

Syllabus:

1. Infinite series
2. Series solution of 2nd order ODE around regular and singular point, Frobenius method.
3. Legendre equation.
4. Bessel equation
5. Tensor analysis
6. Sturm-Liouville problem

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INFINITE SERIES :

An infinite series is the sum of the terms of a sequence $u_1, u_2, u_3 \ldots\ldots$ i.e., $\sum u_n$.

Ordinary add up of infinite no. of terms doesn't necessarily have a meaning. We thus associate sequence of partial sum of series $\{S_n\}$, where $S_n = u_1 + u_2 + \ldots\ldots + u_n$ i.e., $S_1 = u_1$, $S_2 = u_1 + u_2$, etc.

The infinite series $\sum u_n$ is said to be convergent if the sequence $\{S_n\}$ converges. Thus if $\{S_n\}$ tends to a limit, then this limit is sum of infinite series i.e., $\sum u_n = \lim S_n$

If $\{S_n\}$ diverges, $\sum u_n$ diverges

If $\{S_n\}$ oscillates, $\sum u_n$ oscillates.

Necessary condition for convergence -

Statement: A necessary condition for convergence of an infinite series $\sum u_n$ is that $\lim_{n\to\infty} u_n = 0$.

Proof: Let $S_n = u_1 + u_2 + \ldots\ldots + u_n$

Suppose $\sum u_n$ converges. Then $\{S_n\}$ also converges.

Let $\lim_{n\to\infty} S_n = S$

Now, $u_n = S_n - S_{n-1}$ $\quad \therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - S_{n-1} = S - S = 0$.

Hence, for a convergent series, $\lim_{n\to\infty} u_n = 0$.

Note: ① A series may diverge even if $\lim_{n\to\infty} u_n = 0$.

② If $\lim_{n\to\infty} u_n \neq 0$, series diverges.

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Cauchy's general principle of convergence -

Th.1 A series $\sum u_n$ converges iff to each $\epsilon > 0$, $\exists$ a positive integer m s.t. $|u_{n+1} + u_{n+2} + \ldots\ldots + u_{n+p}| < \epsilon$ $\forall\ n \geq m$ and $p \geq 1$

Proof: $\sum u_n$ converges iff the sequence $\{S_n\}$ converges. By Cauchy General principle for sequence, a sequence $\{S_n\}$ converges iff to each $\epsilon > 0$, $\exists$ a positive integer m s.t.

$$|S_{n+p} - S_n| < \epsilon, \quad \forall\ n \geq m \text{ and } p \geq 1$$

$$\Rightarrow |u_{n+1} + u_{n+2} + \ldots\ldots + u_{n+p}| < \epsilon, \quad \forall\ n \geq m \text{ and } p \geq 1$$

Example ① $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots\ldots$ is not convergent.

$$u_n = \frac{n}{n+1}, \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0.$$

$\Rightarrow$ series divergent.

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② $\sum \frac{1}{n}$ does not converge, even though $\lim \frac{1}{n} = 0$

If possible, let $\sum \frac{1}{n}$ converge.

Then, for any given ϵ, say $\epsilon = \frac{1}{4}$, $\exists$ a positive integer m s.t. $\left|\frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+p}\right| < \epsilon$, $\forall\ n \geq m$ and $p \geq 1$

For $n = m$, $p = m$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+p} = \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{2m} > m \cdot \frac{1}{2m} = \frac{1}{2} > \epsilon$$

Thus, contradiction arises. Hence, $\sum \frac{1}{n}$ diverges.

Standard results -

T2. If $\sum u_n = u$, then $\sum c u_n = cu$, c is n-independent no.

Note: Behaviour of a series as regards convergence is not altered by -

(i) the alteration, addition or omission of a finite no. of terms.

(ii) multiplication of all terms by a non-zero number.

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T3. If $\sum u_n = u$, $\sum v_n = v$, then $\sum w_n = u+v$ where, $w_n = u_{n+v}$

Note: If any two of 3 series $\sum u_n$, $\sum v_n$, $\sum w_n$ are convergent, then third is also convergent.

T4. If a series $\sum u_n$ converges to the sum u, then so does any series obtained from $\sum u_n$ by grouping the terms in brackets without altering the order of the terms

Note: Converse is not always true.

e.g. $(1-1)+(1-1)+\ldots$ is convergent, but $1-1+1-1+\ldots$ is divergent.

Series of positive terms - Here $\{S_n\}$ is monotonically increasing (m.i.)

T5. $\sum u_n$ converges iff $\{S_n\}$ is bounded above.

T9. Geometric series $1+r+r^2+\ldots$ converges for $r<1$ and diverges for $r \geq 1$

A comparison series -

T10. $\sum \frac{1}{n^p}$ is convergent iff $p>1$.

Comparison Test -

(1) If $\sum u_n$ and $\sum v_n$ are two positive term series and $u_n \leq k v_n$ $(k \neq 0)$ $\forall\, n \geq m$, then,

(i) $\sum u_n$ converges if $\sum v_n$ converges.

(ii) $\sum v_n$ diverges if $\sum u_n$ diverges.

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(2) (Limit form) If $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, where $l \neq 0, \infty$, then both sequences converge or diverge together.

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(3) If $\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}$, $\forall\, n \geq m$, then same conclusion as (1)

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Prob. $1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots$ is convergent

$$\frac{1}{2!} = \frac{1}{2}, \quad \frac{1}{3!} < \frac{1}{2^2}, \ldots\ldots, \frac{1}{n!} < \frac{1}{2^{n-1}}$$

$\Rightarrow 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots < 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots$, which is a G.P. of common ratio $r = \frac{1}{2} < 1$ and so convergent.

Hence, given series converges.

Cauchy's root test -

(1) If $\lim_{n\to\infty} (u_n)^{1/n} = l$, then $\sum u_n$ (i) converges if $l < 1$, (ii) diverges if $l > 1$. Test fails if $l = 1$.

Prob. $\sum u_n$, $u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ converges.

Sol. Clearly $u_n > 0 \ \forall n$.

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-\frac{1}{n}\cdot n^{3/2}} = \lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

$$= \frac{1}{e} < 1 \quad \therefore \sum u_n \text{ converges.}$$

D'Alembert's ratio test -

(1) If $\sum u_n$ is pos. term series and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then the series (i) converges if $l < 1$, (ii) diverges if $l > 1$, (iii) test fails if $l = 1$.

Prob. Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, $a_n = \frac{n^2-1}{n^2+1}$

Sol. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{x^{n+1}}{x} = \lim_{n\to\infty} \frac{(n+1)^2 - 1}{(n+1)^2 + 1} \cdot \frac{n^2+1}{n^2-1} \cdot x = x$$

$\therefore$ Series converges if $x < 1$, diverges if $x > 1$.

Test fails for $x = 1$.

For $x = 1$, $\lim_{n\to\infty} a_n = 1 \neq 0 \Rightarrow$ series diverges for $x = 1$.

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Raabe's Test -

② If $\sum u_n$ is pos. term series and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=l$, then the series (i) converges if $l>1$

(ii) diverges if $l<1$. Test fails for $l=1$.

Prob. Test the convergence of the series

$$\frac{\alpha}{\beta}+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)}+\cdots\cdots$$

Sol. $u_n = \dfrac{(1+\alpha)(2+\alpha)\cdots\cdots(n-1+\alpha)}{(1+\beta)(2+\beta)\cdots\cdots(n-1+\beta)}$, $n=2,3,\ldots\ldots$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{n+\alpha}{n+\beta}=1 \Rightarrow \text{D'Alembert's ratio test fails}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{n+\beta}{n+\alpha}-1\right)=\lim_{n\to\infty}\frac{\beta-\alpha}{1+\frac{\alpha}{n}}=\beta-\alpha$$

Thus, by Raabe's Test, the series converges if $\beta-\alpha>1$ or, $\beta>\alpha+1$ and, diverges if $\beta<\alpha+1$. Test fails for $\beta=\alpha+1$.

For $\beta=\alpha+1$, the series becomes

$$\frac{\alpha}{1+\alpha}+\frac{1+\alpha}{2+\alpha}+\frac{1+\alpha}{3+\alpha}+\cdots\cdots=\frac{\alpha}{1+\alpha}+\sum\frac{1+\alpha}{n+\alpha}$$

Comparing this series with the divergent series $\sum\frac{1}{n}$,

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1+\alpha}{n+\alpha}\cdot n=1+\alpha\neq 0,\infty$$

$\Rightarrow$ both series converge or diverge together.

$\therefore$ Series diverges for $\beta\leq\alpha+1$.

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Logarithmic Test -

③ If $\sum u_n$ is a pos. term series and $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right)=l$, then the series converges for $l>1$ and diverges for $l<1$.

Prob. $1+\frac{x}{1!}+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\cdots$

Sol. $u_n=\frac{n^n x^n}{n!}$; $n=1,2,3,\ldots$

Now, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n x^n}$

$$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n x = ex.$$

By ratio test, series converges for $ex<1$ or, $x<\frac{1}{e}$ and diverges for $x>\frac{1}{e}$. Test fails for $x=\frac{1}{e}$.

For $x=\frac{1}{e}$, $\frac{u_n}{u_{n+1}}=\frac{n^n}{n!}\cdot\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{1}{e^n}\cdot e^{n+1}=\left(\frac{n}{n+1}\right)^n e$

Now, $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right)=\lim_{n\to\infty} n\left[1-n\log\left(1+\frac{1}{n}\right)\right]$

$$=\lim_{n\to\infty} n\left[1-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots\cdots\right)\right]$$

$$=1/2<1$$

$\therefore$ By logarithmic test, diverges for $x=\frac{1}{e}$.

$\therefore$ Given series converges for $x<\frac{1}{e}$ and diverges for $x\geq\frac{1}{e}$

Gauss Test -

If $\sum u_n$ be a pos. term series and,

$\frac{u_n}{u_{n+1}}=\alpha+\frac{\beta}{n}+\frac{\gamma_n}{n^p}$, $\alpha>0$, $p>1$ and $\{\gamma_n\}$ is bdd. seq.

Then, (i) for $\alpha\neq1$, $\sum u_n$ converges if $\alpha>1$ and diverges if $\alpha<1$ whatever β may be.

(ii) for $\alpha=1$, $\sum u_n$ converges if $\beta>1$ and diverges if $\beta\leq1$.

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Prob. $\frac{2^2}{3^2} + \frac{2^2}{3^2}\cdot\frac{4^2}{5^2} + \frac{2^2}{3^2}\cdot\frac{4^2}{5^2}\cdot\frac{6^2}{7^2} + \cdots$

Sol. $u_n = \dfrac{2^2 . 4^2 . 6^2 \ldots (2n)^2}{3^2 . 5^2 . 7^2 \ldots (2n+1)^2}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{2^2 . 4^2 \ldots (2n)^2}{3^2 . 5^2 \ldots (2n+1)^2} \cdot \frac{3^2 . 5^2 \ldots (2n+3)^2}{2^2 . 4^2 \ldots (2n+2)^2} = \frac{(2n+3)^2}{(2n+2)^2}$$

$$= \left(1 + \frac{3}{2n}\right)^2 \left(1 + \frac{2}{2n}\right)^{-2}$$

$$= \left(1 + \frac{3}{n} + \frac{9}{4n^2}\right)\left(1 - \frac{2}{n} + \frac{3}{n^2} - \cdots\right)$$

$$= 1 + \frac{1}{n} - \frac{3}{4n^2} + \cdots \quad \left(\text{higher powers in } \frac{1}{n}\right)$$

Here $\alpha = 1$, $\beta = 1$ $\Rightarrow$ By Gauss test, series diverges.

Prob. Hypergeometric series :

$$1 + \frac{\alpha . \beta}{1 . \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)} + \ldots$$

for all pos. integral values of x and $\alpha, \beta, \gamma > 0$.

Sol. $1 + \sum u_n$ is a pos. term series.

$$u_n = \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)\beta(\beta+1)(\beta+2)\ldots(\beta+n-1)}{1.2.3\ldots n . \gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)} x^n, \quad n = 1, 2, \ldots$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \frac{1}{x} = \frac{1}{x}$$

$\Rightarrow$ By ratio test, series converges if $x < 1$, diverges if $x > 1$. Test fails for $x = 1$.

For $x = 1$, $\dfrac{u_n}{u_{n+1}} = \dfrac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} = \dfrac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}$

$$= \left(1 + \frac{\gamma+1}{n} + \frac{\gamma}{n^2}\right)\left(1 + \frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}\right)^{-1}$$

$$= \left(1 + \frac{\gamma+1}{n} + \frac{\gamma}{n^2}\right)\left[1 - \left(\frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}\right) + \left(\frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}\right)^2 - \cdots\right]$$

$$= 1 + \frac{-\alpha-\beta+\gamma+1}{n} + \frac{(\alpha+\beta-1)(\alpha+\beta-\gamma)-\alpha\beta}{n^2} + \cdots$$

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Hence, by Gauss test, the series converges if $1-\alpha-\beta+\gamma>1$ or, $\gamma>\alpha+\beta$ and diverges if $\gamma \leq \alpha+\beta$.

Thus, for pos. α, β, γ, x, the hypergeometric series

(i) converges for $x<1$.

(ii) diverges for $x>1$.

(iii) for $x=1$, converges if $\gamma>\alpha+\beta$ and diverges if $\gamma \leq \alpha+\beta$.

Series with arbitrary terms -

(1) Alternating series : $u_1-u_2+u_3-u_4+\ldots, u_n>0$

Leibnitz test -

Th. If (i) $u_{n+1} \leq u_n \ \forall n$ and, (ii) $\lim_{n\to\infty} u_n = 0$.

then, $\sum u_n$ converges.

Prob. $\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\ldots$ converges for $p>0$.

Sol. Let $u_n=\frac{1}{n^p}$. $u_{n+1} \leq u_n$, $\forall n$, $\lim_{n\to\infty} u_n = 0$.

$\therefore \sum u_n$ converges.

Absolute convergence - $\sum u_n$ is said to be absolutely convergent if $\sum |u_n|$ is convergent.

Conditional convergence - A series which is convergent but not absolutely convergent is conditionally convergent.

Th. Every absolute convergent series is convergent.

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Prob. For any fixed x, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is convergent.

Sol. $u_n = \frac{\sin nx}{n^2}$. $|u_n| \leq \frac{1}{n^2}$ and, $\sum \frac{1}{n^2}$ converges.

$\therefore \sum |u_n|$ converges and, so $\sum u_n$ converges.

Prob. Show that the series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ converges absolutely $\forall x$.

Sol. $u_n = \frac{x^n}{n!}$, $\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0, \forall x \neq 0$.

$\because 0 < 1$, by ratio test, $\sum |u_n|$ converges

$\Rightarrow \sum u_n$ converges.

Note. $\because \sum \frac{x^n}{n!}$ is convergent $\forall x$, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

Prob. Show that $\lim_{n\to\infty} \frac{m(m-1)\dots(m-n+1)}{(n-1)!} x^n = 0$, where $|x|<1$, $m \in \mathbb{R}$.

Sol. Consider the series $\sum u_n$, $u_n = \frac{m(m-1)\dots(m-n+1)}{(n-1)!} x^n$

$\therefore \lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \left|\frac{m-n}{n}\right| |x| = |x| < 1$

$\Rightarrow \sum |u_n|$ is convergent.

$\therefore \sum u_n$ convergent $\Rightarrow \lim_{n\to\infty} u_n = 0$.

Test for arbitrary term series -

Abel's test:

① If b_n is positive, monotonic decreasing function, and, if the series $\sum u_n$ is convergent, then the series $\sum b_n u_n$ is convergent.

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Application. Show that the series $0-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{3}+\frac{2}{3^2}-\frac{1}{4}+\frac{3}{4^2}-\cdots$ converges.

Sol. Given series is $\sum b_n u_n$, where

$\sum u_n$ is the series $1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\cdots$ — (1)

and, $\{b_n\}$ is the sequence $0, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, \cdots$ — (2)

$\because u_{n+1} \leq u_n$ and, $\lim u_n = 0$, by Leibnitz test, series (1) is convergent.

Again, $\{b_n\}$ is pos. m.d.

$\therefore$ By Abel's test, $\sum b_n u_n$ is convergent.

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(2) **Dirichlet's test:**

If b_n is a pos. monotonic decreasing function with limit zero, and if, for the series $\sum u_n$, the sequence $\{S_n\}$ of partial sums is bdd. then the series $\sum b_n u_n$ is convergent.

Dirichlet's Th. (Rearrangement of terms)

A series obtained from an absolutely convergent series by a rearrangement of terms converges absolutely and has the same sum as the original series.

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SEQUENCE AND SERIES OF FUNCTIONS -

<u>Pointwise Convergence</u>: Suppose $\{f_n(x)\}$ be a sequence of functions defined on $I = [a,b]$. Then the sequence $\{f_n\}$ is said to be point-wise convergent to $f(x)$ on $[a,b]$, iff for each $\xi \in I$, the corresponding sequence $\{f_n(\xi)\}$ of numbers converges to $f(\xi)$.

We write, $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in I = [a,b]$.

For pointwise convergence, the properties such as boundedness, continuity, integrability, differentiability, etc. are not always transferred to sum function.

<u>Example</u>: • <u>Boundedness</u> – The G.P. $1 + x + x^2 + x^3 + \dots$ converges to $(1-x)^{-1}$ for $-1 < x < 1$. Each term is bdd., but sum is not bdd. at $x = 1$.

• <u>Continuity</u> – $\sum_{n=0}^{\infty} f_n(x)$, $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}$.

At $x = 0$, $f_n(0) = 0$, $\forall n \Rightarrow f(0) = 0$.

For $x \neq 0$, this is a G.P. of common ratio $1/(1+x^2)$, so that its sum function $f(x) = \dfrac{a}{1-r} = \dfrac{x^2}{1 - \frac{1}{1+x^2}} = 1 + x^2$.

$$\therefore f(x) = \begin{cases} 0, & x = 0 \\ 1+x^2, & x \neq 0 \end{cases} \Rightarrow f(x) \text{ is not continuous but each } f_n(x) \text{ is continuous.}$$

• <u>Differentiability</u> – $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$, $x \in \mathbb{R}$.

$$f(x) = \lim_{n\to\infty} f_n(x) = 0 \Rightarrow f'(x) = 0 \Rightarrow f'(0) = 0.$$

But, $f_n'(x) = \sqrt{n}\cos nx \Rightarrow f_n'(0) = \sqrt{n} \to \infty$ as $n \to \infty$.

Thus, at $x = 0$, $\{f_n'(x)\}$ diverges, but the limit function exists, $f'(0) = 0 \Rightarrow$ limit of differentials is not equal to differential of limit.

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• <u>Integrability</u> - $f_n(x) = nx(1-x^2)^n$, $0 \leq x \leq 1$, $n = 1,2,3...$

At $x = 0$, $f_n(0) = 0$, $\forall n \Rightarrow f(0) = 0$.

for $0 < x \leq 1$, $\lim_{n\to\infty} f_n(x) = 0$, since $0 \leq 1-x^2 < 1$.

$$\therefore \quad f(x) = \lim_{n\to\infty} f_n(x) = 0, \quad 0 \leq x \leq 1.$$

$$\Rightarrow \int_0^t f(x).dx = 0, \quad t > 0.$$

Again, $\int_0^t f_n(x).dx = \int_0^t nx(1-x^2)^n.dx = \frac{n}{2n+2}$

$$\Rightarrow \lim_{n\to\infty} \int_0^t f_n(x).dx = \frac{1}{2}$$

$$\therefore \quad \lim_{n\to\infty} \int_0^t f_n(x).dx \neq \int_0^t \lim_{n\to\infty} f_n(x).dx$$

$\Rightarrow$ limit of integrals is not equal to integral of limit.

<u>Note</u>: We thus introduce the notion of <u>uniform convergence</u>.

<u>Uniform Convergence</u>: A sequence of functions $\{f_n(x)\}$ is said to converge uniformly on $[a,b]$ to a function $f(x)$, if for any $\epsilon > 0$, $\exists$ an integer N, depending on ϵ only (not dependent on x) such that -

$$\forall x \in [a,b], \quad |f_n(x) - f(x)| < \epsilon, \quad \forall n \geq N.$$

<u>Note</u>: Uniform Convergence $\Rightarrow$ Pointwise Convergence but not vice-versa.

<u>Cauchy criterion for uniform convergence</u> -

$\{f_n(x)\}$ converges uniformly on $[a,b]$ iff for each $\epsilon > 0$, and $\forall x \in [a,b]$, $\exists$ an integer N such that

$$|f_{n+p}(x) - f_n(x)| < \epsilon, \quad \forall n \geq N, p \geq 1.$$

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Weierstrass M-test -

A series of functions $\sum f_n(x)$ will converge uniformly (and absolutely) on $[a,b]$ if $\exists$ a convergent series $\sum M_n$ of positive numbers such that

$$|f_n(x)| \leq M_n \quad ; \quad \forall\, x \in [a,b]$$

Prob. Show that the series $\sum \frac{x}{n^p + x^2 n^q}$ converges uniformly over any finite interval $[a,b]$, where $q>0$, $p>1$.

Sol. $|f_n(x)| = \left| \frac{x}{n^p + x^2 n^q} \right| < \frac{\alpha}{n^p} \quad ; \quad \alpha \geq \max\{|a|,|b|\}$

The series $\sum \frac{\alpha}{n^p}$ converges ~~uniformly on $[a,b]$~~ for $p>1$.

Hence, by M-test, $\sum f_n(x)$ converges uniformly on $[a,b]$.

Note: Fundamental properties like continuity, differentiability etc. are usually transferred from series to sum function.

POWER SERIES :

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \equiv \sum_{n=0}^{\infty} a_n x^n$$

is called a power series (in x) and a_n are coefficients.

If a definite positive number R exists such that $\sum a_n x^n$ converges (absolutely) for $|x| < R$ and diverges for $|x| > R$, then R is called <u>radius of convergence</u> and $(-R, R)$ is <u>interval of convergence</u> of power series.

The behaviour of the series at $|x| = R$ depends entirely upon the character of the sequence $\{a_n\}$.

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Th. If $\overline{\lim}\, |a_n|^{1/n} = \frac{1}{R}$, then $\sum a_n x^n$ is convergent for $|x| < R$ and diverges for $|x| > R$.

$\overline{\lim} \to$ lim Sup.

Proof - $\overline{\lim\limits_{n\to\infty}}\, |a_n x^n|^{1/n} = \frac{|x|}{R}$

Hence, by Cauchy's root test, the series $\sum a_n x^n$ is absolutely convergent for $|x| < R$ and divergent for $|x| > R$.

Def^n - $R = \dfrac{1}{\overline{\lim}\, |a_n|^{1/n}}$

$R = 0 \Rightarrow$ nowhere convergent except at $x = 0$.

$R = \infty \Rightarrow$ everywhere convergent

Examples. (1) $1 + 2x + 3x^2 + 4x^3 + \dots \equiv \sum\limits_{n=0}^{\infty} a_n x^n$, $a_n = n+1$.

$A = \overline{\lim}\, |a_n|^{1/n} = \overline{\lim}\, (n+1)^{1/n}$

$\log A = \overline{\lim}\, \frac{1}{n} \log(n+1) = \overline{\lim}\, \frac{1}{n+1} = 0 \Rightarrow A = 1$

$\therefore R = 1$

---

(2) $\sum\limits_{n=1}^{\infty} \frac{x^{n-1}}{n^2} = \sum\limits_{n=0}^{\infty} \frac{x^n}{(n+1)^2} = 1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \dots$

$A = \overline{\lim}\, |a_n|^{1/n} = \overline{\lim} \left[\frac{1}{(n+1)^2}\right]^{1/n}$

$\Rightarrow \log A = -\overline{\lim}\, \frac{2}{n} \log(n+1) = 0 \Rightarrow A = 1$

$\therefore R = 1$

Th. If $\sum a_n x^n$ converges for $x = x_0$, then it is absolutely convergent for every $x = x_1$ when $|x_1| < |x_0|$.

Def^n: Standard result.

$$\underline{\lim} \left|\frac{U_{n+1}}{U_n}\right| \leq \underline{\lim}\, |U_n|^{1/n} \leq \overline{\lim}\, |U_n|^{1/n} \leq \overline{\lim} \left|\frac{U_{n+1}}{U_n}\right|$$

provided first and last limit exists.

Thus, if $\underline{\lim} \left|\frac{U_{n+1}}{U_n}\right| = \overline{\lim} \left|\frac{U_{n+1}}{U_n}\right|$, then, $\underline{\lim}\, |U_n|^{1/n} = \overline{\lim}\, |U_n|^{1/n}$

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i.e, if $\lim \left|\frac{U_{n+1}}{U_n}\right|$ exists, then $\lim |U_n|^{1/n}$ also exists and the two are equal.

$\therefore$ Alternative formula for $R = \lim_{n\to\infty} \left|\frac{U_n}{U_{n+1}}\right|$, provided lim exists.

<u>Examples.</u> ① $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \equiv \sum_{n=1}^{\infty} \frac{x^n}{n!}$ or, $\sum_{n=1}^{\infty} a_n x^n$, $a_n = \frac{1}{n!}$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{1}{n!} \cdot (n+1)!\right| = \lim_{n\to\infty} (n+1) = \infty$$

---

② $1 + x + 2!\,x^2 + 3!\,x^3 + 4!\,x^4 + \dots \equiv \sum_{n=0}^{\infty} a_n x^n$, $a_n = n!$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = 0.$$

---

③ $\frac{1}{2}x + \frac{1}{2}\cdot\frac{3}{5}x^2 + \frac{1.3.5}{2.5.8}\cdot x^3 + \dots$

$$R = \lim_{n\to\infty} \frac{1.3.5\dots(2n-1)\,.\,2.5.8\dots(3n+2)}{2.5.8\dots(3n-1)\,.\,1.3.5\dots(2n+1)} = \lim_{n\to\infty} \frac{3n+2}{2n+1} = \frac{3}{2}$$

---

④ $x + \frac{x^2}{2^2} + \frac{2!}{3^3}x^3 + \frac{3!}{4^4}x^4 + \dots$

$$R = \lim_{n\to\infty} \frac{(n-1)!}{n^n} \cdot \frac{(n+1)^{n+1}}{n!} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n+1}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = 1.e = e$$

---

⑤ $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$.

$$\lim |U_n|^{1/n} = \lim \frac{1}{(n^n)^{1/n}} = \lim \frac{1}{n} = 0$$

$$\Rightarrow R = \infty$$

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SERIES SOLUTION OF 2nd ORDER ODE

Ordinary point and singular point –

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0 \qquad \text{——— (1)}$$

A point $x = x_0$ is ordinary point of (1) if it is not a pole of $p(x)$ and $q(x)$, otherwise it is singular point.

A singular point $x = x_0$ is called regular singular point if it is a pole of $p(x)$ of order $\not> 1$ and is a pole of $q(x)$ of order $\not> 2$, otherwise it is irregular singular point.

We shall look for series solution about an ordinary point and regular singular point.

---

(1) $(1-x^2)y'' - xy' + 4y = 0$; $x = \pm 1$ are regular singular pt.

Solution around $x = 0$:

$$y = \sum_{r=0}^{\infty} a_r x^r \; ; \; a_0 \neq 0$$

$$(1-x^2)\sum a_r \cdot r(r-1)x^{r-2} - x\sum a_r \cdot r x^{r-1} + 4\sum a_r x^r = 0$$

$$\Rightarrow \sum_{r=2}^{\infty} r(r-1)a_r\left(x^{r-2} - x^r\right) - \sum_{r=1}^{\infty} r a_r x^r + 4\sum_{r=0}^{\infty} a_r x^r = 0$$

$[r-2 = r' \; ; \; r-1 = r'']$

$$\Rightarrow \sum_{r'=0}^{\infty}(r'+2)(r'+1)a_{r'+2}\left(x^{r'} - x^{r'+2}\right) - \sum_{r''=0}^{\infty}(r''+1)a_{r''+1}x^{r''+1} + 4\sum_{r=0}^{\infty} a_r x^r = 0$$

Dropping prime,

$$\sum_{r=0}^{\infty}\left[(r+1)(r+2)a_{r+2} + 4a_r\right]x^r - \sum_{r=0}^{\infty}(r+1)a_{r+1}x^{r+1} - \sum_{r=0}^{\infty}(r+1)(r+2)a_{r+2}x^{r+2} = 0$$

$\underline{x^0}$: $1.2.a_2 + 4a_0 = 0 \Rightarrow a_2 = -2a_0$

$\underline{x^1}$: $2.3.a_3 + 4a_1 - 1.a_1 \Rightarrow a_3 = -\frac{1}{2}a_1$

....

$\underline{x^{r+2}}$: $(r+3)(r+4)a_{r+4} + 4a_{r+2} - (r+2)a_{r+2} - (r+1)(r+2)a_{r+2} = 0$

$$\therefore \; a_{r+4} = \frac{(r+2)^2 - 4}{(r+3)(r+4)}a_{r+2} = \frac{r}{r+3}a_{r+2} \; ; \; r = 0, 1, \ldots$$

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$\therefore\ a_4 = 0 \quad ; \quad a_6 = \frac{2}{5} a_4 = 0.$

$a_5 = \frac{1}{4} a_3 = -\frac{1}{8} a_1$

$$\therefore\ y = \sum_{r=0}^{\infty} a_r x^r = a_0\left[1 + \frac{a_2}{a_0}x^2 + \frac{a_4}{a_0}x^4 + \dots\right] + a_1\left[x + \frac{a_3}{a_1}x^3 + \frac{a_5}{a_1}x^5 + \dots\right]$$

$$\Rightarrow y = a_0\left[1 - 2x^2\right] + a_1\left[x - \frac{1}{2}x^3 - \frac{1}{8}x^5 + \dots\right]$$

Thus, two linearly independent solutions around $x=0$ are -

$$y_1(x) = 1 - 2x^2, \quad y_2(x) = x - \frac{x^3}{2} - \frac{x^5}{8} + \dots$$

---

(2) $9x(1-x)y'' - 12y' + 4y = 0$

$x = 0, 1$ are regular singular points.

<u>Solution around $x=0$</u> - (Frobenius method)

Assume solution as $y = \sum_{r=0}^{\infty} a_r x^{p+r}$; $a_0 \neq 0$

$$9x(1-x)\sum (p+r)(p+r-1)a_r x^{p+r-2} - 12\sum a_r(p+r)x^{p+r-1} + 4\sum a_r x^{p+r} = 0$$

$$\Rightarrow 9\sum a_r(p+r)(p+r-1)\left(x^{p+r-1} - x^{p+r}\right) - 12\sum a_r(p+r)x^{p+r-1} + 4\sum a_r x^{p+r} = 0$$

$$\Rightarrow 3\sum a_r(p+r)(3p+3r-7)x^{p+r-1} - \sum a_r\left[9(p+r)(p+r-1) - 4\right]x^{p+r} = 0$$

p is to be determined from <u>indicial equation</u>, which is coefficient of lowest power of x equated to zero.

<u>x^{p-1}</u> : $3a_0 p(3p-7) = 0 \Rightarrow p = 0, 7/3$, unequal and not differing by integer.

<u>$p=0$</u> - $3\sum_{r=0}^{\infty} a_r r(3r-7)x^{r-1} - \sum_{r=0}^{\infty} a_r\left[9r(r-1) - 4\right]x^r = 0$

$$\Rightarrow 3\sum_{r=1}^{\infty} a_r r(3r-7)x^{r-1} - \sum_{r=0}^{\infty} a_r\left[9r(r-1) - 4\right]x^r = 0$$

$[r - 1 = r']$

$$\Rightarrow 3\sum_{r'=0}^{\infty} a_{r'+1}(r'+1)(3r'-4)x^{r'} - \sum_{r=0}^{\infty} a_r\left[9r(r-1) - 4\right]x^r = 0$$

Dropping prime,

$$3\sum_{r=0}^{\infty} a_{r+1}(r+1)(3r-4)x^r - \sum_{r=0}^{\infty} a_r\left[9r(r-1) - 4\right]x^r = 0$$

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$\underline{x^r}$: $3a_{r+1}(r+1)(3r-4) = a_r[9r(r-1)-4]$

$$\Rightarrow a_{r+1} = \frac{1}{3}\cdot\frac{9r(r-1)-4}{(r+1)(3r-4)}a_r \quad ; \quad r = 0, 1 \ldots$$

$\therefore a_1 = \frac{1}{3}a_0$

$a_2 = \frac{2}{3}\cdot\frac{1}{3}a_0$

$\therefore$ One solution is - $y_1(x) = 1 + \frac{1}{3}x + \frac{2}{9}x^2 + \ldots$

$\underline{p = 7/3}$ - $3\sum a_r\left(\frac{7}{3}+r\right)3r\, x^{7/3+r-1} - \sum a_r\left[9\left(\frac{7}{3}+r\right)\left(\frac{7}{3}+r-1\right)-4\right]x^{7/3+r} = 0$

$$\Rightarrow 3\sum_{r=1}^{\infty} a_r r(7+3r)x^{r+4/3} - \sum a_r[(7+3r)(4+3r)-4]x^{r+\frac{4}{3}+1} = 0$$

$\underline{x^{r+4/3}}$: $3a_r r(7+3r) = a_{r-1}[\{7+3(r-1)\}\{4+3(r-1)\}-4]$

$= a_{r-1}[(4+3r)(1+3r)-4]$

$= a_{r-1}[9r^2+15r]$

$= 3r\,a_{r-1}(3r+5)$

$$\Rightarrow a_r = \frac{3r+5}{7+3r}a_{r-1}$$

$\therefore a_1 = \frac{8}{10}a_0 \quad ; \quad a_2 = \frac{8}{10}\cdot\frac{11}{13}a_0 \ldots$

$$\therefore y_2 = x^{7/3}\left[1 + \frac{8}{10}x + \frac{8}{10}\cdot\frac{11}{13}x^2 + \ldots\right]$$

Prob. ① $2x^2y'' - xy' + (1-x^2)y = 0$

Sol) $y = \sum_{r=0}^{\infty} a_r x^{r+p} \quad ; \quad a_0 \neq 0$.

$\therefore 2x^2\sum a_r(r+p)(r+p-1)x^{r+p-2} - x\sum a_r(r+p)x^{r+p-1} + (1-x^2)\sum a_r x^{r+p} = 0$

$\Rightarrow 2\sum a_r(r+p)(r+p-1)x^{r+p} - \sum a_r(r+p)x^{r+p} + \sum a_r(x^{r+p} - x^{r+p+2}) = 0$

$\underline{x^p}$: $2a_0p(p-1) - a_0p + a_0 = 0 \Rightarrow 2p^2 - 3p + 1 = 0 \Rightarrow p = 1, 1/2$

$\Rightarrow$ unequal, not differing by integer.

$\underline{x^{r+p}}$: $[2[(r+p)(r+p-1) - (r+p) + 1]a_r = a_{r-2}$

$\Rightarrow 2(r+p-1)(r+p-\frac{1}{2})a_r = a_{r-2} \Rightarrow a_{r+2} = \frac{1}{(p+r+1)(2p+2r+3)}a_r,$

$r = 0, 1, 2 \ldots$

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$\underline{x^{p+1}}$: $\left[2(p+1)p - (p+1) + 1\right] a_1 = 0$

$\Rightarrow 2p(p+\frac{1}{2}) a_1 = 0 \Rightarrow a_1 = 0$.

$\therefore a_1 = a_3 = a_5 = \dots = 0$.

$\underline{p=1}$: $a_{r+2} = \dfrac{a_r}{(r+2)(2r+5)}$

$\therefore a_2 = \dfrac{a_0}{2.5}$; $a_4 = \dfrac{a_2}{4.9} = \dfrac{a_0}{2.4.5.9}$;

$a_6 = \dfrac{a_0}{2.4.5.6.9.13}$; ...

$\therefore y_1(x) = x\left[1 + \dfrac{x^2}{2.5} + \dfrac{x^4}{2.4.5.9} + \dots\right]$

$\underline{p=\frac{1}{2}}$: $a_{r+2} = \dfrac{1}{(r+\frac{3}{2})(2r+4)} a_r$

$\therefore a_2 = \dfrac{a_0}{3.2}$; $a_4 = \dfrac{a_0}{3.7.2.4}$; ...

$\therefore y_2(x) = \sqrt{x}\left[1 + \dfrac{x^2}{2.3} + \dfrac{x^4}{2.3.4.7} + \dots\right]$

---

(2) $2x^2y'' - xy' + (1-x^2)y = x^2$.

G.S. is y = C.F. + P.I.

C.F. is G.S. of $2x^2y'' - xy' + (1-x^2)y = 0$.

Now, P.I. is a particular solution of given equation.

Let it be $y = \sum_{r=0}^{\infty} b_r x^{k+r}$; $b_0 \neq 0$.

$\sum b_r (2k+2r-1)(k+r-1) x^{k+r} - \sum b_r x^{k+r+2} = x^2$.

$\therefore k=2$, $b_0 \cdot 3.1 = 1$, $b_1 . 5.2 = 0 \Rightarrow b_0 = \frac{1}{3}$, $b_1 = 0$

$b_{r+2} = \dfrac{b_r}{(2k+2r+3)(k+r+1)}$; $r = 0, 1, 2 \dots$

$= \dfrac{b_r}{(2r+7)(r+3)}$

$\therefore b_2 = \dfrac{b_0}{7.3} = \dfrac{1}{7.3.3.1}$; $b_4 = \dfrac{b_2}{11.5} = \dfrac{1}{11.5.7.3.3.1}$ ...

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$\therefore\ b_1 = b_3 = b_5 = \dots = 0.$

$$\therefore\ P.I. = \frac{x^2}{1.3} + \frac{x^4}{1.3.3.7} + \frac{x^6}{1.3.3.7.5.11} + \dots$$

ASSIGNMENT : 12/08/2015

Find polynomial solution, if any, of following ODE :

① Legendre equation : $(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$

$\ell = 0, 1 \dots \quad ; \quad -1 \leq x \leq 1.$

② Associated Legendre equation : $(1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]y = 0$

$\ell = 0, 1, \dots \quad ; \quad -\ell \leq m \leq \ell$ are integers, $-1 \leq x \leq 1$

③ Shifted Legendre equation : $x(1-x)y'' - (2x-1)y' + \ell(\ell+1)y = 0,$

$x \in [0, 1] \quad , \quad \ell = 0, 1, 2, \dots$

④ Laguerre equation : $xy'' + (1-x)y' + \alpha y = 0.$

$x \in [0, \infty) \quad , \quad \alpha = 0, 1, 2, \dots$

⑤ Associated Laguerre equation : $xy'' + (\beta+1-x)y' + \alpha y = 0$

$x \in [0, \infty) \quad , \quad \alpha, \beta = 0, 1, 2 \dots$

⑥ Hermite equation : $y'' - 2xy' + 2ky = 0.$

$k = 0, 1, 2, \dots \quad ; \quad -\infty < x < \infty.$

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LEGENDRE EQUATION :

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad ; \quad n \in \mathbb{R}.$$

'n' is called degree of Legendre equation.

$x = \pm 1$ are poles of order 1 of both of the coefficients of y' and y and so these are regular singular points. To show $x=\infty$ is a regular singular point we put $x = \frac{1}{\xi}$, which transforms $x=\infty$ to $\xi = 0$.

$$\frac{dy}{dx} = -\xi^2 \frac{dy}{d\xi} \quad ; \quad \frac{d^2y}{dx^2} = \xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi}.$$

$$\Rightarrow \left(1 - \frac{1}{\xi^2}\right)\xi^3\left(\xi \frac{d^2y}{d\xi^2} + 2\frac{dy}{d\xi}\right) + 2 \cdot \frac{\xi^2}{\xi}\frac{dy}{d\xi} + n(n+1)y = 0$$

$$\Rightarrow \xi^2(\xi^2-1)\frac{d^2y}{d\xi^2} + 2\xi^3\frac{dy}{d\xi} + n(n+1)y = 0.$$

$\therefore$ $\xi = 0$ is not a singularity of coefficients of $\frac{dy}{d\xi}$ and is a pole of order 2 of coefficient of y.

Thus, $\xi = 0$ and hence $x = \infty$ is regular singular point.

<u>We shall now find solution around $x=0$</u> -

$x=0$ is ordinary point. $\therefore$ $y = \sum_{r=0}^{\infty} a_r x^r$; $a_0 \neq 0$.

$$(1-x^2)\sum_{r=0}^{\infty} a_r r(r-1)x^{r-2} - 2x\sum_{r=0}^{\infty} a_r . r . x^{r-1} + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0.$$

$$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)(x^{r-2} - x^r) - 2\sum_{r=1}^{\infty} a_r . r . x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0.$$

$[r' = r-2 \quad ; \quad r'' = r-1]$

$$\Rightarrow \sum_{r'=0}^{\infty} a_{r'+2}(r'+2)(r'+1)(x^{r'} - x^{r'+2}) - 2\sum_{r''=0}^{\infty} a_{r''+1}(r''+1)x^{r''+1} + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_{r+2}(r+1)(r+2)(x^r - x^{r+2}) - 2\sum_{r=0}^{\infty} a_{r+1}(r+1)x^{r+1} + n(n+1)\sum_{r=0}^{\infty} a_r x^r$$

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Equating coefficients of x^r,

$$a_{r+2}(r+2)(r+1) + n(n+1)a_r - a_r(r-1)r - 2a_r r = 0.$$

$$\Rightarrow (r+1)(r+2)a_{r+2} = [r^2+r-n(n+1)]a_r$$
$$= (r+n+1)(r-n)a_r$$

$$\Rightarrow a_{r+2} = \frac{(r-n)(r+n+1)}{(r+1)(r+2)}a_r \quad ; \quad r = 0, 1, 2 \ldots$$

$$\therefore \frac{a_2}{a_0} = -\frac{n(n+1)}{1.2} \quad ; \quad \frac{a_4}{a_2} = -\frac{(n-2)(n+3)}{3.4} \quad ; \quad \frac{a_6}{a_4} = -\frac{(n-4)(n+5)}{5.6} \ldots$$

and, $$\frac{a_3}{a_1} = -\frac{(n-1)(n+2)}{2.3} \quad ; \quad \frac{a_5}{a_3} = -\frac{(n-3)(n+4)}{4.5} \quad ; \quad \frac{a_7}{a_5} = -\frac{(n-5)(n+6)}{6.7} ; \ldots$$

$$\therefore y = a_0\left[1 + \frac{a_2}{a_0}x^2 + \frac{a_4}{a_0}x^4 + \frac{a_6}{a_0}x^6 + \ldots\right] + a_1\left[x + \frac{a_3}{a_1}x^3 + \frac{a_5}{a_1}x^5 + \frac{a_7}{a_1}x^7 + \ldots\right]$$
$$= a_0 y_1(x) + a_1 y_2(x)$$

$$y_1(x) = 1 - \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n-2)(n+3)}{1.2.3.4}x^4 - \frac{n(n+1)(n-2)(n+3)(n-4)(n+5)}{1.2.3.4.5.6}x^6 + \ldots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{2.3}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{2.3.4.5}x^5 - \frac{(n-1)(n+2)(n-3)(n+4)(n-5)(n+6)}{2.3.4.5.6.7}x^7 + \ldots$$

$y_1(x)$ contains only even powers of x, while $y_2(x)$ contains only odd powers of x.

When 'n' is even positive integer, then $y_1(x)$ becomes a polynomial of degree n. When n is odd positive integer, then y_2 becomes a polynomial of degree n. Other remains infinite series.

This polynomial solution is called Legendre polynomial $P_n(x)$.

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(8) If $P_n(x)$ denotes Legendre polynomial of degree n, then $P_n(x) = \sum_{r=0}^{p} \frac{(-1)^r (2n-2r)!}{2^n \cdot r!\,(n-r)!\,(n-2r)!} x^{n-2r}$

where the integer p is $\frac{1}{2}n$ or, $\frac{1}{2}(n-1)$ according as n is even or odd.

Sol.) $a_{r+2} = \frac{(r-n)(r+n+1)}{(r+1)(r+2)} a_r$

$$y(x) = a_n \left[x^n + \frac{a_{n-2}}{a_n} x^{n-2} + \frac{a_{n-2}}{a_n} \cdot \frac{a_{n-4}}{a_{n-2}} x^{n-4} + \dots \right]$$

$$\frac{a_{n-2}}{a_n} = -\frac{(n-1)n}{2(2n-1)} \quad ; \quad \frac{a_{n-4}}{a_{n-2}} = \frac{-(n-3)(n-2)}{4(2n-3)}, \dots$$

$$\therefore y(x) = a_n \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} - \dots \right]$$

Choose $a_n = \frac{(2n)!}{2^n (n!)^2}$.

$$\therefore \frac{(2n)!}{2^n (n!)^2} \cdot \frac{n(n-1)}{2(2n-1)} = \frac{\cancel{2}n\cancel{(2n-1)}(2n-2)!\ \cancel{n}\cancel{(n-1)}}{2^n \cdot \cancel{2}n\cancel{(n-1)}(n-2)!\ \cancel{n}(n-1)!\ \cancel{(2n-1)}}$$

$$\frac{(2n)!}{2^n (n!)^2} \cdot \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} = \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)!\ n(n-1)(n-2)(n-3)}{2^n \cdot 2 \cdot 4 \cdot n(n-1)(n-2)(n-3)(n-4)!\ (2n-1)(2n-3)\ n(n-1)(n-2)!}$$

$$\therefore y(x) = \sum_{r=0}^{p} \frac{(-1)^r (2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

To determine p, for even n last term should be 1.

$$n - 2p = 0 \Rightarrow p = n/2.$$

For odd n, last term must be x,

$$n - 2p = 1 \Rightarrow p = \frac{1}{2}(n-1)$$

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Rodrigue's Formula -

$$P_n(x) = \frac{1}{2^n . n!} \cdot \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

Sol.) $$P_n(x) = \sum_{r=0}^{p} \frac{(-1)^r (2n-2r)!}{2^n . r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

where, $p = \frac{1}{2}n$ or, $\frac{1}{2}(n-1)$ according as n is even or odd.

$$\frac{d^n}{dx^n}\left(x^{2n-2r}\right) = (2n-2r)(2n-2r-1)\ldots\{(2n-2r)-(n-1)\}x^{n-2r}$$

$$= (2n-2r)(2n-2r-1)\ldots(n-2r+1)x^{n-2r}$$

$$= \frac{(2n-2r)!}{(n-2r)!} x^{n-2r}$$

$$\therefore P_n(x) = \sum_{r=0}^{p} \frac{(-1)^r}{2^n . r!(n-r)!} \frac{d^n}{dx^n}\left(x^{2n-2r}\right)$$

$$= \frac{d^n}{dx^n}\left[\sum_{r=0}^{p} \frac{(-1)^r}{2^n\, r!(n-r)!} x^{2n-2r}\right]$$

We can extend the summation over r beyond p up to n, since power of x in these extra terms is less than n and so their nth derivatives vanish.

$$\therefore P_n(x) = \sum_{r=0}^{n} \frac{(-1)^r}{2^n\, r!(n-r)!} (x^2)^{n-r}$$

$$= \frac{1}{2^n . n!} \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r (x^2)^{n-r}$$

$$= \frac{1}{2^n . n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

Orthogonal Property of Legendre Polynomials -

$$\int_{-1}^{1} P_m(x) . P_n(x)\, dx = \frac{2}{2n+1} \delta_{mn}$$

Sol.) $$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_m(x)\right] + m(m+1)P_m(x) = 0 \quad \text{——} \quad (1)$$

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0 \quad \text{——} \quad (2)$$

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② × P_m − ① × P_n ,

$$P_m \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n\right] - P_n \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_m\right] + P_m P_n\left[n(n+1)-m(m+1)\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)\left\{P_m \frac{dP_n}{dx} - P_n \frac{dP_m}{dx}\right\}\right] + P_m P_n\left[n(n+1)-m(m+1)\right] = 0$$

$\therefore$ Integrate w.r.t. x in $[-1,1]$,

$$\left[n(n+1)-m(m+1)\right]\int_{-1}^{1} P_m P_n \, dx = -\left[(1-x^2)\left\{P_m \frac{dP_n}{dx} - P_n \frac{dP_m}{dx}\right\}\right]_{-1}^{1} = 0$$

$\Rightarrow$ for $m \neq n$, $\int_{-1}^{1} P_m P_n \, dx = 0$.

Next, to prove, $\int_{-1}^{1} P_n^2 \, dx = \frac{2}{2n+1}$.

$u_n = (x^2-1)^n$; $u_n^{(m)} = \frac{d^m}{dx^m}(x^2-1)^n$; $P_n(x) = \frac{1}{2^n n!} u_n^{(n)}$.

$$\int_{-1}^{1} P_n^2 \, dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} u_n^{(n)} \cdot u_n^{(n)} \, dx$$

$$= \frac{1}{2^{2n}(n!)^2}\left[u_n^{(n)} \cdot u_n^{(n-1)}\Big|_{-1}^{1} - \int_{-1}^{1} u_n^{(n+1)} u_n^{(n-1)} \, dx\right]$$

$$\Rightarrow \int_{-1}^{1} P_n^2 \, dx = -\frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} u_n^{(n+1)} u_n^{(n-1)} \, dx \quad \text{since } u_n^{(n-1)} = 0 \text{ at } x = \pm 1.$$

To show, general term of $u_n^{(p)} = \frac{d^p}{dx^p}(x^2-1)^n$; $0 < p < n$ vanishes at $x = \pm 1$.

$$u_n^{(p)} = \frac{d^p}{dx^p}\left[(x-1)^n (x+1)^n\right]$$

$$= \frac{d^p}{dx^p}\left\{(x-1)^n\right\}(x+1)^n + {}^pC_1 \frac{d^{p-1}}{dx^{p-1}}\left\{(x-1)^n\right\}\frac{d}{dx}\left\{(x+1)^n\right\} + \dots + (x-1)^n \frac{d^p}{dx^p}\left\{(x+1)^n\right\}$$

General term of $u_n^{(p)}$,

$$= {}^pC_r \frac{d^{p-r}}{dx^{p-r}}\left\{(x-1)^n\right\}\frac{d^r}{dx^r}\left\{(x+1)^n\right\} = \text{const.} \times (x-1)^{n-p+r}(x+1)^{n-r} ; \quad 0 < r < p < n$$

$\because$ $n-p > 0$, $n-p+r > 0$, $n-r > 0$,

general term vanishes at $x = \pm 1$.

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Now, we have,

$$\int_{-1}^{1} u_n^{(n)} u_n^{(n)} . dx = -\int_{-1}^{1} u_n^{(n+1)} u_n^{(n-1)} . dx$$

Similarly, we get successively,

$$\int_{-1}^{1} u_n^{(n+1)} u_n^{(n-1)} . dx = -\int_{-1}^{1} u_n^{(n+2)} u_n^{(n-2)} . dx$$

.....

$$\int_{-1}^{1} u_n^{(n+n-1)} u_n^{(n-n+1)} . dx = \int_{-1}^{1} u_n^{(n+n)} . u_n . dx$$

Multiplying,

$$\int_{-1}^{1} u_n^{(n)} . u_n^{(n)} . dx = (-1)^n \int_{-1}^{1} u_n^{(2n)} u_n . dx$$

$$= (-1)^n \int_{-1}^{1} \frac{d^{2n}}{dx^{2n}} \{(x^2-1)^n\} (x^2-1)^n . dx$$

$$= (-1)^n \int_{-1}^{1} \frac{d^{2n}}{dx^{2n}} \left[x^{2n} - {}^nC_1 . x^{2n-2} + \dots \right] (x^2-1)^n . dx$$

$$= (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n . dx$$

$$= (2n)! \int_{-1}^{1} (1-x^2)^n . dx$$

$$= 2(2n)! \int_{0}^{1} (1-x^2)^n . dx$$

$$\Rightarrow \int_{-1}^{1} u_n^{(n)} u_n^{(n)} . dx = 2(2n)! \int_{0}^{\pi/2} \cos^{2n+1}\theta . d\theta \qquad [x = \sin\theta]$$

$$= 2(2n)! \frac{2n}{2n+1} \cdots\cdots \frac{4}{5} . \frac{2}{3}$$

$$= \frac{2(2n)! (2.4. \dots 2n)^2}{1.2.3.4.5 \dots (2n)(2n+1)}$$

$$= \frac{2 . 2^{2n} (n!)^2}{(2n+1)}$$

$$I_n = \int_0^{\pi/2} \cos^n\theta . d\theta = \int_0^{\pi/2} \sin^n\theta . d\theta$$

$$\Rightarrow I_n = \begin{cases} \frac{n-1}{n} . \frac{n-3}{n-2} \cdots \frac{5}{6} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}, & 'n' \text{ even} \\ \frac{n-1}{n} . \frac{n-3}{n-2} \cdots \frac{4}{5} . \frac{2}{3} . 1, & 'n' \text{ odd} \end{cases}$$

$$\therefore \int_{-1}^{1} P_n^2 . dx = \frac{1}{2^n (n!)^2} . \frac{2 . 2^{2n} (n!)^2}{(2n+1)} = \frac{2}{2n+1}$$

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Recurrence relations -

① $P'_{n+1}(x) - x P'_n(x) = (n+1) P_n(x)$

Proof: $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}\left[(x^2-1)^n\right]$

LHS - $P'_{n+1}(x) - x P_n'(x)$

$$= \frac{1}{2^{n+1}(n+1)!} \frac{d^{n+2}}{dx^{n+2}}\left[(x^2-1)^{n+1}\right] - \frac{x}{2^n n!} \frac{d^{n+1}}{dx^{n+1}}\left[(x^2-1)^n\right]$$

$$= \frac{1}{2^{n+1}(n+1)!}\left[\frac{d^{n+1}}{dx^{n+1}}\left\{\frac{d}{dx}(x^2-1)^{n+1}\right\} - 2(n+1)x \frac{d^{n+1}}{dx^{n+1}}\left\{(x^2-1)^n\right\}\right]$$

$$= \frac{1}{2^{n+1}(n+1)!}\left[\frac{d^{n+1}}{dx^{n+1}}\left\{(n+1)(x^2-1)^n \cdot 2x\right\} - 2(n+1)x \frac{d^{n+1}}{dx^{n+1}}\left\{(x^2-1)^n\right\}\right]$$

$$= \frac{1}{2^{n+1}(n+1)!}\left[\cancel{2(n+1)x \frac{d^{n+1}}{dx^{n+1}}\left\{(x^2-1)^n\right\}} + {}^nC_1 \cdot 2(n+1)\frac{d^n}{dx^n}\left\{(x^2-1)^n\right\} - \cancel{2(n+1)x \frac{d^{n+1}}{dx^{n+1}}\left\{(x^2-1)^n\right\}}\right]$$

$$= \frac{n+1}{2^n \cdot n!} \frac{d^n}{dx^n}(x^2-1)^n = (n+1) P_n(x).$$

---

② $(n+1) P_{n+1}(x) - (2n+1)x P_n(x) + n P_{n-1}(x) = 0$

Proof: LHS - $(n+1)P_{n+1}(x) - (2n+1)x P_n(x)$

$$= \frac{n+1}{2^{n+1}(n+1)!} \frac{d^{n+1}}{dx^{n+1}}\left[(x^2-1)^{n+1}\right] - \frac{(2n+1)x}{2^n n!} \frac{d^n}{dx^n}\left\{(x^2-1)^n\right\} + n P_{n-1}(x)$$

$$= \frac{1}{2^{n+1}(n)!}\left[\frac{d^n}{dx^n}\left\{2(n+1)x(x^2-1)^n\right\} - 2x(2n+1)\frac{d^n}{dx^n}\left\{(x^2-1)^n\right\}\right] + n P_{n-1}(x)$$

$$= \frac{1}{2^{n+1} n!}\left[2(n+1)x \frac{d^n}{dx^n}\left\{(x^2-1)^n\right\} + 2(n+1)n \frac{d^{n-1}}{dx^{n-1}}\left\{(x^2-1)^n\right\} - 2x(2n+1)\frac{d^n}{dx^n}\left\{(x^2-1)^n\right\}\right] + n P_{n-1}(x)$$

$$= \frac{1}{2^n \cdot n!}\left[n(n+1)\frac{d^{n-1}}{dx^{n-1}}\left\{(x^2-1)^n\right\} - nx \frac{d^n}{dx^n}(x^2-1)^n\right] + n P_{n-1}(x)$$

$$= \frac{n}{2^n \cdot n!}\left[(n+1)\frac{d^{n-1}}{dx^{n-1}}\left\{(x^2-1)^n\right\} - \frac{d^n}{dx^n}\left\{x(x^2-1)^n\right\} + n\frac{d^{n-1}}{dx^{n-1}}\left\{(x^2-1)^n\right\}\right]$$

$$+ \frac{n}{2^{n-1}(n-1)!} \frac{d^{n-1}}{dx^{n-1}}\left\{(x^2-1)^n\right\}$$

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$$= \frac{1}{2^n (n-1)!}\left[\frac{d^{n-1}}{dx^{n-1}} \left[(n+1)(x^2-1)^n - \frac{d}{dx}\{x(x^2-1)^n\} + n(x^2-1)^n + 2n(x^2-1)^{n-1} \right]\right]$$

$$= \frac{1}{2^n (n-1)!} \frac{d^{n-1}}{dx^{n-1}} \left[(x^2-1)^{n-1} \{ (n+1)(x^2-1) - (x^2-1) - 2nx^2 + n(x^2-1) + 2n \} \right]$$

$$= 0.$$

---

③ $x P_n'(x) - P_{n-1}'(x) = n P_n(x)$

<u>Proof</u>: $(n+1) P_{n+1}(x) - (2n+1) x P_n(x) + n P_{n-1}(x) = 0$ ——— ①

$P_{n+1}'(x) - x P_n'(x) = (n+1) P_n(x)$ ——— ②

Differentiate ① w.r.t. x,

$(n+1) P_{n+1}'(x) - (2n+1) x P_n'(x) - (2n+1) P_n(x) + n P_{n-1}'(x) = 0$ ——— ③

Eliminate $P_{n+1}'(x)$ from ② and ③,

$\{-(n+1) + (2n+1)\} x P_n'(x) - (n+1)^2 P_n(x) + (2n+1) P_n(x) - n P_{n-1}'(x) = 0$

$\Rightarrow \quad n x P_n'(x) - n^2 P_n(x) - n P_{n-1}'(x) = 0$

$\Rightarrow \quad x P_n'(x) - P_{n-1}'(x) = n P_n(x).$

---

④ $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1) P_n(x)$

(1) $\longrightarrow$ $P_{n+1}'(x) - x P_n'(x) = (n+1) P_n(x)$

(3) $\longrightarrow$ $x P_n'(x) - P_{n-1}'(x) = n P_n(x)$

<u>(1) + (3)</u> $\longrightarrow$ $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1) P_n(x).$

---

⑤ $(x^2-1) P_n'(x) = n x P_n(x) - n P_{n-1}(x)$

Replace n by $n-1$ in (1),

$P_n'(x) - x P_{n-1}'(x) = n P_{n-1}(x)$ ——— (*)

Eliminate $P_{n-1}'(x)$ from (3) and (*),

$-P_n'(x) + x^2 P_n'(x) = n x P_n(x) - n P_{n-1}(x)$

$\Rightarrow (x^2-1) P_n'(x) = n x P_n(x) - n P_{n-1}(x)$

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Solution of Legendre Equation of degree n in the nbd. of $x=$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$x = 1/\xi$, $\quad \xi^2(\xi^2-1)\dfrac{d^2y}{d\xi^2} + 2\xi^2\dfrac{dy}{d\xi} + n(n+1)y = 0$

Solution around $\xi = 0$ – $\quad y = \sum_{r=0}^{\infty} a_r \xi^{p+r}$; $a_0 \neq 0$.

$$\xi^2(\xi^2-1)\sum a_r(p+r)(p+r-1)\xi^{p+r-2} + 2\xi^3\sum a_r(p+r)\xi^{p+r-1} + n(n+1)\sum a_r\xi^{p+r} = 0$$

$$\Rightarrow \sum a_r(p+r)(p+r-1)\left[\xi^{p+r+2} - \xi^{p+r}\right] + 2\sum a_r(p+r)\xi^{p+r+2} + n(n+1)\sum a_r\xi^{p+r} = 0$$

Indicial Eqn. – $\quad a_0 p(p-1) + n(n+1)a_0 = 0$

$$\Rightarrow \left(p-\tfrac{1}{2}\right)^2 = \left(n+\tfrac{1}{2}\right)^2$$

$$\Rightarrow p = n+1, -n, \quad \text{differ by } (2n+1)$$

ξ^{p+1} : $\quad -a_1(p+1)p + n(n+1)a_1 = 0$

$$\Rightarrow a_1\left[\left(p+\tfrac{1}{2}\right) - \left(n+\tfrac{1}{2}\right)\right] = 0 \quad \Rightarrow \underline{a_1 = 0}.$$

ξ^{p+r} : $\quad a_{r-2}(p+r-2)(p+r-3) + 2a_{r-2}(p+r-2) - a_r(p+r)(p+r-1) + n(n+1)a_r = 0$

$$\Rightarrow a_{r-2}(p+r-2)(p+r-1) = a_r\left[\left(p+r-\tfrac{1}{2}\right)^2 - \left(n+\tfrac{1}{2}\right)^2\right]$$

$$\Rightarrow a_r = \frac{(p+r-2)(p+r-1)}{(p+r+n)(p+r-n-1)}a_{r-2} \quad ; \quad r = 2, 3, 4 \ldots$$

$$\therefore \quad a_1 = a_3 = a_5 = \ldots = 0$$

$\underline{p = -n}$: $\quad a_r = \dfrac{(r-n-1)(r-n-2)}{r(r-2n-1)}a_{r-2}$

$\underline{p = n+1}$: $\quad a_r = \dfrac{(r+n-1)(r+n)}{(r+2n+1)r}a_{r-2}$

$$\frac{a_2}{a_0} = -\frac{n(n-1)}{2(2n-1)} \quad ; \quad \frac{a_4}{a_2} = -\frac{(n-2)(n-3)}{4(2n-3)}, \ldots \quad \text{for } \underline{p = -n}$$

$$\frac{a_2}{a_0} = \frac{(n+1)(n+2)}{2(2n+3)} \quad ; \quad \frac{a_4}{a_2} = \frac{(n+3)(n+4)}{4(2n+5)}, \ldots \quad \text{for } \underline{p = n+1}$$

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One soln. around $\xi = 0$ (i.e., near $x = \infty$) -

$$y_1(x) = x^n \left[1 - \frac{n(n-1)}{2(2n-1)} x^{-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{-4} - \dots \right]$$

For $n = 0, 1, 2, \dots$, y_1 becomes a polynomial of degree n, which is Legendre polynomial.

Roots of indicial eqn. equal :

$$xy'' + y' + mxy = 0$$

$x = 0$ is a regular singular point.

$$y = \sum_{r=0}^{\infty} a_r x^{p+r} \quad ; \quad a_0 \neq 0.$$

$$x \sum_{r=0}^{\infty} a_r (p+r)(p+r-1) x^{p+r-2} + \sum_{r=0}^{\infty} a_r (p+r) x^{p+r-1} + mx \sum_{r=0}^{\infty} a_r x^{p+r} = 0$$

$$\Rightarrow \sum a_r (p+r)(p+r-1) x^{p+r-1} + \sum a_r (p+r) x^{p+r-1} + m \sum_{r=0}^{\infty} a_r x^{p+r+1} = 0$$

Indicial Eqn. - x^{p-1}: $a_0 p + a_0 p(p-1) = 0$

$$\Rightarrow p^2 = 0$$

$$\Rightarrow p = 0, 0 \Rightarrow \text{roots equal.}$$

x^p: $a_1 (p+1) p + a_1 (p+1) = 0 \Rightarrow a_1 (p+1)^2 = 0 \Rightarrow \underline{\underline{a_1 = 0}}$.

x^{p+r-1}: $a_r (p+r)(p+r-1) + a_r (p+r) + m a_{r-2} = 0$

$$\Rightarrow a_r = -\frac{m}{(p+r)^2} a_{r-2} \quad ; \quad r = 2, 3, 4, \dots$$

$$\therefore a_1 = a_3 = a_5 = \dots = 0.$$

$$a_2 = -\frac{m}{(p+2)^2} a_0 \quad ; \quad a_4 = (-1)^2 \frac{m^2}{(p+2)^2 (p+4)^2} a_0 \quad ; \quad a_6 = (-1)^3 \frac{m^3 \, a_0}{(p+2)^2 (p+4)^2 (p+6)^2} \dots$$

$$\therefore y(x) = a_0 x^p \left[1 + (-1) \frac{m}{(p+2)^2} x^2 + (-1)^2 \frac{m^2}{(p+2)^2 (p+4)^2} x^4 + (-1)^3 \frac{m^3}{(p+2)^2 (p+4)^2 (p+6)^2} x^6 + \dots \right]$$

Putting $p = 0$,

$$y_1(x) = 1 + (-1)^1 \frac{m}{2^2} x^2 + (-1)^2 \frac{m^2}{2^2 . 4^2} . x^4 + (-1)^3 \frac{m^3}{2^2 . 4^2 . 6^2} x^6 + \dots$$

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To get another solution -

$$\frac{\partial y}{\partial p} = a_0 x^p \ln x \left[1 + (-1)^1 \frac{m}{(p+2)^2} x^2 + (-1)^2 \frac{m^2}{(p+2)^2(p+4)^2} x^4 + (-1)^3 \frac{m^3}{(p+2)^2(p+4)^2(p+6)^2} \right.$$

$$+ a_0 x^p \left[(-1)^1 \frac{m}{(p+2)^2} \left\{ \frac{-2}{p+2} \right\} x^2 + (-1)^2 \frac{m^2 x^4}{(p+2)^2(p+4)^2} \left\{ \frac{-2}{p+2} - \frac{2}{p+4} \right\} + \right.$$

$$\left[y = \frac{1}{(p+2)^2} \Rightarrow \ln y = -2\ln(p+2) \Rightarrow \frac{1}{y}\frac{\partial y}{\partial p} = \frac{-2}{p+2} \Rightarrow \frac{\partial y}{\partial p} = \frac{-2}{(p+2)^2} \right.$$

Putting $p=0$, another soln. is -

$$y_2(x) = \left(\frac{\partial y}{\partial p}\right)_{p=0} = y_1 \ln x - \sum_{n=1}^{\infty} (-1)^n \frac{m^n x^{2n}}{(n!)^2 \cdot 2^{2n}} \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right)$$

Prob. $2x^2 y'' + xy' - (1+x^2)y = 0$

Sol.) $x=0$ is a regular singular point.

Soln. around $x=0$ - $y = \sum_{r=0}^{\infty} a_r x^{p+r}$; $a_0 \neq 0$.

$$2x^2 \sum_{r=0}^{\infty} a_r (p+r)(p+r-1) x^{p+r-2} + x \sum_{r=0}^{\infty} a_r (p+r) x^{p+r-1} - (1+x^2) \sum_{r=0}^{\infty} a_r x^{p+r} = 0$$

$$\Rightarrow 2\sum_{r=0}^{\infty} a_r (p+r)(p+r-1) x^{p+r} + \sum_{r=0}^{\infty} a_r (p+r) x^{p+r} - \sum_{r=0}^{\infty} a_r x^{p+r} - \sum_{r=0}^{\infty} a_r x^{p+r+2} = 0$$

Indicial eqn. - x^p: $2a_0 p(p-1) + a_0 p - a_0 = 0$

$$\Rightarrow 2p^2 - p - 1 = 0$$

$$\Rightarrow \underline{\underline{p = 1, -\tfrac{1}{2}}} \Rightarrow \text{unequal and not differ by integ}$$

Solutions obtained on solving -

$$y_1(x) = x\left[1 + \frac{x^2}{2.7} + \frac{x^4}{2.4.7.11} + \frac{x^6}{2.4.6.7.11.15} + \dots\right]$$

$$y_2(x) = x^{-1/2}\left[1 + \frac{x^2}{\frac{1}{2}.4} + \frac{x^4}{\frac{1}{2}.\frac{5}{2}.4.8} + \frac{x^6}{\frac{1}{2}.\frac{5}{2}.\frac{9}{2}.4.8.12} + \dots\right]$$

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Prob. $xy'' + y' - y = 0$.

Sol.) $y_1(x) = 1 + \frac{x}{(1!)^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \cdots$

$$y_2(x) = y_1(x) \ln x + \left[\frac{x}{1^2}\left(-\frac{2}{1}\right) + \frac{x^2}{1^2 \cdot 2^2}\left(-\frac{2}{1} - \frac{2}{2}\right) + \frac{x^3}{1^2 2^2 3^2}\left(-\frac{2}{1} - \frac{2}{2} - \frac{2}{3}\right) + \cdots \right]$$

$xy'' + y' - y = 0$

$x = 0$ is a regular singular point.

Soln. around $x = 0$ - $y = \sum_{r=0}^{\infty} a_r x^{p+r}$; $a_0 \neq 0$.

$$x \sum_{r=0}^{\infty} a_r (p+r)(p+r-1) x^{p+r-2} + \sum_{r=0}^{\infty} a_r (p+r) x^{p+r-1} - \sum_{r=0}^{\infty} a_r x^{p+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (p+r)(p+r-1) x^{p+r-1} + \sum_{r=0}^{\infty} a_r (p+r) x^{p+r-1} - \sum_{r=0}^{\infty} a_r x^{p+r} = 0.$$

Indicial eqn. - x^{p-1} : $a_0 p(p-1) + a_0 p = 0$

$\Rightarrow p^2 = 0$

$\Rightarrow \underline{\underline{p = 0, 0}}$ $\Rightarrow$ roots equal.

x^p : $a_1 (p+1) p + a_1 (p+1) - a_0 = 0$

$$\Rightarrow a_1 = \frac{a_0}{(p+1)^2}$$

x^{p+r-1} : $a_r (p+r)(p+r-1) + a_r (p+r) - a_{r-1} = 0$

$$\Rightarrow a_r = \frac{a_{r-1}}{(p+r)^2}$$

$\therefore$ $a_2 = \frac{a_1}{(p+2)^2}$; $a_3 = \frac{a_2}{(p+3)^2}$; $\cdots$

$p = 0$: $a_1 = \frac{a_0}{1^2}$; $a_2 = \frac{a_1}{2^2}$; $a_3 = \frac{a_2}{3^2}$ $\cdots$

$$y_1(x) = 1 + \frac{x}{(1!)^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \cdots$$

$$y(x) = a_0 x^p \left[1 + \frac{1}{(p+1)^2} + \frac{1}{\{(p+2)!\}^2} + \frac{1}{\{(p+3)!\}^2} + \cdots \right]$$

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Roots of Indicial Equation differ by integer and coefficient of y become infinite -

Rule : If the indicial eqn. has two roots k_1, k_2 $(k_1 > k_2)$ differing by integer, and if some coefficients of y become infinite when $k = k_2$, we modify the form of y by replacing a_0 by $b_0(k-k_2)$. We thus obtain two independent solns. by putting $k = k_2$ in the modified form of y and $\partial y/\partial k$.

Prob.) $x(1-x)y'' - 3xy' - y = 0$.

$x = 0, 1$ are regular singular point.

Around $x=0$: $y = \sum_{r=0}^{\infty} a_r x^{p+r}$; $a_0 \neq 0$.

$$x(1-x)\sum a_r (p+r)(p+r-1)x^{p+r-2} - 3x\sum a_r (p+r)x^{p+r-1} - \sum a_r x^{p+r} = 0$$

$$\Rightarrow \sum a_r (p+r)(p+r-1)\left[x^{p+r-1} - x^{p+r}\right] - 3\sum a_r (p+r)x^{p+r} - \sum a_r x^{p+r} = 0.$$

Indicial eqn. - x^{p-1} : $a_0 p(p-1) = 0$

$\Rightarrow p = 0, 1 \Rightarrow$ unequal and differ by integer.

x^p : $a_1(p+1)p - a_0 p(p-1) - 3a_0 p - a_0 = 0 \Rightarrow a_1 = \dfrac{a_0(p+1)}{p}$

x^{p+r-1} : $a_r(p+r)(p+r-1) = a_{r-1}\left[(p+r-1)(p+r-2) + 3(p+r-1) + 1\right]$

$= a_{r-1}(p+r)^2$

$$\Rightarrow a_r = \frac{p+r}{p+r-1}a_{r-1} \quad ; \quad r = 1, 2, 3, \ldots$$

$$\therefore a_2 = \frac{p+2}{p+1}a_1 = \frac{p+2}{p}a_0 \quad ; \quad a_3 = \frac{p+3}{p}a_0$$

$$\therefore y(x) = a_0 x^p\left[1 + \frac{p+1}{p}x + \frac{p+2}{p}x^2 + \frac{p+3}{p}x^3 + \ldots\right]$$

For $p=0$, $y(x)$ becomes infinite.

We replace a_0 by $b_0 p$.

$$\therefore y(x) = b_0 x^p\left[p + (p+1)x + (p+2)x^2 + (p+3)x^3 + \ldots\right] \quad \text{——— (1)}$$

For $p=0$, $y_1(x) = x + 2x^2 + 3x^3 + \ldots$

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$$\frac{\partial y}{\partial p} = b_0 x^p \ln x \left[p + (p+1)x + (p+2)x^2 + (p+3)x^3 + \dots \right] + b_0 x^p \left[1 + x + x^2 + x^3 + \dots \right]$$

$$\therefore\ y_2(x) = \left.\frac{\partial y}{\partial p}\right|_{p=0} = \ln x \left[x + 2x^2 + 3x^3 + \dots \right] + \left[1 + x + x^2 + x^3 + \dots \right]$$

$$= y_1(x) \ln x + 1 + x + x^2 + x^3 + \dots$$

HYPERGEOMETRIC EQUATION :

$$x(1-x)y'' + \left[\gamma - (\alpha+\beta+1)x\right] y' - \alpha\beta y = 0 \quad \text{——— (1)}$$

$x = 1, 0$ are regular singular points.

To show $x = \infty$ is also a regular singular point, put

$$x = 1/\xi, \quad \xi^2(\xi - 1)\frac{d^2y}{d\xi^2} + \xi\left[(2-\gamma)\xi + (\alpha+\beta-1)\right]\frac{dy}{d\xi} - \alpha\beta y = 0 \quad \text{——— (2)}$$

$\therefore\ \xi = 0$ (i.e. $x = \infty$) is also a regular singular point.

Soln. in the nbd. of $x = 0$ – $\quad y = \sum_{r=0}^{\infty} a_r x^{p+r}$; $a_0 \neq 0$.

$$\sum a_r (p+r)(p+r-1+\gamma) x^{p+r-1} = \sum a_r (p+r+\alpha)(p+r+\beta) x^{p+r} \quad \text{——— (3)}$$

Indicial eqn – x^{p-1} : $a_0 p(p-1+\gamma) = 0$

$$\Rightarrow \underline{\underline{p = 0,\ 1-\gamma}}$$

We assume that $1-\gamma \neq$ integer or zero.

x^p : $\quad a_1 = \dfrac{(p+\alpha)(p+\beta)}{(p+1)(p+\gamma)} a_0$.

x^{p+r} : $\quad a_{r+1} = \dfrac{(p+r+\alpha)(p+r+\beta)}{(p+r+1)(p+r+\gamma)} a_r$; $r = 0, 1, 2 \dots$

$p = 0$ – $\quad a_{r+1} = \dfrac{(r+\alpha)(r+\beta)}{(r+1)(r+\gamma)} a_r \quad$ ——— (4)

$$\therefore\ a_1 = \frac{\alpha\beta}{1.\gamma} a_0 \quad ; \quad a_2 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)} a_0, \dots$$

$\therefore$ One soln. is –

$$y_1(x) \equiv F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{1.\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)} x^2 + \dots$$

This is hypergeometric series, absolutely convergent for $|x| < 1$.

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$\underline{\rho = 1-\gamma}$: $\quad a_{r+1} = \dfrac{(r+1-\gamma+\alpha)(r+1-\gamma+\beta)}{(r+2-\gamma)(r+1)} a_r \; ; \; r = 0, 1, 2 \ldots$ (5)

(5) is obtained from (4) for $\rho = 0$, by replacing α, β, γ by $1-\gamma+\alpha$, $1-\gamma+\beta$, $2-\gamma$.

$$\therefore \quad y_2(x) = x^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; x)$$

<u>Soln. in the nbd. of $x = 1$</u> –

Put $1 - x = \xi \Rightarrow \xi(1-\xi)\dfrac{d^2y}{d\xi^2} + \left[(\alpha+\beta+1-\gamma) - (\alpha+\beta+1)\xi\right]\dfrac{dy}{d\xi} - \alpha\beta y = 0$

(6) is the same as original hypergeometric eqn. with the replacement of γ by $\alpha+\beta+1-\gamma$.

$\therefore$ Solns. in the nbd. of $x = 1$ are :

$y_3(x) = F(\alpha, \beta; \alpha+\beta+1-\gamma; x)$; $\quad y_4(x) = x^{\gamma-\alpha-\beta} F(\gamma-\beta, \gamma-\alpha; \gamma-\alpha-\beta+1)$

where, $\gamma - \alpha - \beta \neq$ integer or zero.

<u>Soln. in the nbd. of $x = \infty$</u> –

$$x = 1/\xi \Rightarrow \xi^2(\xi-1)\frac{d^2y}{d\xi^2} + \xi\left[(2-\gamma)\xi + (\alpha+\beta-1)\right]\frac{dy}{d\xi} - \alpha\beta y = 0 \quad \text{—— (7)}$$

<u>Soln. around $\xi = 0$</u> : $\quad y = \sum_{r=0}^{\infty} a_r \xi^{\rho+r}$; $a_0 \neq 0$.

$$\sum a_r(\rho+r)(\rho+r-1)\left[\xi^{\rho+r+1} - \xi^{\rho+r}\right] + \sum a_r(\rho+r)\left[(2-\gamma)\xi^{\rho+r+1} + (\alpha+\beta-1)\xi^{\rho+r}\right]$$
$$- \alpha\beta \sum a_r \xi^{\rho+r} = 0$$

<u>Indicial eqn.</u> – <u>ξ^ρ</u> : $\rho = \alpha, \beta$.

Assume $\alpha - \beta \neq 0$ or, integer.

Put $y = \xi^\alpha w(\xi)$ to transform (7) into hypergeometric eqn. :

$$\frac{dy}{d\xi} = \alpha\xi^{\alpha-1} w + \xi^\alpha \frac{dw}{d\xi} \quad ; \quad \frac{d^2y}{d\xi^2} = \xi^\alpha \frac{d^2w}{d\xi^2} + 2\alpha\xi^{\alpha-1}\frac{dw}{d\xi} + \alpha(\alpha-1)\xi^{\alpha-2} w$$

$$\xi(1-\xi)\frac{d^2w}{d\xi^2} + \left[(1+\alpha-\beta) - \{\alpha + (\alpha-\gamma+1) + 1\}\xi\right]\frac{dw}{d\xi} - \alpha(\alpha-\gamma+1)w = 0 \quad \text{—— (8)}$$

(8) is same as original hypergeometric eqn. with the replacement of β by $\alpha-\gamma+1$ and γ by $1+\alpha-\beta$.

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$\therefore$ Two solns. of (8) are -

$$w_1 = F(\alpha, \alpha-\gamma+1; 1+\alpha-\beta; \xi)$$

$$w_2 = \xi^{\beta-\alpha} F(\beta, 1+\beta-\gamma; 1+\beta-\alpha; \xi)$$

Solns. in the nbd of $x=\infty$ of original eqn. are -

$\underline{y = \xi^{\alpha} w}$: $y_5(x) = \frac{1}{x^{\alpha}} F(\alpha, \alpha-\gamma+1; 1+\alpha-\beta; \frac{1}{x})$

$$y_6(x) = \frac{1}{x^{\beta}} F(\beta, 1+\beta-\gamma; 1+\beta-\alpha; \frac{1}{x})$$

Prob.) Show that Legendre eqn. can be transformed into hypergeometric eqn. $\left[\begin{array}{l} x = 1-2t \\ \alpha = -l; \beta = l+1; \gamma = 1 \end{array}\right]$

Sol.) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Let $x = 2t-1 \Rightarrow \frac{dt}{dx} = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{1}{2} \quad ; \quad \frac{d^2y}{dx^2} = \frac{1}{4}\frac{d^2y}{dt^2}$$

(7) $\therefore \frac{(1-2t+1)(1+2t-1)}{4} y'' - \frac{2(2t-1)}{2} y' + n(n+1)y = 0$

$$\Rightarrow t(1-t)y'' - (2t-1)y' + n(n+1)y = 0$$

$$\Rightarrow t(1-t)y'' + [1-2t]y' - (-n)(n+1)y = 0$$

Comparing with,

$$t(1-t)y'' + [\gamma - (\alpha+\beta+1)t]y' - \alpha\beta y = 0$$

$\underline{\underline{\alpha = -n}}$; $\underline{\underline{\beta = n+1}}$; $\underline{\underline{\gamma = 1}}$.

Wronskian of y_1, y_2 :

$$y'' + Py' + Qy = 0$$

$$W(y_1(x), y_2(x)) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \neq 0 \text{ for independence.}$$

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Bessel Eqn. of order γ –

$$x^2y'' + xy' + (x^2-\gamma^2)y = 0 \quad (\gamma \geq 0) \quad \text{——— (1)}$$

(Read modified Bessel eqn.)

$x=0$ is a regular singular point.

Let $y(x) = \sum_{n=0}^{\infty} c_n x^{n+p}$; $c_0 \neq 0$.

$$\therefore \; x^2\left[\sum (p+n)(p+n-1)c_n x^{p+n-2}\right] + x\left[\sum (p+n)c_n x^{p+n-1}\right] + (x^2-\gamma^2)\sum c_n x^{p+n} = 0.$$

$$\Rightarrow \sum (p+n)(p+n-1)c_n x^{p+n} + \sum (p+n)c_n x^{p+n} + \sum c_n (x^{p+n+2} - \gamma^2 x^{p+n}) = 0$$

$$\Rightarrow \left[p(p-1)+p-\gamma^2\right]c_0 x^p + \left[(p+1)p + (p+1) - \gamma^2\right]c_1 x^{p+1} + \cdots + \left[\{(p+n)(p+n-1)+(p+n)-\gamma^2\}c_n + c_{n-2}\right]x^{p+n} = 0$$

Indicial eqn: x^p: $p(p-1)+p-\gamma^2 = 0$

$$\Rightarrow \underline{\underline{p = \pm\gamma}}$$

Suppose γ is not half-integer.

General recurrence relation: x^{p+n}: $c_n = \dfrac{c_{n-2}}{\gamma^2-(p+n)^2}$; $n=2,3,\ldots$

To find c_1: $\left[(p+1)^2-\gamma^2\right]c_1 = 0$

$$\Rightarrow \underline{\underline{c_1 = 0}} \text{ for } \underline{\underline{p = \pm\gamma}}$$

Then, $c_3 = c_5 = \cdots = 0$

$$c_2 = \frac{c_0}{(\gamma+p+2)(\gamma-p-2)} \; ; \quad c_4 = \frac{c_0}{(\gamma+p+2)(\gamma+p+4)(\gamma-p-2)(\gamma-p-4)}$$

$$y_1 = x^{\gamma}\left[1 - \frac{x^2}{4(\gamma+1)} + \frac{x^4}{4.8.(\gamma+1)(\gamma+2)} - \cdots\right]$$

$$y_2 = x^{-\gamma}\left[1 - \frac{x^2}{4(1-\gamma)} + \frac{x^4}{4.8.(1-\gamma)(2-\gamma)} - \cdots\right]$$

Clearly, at origin, y_1 is finite while y_2 is infinite soln.

$\therefore$ G.S. $\Rightarrow y(x) = Ay_1(x) + By_2(x)$; γ is not integer.

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Particular solutions - Bessel function of order γ.

1st kind - $$J_\gamma(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\gamma+m+1)} \left(\frac{x}{2}\right)^{2m+\gamma} ; \quad \gamma \geq 0$$

This is a particular soln. and can be obtained from (5) by choosing $B=0$, $A = 1/2^\gamma \Gamma(\gamma+1)$

Then (4a) $\Rightarrow$ $$AY_1 = \frac{x^\gamma}{2^\gamma \Gamma(\gamma+1)} - \frac{x^{\gamma+2}}{2^{\gamma+2}(\gamma+1)\Gamma(\gamma+1)} + \frac{x^{\gamma+4}}{4.8.2^\gamma(\gamma+1)(\gamma+2)\Gamma(\gamma+1)} - \frac{x^{\gamma+6}}{4.8.12.2^\gamma \Gamma(\gamma+4)} + \dots$$

$$\Rightarrow AY_1 = \frac{1}{\Gamma(\gamma+1)}\left(\frac{x}{2}\right)^\gamma - \frac{(x/2)^{\gamma+2}}{1!\,\Gamma(\gamma+2)} + \frac{(x/2)^{\gamma+4}}{2!\,\Gamma(\gamma+3)} - \frac{(x/2)^{\gamma+6}}{3!\,\Gamma(\gamma+4)} + \dots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+\gamma}}{m!\,\Gamma(\gamma+m+1)} \equiv J_\gamma(x)$$

Similarly, choosing $A=0$, $B = 1/2^\gamma\Gamma(\gamma+1)$,

$$BY_2 \equiv J_{-\gamma}(x)$$

Show : $J_{-\gamma}(x) = (-1)^\gamma J_\gamma(x)$; γ integer.

Proof - Case (i) : γ is pos. integer.

$$J_{-\gamma} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(-\gamma+m+1)} \left(\frac{x}{2}\right)^{2m-\gamma}$$

Note that for $\gamma \leq 0$, $\Gamma(\gamma) = \infty$ and, hence, $1/\Gamma(\gamma) = 0$.

$\because$ γ is pos. integer and m runs through integer values, all the coefficients for $m = 0, 1, \dots, \gamma-1$ must vanish.

$$\therefore J_{-\gamma} = \sum_{m=\gamma}^{\infty} \frac{(-1)^m}{m!\,\Gamma(-\gamma+m+1)} \left(\frac{x}{2}\right)^{2m-\gamma}$$

put $m' = m-\gamma$; $$J_{-\gamma} = \sum_{m'=0}^{\infty} \frac{(-1)^{m'+\gamma}}{(m'+\gamma)!\,\Gamma(m'+1)} \left(\frac{x}{2}\right)^{2(m'+\gamma)-\gamma}$$

$$= \sum_{m'=0}^{\infty} \frac{(-1)^{m'+\gamma}}{m'!\,\Gamma(\gamma+m'+1)} \left(\frac{x}{2}\right)^{2m'+\gamma}$$

$$\Rightarrow J_{-\gamma} = (-1)^\gamma J_\gamma$$

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Bessel function of 2nd kind of order ν -

(Neumann function)

$$Y_\nu(x) = \begin{cases} \dfrac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} & ; \ \nu \neq \text{integer} \\ \lim\limits_{k\to\nu} \dfrac{J_k(x)\cos k\pi - J_{-k}(x)}{\sin k\pi} & ; \ k \neq \text{integer} \end{cases}$$

Two l.i. solns. of Bessel eqn. of order ν are -

$$y(x) = A\,J_\nu(x) + B\,Y_\nu(x). \quad \text{——— (6)}$$

Proof : Case (i) - ν is not integer.

$\sin\nu\pi \neq 0$ and Y_ν is linear combination of J_ν and $J_{-\nu}$ which are themselves l.i. and solns.

Hence, J_ν and Y_ν must be l.i. solns.

Thus, (6) gives G.S.

Case (ii) - ν integer.

$$Y_\nu(x) = \lim_{k\to\nu} \frac{J_k(x)\cos k\pi - J_{-k}}{\sin k\pi} \quad \left[\frac{0}{0} \text{ form, since } \cos k\pi = (-1)^k;\ J_k = (-1)^k J_{-k}\right]$$

$$= \lim_{k\to\nu} \frac{-\pi J_k(x)\sin k\pi + \cos k\pi \frac{\partial}{\partial k}(J_k) - \frac{\partial}{\partial k}(J_{-k})}{\pi\cos k\pi} \quad [\text{L'Hospital's rule}]$$

$$= \frac{1}{\pi}\left[\frac{\partial}{\partial k}(J_k) - (-1)^\nu \frac{\partial}{\partial k}(J_{-k})\right]_{k=\nu}$$

Thus, to show (a) Y_ν is soln., when ν is integer

(b) Y_ν and J_ν are l.i.

To prove (a) : We observe

$$x^2\frac{d^2J_k}{dx^2} + x\frac{dJ_k}{dx} + (x^2-k^2)J_k = 0$$

$$x^2\frac{d^2J_{-k}}{dx^2} + x\frac{dJ_{-k}}{dx} + (x^2-k^2)J_{-k} = 0$$

Differentiating w.r.t. k,

$$x^2\frac{d^2}{dx^2}\left[\frac{\partial}{\partial k}(J_k)\right] + x\frac{d}{dx}\left[\frac{\partial}{\partial k}(J_k)\right] + (x^2-k^2)\left[\frac{\partial}{\partial k}(J_k)\right] - 2kJ_k = 0$$

$$x^2\frac{d^2}{dx^2}\left[\frac{\partial}{\partial k}(J_{-k})\right] + x\frac{d}{dx}\left[\frac{\partial}{\partial k}(J_{-k})\right] + (x^2-k^2)\left[\frac{\partial}{\partial k}(J_{-k})\right] - 2kJ_{-k} = 0$$

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(7) − $(-1)^k \times$ (8) and letting $k \to \gamma$ and using $J_\gamma = (-1)^\gamma J_{-\gamma}$ for integral γ,

$$x^2 \frac{d^2}{dx^2}(\pi Y_\gamma) + x\frac{d}{dx}(\pi Y_\gamma) + (x^2 - \gamma^2)(\pi Y_\gamma) = 0.$$

$\Rightarrow$ Y_γ is a soln. of Bessel eqn.

<u>To prove (b)</u>: We observe that in the explicit expression of Y_γ for integral γ, the term $\ln x$ must appear from differentiation of J_γ w.r.t. γ. Due to this term, Y_γ becomes infinite at $x = 0$. On the other hand, J_γ is ~~closed~~ clearly finite at $x = 0$.

This implies that Y_γ and J_γ must be l.i.

Thus, for all $\gamma \geq 0$, General soln. of Bessel eqn. of order γ is $y(x) = A J_\gamma(x) + B Y_\gamma(x)$.

<u>Hankel function</u> –

<u>1st kind</u>: $H_\gamma^{(1)}(x) = J_\gamma(x) + i Y_\gamma(x)$.

<u>2nd kind</u>: $H_\gamma^{(2)}(x) = J_\gamma(x) - i Y_\gamma(x)$

Another frequently used fundamental system.

$$y(x) = A H_\gamma^{(1)}(x) + B H_\gamma^{(2)}(x)$$

⊛ Solve Bessel eqn. of order 1/2.

⊛ $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

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TENSOR ANALYSIS :

A point in N dimensional space is a set of N numbers $(x^1, x^2, \ldots\ldots, x^N)$.

Coordinate transformation -

Let $(x^1, x^2, \ldots, x^N)$ and $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ be coordinates of a point in two different frames of reference. Suppose there exists N independent relations between the coordinates of the two symbols having the form

$$\bar{x}^k = \bar{x}^k(x^1, x^2, \ldots, x^N) \quad ; \quad k = 1, 2, \ldots, N \qquad (1)$$

Then, conversely to each set of coordinates $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ there will correspond a unique set $(x^1, x^2, \ldots, x^N)$ given by

$$x^k = x^k(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N) \quad ; \quad k = 1, 2, \ldots, N \qquad (2)$$

The relations (1) or (2) define a transformation of coordinates from one frame to another.

The summation convention -

Whenever an index (subscript or superscript) is repeated in a given term we are to sum over that index from 1 to N unless otherwise specified.

Thus, $a_1 x^1 + a_2 x^2 + \ldots + a_N x^N \equiv a_p x^p$, p is dummy index.

This is called summation convention.

Contravariant and covariant vectors -

If N quantities $A^1, A^2, \ldots, A^N$ in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N other quantities $\bar{A}^1, \bar{A}^2, \ldots, \bar{A}^N$ in another system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations-

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} A^q ,$$

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they are called components of a contravariant vector or contravariant tensor of rank 1.

If N quantities $A_1, A_2, \ldots, A_N$ in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N other quantities $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_N$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations -

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} A_q ,$$

they are called components of a covariant vector or covariant tensor of rank 1.

<u>Contravariant</u>, <u>covariant and mixed tensors</u> -

If N^2 quantities A^{qs} in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N^2 other quantities $\bar{A}^{pr}$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations -

$$\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial \bar{x}^r}{\partial x^s} A^{qs} ,$$

they are called contravariant components of a tensor of rank 2.

The N^2 quantities A_{qs} are called covariant components of a tensor of rank 2 if -

$$\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \cdot \frac{\partial x^s}{\partial \bar{x}^r} \cdot A_{qs}$$

Similarly, the N^2 quantities A^q_s are called components of mixed tensor of rank 2 if -

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial x^s}{\partial \bar{x}^r} A^q_s$$

Tensors of higher rank may similarly be defined.
e.g. mixed tensor of rank 5 A^{qst}_{kl} (contravariant of order 3, covariant of order 2)

$$\bar{A}^{prm}_{ij} = \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial \bar{x}^r}{\partial x^s} \cdot \frac{\partial \bar{x}^m}{\partial x^t} \cdot \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A^{qst}_{kl}$$

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The Kronecker Delta -

$$\delta^j_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases} \Rightarrow \text{mixed tensor of rank 2 (prove)}$$

Scalars or invariants -

Suppose ϕ is a function of the coordinates x^k and let $\bar{\phi}$ denote the functional value under a transformation to a new set of coordinates $\bar{x}^k$. Then ϕ is called a scalar or invariant w.r.t. the coordinate transformation if $\phi = \bar{\phi}$.

A scalar or invariant is a tensor of rank 0

Symmetric and skew-symmetric tensor -

A tensor is called symmetric w.r.t. two indices (contravariant or covariant) if its components remain unaltered upon interchange of the indices.

Thus if $A^{mpr}_{qs} = A^{pmr}_{qs}$, the tensor is symmetric in m and p.

A tensor is called skew-symmetric w.r.t. two indices if its components change sign upon interchange of indices.

Thus if $A^{mpr}_{qs} = -A^{pmr}_{qs}$, the tensor is skew-symmetric

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Fundamental operations with tensors -

① **Addition** : The sum of two or more tensors of the same rank and type is also a tensor of the same rank and type.

Thus, if A^{mp}_q and B^{mp}_q are tensors, then $C^{mp}_q = A^{mp}_q + B^{mp}_q$ is also a tensor. Addition of tensors is commutative and associative.

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② <u>Subtraction</u> : The difference of two tensors of same rank and type is also a tensor of same rank and type.

Thus, if A_q^{mp} and B_q^{mp} are tensors, then $C_q^{mp} = A_q^{mp} - B_q^{mp}$ is also a tensor.

③ <u>Outer multiplication</u> : The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors.

e.g., $A_q^{pr} B_s^m = C_{qs}^{prm}$ is the outer product of A_q^{pr} and B_s^m.

Note that not every tensor can be written as a product of two tensors of lower rank. So, division of tensors is not always possible.

④ <u>Contraction</u> : If one contravariant and one covariant index of a tensor are set equal, the result indicates that a summation over the equal indices is to be taken. The resulting sum is a tensor of rank 2 less than that of the original tensor. The process is called contraction.

e.g., in the tensor of rank 5, A_{qs}^{mpr}, set $r=s$ to obtain $A_{qr}^{mpr} = B_q^{mp}$, a tensor of rank 3. Further by setting $p=q$, we obtain $B_p^{mp} = B^m$, a tensor of rank 1.

⑤ <u>Inner multiplication</u> : By the process of outer multiplication of two tensors followed by a contraction, we obtain a new tensor called an inner product of the given tensors. The process is called inner multiplication.

e.g., given the tensors A_q^{mp} and B_{st}^r, the outer product is $A_q^{mp} \cdot B_{st}^r$. Letting $q=r$, we obtain inner product $A_r^{mp} B_{st}^r$. Letting $q=r$ and $p=s$, another inner product $A_r^{mp} B_{pt}^r$ is obtained.

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Inner and outer multiplication of tensors is commutative and associative.

⑥ <u>Quotient Law</u> : Suppose it is not known whether a quantity X is a tensor or not. If an inner product of X with an arbitrary tensor is itself a tensor, then X is also a tensor.

<u>The line element and metric tensor</u> -

In N-dimensional space with coordinates $(x^1, x^2, \ldots, x^N)$ the line element ds is defined by quadratic form, called the metric form or metric.

$$ds^2 = g_{pq}\, dx^p dx^q$$

• The quantities g_{pq} are the components of a covariant tensor of rank two called the <u>metric tensor</u> or <u>fundamental tensor</u>. g_{pq} can be chosen symmetric.

<u>Proof</u> - $ds^2 = g_{pq}\, dx^p dx^q$ is a scalar or invariant.

Changing the dummy indices $p \to q$, $q \to p$,

$$ds^2 = g_{qp}\, dx^q dx^p = g_{qp}\, dx^p dx^q \quad \text{(ordinary product is commutative)}$$

$$\Rightarrow 2\, ds^2 = (g_{pq} + g_{qp})\, dx^p dx^q$$

$$2\, ds^2 = 2\, \tilde{g}_{pq}\, dx^p dx^q$$

$$\Rightarrow ds^2 = \tilde{g}_{pq}\, dx^p dx^q$$

where, $\tilde{g}_{pq} = \frac{1}{2}(g_{pq} + g_{qp})$ is symmetric.

Thus, <u>g_{pq} can be chosen symmetric.</u>

Now, since ds^2 is invariant,

$$\bar{g}_{pq}\, d\bar{x}^p d\bar{x}^q = g_{jk}\, dx^j dx^k = g_{jk} \frac{\partial x^j}{\partial \bar{x}^p} d\bar{x}^p \frac{\partial x^k}{\partial \bar{x}^q} d\bar{x}^q$$

$$= g_{jk} \frac{\partial x^j}{\partial \bar{x}^p} \cdot \frac{\partial x^k}{\partial \bar{x}^q}\, d\bar{x}^p d\bar{x}^q$$

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$\Rightarrow g_{pq}$ is covariant tensor of rank two

The N-dimensional space where $ds^2 = g_{pq}\, dx^p dx^q$ is defined is, in general, called Riemannian space. In the special case where $\exists$ a coordinate transformation from x^j to $\bar{x}^k$ such that the metric form is transformed into diagnol form $(d\bar{x}^1)^2 + (d\bar{x}^2)^2 + \dots + (d\bar{x}^N)^2 \equiv d\bar{x}^k d\bar{x}^k$, then the space is called Euclidean space.

Conjugate or reciprocal tensor -

Let $g = |g_{pq}|$ denote the determinant with elements g_{pq} and suppose $g \neq 0$. Define g^{pq} by -

$$g^{pq} = \frac{G(p,q)}{g}, \quad G(p,q) \text{ is cofactor of } g_{pq}.$$

- g^{pq} is symmetric contravariant tensor of rank 2, called the conjugate or reciprocal tensor.

Proof - Since g_{jk} is symmetric, $G(j,k)$ is symmetric. and so $g^{jk} = G(j,k)/g$ is symmetric.

Let B^p be an arbitrary contravariant vector,

$B_q = g_{pq} B^p$ is an arbitrary covariant vector.

Multiplying by g^{jq},

$$g^{jq} B_q = g^{jq} g_{pq} B^p = \delta^j_p B^p = B^j$$

$$\Rightarrow B^j = g^{jq} B_q$$

Since B^q is an arbitrary vector, g^{jq} is a contravariant tensor of rank two, by quotient law.

- $g_{jk}\, g^{pk} = \delta_j^p$

Proof - $g_{jk}\, G(j,k) = g$, where summation is over k only.

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$$\Rightarrow g_{jk}\frac{G(j,k)}{g} = 1 \qquad \Rightarrow g_{jk}\, g^{jk} = 1$$

Again, $g_{jk}\, G(p,k) = 0 \;,\; p \neq j \;\Rightarrow\; g_{jk}\dfrac{G(p,k)}{g} = 0$

$$\Rightarrow g_{jk}\, g^{pk} = 0 \;,\; p \neq j$$

$$\therefore\; g_{jk}\, g^{pk} = \begin{cases} 0 \;, & p \neq j \\ 1 \;, & p = j \end{cases} \quad \Rightarrow \quad \underline{\underline{g_{jk}\, g^{pk} = \delta_j^{kp}}}$$

Associated Tensor -

Given a tensor, we can derive other tensors by raising or lowering indices. For example, given the tensor A_{pq} we obtain by raising the index p, the tensor $A^{p}_{\cdot q}$, the dot indicating the original position of moved index. Where no confusion can arise we shall often omit the dots.

These derived tensors can be obtained by forming inner products of the given tensor with the metric tensor g_{pq} or its conjugate g^{pq}.

$$A_{rq} \longrightarrow A^{p}_{\cdot q} = g^{rp} A_{rq}$$

$$A^{pq} = g^{rp}\, g^{sq} A_{rs}$$

$$A^{p}_{\cdot rs} = g_{rq} A^{pq}_{\cdot\cdot s} \;,\quad A^{qm.tk}_{\cdot\cdot n} = g^{pk} g_{sn}\, g^{rm} A^{q.st}_{\cdot r\cdot\cdot p}$$

All tensors obtained from a given tensor by forming inner products with the metric tensor and/or its conjugate are called associated tensors of the given tensor.

e.g., A^m and A_m are associated tensors and relation between them is $A_p = g_{pq} A^q \;,\; A^p = g^{pq} A_q$

In rectangular coordinates, $g_{pq} = 1$, if $p = q$ and 0 if $p \neq q$, so that $A_p = A^p$.

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Length of a vector - A^p or A_p is $L^2 = A^p A_p = g^{pq} A_p A_q$
$= g_{pq} A^p A^q$

Angle between two vectors - A^p and B_p is $\cos\theta = \dfrac{A^p B_p}{\sqrt{(A^p A_p)(B^p B_p)}}$

The physical components of a vector A^p or A_p, denoted by A_u, A_v, A_w, are the projections of the vector on the tangents to the coordinate curves and are given in the case of orthogonal coordinates by -

$$A_u = \sqrt{g_{11}}\, A^1 = \frac{A_1}{\sqrt{g_{11}}}; \quad A_v = \sqrt{g_{22}}\, A^2 = \frac{A_2}{\sqrt{g_{22}}}; \quad A_w = \sqrt{g_{33}}\, A^3$$

Similarly, the physical components of a tensor A^{pq} or A_{pq} are -

$$A_{uu} = g_{11} A^{11} = \frac{A_{11}}{g_{11}}; \quad A_{vv} = \sqrt{g_{11} g_{22}}\, A^{12} = \frac{A_{12}}{\sqrt{g_{11} g_{22}}};$$

$$A_{ww} = \sqrt{g_{11} g_{33}}\, A^{13} = \frac{A_{13}}{\sqrt{g_{11} g_{33}}} \quad \text{etc.}$$

Christoffel's symbol -

The symbols - $[pq, r] = \dfrac{1}{2}\left(\dfrac{\partial g_{pr}}{\partial x^q} + \dfrac{\partial g_{qr}}{\partial x^p} - \dfrac{\partial g_{pq}}{\partial x^r}\right)$

$\left\{ {s \atop pq} \right\} = g^{sr} [pq, r]$ are called the Christoffel symbols of the first and second kind respectively.

Transformation Laws for Christoffel symbols -

We denote by a bar a symbol in a coordinate system $\bar{x}^k$.

$$\overline{[jk, m]} = [pq, r] \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^m} + g_{pq} \frac{\partial x^p}{\partial \bar{x}^m} \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}$$

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$$\overline{\left\{ {n \atop jk} \right\}} = \left\{ {s \atop pq} \right\} \frac{\partial \bar{x}^n}{\partial x^s} \cdot \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial \bar{x}^n}{\partial x^q} \cdot \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}$$

These laws show that Christoffel symbols are not tensors unless the second terms on the right are zero.

#

Geodesics :

The distance S between two points t_1 and t_2 on a curve $x^\gamma = x^\gamma(t)$ in a Riemannian space is -

$$S = \int_{t_1}^{t_2} \sqrt{g_{pq} \frac{dx^p}{dt} \frac{dx^q}{dt}} \, dt$$

That curve in the space which makes the distance a minimum is called a geodesic of the space.

Particular Cases -

1. Geodesics on a plane are straight line.
2. Geodesics on a sphere are arcs of great circles.

- A necessary condition that $I = \int_{t_1}^{t_2} F(t, x, \dot{x}) \, dt$ be an extremum is that -

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \quad \text{(Euler's or, Lagrange's eqn.)}$$

Proof: Let I be extremum on the curve $x = X(t)$, $t_1 \leq t \leq t_2$

Suppose $x = X(t) + \epsilon \eta(t)$, ($\epsilon$ independent of t) is a neighbouring curve through t_1 and t_2 so that $\eta(t_1) = \eta(t_2) = 0$. The value of I for the neighbouring curve is - $I(\epsilon) = \int_{t_1}^{t_2} F(t, x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}) \, dt$

A necessary condition that $I(\epsilon)$ be an extremum at $\epsilon = 0$ is that $\frac{dI}{d\epsilon} = 0$. Differentiating under integral sign,

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$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{t_1}^{t_2}\left(\frac{\partial F}{\partial x}\eta + \frac{\partial F}{\partial \dot{x}}\dot{\eta}\right)dt = 0$$

Using by parts on second term,

$$\int_{t_1}^{t_2}\frac{\partial F}{\partial x}\eta\, dt + \underbrace{\left.\frac{\partial F}{\partial \dot{x}}\eta\right|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2}\eta\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\eta\left[\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)\right]dt = 0$$

Since, η is arbitrary, $\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$

- Geodesics in a Riemannian space are found from the differential equation -

$$\frac{d^2x^\gamma}{ds^2} + \left\{ {\gamma \atop pq} \right\}\frac{dx^p}{ds}\frac{dx^q}{ds} = 0$$

<u>Proof</u>: $S = \int_{t_1}^{t_2}\sqrt{g_{pq}\dot{x}^p\dot{x}^q} \Rightarrow$ dist. between two points.

Geodesics is that curve for which dist. S is minimum.

Let $F = \sqrt{g_{pq}\dot{x}^p\dot{x}^q}$ so that $S = \int_{t_1}^{t_2} F\,dt$

For minimum, using Euler's eqn.,

$$\frac{\partial F}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right) = 0$$

$$\frac{\partial F}{\partial x^k} = \frac{1}{2\sqrt{g_{pq}\dot{x}^p\dot{x}^q}}\cdot\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q = \frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q$$

$$\frac{\partial F}{\partial \dot{x}^k} = \frac{1}{2\sqrt{g_{pq}\dot{x}^p\dot{x}^q}}\frac{\partial}{\partial \dot{x}^k}\left[g_{pk}\dot{x}^p\dot{x}^k + g_{kq}\dot{x}^k\dot{x}^q\right]$$

$$= \frac{1}{2\dot{s}}\left[g_{pk}\dot{x}^p + g_{kq}\dot{x}^q\right] = \frac{1}{2\dot{s}}\left[g_{pk}\dot{x}^p + g_{kp}\dot{x}^p\right]$$

$$= \frac{1}{\dot{s}}g_{pk}\dot{x}^p \quad [g_{pk} = g_{kp}]$$

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By Euler's equation,

$$\frac{d}{dt}\left(\frac{g_{pk}\dot{x}^p}{\dot{s}}\right) - \frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q = 0$$

$$\Rightarrow g_{pk}\frac{\ddot{x}^p}{\dot{s}} + \frac{1}{\dot{s}}\frac{\partial g_{pk}}{\partial x^q}\dot{x}^p\dot{x}^q - \frac{1}{\dot{s}^2}g_{pk}\dot{x}^p\ddot{s} - \frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q = 0$$

$$\left[\, A = \frac{\partial g_{pk}}{\partial x^q}\dot{x}^p\dot{x}^q = \frac{\partial g_{qk}}{\partial x^p}\cdot\dot{x}^q\dot{x}^p \Rightarrow 2A = \left(\frac{\partial g_{pk}}{\partial x^q} + \frac{\partial g_{qk}}{\partial x^p}\right)\dot{x}^p\dot{x}^q\right]$$

$$\Rightarrow g_{pk}\ddot{x}^p + \frac{1}{2}\left[\frac{\partial g_{pk}}{\partial x^q} + \frac{\partial g_{qk}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^k}\right]\dot{x}^p\dot{x}^q = \frac{g_{pk}\dot{x}^p\ddot{s}}{\dot{s}}$$

Using arc length as parameter, we take $\dot{s}=1$, $\ddot{s}=0$.
Then above eqn. becomes -

$$g_{pk}\frac{d^2x^p}{ds^2} + [pq,k]\frac{dx^p}{ds}\cdot\frac{dx^q}{ds} = 0$$

Multiplying by g^{rk},

$$\delta^r_p\frac{d^2x^p}{ds^2} + \left\{ {r \atop pq} \right\}\frac{dx^p}{ds}\cdot\frac{dx^q}{ds} = 0$$

$$\Rightarrow \frac{d^2x^r}{ds^2} + \left\{ {r \atop pq} \right\}\frac{dx^p}{ds}\cdot\frac{dx^q}{ds} = 0$$

<u>The covariant derivative</u> - of a tensor A_p w.r.t. x^q is denoted by $A_{p,q}$ and is defined by

$$A_{p,q} = \frac{\partial A_p}{\partial x^q} - \left\{ {s \atop pq} \right\}A_s$$

The covariant derivative of A^p w.r.t. x^q is denoted by $A^p_{,q}$ and is defined by

$$A^p_{,q} = \frac{\partial A^p}{\partial x^q} + \left\{ {p \atop qs} \right\}A^s$$

<u>Note</u>: (1) For rectangular system, the Christoffel's symbols are zero and covariant derivative reduces to usual partial derivative.

(2) The rules of covariant differentiation for sums and product of tensors are the same as those for ordinary differentiation.

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③ Covariant derivatives express rates of change of physical quantities independent of any frames of reference.

Generalizing for high rank tensors, the covariant derivative of $A^{p_1 \dots p_n}_{r_1, \dots, r_n}$ w.r.t. x^q is $A^{p_1 \dots p_n}_{r_1 \dots r_n, q} = \dfrac{\partial A^{p_1 \dots p_n}_{r_1 \dots r_n}}{\partial x^q}$

$$- \left\{ {s \atop r_1 q} \right\} A^{p_1 \dots p_n}_{s r_2 \dots r_n} - \left\{ {s \atop r_2 q} \right\} A^{p_1 \dots p_n}_{r_1 s r_3 \dots r_n} - \dots - \left\{ {s \atop r_n q} \right\} A^{p_1 \dots p_n}_{r_1 r_2 \dots r_{n-1} s}$$

$$+ \left\{ {p_1 \atop q s} \right\} A^{s p_2 \dots p_n}_{r_1 \dots r_n} + \left\{ {p_2 \atop q s} \right\} A^{p_1 s p_3 \dots p_n}_{r_1 \dots r_n} + \dots + \left\{ {p_n \atop q s} \right\} A^{p_1 \dots p_{n-1} s}_{r_1 \dots r_n}$$

• Covariant Derivative of a tensor is also a tensor.

If A_p and A^p are tensors, then $A_{p,q}$ and $A^p_{,q}$ are tensors.

<u>Proof</u> – <u>To show $A_{p,q}$ is a tensor</u>:

$$A_{p,q} = \frac{\partial A_p}{\partial x^q} - \left\{ {s \atop pq} \right\} A_s$$

$$\bar{A}_j = \frac{\partial x^r}{\partial \bar{x}^j} A_r \Rightarrow \frac{\partial \bar{A}_j}{\partial \bar{x}^k} = \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial A_r}{\partial x^t} \frac{\partial x^t}{\partial \bar{x}^k} + \frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k} A_r \quad \text{—— (1)}$$

We now express $\dfrac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k} A_r$ in terms of Christoffel symbol of 2nd kind. The transformation law for Christoffel symbol of 2nd kind is –

$$\left\{ {\bar{n} \atop jk} \right\} = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial \bar{x}^n}{\partial x^s} \left\{ {s \atop pq} \right\} + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial \bar{x}^n}{\partial x^p}$$

Multiplying by $\dfrac{\partial x^m}{\partial \bar{x}^n}$,

$$\frac{\partial x^m}{\partial \bar{x}^n} \left\{ {\bar{n} \atop jk} \right\} = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \delta^m_s \left\{ {s \atop pq} \right\} + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k} \delta^m_p \quad \left[\text{Using } \frac{\partial x^m}{\partial \bar{x}^n} \cdot \frac{\partial \bar{x}^n}{\partial x^s} = \delta^m_s \right]$$

$$= \frac{\partial x^p}{\partial \bar{x}^j} \cdot \frac{\partial x^q}{\partial \bar{x}^k} \left\{ {m \atop pq} \right\} + \frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k}$$

$$\Rightarrow \frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} = \left\{ {\bar{n} \atop jk} \right\} \frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \left\{ {m \atop pq} \right\} \quad \text{—— (2)}$$

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Replacing m by r in (2) and substituting into (1),

$$\frac{\partial \bar{A}_j}{\partial \bar{x}^k} = \frac{\partial x^r}{\partial \bar{x}^j}\frac{\partial A_r}{\partial x^t}\frac{\partial x^t}{\partial \bar{x}^k} + \left\{ {\bar{n} \atop jk} \right\} \frac{\partial x^r}{\partial \bar{x}^n} A_r - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\left\{ {r \atop pq} \right\} A_r$$

$$\Rightarrow \frac{\partial \bar{A}_j}{\partial \bar{x}^k} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial A_p}{\partial x^q} + \left\{ {\bar{n} \atop jk} \right\} \bar{A}_n - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\left\{ {r \atop pq} \right\} A_r$$

$$\therefore \frac{\partial \bar{A}_j}{\partial \bar{x}^k} - \left\{ {\bar{n} \atop jk} \right\} \bar{A}_n = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\left[\frac{\partial A_p}{\partial x^q} - \left\{ {r \atop pq} \right\} A_r \right]$$

$$\Rightarrow \bar{A}_{j,k} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k} A_{p,q}$$

$\therefore$ $A_{p,q}$ is a covariant tensor of rank 2.

<u>To show</u> : <u>$A^p_{,q}$ is a tensor</u> -

$$A^p_{,q} \equiv \frac{\partial A^p}{\partial x^q} + \left\{ {p \atop qs} \right\} A^s$$

$$\bar{A}^j = \frac{\partial \bar{x}^j}{\partial x^r} A^r \Rightarrow \frac{\partial \bar{A}_j}{\partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial x^r}\frac{\partial A^r}{\partial x^t}\frac{\partial x^t}{\partial \bar{x}^k} + \frac{\partial^2 \bar{x}^j}{\partial x^r \partial x^t}\frac{\partial x^t}{\partial \bar{x}^k} A^r \quad \text{——— (1)}$$

We now express $\frac{\partial^2 \bar{x}^j}{\partial x^r \partial x^t}$ in terms of Christoffel symbol of 2nd kind. Transformation law for Christoffel symbol of 2nd kind is -

$$\left\{ {n \atop jk} \right\} = \frac{\partial \bar{x}^p}{\partial x^j} \cdot \frac{\partial \bar{x}^q}{\partial x^k} \cdot \frac{\partial x^n}{\partial \bar{x}^s}\left\{ {s \atop pq} \right\} + \frac{\partial^2 \bar{x}^p}{\partial x^j \partial x^k}\frac{\partial x^n}{\partial \bar{x}^p}$$

Multiplying by $\frac{\partial \bar{x}^m}{\partial x^n}$,

$$\frac{\partial \bar{x}^m}{\partial x^n}\left\{ {n \atop jk} \right\} = \frac{\partial \bar{x}^p}{\partial x^j} \cdot \frac{\partial \bar{x}^q}{\partial x^k}\delta^m_s \left\{ {s \atop pq} \right\} + \frac{\partial^2 \bar{x}^p}{\partial x^j \partial x^k} \cdot \delta^m_p$$

$$= \frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial \bar{x}^q}{\partial x^k}\left\{ {m \atop pq} \right\} + \frac{\partial^2 \bar{x}^m}{\partial x^j \partial x^k}$$

$$\Rightarrow \frac{\partial^2 \bar{x}^m}{\partial x^j \partial x^k} = \left\{ {n \atop jk} \right\}\frac{\partial \bar{x}^m}{\partial x^n} - \frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial \bar{x}^q}{\partial x^k}\left\{ {m \atop pq} \right\} \quad \text{——— (2)}$$

Replacing m, j, k by j, r, t in (2) and substituting into (1),

$$\frac{\partial \bar{A}^j}{\partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial x^r}\cdot\frac{\partial x^t}{\partial \bar{x}^k}\cdot\frac{\partial A^r}{\partial x^t} + \left\{ {n \atop rt} \right\} \frac{\partial \bar{x}^j}{\partial x^n}\cdot\frac{\partial x^t}{\partial \bar{x}^k} A^r - \frac{\partial \bar{x}^p}{\partial x^r}\frac{\partial \bar{x}^q}{\partial x^t}\left\{ {j \atop pq} \right\}\frac{\partial x^t}{\partial \bar{x}^k} A^r$$

$$= \frac{\partial \bar{x}^j}{\partial x^r}\cdot\frac{\partial x^t}{\partial \bar{x}^k}\cdot\frac{\partial A^r}{\partial x^t} + \left\{ {n \atop rt} \right\} \frac{\partial \bar{x}^j}{\partial x^n}\cdot\frac{\partial x^t}{\partial \bar{x}^k} A^r - \frac{\partial \bar{x}^p}{\partial x^r}\delta^q_k \overline{\left\{ {j \atop pq} \right\}} A^r$$

$$= \frac{\partial \bar{x}^j}{\partial x^r}\cdot\frac{\partial x^t}{\partial \bar{x}^k}\cdot\frac{\partial A^r}{\partial x^t} + \left\{ {r \atop st} \right\} \frac{\partial \bar{x}^j}{\partial x^r}\cdot\frac{\partial x^t}{\partial \bar{x}^k} A^s - \overline{\left\{ {j \atop pk} \right\}} \bar{A}^p$$

$$\Rightarrow \frac{\partial \bar{A}^j}{\partial \bar{x}^k} + \overline{\left\{ {j \atop pk} \right\}} \bar{A}^p = \frac{\partial \bar{x}^j}{\partial x^r}\frac{\partial x^t}{\partial \bar{x}^k}\left[\frac{\partial A^r}{\partial x^t} + \left\{ {r \atop st} \right\} A^s\right]$$

$$\Rightarrow \overline{A^j_k} = \frac{\partial \bar{x}^j}{\partial x^r}\frac{\partial x^t}{\partial \bar{x}^k} A^r_{,t}$$

$\therefore$ $A^p_{,q}$ is a mixed tensor of rank 2.

(i) $\dfrac{\partial x^p}{\partial x^q} = \delta^p_q$; (ii) $\dfrac{\partial x^p}{\partial \bar{x}^q}\dfrac{\partial \bar{x}^q}{\partial x^r} = \delta^p_r$

<u>Proof</u> - (i) for $p = q$, $\dfrac{\partial x^p}{\partial x^q} = 1$, since $x^p = x^q$.

for $p \neq q$, $\dfrac{\partial x^p}{\partial x^q} = 0$, since x^p and x^q are independent.

$$\therefore \frac{\partial x^p}{\partial x^q} = \delta^p_q$$

(ii) Coordinates x^p are functions of coordinates $\bar{x}^q$, which are in turn functions of coordinates x^r.

Then by chain rule,

$$\frac{\partial x^p}{\partial x^r} = \frac{\partial x^p}{\partial \bar{x}^q}\cdot\frac{\partial \bar{x}^q}{\partial x^r} = \delta^p_r \quad \left[\because \frac{\partial x^p}{\partial x^r} = \delta^p_r\right]$$

• The covariant derivatives of (a) g_{jk} ; (b) g^{jk} ; (c) δ^j_k are zero so that these quantities may be treated as constants in performing covariant differentiation.

<u>Proof</u> - (a) $g_{jk,q} = \dfrac{\partial g_{jk}}{\partial x^q} - \left\{ {s \atop jq} \right\} g_{sk} - \left\{ {s \atop kq} \right\} g_{sj}$

By def^n, $\left\{ {s \atop pq} \right\} = g^{sr}[pq, r] \Rightarrow g_{sk}\left\{ {s \atop pq} \right\} = \delta^r_k [pq, r]$

$$\Rightarrow [pq, k] = g_{sk}\left\{ {s \atop pq} \right\}$$

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Using this result,

$$g_{jk,q} = \frac{\partial g_{jk}}{\partial x^q} - [jq,k] - [kq,j] \quad \text{———} \quad (1)$$

By def^n, $[jq,k] + [kq,j] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^q} + \cancel{\frac{\partial g_{qk}}{\partial x^j}} - \cancel{\frac{\partial g_{jq}}{\partial x^k}}\right) + \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^q} + \cancel{\frac{\partial g_{kq}}{\partial x^j}} - \cancel{\frac{\partial g_{jq}}{\partial x^k}}\right)$

$$= \frac{\partial g_{jk}}{\partial x^q} \quad [\because g_{ij} = g_{ji}] \quad \text{———} \quad (2)$$

Using (1) and (2), $g_{jk,q} = 0$

~~Using this result~~

(b) $$g^{jk}_{,q} = \frac{\partial g^{jk}}{\partial x^q} + \left\{ {j \atop qs} \right\} g^{sk} + \left\{ {k \atop qs} \right\} g^{js} \quad \text{———} \quad (1)$$

$$\frac{\partial}{\partial x^m}\left(g^{jk} g_{ij}\right) = \frac{\partial}{\partial x^m}\left(\delta_i^k\right) = 0$$

$$\Rightarrow g^{jk}\frac{\partial g_{ij}}{\partial x^m} + \frac{\partial g^{jk}}{\partial x^m} g_{ij} = 0 \quad \Rightarrow g_{ij}\frac{\partial g^{jk}}{\partial x^m} = -g^{jk}\frac{\partial g_{ij}}{\partial x^m}$$

Multiplying by g^{ir},

$$\delta_j^r \frac{\partial g^{jk}}{\partial x^m} = -g^{ir} g^{jk}\left([im,j] + [jm,i]\right)$$

$$\Rightarrow \frac{\partial g^{rk}}{\partial x^m} = -g^{ir}\left\{ {k \atop im} \right\} - g^{jk}\left\{ {r \atop jm} \right\}$$

Replacing r, m, i, j by j, q, s, s,

$$\frac{\partial g^{jk}}{\partial x^q} = -g^{sj}\left\{ {k \atop sq} \right\} - g^{sk}\left\{ {j \atop sq} \right\} \quad \text{———} \quad (2)$$

From (1) and (2), $g^{jk}_{,q} = 0$

(c) $$\delta^j_{k,q} = \frac{\partial \delta^j_k}{\partial x^q} - \left\{ {s \atop kq} \right\}\delta^j_s + \left\{ {j \atop qs} \right\}\delta^s_k$$

$$= 0 - \left\{ {j \atop kq} \right\} + \left\{ {j \atop qk} \right\} = 0$$

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• Derive transformation laws for the Christoffel symbols of (a) the first kind, (b) the second kind.

Proof - (a) $\bar{g}_{jk} = \dfrac{\partial x^p}{\partial \bar{x}^j}\dfrac{\partial x^q}{\partial \bar{x}^k} g_{pq}$

$$\therefore \frac{\partial \bar{g}_{jk}}{\partial \bar{x}^m} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial g_{pq}}{\partial x^r}\frac{\partial x^r}{\partial \bar{x}^m} + \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial^2 x^q}{\partial \bar{x}^m \partial \bar{x}^k} g_{pq} + \frac{\partial^2 x^p}{\partial \bar{x}^m \partial \bar{x}^j}\cdot\frac{\partial x^q}{\partial \bar{x}^k} g_{pq} \quad \text{—— (1)}$$

By cyclic permutation of indices j, k, m by p, q, r,

$j \to k$, $k \to m$, $m \to j$; $p \to q$, $q \to r$, $r \to p$

$$\frac{\partial \bar{g}_{km}}{\partial \bar{x}^j} = \frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}\frac{\partial g_{qr}}{\partial x^p}\frac{\partial x^p}{\partial \bar{x}^j} + \frac{\partial x^q}{\partial \bar{x}^k}\cdot\frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^m} g_{qr} + \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m} g_{qr} \quad \text{—— (2)}$$

$j \to m$, $m \to k$, $k \to j$; $p \to r$, $r \to q$, $q \to p$

$$\frac{\partial \bar{g}_{mj}}{\partial \bar{x}^k} = \frac{\partial x^r}{\partial \bar{x}^m}\cdot\frac{\partial x^p}{\partial \bar{x}^j}\cdot\frac{\partial g_{rp}}{\partial x^q}\frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial x^r}{\partial \bar{x}^m}\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^j} g_{rp} + \frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^m}\frac{\partial x^p}{\partial \bar{x}^j} g_{rp} \quad \text{—— (3)}$$

(2) + (3) − (1), multiply by $1/2$ and using def^n of Christoffel symbol of 1st kind,

$$[jk,m] = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}[pq,r] + \frac{1}{2}\left[\frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m} g_{qr} + \frac{\partial x^r}{\partial \bar{x}^m}\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^j} g_{rp}\right]$$
$$+ \frac{1}{2}\left[\frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^m} g_{qr}\frac{\partial x^q}{\partial \bar{x}^k} - \frac{\partial^2 x^p}{\partial \bar{x}^m \partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k} g_{pq}\right] + \frac{1}{2}\left[\frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^m}\frac{\partial x^p}{\partial \bar{x}^j} g_{rp} - \frac{\partial^2 x^q}{\partial \bar{x}^m \partial \bar{x}^k}\frac{\partial x^p}{\partial \bar{x}^j} g_{pq}\right]$$

We replace the indices as follows.

In 2nd term, $q \to p$, $r \to q$; in 3rd term, $r \to p$; in 4th term $r \to q$. Then we get -

$$[jk,m] = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}[pq,r] + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^m} g_{pq}$$

(b) The transformation law for Christoffel symbol of 1st kind is:

$$\overline{[jk,m]} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}[pq,r] + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^m} g_{pq} \quad \text{—— (1)}$$

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Multiplying (1) by $\bar{g}^{nm} = \frac{\partial \bar{x}^n}{\partial x^s}\frac{\partial \bar{x}^m}{\partial x^t} g^{st}$,

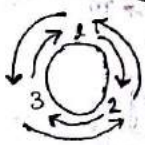
$$\bar{g}^{nm}[\overline{jk,m}] = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}\frac{\partial \bar{x}^n}{\partial x^s}\frac{\partial \bar{x}^m}{\partial x^t} g^{st}[pq,r] + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^m}\frac{\partial \bar{x}^n}{\partial x^s}\frac{\partial \bar{x}^m}{\partial x^t} g^{st} g_{pq}$$

$$\Rightarrow \left\{\overline{\substack{n\\jk}}\right\} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial \bar{x}^n}{\partial x^s}\delta_t^r g^{st}[pq,r] + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial \bar{x}^m}{\partial x^s}\delta_t^q g^{st} g_{pq}$$

$$= \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial \bar{x}^n}{\partial x^s}\left\{\substack{s\\pq}\right\} + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial \bar{x}^n}{\partial x^p}$$

$$\left[\because \delta_t^r g^{st}[pq,r] = g^{sr}[pq,r] = \left\{\substack{s\\pq}\right\} \text{ and, } \frac{\partial \bar{x}^n}{\partial x^s}\delta_t^q g^{st} g_{pq} = \frac{\partial \bar{x}^n}{\partial x^s} g^{sq} g_{pq} = \frac{\partial \bar{x}^n}{\partial x^s}\delta_p^s = \frac{\partial \bar{x}^n}{\partial x^p}\right]$$

Permutation Symbols -



We define e_{pqr} as follows

$e_{123} = e_{231} = e_{312} = +1$, $e_{213} = e_{132} = e_{321} = -1$.

$e_{pqr} = 0$ if two or more indices are equal.

e^{pqr} is defined in the same manner.

The symbols e_{pqr} and e^{pqr} are called permutation symbols in 3-dimensional space. Further we define,

$$\epsilon_{pqr} = \frac{1}{\sqrt{g}} e_{pqr} \quad ; \quad \epsilon^{pqr} = \sqrt{g}\, e^{pqr}$$

- ϵ_{pqr} and ϵ^{pqr} are covariant and contravariant tensor respectively. These are called <u>permutation tensors</u> in 3-D space.

<u>Note</u>: Generalization to higher dimensions are possible.

Tensor form of gradient, divergence and curl -

① **Gradient**: of an invariant (or scalar) ϕ is a covariant vector defined by $\vec{\nabla}\phi \equiv \text{grad}\,\phi = \phi_{,p} = \frac{\partial \phi}{\partial x^p}$

where, $\phi_{,p}$ is the covariant derivative of ϕ w.r.t. x^p.

② **Divergence**: of A^p is the contraction of its covariant derivative w.r.t. x^q, i.e., the contraction of $A^p_{,q}$ i.e.,

$$\text{div}\, A^p = A^p_{,p} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g}\, A^k\right)$$

Note - $A^p_{,q} = \frac{\partial A^p}{\partial x^q} + \left\{ {p \atop qs} \right\} A^s$

$$\Rightarrow A^p_{,q} = \frac{\partial A^p}{\partial x^q} + g^{pr} A^s \frac{1}{2}\left(\frac{\partial g_{qr}}{\partial x^s} + \frac{\partial g_{sr}}{\partial x^q} - \frac{\partial g_{qs}}{\partial x^r}\right)$$

Setting $q = p$,

$$A^p_{,p} = \frac{\partial A^p}{\partial x^p} + g^{pr} A^s \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^s} + \frac{\partial g_{sr}}{\partial x^p} - \frac{\partial g_{ps}}{\partial x^r}\right)$$

$$A^s g^{pr} \frac{\partial g_{pr}}{\partial x^s} = \frac{1}{g} A^s \frac{\partial g}{\partial x^s} = \frac{1}{g} A^k \frac{\partial g}{\partial x^k} \quad \left[\because \frac{\partial g}{\partial x^m} = g g^{jr} \frac{\partial g_{jr}}{\partial x^m}\right]$$

$$g^{pr} \frac{\partial g_{sr}}{\partial x^p} - g^{pr} \frac{\partial g_{ps}}{\partial x^r} = g^{rp} \frac{\partial g_{sp}}{\partial x^r} - g^{pr} \frac{\partial g_{ps}}{\partial x^r} = 0$$

$$\therefore A^p_{,p} = \frac{\partial A^k}{\partial x^k} + \frac{1}{2g} A^k \frac{\partial g}{\partial x^k} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g}\, A^k\right)$$

③ **Curl**: of A_p is a tensor of rank 2 and is defined by-

$$\text{curl}\, A_p = A_{p,q} - A_{q,p} = \frac{\partial A_p}{\partial x^q} - \frac{\partial A_q}{\partial x^p}$$

Note - $A_{p,q} = \frac{\partial A_p}{\partial x^q} - \left\{ {s \atop pq} \right\} A_s$, $A_{q,p} = \frac{\partial A_q}{\partial x^p} - \left\{ {s \atop qp} \right\} A_s$

$$\Rightarrow A_{p,q} - A_{q,p} = \frac{\partial A_p}{\partial x^q} - \frac{\partial A_q}{\partial x^p}$$

④ **Laplacian**: The laplacian of ϕ is the divergence of grad ϕ i.e., $\nabla^2 \phi = \text{div}\, \phi_{,p} = \text{div}\left(g^{pq} \phi_{,p}\right) = \left(g^{pq} \phi_{,p}\right)_{,q} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{jk} \frac{\partial \phi}{\partial x^k}\right)$

Note - $\phi_{,p} = \frac{\partial \phi}{\partial x^p}$, $\text{div}\left(g^{pq} \phi_{,p}\right) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g}\, g^{pk} \phi_{,p}\right)$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ki} \phi_{,k}\right) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{jk} \frac{\partial \phi}{\partial x^k}\right)$$

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The Intrinsic ~~and~~ or Absolute Derivative - of A_p along a curve $x^q = x^q(t)$, denoted by $\frac{\delta A_p}{\delta t}$, is defined as the inner product of the covariant derivative of A_p and $\frac{\partial x^q}{\partial t}$ i.e., $A_{p,q}\frac{\partial x^q}{\partial t}$ and is given by -

$$\frac{\delta A_p}{\delta t} \equiv \frac{dA_p}{dt} - \left\{ {r \atop pq} \right\} A_r \frac{dx^q}{dt}$$

Similarly, $$\frac{\delta A^p}{\delta t} \equiv \frac{dA^p}{dt} + \left\{ {p \atop qr} \right\} A^r \frac{dx^q}{dt}$$

The vectors A_p or A^p are said to move parallel, along a curve if their intrinsic derivatives along the curve are zero respectively.

Intrinsic derivatives of higher rank tensors are similarly defined.

<u>Note</u>: $$\frac{\delta A_p}{\delta t} = A_{p,q}\frac{dx^q}{dt} = \left(\frac{\partial A_p}{\partial t} - \left\{ {s \atop pq} \right\} A_s\right)\frac{dx^q}{dt}$$

$$= \frac{\partial A_p}{\partial x^q}\frac{dx^q}{dt} - \left\{ {s \atop pq} \right\} A_s \frac{dx^q}{dt} = \frac{dA_p}{dt} - \left\{ {r \atop pq} \right\} A_r \frac{dx^q}{dt}$$

$$\frac{\delta A^p}{\delta t} = A^p{}_{,q}\frac{dx^q}{dt} = \left(\frac{\partial A^p}{\partial x^q} + \left\{ {p \atop qs} \right\} A^s\right)\frac{dx^q}{dt} = \frac{dA^p}{dt} + \left\{ {p \atop qr} \right\} A^r.$$

Relative and Absolute Tensors -

A tensor $A^{p_1 \dots p_m}_{r_1 \dots r_n}$ is called a relative tensor of weight w if its components transform according to the following equation -

$$A^{q_1 \dots q_m}_{s_1 \dots s_n} = \left|\frac{\partial x}{\partial \bar{x}}\right|^w \frac{\partial \bar{x}^{q_1}}{\partial x^{p_1}} \cdots \frac{\partial \bar{x}^{q_m}}{\partial x^{p_m}} \frac{\partial x^{r_1}}{\partial \bar{x}^{s_1}} \cdots \frac{\partial x^{r_n}}{\partial \bar{x}^{s_n}} A^{p_1 \dots p_m}_{r_1 \dots r_n}$$

where, $J = \left|\frac{\partial x}{\partial \bar{x}}\right|$ is the Jacobian of the transformation.

If $w = 0$, the tensor is called absolute. If $w = 1$, the relative tensor is called a tensor density.

<u>So far we have dealt with absolute tensors.</u>

The operations of addition, multiplication, etc. of relative tensors are similar to those of absolute tensors.

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PROBLEMS :

① $\left\{ {p \atop pq} \right\} = \frac{\partial}{\partial x^q} \ln \sqrt{g}$

Sol) $g = g_{jk}\, G(j,k)$, sum over k only, where $G(j,k)$ is cofactor of g_{jk} in $g = |g_{jk}|$. Since, $G(j,k)$ does not contain g_{jk} explicitly, $\frac{\partial g}{\partial g_{jr}} = G(j,r)$.

g is a function of g_{jk}, where g_{jk} is function of x^q.

Then by Chain rule,

$$\frac{\partial g}{\partial x^m} = \frac{\partial g}{\partial g_{jr}} \cdot \frac{\partial g_{jr}}{\partial x^m} = G(j,r)\frac{\partial g_{jr}}{\partial x^m} = g g^{jr} \frac{\partial g_{jr}}{\partial x^m}$$

$$= g g^{jr}\left([jm,r] + [rm,j]\right) \qquad \left[g^{jk} = \frac{G(j,k)}{g}\right]$$

$$= g\left(\left\{ {j \atop jm} \right\} + \left\{ {r \atop rm} \right\}\right) = 2g \left\{ {j \atop jm} \right\}$$

$$\Rightarrow \left\{ {j \atop jm} \right\} = \frac{1}{2g}\frac{\partial g}{\partial x^m} = \frac{\partial}{\partial x^m} \ln \sqrt{g}$$

Homework :

② Write the law of transformation for the tensors :

ⓐ A^{i}_{jk} ; ⓑ B^{mn}_{ijk} ; ⓒ C^{m}

③ A quantity $A(j,k,l,m)$ which is a function of coordinates x^i transforms to another coordinate system $\bar{x}^i$ according to the rule $\bar{A}(p,q,r,s) = \frac{\partial x^j}{\partial \bar{x}^p}\frac{\partial \bar{x}^q}{\partial x^k}\frac{\partial \bar{x}^r}{\partial x^l}\frac{\partial \bar{x}^s}{\partial x^m} A(j,k,l,m)$

Is the quantity a tensor? If so write the tensor in suitable notation. Give the contravariant and covariant order and rank.

④ Is dx^k a tensor? If so, state whether it is contravariant or covariant.

---

⑤ grad ϕ is a tensor.

Proof- ϕ is a function of x^k, where x^k is a function of $\bar{x}^j$. Therefore, $\frac{\partial \phi}{\partial \bar{x}^j} = \frac{\partial \phi}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^j}$

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But ϕ is a scalar. Therefore $\phi(x^1, x^2, \ldots, x^n) = \bar{\phi}(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$

Therefore, $\dfrac{\partial \phi}{\partial \bar{x}^j} = \dfrac{\partial \bar{\phi}}{\partial \bar{x}^j}$

$\Rightarrow \dfrac{\partial \bar{\phi}}{\partial \bar{x}^j} = \dfrac{\partial x^k}{\partial \bar{x}^j} \cdot \dfrac{\partial \phi}{\partial x^k} \Rightarrow \dfrac{\partial \phi}{\partial x^k}$ is a covariant tensor of rank one. Therefore,

grad $\phi \equiv \dfrac{\partial \phi}{\partial x^k}$ is a covariant tensor of rank 1.

---

(6) A covariant tensor has components xy, $2y - z^2$, xz in rectangular coordinates. Find its covariant components in spherical coordinates.

Sol) Let [redacted] A_j be covariant tensor in rectangular coordinates $x^1 = x$; $x^2 = y$, $x^3 = z$.

$A_1 = xy = x^1 x^2$; $A_2 = 2y - z^2 = 2x^2 - (x^3)^2$; $A_3 = xz = x^1 x^3$

Let $\bar{A}_k$ be transformed tensor in spherical coordinates $\bar{x}^1 = r$, $\bar{x}^2 = \theta$, $\bar{x}^3 = \phi$.

Now, $\bar{A}_k = \dfrac{\partial x^j}{\partial \bar{x}^k} A_j$

We know, $x^1 = \bar{x}^1 \sin \bar{x}^2 \cos \bar{x}^3$, $x^2 = \bar{x}^1 \sin \bar{x}^2 \sin \bar{x}^3$,
$x^3 = \bar{x}^1 \cos \bar{x}^2$

$$\therefore \bar{A}_1 = \frac{\partial x^1}{\partial \bar{x}^1} A_1 + \frac{\partial x^2}{\partial \bar{x}^1} A_2 + \frac{\partial x^3}{\partial \bar{x}^1} A_3$$

$$= (\sin \bar{x}^2 \cos \bar{x}^3)\, x^1 x^2 + (\sin \bar{x}^2 \sin \bar{x}^3)\left[2x^2 - (x^3)^2\right] + \cos \bar{x}^2 \, x^1 x^3$$

$$= \sin\theta \cos\phi \, r^2 \sin^2\theta \sin\phi \cos\phi + \sin\theta \sin\phi \left(2r \sin\theta \sin\phi - r^2 \cos^2\theta\right)$$
$$+ \cos\theta \, r^2 \sin\theta \cos\theta \cos\phi.$$

$$\bar{A}_2 = \frac{\partial x^1}{\partial \bar{x}^2} A_1 + \frac{\partial x^2}{\partial \bar{x}^2} A_2 + \frac{\partial x^3}{\partial \bar{x}^2} A_3$$

$$= r\cos\theta \cos\phi \cdot r \sin^2\theta \cos\phi \sin\phi + r\cos\theta \sin\phi \left(2r \sin\theta \sin\phi - r^2 \cos^2\theta\right)$$
$$+ (-r\sin\theta)\, r^2 \sin\theta \cos\theta \cos\phi$$

$$\bar{A}_3 = \frac{\partial x^1}{\partial \bar{x}^3} A_1 + \frac{\partial x^2}{\partial \bar{x}^3} A_2 + \frac{\partial x^3}{\partial \bar{x}^3} A_3$$

$$= (-r\sin\theta \sin\phi)\, r^2 \sin^2\theta \sin\phi \cos\phi + r\sin\theta \cos\phi \left(2r \sin\theta \sin\phi - r^2 \cos^2\theta\right)$$

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⑦ ⓐ Show that $\frac{\partial A_p}{\partial x^q}$ is not a tensor even though A_p is a covariant tensor of rank one.

ⓑ Add suitable term to make it a tensor. What is it called ?

Sol.) ⓐ $\bar{A}_j = \frac{\partial x^p}{\partial \bar{x}^j} A_p \Rightarrow \frac{\partial \bar{A}_j}{\partial \bar{x}^j} \cdot \frac{\partial A_p}{\partial x^q} \frac{\partial x^q}{\partial x^k} + \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^j} A_p$

Due to 2nd term on rhs, $\frac{\partial A_p}{\partial x^q}$ is not a tensor.

ⓑ If we add $-\left\{ {s \atop pq} \right\} A_s$, then it is called covariant derivative of A_p with respect to q, denoted by $A_{p,q}$ i.e.,

$$A_{p,q} = \frac{\partial A_p}{\partial x^q} - \left\{ {s \atop pq} \right\} A_s$$

We have already proved that $A_{p,q}$ is a covariant tensor of rank 2.

---

⑧ Show that the velocity of a fluid at any point is a contravariant tensor of rank one.

Sol.) The velocity of a fluid at any point has components $\frac{dx^k}{dt}$ in the coordinate system x^k. In the coordinate system $\bar{x}^j$ the velocity is $\frac{d\bar{x}^j}{dt}$. By chain rule,

$$\frac{d\bar{x}^j}{dt} = \frac{\partial \bar{x}^j}{\partial x^k} \frac{dx^k}{dt}$$

This shows that the $\frac{dx^k}{dt}$ is a contravariant tensor of rank one.

---

⑨ Show that δ^p_q is a mixed tensor of rank 2.

Sol.) $\bar{\delta}^j_k = \delta^j_k = 1$ if $j=k$ and 0 if $j \neq k$.

Now, $\delta^j_k = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^p}{\partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial \bar{x}^p} \frac{\partial x^q}{\partial \bar{x}^k} \delta^p_q$

$$\therefore \bar{\delta}^j_k = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^k} \delta^p_q$$

This shows that δ^p_q is a mixed tensor of rank 2.

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Homework :

(10) If A^{pq}_{r} and B^{pq}_{r} are tensors, prove that their sum and difference are tensors.

(11) If A^{pq}_{r} and B^{s}_{t} are tensors, prove that $C^{pqs}_{rt} = A^{pq}_{r} B^{s}_{t}$ is also a tensor.

(12) Let A^{pq}_{rst} be a tensor.

(a) Choose $p=t$ and show that A^{pq}_{rsp}, where the summation convention is [redacted] employed, is a tensor. What is its rank?

(b) Choose $p=t$ and $q=s$ and show similarly that A^{pq}_{rqp} is a tensor. What is its rank?

(13) Prove that the contraction of the tensor A^{p}_{q} is a scalar or invariant.

(14) Show that the contraction of the outer product of the tensors A^{p} and B_{q} is an invariant.

---

(15) Show that an inner product of the tensors A^{p}_{r} and B^{qs}_{t} is a tensor of rank 3.

Sol) Outer product of A^{p}_{r} and $B^{qs}_{t} = A^{p}_{r} B^{qs}_{t}$.

Let us contract with respect to indices p and t. So, we set $t=p$. Now, $\bar{A}^{j}_{k} = \dfrac{\partial \bar{x}^{j}}{\partial x^{p}} \dfrac{\partial x^{r}}{\partial \bar{x}^{k}} A^{p}_{r}$.

$$\bar{B}^{lm}_{n} = \frac{\partial \bar{x}^{l}}{\partial x^{q}} \frac{\partial \bar{x}^{m}}{\partial x^{s}} \frac{\partial x^{t}}{\partial \bar{x}^{n}} B^{qs}_{t}$$

$$\therefore \bar{A}^{j}_{k} \bar{B}^{lm}_{n} = \frac{\partial \bar{x}^{j}}{\partial x^{p}} \frac{\partial x^{r}}{\partial \bar{x}^{k}} \frac{\partial \bar{x}^{l}}{\partial x^{q}} \frac{\partial \bar{x}^{m}}{\partial x^{s}} \frac{\partial x^{t}}{\partial \bar{x}^{n}} A^{p}_{r} B^{qs}_{t}$$

Setting $j=n$,

$$\bar{A}^{j}_{k} \bar{B}^{lm}_{j} = \frac{\partial \bar{x}^{j}}{\partial x^{p}} \frac{\partial x^{r}}{\partial \bar{x}^{k}} \frac{\partial \bar{x}^{l}}{\partial x^{q}} \frac{\partial \bar{x}^{m}}{\partial x^{s}} \cdot \frac{\partial x^{t}}{\partial \bar{x}^{j}} A^{p}_{r} B^{qs}_{t}$$

$$= \frac{\partial x^{r}}{\partial \bar{x}^{k}} \frac{\partial \bar{x}^{l}}{\partial x^{q}} \frac{\partial \bar{x}^{m}}{\partial x^{s}} \delta^{t}_{p} A^{p}_{r} B^{qs}_{t} = \frac{\partial x^{r}}{\partial \bar{x}^{k}} \frac{\partial \bar{x}^{l}}{\partial x^{q}} \frac{\partial \bar{x}^{m}}{\partial x^{s}} A^{p}_{r} B^{qs}_{p}$$

This shows that $A^{p}_{r} B^{qs}_{p} = C^{qs}_{r}$ is a tensor of rank three.

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(16) A quantity $A(p,q,r)$ is such that in the coordinate system x^i, $A(p,q,r)\, B^{qs}_{r} = C^{s}_{p}$, where B^{qs}_{r} is an arbitrary tensor and C^{s}_{p} is a tensor. Prove that $A(p,q,r)$ is a tensor.

Sol.) In the transformed coordinate $\bar{x}^i$.

$$\bar{A}(j,k,l)\, \bar{B}^{km}_{l} = \bar{C}^{m}_{j}$$

Then, $\bar{A}(j,k,l) \dfrac{\partial \bar{x}^k}{\partial x^q} \dfrac{\partial \bar{x}^m}{\partial x^s} \dfrac{\partial x^r}{\partial \bar{x}^l} B^{qs}_{r} = \dfrac{\partial \bar{x}^m}{\partial x^s} \dfrac{\partial x^p}{\partial \bar{x}^j} C^{s}_{p} = \dfrac{\partial \bar{x}^m}{\partial x^s} \dfrac{\partial x^p}{\partial \bar{x}^j} A(p,q,r)\, B^{qs}_{r}$

$$\Rightarrow \frac{\partial \bar{x}^m}{\partial x^s}\left[\frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} \bar{A}(j,k,l) - \frac{\partial x^p}{\partial \bar{x}^j} A(p,q,r)\right] B^{qs}_{r} = 0$$

Inner multiplication by $\dfrac{\partial x^n}{\partial \bar{x}^m}$,

$$\delta^n_s \left[\frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} \bar{A}(j,k,l) - \frac{\partial x^p}{\partial \bar{x}^j} A(p,q,r)\right] B^{qs}_{r} = 0$$

$$\Rightarrow \left[\frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} \bar{A}(j,k,l) - \frac{\partial x^p}{\partial \bar{x}^j} A(p,q,r)\right] B^{qn}_{r} = 0$$

Since, B^{qn}_{r} is arbitrary tensor, we have,

$$\frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} \bar{A}(j,k,l) - \frac{\partial x^p}{\partial \bar{x}^j} A(p,q,r) = 0$$

Inner multiplication by $\dfrac{\partial x^q}{\partial \bar{x}^m} \dfrac{\partial \bar{x}^n}{\partial x^r}$,

$$\delta^k_m \delta^n_l\, \bar{A}(j,k,l) - \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^m} \frac{\partial \bar{x}^n}{\partial x^r} A(p,q,r) = 0$$

$$\Rightarrow \bar{A}(j,m,n) = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^m} \frac{\partial \bar{x}^n}{\partial x^r} A(p,q,r)$$

This shows that $A(p,q,r) \equiv A^{r}_{pq}$ is a tensor.

<u>Note</u>- In the above problem, a special case of quotient law is established which states that if an inner product of a quantity X with an arbitrary tensor B is a tensor C, then X is a tensor.

---

(17) If a tensor A^{pqr}_{st} is symmetric or skew-symmetric w.r.t. indices p and q in one coordinate system, show that it remains so w.r.t. p and q in any coordinate system.

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(18) Show that every tensor can be expressed as the sum of two tensors of which one is symmetric and the other skew-symmetric in a pair of covariant or contravariant indices.

(19) If $\phi = a_{jk} A^j A^k$, show that we can always write $\phi = b_{jk} A^j A^k$, where b_{jk} is symmetric.

(20) Express in matrix notation the transformation equations for (a) a covariant vector, (b) a contravariant tensor of rank 2, assuming $N = 3$.

(21) Determine the metric tensor in (a) cylindrical and (b) spherical coordinates.

Note: In the above problem for both cases g is diagonal, because both coordinate systems are orthogonal.

For any orthogonal system, $g_{jk} = 0 \; ; j \neq k$

(22) Determine the conjugate metric tensor in (a) cylindrical and (b) spherical coordinates.

(23) Find (a) g and (b) g^{jk} corresponding to

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6dx^1dx^2 + 4dx^2dx^3.$$

(24) If $A_j = g_{jk} A^k$, then show that $A^k = g^{jk} A_j$

(25) (a) Show that $L^2 = g_{pq} A^p A^q$ is an invariant.

(b) Show that $L^2 = g^{pq} A_p A_q$

Note: Length of vector A_p or A^p is $L = \sqrt{A_p A^p}$

(26) (a) If A^p and B^q are vectors, show that $g_{pq} A^p B^q$ is an invariant.

(b) Show that $\dfrac{g_{pq} A^p B^q}{\sqrt{(A^p A_p)(B^q B_q)}}$ is an invariant.

Note: Angle between 2 vectors A^p and B^q is $\cos\theta = \dfrac{g_{pq} A^p B^q}{\sqrt{(A^p A_p)(B^q B_q)}}$

if $\cos\theta = 0$ i.e., $g_{pq} A^p B^q = A^p B_p = 0$, vectors are orthogonal.

(27) Express
(a) A^{jkl} an

(28) Prove
curves
given by

(29) Prove
(a) $g_{12} =$

(30) Prove

(31) Evalu
(b) the

(32) Dete
in (a

(33) Write
of the
(e) A^{jk}_{m}

(34) Find
Note:
rules

(35) Prov

(36) Prov

(37) Prov

(38) Prov

(39) Expr
compo

(40) Expr

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(27) Express the relationship between the associated tensors (a) A^{jkl} and A_{pqr} ; (b) $A^{k}_{j\cdot l}$ and A^{qkr} ; (c) $A^{p\cdot rs}_{\cdot q\cdot\cdot t}$ and $A^{\cdots sl}_{jqk}$

(28) Prove that the angles $\theta_{12}, \theta_{23}, \theta_{31}$ between the coordinate curves in a three dimensional coordinate system are given by - $\cos\theta_{12} = \dfrac{g_{12}}{\sqrt{g_{11}g_{22}}}$; $\cos\theta_{23} = \dfrac{g_{23}}{\sqrt{g_{22}g_{33}}}$; $\cos\theta_{31} = \dfrac{g_{31}}{\sqrt{g_{33}g_{11}}}$

(29) Prove that for an orthogonal coordinate system, (a) $g_{12} = g_{23} = g_{31} = 0$; (b) $g_{11} = \dfrac{1}{g^{11}}$, $g_{22} = \dfrac{1}{g^{22}}$, $g_{33} = \dfrac{1}{g^{33}}$

(30) Prove (a) $[pq,r] = [qp,r]$; (b) $\left\{ {s \atop pq} \right\} = \left\{ {s \atop qp} \right\}$; (c) $[pq,r] = g_{rs}\left\{ {s \atop pq} \right\}$

(31) Evaluate the Christoffel symbols of (a) the first kind, (b) the second kind, for spaces where $g_{pq} = 0$ if $p \neq q$.

(32) Determine the Christoffel symbols of the second kind in (a) rectangular, (b) cylindrical, (c) spherical coordinates.

(33) Write the covariant derivative with respect to x^q of each of the following tensors: (a) A_{jk}, (b) A^{jk}, (c) A^{j}_{k}, (d) A^{j}_{kl}, (e) A^{jkl}_{mn}.

(34) Find the covariant derivative of $A^{j}_{k} B^{lm}_{n}$ with respect to x^q.

<u>Note</u>: Thus covariant derivative of a product of tensors obey rules like those of ordinary derivatives of products.

(35) <u>Prove</u>: $(g_{jk} A^{km}_{n})_{,q} = g_{jk} A^{km}_{nq}$

(36) Prove that $\operatorname{div} A^{p} = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^{k}} (\sqrt{g}\, A^{k})$

(37) Prove that $\nabla^2 \phi = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^{k}} \left(\sqrt{g}\, g^{kr} \dfrac{\partial \phi}{\partial x^{r}}\right)$

(38) Prove that $A_{p,q} - A_{q,p} = \dfrac{\partial A_p}{\partial x^q} - \dfrac{\partial A_q}{\partial x^p}$

(39) Express the divergence of a vector A^p in terms of its physical components for: (a) Cylindrical, (b) Spherical coordinates.

(40) Express the Laplacian of ϕ, $\nabla^2\phi$ in: (a) Cylindrical, (b) Spherical coordinates.

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STURM - LIOUVILLE THEORY -

Consider the operator-

$$L = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$$

Suppose it acts on $u(x) \in [a, b]$. We assume that the first (2-i) derivatives of $p_i(x)$ are continuous and $p_0(x) \neq 0$ in $[a, b]$.

For any two functions $u(x)$, $v(x)$, the inner product is defined as-

$$\langle v|u\rangle = \int_a^b v^*(x)\, w(x)\, u(x)\, dx$$

where, $w(x)$ is some weight function related to the eigenvalue equation.

$$Lu(x) = -\lambda\, w(x)\, u(x)$$, λ is eigenvalue of L and $u(x)$ is eigenfunction of L corresponding to eigenvalue λ.

The operator L is called self-adjoint if -

$\langle v|Lu\rangle = \langle Lv|u\rangle$, holds $\forall\, u, v$.

Let us find the condition for self-adjointness -

$$\langle v|Lu\rangle = \int_a^b v^*(x)\, w(x)\left(p_0 u''(x) + p_1 u'(x) + p_2\right) dx$$

$$= p_0 v^* w u'\Big|_a^b - \int_a^b (p_0 v^* w)' u'\, dx + p_1 v^* w u\Big|_a^b - \int_a^b (p_1 v^* w)' u\, dx + \int_a^b p_2 v^* w u\, dx$$

$$\Rightarrow \langle v|Lu\rangle = v^* w\left(p_0 u' + p_1 u\right)\Big|_a^b - (p_0 v^* w)' u\Big|_a^b + \int_a^b (p_0 v^* w)'' u\, dx$$

$$- \int_a^b (p_1 v^* w)' u\, dx + \int_a^b p_2 v^* w u\, dx$$

$$= \left[v^* w(p_0 u' + p_1 u) - (p_0 v^* w)' u\right]_a^b + \int_a^b \left[(p_0 v^* w)'' - (p_1 v w)' + p_2 v^* w\right] u\, dx$$

$$\left[\begin{aligned} (p_0 v^* w)'' &= p_0 w v^{*''} + 2(p_0 w)' v^{*'} + (p_0 w)'' v^* \\ (p_1 v^* w)' &= (p_1 w)' v^* + (p_1 w) v^{*'} \end{aligned}\right]$$

$$\langle Lv|u\rangle = \int_a^b \left(p_0 v^{*''} + p_1 v^{*'} + p_2 v^*\right) w u\, dx$$, assuming p_i's are real.

If two integrands are equal,

$$2(p_0 w)' v^{*'} + (p_0 w)'' v^* - (p_1 w)' v^* - (p_1 w) v^{*'} = p_1 v^{*'} w$$

$$2\left[(p_0 w)' - (p_1 w)\right] v^{*'} + \left[(p_0 w)'' - (p_1 w)'\right] v^* = 0$$

v is arbitrary $\Rightarrow (p_0 w)' = p_1 w$.

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The non-integral term in $\langle v|Lu\rangle$

$= p_0 v^* w u' + p_1 u v^* w - (p_0 w)' v^* w - (p_0 w) v^{*\prime} u \quad [\because (p_0 w)' = p_1 w]$

$= p_0 w (v^* u' - v^{*\prime} u)$, which vanishes if we assume a boundary condition -

$$p_0 w v^* \frac{du}{dx}\Big|_{x=a} = 0 \quad , \quad p_0 w v^* \frac{du}{dx}\Big|_{x=b} = 0$$

or, more generally,

$$v^* p_0 w \frac{du}{dx}\Big|_{x=a} = v^* p_0 w \frac{du}{dx}\Big|_{x=b}$$

for all solutions u, v of eigenvalue equation.

<u>The adjoint of the operator L</u> is L^+ and is defined by

$$L^+ = \frac{d^2}{dx^2} p_0 - \frac{d}{dx} p_1 + p_2$$

$$\left[\begin{array}{l} \underline{\text{Remember}} - \quad (AB)^+ = B^+ A^+ \\ \left(\frac{d}{dx}\right)^+ = -\frac{d}{dx} \ , \ \left(\frac{d^2}{dx^2}\right)^+ = \frac{d^2}{dx^2} \ , \ \cdots , \ \left(\frac{d^n}{dx^n}\right)^+ = (-1)^n \frac{d^n}{dx^n} \end{array}\right]$$

and in general,

$$\langle v|Lu\rangle = \langle L^+ v|u\rangle$$

We have just shown that if -

$$(p_0 w)' = p_1 w$$

holds, then $L^+ = L$ (self-adjoint operator)

For self-adjoint case,

$$L^+ wu = Lwu = \frac{d^2}{dx^2}(p_0 wu) - \frac{d}{dx}(p_1 wu) + p_2 wu$$

$$= [p_0 wu' + (p_0 w)' u]' - (p_1 w)' u - (p_1 w) u' + p_2 wu$$

$$= (p_0 w) u'' + 2(p_0 w)' u' + \cancel{(p_0 w)'' u} - \cancel{(p_1 w)' u} - (p_1 w) u' + p_2 wu$$

$$= (p_0 w) u'' + (p_0 w)' u' + p_2 wu$$

$$= \frac{d}{dx}\left(p_0 w \frac{du}{dx}\right) + p_2 wu$$

When the weight function $w(x) = 1$, the self-adjoint form of linear homogeneous 2nd order ODE is -

$$Lu \equiv \frac{d}{dx}\left(p_0 \frac{du}{dx}\right) + p_2 u = 0$$

This self-adjoint operator L is also called Sturm-Liouville operator.

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Note: A given operator is not inherently self-adjoint. Its self-adjointness depends on the properties of the function space in which it acts and on the boundary conditions.

A non-self-adjoint operator can always be transformed into self-adjoint form as follows -

Consider the non-self-adjoint operator L :

$$Lu = p_0 \frac{d^2u}{dx^2} + p_1 \frac{du}{dx} + p_2 u \quad ; \quad p_0' \neq p_1$$

Multiply L by f/p_0,

$$\frac{f}{p_0} Lu = f \frac{d^2u}{dx^2} + f \frac{p_1}{p_0} \frac{du}{dx} + \frac{f p_2}{p_0} u$$

We need for self adjointness,

$$f' = \frac{f p_1}{p_0} \Rightarrow \ln f = \int \frac{p_1}{p_0} dx \Rightarrow f = e^{\int^x \frac{p_1}{p_0} dt}$$

Then,

$$\frac{1}{p_0} e^{\int^x p_1/p_0 \, dt} Lu = e^{\int^x p_1/p_0 \, dt} u'' + \frac{p_1}{p_0} e^{\int^x p_1/p_0 \, dt} u' + \frac{p_2}{p_0} e^{\int^x p_1/p_0 \, dt} u .$$

$$= \frac{d}{dx}\left(e^{\int^x p_1/p_0 \, dt} \frac{du}{dx}\right) + \frac{p_2}{p_0} e^{\int^x p_1/p_0 \, dt} u .$$

which is in self-adjoint form.

Some Well-known 2nd order ODE -

Compare the following equations with $Lu = -\lambda w u$, where $Lu = p_0 u'' + p_1 u' + p_2 u$, check whether given equation is in self-adjoint form. If not, transform it to self-adjoint form and find p_0, p_2, λ, w.

---

① Legendre Equation :

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0 \quad ; \quad l = 0, 1, 2 \ldots \quad (-1 \leq x \leq 1)$$

This equation can be written as -

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + l(l+1)y = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] = -l(l+1)y$$

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$\therefore$ It is self-adjoint form.

$p_0 = (1-x^2)$; $p_2 = 0$, $w = 1$, $\lambda = l(l+1)$

<u>Note</u>: $y_l \equiv P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}\left[(x^2-1)^l\right]$ are orthogonal in $[-1,1]$.

$$\langle y_l | y_{l'} \rangle = \int_{-1}^{1} P_l(x) . P_{l'}(x)\, dx = \frac{2}{2l+1} \delta_{ll'}$$

---

② Associated Legendre Equation :

$$(1-x^2)\, y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0 \quad ; \quad l = 0, 1 \ldots$$

$-l \leq m \leq l$ are integers, $(-1 \leq x \leq 1)$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] - \frac{m^2}{1-x^2} y = -l(l+1)\, y$$

$\Rightarrow$ It is self-adjoint form.

$p_0 = (1-x^2)$; $p_2 = -\frac{m^2}{1-x^2}$; $w = 1$, $\lambda = l(l+1)$

<u>Note</u>: $$y_l^m \equiv P_l^m(x) = \begin{cases} \frac{(-1)^m}{2^l l!} \frac{d^{l+m}}{dx^{l+m}}\left[(x^2-1)^l\right] , & m \geq 0 \\ (-1)^m \frac{(l+m)!}{(l-m)!} P_l^{-m}(x) , & m \leq 0 \end{cases}$$

These are orthogonal with respect to l with weight function 1 : $$\int_{-1}^{1} P_l^m(x)\, P_{l'}^m(x)\, dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}$$

These are also orthogonal with respect to m with weight function $1/(1-x^2)$.

$$\int_{-1}^{1} P_l^m(x)\, P_l^{m'}(x) \frac{dx}{1-x^2} = \frac{(l+m)!}{m\,(l-m)!} \delta_{mm'}$$

---

③ Shifted Legendre Equation :-

$\tilde{P}_l = P_l(2x-1)$

put $x = 2x' - 1$ in Legendre Equation,

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\frac{dy}{dx} \quad ; \quad \frac{d^2y}{dx^2} = \frac{1}{4}\frac{d^2y}{dx^2}$$

$$(1-x^2) = 4x'(1-x')$$

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Dropping prime,

$$x(1-x)y'' - (2x-1)y' + \ell(\ell+1)y = 0 \quad ; \quad x \in [0,1], \quad \ell = 0,1,2\ldots$$

$$\Rightarrow \frac{d}{dx}\left[x(1-x)\frac{dy}{dx}\right] = -\ell(\ell+1)y$$

$\Rightarrow$ It is self-adjoint form.

$$p_0 = x(1-x) \quad ; \quad p_2 = 0 \quad ; \quad w = 1 \quad ; \quad \lambda = \ell(\ell+1)$$

<u>Note</u>: $y_1 = \tilde{P}_\ell(x) = \frac{1}{\ell!}\frac{d^\ell}{dx^\ell}(x^2-x)^\ell$

<u>Orthogonality</u> - $\int_{-1}^{1} \tilde{P}_\ell(x)\tilde{P}_{\ell'}(x)\,dx = \frac{1}{2\ell+1}\delta_{\ell\ell'}$

---

(4) Laguerre Equation:

$$xy'' + (1-x)y' + \alpha y = 0 \quad ; \quad 0 \leq x < \infty, \quad \alpha = 0,1,2,\ldots$$

This is non-self-adjoint.

Multiply by $\frac{1}{x}e^{\int \frac{1-x}{x}dx} = \frac{1}{x}e^{\ln x - x} = e^{-x}$

$$xe^{-x}y'' + (1-x)e^{-x}y' + \alpha y e^{-x} = 0$$

$$\frac{d}{dx}\left[xe^{-x}\frac{dy}{dx}\right] = -\alpha e^{-x}y.$$

$\Rightarrow$ It is self-adjoint form.

$$p_0 = xe^{-x} \quad ; \quad p_2 = 0 \quad ; \quad w = e^{-x} \quad ; \quad \lambda = \alpha.$$

<u>Note</u>: $y_\alpha \equiv L_\alpha(x) = \frac{e^x}{\alpha!}\frac{d^\alpha}{dx^\alpha}(e^{-x}.x^\alpha)$

Orthogonality w.r.t. weight function $w(x)$:

$$\int_0^\infty L_\alpha(x)L_{\alpha'}(x)e^{-x}.dx = 0 \quad ; \quad \alpha \neq \alpha'$$

---

(5) Associated Laguerre Equation:

$$xy'' + (\beta+1-x)y' + \alpha y = 0, \quad x \in [0,\infty), \quad \alpha,\beta = 0,1,2,\ldots$$

This is non-self-adjoint.

Multiply by $\frac{1}{x}e^{\int \frac{\beta+1-x}{x}dx} = x^\beta e^{-x}$

$$\frac{d}{dx}\left(x^{\beta+1}e^{-x}\frac{dy}{dx}\right) = -\alpha y x^\beta e^{-x} \Rightarrow \text{self-adjoint form.}$$

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$p_0 = x^{\beta+1} e^{-x}$, $p_2 = 0$, $w = x^{\beta} e^{-x}$, $\lambda = \alpha$

<u>Note</u>: $y \equiv L_\alpha^\beta(x) = \dfrac{x^{-\beta} e^{x}}{\alpha!} \dfrac{d^\alpha}{dx^\alpha}\left(e^{-x} x^{\alpha+\beta}\right)$

<u>Orthogonality</u> - $\displaystyle\int_0^\alpha e^{-x} x^{\beta} L_\alpha^\beta(x)\, L_{\alpha'}^\beta(x)\, dx = \frac{(\alpha+\beta)!}{\alpha!}\, \delta_{\alpha\alpha'}$

---

⑥ Hermite Equation :

$y'' - 2xy' + 2ky = 0$, $k = 0, 1, 2, \ldots$, $-\infty < x < \infty$.

This is non-self-adjoint.

Multiply by $e^{-\int 2x\,dx} = e^{-x^2}$,

$$\frac{d}{dx}\left(e^{-x^2}\frac{dy}{dx}\right) = -2ky\, e^{-x^2}$$

$p_0 = e^{-x^2}$, $p_2 = 0$, $w = e^{-x^2}$, $\lambda = 2k$.

<u>Note</u>: $y \equiv H_k(x) = (-1)^k e^{x^2} \dfrac{d^k}{dx^k}\left(e^{-x^2}\right)$

<u>Orthogonality</u> - $\displaystyle\int_{-\infty}^{\infty} H_k(x) H_{k'}(x) e^{-x^2} dx = \sqrt{\pi}\, 2^k k!\, \delta_{kk'}$

---

⑦ Simple Harmonic Oscillator :

$y'' + n^2 y = 0$, $x \in [0, \infty)$

This is in self-adjoint-form.

$p_0 = 1$, $p_2 = 0$, $w = 1$, $\lambda = n^2$.

<u>Note</u>: $y_n(x) = A\cos(nx - \phi)$, A, ϕ arbitrary const.

---

⑧ Bessel Equation :

$x^2 y'' + x y' + (k^2 x^2 - \gamma^2) y = 0$, $\gamma, k \in \mathbb{R}$, $x \in [0, a]$, $a \in \mathbb{R}$.

This is non-self-adjoint.

Multiply by $\dfrac{1}{x^2} e^{\int \frac{x}{x^2} dx} = \dfrac{1}{x}$,

$$\frac{d}{dx}\left(x \frac{dy}{dx}\right) - \frac{\gamma^2}{x} y = k^2 x y \Rightarrow p_0 = x,\ p_2 = -\frac{\gamma^2}{x},\ w(x) = x,\ \lambda = -k^2.$$

<u>Note</u>: $y \equiv J_\gamma(kx)$, $J_\gamma(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\gamma+1)} \left(\frac{x}{2}\right)^{\gamma+2n}$, $|\gamma| \neq 1/2$

and, $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \cos x$, $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$

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Orthogonality - $\int_0^a J_\gamma\left(\alpha_{\gamma m}\frac{x}{a}\right) J_\gamma\left(\alpha_{\gamma n}\frac{x}{a}\right) x\,dx = \frac{1}{2}a^2\left[J_{\gamma+1}(\alpha_{\gamma m})\right]^2\delta_{mn}$

where, $\alpha_{\gamma i}$ is i-th zero of $J_\gamma(x)$.

---

⑨ Chebyshev Equation :

$$(1-x^2)y'' - xy' + n^2 y = 0 \quad , \quad n = 0, 1, 2, \ldots \quad , \quad x \in [-1, 1]$$

This is non-self-adjoint.

Multiply by $\frac{1}{1-x^2} e^{-\int \frac{x}{1-x^2}dx} = \frac{1}{\sqrt{1-x^2}}$,

$$\frac{d}{dx}\left(\sqrt{1-x^2}\,\frac{dy}{dx}\right) = \frac{n^2}{\sqrt{1-x^2}}\,y$$

$p_0 = \sqrt{1-x^2}$, $p_2 = 0$, $w(x) = \frac{1}{\sqrt{1-x^2}}$, $\lambda = -n^2$

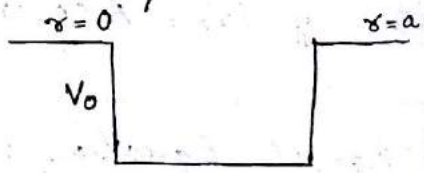
Note : $y \equiv T_n(x) = \cos(n\cos^{-1}x)$

Orthogonality - $\int_{-1}^{1} T_m(x)\,T_n(x)\,\frac{dx}{\sqrt{1-x^2}} = \begin{cases} \pi/2\,\delta_{mn} & , m \neq 0, n \neq 0 \\ \pi & , m = n = 0 \end{cases}$

A physical model -

Consider the problem of deuteron which is a bound state of a neutron and proton. Due to the short range of nuclear force, the deuteron properties do not depend much on the detailed shape of the interaction potential. Thus, the neutron-proton nuclear interaction is modelled by a spherically symmetric square well potential

$$V = \begin{cases} V_0 < 0 & , 0 \leq r < a \\ 0 & , r > a \end{cases}$$



Schrödinger eqn. describing the stationary states is

$$H\Psi \equiv -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = E\Psi \quad \text{——} \quad (1)$$

$H = T + V$ is Hamiltonian of the system.

$T = \frac{p^2}{2m}$ is K.E., V is potential energy, m = common mass of proton and neutron

$p \equiv -\frac{1}{2}\hbar\vec{\nabla}$ is momentum, $\hbar = h/2\pi$, h = Planck's constant.

In quantum mechanics, H, T, p, V, etc. are operators.

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① is an eigenvalue equation, E is energy eigenvalue and Ψ is wave function.

In spherical polar coordinates (r, θ, ϕ), the Laplacian is -

$$\nabla^2 = \frac{1}{r^2 \sin^2\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial \phi^2}\right]$$

[Obtain this form from $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$]

Since potential depends on radial coordinate only $(V = V(r))$, we can seperate: $\Psi(r, \theta, \phi) = \Psi(r)\,\Theta(\theta)\,\Phi(\phi)$

[Do the separation.]

Radial part of S.E., $-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(r\Psi(r)) = (E - V)\Psi(r)$

Substitute $u(r) = r\Psi(r)$.

Then, $0 \leq r \leq a$: $\frac{d^2u}{dr^2} = -k_1^2 u$, $k_1^2 = \frac{2m}{\hbar^2}(E - V_0) > 0$ —— ②

$a \leq r < \infty$: $\frac{d^2u}{dr^2} = k_2^2 u$, $k_2^2 = -\frac{2m}{\hbar^2}E > 0$ —— ③

B.C. : Ψ is finite at $r = 0$ and at $r = \infty$.

Normalization condition : $\int_0^\infty u^2(r)\,dr = 1$

Therefore, acceptable solns. of ② and ③ are -

$$u(r) = \begin{cases} \sin k_1 r, & 0 \leq r \leq a \\ \cancel{Ae^{k_2 r}} + Be^{-k_2 r}, & a \leq r < \infty \end{cases}$$

Matching condition : Wave function must be smooth.

$\therefore$ u and u' must be continuous at $r = a$.

$\therefore$ $\sin k_1 a = Be^{-k_2 a}$, $k_1 \cos k_1 a = -k_2 B e^{-k_2 a}$ —— ⑤

$\Rightarrow \tan k_1 a = -\frac{k_1}{k_2} = -\sqrt{\frac{V_0 - E}{E}}$ —— ⑥

⑥ is energy eqn. This transcendental eqn. gives discrete solutions $E = E_n$, $n = 0, 1, 2 \ldots$ These are energy eigenvalues which corresponds discrete quantum state Ψ_n.

Deuteron has only one bound state i.e., eqn. ⑥ gives only one value of E i.e., $E = E_0$

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Boundary condition and integration interval -

Consider again the eigenvalue equation:

$$L\, u(x) = -\lambda w(x) u(x)$$

where, L is a 2nd order linear differential operator.

$$L = p_0 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_2$$

We assume that eigenfunctions satisfy following B.C.

$$v^* p_0 w \frac{du}{dx}\Big|_{x=a} = v^* p_0 w \frac{du}{dx}\Big|_{x=b}$$

where $u(x)$, $v(x)$ are any two solutions of eigenvalue equation. Sometimes vanishing B.C. at both ends are taken.

The operator L will be self-adjoint if

$$(p_0 w)' = p_1 w$$

holds. In this case, the eigenfunctions are orthogonal i.e, following inner product vanishes

$$\langle u|v\rangle = \int_a^b u^*(x)\, w(x)\, v(x)\, dx = 0$$

for all eigenfunctions u, v $(u \neq v)$.

In quantum mechanics, eigenvalues represent quantum state which is normalized as follows to preserve probability interpretation $\langle u|u\rangle = \int_a^b w(x)|u|^2 dx = 1$

In this case, eigenfunctions are orthonormal.

Integration interval is so chosen that orthogonality condition and boundary conditions will be satisfied in that interval.

For example, consider $L = d^2/dx^2$ with

$$Lu = -n^2 u$$

$\therefore$ Eigenfunctions are -

$$u_n = \cos nx, \quad v_m = \sin mx.$$

B.C. give

$$-n \sin nx \sin mx \Big|_a^b = 0 \quad \text{or,} \quad m \cos mx \cos nx \Big|_a^b = 0$$

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Now, $\sin x$ and $\cos x$ are periodic function of period 2π. Thus, above B.C. will be satisfied if $b-a = 2\pi$.

These eigenfunctions form the basis of Fourier series. For this case, usual choices for integration interval are $a=0$, $b=2\pi$, or $a=-\pi$, $b=\pi$.

Hermitian Operators -

Consider self-adjoint eigen value eqn

$$Lu \equiv (p_0 u')' + p_2 u = -\lambda w u \quad , \quad v^* p u'\big|_{x=a} = v^* p u'\big|_{x=b}$$

Then, Hermiticity property of the operator L is

$$\int_a^b v^* L u \, dx = \int_a^b u (Lv)^* dx$$

We now prove this relation:

$$\int_a^b v^* L u \, dx = \int_a^b v^* (p_0 u')' dx + \int_a^b v^* p_2 u \, dx$$

$$= \underbrace{v^* p_0 u'\Big|_a^b}_{=0} - \int_a^b v^{*\prime} p_0 u' dx + \int_a^b u p_2 v^* dx$$

$$= \underbrace{-v^* p_0 u\Big|_a^b}_{=0} + \int_a^b (v^{*\prime} p_0) u \, dx + \int_a^b u p_2 v^* dx$$

$$\Rightarrow \int_a^b v^* L u \, dx = \int_a^b u \left[(p_0 v^{*\prime})' + p_2 v^* \right] dx = \int_a^b u (Lv)^* dx$$

Hermitian operators in Quantum Mechanics:

So far we have focussed on the classical second order differential operators of mathematical physics. In quantum mechanics, the operators need to be neither second-order differential operators nor real, e.g., $\vec{p} = -\frac{1}{2}\hbar\vec{\nabla}$ is momentum operator, which is Hermitian. Here the eigenfunctions, popularly called wavefunctions, satisfy appropriate boundary conditions, namely vanishing sufficiently strongly at infinity or having periodic behaviour (as in a crystal lattice, or unit intensity for scattering problems)

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A complex operator A is called Hermitian if

$$\int_\Omega \psi_1^* A \psi_2 \, d\tau = \int_\Omega (A\psi_1)^* \psi_2 \, d\tau$$

The adjoint of operator A is A^+ defined by

$$\int_\Omega \psi_1^* A^+ \psi_2 \, d\tau \equiv \int_\Omega (A\psi_1)^* \psi_2 \, d\tau$$

Thus, if $A = A^+$ (self-adjoint) and satisfies the boundary conditions, then A is Hermitian.

In quantum mechanics, the eigenvalues of an operator is observable and so must be real. Note that an operator corresponds to a physical quantity, Then e.g., $p \to -\frac{1}{2}\hbar \frac{d}{dx}$. We shall later show that eigenvalues of an Hermitian operator are always real. But remember that Hermiticity is too restrictive constraint to guarantee the reality of the spectra. There exists a broader class of non-Hermitian operators satisfying some symmetry condition which possess real eigenvalues. This leads to non-Hermitian quantum mechanics including as a special case PT-symmetric quantum mechanics. The study of such non-Hermitian operators is beyond the scope of this course.

Expectation value of an operator A is

$$\langle A \rangle = \int_\Omega \psi^* A \psi \, d\tau$$

In QM, $\langle A \rangle$ is result of measurement of a physical quantity represented by A when the physical system is in a state described by the wave function ψ. Thus, $\langle A \rangle$ must be real in QM.

We now show that if A is Hermitian, then $\langle A \rangle$ is always real.

$$\langle A \rangle^* = \int_\Omega \psi A^* \psi^* d\tau = \int_\Omega (A\psi)^* \psi \, d\tau$$

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$$= \int_{\Omega} \psi^* A \psi \, d\tau = \langle A \rangle, \text{ since } A \text{ is Hermitian.}$$

$\therefore \langle A \rangle$ is real.

• <u>Eigenvalues of a Hermitian operator is real.</u>

Let $\{\psi_n, \lambda_n\}_{n=0}^{N}$ be eigenfunctions and eigenvalues of following eigenvalue problem on $[a,b]$:

$$A\psi = -\lambda w \psi$$

where, $w(x)$ is a non-negative real function.

Then for arbitrary n,

$$A\psi_n = -\lambda_n w \psi_n \quad \text{——} \quad (1)$$

$$\Rightarrow \psi_n^* A \psi_n = -\lambda_n w |\psi_n|^2 \quad \text{——} \quad (2)$$

Taking complex conjugate of (1),

$$A^* \psi_n^* = -\lambda_n^* w \psi_n^*$$

$$\Rightarrow \psi A^* \psi_n^* = -\lambda_n^* w |\psi_n|^2 \quad \text{——} \quad (3)$$

(2) − (3) and integrating over $[a,b]$,

$$\int_a^b \psi_n^* A \psi_n \, dx - \int_a^b \psi_n (A\psi_n)^* dx = (\lambda_n^* - \lambda_n) \int_a^b w |\psi_n|^2 dx$$

$\Rightarrow 0 = (\lambda_n^* - \lambda_n) \int_a^b w |\psi_n|^2 dx$, since A is Hermitian.

$\Rightarrow \lambda_n^* = \lambda_n$, since integrand is non-negative and so integral cannot vanish unless $\psi_n = 0$, which is trivial.

[In QM, integral = 1 (normalizability)]

$\Rightarrow \lambda_n$ s are real.

<u>Note</u> : (1) For any two distinct eigenvalues λ_m, λ_n, we have,

$$(\lambda_m^* - \lambda_n) \int_a^b \psi_n \psi_m^* w \, dx = 0 \quad \text{(prove it)}$$

(2) Spectral decomposition of operator A :

$$A = \sum_n \lambda_n |\psi_n\rangle\langle\psi_n| \;, \quad f(A) = \sum_n f(\lambda_n) |\psi_n\rangle\langle\psi_n|$$

$|\rangle, \langle|$ are Dirac's bra-ket notation used in QM.

$|\psi\rangle \rightarrow$ quantum state, $\langle\psi| \rightarrow$ adjoint of $|\psi\rangle$

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• A Hermitian operator possesses an orthogonal set of eigenfunctions.

Consider again the eigenvalue equation

$$A\Psi = -\lambda W \Psi \quad \text{on } [a,b], \; W \text{ is real function.}$$

Let Ψ_m, Ψ_n be two distinct eigenfunctions corresponding to eigenvalues λ_m, λ_n.

Case 1 : Non-degenerate case $(\lambda_m \neq \lambda_n)$

$$\therefore \; A\Psi_m = -\lambda_m \Psi_m W \;, \quad A\Psi_n = -\lambda_n W \Psi_n$$

Multiplying first by Ψ_n^* and complex conjugate of second by Ψ_m and integrating the difference over $[a,b]$.

$$\int_a^b \Psi_n^* A \Psi_m \, dx - \int_a^b \Psi_m A^* \Psi_n^* dx = (\lambda_n^* - \lambda_m) \int_a^b W(x) \Psi_m \Psi_n^* dx$$

$$\Rightarrow (\lambda_n^* - \lambda_m) \int_a^b \Psi_n^* \Psi_m W dx = 0 \quad \left[\because A \text{ is Hermitian}, \; \int_a^b \Psi_n^* A \Psi_m dx = \int_a^b (A\Psi_n)^* \Psi_m dx\right]$$

$$\Rightarrow (\lambda_n - \lambda_m) \int_a^b \Psi_n^* \Psi_m W dx = 0 \quad [\because \lambda_n \neq \lambda_m]$$

$$\Rightarrow \int_a^b \Psi_n^* \Psi_m W dx = 0 \quad [\because \lambda\text{'s are real}]$$

$$\Rightarrow \langle \Psi_n | \Psi_m \rangle = 0 \;, \quad \text{for } m \neq n \text{ (orthogonality)}$$

Case 2 : Degenerate case $(\lambda_m = \lambda_n)$

In this case, two l.i. eigen~~value~~ functions Ψ_m, Ψ_n need not be automatically orthogonal. But we can always construct an orthogonal set by Gram-Schmidt orthogonalization method as follows:

Suppose $\{\Psi_n, \lambda_n\}_{n=0}^{N}$ be the eigenfunctions and eigenvalues of operator A. Eigenfunction are linearly independent, some of which may be degenerate.

We show that an orthonormal set $\{\phi_n, \lambda_n\}$ can be constructed, which are eigenfunctions and eigenvalues

$$\tilde{\phi}_0 = \Psi_0 \;, \quad \phi_0 = \frac{\Psi_0}{\sqrt{\int |\Psi_0|^2 W dx}} \quad \text{so that} \quad \int_a^b |\phi_0|^2 W dx = 1$$

$\tilde{\phi}_2 = \Psi_1$

$\Rightarrow \langle \Psi_2 | \phi$

$\therefore \; \phi_2$

$\tilde{\phi}_2 = \Psi_2$

$\Rightarrow \langle \Psi_2,$

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$\tilde{\phi}_1 = \psi_1 + a_{1,0}\phi_0$ s.t. $\langle \tilde{\phi}_1, \phi_0 \rangle = 0$

$\Rightarrow \langle \psi_1 | \phi_0 \rangle + a_{1,0} \langle \phi_0, \phi_0 \rangle = 0 \Rightarrow a_{1,0} = -\int_a^b \psi_1^* \phi_0 \, w \, dx$

$$\left[\langle \phi_0, \phi_0 \rangle \equiv \| \phi_0 \|^2 = \int_a^b |\phi_0|^2 w \, dx = 1 \right]$$

$\therefore \phi_1 = \dfrac{\tilde{\phi}_1}{\| \tilde{\phi}_1 \|}$

$\tilde{\phi}_2 = \psi_2 + a_{2,0}\phi_0 + a_{2,1}\phi_1$ s.t. $\langle \tilde{\phi}_2, \phi_j \rangle = 0$, $j = 0, 1$.

$\Rightarrow \langle \psi_2, \phi_0 \rangle + a_{2,0} = 0 \Rightarrow a_{2,0} = -\int_a^b \psi_2^* \phi_0 \, w \, dx$

and, $\langle \psi_2, \phi_1 \rangle + a_{2,1} = 0 \Rightarrow a_{2,1} = -\int_a^b \psi_2^* \phi_1 \, w \, dx$.

Then, $\phi_2 = \dfrac{\tilde{\phi}_2}{\| \tilde{\phi}_2 \|}$

We can carry out this procedure until all eigenfunctions are constructed.

<u>General result</u> : $\phi_i = \tilde{\phi}_i / \| \tilde{\phi}_i \|$

$\tilde{\phi}_0 = \psi_0$, $\tilde{\phi}_i = \psi_i + \sum_{j=0}^{i-1} a_{i,j} \phi_{i-1}$, $i = 1, 2, \ldots, N$

$$\Rightarrow a_{i,j} = -\int_a^b \psi_i^* \phi_j \, w \, dx$$

Thus, in degenerate case too, an orthogonal set of eigenfunctions exist.

• <u>Eigenfunctions of a Hermitian operator form a complete set.</u>

By Completeness, we mean that any well-behaved function $F(x)$ can be approximated by a series

$$F(x) = \sum_n a_n \psi_n(x) \quad , \quad \{\psi_n\} \longrightarrow \text{orthogonal eigenfunctions}.$$

<u>Legendre polynomials by Gram-Schmidt Orthogonalization</u> -

Consider $\{\psi_n\} = \{x^n\}$, $n = 0, 1, 2 \ldots$ in $[-1, 1]$ with $w(x) = 1$.

Note that ψ_n s are non-orthogonal.

$\tilde{\phi}_0 = \psi_0 = 1$, $\phi_0 = \dfrac{\tilde{\phi}_0}{\| \tilde{\phi}_0 \|} = \dfrac{1}{\sqrt{\left[\int_{-1}^{1} 1^2 dx\right]}} = \dfrac{1}{\sqrt{2}} \equiv \dfrac{1}{\sqrt{2}} P_0(x)$

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$$\tilde{\phi}_1 = \psi_1 + a_{1,0}\phi_0 \quad ; \quad a_{1,0} = -\int_{-1}^{1} \psi_1 \phi_0 \, dx = -\int_{-1}^{1} \frac{x}{\sqrt{2}} dx = 0$$

$$\Rightarrow \tilde{\phi}_1 = x \quad , \quad \phi_1 = \frac{\tilde{\phi}_1}{\|\tilde{\phi}_1\|} = \frac{x^{+1}}{\sqrt{\left[\int_{-1}^{1} x^2 dx\right]}} = \sqrt{\frac{3}{2}}\, x \equiv \sqrt{\frac{3}{2}}\, P_1(x)$$

$$\tilde{\phi}_2 = \psi_2 + a_{2,0}\phi_0 + a_{2,1}\phi_1 \quad ; \quad a_{2,0} = -\int_{-1}^{1} \psi_2 \phi_0 \, dx = -\int_{-1}^{1} \frac{x^2}{\sqrt{2}} dx = -\sqrt{\frac{2}{3}}$$

$$a_{2,1} = -\int_{-1}^{1} \psi_2 \phi_1 \, dx = -\int_{-1}^{1} \frac{x^2}{\sqrt{2}} \sqrt{3}\, x \, dx = 0$$

$$\Rightarrow \tilde{\phi}_2 = x^2 - \frac{\cancel{\sqrt{2}}}{3} \cdot \frac{1}{\cancel{\sqrt{2}}} \quad ; \quad \phi_2 = \frac{\tilde{\phi}_2}{\|\tilde{\phi}_2\|} = \frac{x^2 - 1/3}{\sqrt{\int_{-1}^{1} (x^2 - 1/3)^2 dx}} = \frac{\sqrt{5}}{2\sqrt{2}} (3x^2 - 1)$$

$$\equiv \sqrt{\frac{5}{2}}\, P_2(x)$$

Similarly, show $\phi_3(x) = \sqrt{\frac{7}{2}} \cdot \frac{1}{2}(5x^3 - 3x) \equiv \sqrt{\frac{7}{2}}\, P_3(x)$

In general, we get, $\phi_n(x) = \sqrt{\frac{2n+1}{2}}\, P_n(x)$ $\left[\underline{\text{Remember}}: \int_{-1}^{1} P_n^2 dx = \frac{2}{2n+1}\right]$

Do the following exercise for $\{\psi_n\} = \{x^n\}$, $n = 0, 1, 2, \ldots$

<u>Apply Gram - Schmidt procedure.</u>

Polynomials	Interval	$w(x)$	Normalization
Legendre	$[-1, 1]$	1	$\int_{-1}^{1} P_n^2 dx = \frac{2}{2n+1}$
Shifted Legendre	$[0, 1]$	1	$\int_{-1}^{1} \tilde{P}_n^2 dx = \frac{1}{2n+1}$
Chebyshev	$[-1, 1]$	$(1-x^2)^{-1/2}$	$\int_{-1}^{1} \frac{T_n^2 dx}{\sqrt{1-x^2}} = \begin{cases} \pi/2, & n \neq 0 \\ \pi, & n = 0 \end{cases}$
Laguerre	$[0, \infty)$	e^{-x}	$\int_{0}^{\infty} L_n^2 e^{-x} dx = 1$
Associated Laguerre	$[0, \infty)$	$x^k e^{-x}$	$\int_{0}^{\infty} (L_n^k)^2 x^k e^{-x} dx = \frac{(n+k)!}{n!}$
Hermite	$(-\infty, \infty)$	e^{-x^2}	$\int_{-\infty}^{\infty} H_n^2 e^{-x^2} dx = 2^n \sqrt{\pi}\, n!$

<u>Bessel's Inequality.</u> -

We know that if a set of eigenfunctions $\{\psi_n\}$ is complete, then any well-behaved function $\Psi(x)$ may be expressed as

$$\Psi(x) = \sum_n a_n \psi_n$$

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[In QM, Ψ may be general solution as a consequence of superposition principle.]

Now, if $\{\psi_n\}$ is not complete set, we have Bessel's inequality

$$\int_\Omega |\Psi(x)|^2 w(x)\,dx \geq \sum_n a_n^2 \,, \quad w(x) \text{ is real function.}$$

Proof- Consider the integral

$$\int_\Omega \left|\Psi(x) - \sum_n a_n \psi_n(x)\right|^2 w(x)\,dx \geq 0 \qquad [\because w(x) > 0]$$

Clearly, equality holds for complete set.

Expanding, $\int_\Omega \left[|\Psi|^2 + \sum_{n,m} a_n a_m^* \psi_n \psi_m^* - \sum_n a_n^* \Psi \psi_n^* - \sum_n a_n \Psi^* \psi_n\right] w(x)\,dx \geq 0$

$$\Rightarrow \int_\Omega |\Psi|^2 w(x)\,dx + \sum_n |a_n|^2 - \sum_n a_n^* \int_\Omega \left[\sum_m a_m \psi_m \psi_n^*\right] w(x)\,dx$$

$$- \sum_n a_n \int_\Omega \left[\sum_m a_m^* \psi_m^* \psi_n\right] w(x)\,dx \geq 0 \qquad \left[\because \int_\Omega \psi_n \psi_m^* w(x)\,dx = \delta_{mn} \text{ (orthogonality)}\right]$$

$$\Rightarrow \int_\Omega |\Psi|^2 w\,dx + \sum_n |a_n|^2 - 2\sum_n |a_n|^2 \geq 0$$

$$\Rightarrow \int_\Omega |\Psi|^2 w\,dx \geq \sum_n |a_n|^2$$

equality holds for complete set. The relation for equal sign is $\int_\Omega |\Psi|^2 w(x)\,dx = \sum_n |a_n|^2$

which is called Parseval relation.

$$\int_\Omega |\Psi(x)|^2 w(x)\,dx \geq \sum_n a_n^2 \,, \quad w(x) \text{ is real function.}$$

Schwartz Inequality - (Dirac Bra-ket notation)

$$|\langle \Psi | \Phi \rangle|^2 \leq \|\Psi\|^2 \, \|\Phi\|^2$$

or, $|\langle \Psi | \Phi \rangle|^2 \leq \langle \Psi | \Psi \rangle \langle \Phi | \Phi \rangle$

(Normal notation)

$$\left|\int_\Omega \Psi^* \Phi\, w\,dx\right|^2 \leq \left(\int_\Omega |\Psi|^2 w\,dx\right)\left(\int_\Omega |\Phi|^2 w\,dx\right)$$

equality holds if Ψ, Φ are linearly dependent.

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Proof - Construct $F(x) = \Psi(x) + \lambda\,\Phi(x)$, $\lambda \in \mathbb{C}$

Then $0 \leq \int_\Omega |F|^2 w\,dx = \int_\Omega \left[|\Psi|^2 + |\lambda|^2 |\Phi|^2 + \lambda^* \Psi \Phi^* + \lambda \Psi^* \Phi \right] w\,dx$ ——— (1)

equality holds if $F \equiv 0$ i.e., Ψ, Φ are linearly dependent.

Considering $I(\lambda) = \int_\Omega |F(x;\lambda)|^2 w(x)\,dx$, we wish to find value of λ for which $I(\lambda)$ is minimum.

$$\frac{dI}{d\lambda} = 0 \Rightarrow \lambda^* \int_\Omega |\Phi|^2 w\,dx + \int_\Omega \Psi^* \phi w\,dx = 0 \quad [\lambda, \lambda^* \text{ are independent}]$$

$$\Rightarrow \lambda^* = \frac{-\langle \Psi, \Phi \rangle}{\langle \Phi, \Phi \rangle} \quad \text{——— (2a)}$$

$$\therefore \lambda = -\frac{\langle \Psi, \Phi \rangle^*}{\langle \Phi, \Phi \rangle} \quad \left[\begin{array}{l} \langle \Phi | \Phi \rangle \equiv \|\Phi\|^2 \text{ is real.} \\ \langle \Psi | \Phi \rangle^* \equiv \int_\Omega \Psi \Phi^* w\,dx \end{array} \right]$$

(1) is valid $\forall \lambda$.

Substituting values of λ, λ^* from (2a) into (1),

$$\int_\Omega \left[|\Psi|^2 + \frac{|\langle \Psi | \Phi \rangle|^2}{\langle \Phi | \Phi \rangle^2} |\Phi|^2 - \frac{\langle \Psi | \Phi \rangle}{\langle \Phi | \Phi \rangle} \Psi \Phi^* - \frac{\langle \Psi | \Phi \rangle^*}{\langle \Phi | \Phi \rangle} \Psi^* \Phi \right] w\,dx \geq 0$$

$$\Rightarrow \langle \Psi | \Psi \rangle + \cancel{\frac{\langle \Psi | \Phi \rangle^2}{\langle \Phi | \Phi \rangle}} - \frac{\langle \Psi | \Phi \rangle \langle \Phi | \Psi \rangle}{\langle \Phi | \Phi \rangle} - \cancel{\frac{|\langle \Psi | \Phi \rangle|^2}{\langle \Phi | \Phi \rangle}} \geq 0$$

[Note: $\langle \Psi | \Psi \rangle \equiv \int_\Omega |\Psi|^2 w\,dx$, $\langle \Psi | \Phi \rangle \equiv \int_\Omega \Psi^* \Phi w\,dx$, etc.

Also, $\langle \Psi | \Phi \rangle^* = \langle \Phi | \Psi \rangle$]

$$\Rightarrow \langle \Psi | \Psi \rangle \langle \Phi | \Phi \rangle \geq |\langle \Psi | \Phi \rangle|^2$$

Green's Function -

Dirac's Delta function $\delta(x-y)$ is defined by the properties

(1) $\delta(x-y) = 0$; $x \neq y$ and infinite at $x = y$.

(2) $\int_{y-\epsilon}^{y+\epsilon} \delta(x-y) f(x)\,dx = f(y)$, f is well-behaved.

which includes $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

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Representations of Delta function -

① By Orthogonal functions $\{\psi_n\}$, $\int_\Omega \psi_n \psi_m w \, dx = \delta_{mn}$

Assuming $\delta(x-y) = \sum_{n=0}^{\infty} a_n(y)\, \psi_n(x)$,

Multiply by ψ_m and integrate,

$$a_m(y) = \int_\Omega \delta(x-y)\, \psi_m(x)\, dx = \psi_m(y)$$

$$\Rightarrow \delta(x-y) = \sum_n \psi_n(y)\, \psi_n(x)$$

② Integral Representation -

$$\delta(x-y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{2w(x-y)}\, dw$$

$$\delta(x-y) = \delta(y-x) \quad \text{(even)}$$

Consider following eq$^{\underline{n}}$ with a non-homogeneous term on a 3-dimensional space:

$$A\psi + \lambda w \psi = -P, \quad A \text{ is Hermitian operator} \quad \text{—— ①}$$

Then solution may be expressed as an integral:

$$\psi(\vec{r}) = \int_\Omega G(\vec{r}, \vec{r}\,') P(\vec{r}\,')\, d^3\vec{r}\,' \quad \text{—— ②}$$

where, $G(\vec{r}, \vec{r}\,')$ is Green's function satisfying

$$A G(\vec{r}, \vec{r}\,') + \lambda w(\vec{r})\, G(\vec{r}, \vec{r}\,') = -\delta(\vec{r} - \vec{r}\,') \quad \text{—— ③}$$

$G(\vec{r}, \vec{r}\,')$ may be represented in terms of orthogonal set of eigenfunctions $\{\psi_n\}$, where

$$A\psi_n = -\lambda_n w \psi_n, \quad n = 0, 1, 2, \ldots$$

Assume $G(\vec{r}, \vec{r}\,') = \sum_n C_n(\vec{r}\,')\, \psi_n(\vec{r})$

Substitute into ③,

$$(A + \lambda w) \sum_n C_n(\vec{r}\,')\, \psi_n(\vec{r}) = -\sum_n \psi_n(\vec{r})\, \psi_n(\vec{r}\,')$$

$$\Rightarrow \sum_n C_n(\vec{r}\,') \left[-\lambda_n w \psi_n(\vec{r}) + w \lambda \psi_n(\vec{r}) \right] = -\sum_n \psi_n(\vec{r})\, \psi_n(\vec{r}\,')$$

Multiply by $\psi_m(\vec{r})$ and integrate,

$$C_m(\vec{r}\,')\, w(\lambda - \lambda_m) = -\psi_m(\vec{r}\,') \Rightarrow C_n(\vec{r}\,') = -\frac{\psi_n(\vec{r}\,')}{w(\lambda - \lambda_n)}$$

$$\therefore G(\vec{r}, \vec{r}\,') = \sum_n \frac{\psi_n(\vec{r})\, \psi_n(\vec{r}\,')}{w(\vec{r})(\lambda_n - \lambda)}$$

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Since, f is continuous on $[a,b]$, we have $f(x+\theta h) \to f(x)$ as $h \to 0$. $\therefore f(x+h) - f(x) = h f(x+\theta h)$.

$$\Rightarrow \frac{F(x+h)-F(x)}{h} = f(x+\theta h) \Rightarrow \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(x+\theta h) = f(x)$$

This theorem states that integration process improves the function i.e., if f is continuous implies F has the stronger property of differentiability.

<u>Theorem</u>: Let $f \in R[a,b]$. $x \in [a,b]$, define $F(x) = \int^x f(t)\,dt$. Then F is continuous on $[a,b]$.

<u>Proof</u> - Let $c \in [a,b]$ and $\epsilon > 0$. Then for all $x \in [a,b]$ with $x < c$, $|F(x) - F(c)| = \left| \int_a^x f(t).dt - \int_a^c f(t)\,dt \right| = \left| \int_c^x f(t)\,dt \right|$

$$|F(x) - F(c)| \leq \sup_{t\in[a,b]} |f(t)| \left| \int_c^x 1.dt \right|$$

$$f \leq M, \int_a^b f \leq \int_a^b M\,dt = M\int_a^b dt = M(b-a)$$

Given $\epsilon > 0$, choose $\delta = \epsilon/M$, then we have,

$$|F(x) - F(c)| \leq M|x-c| \leq M\epsilon/M < \epsilon \quad \forall\ |x-c| < \delta \quad x \in (c-\delta, c+\delta)$$

$\Rightarrow F \in C[a,b]$ when every $|x-c| < \delta = \epsilon/M$

(*) Given a continuous function f on $[a,b]$, let ϕ be an antiderivative of f i.e., $\phi' = f$. Then if we define-

$$F(x) = \int_a^x f(t)\,dt \quad \text{we have, } F'(x) = f(x) = \phi'(x);\ \forall\ x \in [a,b]$$

$$\Rightarrow [F - \phi]'(x) = 0 \quad \forall\ x \in [a,b]$$

$\Rightarrow F - \phi = c$, constant $\Rightarrow F = c + \phi$

and it follows that, $\int_a^x f(t)\,dt = F(x) = \phi(x) + c$ for some constant c.

Put $x = a$ then, $0 = \int_a^a f(t)\,dt = F(a) = \phi(a) + c \Rightarrow \phi(a) = -c$

Put $x = b$ then, $\int_a^b f(t)\,dt = \phi(b) + c \Rightarrow \phi(b) - \phi(a)$.

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Theorem: If $f \in R[a,b]$, $M = \sup_{x\in[a,b]} f(x)$ and $m = \inf_{x\in[a,b]} f$,

then $m(b-a) \leq \int_a^b f \leq M(b-a)$

Proof- Let $P := \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a,b]$.

Then, $m\delta_i \leq m_i\delta_i \leq M_i\delta_i \leq M\delta_i$

$\Rightarrow \sum_{i=1}^{n} m\delta_i \leq \sum_{i=1}^{n} m_i\delta_i \leq \sum_{i=1}^{n} M_i\delta_i \leq \sum_{i=1}^{n} M\delta_i$

$\Rightarrow m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$ $\left[\because \sum_{i=1}^{n} \delta_i = b-a\right]$

$\forall P \in P[a,b]$

$\Rightarrow m(b-a) \leq L(P,f) \leq \int_a^b f \leq U(P,f) \leq M(b-a)$ $\left[\because \int_a^b f = \sup_{P\in P[a,b]}\{L(P,f)\} = \inf_{P\in P[a,b]}\{U(P,f)\}\right]$

$m(b-a) \leq \int_a^b f \leq M(b-a)$

Theorem: (Mean Value theorem for integrals)-

Let f be a continuous function on $[a,b]$. Then $\exists\ c \in [a,b]$

$\ni \int_a^b f(x)\,dx = f(c)(b-a)$. f continuous on $[a,b]$.

$f(a) \leq \gamma \leq f(b)$ $\exists\ c \in [a,b]$. $\ni f(c) = \gamma$.

Proof- Since, f is continuous on $[a,b]$, we have $f \in R[a,b]$.

$\Rightarrow m(b-a) \leq \int_a^b f \leq M(b-a) \Rightarrow m \leq \frac{1}{b-a}\int_a^b f \leq M$

where, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$

Then, by intermediate value theorem, $\exists\ c \in [a,b] \ni$

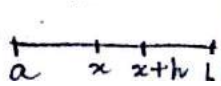
$f(c) = \frac{1}{b-a}\int_a^b f \Rightarrow \int_a^b f = f(c).(b-a)$.

Theorem: (Fundamental theorem of integral calculus)-

Let f be a continuous function on $[a,b]$. For each $x \in [a,b]$, define $F(x) = \int_a^x f(t)\,dt$. Then, F is differentiable in $[a,b]$ and $F'(x) = f(x)$; $\forall\ x \in [a,b]$.

Proof- Let $x, x+h \in [a,b]$. Then, $F(x+h) - F(x) = \int_a^{x+h} f(t).dt - \int_a^x f(t)\,dt$

$= \int_x^{x+h} f(t).dt = (x+h-x)f(x+\theta h)$; $0 \leq \theta \leq 1$



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$\Rightarrow |f|(x) \leq M_i - m_i + |f|(y) \quad \forall \; x, y \in I_i$

$\Rightarrow A_i \leq M_i - m_i + |f|(y) \quad \forall \; y \in I_i$

$\Rightarrow A_i \leq M_i - m_i + a_i \quad \Rightarrow A_i - a_i \leq M_i - m_i \quad \forall \; i = 1, 2, \ldots, n$

$\because f \in R[a,b]$ for given $\epsilon > 0$, choose $P \in P[a,b]$

$\ni U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\delta_i < \epsilon$

Now, $U(P,|f|) - L(P,|f|) = \sum_{i=1}^{n} (A_i - a_i)\delta_i \leq \sum_{i=1}^{n} (M_i - m_i)\delta_i < \epsilon$

$\therefore |f| \in R[a,b]$

$\because f \leq |f|$ and, $-f = (-1)f < |f|$

$\Rightarrow \int_a^b f \leq \int_a^b |f|$ and, $-\int_a^b f \leq \int_a^b |f|$

$\Rightarrow \left| \int_a^b f \right| \leq \int_a^b |f|$

(iv) We show that if $f \in R[a,b]$, then $f^2 \in R[a,b]$.

Since, $f^2 = |f|^2$, we assume $f(x) \geq 0 \quad \forall \; x \in [a,b]$.

If $f(x) = 0 \quad \forall \; x \in [a,b]$, then nothing to prove.

So assume $f(x) \neq 0$, then $M = \sup_{x \in [a,b]} f(x) > 0$.

$P_i = \sup_{x \in I_i} f^2(x)$; $p_i = \inf_{x \in I_i} f^2(x)$; Then, $P_i = M_i^2$ and, $p_i = m_i^2$

$$P_i - p_i = M_i^2 - m_i^2 = (M_i - m_i)(M_i + m_i) \leq 2M_i(M_i - m_i)$$
$$\leq 2M(M_i - m_i)$$

Since, $f \in R[a,b]$, given $\epsilon > 0$, choose $P \in P[a,b] \ni$

$$U(P,f) - L(P,f) < \epsilon/2M$$

Then, $U(P,f^2) - L(P,f^2) = \sum (P_i - p_i)\delta_i \leq 2M \sum (M_i - m_i)\delta_i$

$$= 2M[U(P,f) - L(P,f)] \leq \epsilon.$$

$\Rightarrow f^2 \in R[a,b]$

Since, $f \cdot g = \dfrac{(f+g)^2 - (f-g)^2}{4}$

and, $f \neq g \in R[a,b]$, $(f+g)^2, (f-g)^2 \in R[a,b]$.

implies that, $f \cdot g \in R[a,b]$

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(ii) Proof - If $\underline{k=0}$, then, $\int_a^b kf = 0 = k\int_a^b f$

If $\underline{k \neq 0}$, assume $k > 0$,

then, $\sup_{I_i} kf = kM_i$ and $\inf_{I_i} kf = km_i$ and,

$$U(P, kf) = \sum_{i=1}^{n} kM_i\delta_i = k\sum_{i=1}^{n} M_i\delta_i = k\,U(P,f)$$

Similarly, $L(P, kf) = \sum km_i\delta_i = k\sum m_i\delta_i = k\,L(P,f)$

Now, Since, $f \in R[a,b]$, choose $P \in P[a,b]$

$\ni U(P,f) - L(P,f) < \epsilon/k \Rightarrow k[U(P,f) - L(P,f)] < \epsilon$

$\Rightarrow U(P,kf) - L(P,kf) < \epsilon \Rightarrow kf \in R[a,b]$, for $k>0$.

Now, $\int_a^b kf = \inf_P U(P,kf) = \inf_P k\,U(P,f)$; $P \in P[a,b]$

$\Rightarrow \int_a^b kf = k \inf U(P,f)$ $(\because k>0)$

$= k\int_a^b f$; for $k>0$.

Now, if $\underline{k<0}$,

If $k<0$, say $k = -l$, $l>0$.

$0 = \int_a^b [(-l)f + lf] = \int_a^b (-l)f + \int_a^b lf$

$\therefore \int_a^b kf = \int_a^b (-l)f = -l\int_a^b f = k\int_a^b f$

$\therefore \int_a^b kf = k\int_a^b f$ for all $k \in \mathbb{R}$

Now, $f-g = f + (-1)g$

$g \in R[a,b] \Rightarrow (-1)g \in R[a,b] \Rightarrow f + (-1)g \in R[a,b]$

and, $\int_a^b f + (-1)g = \int_a^b f + \int_a^b (-1)g = \int_a^b f + (-1)\int_a^b g$

$\Rightarrow \int_a^b (f-g) = \int_a^b f - \int_a^b g$

(iii) Now, we prove $f \in R[a,b] \Rightarrow |f| \in R[a,b]$

Let $A_i = \sup_{I_i} |f|(x)$; $a_i = \inf_{I_i} |f|(x)$

Now, for all $x, y \in I_i$, we have,

$|f|(x) - |f|(y) = |f(x)| - |f(y)| \leq |f(x) - f(y)| \leq M_i - m_i$

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Let $\epsilon > 0$ be given, since $f, g \in R[a,b]$. Choose $P \in P[a,b]$ such that $U(P,f) - L(P,f) < \epsilon/2$ and, $U(P,g) - L(P,g) < \epsilon/2$.

Then, $U(P, f+g) - L(P, f+g) = \sum_{i=1}^{n} (Q_i - q_i)\delta_i$

$$\leq \sum_{i=1}^{n} (M_i - m_i)\delta_i + \sum_{i=1}^{n} (P_i - p_i)\delta_i \quad [\text{by } (*)]$$

$$= [U(P,f) + U(P,g)] - [L(P,f) + L(P,g)] < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus, given $\epsilon > 0$ $\exists$ $P \in P[a,b]$ such that

$$U(P, f+g) - L(P, f+g) < \epsilon \Rightarrow f+g \in R[a,b].$$

Now, the def. of upper and lower Riemann sums, we have,

$$\left.\begin{aligned} L(P,f) &> \int_a^b f - \epsilon/2 \quad \text{and,} \\ L(P,g) &> \int_a^b g - \epsilon/2 \\ U(P,f) < \int_a^b f + \epsilon/2 \quad &\text{and,} \quad U(P,g) < \int_a^b g + \epsilon/2 \end{aligned}\right\} \text{---} (**)$$

Then from, $U(P, f+g) = \sum_{i=1}^{n} Q_i \delta_i \leq \sum_{i=1}^{n} (M_i\delta_i + P_i\delta_i)$

$$= U(P,f) + U(P,g)$$

We have, $\int_a^b (f+g) \leq U(P, f+g) \leq U(P,f) + U(P,g)$

$$< \int_a^b f + \epsilon/2 + \int_a^b g + \epsilon/2 \text{ ——— by } (**)$$

$$= \int_a^b f + \int_a^b g + \epsilon \text{ ——— } (\#)$$

Similarly, $\int_a^b (f+g) \geq L(P, f+g) > L(P,f) + L(P,g)$

$$> \int_a^b f - \epsilon/2 + \int_a^b g - \epsilon/2 = \int_a^b f + \int_a^b g - \epsilon \text{ ——— } (\#\#)$$

Now, from $(\#)$ and $(\#\#)$,

$$\left| \int_a^b f+g - \left[\int_a^b f + \int_a^b g\right] \right| < \epsilon$$

Since, ϵ is arbitrary, we have,

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

(3) Theorem

$\int_a^b f$

Proof - Let

of $[a,b]$

$\Rightarrow M_i \leq P_i$

Now, $U(P,$

$\Rightarrow U(P,$

$\Rightarrow \int_a^b f$

$\Rightarrow \int_a^b f$

Since, the

follows th

(4) Theorem

(i) f

(ii) kf

(iii) $|f|$

(iv) f

Proof - (i

$M_i =$

x

$P_i =$

x

Then, Q

$\therefore (Q_i$

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③ <u>Theorem</u>: If $f, g \in R[a,b]$ such that $f \leq g$ then
$$\int_a^b f \leq \int_a^b g$$

<u>Proof</u> - Let $P = \{a = x_0, x_1, \ldots, b = x_n\}$ be any partition of $[a,b]$. $M_i = \sup_{x \in I_i} f(x)$; $P_i = \sup_{x \in I_i} g(x)$ $\forall\, i = 1, 2, \ldots n$
where, $I_i = [x_{i-1}, x_i]$

$\Rightarrow M_i \leq P_i \quad \forall\, i$

Now, $U(P,f) = \sum_{i=1}^{n} M_i \delta_i \leq \sum_{i=1}^{n} P_i \delta_i = U(P,g)$

$\Rightarrow U(P,f) < U(P,g) \quad \forall\, P = P[a,b]$

$\Rightarrow \int_a^b f = \inf\{U(P,f) \mid P \in P[a,b]\} < U(P,f) < U(P,g)$
$\forall\, P \in P[a,b]$

$\Rightarrow \int_a^b f < U(P,g) \quad \forall\, P \in P[a,b]$

Since, this is true for any partition P of $[a,b]$, it follows that $\int_a^b f \leq \inf_{P \in P[a,b]} U(P,g) = \int_a^b g$

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④ <u>Theorem</u>: Let $f, g \in R[a,b]$. Then

(i) $f \pm g \in R[a,b]$ and, $\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$

(ii) $kf \in R[a,b]$, for any $k \in \mathbb{R}$ and $\int_a^b kf = k \int_a^b f$.

(iii) $|f| \in R[a,b]$ and, $\left|\int_a^b f\right| \leq \int_a^b |f|$

(iv) $f \cdot g \in R[a,b]$.

<u>Proof</u> - (i) Let $P = \{a = x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$.

$M_i = \sup_{x \in I_i} f(x)$; $m_i = \inf_{x \in I_i} f(x)$; $P_i = \sup_{x \in I_i} g(x)$;

$p_i = \inf_{x \in I_i} g(x)$; $Q_i = \sup_{x \in I_i} (f+g)(x)$; $q_i = \inf_{x \in I_i} (f+g)(x)$

Then, $Q_i \leq M_i + P_i$ and, $q_i \geq m_i + p_i$

$\therefore (Q_i - q_i) \leq (M_i - m_i) + (P_i - p_i)$; $i = 1, 2, \ldots, n$ ——— (*)

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$$L(P_2, f) > \int_b^c f - \epsilon/2 \quad \text{and,} \quad U(P_2, f) < \int_b^c f + \epsilon/2$$

$$\Rightarrow -L(P_1, f) - \epsilon/2 < -\int_a^b f < -U(P_1, f) + \epsilon/2 \quad \text{——— (ii)}$$

$$-L(P_2, f) - \epsilon/2 < -\int_b^c f < -U(P_2, f) + \epsilon/2 \quad \text{——— (iii)}$$

Consider (i) + (ii) + (iii),

$$U(P, f) - [L(P_1, f) + L(P_2, f)] - 2\epsilon < \left[\int_a^c f - \int_a^b f - \int_b^c f\right]$$

$$< L(P, f) - [U(P_1, f) + U(P_2, f)] + 2\epsilon$$

$$\Rightarrow U(P, f) - L(P, f) - 2\epsilon < \left[\int_a^c f - \int_a^b f - \int_b^c f\right] < L(P, f) - U(P, f) + 2\epsilon$$

$$\Rightarrow -2\epsilon < \int_a^c f - \int_a^b f - \int_b^c f < 2\epsilon$$

Since, this is true for any $\epsilon > 0$, we have

$$\int_a^c f = \int_a^b f + \int_b^c f \quad \text{———} \quad (*)$$

$$\left[\text{From } \left| \int_a^c f - \left[\int_a^b f + \int_b^c f\right] \right| < 2\epsilon \quad ; \quad 0 \leq a_1 < \epsilon \right]$$

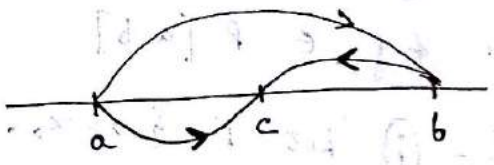
(*) For $a < b$, if we define,

$$\int_a^a f = 0 \quad ; \quad \int_b^a f = -\int_a^b f \quad , \text{ we have}$$

Then we extend the validity of integral (*) as: (a) if $a \leq c \leq b$.

$$\int_a^b f = \int_a^c f + \int_c^b f = \int_a^c f - \int_b^c f$$

$$\Rightarrow \int_a^c f = \int_a^b f + \int_b^c f$$



<u>Note</u>: If f and g are two functions on $[a, b]$ such that $f(x) \leq g(x)$ $\forall$ $x \in [a, b]$, then we write $f \leq g$ on $[a, b]$.

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Let P_1

P_2

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In either case, $U(P',f) - L(P',f) \leq U(P,f) - L(P,f) < \epsilon$.

Let $P_1 = P' \cap [a,b] = \{a = x_0, x_1, \ldots\ldots, x_m = b\}$

$P_2 = P' \cap [b,c] = \{b = x_m, x_{m+1}, \ldots\ldots, x_n = c\}$.

Then, $0 \leq U(P_1, f) - L(P_1, f) = \sum_{i=1}^{m} (M_i - m_i)(x_i - x_{i-1})$

where, $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$; $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$

$$0 \leq U(P_1, f) - L(P_1, f) \leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

$$= U(P', f) - L(P', f) < \epsilon.$$

$\Rightarrow f \in R[a,b]$.

Similarly, $0 \leq U(P_2, f) - L(P_2, f) < \epsilon$.

Conversely, suppose that $f \in R[a,b]$ and $f \in R[b,c]$.

This implies $\exists$ partitions P_1 of $[a,b]$ and P_2 of $[b,c]$

$\Rightarrow U(P_1, f) - L(P_1, f) < \epsilon/2$

$U(P_2, f) - L(P_2, f) < \epsilon/2$.

Let $P = P_1 \cup P_2$. Then $P \in [a,c]$.

Then, $U(P,f) - L(P,f) = [U(P_1,f) + U(P_2,f)] - [L(P_1,f) + L(P_2,f)]$

$= [U(P_1,f) - L(P_1,f)] + [U(P_2,f) - L(P_2,f)]$

$< \epsilon/2 + \epsilon/2 = \epsilon$.

$\Rightarrow f \in R[a,c]$.

---

(2) Now, with the partitions P, P_1 and P_2 as above, we have, $U(P,f) < \int_a^c f + \epsilon$ and $L(P,f) > \int_a^c f - \epsilon$.

$\Rightarrow U(P,f) - \epsilon < \int_a^c f < L(P,f) + \epsilon$ ——— (i)

Similarly, $L(P_1,f) > \int_a^b f - \epsilon/2$ and,

$U(P_1,f) < \int_a^b f + \epsilon/2$

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In each of these subintervals, ~~in $[a,b]$ that are formed by deleting the above p-subint~~ f has only finite number of discontinuity. (Otherwise for the interval which contains an infinite number of points of discontinuities of f will have a limit point by Bolzano-Weierstrass theorem.)

Since, f is Riemann integrable in each of these $(p+1)$ subintervals I_i', $i = 1, 2, \ldots p+1$, $\exists$ a partition P_i of I_i' $\ni$ $U(P_i, f) - L(P_i, f) < \epsilon/2(p+1)$ $\quad i = 1, 2, \ldots, p+1$.

Therefore, total contribution of these $(p+1)$ subintervals I_i', $i = 1, \ldots, p+1$ to the oscillatory sum is

$$\sum_{i=1}^{p+1} \left[U(P_i, f) - L(P_i, f)\right] < \sum_{i=1}^{p+1} \frac{\epsilon}{2(p+1)} = \frac{\epsilon}{2}.$$

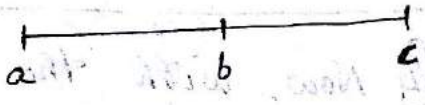
Now, let $P = \{a, \ldots, a_1, b_1 \ldots a_2, b_2 \ldots a_3, \ldots, b_{p-1} \ldots a_p, b_p \ldots b\}$

$$U(P, f) - L(P, f) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\Rightarrow f \in R[a,b]$.

Properties of Riemann Integration –

(1) **Theorem:** Let $a, b, c \in R$ and suppose that $a < b < c$. Let $f : [a,c] \to R$ be a bounded function. Then, $f \in R[a,c]$ iff $f \in R[a,b]$ and $f \in [b,c]$. If the integral exists, then $\int_a^c f = \int_a^b f + \int_b^c f$.



Proof – Let $\epsilon > 0$ be given.

Suppose $f \in R[a,c]$. Then $\exists$ $P \in P[a,c] \ni U(P,f) - L(P,f) < \epsilon$.

The partition P may or may not contain the point b.

Let $P' = P \cup \{b\} = P$ if $b \in P$

otherwise, $P \subset P'$

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We have,

$$U(P,f) - L(P,f) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, for each $\epsilon > 0$ $\exists$ $P \in \mathcal{P}[a,b]$ $\ni$ $U(P,f) - L(P,f) < \epsilon$

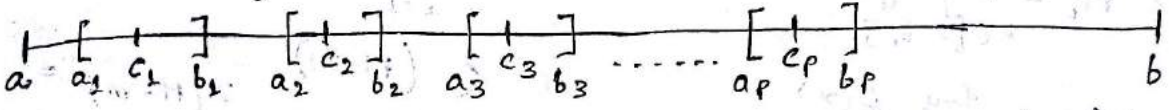
$\Rightarrow f \in \mathcal{R}[a,b]$.

<u>Theorem</u>: If the set of points of discontinuity of a bounded function $f : [a,b] \to \mathbb{R}$ has finite number of limit points, then $f \in \mathcal{R}[a,b]$.

<u>Proof</u>- Let $c_1, c_2, \ldots, c_p$ be the finite number of limit points of the set of discontinuity of f on $[a,b]$

$\ni c_1 < c_2 < \ldots\ldots < c_p$.

Let $\epsilon > 0$ be given.



Enclose $c_1, c_2, \ldots, c_p$ by non-overlapping sub-intervals $I_i = [a_i, b_i]$; $i = 1, 2, \ldots, p$ such that

$$\sum_{i=1}^{p} (b_i - a_i) < \frac{\epsilon}{2(M-m)} \quad \text{———} \quad (*)$$

where, $M = \sup\limits_{x \in [a,b]} f(x)$; $m = \inf\limits_{x \in [a,b]} f(x)$

Let $M_i = \sup\limits_{x \in [a_i, b_i]} f(x)$; $m_i = \inf\limits_{x \in [a_i, b_i]} f(x)$

Now, contribution of these p sub intervals $I_i = [a_i, b_i]$ $i = 1, 2, \ldots, p$ to the oscillatory sum is

$$\sum_{i=1}^{p} (M_i - m_i)(b_i - a_i) < (M-m) \sum_{i=1}^{p} (b_i - a_i)$$

$$< (M-m) \cdot \epsilon / 2(M-m) \quad \text{———} \quad \text{by } (*)$$

$$= \epsilon/2.$$

The $(p+1)$ subintervals in $[a,b]$ that are formed by deleting the above p-subintervals are $I_1' = [a, a_1]$, $I_2' = [b_1, a_2]$, ..., $I_p' = [b_{p-1}, a_p]$, $I_{p+1}' = [b_p, b]$.

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Let $\epsilon > 0$ be given.

Enclose p points $c_1, c_2, \ldots c_p$ by p-non-overlapping sub-intervals - $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2] \ldots I_p = [a_p, b_p]$

$\ni$ sum of their lengths $\sum_{i=1}^{p} (b_i - a_i) < \dfrac{\epsilon}{2(M-m)}$

where, $M = \sup_{x \in [a,b]} f(x)$; $m = \inf_{x \in [a,b]} f(x)$

The total contribution of these p-sub intervals $I_1, I_2, \ldots, I_p$ to the oscillatory sum $= \sum_{i=1}^{p} (M_i - m_i)(b_i - a_i)$

$$< \sum_{i=1}^{p} (M-m).(b_i - a_i) = (M-m)\sum_{i=1}^{p}(b_i -$$

where, $M_i = \sup_{x \in [a_i, b_i]} f(x)$

$$< (M-m).\frac{\epsilon}{2(M-m)} = \epsilon/2.$$

$m_i = \inf_{x \in [a_i, b_i]} f(x)$

The remaining $(p+1)$ subintervals in $[a,b]$ that are formed by deleting the above p-subintervals are

$I_1' = [a, a_1]$; $I_2' = [b_1, a_2]$; $I_3' = [b_2, a_3] \ldots$; $I_p' = [b_{p-1}, a_p]$

$I_{p+1}' = [b_p, b]$

Since, f is continuous in each of these sub intervals, $\exists$ a partition P_r of I_r', $r = 1, 2, 3, \ldots, p+1$ $\ni$ part of the oscillatory sum arising from each of these $(p+1)$ subintervals is $< \dfrac{\epsilon}{2(p+1)}$

i.e., $\exists$ a partition of I_r, $r = 1, 2, \ldots, p+1$ $\ni$

$$U(P_r, f) - L(P_r, f) < \frac{\epsilon}{2(p+1)}$$

Total contribution

$$\therefore \sum_{r=1}^{p+1} [U(P_r, f) - L(P_r, f)] < \sum_{r=1}^{p+1} \frac{\epsilon}{2(p+1)} = \frac{\epsilon}{2\cancel{(p+1)}}.\cancel{(p+1)} = \frac{\epsilon}{2}.$$

~~Let~~ Thus, for the partition -

$P = \{a, \ldots a_1; b_1, \ldots a_2; b_2, \ldots a_3; \ldots; b_p \ldots b\}$ of $[a,b]$,

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Theorem: If $f:[a,b]\to R$ is monotonic, then $f\in R[a,b]$.

Proof - Let f be monotonically increasing on $[a,b]$.

Then $f(a)\leq f(x)\leq f(b)$, $\forall\ x\in[a,b]$

$\Rightarrow f(x)\in[f(a),f(b)]$, $\forall\ x\in[a,b]$.

$\therefore \inf\limits_{x\in[a,b]} f(x) = f(a)$; $\sup\limits_{x\in[a,b]} f(x) = f(b)$.

Let $\epsilon>0$ be given and $P=\{a=x_0,x_1,x_2,\ldots,x_n=b\}$ be a partition of $[a,b]$, such that $\delta_r<\dfrac{\epsilon}{(f(b)-f(a)+1)}$

$r=1,2,\ldots,n$.

Now, $m_r=\inf\limits_{x\in I_r} f(x)=f(x_{r-1})$; $M_r=\sup\limits_{x\in I_r} f(x)=f(x_r)$

Consider, $U(P,f)-L(P,f)=\sum\limits_{r=1}^{n}(M_r-m_r)\delta_r$

$$=\sum_{r=1}^{n}\left(f(x_r)-f(x_{r-1})\right)\delta_r$$

$$<\sum_{r=1}^{n}\left(f(x_r)-f(x_{r-1})\right)\times\frac{\epsilon}{f(b)-f(a)+1}\quad(\text{by }\circledast)$$

$$=\frac{\epsilon}{f(b)-f(a)+1}\cdot\sum_{r=1}^{n}\left(f(x_r)-f(x_{r-1})\right)$$

$$=\frac{f(b)-f(a)}{f(b)-f(a)+1}\times\epsilon<\epsilon.$$

Therefore, $f\in R[a,b]$.

Similarly, we can prove if f is monotonically decreasing $U(P,f)-L(P,f)<\epsilon$.

Theorem: If the set of points of a discontinuity of a bounded function $f:[a,b]\to R$ is finite, then $f\in R[a,b]$.

Proof - Let $c_1,c_2,\ldots,c_p$ be a finite number of discontinuity of f on $[a,b]$ $\ni$ $c_1<c_2<\ldots<c_p$.

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Classes of some bounded integrable functions -

Theorem: If $f : [a,b] \longrightarrow R$ be a continuous function, then $f \in R[a,b]$.

Proof - Let f be continuous on $[a,b]$.

$\Rightarrow f$ is uniformly continuous on $[a,b]$.

Hence, for each $\epsilon > 0$, $\exists\ \delta > 0 \ni$

$|f(x') - f(x'')| < \epsilon/(b-a)$ —— (1) $\forall\ x', x'' \in [a,b]$ and, $|x' - x''| < \delta$

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be a partition of $[a,b]$ such that $\|P\| < \delta$.

Since, f is continuous on $[a,b]$, it is bounded.

$\Rightarrow f$ is also bounded in each sub interval $I_r = [x_{r-1}, x_r]$ and attains its bounds at some $c_r, d_r \in I_r \ni$

$$m_r = f(c_r) = \inf_{x \in I_r} f(x)$$

$$M_r = f(d_r) = \sup_{x \in I_r} f(x)$$

Now, $|c_r - d_r| \leq |x_r - x_{r-1}| < \delta$.

Then, by (1), we have $|f(c_r) - f(d_r)| < \epsilon/(b-a)$

Also, $M_r - m_r = |f(d_r) - f(c_r)| < \epsilon/(b-a)$

Consider, $U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$

$$< \sum_{r=1}^{n} \epsilon/(b-a) \cdot \delta_r = \frac{\epsilon}{(b-a)} \cdot \sum_{r=1}^{n} \delta_r$$

$$= \frac{\epsilon}{\cancel{(b-a)}} \cdot \cancel{(b-a)} = \epsilon$$

Thus, given $\epsilon > 0$ $\exists\ P \in P[a,b] \ni U(P,f) - L(P,f) < \epsilon$

$\Rightarrow f \in R[a,b]$

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Theorem: A bounded function $f \in R[a,b]$ iff for each $\epsilon > 0$, $\exists$ a partition $P \in P[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.

Proof- Let $f \in R[a,b]$

Then, $\underline{\int_a^b} f(x).dx = \overline{\int_a^b} f(x).dx = \int_a^b f(x).dx$

Let $\epsilon > 0$ be given.

Since, $\overline{\int_a^b} f(x).dx = \inf\{U(P,f) \mid P \in P[a,b]\}$

and, $\underline{\int_a^b} f(x).dx = \sup\{L(P,f) \mid P \in P[a,b]\}$

implies $\exists$ $P_1, P_2 \in P[a,b]$ such that -

$$U(P_1,f) < \overline{\int_a^b} f(x).dx + \epsilon/2 = \int_a^b f(x).dx + \epsilon/2 \quad \text{------ (i)}$$

and, $$L(P_2,f) > \underline{\int_a^b} f(x).dx - \epsilon/2 = \int_a^b f(x).dx - \epsilon/2 \quad \text{------ (ii)}$$

Let $P = P_1 \cup P_2$. then $P_1, P_2 \subset P$

Then from (i) and (ii), we have,

$$U(P,f) \leq U(P_1,f) \leq \int_a^b f(x).dx + \epsilon/2 < L(P_2,f) + \epsilon/2 + \epsilon/2$$

$$\Rightarrow U(P,f) - L(P,f) < \epsilon.$$

Conversely, let $\epsilon > 0$ be given. Let the partition $P \in P[a,b]$ such that - $U(P,f) - L(P,f) < \epsilon$ ------ (*)

$$\because L(P,f) < \underline{\int_a^b} f(x).dx \leq \overline{\int_a^b} f(x).dx \leq U(P,f)$$

$$\Rightarrow \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P,f) - L(P,f) < \epsilon \quad (\text{by } (*))$$

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Note: No lower Riemann Sum can exceed any Upper Riemann Sum.

Proof - Let $P_1, P_2 \in P[a,b]$ and $P = P_1 \cup P_2$

Then, $L(P_1, f) \leq L(P, f)$

$U(P, f) \leq U(P_2, f)$

also, $L(P, f) \leq U(P, f)$

$\Rightarrow L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f)$

$\Rightarrow L(P_1, f) \leq U(P_2, f) \quad \forall \; P_1, P_2 \in P[a,b]$

Theorem: If $f : [a,b] \longrightarrow \mathbb{R}$ is bounded function, then

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x).dx$$

Proof - Let $P_1, P_2 \in P[a,b]$. Then

$L(P_1, f) \leq U(P_2, f)$

$$\left[\begin{array}{l} \{L(P_1, f) \mid P_1 \in P[a,b]\} \leq U(P_2, f) \\ \Rightarrow \sup\{L(P_1, f) \mid P_1 \in P[a,b]\} \leq U(P_2, f) \end{array}\right]$$

Since, this is true for each partition $P_1 \in P[a,b]$ keeping the partition P_2 fixed and the set $\{L(P_1, f) \mid P_1 \in P[a,b]\}$ has an upper bound $U(P_2, f)$. It follows that - $\underline{\int_a^b} f(x).dx = \sup\{L(P_1, f) \mid P_1 \in P[a,b]\} \leq U(P_2, f)$ $\forall \; P_2 \in P[a,b]$.

This implies the set - $\{U(P_2, f) \mid P_2 \in P[a,b]\}$ has a lower bound $\underline{\int_a^b} f(x).dx$.

Since, this is true for $\forall \; P_2 \in P[a,b]$, it follows that

$$\underline{\int_a^b} f(x).dx \leq \inf\{U(P_2, f) \mid P_2 \in P[a,b]\} = \overline{\int_a^b} f(x).dx$$

$$\Rightarrow \boxed{\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x).dx}$$

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A one-dimensional problem using Green's function -

(A different approach)

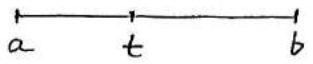
$$L\Psi(x) = -f(x) \quad , \quad L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \quad \text{(self-adjoint)}$$

on $x \in [a,b]$ with b.c.

$$\Psi(a) = 0 \quad , \quad \Psi'(a) = 0 \quad \text{or,} \quad \alpha\Psi(a) + \beta\Psi'(a) = 0$$
$$\Psi(b) = 0 \quad , \quad \Psi'(b) = 0 \quad \text{or,} \quad \alpha\Psi(b) + \beta\Psi'(b) = 0$$

We define Green's function $G(x,t)$ on $[a,b]$ with the properties

① $$G(x,t) = \begin{cases} G_1(x) & , a \leq x < t \\ G_2(x) & , t < x \leq b \end{cases}$$



② $$L\,G_1(x) = 0 \quad , \quad a \leq x < t$$
$$L\,G_2(x) = 0 \quad , \quad t < x \leq b$$

③ $$G_1(a) = 0 \quad , \quad G_1'(a) = 0 \quad \text{or,} \quad \alpha G_1(a) + \beta G_1'(a) = 0$$
$$G_1(b) = 0 \quad , \quad G_1'(b) = 0 \quad \text{or,} \quad \alpha G_1(b) + \beta G_1'(b) = 0$$

④ $G(x,t)$ is continuous: $\lim_{x\to t^-} G_1(x) = \lim_{x\to t^+} G_2(x)$

⑤ $G'(x,t)$ is discontinuous: $\frac{d}{dx}G_2(x)\Big|_t - \frac{d}{dx}G_1(x)\Big|_t = -\frac{1}{p(t)}$

To construct Green's function, let $u(x)$ and $v(x)$ be two linearly independent solutions of homogeneous equation.

$L\Psi(x) = 0$, where $W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \neq 0$ is the Wronskian

and $u(x), v(x)$ satisfy B.C. $\alpha u(a) + \beta u'(a) = 0$, $\alpha v(b) + \beta v'(b) = 0$.

We take, $$G(x,t) = \begin{cases} c_1 u(x) & , a \leq x < t. \\ c_2 v(x) & , t < x \leq b. \end{cases}$$

c_1, c_2 are to be determined.

Using continuity condition, $c_2 v(t) - c_1 u(t) = 0$ —— (i)

Using discontinuity condition, $c_2 v'(t) - c_1 u'(t) = -\frac{1}{p(t)}$ —— (ii)

(i) and (ii) has non-trivial solution for c_1, c_2 since $W(u,v) \neq 0$.

We see that,

$$p(x)u'' + p'(x)u' + q(x)u = 0 \quad ; \quad p(x)v'' + p'(x)v' + q(x)v = 0$$

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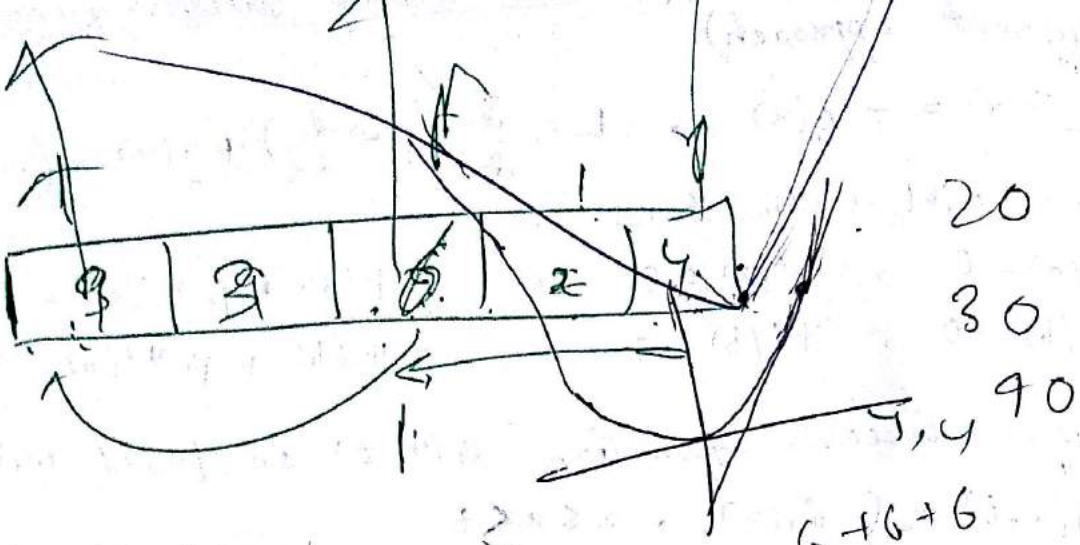

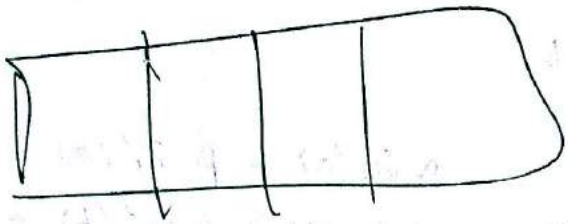

6+6+6
1186
226
336
446
556
666

Now, $W'(u,v) = (uv' - u'v)' = uv'' - u''v$

$$= \frac{u}{p}(-p'v' - qv) - \frac{1}{p}v(-p'u' - qu)$$

$$= \frac{p'}{p}(vu' - uv') = -\frac{p'}{p}W(u,v)$$

$$\Rightarrow \ln W(u,v) = -\ln p + A \Rightarrow W(u,v) = \frac{A}{p}$$

Then, $c_1 = -\dfrac{v(t)}{A}$, $c_2 = -\dfrac{u(t)}{A}$

$$\therefore G(x,t) = \begin{cases} -\frac{1}{A}u(x)v(t), & a \le x < t \\ -\frac{1}{A}u(t)v(x), & t < x \le b \end{cases}$$

Then, Solution of inhomogeneous eqn. $L\psi(x) = -f(x)$ is

$$\psi(x) = \int_a^b G(x,t)\, f(t)\, dt$$

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