Fluid Mechanics.

1. Fluid Mechanics by Kundu & Cohen
   : Academic Press Publication

* Chapter 1: Basic Concepts in Fluid Mechanics.

Fluid

incompressible compressible

(i) Liquids are usually incompressible because their volumes do not change, when pressure changes.

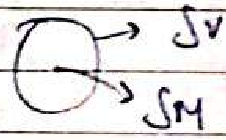
(ii) Gases are compressible. their volume changes when pressure changes.

→ Continuum Hypothesis: We assume that fluid is uniformly / macroscopically distributed in a region.

→ Isotropy: A fluid is said to be isotropic w.r.t some properties (pressure, velocity, density) if that property remains unchanged in all directions. If these properties change it is called anisotropic. (wrt Time)

→ Density: of a fluid is mass per unit volume.

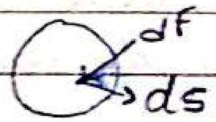
Mathematically, $\rho = \lim_{\delta V \to 0} \frac{\delta M}{\delta V}$



specific weight $\gamma = \rho g$

→ Pressure: is defined as Force per unit area.

$$P = \lim_{\delta S \to 0} \frac{\delta F}{\delta S}$$



→ Temperature:

→ Thermal Conductivity: It is given by Fourier's law

$$q_x \propto \frac{\partial T}{\partial x}$$

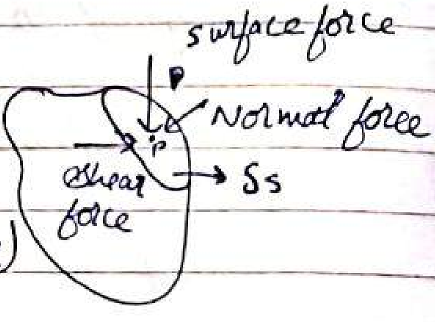
heat $\leftarrow q_x = -k \frac{\partial T}{\partial x}, \xrightarrow{3D} q_x = -k \nabla T$

where q_x is the conductive heat flow per unit area, k is the thermal conductivity

etc.)

→ Viscous & inviscous fluid :-

An infinitessimal fluid element is acted upon by 2 types of forces. Body force (contact force) & surface force



Body force is proportional to the mass of the body & surface force is proportional to surface of the body

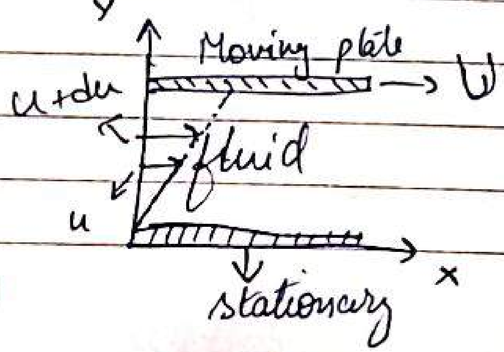
Forces of element can be resolved into:-

(i) Surface force
(ii) Normal force
(iii) Shear force / Shear Stress

(a) A fluid is said to be viscous if both normal & shear forces are present.

(b) A fluid is inviscous / inviscal / friction less / perfect / ideal if it does not exert any shear stress

→ Viscosity :- Viscosity of a fluid is that property which exhibits a certain resistance to alteration of form.



Upper plate moves with velocity U in x-direction whereas lower plate is stationary.

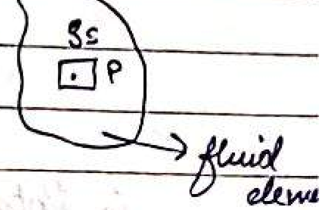
The fluid at $Y=0$ is at rest & $Y=h$ is in motion and moving with plate.

Then the shear stress τ is given by $\tau = \mu \frac{du}{dy}$ where μ is a constant of proportionality & is called a viscosity of the fluid.

For:

Ideal fluid $\tau = 0 \Rightarrow \mu = 0$

Two types of forces exist on the fluid element :-



1. Body Force : It is distributed over the entire mass or volume of the element. It is expressed as the per unit ~~force~~ mass of the element. Ex: Gravity

2. Surface Force : Forces exerted on the fluid element by it's surrounding ~~thorough~~ through direct contact with the surface.

$$\text{Surface Force} \begin{cases} \text{Normal Force} \\ \text{Shear Force} \end{cases}$$

a. Normal Force : along the normal to the surface area
b. Shear Force : along the plane of the surface area
$\tau = \mu \, du/dy$

3. Laminar / streamline flow : A flow in which each fluid particle traces out a definite curve and the curves traced by any 2 different particles do not intersect.

$\boxed{\text{Reynold's No. Low}}$

4. Turbulent Flow : Curves intersect & are not definite.

$\boxed{\text{Reynold's No. High}}$

5. Steady & Unsteady Flow : A fluid in which properties such as pressure, velocity etc. are independent of time t; i.e. $\frac{d(\text{property})}{dt} = 0$. Such flows are called steady flows.

6. Rotational & Irrotational Flow : A flow in which the fluid is rotating about their own axis.

Fluid Flow :-

$$\bar{q} \quad \frac{d\vec{q}}{dt}$$

Initially at $t=0$ let the fluid particle is at P_0. The current position,

$$x = f(x_0, y_0, z_0, t)$$
$$y = g(x_0, y_0, z_0, t)$$
$$z = h(x_0, y_0, z_0, t)$$

$\longrightarrow$ Lagrangian Approach
-Requires history or initial position.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

(Material Description)

Eulerian Approch :-

We fix a point in space, P and we look how a fluid particles is behaving at that point.

(Spatial Description)

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Relationship between Lagrangian and Eulerian Approach :-

1. Lagrangian $\rightarrow$ Eulerian

Suppose $\phi(x_0, y_0, z_0, t)$ be a Lagrangian description of a physical quantity ϕ. $\phi = \phi(x_0, y_0, z_0, t)$

Since Lagrangian gives current position as

$$\left.\begin{aligned} x &= f_1(x_0, y_0, z_0, t) \\ y &= f_2(x_0, y_0, z_0, t) \\ z &= f_3(x_0, y_0, z_0, t) \end{aligned}\right\} \quad \text{Find} \quad \begin{aligned} x_0 &= g_1(x, y, z, t) \\ y_0 &= g_2(x, y, z, t) \\ z_0 &= g_3(x, y, z, t) \end{aligned}$$

Therefore $\phi = \phi(x_0, y_0, z_0, t)$
$= \phi(g_1, g_2, g_3, t)$

is the corresponding Eulerian Description

Ex: $x = x_0^2$; $y = y_0^2$, $z = z_0^2$

$$\Rightarrow \phi = \phi(\sqrt{x}, \sqrt{y}, \sqrt{z}, t)$$

2. Eulerian → Lagrangian

Suppose $\psi(x, y, z, t)$ be a physical quantity associated with the flow

$$\psi = \psi(x, y, z, t)$$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$= F_1(x, y, z, t)\,\hat{i} + F_2(x, y, z, t)\,\hat{j} + F_3(x, y, z, t)\,\hat{k}$$

$$\rightarrow \quad \frac{dx}{dt} = F_1(x, y, z, t)$$

$$\Rightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} F_1(x, y, z, t)\,dt$$

$$\Rightarrow x(t) = x(t_0) + \int_{t_0}^{t} F_1(x, y, z, t)$$

|||ly, $y(t)$, $z(t)$

Ex: The velocity components for a 2-dimensional fluid system can be given by Eulerian description as

$$u = 2x + 3y + 3t$$
$$v = x + y + t/2$$

Find Lagrangian desc. of the fluid.

$$\frac{dx}{dt} = 2x + 3y + 3t \quad = x'$$

$$\frac{dy}{dt} = x + y + t/2 \quad = y'$$

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$\Rightarrow$ ~~$x'' = 2x' + 3y' + 3$~~

$\Rightarrow$ $(D-2)x = 3y + 3t$ — (4)

$(D-1)y = x + t/2$ — (5) (5)×(D-2)

$\Rightarrow$ $(D^2 - 3D + 2)y - (D-2)x = \frac{1}{2} - t$ — (6)

~~$(D^2 - 3D + 2)y - (D-2)x =$~~

$\Rightarrow$ $(D^2 - 3D + 2)y = 3y + 2t - 1/2$

$\Rightarrow$ $(D^2 - 3D)y = 2t - 1/2$

$\Rightarrow$ $y(t) = C_0 + C_2 e^{3t} - t^2/3 - 7t/18$

↳ PI + CF

$\Rightarrow$ $dy/dt = 3C_2 e^{3t} - 2/3\, t - 7/18$

$dy/dt = x + y + t/2$

$\Rightarrow$ $x = 3C_2 e^{3t} - 4/3\, t - 7/18 - y - t/2$

$= -C_1 + 2C_2 e^{3t} + t^2/3 - 7t/9 - 7/18$

Initially, $t = t_0$, $x(t_0) = x_0$, $y(t_0) = y_0$ | $t_0 = 0$

$y_0 = C_1 + C_2$

$x_0 = -C_1 + 2C_2 - 7/18$

$\Rightarrow$ $C_1 = (2y_0 - x_0)/3 - 7/54$

$C_2 = (x_0 + y_0)/3 + 7/54$

* ~~Problem~~ :- Common Operators :-

$$\nabla = \partial/\partial x\,\hat{i} + \partial/\partial y\,\hat{j} + \partial/\partial z\,\hat{k} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$$

$$\vec{\nabla}\cdot\vec{\nabla} = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

$$\vec{\nabla}(f\cdot\vec{g}) = \vec{\nabla} f\,\vec{g} + f\,\vec{\nabla}\cdot\vec{g}$$

*

(i) Gauss Div Theorem : Let S be a surface bounding a volume V, and $\hat{n}$ be the unit outward drawn normal then.

$$\int_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dV$$

(ii) Stoke's Theorem : Let S be the surface bounded by a closed curve C, then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\,\hat{n}\,ds$$

(iii) Green's Theorem : Let ϕ & ψ be continuously differentiable function then,

$$\int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\psi}{\partial n}\right)ds$$

$$= \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dv$$

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* 1. Cartesian System :-

$$\nabla = (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$$

2. Spherical Polar System :-

$$x = r\sin\theta\cos\phi \ ,\quad y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r \geq 0\ ,\quad 0 \leq \theta \leq \pi\ ,\quad 0 \leq \phi \leq 2\pi$$

3. Cylindrical System :-

$$x = r\cos\theta\ ,\ y = r\sin\theta\ ,\quad z = z$$

$$r \geq 0\ ,\quad 0 \leq \theta \leq 2\pi\ ,\quad -\infty \leq z \leq \infty$$

* Material, local & Convective derivatives :-

Suppose a fluid particle moves from $P(x,y,z)$ to $Q(x+\delta x,\ y+\delta y,\ z+\delta z)$ in time $t+\delta t$. Further let be some fluid property associated with the flow. Let the total change of the fluid property from P to Q be

$$\delta f = \partial f/\partial x \cdot \delta x + \partial f/\partial y \cdot \delta y + \partial f/\partial z \cdot \delta z + \partial f/\partial t \cdot \delta t$$

$$\Rightarrow \delta f/\delta t = \sum_{x,y,z} \partial f/\partial x \cdot \delta x/\delta t + \partial f/\partial t$$

$$\Rightarrow \lim_{\delta t \to 0} \frac{\delta f}{\delta t} = \lim_{\delta t \to 0} [\quad]$$

$$\Rightarrow df/dt = \sum_{x,y,z} \partial f/\partial x \cdot dx/dt + \partial f/\partial t$$

$$= \left(\partial f/\partial x \,\hat{i} + \partial f/\partial y \,\hat{j} + \partial f/\partial z \,\hat{k}\right) \cdot \left(dx/dt \,\hat{i} + dy/dt \,\hat{j} + dz/dt \,\hat{k}\right) + \partial f/\partial t$$

$$\vec{\nabla} \cdot f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \cancel{\vec{\nabla} \cdot f \cdot \vec{q}} \;\; \vec{q} \cdot (\vec{\nabla} \cdot f) + \partial f/\partial t \qquad \vec{q} = \frac{d\vec{r}}{dt}$$

$$= \cancel{df/dt} \left(\frac{\partial}{\partial t} + \cancel{\vec{\nabla} \cdot \vec{q}} \;\; \vec{q} \cdot \vec{\nabla}\right) f$$

$\therefore$ Material Derivative

$$\frac{Df}{Dt} = \frac{df}{dt} = \left(\frac{\partial}{\partial t} + \cancel{\vec{\nabla} \cdot \vec{q}} \;\; \vec{q} \cdot \vec{\nabla}\right) f$$

Local Derivative

Convective flux

Material Derivative (D/Dt, d/dt) is ~~called~~ spoken as of differentiation following the motion of the fluid. $\partial/\partial t$ is the local derivative, and it is associated with time variation at a fixed point.

$\vec{q} \cdot \vec{\nabla}$ is called Convective Derivative and it is associated with the change in the fluid property f due to motion of the fluid.

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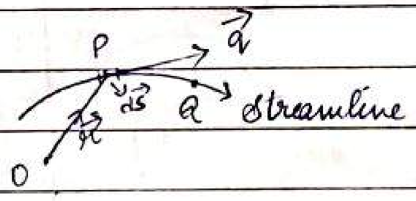
* Line of Flow :-

A line whose direction coincides with the direction of the resultant velocity of the fluid

* Stream lines :-

A streamline is a continuous line of flow drawn in the fluid so that the tangent at every point of it at any instant of time coincides with direction of motion of fluid.



Consider ds as an element of the stream line passing through the point P, st $\vec{OP} = \vec{r}$. Let $\vec{q}$ be the fluid velocity at that point.

The direction of the tangent and direction of velocity are same, i.e. parallel, i.e

$$\vec{ds} \times \vec{q} = \vec{0}$$

$\Rightarrow \quad (\hat{i}dx + \hat{j}dy + \hat{k}dz) \times (u\hat{i} + v\hat{j} + w\hat{k}) = \vec{0}$

$\Rightarrow \quad (wdy - vdz)\hat{i} + (udz - wdx)\hat{j} + (vdx - udy)\hat{k} = \vec{0}$

$\Rightarrow \quad wdy - vdz = 0 \quad \Rightarrow \quad dy/v = dz/w$

$\quad udz - wdx = 0 \quad \Rightarrow \quad dx/u = dz/w \quad \Rightarrow \quad \dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w}$

$\quad vdx - udy = 0 \quad \Rightarrow \quad dx/u = dy/v$

Streamline of the fluid at any point !

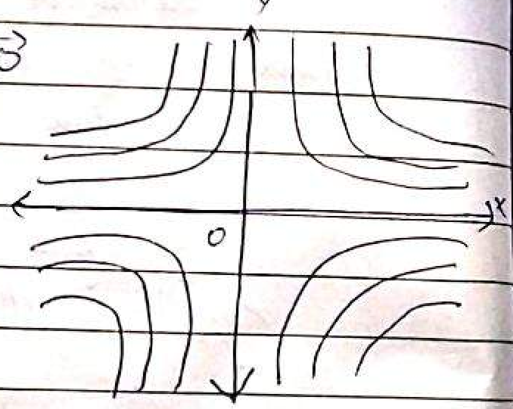
Ex: The velocity vector $\vec{q}$ is given by

$$\vec{q} = x\hat{i} - y\hat{j}$$

Determine streamline equation.

Soln: Here $\vec{q} = x\hat{i} - y\hat{j} + 0\hat{k}$.

$$\therefore \vec{q} \times d\vec{r} = 0$$
$$\Rightarrow (x\hat{i} - y\hat{j}) \times (dx\hat{i} + dy\hat{j}) = \vec{0}$$
$$\Rightarrow (xdy + ydx)\hat{k} = \vec{0}$$
$$\Rightarrow d(xy) = 0$$
$$\Rightarrow xy = c$$



Ex: The velocity of fluid is given by

$$\vec{q} = 2x\hat{i} - y\hat{j} - z\hat{k}.$$

Determine the equation of streamline passing through (1, 1, 1)

Soln: $\vec{q} \times d\vec{r} = 0$

$$\begin{vmatrix} i & j & k \\ 2x & -y & -z \\ dx & dy & dz \end{vmatrix} = 0$$

$$i(-ydz + zdy) - j(2xdz + zdx) + k(2xdy + ydx) = 0$$

① $\Rightarrow$ $\dfrac{z\,dy - y\,dz}{z^2} = 0/z^2$

$\Rightarrow$ $d(y/z) = 0$

$\Rightarrow$ $y = c_1 z$

② $\Rightarrow$ $2x\,dz + z\,dx = 0$

$\Rightarrow$ $\dfrac{2\,dz}{z} = -\dfrac{dx}{x}$

$\Rightarrow$ $2\ln z = -\ln x + C$

$\Rightarrow$ $\ln z^2 = -\ln c_2 x$

$\Rightarrow$ $xz^2 = c_2$

At $(1,1,1)$: $c_1 = 1$, $c_2 = 1$

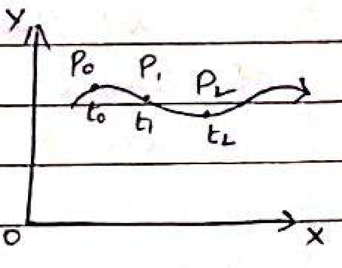
$\Rightarrow$ $y = z$; $xz^2 = 1$ [(or) $xy^2 = 1$]

represents the streamline at $(1,1,1)$

* Path lines :-

A curve described in space by a moving fluid ~~particle~~ (or) a line traced by a fluid particle

The path line is obtained by



$\vec{q} = \dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j} + \dfrac{dz}{dt}\hat{k}$

Ex: The velocity field at a point P in a fluid is given by

$$\vec{q} = (x/t, y, 0)$$ Obtain the path lines.

Soln: Here $u = x/t$, $v = y$, $z = 0$.

$\therefore$ The equation of the path lines :-

$$dx/dt = x/t \quad ; \quad dy/dt = y$$
$$\Rightarrow x = c_1 t \quad ; \quad y = c_2 e^t$$

At $t = t_0$, $x(t_0) = x_0$, $y(t_0) = y_0$

$$\Rightarrow c_1 = x_0/t_0$$
$$c_2 = y_0/e^{t_0}$$

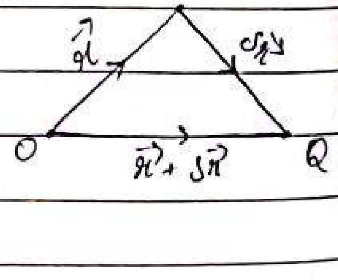
$\therefore$ The required eq^n of the path lines :-

$$x = \frac{x_0}{t_0} t \quad , \quad y = y_0 e^{t - t_0}$$

* Velocity & Acceleration of Fluid Particle :-

Let P be the position of the fluid particle at time t & Q be its position at $t + \delta t$.

ST $\vec{OP} = \vec{r}$, $\vec{OQ} = \vec{r} + \delta\vec{r}$, $\vec{PQ} = \delta\vec{r}$



The velocity $\vec{q}$ at P is given by

$$\vec{q} = \lim_{\delta t \to 0} \frac{\vec{r} + \delta\vec{r} - \vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t}$$

$$\vec{q} = \frac{d\vec{r}}{dt}$$

Acc. $\vec{a} = \frac{d\vec{q}}{dt} = \frac{d^2\vec{r}}{dt^2}$

Acceleration of fluid particle at P :-

$$\vec{a} = d/dt(\vec{q})$$

$$D/Dt = d/dt = \left(\partial/\partial t + \vec{q}\cdot\vec{\nabla}\right)$$

$$\therefore \vec{a} = d\vec{q}/dt$$

$$= \left(\partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z\right)\vec{q}$$

$$= \partial\vec{q}/\partial t + u\,\partial\vec{q}/\partial x + v\,\partial\vec{q}/\partial y + w\,\partial\vec{q}/\partial z$$

$$= \partial\vec{q}/\partial t + u\cdot\partial\vec{q}/\partial x + v\,\partial\vec{q}/\partial y + w\,\partial\vec{q}/\partial z$$

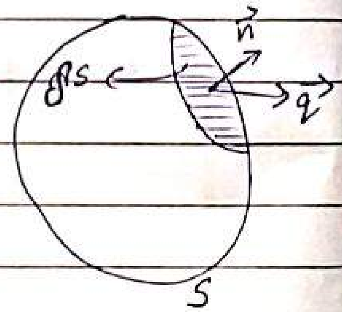
Ex: $\vec{q} = (Ax^2yt)\hat{i} + (Bxy^2t)\hat{j} + (Cxyz)\hat{k}$

$$\rightarrow d\vec{q}/dt = \partial\vec{q}/\partial t + u\,\partial\vec{q}/\partial x + v\,\partial\vec{q}/\partial y + w\,\partial\vec{q}/\partial z$$

* Equation of Continuity / Conservation of Mass

By continuity / conservation of mass, we mean that the fluid always remains a continuum, i.e. a continuously distributed matter. When a region of fluid containing no source or sink then the amount of fluid within the region is conserved in accordance with principle of conservation of mass.

fluid in - fluid out + source - sink = fluid accumulation



Consider an infinitesimal fluid element δs of volume δV and density ρ, which is situated a point whose position vector $\vec{r}$. Let $\vec{q}$ be the fluid velocity at the element δs then the normal component $\vec{q}$ measured outward from the volume V is $= \vec{q} \cdot \hat{n}$, where $\hat{n}$ is the unit vector drawn outward normal & V is the volume of the fluid in the closed region S fixed in space.

The mass of the fluid element = $\rho \delta V$. Throughout the motion, the mass of any fluid element must be conserved, hence the mass of any fluid element must remain unchanged as it moves about. This means,

$$\frac{D}{Dt}(\rho \delta V) = 0 \quad \text{—} (1)$$

Rate of mass flow across δS

~~per unit mass~~ $= \underbrace{\rho(\hat{n}\cdot\vec{q})}_{\cancel{\text{per unit}}}\,\delta S$

Total rate of mass flow through $V = -\int_S \rho(\hat{n}\cdot\vec{q})\,dS$

[-ve for inward flow as $\hat{n}$ represents outward normal]

By Gauss Divergence Theorem :-

$$= -\int_V \vec{\nabla}\cdot(\rho\vec{q})\,dV$$

Also, the rate of increase of mass within $V = \frac{\partial}{\partial t}\left[\int_V \rho\,dv\right]$

$$V = \int_V \partial\rho/\partial t\,dV$$

Now by principle of conservation of mass :-

$$\int_V \partial\rho/\partial t\,dV = -\int_V \vec{\nabla}\cdot(\rho\vec{q})\,dV$$

$$\Rightarrow \int_V \left(\partial\rho/\partial t + \vec{\nabla}\rho.\vec{q}\right)dV = 0 \quad \text{—— (2)}$$

(2) is valid for any arbitrary V provided it lies entirely in the fluid. $\therefore$ The integrand must be 0.

$$\Rightarrow \partial\rho/\partial t + \vec{\nabla}(\rho\cdot\vec{q}) = 0$$

— Equation of continuity / conservation of mass free from any source or sink.

$$\Rightarrow \partial\rho/\partial t + \vec{q}\,\vec{\nabla}\cdot\rho + \rho\,\vec{\nabla}\cdot\vec{q} = 0$$

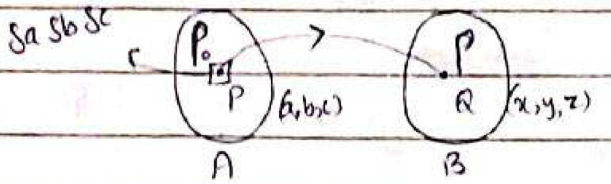
$$\Rightarrow D/Dt(\rho) + \rho\vec{\nabla}\cdot\vec{q} = 0$$

D/Dt : Material Derivative $\frac{\partial}{\partial t} + \vec{q}\cdot\vec{\nabla}$

For incompressible fluid, ρ is unchanged $\frac{D\rho}{Dt} = 0$

$$\Rightarrow \vec{\nabla}\cdot\vec{q} = 0$$

* Lagrange's Method (Eq. of continuity)



Let A be the fluid region occupied by a fluid at time $t=0$, and B is the region occupied by ~~same~~ the fluid at time t. Let ρ_0 & ρ be their densities respectively. Mass of the fluid at P at $t=0$ is $= \rho_0(\delta a\,\delta b\,\delta c)$ & at Q at t $= \rho(\delta x\,\delta y\,\delta y)$

Now, total mass should be same in both regions.

$$\rho_0(\delta a\,\delta b\,\delta c) = \rho(\delta x\,\delta y\,\delta z)$$

$$\Rightarrow \int_A \rho_0(\delta a\,\delta b\,\delta c) = \int_B \rho(\delta x\,\delta y\,\delta z)$$

$$\Rightarrow \iiint_A \rho_0 \, dadbdc = \iiint_B \rho \, dxdydz$$

$$\Rightarrow \iiint_A \rho_0\,\delta a\,\delta b\,\delta c = \iiint_A \rho \frac{\partial(x,y,z)}{\partial(a,b,c)}\,\delta a\,\delta b\,\delta c$$

$$\Rightarrow \iiint_A (\rho_0 - \rho J)\,dadbdc \qquad J = \frac{\partial(x,y,z)}{\partial(a,b,c)}$$

As A is arbitrary, we obtain

$$\rho_0 - \rho J = 0 \quad — (1)$$

$$\Rightarrow J = \rho_0/\rho$$

This is the required continuity equation in Lagrangian form.

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Differentiating ① w.r.t time :-

$$\frac{d\rho_0}{dt} = \rho \frac{dJ}{dt} + J\frac{d\rho}{dt}$$ ρ_0 is constant

$\Rightarrow \rho \, dJ/dt + J \, d\rho/dt = 0$ — ②

Now we change the variable from Lagrangian to Eulerian form.

$\partial u/\partial a = \partial/\partial a (dx/dt) = d/dt(\partial x/\partial a)$

Slly. $\partial v/\partial a = d/dt(\partial y/\partial a)$; $\partial w/\partial a = d/dt(\partial w/\partial a)$
+6 terms

Now, $d\rho/dt = (\partial/\partial t + \vec{q}\cdot\vec{\nabla})\rho$
$= \partial\rho/\partial t + u\,\partial\rho/\partial x + v\,\partial\rho/\partial y + w\,\partial\rho/\partial y$

Since, $J = \begin{vmatrix} \partial x/\partial a & \partial y/\partial a & \partial z/\partial a \\ \partial x/\partial b & \partial y/\partial b & \partial z/\partial b \\ \partial x/\partial c & \partial y/\partial c & \partial z/\partial c \end{vmatrix}$

$dJ/dt = \begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix} + \begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix} + \begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix}$

$= \begin{vmatrix} \partial u/\partial a & \partial y/\partial a & \partial z/\partial a \\ \partial u/\partial b & \partial y/\partial b & \partial z/\partial b \\ \partial u/\partial c & \partial y/\partial c & \partial z/\partial c \end{vmatrix} + \begin{vmatrix} & \partial v/\partial a & \\ & \partial v/\partial b & \\ & \partial v/\partial c & \end{vmatrix}$

$+ \begin{vmatrix} & & \partial w/\partial a \\ & & \partial w/\partial b \\ & & \partial w/\partial c \end{vmatrix}$

$$\Rightarrow dJ/dt = \frac{\partial(u,y,z)}{\partial(a,b,c)} + \frac{\partial(x,v,z)}{\partial(a,b,c)} + \frac{\partial(x,y,w)}{\partial(a,b,c)}$$

But, $\partial u/\partial a = \sum_{x,y,z} \left(\partial u/\partial x \cdot \partial x/a \right)$

$\partial w/\partial c$

$$\therefore \frac{\partial(u,y,z)}{\partial(a,b,c)} = \partial u/\partial x \times J$$

$$\therefore dJ/dt = \left(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z \right) J$$

$\therefore$ Substituting in ② :-

$$dP/dt \, J + P \times J \left(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z \right) = 0$$

$$\Rightarrow dP/dt + P \, \vec{\nabla} \cdot \vec{q} = 0$$

$\rightarrow$ Eulerial Eq of Continuity

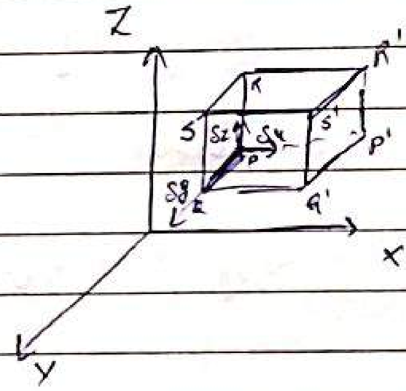
* Equation of Continuity :- (In different co-ordinate systems)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{q}) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{q}\,\nabla \rho + \rho\, \vec{\nabla}\cdot\vec{q} = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho\, \vec{\nabla}\cdot\vec{q} = 0$$

1. ⇒ Cartesian Coordinate System :-



Consider ρ be the density of the fluid at $P(x,y,z)$ and $\vec{q} = (u, v, w)$ be the fluid velocity. Construct a small rectangle ||piped in the fluid with length, breadth & height as $\delta x, \delta y, \delta z$ respectively.

Mass of the fluid that passes through PQRS $= \rho(\delta y\, \delta z)u$ per unit time

$= f(x,y,z)$ — (1)

Mass of the fluid that passes through the opposite side $= f(x+\delta x, y, z)$

$$= f + \delta x\, f' + \frac{\delta x^2}{2} f'' + \cdots$$

— (2) $\quad f' = \partial f/\partial x$, $\quad f'' = \partial^2 f/\partial x^2$

Mass of excess fluid within region $= -f(x+\delta x, y, z) + f(x,y,z)$

$$= -\delta x\, f' - \frac{\delta x^2}{2} f'' + \cdots$$

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$$\cong -\delta x \cdot \frac{\partial f}{\partial x} \quad \text{(Truncation)}$$

$$= -\delta x \frac{\partial}{\partial x}(\rho u \, \delta y \, \delta z)$$

$$= -\frac{\partial}{\partial x}(\rho u) \, \delta x \cdot \delta y \cdot \delta z$$

∴ Excess fluid along y-axis & z-axis :-

$$y\text{ axis} = -\partial/\partial y \, (\rho v) \, \delta x \cdot \delta y \cdot \delta z$$

$$z\text{-axis} = -\partial/\partial z \, (\rho w) \, \delta x \cdot \delta y \cdot \delta z$$

∴ The excess fluid along all directions

$$= -\left[\partial/\partial x (\rho u) + \partial/\partial y (\rho v) + \partial/\partial z (\rho w) \right] \delta x \, \delta y \, \delta z$$

The Total mass inside parallelopiped $= \rho \, (\delta x \cdot \delta y \, \delta z)$

The rate of change/increase mass inside the parallelopiped $= \partial/\partial t \, (\rho \, \partial x \cdot \partial y \cdot \partial z)$

By conservations laws,

Rate of mass accumulation = Rate of mass in − Rate of mass out

$$\Rightarrow \partial/\partial t \, (\rho \, \delta x \, \delta y \, \delta z) = -\left[\partial/\partial x (\rho u) + \partial/\partial y (\rho v) + \partial/\partial z (\rho w) \right] \delta x \, \delta y \, \delta z$$

$$\Rightarrow \partial\rho/\partial t\,(\delta x \cdot \delta y \cdot \delta z) = -\vec{\nabla}\cdot(\rho\vec{q})\,\delta x \cdot \delta y \cdot \delta z$$

$$\cancel{\Rightarrow \partial\rho/\partial t\,(\delta x \cdot \delta y \cdot \delta z) = -\nabla}$$

$$\Rightarrow \left(\partial\rho/\partial t + \vec{\nabla}\cdot(\rho\vec{q})\right)\delta x\,\delta y\,\delta z = 0$$

$$\Rightarrow \partial\rho/\partial t + \vec{\nabla}\cdot(\rho\vec{q}) = 0$$

2. In spherical coordinate system :-

$$\partial\rho/\partial t\ \cancel{\partial\rho/\partial\phi} + 1/r^2\ \partial/\partial r\,(\rho r^2 q_r)$$

$$+ 1/r\sin\theta\ \partial/\partial\theta\,(\rho q_\theta \cdot \sin\theta) + 1/r\sin\theta\ \partial/\partial\phi\,(\rho q_\phi) = 0$$

δz

$$\partial\rho/\partial t + \vec{\nabla}\cdot(\rho\vec{q}) = 0$$

$x = r\cos\theta\sin\phi$
$y = r\cos\theta\cos\phi$
$z = r\sin\theta$

3. In cylindrical polar system :-

$$\partial\rho/\partial t + 1/r\ \partial/\partial r\,(\rho r q_r) + 1/r\ \partial/\partial\theta\,(\rho q_\theta) + \partial/\partial t\,(\rho q_z) = 0$$

$x = r\sin\theta$, $y = r\cos\theta$, $z = z$

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Velocity potential :-

$$\vec{q} = -\nabla \phi$$

If $\vec{q}$ is the fluid velocity at any instant of time t, then the eqns of the streamline is given by

$$dx/u = dy/v = dz/w \quad —— (1)$$

These surfaces from (1) will intersect the surfaces

$$u\,dx + v\,dy + w\,dz = 0$$ orthogonally.

Consider a scalar function $\phi(x, y, z, t)$ at any instant of time t such that

$$u\,dx + v\,dy + w\,dz = -d\phi$$

$$= -\left(\partial\phi/\partial x \cdot dx + \partial\phi/\partial y \cdot dy + \partial\phi/\partial z \cdot dz\right)$$

$$\Rightarrow u = -\partial\phi/\partial x \quad v = -\partial\phi/\partial y \quad w = -\partial\phi/\partial z$$

$$\Rightarrow u\hat{i} + v\hat{j} + w\hat{k} = -\partial\phi/\partial x\,\hat{i} - \partial\phi/\partial y\,\hat{j} - \partial\phi/\partial z\,\hat{k}$$

$$\Rightarrow \vec{q} = -\vec{\nabla}\phi$$

If fluid is incompressible,

$$\vec{\nabla}\cdot\vec{q} = -\vec{\nabla}^2\phi = 0$$ } Laplace Equation

Laplacian of ϕ

Also, known as harmonic.

* The flow is called irrotational if $\vec{\nabla} \times \vec{q} = \vec{0}$

Proposition : The necessary & sufficient condition for $\vec{q} = -\nabla \phi$ iff

$$\vec{\nabla} \times \vec{q} = 0 \quad (\text{Curl of } q \text{ is } 0)$$

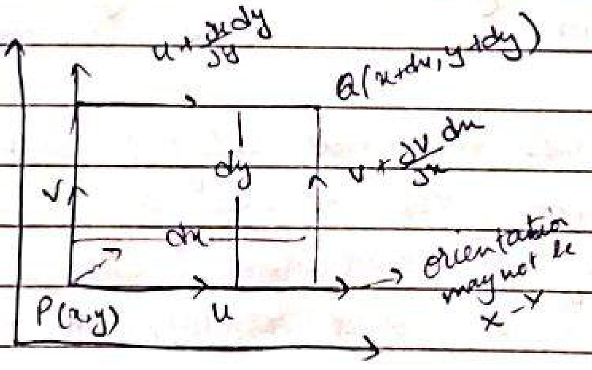
Proof :-

$$\vec{q} = -\vec{\nabla}\phi$$

$$\vec{\nabla} \times \vec{q} = -\left[\vec{\nabla} \times (\vec{\nabla}\phi)\right] = \vec{0}$$

* Rotational Flow :-

Consider a 2-dimensional flow in XY plane and at every point ~~p~~ $P(x,y)$. the velocity has 2 components u & v, i.e. $\vec{q} = (u,v)$. Let Q be the displaced position of P at any time t. Then the velocity components at Q $u + \partial u/\partial y \, dy$ & $v + \partial v/\partial x \, dx$



The velocities cause the fluid to rotate about an axis $\perp^r$ to xy-plane. The rotation ω_z about z-axis of the fluid element is determined as:-

$$\omega_z\Big|_y = \frac{\left(v + \partial v/\partial x \cdot dx\right) - v}{dx} = \partial v/\partial x \quad (\text{anticlockwise})$$

$$\omega_z\Big|_x = \frac{u - \left(u + \partial u/\partial y \, dy\right)}{dy} = -\partial u/\partial y \quad (\text{clockwise without } -ve)$$

The rotation ω_z is given by mean angular velocity

$$\omega_z = \frac{\left(\partial v/\partial x\right) + \left(-\partial u/\partial y\right)}{2}$$

$$= \frac{1}{2}\left(\partial v/\partial x - \partial u/\partial y\right)$$

If $\omega_z = 0 \rightarrow$ Irrotational

$\neq 0 \rightarrow$ Rotational $\quad$ +ve $\rightarrow$ anticlockwise

-ve $\rightarrow$ clockwise

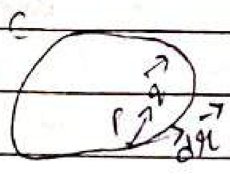
Hence, for Irrotational flow :-

$$\partial v/\partial x = \partial u/\partial y$$

* ~~Vorticity~~ :- <u>Circulation</u> & <u>Vorticity</u> :-

→ Circulation & Vorticity are 2 primary measures of rotation of a fluid.

Circulation : It is a scalar integral quantity which measures the rotation of a fluid particle along a closed curve. The circulation, C, about a closed contour/curve is given by :-

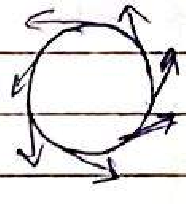


$$C = \oint_C \vec{q} \cdot d\vec{r}$$

→ Vorticity : The vorticity is a type of pseudovector field that describes the local spinning motion of a fluid element / continuum near some point (the tendency of fluid to rotate).

The vorticity vector is defined as :-

$$\vec{\omega} = \vec{\nabla} \times \vec{q}$$



→ Applying Stoke's theorem :-

$$\oint_C \vec{q} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{q}) \hat{n} \cdot dS$$

Stoke's theorem states that the circulation about any closed curve / contour is equal to the integral of the normal component of vorticity over the area enclosed by the contour.

→ Vortex lines : A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector $\vec{\omega}$.

Let $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ be the vorticity vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point P on a vortex line. Since the velocity will be in the same direction as the tangent, dr at P,

$$\vec{\omega} \times \vec{dr} = 0$$

$$\Rightarrow \omega_2 dz = \omega_3 dy$$

$$\omega_3 dx = \omega_1 dz \quad \Rightarrow \quad \frac{dx}{\omega_1} = \frac{dy}{\omega_2} = \frac{dz}{\omega_3}$$

$$\omega_1 dy = \omega_2 dx$$

Vortex lines

~~Examples~~

* Examples :-

1) Determine whether the motion

$$\vec{q} = \frac{A(x\hat{j} - y\hat{i})}{x^2+y^2}, \quad A = \text{const}$$

is incompressible or not. If it is incompressible, then determine streamlines. Also check whether it is irrotational. If yes, then find the potential function.

Incompressibility : $\vec{\nabla} \cdot \vec{q} = 0$

$$\vec{\nabla} \cdot \vec{q} = A \frac{(x^2+y^2)0 - y(2x)}{(x^2+y^2)^2} - A \frac{(x^2+y^2)0 - x(2y)}{(x^2+y^2)^2}$$

$$= 0$$

$\Rightarrow$ Incompressible

$\partial v/\partial x = \partial u/\partial y$

& ~~$\partial u/\partial x + \partial v/\partial y = 0$~~

→ Irrotational

Eq^n of streamline is $dx/u = dy/v = dz/0$

$\Rightarrow dz = 0 \Rightarrow z = \text{constant} = C_1$

Considering 1st 2 eqn's :-

$$dx/u = dy/v$$

$$dx/(-yA/x^2+y^2) = dy/(xA/x^2+y^2)$$

$$\Rightarrow x^2+y^2 = C_2$$

Stream line equations :-

$$x^2+y^2 = C_2$$
$$z = C_1$$

As it is irrotational, $\vec{q} = -\nabla\phi$

$$\cancel{\phi = \frac{+Ay}{x^2+y^2}}$$

$$\frac{-Ay}{x^2+y^2}\hat{i} + \frac{Ax}{x^2+y^2}j = -\frac{\partial\phi}{\partial x}i - \frac{\partial\phi}{\partial y}j$$

$$\Rightarrow \phi = \int\frac{Ay}{x^2+y^2}\partial x + f(y) \quad \& \quad \phi = \int\frac{-Ax}{x^2+y^2}\partial y + g(x)$$

$$= A\tan^{-1}(x/y) + f(y) \quad\quad = -A\tan^{-1}(y/x) + g(x)$$

$$\phi = A\tan^{-1}(x/y) + f(y)$$

$$\frac{\partial\phi}{\partial y} = \frac{-Ax}{x^2+y^2} + f'(y) = \frac{-Ax}{x^2+y^2}$$

$$\Rightarrow f'(y) = 0$$

$$\Rightarrow f(y) = C_1$$

$$\therefore \phi = A\tan^{-1}(x/y)$$

~~Ex~~ 2) Determine whether the velocity potential $\phi(x,y,z) = \frac{1}{2}a(x^2+y^2-2z^2)$ satisfies Laplace Equation. Also determine streamlines

$$\vec{q} = -\nabla\cdot\phi$$

$$= -\cancel{\tfrac{1}{2}}a\left(x\hat{i} + y\hat{j} - 2zK\right)$$

$$\vec{\nabla}\cdot\vec{q} = -a\left(1 + 1 - 2\right)$$

$$= 0$$ (Satisfies Laplace equation)

~~Solⁿ i) verify~~

Ex. 1) Determine whether the motion

$$\vec{q} = \frac{A(x\hat{j} - y\hat{i})}{(x^2+y^2)}$$

is incompressible or not. If it is incompressible then determine streamlines. Also check whether it's irrotational. If yes, then ~~fluid~~ find the potential function.

i) incompressible if $\vec{\nabla}\cdot\vec{q} = 0$

$$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{Ax\hat{j}}{x^2+y^2} - \frac{Ay\hat{i}}{x^2+y^2}\right)$$

$$= 0$$

The eqⁿ of stream line is.

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{0} \rightarrow z = \text{Constant} = C_1$$

$$\frac{dx}{\left(\frac{-Ay}{x^2+y^2}\right)} = \frac{dy}{\left(\frac{Ax}{x^2+y^2}\right)}$$

$$\boxed{x^2+y^2 = C_2, \quad z = C_1}$$

ii) Irrotational, $\vec{\nabla}\times\vec{q} = 0$

$$\vec{\nabla}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-Ay}{x^2+y^2} & \frac{Ax}{x^2+y^2} & 0 \end{vmatrix} = 0$$

iii) Velocity potential :

$\vec{q} = -\nabla\phi$ where ϕ is potential

$$\frac{-Ay}{x^2+y^2}\hat{i} + \frac{Ax}{x^2+y^2}\hat{j} = -\frac{\partial\phi}{\partial x}\hat{i} - \frac{\partial\phi}{\partial y}\hat{j}$$

$$\frac{\partial\phi}{\partial x} = \frac{Ay}{x^2+y^2}, \quad \frac{\partial\phi}{\partial y} = \frac{-Ax}{x^2+y^2}$$

$$\phi = A\tan^{-1}\frac{x}{y} + A(y)$$

$$\frac{\partial\phi}{\partial y} = A\cdot\frac{y^2}{x^2+y^2}\left(-\frac{x}{y^2}\right) + A'(y)$$

$$\frac{-Ax}{x^2+y^2} = \frac{-Ax}{x^2+y^2} + A'(y)$$

$\Rightarrow A'(y) = 0$

$\Rightarrow A(y) = C_3$

$$\boxed{\phi(x,y) = A\tan^{-1}\left(\frac{x}{y}\right) + C_3}$$

Ex.2. Determine whether, $\phi(x,y,z) = \frac{1}{2}a(x^2+y^2-2z^2)$ satisfies Laplace's equation. Also, determine streamline.

Laplace equation $\nabla^2\phi = 0$ (Harmonic Eqn)

$\Rightarrow \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$

$$\nabla^2\phi = \frac{1}{2}a(2+2-4) = 0$$

$\vec{q} = -\nabla\phi$ (velocity vector)

(Vorticity is a measurement of rotation)
Curl, tendency of rotation

$$\vec{q} = -\left[\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right]$$

$$\vec{q} = -\left[ax\hat{i} + ay\hat{j} + (-2az)\hat{k}\right]$$

$$\boxed{\vec{q} = -ax\hat{i} - ay\hat{j} + 2az\hat{k}}$$

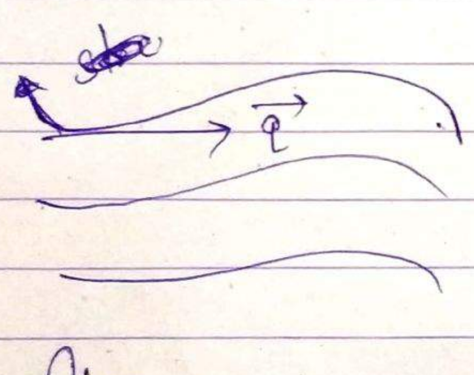
the stream lines are

$$\frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az}$$

$$C_1 x = y \qquad \frac{dy}{y} = -\frac{dz}{2z}$$

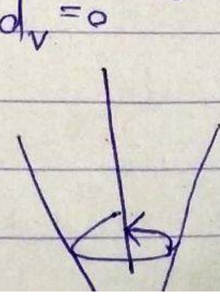
$$\ln(y) = -\frac{1}{2}\ln z + \ln c$$

$$\boxed{\begin{array}{l} y = C_1 x \\ y^2 z = C_2 \end{array}}$$



Vortex line :

a line to which vorticity vectors are tangent at all its points is called vortex lines $\vec{\omega} \times \vec{d_v} = 0$



stream line ki velocity zaruri nhi h
while Vorticity line have a velocity.

Find the vorticity components of a fluid particle whose velocity is given by-

$$\vec{q} = (K_1 x^2 y t)\hat{i} + (K_2 y^2_z t)\hat{j} + (K_3 z t^2)\hat{k}$$

$$\vec{\omega} = \vec{\nabla} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ K_1 x^2 y t & K_2 y^2_z t & K_3 z t^2 \end{vmatrix}$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(-K_1 x^2 t)$$

$-K_2 y^2 t$

$$-K_2 y^2 t\, \hat{i} - K_2 x^2 t\, \hat{j} \neq \vec{0}$$

$\Rightarrow$ the flow is rotational.

The vortex lines,

$$\vec{\omega} \times d\vec{r} = 0$$

$$\frac{dx}{\omega_1} = \frac{dy}{\omega_2} = \frac{dz}{\omega_3}$$

$$\Rightarrow \frac{dx}{-K_2 y^2 t} = \frac{dy}{0} = \frac{dz}{-K_1 x^2 t}$$

$dy = 0$

$$\boxed{y = C_1 = \text{Const}}$$

$$\frac{dx}{-K_2 y^2 t} = \frac{dz}{-K_1 x^2 t}$$

$$\int x^2 dx = \frac{K_2}{K_1} \int y^2 dz$$

$$\left.\begin{aligned} \frac{x^3}{3} &= \frac{K_2}{K_1} y^2 z + C_2 \\ y &= c_1 \end{aligned}\right\} - (1)$$

Determine the vortex lines of the flow where velocity is given by-

$\vec{q}(x,y,z) = \hat{i}(Az - By) + \hat{j}(Bx - Cz) + \hat{k}(Cy - Ax)$, where A, B, C are constants.

$$\vec{\nabla}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ Az-By & Bx-Cz & Cy-Ax \end{vmatrix} \quad \begin{array}{c} \partial/\partial x \\ Az-By \end{array}$$

$$= \hat{i}(C+C) + \hat{j}(A+A) + \hat{k}(2B)$$
$$= 2C\hat{i} + 2A\hat{j} + 2B\hat{k} \neq \vec{0}$$

$$\Rightarrow \frac{dx}{\cancel{2}C} = \frac{dy}{\cancel{2}A} = \frac{dz}{\cancel{2}B}$$

$\int A\,dx = \int C\,dy$ $\qquad$ $\int B\,dy = \int A\,dz$

$\boxed{Ax = Cy + C_1}$ $\qquad$ $\boxed{By = Az + C_2}$

$\boxed{Ax - Cy = C_1}$, $\boxed{By - Az = C_2}$

Streak lines: A streak line is defined as the locus of different particles passing through a fixed point, A streak line is a line on which lie all the fluid elements that at some earlier instant passed through a particle point isspace,

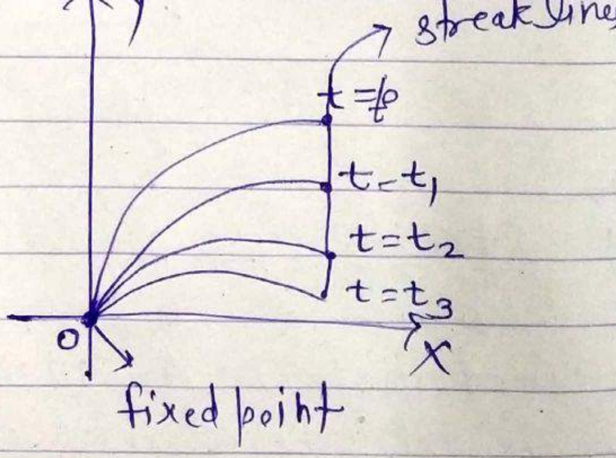
Consider fluid particle $P(x_0, y_0, z_0)$ passes through a fixed point $\vec{r_1}(x, y, z)$ in the course of time By Lagrangian description,

$$x_1 = f_1(x_0, y_0, z_0, t)$$
$$x_2 = f_2(x_0, y_0, z_0, t)$$
$$x_3 = f_3(x_0, y_0, z_0, t)$$

Solving for x_0, y_0, z_0

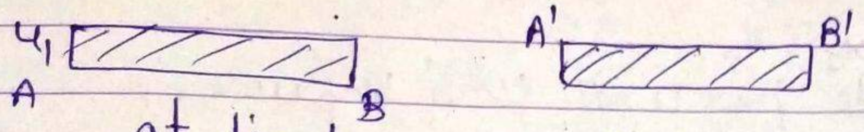
$$x_0 = F_1(x, y, z, t)$$
$$y_0 = F_2(x, y, z, t)$$
$$z_0 = F_3(x, y, z, t)$$

since the streak lines is the locus of the positions of the particles which have passes through the fixed point (x, y, z).
therefore, the streak lines at any instant of time.



linear strain rate :

A study of dynamics of fluid involves determination of forces on a fluid element which depends on the amount and nature of its deformation.



at time t $\qquad$ At t+dt.

$$A'B' = AB + BB' - AA'$$

The deformation of the fluid is similar to deformation in solid where one defines the normal strain as a change in length per unit length of the linear fluid element, and sheer strain as the change as of angle 90°.

Consider first the normal strain rate of a fluid element in x_1 direction is given,

$$\frac{1}{\delta x_1}\frac{D}{Dt}(\delta x_1) = \frac{1}{dt}\left(\frac{A'B' - AB}{AB}\right)$$

$$= \frac{1}{dt}\frac{1}{\delta x_1}\left[\delta x_1 + \frac{\partial u_1}{\partial x_1}\delta x_1 dt - \delta x_1\right] = \frac{\partial u_1}{\partial x_1}$$

$$\frac{1}{x_2}\frac{D}{Dt}(\delta x_2) = \frac{\partial u_2}{\partial x_2}$$

The general formulla for normal strain along x_α

$$= \frac{\partial u_\alpha}{\partial x_\alpha}, \quad \alpha = 1,2,3$$

The total normal strain $= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_\beta}{\partial x_\beta}$,

β is the dummy sumation convention

Alternatively,
consider a volume element of sides $\delta x_1, \delta x_2, \delta x_3$.
Define $\delta V = \delta x_1 \delta x_2 \delta x_3$.
Then the volume rate per unit unit volume is

$$\frac{1}{\delta x_1 \delta x_2 \delta x_3} \frac{D}{Dt}\left(\delta x_1 \delta x_2 \delta x_3\right)$$

$$= \frac{1}{\delta x_1 \delta x_2 \delta x_3} \frac{D(\delta x_1)}{Dt} \delta x_2 \delta x_3 + \frac{1}{\delta x_1 \delta x_2 \delta x_3} \frac{D}{Dt}(\delta x_2) \delta x_1 \delta x_3 + \frac{1}{\delta x_1 \delta x_2 \delta x_3} \frac{D}{Dt}(\delta x_3) \delta x_1 \delta x_2$$

$$= \frac{1}{\delta x_1} \frac{D}{Dt}(\delta x_1) + \frac{1}{\delta x_2} \frac{D}{Dt}(\delta x_2) + \frac{1}{\delta x_3}\left(\frac{D}{Dt}\right)(\delta x_3)$$

$$\frac{1}{\delta V} \frac{D}{Dt}(\delta V) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \qquad i = 1,2,3$$

In addition to undergoing normal strain rates, a fluid element may also simply deform in ~~space~~ shape. The shear strain rate of an element is ~~in~~ defined as the rate of decrese / increase of the angle formed by two mutually perpendicular lines on the element. The Deformation of the fluid element due strain rate tension is denoted by

$$\varepsilon_{ij} = e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) ; \quad i,j = 1,2,3.$$

... strain rate is given by -

$$\frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i}$$

$i=j$, $\varepsilon_{ii} \rightarrow$ principal stresses.

Boundary Conditions: Stream function

we used $\frac{D}{Dt}$ because it signifies that a specific fluid particle is followed, so that the volume of the particle is $\propto \frac{1}{\rho}$ i.e. $\delta V \propto \frac{1}{\rho}$

$$-\frac{1}{\rho}\cdot\frac{D\rho}{Dt} = \sum \frac{\partial u_i}{\partial x_i} \quad \text{——} \ (1)$$

An another type of Continuity Eq^n, which means, that the fluid flow has no void in it. The density of the fluid does not change appreciably/drastically throughout the flow under several Condition, the most important one is that the flow speed should be small Compared speed of sound in that medium. This assumption is Called Boussinesq approximation.

from (1), $\frac{1}{\rho}\frac{D\rho}{Dt} = 0$

$$\Rightarrow \sum_{i=1}^{2}\frac{\partial u_i}{\partial x_i} = 0$$

$u_1 = u$, $u_2 = v$ | $x_1 = x$, $x_2 = y$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v}{\partial x} = 0 \Rightarrow \quad \text{——} \ (2)$$

The streamlines of the flow is given by,

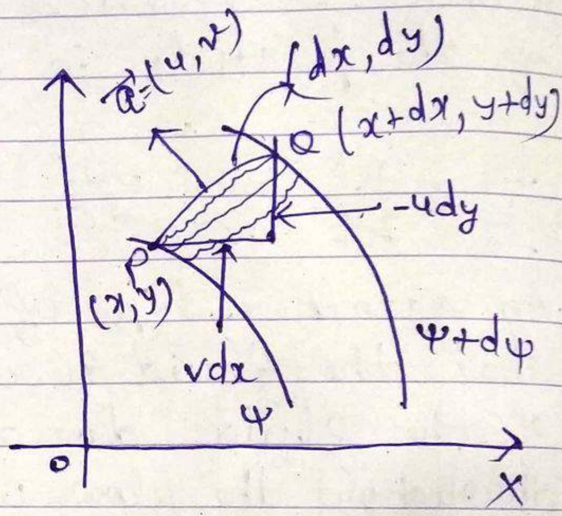
$$\frac{dx}{u} = \frac{dy}{v}$$

$$\Rightarrow v\,dx - u\,dy = 0$$

$$\Rightarrow \frac{\partial \psi}{\partial x}.dx + \frac{\partial \psi}{\partial y}.dy = 0$$

$$d\psi = 0$$

$$\psi(x,y,t) = C$$



#

Consider an arbitrary fluid element (line element), $(dx, dy) = d\vec{r}$ in the flow. The volume rate of ~~the~~ flow across such element will be,

$$v\,dx + (-u)\,dy = v\,dx - u\,dy$$

$$= -\frac{\partial \psi}{\partial x}.dx - \frac{\partial \psi}{\partial y}.dy = -d\psi.$$

this shows that the volume flow rate between a pair of streamlines is numerically equal difference of streamlines, $d\psi$.

The -ve sign of ψ is such that, facing the direction of motion ψ.

Conservation laws:

(1) Gauss-Div. theorem:

$$\vec{F}(\vec{x},t) = \vec{F}(x,y,z,t) : \mathbb{R}^4 \to \mathbb{R}^3$$
$$\vec{F}(\vec{x}) = \vec{F}(x,y,z) : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\iiint_V \operatorname{div}\vec{F} \cdot dv = \iint_S \vec{F}\cdot\vec{n}\, ds$$

(2) Let $f: [a,b] \to \mathbb{R}$ be a diff function -

$$\frac{d}{dx}\int f(x)\, dx = 0$$

$$\frac{d}{dx}\int_c^d f(x,y)\cdot dy = \int_c^d \frac{\partial f}{\partial x}\cdot dy$$

$$\frac{d}{dx}\int_{c(x)}^{d(x)} f(x,y)\cdot dy = f(x,dx)\frac{d}{dx}d(x) - f(x,c(x))\cdot\frac{d}{dx}c(x) + \int_{c(x)}^{d(x)} \frac{\partial f}{\partial x}\cdot dy$$

(Leibnitz rule of diff)

Generalization:

$$\frac{d}{dt}\int_{v(t)} \vec{F}(\vec{x},t)\cdot dv = \iint_{S(t)} d\vec{s}\cdot \vec{u_s}\vec{F} + \int_{V(t)} \frac{\partial F}{\partial t}\cdot dV \quad \text{--- (1)}$$

where, $\vec{u_s}$ is the velocity of the boundary and $S(t)$ is the surface of $V(t)$.

for a material value $V(t)$, the surface of the fluid moves with the velocity $\vec{u}$, then (1) reduce to

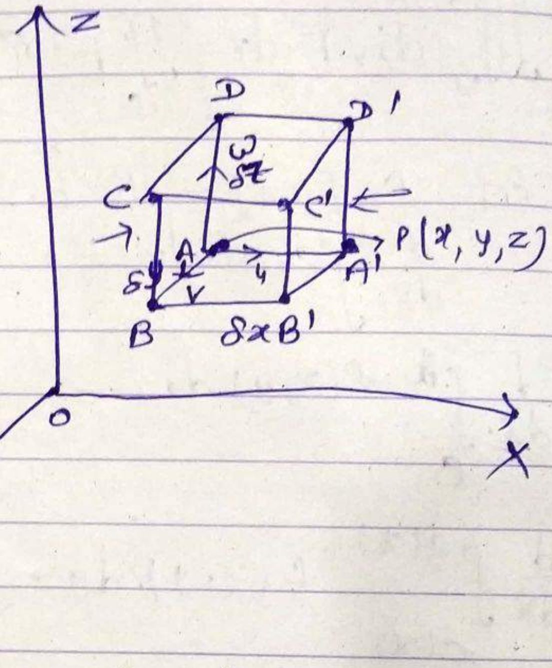
$$\frac{D}{Dt}\int_{v(t)} \vec{F}(\vec{x},t)\cdot dv = \int_V \frac{\partial F}{\partial t}\cdot dv + \int_S d\vec{s}\cdot\vec{u}\cdot\vec{F}.$$

This this is Called Reynold's transportation rule.

for fixed volume,

$$\frac{d}{dt}\int_{v(t)} F(x,t).dv = \int_{v} \frac{\partial F}{\partial t}.dv$$

Euler's Eqn of motion:



Let p be the pressure and ρ be the density at a point $P(x,y,z)$ in an inviscid (Perfect fluid). Consider an elementry fluid element $\delta x \delta y \delta z$

Let $\vec{q} = (u, v, w)$ be the fluid velocity and (X, Y, Z) be the Components of the external force per unit mass at a time t at P.

Then if $p = f(x,y,z)$, we have, Force on the plane through P ~~parallel~~ perpendicular to the force $ABCD. = p\,\delta y\,\delta z$

Force on the plane $\_A'B'C'D' = f\left(x - \frac{1}{2}\delta x, y, z\right).\delta y \delta z.$

$$= \left\{ f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \left(\frac{\delta x}{2}\right)^2 \frac{\partial^2 f}{\partial x^2}\cdot\frac{1}{2!} + \dots \right\} \delta y \delta z$$

taylor series

$$= \left(f - \frac{1}{2}\delta x \frac{\partial f}{\partial x}\right)\delta y \delta z \quad \text{--- (1)}$$

similarly force on the face $A'B'C'D'$

$$= f\left(x + \frac{\delta x}{2}, y, z\right) \delta y \delta z$$

$$= \left(f(x) + \frac{1}{2} \frac{\delta x}{} \frac{\partial f}{\partial x}\right) \delta y \delta z \quad \text{—} (2)$$

The net force in X-direction due to force on ABCD and $A'B'C'D'$

$$= f(x) - \frac{1}{2}\frac{\partial f}{\partial x} \delta x \delta y \delta z - f(x) - \frac{1}{2}\frac{\partial f}{\partial x} \delta x \delta y \delta z.$$

$$= -\frac{\partial f}{\partial x} \cdot \delta x \delta y \delta z \quad \text{—} (3)$$

the mass of the fluid element $= \rho \delta x \delta y \delta z$.
Hence the external force on the element along X-axis

$$= X \rho \delta x \delta y \delta z$$

we know that $\frac{Du}{Dt}$ is the total acceleration of the element in X-direction.

By Newton's 2^{nd} law of motion,

Max. acc^n = sum of the components of external forces,

$$\Rightarrow \rho \delta x \delta y \delta z \frac{Du}{Dt} = X \rho \delta x \delta y \delta z - \frac{\partial f}{\partial x} \delta x \delta y \delta z.$$

$$\rho \frac{Du}{Dt} = X\rho - \frac{\partial f}{\partial x}$$

$$\rho \frac{Du}{Dt} = \rho X - \frac{\partial p}{\partial x} \quad \text{—} (4)$$

Preceding similarly, $\rho \frac{Dv}{Dt} = \rho Y - \frac{\partial p}{\partial y}$ — (5), $\rho \frac{Dw}{Dt} = \rho Z - \frac{\partial p}{\partial z}$ — (6)

By (4), (5) and (6)

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\Rightarrow \left(\frac{Du}{Dt}\hat{i} + \frac{Dv}{Dt}\hat{j} + \frac{Dw}{Dt}\hat{k}\right) = (X\hat{i} + Y\hat{j} + Z\hat{k}) - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right)$$

$$\frac{D}{Dt}\vec{q} = \vec{F} - \frac{1}{\rho}\nabla.\vec{p}$$

$$\boxed{\frac{D\vec{q}}{Dt} + \frac{1}{\rho}\vec{\nabla}.\vec{p} = \vec{F}} \text{ — (1)} \quad \text{Euler's equation} \quad (I)$$

$$\boxed{\frac{D\vec{q}}{Dt} = \frac{\partial\vec{q}}{\partial t} + (\vec{q}.\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla}.\vec{p}}$$

Again, $\vec{\nabla}(\vec{q}.\vec{q}) = 2[\vec{q}\times(\vec{\nabla}\times\vec{q}) + (\vec{q}.\vec{\nabla})\vec{q}]$

$$\Rightarrow (\vec{q}.\vec{\nabla})\vec{q} = \frac{1}{2}\vec{\nabla}|\vec{q}|^2 - \vec{q}\times(\vec{\nabla}\times\vec{q}) \text{ where } |\vec{q}| = q \text{ — (ii)}$$

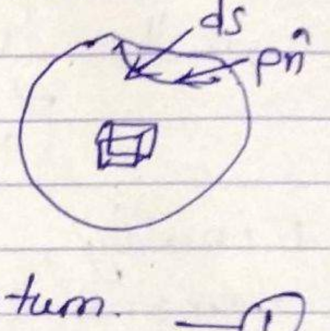
from (1) and (ii)

$$\boxed{\frac{\partial\vec{q}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}q^2\right) - \vec{q}\times(\vec{\nabla}\times\vec{q}) = \vec{F} - \frac{1}{\rho}\nabla\vec{p}} \text{ — (II)}$$

$$\frac{\partial\vec{q}}{\partial t} - \vec{q}\times(\vec{\nabla}\times\vec{q}) = \vec{F} - \frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\frac{1}{2}q^2$$

Euler's eq^n of an inviscus fluid (in vector form/ method):

Consider any arbitrary closed surface S. drawn in the region occupied by an incompressible fluid and it is ~~nothi~~ moving with it, so that it contain some fluid particles at every ~~point~~. instant.



By Newton's 2nd law
external force acting on the mass of the fluid
= rate of change of linear momentum. —①

The mass of the fluid under consideration experiences the following forces.

i) The normal pressure thrusts on the boundary.
ii) The external force $\vec{F}$, per unit mass.

Let ρ be the density of the fluid particle at P whose volume is δV. Then the mass of the fluid particle is ρdV. Therefore the mass of entire region S whose volume is V, $M = \int_{V} \rho dv$.
So, the momentum $\vec{M}$ of the volume V is given by. $\vec{M} = \iiint \vec{q}\, dv$

$$\frac{D\vec{M}}{Dt} = \vec{q}\int_V \cancel{\frac{D}{Dt}\rho\, dv}^{\,0} + \frac{D\vec{q}}{Dt}\int_V \rho\, dv$$

$$\frac{D\vec{M}}{Dt} = \frac{D\vec{q}}{Dt}\int_V \rho\, dv \quad \text{--- (ii)}$$

If $\vec{F}$ be the external force per unit mass acting on the fluid particle P. then the total force on the region S,

$$\vec{F}_{total} = \int_V \vec{F}\rho\, dv \quad \text{--- (iii)}$$

Finally, if p be the pressure at a point of a surface element δs, then the total pressure on the regions, p total $= \int_S p.(\vec{n})\, ds. = \int_V \vec{\nabla}\vec{p}\, dv$

$$\int \frac{D\vec{q}}{Dt}\rho\, dv = \int_V \vec{F}\rho\, dv - \int_V \nabla p.\, dv$$

$$\Rightarrow \int_V \left(\frac{D\vec{q}}{Dt} - \vec{F} + \frac{1}{\rho}\vec{\nabla}\vec{p}\right) dv = 0, \ \rho \neq 0$$

Since V is arbitrary,

$$\boxed{\frac{Dq}{Dt} = \vec{F} + \frac{1}{\rho}\nabla p = 0}$$

Eqⁿ of perfect fluid. (Euler's eqⁿ)

$$\frac{D\vec{q}}{Dt} = \vec{F} = -\frac{1}{\rho}\nabla\vec{p}$$

we know, $\frac{\partial\vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\nabla\vec{p}$ — (i)

$$\vec{\nabla}(\vec{q}\cdot\vec{q}) = 2[\vec{q}\times(\vec{\nabla}\times\vec{q}) + (\vec{q}\cdot\vec{\nabla})\vec{q}]$$

$$\Rightarrow (\vec{q}\cdot\vec{\nabla})\vec{q} = \frac{1}{2}[\vec{\nabla}(\vec{q}\cdot\vec{q}) - \vec{q}\times(\vec{\nabla}\times\vec{q})] \quad \text{— (ii)}$$

from relation (i) and (ii)

$$\frac{\partial\vec{q}}{\partial t} + \vec{\nabla}\left(\frac{\vec{q}^2}{2}\right) + (\vec{\nabla}\times\vec{q})\times(\vec{q}) = \vec{F} - \frac{1}{\rho}\vec{\nabla}p$$

Define, $\vec{\omega} = \vec{\nabla}\times\vec{q}$, the vorticity vector

$$\frac{\partial\vec{q}}{\partial t} + \vec{\nabla}\left(\frac{\vec{q}^2}{2}\right) + \vec{\omega}\times\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p \quad \text{— (iii)}$$

This equation is known as Lamb's Hydrodynamical Eqⁿ. The relation remains invariant under Co-ordinate transformation.

suppose the force field is conservative
In a conservative force field the work done by the force $\vec{F}$ is independent of the path

e If $\vec{F} = X\hat{i} + Y\hat{j} + Z\hat{k}$ and $\vec{F}$ is conservative, then $\exists\, \phi$ s.t $\vec{F} = -\nabla\phi$, ϕ is scalar
— (iv)

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* Equation of perfect fluid (Euler's Eq.)

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla\vec{p}$$

We know,

$$\frac{\partial\vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\nabla\vec{p} \quad -①$$

$$\vec{\nabla}(\vec{q}\cdot\vec{q}) = 2\left[\vec{q}\times(\vec{\nabla}\times\vec{q}) + (\vec{q}\cdot\vec{\nabla})\vec{q}\right]$$

$$\Rightarrow (\vec{q}\cdot\vec{\nabla})\vec{q} = \frac{1}{2}\left[\vec{\nabla}(\vec{q}\cdot\vec{q})\right] - \vec{q}\times(\vec{\nabla}\times\vec{q}) \quad -②$$

From relation ①, ② :-

$$\partial\vec{q}/\partial t + \vec{\nabla}\left(\frac{\vec{q}^2}{2}\right) + (\vec{\nabla}\times\vec{q})\times\vec{q} = F - \frac{1}{\rho}\vec{\nabla}.\vec{p}$$

Denote,

$\vec{\omega} = \vec{\nabla}\times\vec{q}$, the vorticity vector

$$\partial\vec{q}/\partial t + \vec{\nabla}\left(\vec{q}^2/2\right) + \vec{\omega}\times\vec{q} = \vec{F} - \frac{1}{\rho}\nabla\vec{p}$$

This equation is known as Lamb's Hydrodynamical eq^n.
This equation remains invariant under coordinate transformation

Suppose the force field is conservative,
then the work done by force $(\vec{F})$ is independent of the path.

Also, if $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ is conservative, then $\exists\ \phi$ s.t
$\vec{F} = -\vec{\nabla}\phi$, ϕ is a scalar

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From ③, ④ :-

$$\partial \vec{q}/\partial t + \vec{\nabla}(\vec{q}^{\,2}/2) + \vec{\omega} \times \vec{q} + \vec{\nabla}\phi + \frac{1}{\rho}\nabla P = 0$$

(Stress at a point)

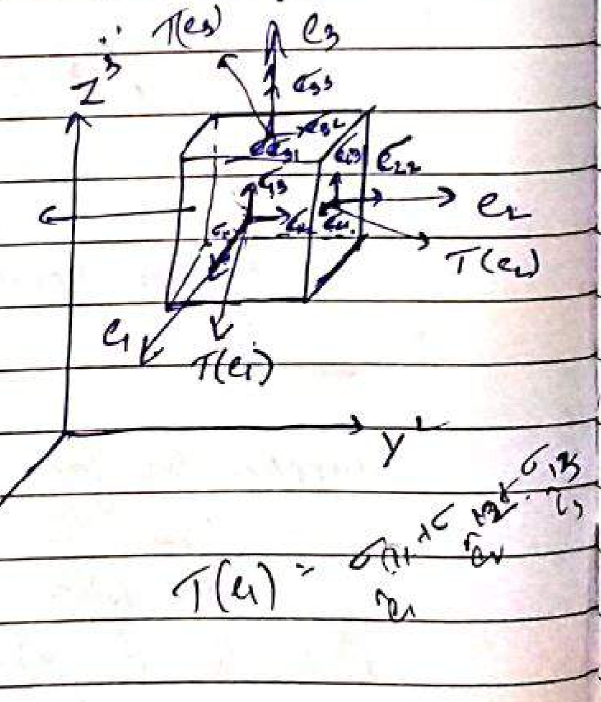
* Cauchy Stress Tensors :-

In continuum mechanics, the cauchy stress tensor or simply a stress tensor is a second order tensor consisting 9 components σ_{ij} where $i = 1, 2, 3, \ldots$, $j = 1, 2, 3, \ldots$

This tensor completely defines the stress at a point inside a material in the deformed state, placement or configuration. This relates a unit length direction vector $\hat{n}$ to the stress tensor $T^{(n)}$.

$$T^{(n)} = \hat{n}\,\vec{\sigma} \quad (\text{or}) \quad T_i^{(n)} = \sigma_{ij}\, n_i$$

where $\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$



$$T^{(n)} = \left(T^{(e_1)}, T^{(e_2)}, T^{(e_3)}\right)$$

$$T_i^{(e_1)} = \sum_{j=1}^{3} \sigma_{ji} \cdot e_j$$

$$= \sigma_{1i} e_1 + \sigma_{2i} e_2 + \sigma_{3i} \sigma_3.$$

$$T^{(n)} = \underset{[3\times3]}{\sigma} \cdot \underset{[3\times1]}{\hat{n}}$$

$$T_i^{n} = \sigma_{ij} \cdot n_j$$

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* Conservation of mass → Equation of Continuity
of Linear Momentum ⇒ Euler Equation

Now, conservation of angular momentum.

* Cauchy postulates :-

According to Cauchy postulates, the stress vector remains unchanged for all surfaces passing through a point 'p' and having the same normal $\hat{n}$. This means, stress vector is a function of $\hat{n}$ only and not influenced by the curvature of the internal surface. Hence $T(n)$
→ Linear Elasticity

If above doesn't hold true → Unlinear Elasticity

Cauchy's Fundamental lemma :-

The stress vectors acting on the opposite sides of the sides of the same surface are equal in magnitude & opposite in sign, i.e. $T^{(-n)} = -T^{(n)}$

* Torque & Angular Momentum :-

(Moment / Moment of force / Turning effect)

The rotational equivalent of the linear force. It is the product of the ~~force~~ magnitude of the force and $\perp^r$ distance of the line of action of the force from the axis of rotation.

Torque $\vec{\tau} = \vec{r} \times \vec{F}$
$= m(\vec{r} \times d\vec{v}/dt)$

Angular Momentum $\vec{L} = \vec{r} \times m\vec{v}$
$= m(\vec{r} \times \vec{v})$

$\frac{d\vec{L}}{dt} = m\frac{d\vec{r}}{dt} \times \vec{v} + m\,\vec{r} \times \frac{d\vec{v}}{dt} = \vec{\tau}$

$\vec{v} \times \vec{v} = 0$

$\tau = |r|\,|F| \sin\theta$

$L = |r|\,|v| \sin\theta$

* **Moment of Inertia :-**

The ratio of angular momentum L of a system to it's angular velocity ω. $I = L/\omega$

* **Conservation of Angular Momentum** $\Rightarrow \sigma_{ij} = \sigma_{ji}$ (Symmetric axis)

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* Torque & Angular Momentum :-

$\vec{\tau} = \vec{r} \times \vec{F}$ — $\vec{F}$ is external force

$\vec{L} = \vec{r} \times \vec{P}$ — $\vec{P}$ is linear momentum

* Principal of conservation of angular momentum :-

If $\tau_{ij} = \tau_{ji}$ (or) stress tensor is symmetric, there is no change is angular momentum otherwise, there is torque (net)

* Levi-Civita system :-

This symbol is used quite often in tensor analysis, vector analysis

$$\varepsilon_{i_1 i_2 i_3 \ldots i_n} = (-1)^p \varepsilon_{123\ldots n}$$

where $i_1, i_2, i_3 \ldots i_n$ are distinct and in ordered fashion $1, 2, 3 \ldots n$ & $\varepsilon_{123\ldots n} = 1$

→ In 2-dimension

$\varepsilon_{ij} = 1$, $(i,j) = (1,2)$

$= -1$, $(i,j) = (2,1)$

$= 0$, $i = j$

→ In 3-dimension

$\varepsilon_{ijk} = 1$, $(i,j,k) = (1,2,3)$ $(2,3,1)$ $(3,1,2)$

$= -1$, $(i,j,k) = (2,1,3)$, $(1,3,2)$, $(3,2,1)$

$= 0$, if any 2 are equal

$i = j \,||\, j = k \,||\, k = i$

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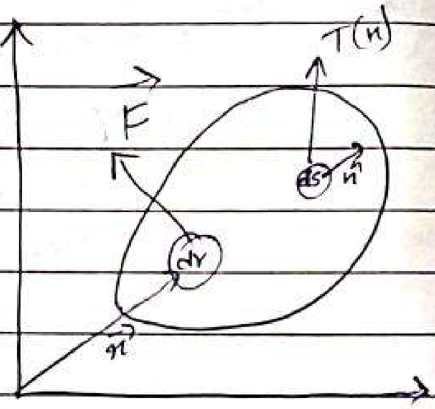
* Statement (Conservation of Angular Momentum) :-

According to principal of conservation of Angular Momentum, the moment of a force acting on a body with respect to an arbitrary point is 0. (if angular momentum is constant)

Consider a continuum body of Volume V & surface area S. Let $\vec{F}$ be the body force and $T^{(n)}$ be the stress /surface force per unit area.



$\vec{r} \times (\vec{T}^{(n)} + \vec{F})$ → for a point, ∴ Integrating

(stress vector: $\vec{T}^{(n)}$)

$$\int \vec{r} \times T(n) + \vec{r} \times \vec{F} = \vec{0}$$

$$\Rightarrow \int \vec{r} \times \vec{T}^{(n)} + \int \vec{r} \times \vec{F} = \vec{0}$$

$$\Rightarrow \int_S \vec{r} \times \vec{T}^{(n)} \, dS + \int_V \vec{r} \times \vec{F} \, dV = \vec{0}$$

where $\vec{r} = x_j e_j$ is the position vector

$$\Rightarrow \int_S \varepsilon_{ijk} x_j T_k^{(n)} \, dS$$

$$+ \int_V \varepsilon_{ijk} x_j F_k \, dV = 0$$

$\varepsilon_{ijk} a_j b_k$

$i=1$, $\varepsilon_{1jk} a_j b_k$

$= \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2$

$= a_2 b_3 - a_3 b_2$

$\Rightarrow \hat{i}\,(a_2 b_3 - a_3 b_2)$

$\varepsilon_1 \to i$, $\varepsilon_2 \to j$, $\varepsilon_3 \to k$

Hence $\vec{a} \times \vec{b} = \varepsilon_{ijk} a_j b_k$

$\bar{a} \times \bar{b} = i(a_2 b_3 - a_3 b_2)$

$+ j(a_3 b_1 - a_1 b_3)$

$+ k(a_1 b_2 - a_2 b_1)$

$\varepsilon_{ijk} x_j(\sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3)$

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$$\Rightarrow \int_S \varepsilon_{ijk} x_j \sigma_{mk} n_m \, dS + \int_V \varepsilon_{ijk} x_j F_k \, dV = 0$$

By divergence Theorem :- $\partial/\partial x_m (\varepsilon_{ijk} x_j \sigma_{mk})$

$$0 = \int_V (\varepsilon_{ijk} x_j \sigma_{mk})_{,m} \, dV + \int_V \varepsilon_{ijk} x_j F_k \, dV$$

In m-coordinate system :-

$(x_1, x_2, x_3) \Rightarrow \dfrac{\partial x_m}{\partial x_m} = 1$

$\dfrac{\partial x_{i \neq m}}{\partial x_m} = 0$

& $\dfrac{\partial x_j}{\partial x_m} = x_{j,m} = \delta_{jm} = \begin{cases} 1, & j = m \\ 0, & \text{otherwise} \end{cases}$

$$\int_V \Big[(\underbrace{\varepsilon_{ijk} x_{j,m}}_{\frac{\partial x_j}{\partial x_m}} \sigma_{mk}) + (\varepsilon_{ijk} x_j \underbrace{\sigma_{mk,m}}_{\frac{\partial \sigma_{mk}}{\partial x_m}}) \Big] dV + \int_V \varepsilon_{ijk} x_j F_k \, dV = 0$$

$$\Rightarrow \int_V \varepsilon_{ijk} x_{j,m} \sigma_{mk} \, dV + \int_V \varepsilon_{ijk} x_j (\sigma_{mk,m} + F_k) \, dV = 0 \quad \text{--- (2)}$$

By Cauchy law of motion, (Discussed immediately after)

$$\sigma_{mk,m} + F_k = 0 \quad \text{--- (3)}$$

From (2), (3) :-

$$\int_V \varepsilon_{ijk} x_{j,m} \sigma_{mk} \, dV = 0$$

$$\Rightarrow \int_V \varepsilon_{ijk} \delta_{jm} \sigma_{mk} \, dV = 0$$

$$\Rightarrow \int_V \varepsilon_{ijk} \sigma_{jk} \, dV = 0 \quad (\delta_{jm} = 1 \text{ when } m = j)$$

Since V is arbitrary :-

$$\varepsilon_{ijk} \sigma_{jk} = 0$$

$$\Rightarrow \varepsilon_{jk} \sigma_{jk} \hat{e}_1 + \varepsilon_{jk} \sigma_{jk} \hat{e}_2 + \varepsilon_{jk} \sigma_{jk} \hat{e}_3 = 0$$

$$\Rightarrow \quad \Rightarrow \sigma_{12} - \sigma_{21} = 0 \quad \Rightarrow \sigma_{12} = \sigma_{21}$$

$$\sigma_{23} - \sigma_{32} = 0 \quad \Rightarrow \sigma_{23} = \sigma_{32}$$

$$\sigma_{31} - \sigma_{13} = 0 \quad \Rightarrow \sigma_{31} = \sigma_{13}$$

(σ)

Symmetric stress tensor.

* Cauchy ~~Stress~~ Laws of Motion :-

→ First Law : According to the principal of conservation of linear momentum, if a continuum body is in ~~state of~~ static equillibrium, it can be demonstrated that the components of Cauchy stress tensor at every point satisfies equillibrium equation.

$$\sigma_{ji,j} + F_i = 0$$

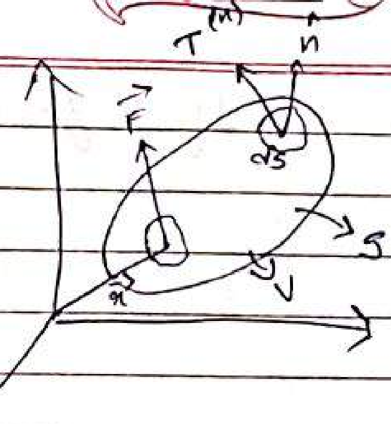
$$\text{i.e. } \sigma_{ji,j} = -F_i$$

In particular, for a hydrostatic fluid in equillibrium, we have

$$\sigma_{ij} = -p\delta_{ij}$$

Proof :-

Consider a body (continuum) occupying a volume V with surface area S, with surface forces $T^{(n)}$ per unit area acting at every pt. & $\vec{F}$ be the body force per unit volume



$$\int_S T_i^{(n)}\, dS + \int F_i\, dV = 0$$

$$\Rightarrow \int_S \sigma_{ji} n_j dS + \int_V F_i dV = 0$$

$$\Rightarrow \int_V (\sigma_{ji})_{,j}\, dV + \int_V F_i\, dV = 0$$

$$\Rightarrow \int_V (\sigma_{ji,j} + F_i)\, dV = 0$$

$$\Rightarrow \sigma_{ji,j} + F_i = 0$$

$\vec{T}^{(n)} = \sigma \vec{n}$

$T_i^{(n)} = \sigma_{ji} n_j$

$i = 1,2,3$

Eg: $T^{(n)} = (f, g)$

$\sigma = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\hat{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T^{(n)} = \sigma n$

$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

→ Principal stress, stress invariants, Deviatoric stress tensors

At every point in a stressed body there are atleast 3 planes called principal planes with normal $\hat{n}$ called principal directions, where the corresponding stress vector is perpendicular to the plane and the shear stresses are zero. Then the 3 stresses are called principal stresses.

$$\sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

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By Definition,

$$T^{(n)} = \sigma_n \hat{n}$$

Also, $T_i^{(n)} = \sigma_{ij} n_j$ & $n_i = \delta_{ij} n_j$

$$n_i = 1 \cdot n_i + 0 \cdot \sum_{i \neq j} n_j$$

As ~~As only principal~~ stresses ~~exist that~~ are parallel to normal components

$$\left[\cancel{T^{(n)} = \lambda \hat{n}} \right.$$

$$\sigma_{ij} n_j = \lambda n_j$$

$$\left. = \lambda \delta_{ij} n_j \right]$$

in n_j direction

$$(\sigma_{ij} - \lambda \delta_{ij})\, n_j = 0 \quad \text{—— (1)}$$

(1) is a homogeneous equation in n_j. This eq. has non-trivial solutions when,

$$\det(\sigma_{ij} - \lambda \delta_{ij}) = 0$$

(or)

$$\begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0$$

$tr(\sigma) =$ $I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$

$\frac{1}{2}\left[tr^2(\sigma) - tr(\sigma^2)\right]$ $I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$

$I_3 = \det(\sigma_{ij}) = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} - \sigma_{31}^2\sigma_{22}$

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Then, Principal stresses :-

$$\sigma_{11} = \max(\lambda_1, \lambda_2, \lambda_3)$$
$$\sigma_{33} = \min(\lambda_1, \lambda_2, \lambda_3)$$
$$\sigma_{22} = I_1 - \sigma_{11} - \sigma_{33}$$

We are finding the values of principal stresses obtained if coordinate axes are rotated to make shear stresses 0.

→ Stress Deviatoric Tensor :- 22/8/19

The stress tensor σ_{ij} can be expressed as the sum of 2 other stress tensors:-

1) mean hydrostatic stress tensor or volumetric stress tensor or mean normal stress tensor $\pi \delta_{ij}$ which tends to change the volume of the stressed body &

2) A deviatoric component called stress deviatoric tensor S_{ij} which tends to distort it so.

$$\sigma_{ij} = \pi \delta_{ij} + S_{ij}$$

where π is the mean stress given by

$$\pi = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$
$$= \frac{1}{3}(\sigma_{kk}) = \frac{1}{3} I_1$$

Ex: 1 : The stress tensor at P is given by

$$\sigma = \begin{bmatrix} 7+a & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4+6 \end{bmatrix} \quad \& \quad \hat{n} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

Then find stress vector at the point P.

Soln: We know.

$$\vec{T}^{(n)} = \sigma \cdot \hat{n}$$

$$\vec{T}^{(n)} = \begin{bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{bmatrix} = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -10/3 \\ 0 \end{bmatrix}$$

Ex 2: Find out the principal stress :-

$$\sigma = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Soln: Principal stresses are given by :-

$$|(\sigma - \lambda I)| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = 0$$

$$(\lambda+2)(\lambda-4)(\lambda-1) = 0$$

$\sigma_{11} = 4$

$\sigma_{22} = 1$

$\sigma_{33} = -2$

$$\lambda = 1, -2, 4$$

→ 2^{nd} Law of Motion :- Bernoulli Equation :-

When the velocity exists (so that the flow/motion of the fluid may be irrotational) and external forces are given in the form of a potential function, then equations of motion can always be integrated

For any ϕ, $\vec{q} = -\vec{\nabla}\phi \Rightarrow \overset{(u)}{\vec{f_1}} = -\partial\phi/\partial x$, $\overset{(v)}{\vec{f_2}} = -\partial\phi/\partial y$,
$(w)\vec{f_3} = -\partial\phi/\partial z$

For any V, $\vec{F} = -\vec{\nabla} V \Rightarrow f_1 = -\partial V/\partial x$ → potential fn, $f_2 = -\partial V/\partial y$
$f_3 = -\partial V/\partial z$

$$\partial u/\partial y = \partial/\partial y\left(-\partial\phi/\partial x\right) = \frac{-\partial^2\phi}{\partial y\partial x} = \partial/\partial x\left(\frac{-\partial\phi}{\partial y}\right) = \partial v/\partial x \quad -(1)$$

Slly, $\partial v/\partial z = \partial w/\partial y$, From Euler's Equation ✓
$-(2)$ $\quad \partial w/\partial x = \partial u/\partial z \quad -(3)$

From Euler's Equation we have :-

$$D\vec{q} = \vec{F} - 1/\rho\, \nabla\vec{p}$$

$$\Rightarrow \left(\partial\vec{q}/\partial t + \vec{q}\cdot\vec{\nabla}\cdot\vec{q}\right) = \vec{F} - 1/\rho\, \nabla p$$

$$\partial u/\partial t + u\,\partial u/\partial x + v\,\partial u/\partial y + w\,\partial u/\partial z = \overset{\text{Force}}{X} - \frac{1}{\rho}\,\partial p/\partial x$$

$$\partial v/\partial t + u\,\partial v/\partial x + v\,\partial v/\partial y + w\,\partial v/\partial z = Y - 1/\rho\,\partial p/\partial y$$

$$\partial z/\partial t + u\,\partial w/\partial x + v\,\partial w/\partial y + w\,\partial w/\partial z = Z - 1/\rho\,\partial p/\partial z$$

Now we use 1, 2, 3 to obtain

$$-\frac{\partial^2\phi}{\partial t\,\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\rightarrow\ -\frac{\partial^2\phi}{\partial t\,\partial y} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\rightarrow\ -\frac{\partial^2\phi}{\partial t\,\partial z} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right) + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right) + u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right) + u\frac{\partial w}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right) = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right) = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right) = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\rightarrow\ -\partial\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\partial\left(u^2+v^2+w^2\right) = X\,\partial x - \frac{1}{\rho}\frac{\partial p}{\partial x}\,\partial x$$

$$-\partial\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\partial\left(u^2+v^2+w^2\right) = Y\,\partial y - \frac{1}{\rho}\frac{\partial p}{\partial y}\,\partial y$$

$$-\partial\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\partial\left(u^2+v^2+w^2\right) = Z\,\partial z - \frac{1}{\rho}\frac{\partial p}{\partial z}\,\partial z$$

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* Bernoulli's Equation :-

$$1.\quad \vec{\nabla} \times \vec{q} = 0$$
$$\vec{q} = -\vec{\nabla}\phi$$
$$2.\quad \vec{F} = -\vec{\nabla}\cdot\vec{v}$$

$$-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2) = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \Big| \times dx$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2+w^2) = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad \Big| \times dy$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2+w^2) = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial z} \quad \Big| \times dz$$

Adding above :-

$$-\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right)dy + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right)dz\right]$$
$$+ \frac{1}{2}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)(u^2_{dx}+v^2_{dy}+w^2_{dz})\right]$$
$$= -\frac{\partial v}{\partial x}dx - \frac{\partial v}{\partial y}dy - \frac{\partial v}{\partial z}dz$$
$$- \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}d\right.$$

$$\Rightarrow -\int d\left(\frac{\partial\phi}{\partial t}\right) + \int \frac{1}{2}\, d(u^2+v^2+w^2) = -\int dV - \int \frac{1}{\rho}\, dp$$

$\rho \to$ density, $p \to$ pressure

$$\Rightarrow -\frac{\partial\phi}{\partial t} + \frac{u^2+v^2+w^2}{2} = -V - \int\frac{dp}{\rho} + \psi(t)$$

$$\Rightarrow -\frac{\partial\phi}{\partial t} + \frac{u^2+v^2+w^2}{2} + V + \int\frac{dP}{\rho} = \psi(t)$$

$\rightarrow$ Bernoulli's eq^n of motion for a irrotational fluid in cons. force field. Special case ρ = constant.

$dP \rightarrow$ pressure
$\rho \rightarrow$ density

$$\rightarrow \int\frac{dP}{\rho} = \frac{1}{\rho}\int dP$$

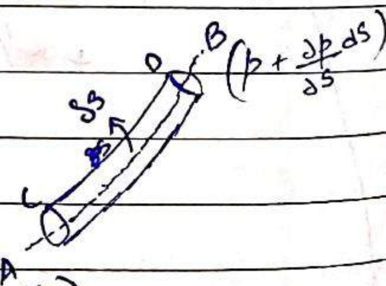
* Bernoulli's Theorem for steady flow :-
(with no velocity potential & conservative force field)

Statement : When a motion is steady & the velocity potential does not exist, then

$$\frac{\vec{q}^2}{2} + V + \int \frac{dP}{\rho} = \text{constant}.$$

where p is the pressure, $\vec{q}$ is the velocity & V is the force potential from which the external forces are derivable.

Proof : Let us consider a streamline AB in the fluid. Let δs be an element of this stream line and CD be a small cylinder of cross-sectional area α and δs as the axis.



If $\vec{q}$ be the fluid velocity and S be the ~~continuous~~ component of external forces per unit mass in the direction of streamline, then by Newton's Law, $\rho \alpha \delta s \frac{D\vec{q}}{Dt} = \rho \alpha \delta s \cdot S + p\alpha - \left(p + \frac{\partial p}{\partial s}\delta s\right)\alpha$

$$\Rightarrow \rho \frac{D\vec{q}}{Dt} = \rho S - \frac{\partial p}{\partial s}$$

$$\Rightarrow \frac{D\vec{q}}{Dt} = S - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

$$\Rightarrow \overbrace{\frac{\partial \vec{q}}{\partial t}}^{=0} + q \cdot \frac{\partial q}{\partial s} = S - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

$$\Rightarrow q \cdot \partial q/\partial s = S - 1/\rho \, \frac{\partial p}{\partial s}$$

$$\Rightarrow \int q\,dq = \int S\,\partial s - \int \frac{1}{\rho}\,\partial p$$

~~$\Rightarrow q^2/2 = S.s - \int dp/\rho = \text{constant}$~~

Force potential

$S = -\frac{\partial V}{\partial s}$

$\Rightarrow -\partial V = S\,\partial s$

external forces

$$\Rightarrow \frac{q^2}{2} + V + \int \frac{\partial p}{\rho} = \text{constant}$$

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Equations in other Coordinate System:-

1) Cartesian Coordinates : $\nabla \equiv (\partial/\partial x, \partial/\partial y)$ in 2D

$\equiv (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ in 3D

2) Polar Coordinates :

2D: Let $u : R^2 \to R$ | $u(x,y) = u(r\cos\theta, r\sin\theta)$

$x = r\cos\theta$, $y = r\sin\theta$

is the substitution where.

$r > 0$. $0 \le \theta \le 2\pi$

$$\Rightarrow \partial u/\partial x = \partial/\partial x\, u(\overset{r,\theta}{\cancel{r\cos\theta, r\sin\theta}})$$

$$= \partial u/\partial r \cdot \partial r/\partial x + \partial u/\partial \theta \cdot \partial \theta/\partial x$$

• $\partial r/\partial x = 1/\cos\theta$, $\partial\theta/\partial x = -1/r\sin\theta$

$$\Rightarrow \partial u/\partial x = \partial u/\partial r \cdot \frac{1}{\cos\theta} - \partial u/\partial\theta \cdot 1/r\sin\theta \quad -(1)$$

Illy, $\partial u/\partial y = \partial u/\partial r \cdot 1/\sin\theta + \partial u/\partial\theta\; 1/r\cos\theta \quad -(2)$

Now,

$$\partial u/\partial x\,\hat{i} + \partial u/\partial y\,\hat{j} = \left(1/\cos\theta\ \partial u/\partial r - 1/r\sin\theta\ \partial u/\partial\theta\right)\hat{i}$$

$$+ \left(\partial u/\partial r\ 1/\sin\theta + \partial u/\partial\theta\ \frac{1}{r\cos\theta}\right)\hat{j}$$

$$= \partial u/\partial r\left(\frac{\hat{i}}{\cos\theta} + \frac{\hat{j}}{\sin\theta}\right)$$

$$+ \partial u/\partial\theta\left(\frac{-\hat{i}}{r\sin\theta} + \frac{\hat{j}}{r\cos\theta}\right)$$

$$\partial u/\partial x = \partial u/\partial r \cdot \partial r/\partial x + \partial u/\partial\theta \cdot \partial\theta/\partial x$$

$$x = r\cos\theta, \quad y = r\sin\theta \qquad \theta = \tan^{-1}(y/x)$$

$$x^2 + y^2 = r^2$$

$$x = r\frac{\partial r}{\partial x} \Rightarrow \frac{\partial r}{\partial x} = \cos\theta = \frac{x}{r}$$

$$\partial\theta/\partial x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}$$

$$-y/r^2$$

$$\partial r/\partial y = \sin\theta = y/r$$

$$\cancel{\partial u/\partial x =}$$

$$= -y/x^2+y^2 = -y/r^2 = -\sin\theta/r$$

$$\partial u/\partial x = \partial u/\partial r \cdot \cos\theta + \partial u/\partial\theta \cdot \left(-\sin\theta/r\right)$$

$$\Rightarrow \partial u/\partial x + \partial u/\partial y = \partial u/\partial r\,(\cos\theta + \sin\theta)$$

$$+ \frac{1}{r}(\cos\theta - \sin\theta)\,\partial u/\partial\theta$$

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3D: $x = r\cos\theta\cos\phi \quad y = r\cos\theta\sin\phi, \quad z = r\sin\theta$

$$u(x,y,z) = u(r,\theta,\phi)$$

$$\partial u/\partial x = \partial u/\partial r \cdot \partial r/\partial x + \partial u/\partial\theta \cdot \partial\theta/\partial x + \partial u/\partial\phi \, \partial\phi/\partial x$$

$\partial u/\partial y =$

$\partial u/\partial z =$

$$x^2 + y^2 + z^2 = r^2$$
$$\phi = \tan^{-1}(y/x)$$
$$\theta = \sin^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

3) Cylindrical Coordinates :-

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$u(x,y,z) = u(r,\theta,z)$$
$$\Rightarrow \partial u/\partial x = \partial u/\partial r \, \partial r/\partial x + \partial u/\partial\theta \cdot \partial\theta/\partial x + \partial u/\partial z \cdot \partial z/\partial x$$

$$x/x^2+y^2 = x/r^2 = \cos\theta/r$$

§ The Energy Equation :-

Statement : The rate of change of total energy (Kinetic + Potential + Intrinsic) of any portion of a compressible inviscid fluid as it moves about is equal to the rate at which work is being done by pressure on the boundary provided ~~by~~ the forces are conservative. The potential due to extraneous forces is supposed as independent of time.

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Definitions :-

1) Internal Energy :-

In thermodynamics, the internal energy of a system is the total energy contained within the system. It is the energy necessary to create / prepare the system in a given state.

2) Potential Energy :-

In physics, potential energy is the energy held by an object because of ~~the~~ its position

[illegible]

Let us consider any arbitrary closed surface / region occupied by an inviscid fluid and let V be the volume of the fluid within S. Let ρ be the density of the fluid particle P within S with volume dv surrounding P. Let $\vec{q}(\vec{x}, t)$ be the velocity of the particle at P. Then Euler's equation is :-

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{\nabla \overset{\text{pressure}}{p}}{\rho} \quad \text{—} \ (1)$$

Let the external forces be conservative so that $\exists$ a potential Ω independent of time s.t

$$\vec{F} = -\vec{\nabla}\Omega \quad ; \quad \frac{\partial \Omega}{\partial t} = 0 \quad \text{—} \ (2)$$

Now multiply (1) by $\vec{q}$, then

$$\vec{q} \cdot \frac{d\vec{q}}{dt} = -\vec{q}\, \nabla p \cdot 1/\rho + [-\vec{\nabla}\Omega \cdot \vec{q}]$$

— $\times\, \rho$

$$\rho\, \vec{q} \cdot d\vec{q}/dt = -\vec{q}\, \nabla p - \rho\, [\vec{\nabla}\Omega \cdot \vec{q}] \quad \text{—} \ (3)$$

For Ω,

$$\frac{d\Omega}{dt} = \frac{\partial \Omega}{\partial t} + \vec{q} \cdot \nabla \Omega$$

- From (2)

$$d\Omega/dt = \vec{q} \cdot \vec{\nabla}\Omega \quad \text{—} \ (4)$$

From ③. ④ :-

$$\rho \vec{q} \cdot d\vec{q}/dt + \rho \, d\Omega/dt = -\vec{q} \cdot \nabla \vec{P} \rightarrow \text{Pressure}$$

$$\Rightarrow \rho\left(\frac{1}{2} \, d\vec{q}^2/dt\right) + \rho \, d\Omega/dt = -\vec{q} \cdot \nabla \vec{P}$$

$$\Rightarrow \rho \, d/dt\left(1/2 \, \vec{q}^2\right) + d/dt\,(\rho\Omega) = -\vec{q} \cdot \nabla \vec{P}$$

Integrating over volume :-

$$\int_v \rho \, d/dt\,(1/2 \, \vec{q}^2)\,dv + \int_v \rho \, d/dt\,(\Omega)\,dv = -\int_v \vec{q} \, \nabla \vec{P}\,dv$$

$$\Rightarrow \int_v 1/2 \, d/dt\,(\rho \vec{q}^2)\,dv + \int_v d/dt\,(\rho\Omega)\,dv = -\int_v \vec{q} \, \nabla \vec{P}\,dv$$

$$\Rightarrow d/dt\left(\underbrace{\int_v (1/2 \, \rho \vec{q}^2)\,dv}_{K.E} + \underbrace{\int_v (\rho\Omega)\,dv}_{P.E}\right) = -\int_v \vec{q} \, \nabla \vec{P}\,dv \quad - ⑤$$

Let $T = K.E$, $P = P.E$, $I = I.E = \int_v \rho E \, dv$, where E is internal per unit mass.

~~Since gradient of~~ Since,

$$\vec{\nabla}(P\vec{q}) = \underbrace{\vec{q} \cdot \vec{\nabla} \vec{P}}_{\text{gradient}}^{\text{scalar}} + \underbrace{P \, \vec{\nabla} \cdot \vec{q}}_{\text{divergence}}^{\text{Pressure}} \rightarrow \text{vector}$$

$$\therefore \int_v -\vec{q} \cdot \vec{\nabla} P \, dv = -\int_v \left[\vec{\nabla}(P\vec{q}) - P(\vec{\nabla} \cdot \vec{q})\right] dv$$

$$= \int_v P \, \vec{\nabla} \vec{q} \, dv - \int_v \vec{\nabla}(P\vec{q})\,dv$$

$$= \int_s P\vec{q} \cdot \vec{n} \, ds + \int_v P \, \vec{\nabla} \cdot \vec{q} \, dv \quad ⑥$$

where $\hat{n}$ is normal & ds is the surface

Now,

we try to show that $\int_V P\,\vec{\nabla}\cdot\vec{q}\,dv = -\frac{dI}{dt}$

Since E is the internal energy, E is defined as the work done by the unit mass of the fluid,

$$E = \text{force} \times dl$$
$$= \text{force}/\text{area} \times (\text{area}) \times dl$$
$$= +\int_{V}^{V_0} P \times dv$$

For E as the work done per unit mass

$1/\rho = t$

$-1/\rho^2 \, d\rho = dt$

$$V\rho = 1 \Rightarrow V = 1/\rho$$
$$dV = -1/\rho^2 \, d\rho$$

$$\therefore E = -\int_{\rho}^{\rho_0} P \, d\rho/\rho^2$$

$$= -\int_{\rho}^{\rho_0} P/\rho^2 \, d\rho$$

$$\Rightarrow \frac{dE}{d\rho} = -\frac{d}{d\rho}\int_{\rho}^{\rho_0} P/\rho^2 \, d\rho$$

$$\Rightarrow dE/d\rho = P/\rho^2$$

$$\Rightarrow dE/dt = dE/d\rho \times d\rho/dt$$

$$= P/\rho^2 \, d\rho/dt$$

$$\Rightarrow dE/dt \, \rho dV = \frac{P}{\rho}\frac{d\rho}{dt} dV \quad \text{— (9)}$$

But, $\frac{d}{dt}(E\rho dv) = \frac{dE}{dt}\rho dv + E\underbrace{\frac{d}{dt}(\rho dv)}_{\frac{d}{dt}(\text{mass}) = 0}$ — (11)

By (9), (10) :-

$$\frac{d}{dt}(E\rho dv) = \frac{p}{\rho}\frac{d\rho}{dt}dv$$

$$\Rightarrow \frac{d}{dt}(E\rho dv) = \frac{p}{\rho}\times(-\rho\vec{\nabla}\cdot\vec{q})dv \qquad \left(\frac{d\rho}{dt}+\rho\vec{\nabla}\cdot\vec{q}=0\right)$$

$$\Rightarrow \underbrace{\int_V \frac{d}{dt}(E\rho dv)}_{I} = \int_V -p\,\vec{\nabla}\cdot\vec{q}\,dv$$

$$\Rightarrow \frac{dI}{dt} \Rightarrow \frac{d}{dt}\left(\int_V E\rho dv\right) = \int_V -p\,\vec{\nabla}\cdot\vec{q}\,dv$$

$$\Rightarrow \frac{dI}{dt} = \int_V -p\,\vec{\nabla}\cdot\vec{q}\,dv \quad \text{— (10)}$$

From (5), (6), (10)'s

$$\frac{dT}{dt} + \frac{dP}{dt} + \frac{dI}{dt} = \int_S p\vec{q}\cdot\vec{n}\,dS \quad \text{— (11)}$$

Rate of work done by the fluid pressure on an element dS of S = $p\,dS\,\hat{n}\cdot\vec{q}$

$$\Rightarrow \text{work done (total) } W = \int_S p(\vec{n}\cdot\vec{q})dS \quad \text{— (12)}$$

By (11), (12) :-

$$\frac{dT}{dt} + \frac{dP}{dt} + \frac{dI}{dt} = W$$

$$\Rightarrow \frac{d}{dt}(I+P+T) = W$$

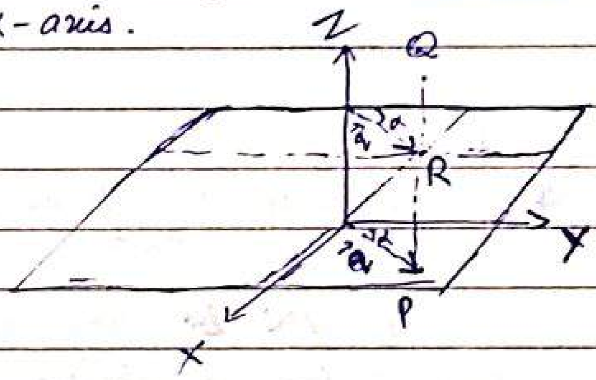
For incompressible fluid,

$\frac{d}{dt}(P+T) = W$

I = constant

* 2-Dimensional Flow :- Flow along x, y for all z.

Suppose the plane under consideration is XY-plane and let P be any arbitrary point in the plane. Draw a straight line from $Q \to P$ (PQ) and let R be another point on the plane ||lel to XY plane and lies on PQ. If $\vec{q}$ is the velocity of the fluid in XY plane which makes an angle α, then $\vec{q}$ is be the velocity in the ||lel plane which is ~~and~~ of the same magnitude and makes the same angle α w.r.t x-axis.



* Stream Function / Current function :-

Let $\vec{q} = (u, v)$ be the fluid velocity. Then the equation of the stream function / lines of flow.

$$\frac{dx}{u} = \frac{dy}{v}$$

$$\Rightarrow v\,dx - u\,dy = 0 \quad \text{—(1)}$$

From eq. of continuity,

$$\vec{\nabla} \cdot \vec{q} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad \text{—(2)}$$

$\Rightarrow$ ② is the exactness condition of ①.

$\Rightarrow \exists\ \psi$ s.t $d\psi = v\,dx - u\,dy$ (Exact Differential)

$M\,dx + N\,dy = 0$

$\frac{\partial M}{\partial y} = \partial N/\partial x$

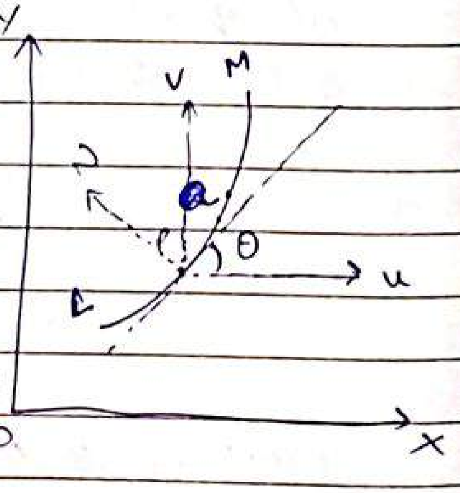
$\Rightarrow \partial\psi/\partial x \cdot dx + \partial\psi/\partial y \cdot dy = v\,dx - u\,dy$ $\rightarrow$ exact differential

$$\Rightarrow \underbrace{u = -\partial\psi/\partial y\ ,\ v = \partial\psi/\partial x}_{} \quad \text{— I}$$

$\rightarrow \psi$ can be determined

* Physical significance of stream function in 2D Flow:-

Let LM be any curve on xy plane and let ψ_1 & ψ_2 be the 2 stream functions at L & M. Let P be any ~~plane st LP = s~~ point on LM s.t LP = s & LQ = s + δs where Q is another pt. on LM. Let θ be the angle between x-axis & the target at P. Let $\vec{q} = (u,v)$ be the fluid velocity.



$\vec{OP} = \vec{r} = x\hat{i} + y\hat{j}$
$= r\cos\theta\ \hat{i} + r\sin\theta\ \hat{j}$

$\hat{PN} = | r\cos(\theta + \pi/2)\hat{i} + r\sin(\theta + \pi/2)\hat{j} | / |r|$

$= (-\sin\theta\ \hat{i} + \cos\theta\ \hat{j})$

Now, velocity component along normal
$= \vec{q}\cdot\hat{n}$

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$$= (u, v) \cdot (-\sin\theta, \cos\theta)$$

$$= -u\sin\theta + v\cos\theta$$

The flux across PQ $= \vec{q} \cdot \hat{n}\, ds = (v\cos\theta - u\sin\theta)\, dS$

Total flux in LM $= \int_{LM} (v\cos\theta - u\sin\theta)\, ds$ — (3)

$$\Rightarrow LM_{flux} = \int_{LM} \left(v\frac{dx}{ds} - u\frac{dy}{ds} \right) ds$$

$$u = -\partial\psi/\partial y \quad v = \partial\psi/\partial x$$

$$\Rightarrow LM_{flux} = \int_{LM} \left(\partial\psi/\partial x\, dx + \partial\psi/\partial y\, dy \right)$$

$$= \int_{LM} d\psi$$

$$= \psi_2 - \psi_1 \quad \text{— (A)}$$

Now,

$$dy/dx = \tan\theta$$

$$\Rightarrow \frac{dy/ds}{dx/dx} = \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \frac{dy/ds}{\sin\theta} = \frac{dx/ds}{\cos\theta} = \frac{(dy/ds)^2 + (dx/ds)^2}{\sin^2 + \cos^2}$$

$$= 1$$

$$(ds^2 = dy^2 + dx^2)$$

$$\Rightarrow dy/ds = \sin\theta, \quad dx/ds = \cos\theta$$

Thus the difference of the values of the stream f^n at 2 points represents the flow across any line joining those points.

\# Since the normal velocity is the velocity normal to δs contributes to the flux across δs, therefore the flux across $\delta s = \delta s \times$ normal velocity.

$$\psi + d\psi - \psi = \delta s \times (\text{velocity from right to left})$$

$$\Rightarrow \text{velocity from right to left} = \frac{\delta\psi}{\delta s} \quad \text{(III)}$$

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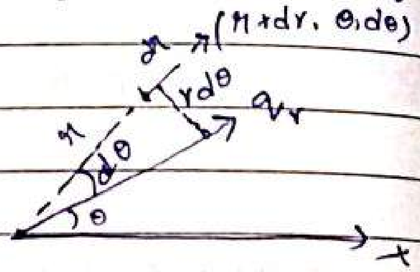
* Stream Function / Velocity Component in polar-coordinate

Let the stream function ψ is given in terms of polar coordinates (r, θ) and velocity components $\vec{q} = (q_r, q_\theta)$



q_r : velocity from right to left across $r\,d\theta$

$$= \lim_{d\theta \to 0} \frac{\partial\psi}{r\,d\theta} = \frac{1}{r}\frac{d\psi}{d\theta}$$

q_θ : velocity from ~~left~~ right to ~~right~~ left across dr.

$$\therefore \quad q_r = \frac{1}{r}\,\partial\psi/\partial\theta, \quad q_\theta = \partial\psi/\partial r$$

* Complex Potential :-

$$\left.\begin{aligned} \vec{F} &= -\nabla\phi \\ \vec{\nabla}\times\vec{q} &= 0 \\ \vec{q} &= -\nabla\psi \end{aligned}\right\} \text{Conservation Field}$$

$w = f(z)$, $z = x+iy$

$$= \phi(x,y) + i\,y(x,y)$$

$$df/dz = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}, \quad \frac{\partial\phi}{\partial x} = \partial\psi/\partial y$$

$$\partial\phi/\partial y = -\partial\psi/\partial x$$

$$\vec{\nabla}\cdot\vec{q} = 0$$

$$\Rightarrow \partial u/\partial x + \partial v/\partial y = 0$$

$$\Rightarrow \quad u = -\partial\psi/\partial y, \quad v = \partial\psi/\partial x \qquad -(1)$$

Irrotational,

$$\vec{q} = -\nabla \phi$$

$$\Rightarrow u = -\partial\phi/\partial x, \quad v = -\partial\phi/\partial y \quad \text{—(II)}$$

$$\text{(I), (II)} \Rightarrow \partial\phi/\partial x = -\partial\psi/\partial y, \quad \partial\phi/\partial y = -\partial\psi/\partial x$$

Complex potential w = potential f^n + i stream f^n

* Spin component of ψ : The spin / rotation in a flow is given by

$$\Omega = \tfrac{1}{2} \operatorname{curl} \vec{q} = \tfrac{1}{2} \vec{\nabla} \times \vec{q}$$

For a 2-dimensional flow :-

$$\vec{q} = (u, v, 0)$$

$$\operatorname{curl} \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & 0 \end{vmatrix} = \hat{i}\, \partial v/\partial z + \hat{j}\, \partial u/\partial z + \hat{k}\left(\partial v/\partial x - \partial u/\partial y\right)$$

$$= \hat{k}\left(\partial v/\partial x - \partial u/\partial y\right)$$

Spin of the flow = $\frac{1}{2}$ curl $\vec{q}$

$$= \frac{1}{2}\,\hat{k}\left(\partial v/\partial x - \partial v/\partial y\right)$$

If the flow is irrotational, then,

$$\text{Curl } q = \frac{1}{2}\left(\partial v/\partial x - \partial u/\partial y\right)\hat{k} = \vec{0}$$

$$\Rightarrow \partial v/\partial x - \partial u/\partial y = 0$$

$$\Rightarrow \underbrace{\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0}_{\text{Laplace Equation}}, \quad \left(u = -\partial\psi/\partial y,\ v = \partial\psi/\partial x\right)$$

Ex: Find out the velocity of the flow $w = ikz$

Let $w = f(z) = \phi + i\psi$

$$\partial\phi/\partial x = \partial\psi/\partial y, \quad \partial\phi/\partial y = -\partial\psi/\partial x$$

For 2 dimensional flow, $u = -\partial\phi/\partial x$, $v = -\partial\phi/\partial y$

$$\partial w/\partial z = \partial\phi/\partial x \cdot \partial x/\partial z + i\left(\partial\psi/\partial x \cdot \partial x/\partial z\right)$$

$$= -u + iv$$

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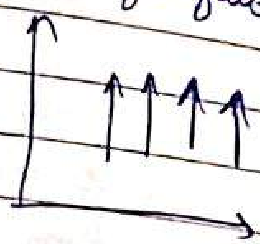
$w = ikz$

uniform flow



$\partial w / \partial z = \partial / \partial z \, (ikz)$

$= ik = -u + iv$

$\Rightarrow u = 0, \quad v = k$

Ex 2: $w = -ke^{-i\alpha} z$

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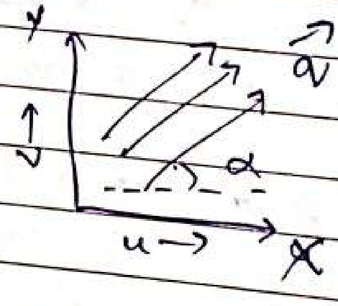
$\partial w / \partial z = -ke^{-i\alpha} = -u + iv$

~~$u = ke^{-i\alpha}$, $v = 0$~~

$\therefore \; -k(\cos\alpha - i\sin\alpha) = -u + iv$

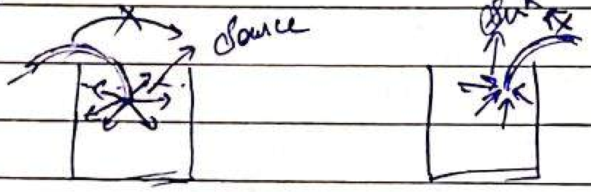
$\Rightarrow u = k\cos\alpha, \quad v = k\sin\alpha$

$\therefore$



* Source and Sink :-

If the motion of a fluid consists of cylindrical radial flow in all direction proceeding / emanating from a point, the point is known as simple source. If however, the flow is directed radially inwards to a point from all directions in a symmetrical manner, then the point is known as simple sink.



Source + Sink together = Doublet / Dipole

* Strength of a source / sink :-

Consider a source at the origin. Then the mass of the fluid coming out from the origin in a unit time is known as the strength of the source.

Similarly, the mass of fluid going into the sink per unit time at the origin is called strength of the sink.

Sink is a source of negative strength

Strength of Source = m

Strength of Sink = $-m$

Sources and sinks in 2-dimension :-

In 2D, a source of strength m is such that the flow across any small curve surrounding it is $2\pi m$.

Consider a circle of radius r with source at its center. Then the radial velocity

$$q_r = \frac{-1}{r} \partial\psi/\partial\theta \quad (1) \qquad \psi(r,\theta) \text{ is the stream fn.}$$

Also, $\partial\phi/\partial r = \partial\phi/\partial y \cdot \partial y/\partial r = \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \partial\phi/\partial x$

$$= \frac{\sin\theta}{\cancel{\cos\theta}} \times \partial\psi/\partial x \cancel{\times \partial\psi/\partial y}$$

$$= \sin\theta \times \partial\psi/\partial r \times \partial r/\partial x$$

$$=$$

$$\partial\phi/\partial r = 1/r \; \partial\psi/\partial\theta$$

Also, $q_r = -\partial\phi/\partial r \quad (2)$

Then the flow across the circle $2\pi r q_r$

Hence we have,

$$2\pi m = 2\pi r q_r$$

$$\Rightarrow r q_r = m \quad (3)$$

$$\Rightarrow -r \cdot 1/r \; \partial\psi/\partial\theta = m \text{ . by relation (1)}$$

$$\Rightarrow \partial\psi = -m\,\partial\theta$$

$$\Rightarrow \psi = -m\theta \quad (4)$$

From (3), (4)

$$-r \cdot \partial\phi/\partial r = m$$
$$\Rightarrow \partial\phi = -m\,\partial r/r$$
$$\Rightarrow \phi = -m \log_e r \quad \text{---(5)}$$

And for sink, $\phi = m \log_e r$

From (4) $\psi = -m\theta$ = constant

i.e. Stream functions are always constant.

$\phi = -m \log_e r$ = constant

$$\Rightarrow r = e^{-\phi/m} = \text{Konstant}$$

This shows that the ~~curves~~ circle curves of equi-potential (velocity) are r = constant, i.e constant circles with centre as origin.

* Complex potential w with source / sink :-

By definition :-

$$w = f(r,\theta)$$
$$= \phi(r,\theta) + i\,\psi(r,\theta)$$

$$= -m \log_e r - i\,m\theta$$

$$= -m\left(\log_e r + \log_e e^{i\theta}\right)$$

$$= -m \log_e re^{i\theta}$$

$$= -m \log_e z$$

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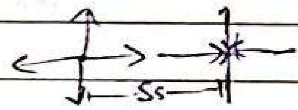
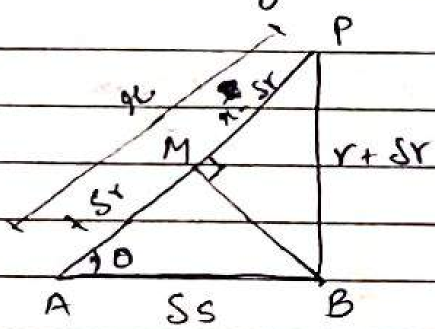
* Doublet (Dipole) in a fluid :-

A combination of a source of strength m and a sink of strength $-m$ at a small distance δs apart where in limit m is taken to be infinitely great & δs infinitely small but the product $m\delta s$ is finite and equal to μ. is called a doublet of strength μ, and the line δs is taken in the ~~same~~ sense of axis between poles $+m, -m$ of doublet.



Complex potential for a doublet. Let A & B denote the positions of a source & a sink and P be any point in the flow.

Let, $AP = r$, $BP = r + \delta r$
$\angle PAB = \theta$.



Let ϕ be the velocity potential due to doublet :- (at P)

Then $\phi = -m\log r + m\log(r+\delta r)$

$$= -m\log\left(\frac{r}{r+\delta r}\right) = +m\log\left(1+\frac{\delta r}{r}\right)$$

$$\phi = +m\left[\delta r/r + (\delta r/r)^2 + \cdots\right]$$

when δr is small, truncation

$$\phi = +m\,\delta r/r \quad -(1)$$

Let BM be the $\perp^r$ drawn from B to AP at M

Then, $AM = AP - MP = -r + (r+\delta r) = +\delta r$

$$\cos\theta = \frac{AM}{AB} = +\delta r/\delta s \Rightarrow \delta r = +\delta s\cos\theta \quad -(2)$$

From (1), (2) :-

$$\phi = -m\,\delta r/r = -m\frac{\delta s\cos\theta}{r} = -\mu\frac{\cos\theta}{r}$$

Sign issues

And,

$$\partial\phi/\partial r = \frac{-\mu\cos\theta}{r^2}$$

$$\Rightarrow \frac{1}{r}\partial\psi/\partial\theta = -\mu/r^2\cos\theta \quad \text{(By CR equation)}$$

$$\frac{\partial\phi}{\partial r} = \frac{1}{r}\partial\psi/\partial\theta$$

$$\Rightarrow \psi = -\frac{\mu}{r}\sin\theta + f(r) \quad -(3)$$

$$\frac{1}{r}\partial\phi/\partial\theta = \frac{-\partial\psi}{\partial r}$$

Again, $\frac{1}{r}\partial\phi/\partial\theta = -\partial\psi/\partial r$

$$\Rightarrow -1/r \times \mu/r\sin\theta = -\mu/r^2\sin\theta + f'(r)$$

$$\Rightarrow f'(r) = 0$$

$$\Rightarrow f(r) = \text{constant}$$

Omitting constant,

$$\phi = \mu \cos\theta / r, \quad \psi = -\mu/r \sin\theta$$

The required complex potential =

$$w = \phi + i\psi = \frac{\mu\cos\theta}{r} - i\frac{\mu}{r}\sin\theta$$

$$= \mu/r \, (\cos\theta - i\sin\theta)$$

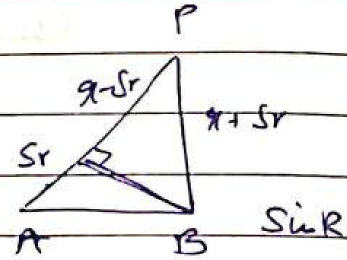
$$= \mu/r \, e^{-i\theta}$$

$$= \mu/z$$

→ Motion in 2D

Doublet :-

$\phi = m\log(r+\delta r) - m\log r$



$\phi = \frac{\mu}{r}\cos\theta \ , \ \psi = -\frac{\mu}{r}\sin\theta$

$\Rightarrow \ w = \phi + i\psi = \mu/z$

Remark 1: If the doublet makes an angle with x-axis then,

$w = \frac{\mu}{re^{i(\theta-\alpha)}}$ (replacing θ by $\theta-\alpha$)

2: If the doublet is located at a point z_1 then

$w = \frac{\mu}{z-z_1}$

3: Let $\mu_1, \mu_2, \ldots, \mu_m$ be the strength of a doublet at $z_1, z_2, \ldots$

$\Rightarrow \ w = \frac{\mu_1}{z-z_1} + \frac{\mu_2}{z-z_2} + \cdots + \frac{\mu_m}{z-z_m}$

Ex: There is a source of strength m at (0,0) and equal sinks (1,0) & (−1,0). Discuss the 2-dimensional ~~flow~~ motion

Soln: The given complex potential for the given flow

$w = \phi + i\psi = m\log(z-1) - m\log z + m\log(z+1)$

$\phi + i\psi = m[\log((x-1)+iy) - \log(x+iy) + \log((x+1)+i$

$\phi = \frac{m}{2}\left[\log((x-1)^2+y^2) - \log(x^2+y^2) + \log((x+1)^2+y^2)\right]$

$\psi = m\left[\tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x+1}\right]$

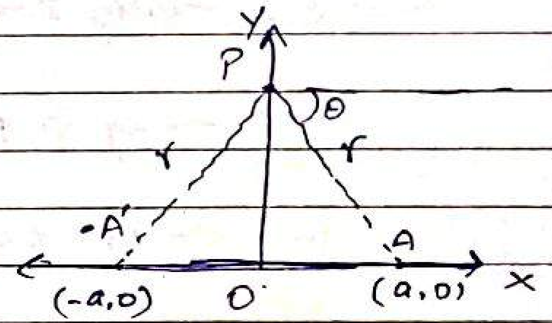
* Image of a source :-

If in a fluid a surface S can be drawn across which there is no flow, then ~~the~~ any ~~image~~ system of sources, sinks and doublets on opposite sides of this surface is known as the image of the system w.r.t the surface. Moreover if this surface is solid/impermeable and the fluid on one side is removed, the motion on the other side is unaffected/unchanged.

→ Complex Potential of an Image w.r.t source :-

Suppose that image of a source of strength m at $A(a,0)$ on the x-axis is required w.r.t OY.

Take an equal source at $A'(-a,0)$. Let P be any point on OY such that $AP = A'P = r$. Then the velocity at P due to A is m/r.



Also, velocity due to A' source is m/r.

$$\phi = -m\log r$$

$$= -m \log\sqrt{x^2+y^2}$$

$$\partial\phi/\partial x = \frac{-mx}{x^2+y^2}$$

$$\vec{q} = -\left(\partial\phi/\partial x\,\hat{i} + \partial\phi/\partial y\,\hat{j}\right)$$

$$= \frac{m}{r^2}\left(x\hat{i} + y\hat{j}\right)$$

$$|\vec{q}| = \frac{m\,|r|}{r^2} = m/r$$

Resultant velocity at $P = \frac{m}{r}\cos\theta - \frac{m}{r}\cos\theta = 0$ ("⊥" to OY, across surface) which shows that there is no flow/velocity along OY. Hence the image of ~~the~~ a source w.r.t a line in 2D is an equal source equidistant from the line opposite to the source.

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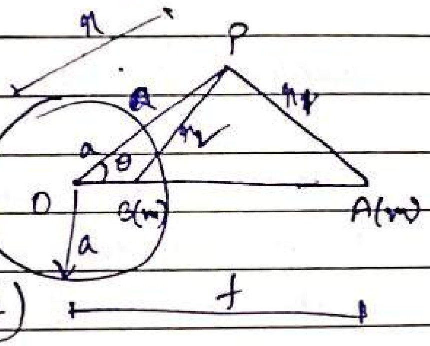
* Image of a source w.r.t a circle :-

Statement : The image system for a source outside circle consists of an equal source at the inverse point and ~~the~~ an equal sink at the centre of the circle

Proof :

Let us consider a source of strength m at A outside the circle of radius 'a' and centre at O. Let OA = f. B be the inverse pt. of A w.r.t the circle. Thus OA · PB = a^2 so that

OB = a^2/f , by Δ^{le} property



Then the complex potential due to the source at A & B

$$w = -m\log(z-a) - m\log(z-a^2/f)$$

$$= -m\log(x-f+iy) - m\log(x-a^2/f+iy)$$

$$= -m\log(r\cos\theta - f + ir\sin\theta) - m\log(r\cos\theta - a^2/f + ir\sin\theta) = \phi + i\psi$$

The velocity potential will be

$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1} y/x$

$$\phi = -m/2 \log\left[(r\cos\theta - f)^2 + (r\sin\theta)^2\right]$$

$$- m/2 \log\left[\left(r\cos\theta - a^2/f\right)^2 + (r\sin\theta)^2\right]$$

$$= -m/2 \log\left[r^2 + f^2 - 2rf\cos\theta\right] - m/2\log\left[r^2 + \frac{a^4}{f^2} - \frac{2a^2}{f}r\cos\theta\right]$$

$\vec{q} = -\nabla\phi \qquad \vec{q}_r = -\partial\phi/\partial r \; ; \; \vec{q}_\theta = \frac{-1}{r}\,\partial\phi/\partial\theta$

For The given system to be image, when $r \to a$ (radius of circle) the radial velocity must become 0.

$\therefore \; \cancel{\left(\partial\phi/\partial r\right)_{r=a}} = \left(\vec{q}_r\right)_{r=a} = 0$

$\to \; \left(-\partial\phi/\partial r\right)_{r=a} = 0$

$$-\partial\phi/\partial r = m/2 \left[\frac{2(r - f\cos\theta)}{r^2 + f^2 - 2rf\cos\theta} + \frac{2(r - \frac{a^2}{f}\cos\theta)}{r^2 + \frac{a^4}{f^2} - 2r\frac{a^2}{f}\cos\theta} \right]$$

$$\left[-\partial\phi/\partial r\right]_{r=a} = m/a$$

which is not 0. But if a sink of strength $-m$ is kept at center,

Then $\vec{q} = -m/a$ cancels out about radial velocity.

$\therefore$ Image of a source outside a circle w.r.t circle is a doublet with source at inverse pt. & sink in center of circle

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Ex 1: Image of a doublet

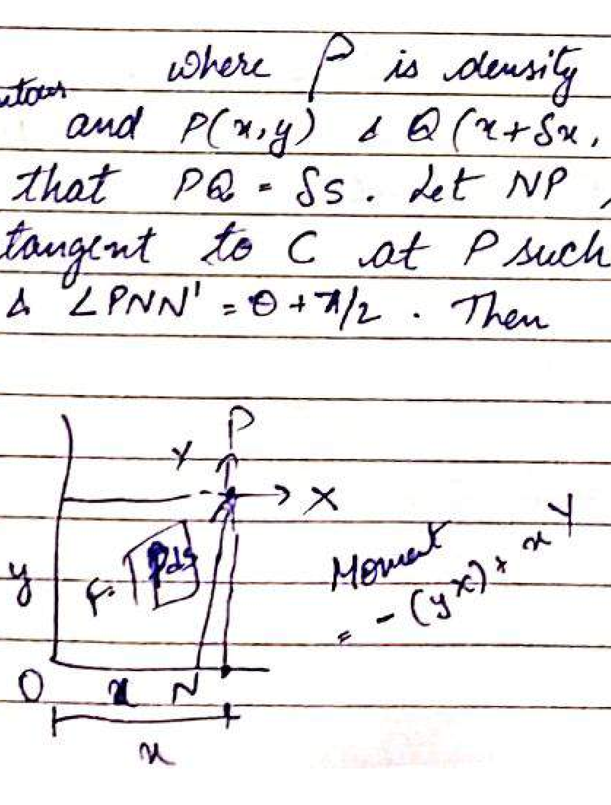
Ex 2: Milne - Thompson Theorem : Let $f(z)$ be the complex potential for a flow having no rigid boundaries such that there is no singularities of flow within the circle $|z| = a$. Then on introducing the solid cylinder $|z| = a$ into the flow, the new complex potential is given by $w = f(z) + \bar{f}(a^2/z)$, $|z| \geq a$

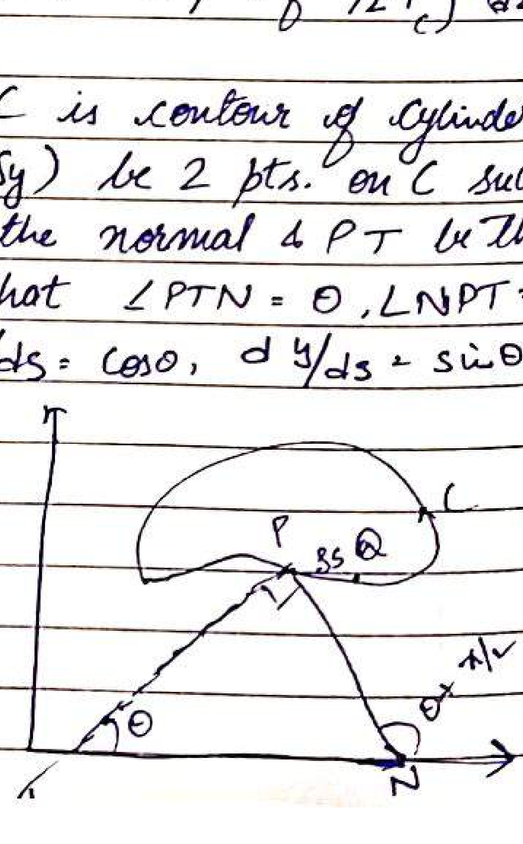
* The Blasius Theorem : In a steady 2D irrotational motion of an incompressible fluid under no external forces given by a complex potential $w = f(z)$. If the pressure thrusts on a fixed cylinder of any shape is represented by a force (X, Y) and a couple of moment ~~M~~ M about the origin of coordinates then,

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz \quad \& \quad M = \text{Real part of } -\frac{1}{2} i\rho \int_C z\left(\frac{dw}{dz}\right)^2 dz$$

where ρ is density & C is contour of cylinder

Proof :- Let C be contour and $P(x,y)$ & $Q(x+\delta x, y+\delta y)$ be 2 pts. on C such that $PQ = \delta s$. Let NP be the normal & PT be the tangent to C at P such that $\angle PTN = \theta$, $\angle NPT = \pi/2$ & $\angle PNN' = \theta + \pi/2$. Then $dx/ds = \cos\theta$, $dy/ds = \sin\theta$





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~~tot~~ Now if p denotes the pressure ~~at~~ P, the force per unit length of the section δs is $p\,\delta s$ normal to C. By figure (1a),

$$X = \int_C p\cos(\pi/2+\theta)\,ds,$$ (As Force acts along normal)

$$Y = \int_C p\sin(\pi/2+\theta)\,ds$$

normal makes $\pi/2+\theta$ with x-axis.

$$\left.\begin{aligned} \Rightarrow X &= \int_C -p\sin\theta\,ds = -\int_C p\,dy \\ Y &= \int_C p\,dx \end{aligned}\right\} \quad (1)$$

$$M = \int_C \left[xp\sin(\pi/2+\theta)\,ds - yp\cos(\pi/2+\theta)\,ds\right]$$

$$= \int_C \left[xp\cos\theta\,ds + yp\sin\theta\,ds\right]$$

$$= \int_C \left[xp\,dx + yp\,dy\right] = \int_C p(x\,dx + y\,dy) \quad -(2)$$

Now the bernouli's equation :-

$$\tfrac{1}{2}q^2 + p/\rho = \text{constant} = C$$

$$\Rightarrow \tfrac{1}{2}\rho q^2 + p = C\rho = A \quad - (3)$$

From (1), (3) :-

$$X = -\int_C \left(A - \tfrac{1}{2}\rho q^2\right) dy$$

$$Y = \int_C \left(A - \tfrac{1}{2}\rho q^2\right) dx$$

$$X - iY = -\left[\int_c (A - \tfrac{1}{2}\rho q^2)\,dy + i\,(A - \tfrac{1}{2}\rho q^2)\,dx\right]$$

$$= -\left[\int_c A\,(dy + i\,dx)\right] + \frac{1}{2}\int_c (\rho q^2 dy + i\rho q^2 dx)$$

$\downarrow$

$= 0$ by Cauchy residual Theorem
Analytic f^n over closed contour

$$= \tfrac{1}{2}\int_c (\rho q^2 dy + i\rho q^2 dx)$$

$$= \tfrac{1}{2}\rho \int iq^2\left(\frac{dy}{i} + dx\right)$$

$$= \tfrac{1}{2}i\rho \int q^2 (dx - i\,dy) = \tfrac{1}{2}i\rho \int (u^2+v^2)(dx - i\,dy)$$

$$= \tfrac{1}{2}i\rho \int (u - iv)^2 (dx + i\,dy)$$

$$= \frac{i\rho}{2}\int \left(\frac{dw}{dz}\right)^2 dz \qquad [w = f(z)]$$

H.W $M = ?$

To Prove :-

$$u^2+v^2\,(dx - i\,dy) = (u - iv)^2\,(dx + i\,dy)$$

Now, the contour is the streamline,

$$dx/u = dy/v$$

$$\Rightarrow dx/u = dy/v = \frac{dx + i\,dy}{u + iv} = \frac{dx - i\,dy}{u - iv}$$

$$\Rightarrow \frac{dx + i\,dy}{dx - i\,dy} = \frac{u + iv}{u - iv} \Rightarrow \frac{u^2 + v^2}{(u - iv)^2}$$

* General Theory of Irrotational Flow

1. Connected region :-

A region of space is said to be a connected region if a continuous curve joining any 2 pts. of the region lies entirely in the region.

2. Flow & Circulation :-

If A & P be any 2 points in fluid then the value of the integral $\int_A^P (u\,dx + v\,dy + w\,dz)$ taken along the path from A to P is called flow along the path from A to P.

When the velocity potential exists then $\vec{q} = -\nabla\phi$; the flow from A to P is :-

$$= \int_A^P \left(-\frac{\partial\phi}{\partial x}dx - \frac{\partial\phi}{\partial y}dy - \frac{\partial\phi}{\partial z}dz\right)$$

$$= -\int_A^P d\phi = \phi_A - \phi_P$$

The flow around a closed curve is called circulation around the curve

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3. Stoke's Theorem :-

a) Let $\vec{q}$ be the velocity vector, Ω is the velocity & S be the surface bounded by a closed curve C. Then

$$\Gamma = \int_C \vec{q}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{q})\cdot \hat{n}\,ds = \int_S \vec{\Omega}\cdot\hat{n}\,ds$$

$$\Rightarrow \Gamma = \int_S \vec{\Omega}\cdot\hat{n}\,ds$$

where Γ is the circulation around C and ~~is~~ $\hat{n}$ is unit outward drawn normal.

In other words, the circulation Γ around any closed

b) curve C drawn in a fluid is equal to the surface integral of the normal component of spin (i.e. vorticity vector) taken over any surface provided the surface lies wholly in the fluid.

4. Kelvin Circulation Theorem :-

When the external forces are conservative and derivated from a single valued potential function and the density is a function of pressure only, i.e. $\rho = \rho(P)$, then the circulation in a particular closed circuit moving with the fluid is constant at all time.

single valued : ~~injective~~ scalar valued

Proof :- Given ~~a~~ $\exists$ a V such that $\vec{F} = -\vec{\nabla}V$.

$$F_x = -\partial V/\partial x, \quad F_y = -\partial V/\partial y, \quad F_z = -\partial V/\partial z \quad -(1)$$

Let $\vec{q}$ be the velocity vector, then

$\Gamma = \int_C \vec{q}\cdot d\vec{r}$, where C is the curve ~~along~~ enclosing S in such a way that orientation of

C is anti-clockwise.

$$\Rightarrow \frac{D\Gamma}{Dt} = \frac{D}{Dt} \oint_C \vec{q}\, d\vec{r}$$

$$= \oint_C \frac{D}{Dt} (\vec{q}\, d\vec{r})$$

$$= \oint_C \left[\frac{D\vec{q}}{Dt} \cdot d\vec{r} + \vec{q}\, d\left(\frac{D}{Dt}\vec{r}\right) \right] \quad -(2)$$

By Euler's Equation :-

$$D\vec{q}/Dt = F - \frac{\nabla p}{\rho}$$

$$= -\vec{\nabla} V - \frac{\vec{\nabla} p}{\rho}$$

$$D\vec{q}/Dt\, d\vec{r} = -\vec{\nabla} V \cdot d\vec{r} - \frac{\vec{\nabla} p}{\rho} \cdot d\vec{r}$$

$$\Rightarrow (2) = \oint_C \left(-\vec{\nabla} V d\vec{r} - \frac{\vec{\nabla} p}{\rho} d\vec{r} + \vec{q} \cdot d\vec{q} \right)$$

$$\vec{\nabla} V d\vec{r} = \frac{\partial V}{\partial x_i} dx_i = dV$$

$$\vec{\nabla} p\, d\vec{r} = dp$$

$$= \oint_C \left[-dV - \frac{dp}{\rho} + \frac{1}{2} d(\vec{q}^2) \right]$$

$$= \left[\frac{1}{2} q^2 - V - \int \frac{dp}{\rho} \right]_C \quad -(3)$$

Here $[\cdot]_C$ ~~indicates~~ denotes the change in the quantity enclosed within brackets on moving once around C. Since $\vec{q}$, V & ρ are single valued functions then the R.H.S of (3) is 0.

$$\frac{d\Gamma}{dt} = 0 \quad \Rightarrow \Gamma = \text{constant}$$

~~* Irrotational Motion :-~~

5. → Permeance of irrotational motion :-

When the external forces are are conservative & derivable from a single valued potential function & density depends on pressure only, then the motion of an inviscid fluid, if once irrotational, remains irrotational even afterwards.

• Proof :-

From Stoke's Theorem,

$$\Gamma = \int_C \vec{q}\cdot d\vec{r} = \int_S \text{curl}\,\vec{q}\cdot\hat{n}\,ds$$

At any time t_1, let the motion be irrotational i.e.

$$\text{curl}\,\vec{q} = 0 \;\Rightarrow\; \underset{(\text{circulation})}{\Gamma} = 0 \text{ at time } t_1$$

By Kelvin Circulation Theorem $\left(\frac{D\Gamma}{Dt} = 0 \Rightarrow \Gamma = \text{constant}\right)$,

$$\Rightarrow \Gamma = 0 \quad \forall t.$$

$$\Rightarrow \int_S \text{curl}\,\vec{q}\cdot d\vec{s} = 0 \quad \text{at any } t$$

Since S is arbitrary, $\text{curl}\,\vec{q} = \vec{0}$ at all time

$\Rightarrow$ The flow is irrotational at all time

6. Green's Theorem :-

Let ϕ & ϕ' be 2 single valued functions (continuously differentiable) then,

$\partial\phi/\partial n = \nabla\phi \cdot \hat{n}$

$$\int_V (\nabla\phi \cdot \nabla\phi')\, dV = +\int_S \phi \frac{\partial\phi'}{\partial n} dS - \int_V \phi \nabla^2\phi'\, dV$$

(Div. Th. for product of 2 f'n)

$$= \int_S \phi' \frac{\partial\phi}{\partial n} dS - \int_V \phi' \nabla^2\phi\, dV$$

where S is a closed surface bounding a simply connected region V.

Also,

$$\int_S \phi \frac{\partial\phi'}{\partial n} dS - \int_V \phi \nabla^2\phi'\, dV = \int_S \phi' \frac{\partial\phi}{\partial n} dS - \int_V \phi' \nabla^2\phi\, dV$$

Let ϕ, ϕ' be the velocity then $\nabla^2\phi = \nabla^2\phi' = 0$

$$\Rightarrow \int_S \phi \frac{\partial\phi'}{\partial n} dS = \int_S \phi' \frac{\partial\phi}{\partial n} dS \quad \text{— (1)}$$

By (1) it shows that for an irrotational flow if there are 2 possible motions inside S by means of 2 different impulsive pressures on the boundary, then work done by the first 'p' acting through the displacement produced by the second 'p' must be equal to the work done by second 'p' acting through the displacement produced by first 'p'.

Now,

let ϕ be the velocity potential of fluid s.t $\nabla^2\phi = 0$ & let $\phi = \phi'$. then

$$\vec{q} = -\nabla\phi$$

$$\int_V (\nabla\phi)^2 dv = \int_S \phi \, \partial\phi/\partial n \, ds$$

$$\Rightarrow \int_V |\vec{q}|^2 dv = \int_S \phi \, \partial\phi/\partial n \, ds$$

$$\Rightarrow \rho \int_V |\vec{q}|^2 dv = \rho \int_S \phi \, \partial\phi/\partial n \, ds$$

$$\Rightarrow \underbrace{\frac{1}{2}\int_V \rho \, |\vec{q}|^2 dv}_{K.E} = \rho/2 \int_S \phi \, \partial\phi/\partial n \, ds$$

$$\left(\partial\phi/\partial n = \nabla\phi \cdot \hat{n} = -\vec{q} \cdot \hat{n} \right)$$

$$= \rho/2 \int_S \phi \, (\underbrace{-\vec{q}\cdot\hat{n}}_{\text{Inflow}}) \, ds$$

If there is no activity in boundary (no inflow / outflow)

$$\Rightarrow \vec{q} \cdot \hat{n} = 0$$

$$\Rightarrow K.E = 0$$

$$\Rightarrow \int_V |\vec{q}|^2 dv = 0$$

As V is arbitrary,

$$|\vec{q}|^2 = 0$$

$$\Rightarrow \vec{q} = \vec{0}$$

$\Rightarrow$ No fluid velocity

1. Cyclic & Acyclic Irrotational-motion :-

A motion in which the velocity potential is a single-valued function is called acyclic & if it is not single-valued, it is called cyclic.

→ Theorem: Acyclic irrotational motion is impossible if the liquid is entirely bounded by a fixed rigid wall.

Proof
Soln: Since for a rigid wall, then $\vec{q}\cdot\hat{n} = 0 \Rightarrow -\nabla\phi\,\hat{n} = 0$
$\Rightarrow \partial\phi/\partial n = 0$

$$T = \tfrac{1}{2}\int_V \rho|\vec{q}|^2 dv = \int_S \phi\,\partial\phi/\partial n\; ds$$

$$\Rightarrow \int_V \rho\,|\vec{q}|^2 dV = 0$$

$$\Rightarrow \vec{q} = \vec{0}$$

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8. Kelvin Minimum Energy Theorem :-

The irrotational motion of a liquid occupying a simply converted region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary.

Proof :-

Let T_1 be the kinetic energy, $\vec{q_1}$ be the fluid velocity of the actual irrotational motion then $\exists$ a potential function ϕ s.t $\vec{q_1} = -\vec{\nabla}\phi$

Let T_2 be the kinetic energy, $\vec{q_2}$ be the fluid velocity of any other possible ~~motion~~ state of motion consistent with the same normal velocity of the boundary S. In other words

$\vec{q_1}\cdot\hat{n} = \vec{q_2}\cdot\hat{n}$, where $\hat{n}$ is the normal compont

Since incompressibility condition gives,

$$\vec{\nabla}\vec{q_1} = \vec{\nabla}\vec{q_2} = 0$$

Also,

$$T_1 = \frac{1}{2}\int_V \rho|\vec{q_1}|^2 dV \quad T_2 = \frac{1}{2}\int_V \rho|q_2|^2 dV$$

To show $T_1 \leq T_2$

$$T_2 - T_1 = \frac{1}{2}\int_V \rho\left(|\vec{q_2}|^2 - |\vec{q_1}|^2\right)dV$$

$$= \frac{1}{2}\int_V \rho\left[2\vec{q_1}[\vec{q_2}-\vec{q_1}] + (\vec{q_2}-\vec{q_1})^2\right]dV$$

$$= \rho\int_V \vec{q_1}[\vec{q_2}-\vec{q_1}]\,dV + \frac{1}{2}\rho\int_V (\vec{q_2}-\vec{q_1})^2 dV$$

$$= -\rho\int_V \nabla\phi(\vec{q_2}-\vec{q_1})dV + \frac{1}{2}\rho\int_V (\vec{q_2}-\vec{q_1})^2 dV$$

Now,

$$\vec{\nabla}\left[\phi(\vec{q_2}-\vec{q_1})\right] = \vec{\nabla}\phi(\vec{q_2}-\vec{q_1}) + \phi\,\cancel{\vec{\nabla}(\vec{q_2}-\vec{q_1})}^{\,0}$$

$$= \nabla\phi(\vec{q_1}-\vec{q_2})$$

$$\therefore T_2 - T_1 = -\rho\int_V \vec{\nabla}\left[\phi(\vec{q_2}-\vec{q_1})\right] dV + \frac{1}{2}\int_V \rho(\vec{q_1}-\vec{q_2})^2 dV$$

$$= -\rho\int_S \underbrace{\phi(\vec{q_2}-\vec{q_1})\,\hat{n}}_{=0}\, dS + \frac{1}{2}\int_V \rho(\vec{q_1}-\vec{q_2})^2 dV$$

$$\therefore T_2 - T_1 \geq 0$$

$$\Rightarrow T_2 \geq T_1$$

Motion of Cylinders

We study 2D Irrotational Flow produced by motion of a cylinder in an infinite mass of a liquid at rest. Or when cylinder is inserted in the stream.

Let ϕ be velocity potential & ψ stream fn then.

$$\partial\phi/\partial x = \partial\psi/\partial y \; , \; \partial\phi/\partial y = -\partial\psi/\partial x \quad \&$$

$$w = \phi + i\psi \quad \&$$

ψ satisfies $\nabla^2\psi = 0$

(i) Since the liquid is at rest at infinity.

$$v = \partial\psi/\partial x \; , \; u = \partial\psi/\partial y$$
$$= 0 \qquad\qquad = 0$$

(ii) At any fixed boundary the normal velocity is 0.

$$\vec{q}\cdot\hat{n} = 0$$
$$\Rightarrow \nabla\phi\cdot\hat{n} = 0$$
$$\Rightarrow \partial\psi/\partial x = 0 \; , \; \partial\psi/\partial y = 0$$
$$\Rightarrow \psi = \text{const. on boundary}$$

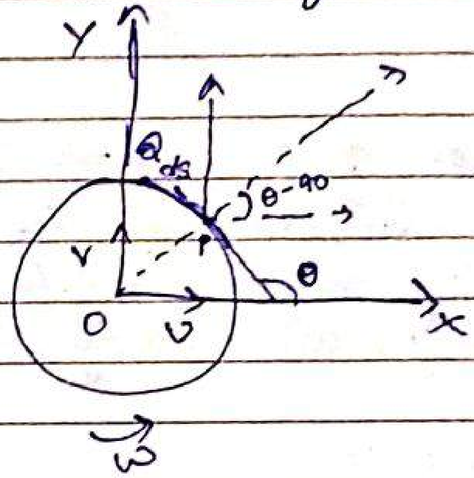
(iii) At the boundary of the moving cylinder the normal velocity of the fluid must be equal to normal component of the velocity of the ~~system~~ cylinder

If ~~the~~ a point on cylinder has a particular normal velocity. the surrounding fluid also has same ~~normal~~ velocity in the direction of the normal.

1. Stream Function

Let a point O of the cross-section of the cylinder be taken as origin. Let U & V be components of the velocities at O. & let the cylinder turned with angular velocity $\vec{\omega}$. If $P(x,y)$ & $Q(x,y)$ be any 2 points on the cylinder, then the velocity component at P is $U-\omega y$ & $V+\omega x$.

Let 'θ' be the angle made by the tangent at P, then $\sin\theta = dy/ds$, $\cos\theta = dx/ds$



∴ The normal component of velocity at P

$= (U-\vec{\omega}y)\cos(\theta-90) + (V+\omega y)\cos(180-\theta)$

$= (U-\omega y)\sin\theta - (V+\omega x)\cos\theta$

$= (U-\omega y)\, dy/ds - (V+\omega x)\, dx/ds \quad — (1)$

From stream function we know, the normal component of velocity $= -\partial\psi/\partial s \quad — (2)$

$$-\partial\psi/\partial s = (U-\omega y)\, dy/ds - (V+\omega x)\, dx/ds$$

$$\Rightarrow d\psi = -(U-\omega y)\, dy + (V+\omega x)\, dx$$

$$\Rightarrow \psi(x,y) = (Vx - Uy) + \tfrac{1}{2}\omega(x^2+y^2) + C$$

Case 1: If the cylinder is rotated about fixed axes and fluid is at rest $U = V = 0$

$$\Rightarrow \psi(x,y) = \tfrac{1}{2}\omega(x^2+y^2)$$

Case 2: $V = 0$, $\psi(x,y) = \tfrac{1}{2}\omega(x^2+y^2) - Uy + c$

2. Kinetic Energy :-

In any type of cylinder moving in a fluid at rest at infinity, the the Kinetic energy is given by

$$T = \frac{\rho}{2} \iint_S \phi \, \partial\phi/\partial n \, dS$$

3. Motion of a Cylinder :-

To determine the motion of a circular cylinder moving in an infinite mass of the liquid at rest at infinity, with velocity U in the direction of X-axis.

Solution :-

Let ϕ & ψ be the velocity potential and the stream function of the flow. $\vec{q} = (U, 0)$ $\quad \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$

$y = r\sin\theta$

$x = r\sin\theta$

Then,

$$\nabla^2 \phi = 0$$

$$\Rightarrow \partial^2\phi/\partial r^2 + 1/r \, \partial\phi/\partial r + 1/r^2 \, \partial^2\phi/\partial\theta^2 = 0$$

The solution of ① is given by $A_n r^n \sin n\theta$ (or) $B_n r^n \cos n\theta$ where $n \in \mathbb{Z}$. Now, the normal velocity at any pt. of the cylinder $= -\partial\phi/\partial r = U\cos\theta$, where $r = a$

$$\Rightarrow \partial\phi/\partial r = -U\cos\theta \quad \text{at } r = a.$$

$u = v = 0$ at $r \to \infty$

$$\Rightarrow \partial\phi/\partial r = 0 \ \ \& \ \ 1/r \, \partial\phi/\partial\theta = 0 \text{ at } r \to \infty$$

The above conditions suggest that

① — $\phi(r, \theta) = Ar\cos\theta + B/r \cos\theta$ s.t

$$\partial\phi/\partial r = A\cos\theta - B/r^2 \cos\theta, \quad \partial\phi/\partial\theta = -Ar\sin\theta + \frac{B}{r}\cos\theta$$

$$\Rightarrow \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -A\sin\theta + \frac{B}{r^2}\cos\theta$$

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Putting $r = a$,

$$\left.\partial\phi/\partial r\right|_{r=a} = (A - B/a^2)\cos\theta = -U\cos\theta \Rightarrow A - B/a^2 = -U$$

$$\Rightarrow B = a^2 U \quad - (5)$$

$r = \infty$,

$$\left.\partial\phi/\partial r\right|_{r=\infty} = A\cos\theta = 0 \Rightarrow A = 0 \quad - (4)$$

$\therefore$ From (1), (5), (4) :-

$$\phi(r,\theta) = \frac{Ua^2}{r}\cos\theta \quad - (*)$$

If $w = \phi + i\psi$ be the complex potential then ϕ, ψ satisfies CR equation.

$$\left.\begin{aligned} \partial\psi/\partial r &= -1/r\, \partial\phi/\partial\theta \\ \partial\psi/\partial\theta &= \partial\phi/\partial r \end{aligned}\right\} \quad (5)$$

From (5),

$$\partial\psi/\partial r = -1/r \times -\frac{Ua^2}{r}\sin\theta$$
$$= Ua^2/r^2 \sin\theta$$
$$\Rightarrow \psi = -Ua^2/r \sin\theta + \underline{f(\theta)} \quad - (6)$$

term can be ignored

From (*), (6)

$$w = \phi + i\psi = Ua^2/r\cos\theta - i\frac{Ua^2}{r}\sin\theta$$
$$= Ua^2/r\, e^{-i\theta} \quad \text{(Demoivre's theorem)}$$
$$= Ua^2/re^{i\theta}$$
$$= Ua^2/z$$

Also note,

Stream function :-

$$\psi = \text{constant}$$

$$\Rightarrow \frac{Ua^2}{r}\sin\theta = c$$

$$\Rightarrow \frac{Ua^2}{c} r \sin\theta = r$$

$$\Rightarrow c' r \sin\theta = r^2$$

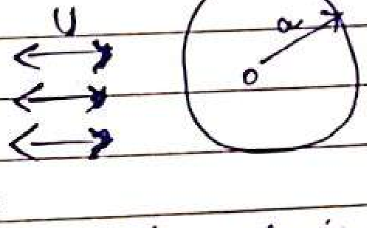
$$\Rightarrow c' y = x^2 + y^2$$

$$\Rightarrow x^2 + (y - c'/2)^2 = c'^2/4$$

This'' the equation of stream function.

* Motion of a cylinder in liquid ~~scheme~~ stream :-

(Flow past a cylinder)



Let the cylinder be at rest & let the liquid flow past the cylinder with velocity U in the $-ve$ direction of x-axis.
This motion may be deduced from the previous motion (previous result) by imposing a velocity $-U$ $\parallel$ to x-axis on both the cylinder & liquid. The cylinder is then reduced to rest. This will give

$$\partial\phi/\partial r = U\cos\theta - \frac{U}{a^2}\cancel{\sin}\cos\theta \quad \text{at } r = a$$

$$\partial\phi/\partial r = -U\cos\theta \text{ at } r = \infty$$

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$\Rightarrow \quad \phi(r,\theta) = \frac{Ua^2}{r}\cos\theta + Ur\cos\theta$

$= U\left(a^2/r + r\right)\cos\theta$

From CR-equation,

$\partial\psi/\partial r = -1/r \; \partial\phi/\partial\theta = -1/r \times U \times \left(a^2/r + r\right)\sin\theta$

$= U/r \left(a^2/r + r\right)\sin\theta$

$\psi(r,\theta) = U\left(r - a^2/r\right)\sin\theta$

Now.

$w = U\left(\frac{a^2}{r} + r\right)\cos\theta + iU\left(r - \frac{a^2}{r}\right)\sin\theta$

$= \frac{Ua^2}{Z} + UZ$

Stream f^n:-

$\psi = C$

$U(r - a^2/r)\sin\theta = C$

$Ur\sin\theta - \frac{Ua^2}{r}\sin\theta = C$

$U(r^2 - a^2)\sin\theta = rC$

$Ur^2\sin\theta - rC = Ua^2\sin\theta$

$\vdots$

The velocity distribution at any pt. $z = re^{i\theta}$ will be

$q = \left|dw/dz\right|$

$= \left|U\left(1 - a^2/z^2\right)\right|$

$= \left|U - Ua^2/z^2\right|$

$= \left|U - Ua^2r^{-2}e^{-2i\theta}\right|$

at $r = a$

$q|_a = \left|U - Ue^{-2i\theta}\right| = |U|\,\left|1 - e^{-2i\theta}\right|$

$= 2|U|\,|\sin\theta|$

$\Rightarrow q_{max} = 2|U|$ if $\theta = \pi/2$

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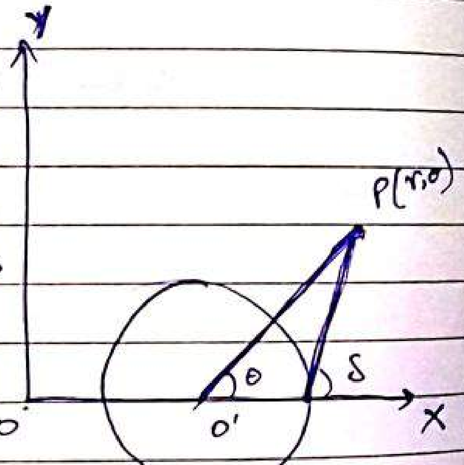
Problem) A cylinder of radius 'a' is moving with velocity U along x-axis, show that the motion produced by the cylinder in a mass of fluid at rest is given by the complex potential. $w = \dfrac{Ua^2}{z-Ut}$, $z = x+iy$. Find the magnitude & direction of velocity. Show that for a marked particle of the fluid we have,

$$\frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = \frac{U}{r^2}\left(\frac{a^2}{r^2}e^{i\theta} - e^{-i\theta}\right)$$

&

$$\left(r - \frac{a^2}{r}\right)\sin\theta = b$$

Soln:- Let O' be the centre of the cylinder cross-section at any 'time t'. Then the coordinates at O' is $(Ut, 0)$. We know that w.r.t origin $O(0,0)$ the complex potential is $\dfrac{Ua^2}{z}$. Then the complex at O is



$$w = \frac{Ua^2}{z - Ut} \quad \text{(translation of axis in x-direction)}$$

$$= \phi + i\psi$$

$$\frac{dw}{dz} = \frac{-Ua^2}{(z-Ut)^2} = \frac{-Ua^2}{[(z-Ut)+iy]^2}$$

Let $z - Ut = re^{i\theta}$

$$= \frac{-Ua^2}{a^2 e^{2i\theta}} = \frac{-Ua^2}{r^2}e^{-2i\theta}$$

$$= \frac{-Ua^2}{r^2}(\cos 2\theta - i\sin 2\theta)$$

$$= -u + iv$$

$$\Rightarrow \quad \frac{-Ua^2}{r^2}\cos 2\theta = -u$$

$$+\frac{Ua^2}{r^2}\sin 2\theta = v$$

$$\therefore \quad |q| = \sqrt{u^2+v^2} = Ua^2/r^2$$

$$dy/ds = \sin\delta \qquad dx/ds = \cos\delta$$

$$dy/dx = \tan\delta$$

$$\frac{dy/dr}{dx/dr} = \tan\delta \quad \Rightarrow \quad v/u = \tan\delta$$

$$\Rightarrow \quad \tan 2\theta = \tan\delta$$

$$\Rightarrow \quad 2\theta = \delta$$

Next, to find the motion of the fluid particle we consider fixed axes Ox & Oy at the instant where the center of the cylinder is at O'. To find the path of any particle referred to the cylinder we reduce the cylinder to rest. Hence relative to the cylinder the complex potential is given by,

$$w = Uz + \frac{Ua^2}{z}$$

$$= U\left(re^{i\theta} + \frac{a^2e^{-i\theta}}{r}\right)$$

$$\phi + i\psi = U(r + a^2/r)\cos\theta + iU(r - a^2/r)\sin\theta$$

Then,

$$\psi = \text{constant} = b'$$

$$\frac{r-a^2}{r}(\sin\theta) = b'/U = b \text{ (another constant)}$$

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$1 - a^2/r^2$

$\rightarrow dr/dt = -\partial\phi/\partial r = -U\cos\theta\left(1 - a^2/r^2\right)$

$\rightarrow r\,d\theta/dt = -1/r\ \partial\phi/\partial\theta = -1/r\ U\left(r + \frac{a^2}{r}\right)\sin\theta$

$= -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$

$\Rightarrow \frac{1}{r}\,dr/dt + i\,d\theta/dt$

$= -U/r\cos\theta\left(1 - a^2/r^2\right)$

$\quad - iU/r\sin\theta\left(1 + a^2/r^2\right)$

$= -U/r\ e^{i\theta} + U/r\ a^2/r^2\ e^{-i\theta}$

$= U/r\left(a^2/r^2\ e^{-i\theta} - e^{i\theta}\right)$

4. To find the complex potential due to circulation about a circular ~~the~~ cylinder. Let K be the constant circulation about the cylinder. Then the suitable form of ϕ may be obtained by equating to the circulation around a circle of radius r. Thus,

$-1/r\ \partial\phi/\partial\theta\,(2\pi r) = K.$

$\Rightarrow \partial\phi/\partial\theta = -K/2\pi \quad - (1)$

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By CR Eq^n :-

$\partial\phi/\partial r = 1/r\ \partial\psi/\partial\theta,\ 1/r\ \partial\phi/\partial\theta = -\partial\psi/\partial r \quad - (2)$

By (1) : $\phi(r,\theta) = -K/2\pi\ \theta$

By (2) : $\cancel{\phi}\,\psi(r,\theta) = K/2\pi\ \log_e r$

The required complex potential :-

$w = \phi + i\psi = -K/2\pi\ \theta + i\ K/2\pi\ \ln r$

$$\phi + i\psi = -\frac{K}{2\pi}(\theta - i\log r)$$

$$= \frac{Ki}{2\pi}(\log r + i\theta)$$

$$= \frac{Ki}{2\pi}(\log r + \log e^{i\theta})$$

$$= \frac{Ki}{2\pi}\log re^{i\theta} = \frac{Ki}{2\pi}\log z$$

The stream function :-

$$\frac{K}{2\pi}\log_e r = C$$

$$\Rightarrow r = \text{constant} = D$$

$$\Rightarrow x^2+y^2 = D^2 = E$$

$$\frac{dw}{dz} = \frac{Ki}{2\pi} \times \frac{1}{z} = \frac{Ki}{2\pi r} \cdot e^{-i\theta}$$

$$= \frac{K}{2\pi r}(i\cos\theta + \sin\theta)$$

$$\Rightarrow -u + iv = \frac{K}{2\pi r}(i\cos\theta + \sin\theta)$$

$$\Rightarrow u = -\frac{K}{2\pi r}\sin\theta$$

$$v = \frac{K}{2\pi r}\cos\theta$$

$$\Rightarrow |\vec{q}| = \frac{K}{2\pi r}$$

5. Streaming & Circulation around a fixed cylinder :-

The complex potential w_1 due to the circulation of strength 'K' about the cylinder

$$w_1 = \frac{Ki}{2\pi} \log z \quad -(1)$$

The complex potential w_2 due to streaming past a fixed ~~point~~ cylinder of radius 'a' and velocity 'U' is given by,

$$w_2 = Uz + \frac{Ua^2}{z} \quad -(2)$$

$$\therefore \quad w = w_1 + w_2 = \frac{Ki}{2\pi} \log z + Uz + \frac{Ua^2}{z} \quad -(3)$$

Do calculation

$$= \phi + i\psi$$

By comparision : $\phi(r,\theta) = U(r + a^2/r)\cos\theta - K\theta/2\pi$

$\psi(r,\theta) = U(r - a^2/r)\sin\theta + K \log r/2\pi \quad -(4)$

Around the cylinder, the normal component of velocity is zero, i.e. $\partial\phi/\partial r = 0$. ($q_r = 0$; $q = -\nabla\phi = -(\partial\phi/\partial r, \frac{1}{r}\partial\phi/\partial\theta)$)
Only tangential velocity exists, $-\frac{1}{r}\partial\phi/\partial\theta$ exists q_r q_θ

$$q_r = 0 \qquad q_\theta = \frac{1}{r} U(r + a^2/r)\sin\theta + K/2\pi r$$

$$\vec{q} = (q_r, q_\theta)$$

$$|\vec{q}|_{r=a} = \left[\frac{U}{r}(r + a^2/r)\sin\theta + K/2\pi r\right]_{r=a}$$

$$= \left|\frac{U}{a}(a+a)\sin\theta + K/2\pi a\right|$$

$$= \left|2U\sin\theta + K/2\pi a\right| \quad -(5)$$

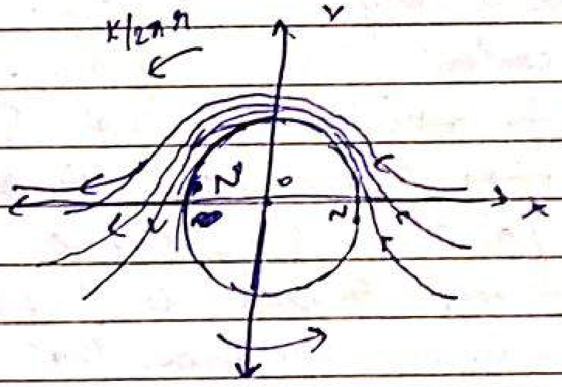
If there were no circulation i.e., $K=0$, there would be points of '0' velocity on the cylinder at $\theta = 0$ & π. However in the presence of circulation, the ~~velocity~~ stagnant points are attained i.e $\vec{q} = \vec{0}$ when

$$2U\sin\theta + K/_{2\pi a} = 0$$

$$\Rightarrow \quad \sin\theta = -K/_{4\pi U a}$$

$$\Rightarrow \quad |K| \leq |4\pi U a|$$ for 'θ' to exist for stagnant pt.

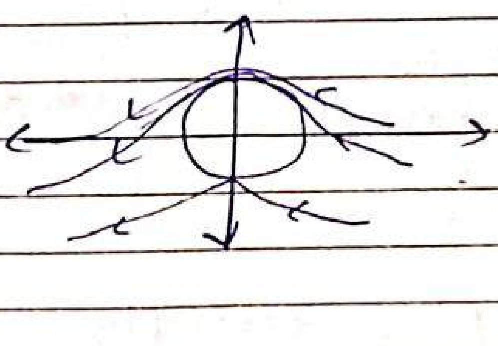
Case 1: $|K| < 4\pi a U$



$\theta = \pm\sin^{-1}\left(\frac{K}{4\pi U a}\right)$

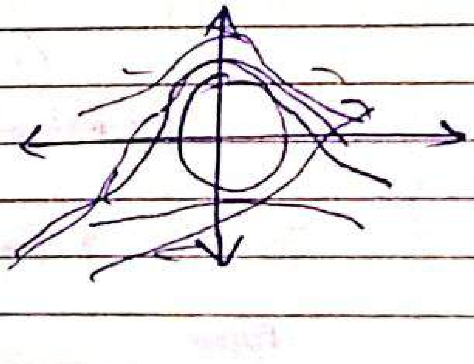
Case 2:

$K = 4\pi a U$



Case 3: $K > 4\pi a U$

No stagnant points



Aerodynamics :

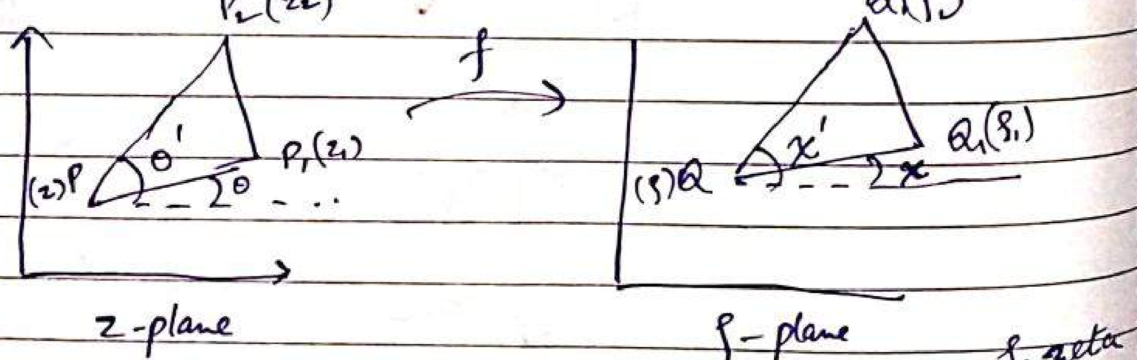
1. Aerofoils & Kutta-Joukowski Theorem :-

Conformal mapping : A mapping that preserves angle in both magnitude & orientation.

$$w = \phi + i\psi = f(Z) \xrightarrow[\text{conformal mapping}]{} g(\varsigma) = \overset{\xi + i\eta}{\cancel{\alpha + i\beta}}$$

simpler form

Let $f(z)$ be a single valued diff. complex function within a closed contour C in xy plane. Let $\varsigma = \xi + i\eta$ be another complex variable such that $\varsigma = f(z)$. Then corresponding ~~function~~ to each point in Z-plane or in C. There will be a point in ς-plane or in C'. The necessary condition for such map to exist is $f'(z) \neq 0$ at any pt. in Z-plane or in C. This also means that $d\varsigma/dz$ must exist independently of the direction of δz. Let P, P_1, P_2 & Q, Q_1, Q_2 be the '2' sets of points in Z-plane & ς-plane, respectively



z-plane $\qquad\qquad$ ς-plane

ς: zeta

Now,
$$\frac{\varsigma_1 - \varsigma}{z_1 - z} = \frac{f(z_1) - f(z)}{z_1 - z}$$

$$\frac{\varsigma_2 - \varsigma}{z_2 - z} = \frac{f(z_2) - f(z)}{z_2 - z}$$

When $P_1 \to P$, $P_2 \to P$

$$\frac{\zeta_1 - \zeta}{z_1 - z} = f'(z) \quad ; \quad \frac{\zeta_2 - \zeta}{z_2 - z} = f'(z)$$

$$\Rightarrow \frac{\zeta_1 - \zeta}{z_1 - z} = \frac{\zeta_2 - \zeta}{z_2 - z} = f'(z) = \frac{d\zeta}{dz} \quad \text{--- (1)}$$

Next,

$$\frac{QQ_1 e^{i\chi}}{PP_1 e^{i\theta}} = \frac{QQ_2 e^{i\chi'}}{PP_2 e^{i\theta'}}$$

$$\Rightarrow \frac{QQ_1}{PP_1} e^{i(\chi - \theta)} = \frac{QQ_2}{PP_2} e^{i(\chi' - \theta')}$$

When $P_1 \to P$ & $P_2 \to P$

$$\Rightarrow e^{i(\chi - \theta)} = e^{i(\chi' - \theta')}$$

$$\Rightarrow \chi - \theta = \chi' - \theta' \Rightarrow \chi' - \chi = \theta' - \theta$$

i.e $\angle Q_1 Q Q_2 = \angle P_1 P P_2$

Also, $\dfrac{QQ_1}{PP_1} = \dfrac{QQ_2}{PP_2} = |f'(z)| = \dfrac{d\zeta}{dz}$

$\Rightarrow \Delta P_1 P P_2$ & $\Delta Q_1 Q Q_2$ are similar

This establishes the similarity of the corresponding infinitesimal element of the 2 planes. Such a relation between the 2 planes is called conformal representation of either one of the other.

$$\zeta = f(z) \quad , \quad z = x+iy$$
$$= \xi + i\eta$$

$$\frac{d\zeta}{dz} = f'(z) = \frac{d}{dz}(\xi + i\eta)$$

$$= \frac{d}{dx}(\xi + i\eta)\frac{dx}{dz}$$

$$= \frac{d\xi}{dx} + i\frac{d\eta}{dx} \quad \text{— (1)}$$

Also, $$\frac{d\zeta}{dz} = \frac{d}{dy}(\xi + i\eta)\frac{dy}{dz}$$

$$= \frac{d\xi}{dy} + i\frac{d\eta}{dy} \quad \text{— (2)}$$

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Conformal Mapping :-

$$\zeta = f(z) = f(x+iy) = \xi + i\eta$$

From complex analysis :-

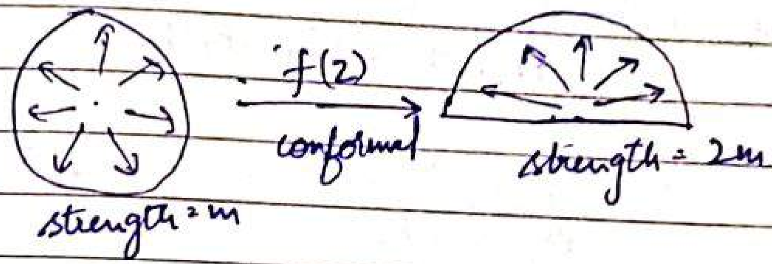
$$\frac{d\zeta}{dz} = \xi_x + i\eta_x$$

From CR-equation :-

$$\frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial y} \quad , \quad \frac{\partial\xi}{\partial y} = -\frac{\partial\eta}{\partial x}$$

$$\frac{d\zeta}{dz} = \frac{\partial\eta}{\partial y} + i\,\eta_x = i\left(\frac{\partial\eta}{\partial x} - i\frac{\partial\eta}{\partial y}\right)$$

$$\Rightarrow f'(z) = \sqrt{\left(\partial\eta/\partial x\right)^2 + \left(\partial\eta/\partial y\right)^2}$$



$w = F_1(z) = \phi + i\psi$, where ϕ & ψ are velocity & current functions of any motion within the contour C' in ξ-plane.

Then within C',

$$\phi + i\psi = F_1(\xi + i\eta) = f_1(\xi, \eta) + i\bar{f}_1(\xi, \eta)$$

& C' given by,

$$\psi = \bar{f}_1(\xi, \eta)$$

Similarly,

$$w = F_1(\zeta)$$
$$= F_1(\xi + i\eta) = F_1(\xi(x,y) + i\eta(x,y))$$

$$\Rightarrow \phi + i\psi = F_2(x,y) = f_2(x,y) + i\bar{f}_2(x,y)$$

In this case the contour is :-

$$\psi(x,y) = \bar{f}_2(x,y) = \text{constant}$$

This shows that ψ & ϕ are same in both the planes.

$$q_1 = dw/dz \quad , \quad q_2 = \cancel{\frac{d\xi}{dz}}\, dw/d\zeta$$

$$|q_1|^2 = |dw/dz|^2 \quad , \quad |q_2|^2 = \left|\frac{\cancel{d\xi}/dz}{dw/d\zeta}\right|^2$$

$$\Rightarrow |q_1|^2 = |q_2|^2$$

$$|dw/dz|^2 = |d\zeta/dz|^2 \quad -(1)$$

$$\frac{\Delta Q_1 Q Q_2}{\Delta P_1 P P_2} = \frac{\frac{1}{2} QQ_1 \cdot QQ_2 \cdot \sin \angle Q_1 Q Q_2}{\frac{1}{2} \cdot PP_1 \cdot PP_2 \cdot \sin \angle P_1 P P_2}$$

$$= \frac{QQ_1 \cdot QQ_2}{PP_1 \cdot PP_2} = |f'(z)|^2$$

$$= (\partial \xi / \partial x)^2 + (\partial \eta / \partial y)^2$$

$$=: h^2$$

$$\Rightarrow \Delta Q_1 Q Q_2 = h^2 \Delta P_1 P P_2$$

Area of the Δ^{le} in ζ & Z planes are of ratio h^2 to 1

$$\Rightarrow d\xi \, d\eta = h^2 dx dy$$

From ①, $|q_2|^2 = |dw/d\zeta|^2 = |dw/dz|^2 \, |dz/d\zeta|^2$

$$= q_1^2 / h^2$$

$$\Rightarrow |q_2|^2 d\xi d\eta = \frac{|q_1|^2}{h^2} \cdot h^2 dx dy$$

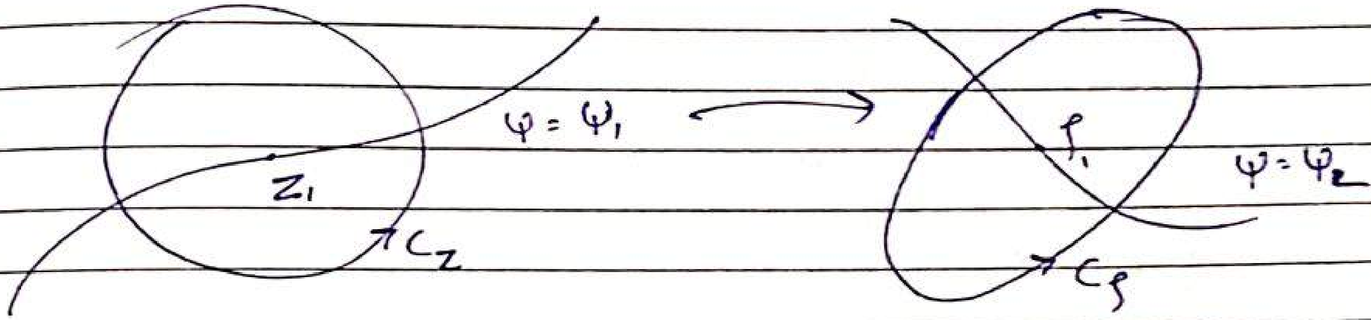
$$\Rightarrow |q_2|^2 d\xi d\eta = |q_1^2| \, dx dy$$

$$\Rightarrow \frac{1}{2} \iint \rho |q_2|^2 d\xi d\eta = \frac{1}{2} \iint \rho |q_1|^2 dx dy$$

$$\Rightarrow T_\zeta = T_Z$$

K.E is same in both planes.

Transformation of a source :-



Let there be a source of strength m at z_1 & ζ_1 be the corresponding pt in ζ-plane. Let these be regular points of transformation. Then a small curve C_z may be drawn to exclude z_1 & similarly C_ζ to exclude ζ_1. Since we know the value of stream function is independent of the domain considered, we have

$$\int_{C_z} d\psi = \int_{C_\zeta} d\psi$$

But, $$\int_{C_z} d\psi = \int_{C_z} \left(\partial\psi/\partial x \, dx + \partial\psi/\partial y \, dy \right) = \int_{C_z} \left(-v\,dx + u\,dy \right)$$

= the total flow across the contour C_z

= sum of strength of sources " " C_z.

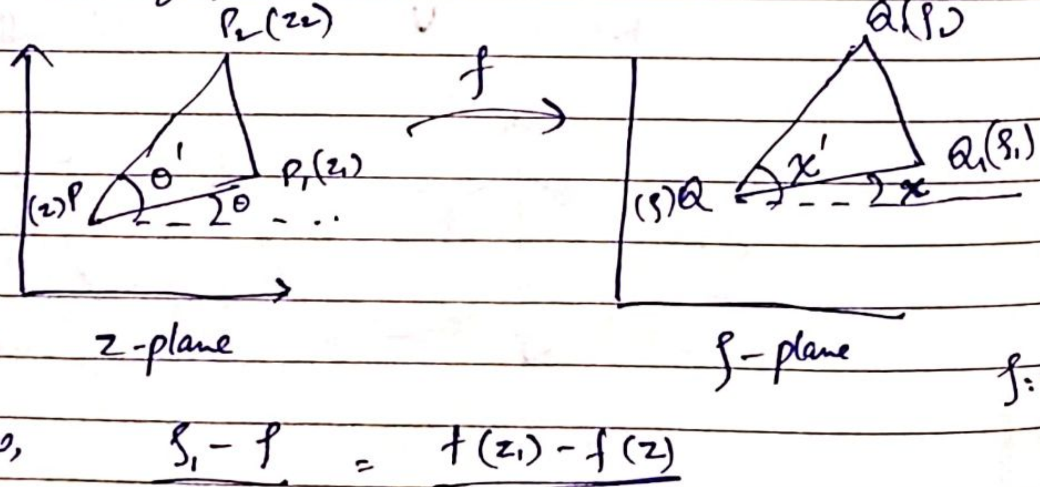
Aerodynamics

1. ## Aerofoils & Kutta-Joukowski Theorem :-

Conformal mapping : A mapping that preserves angle in both magnitude & orientation.

$$w = \phi + i\psi = f(z) \xrightarrow[\text{conformal mapping}]{} g(\zeta) = \underbrace{\xi + i\eta}_{\text{simpler form}} \quad (\cancel{\alpha + i\beta})$$

Let $f(z)$ be a single valued diff. complex function within a closed contour C in xy plane. Let $\zeta = \xi + i\eta$ be another complex variable such that $\zeta = f(z)$. Then corresponding ~~function~~ to each point in z-plane or in C, there will be a point in ζ-plane or in C'. The necessary condition for such map to exist is $f'(z) \neq 0$ at any pt. in z-plane or in C. This also means that $d\zeta/dz$ must exist independently of the direction of δz. Let P, P_1, P_2 & Q, Q_1, Q_2 be the '2' sets of points in z-plane & ζ-plane, respectively



ζ: zeta

Now,
$$\frac{\zeta_1 - \zeta}{z_1 - z} = \frac{f(z_1) - f(z)}{z_1 - z}$$

$$\frac{\zeta_2 - \zeta}{z_2 - z} = \frac{f(z_2) - f(z)}{z_2 - z}$$

When $P_1 \to P$, $P_2 \to P$

$$\frac{\zeta_1 - \zeta}{z_1 - z} = f'(z) \quad ; \quad \frac{\zeta_2 - \zeta}{z_2 - z} = f'(z)$$

$$\Rightarrow \frac{\zeta_1 - \zeta}{z_1 - z} = \frac{\zeta_2 - \zeta}{z_2 - z} = f'(z) = \frac{d\zeta}{dz} \quad \text{— (1)}$$

Next,

$$\frac{QQ_1 e^{i\chi}}{PP_1 e^{i\theta}} = \frac{QQ_2 e^{i\chi'}}{PP_2 e^{i\theta'}}$$

$$\Rightarrow \frac{QQ_1}{PP_1} e^{i(\chi - \theta)} = \frac{QQ_2}{PP_2} e^{i(\chi' - \theta')}$$

When $P_1 \to P$ & $P_2 \to P$

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$$\varsigma = f(z) \quad , \quad z = x+iy$$
$$= \xi + i\eta$$

$$\frac{d\varsigma}{dz} = f'(z) = \frac{d}{dz}(\xi + i\eta)$$

$$= \frac{d}{dx}(\xi + i\eta)\, \frac{dx}{dz}$$

$$= \frac{d\xi}{dx} + i\frac{d\eta}{dx} \quad \text{—(1)}$$

Also, $$\frac{d\varsigma}{dz} = \frac{d}{dy}(\xi + i\eta)\,\frac{dy}{dz}$$

$$= \frac{d\xi}{dy} + i\frac{d\eta}{dy} \quad \text{—(2)}$$

24/10/19

Conformal Mapping :-

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From complex analysis :-

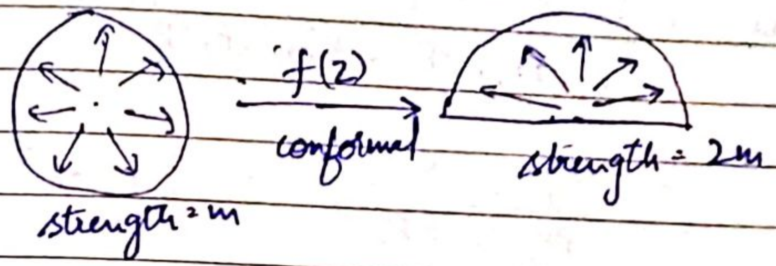
$$\frac{d\varsigma}{dz} = \xi_x + i\eta_x$$

From CR-equation :-

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} \quad , \quad \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x}$$

$$\frac{d\varsigma}{dz} = \frac{\partial \eta}{\partial y} + i\,\eta_x = i\left(\frac{\partial \eta}{\partial x} - i\,\frac{\partial \eta}{\partial y}\right)$$

$$\rightarrow f'(z) = \sqrt{\left(\partial\eta/\partial x\right)^2 + \left(\partial\eta/\partial y\right)^2}$$



$w = F_1(z) = \phi + i\psi$, where ϕ & ψ are velocity & current functions of any motion within the contour C' in $\xi\,\zeta$-plane.

Then within C',

$$\phi + i\psi = F_1(\xi + i\eta) = f_1(\xi, \eta) + i\bar{f}_1(\xi, \eta)$$

& C' given by,

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$$q_1 = dw/dz \quad , \quad q_2 = \cancel{\frac{d\zeta}{dz}}\, dw/d\zeta$$

$$|q_1|^2 = |dw/dz|^2 \quad , \quad |q_2|^2 = \left|\frac{\cancel{d\zeta}/dz}{dw/d\zeta}\right|^2$$

$$\Rightarrow |q_1|^2 = |q_2|^2$$

$$|dw/dz|^2 = |d\zeta/dz|^2 \quad -(1)$$

$$\frac{\Delta Q_1 Q Q_2}{\Delta P_1 P P_2} = \frac{\tfrac{1}{2} QQ_1 \cdot QQ_2 \cdot \sin \angle Q_1 Q Q_2}{\tfrac{1}{2} \cdot PP_1 \cdot PP_2 \cdot \sin \angle P_1 P P_2}$$

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$$= (\partial \xi/\partial x)^2 + (\partial \eta/\partial y)^2$$

$$=: h^2$$

$$\Rightarrow \Delta Q_1 Q Q_2 = h^2 \Delta P_1 P P_2$$

Area of the Δ^{le} in ζ & Z planes are of ratio h^2 to 1

$$\Rightarrow d\xi\, d\eta = h^2 dx\, dy$$

From ①, $|q_2|^2 = |dw/d\zeta|^2 = |dw/dz|^2 \, |dz/d\zeta|^2$

$$= q_1^2/h^2$$

$$\Rightarrow |q_2|^2 d\xi\, d\eta = \frac{|q_1|^2}{h^2} \cdot h^2 dx\, dy$$

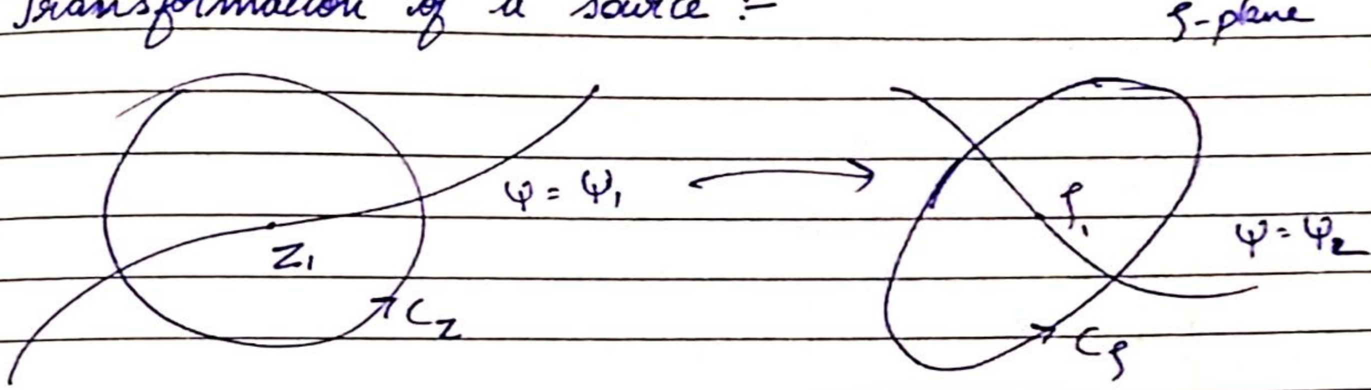
$$\Rightarrow |q_2|^2 d\xi\, d\eta = |q_1|^2 \, dx\, dy$$

$$\Rightarrow \tfrac{1}{2}\int \rho |q_2|^2 d\xi\, d\eta = \tfrac{1}{2}\int \rho |q_1|^2 dx\, dy$$

$$\Rightarrow T_\zeta = T_Z$$

K.E is same in both planes.

Transformation of a source :-



Let there be a source of strength m at z_1 & ζ_1 be the corresponding pt in ζ-plane. Let these be regular points of transformation. Then a small curve C_Z may be drawn to exclude z_1 & similarly for C_ζ to exclude ζ_1. Since we know the value of stream function is independent of the domain considered, we have

$$\int_{C_Z} d\psi = \int_{C_\zeta} d\psi$$

But, $\displaystyle\int_{C_Z} d\psi = \int_{C_Z} \left(\partial\psi/\partial x\, dx + \partial\psi/\partial y\, dy\right) = \int_{C_Z} \left(-v\,dx + u\,dy\right)$

= the total flow across the contour C_Z

= sum of strength of sources " " C_Z.

4.
Theorem: Under conformal ~~definition~~ representation a uniform line source maps into another uniform line source of the same strength.

Proof:-

Let there be a uniform line source of strength 'm' per unit length through the pt. $z = z_0$ & suppose the conformal transformation $\zeta = f(z)$. Let C_{z_0} be the curve around z_0 in z-plane & C_{z_0} is mapped into C_{ζ_0} in ζ-plane. Then $\zeta = \zeta_0$ lies in C_{ζ_0}.

The complex potential in z-plane & ζ-plane is same and has the form

$$w = \phi + i\psi, \text{ in } z\text{-plane}$$
$$w = \phi' + i\psi', \text{ in } \zeta\text{-plane}$$
$$\Rightarrow \quad \phi = \phi', \quad \psi = \psi'$$

Since 'ψ' is the same at the corresponding pts. $z = z_0$ & $\zeta = \zeta_0$ we have $\int_{C_{z_0}} d\psi = \int_{C_{\zeta_0}} d\psi'$ —(1)

But in z-plane,

$$w = -m\log(z - z_0)$$

$$\Rightarrow dw = -m\,dz/(z - z_0)$$

$$\Rightarrow \int_{C_{z_0}} dw = -\int_{C_{z_0}} m\,dz/(z - z_0)$$

$$\Rightarrow \int_{C_{z_0}} (d\phi + d\psi) = -m \cdot 2\pi i \quad \text{by Cauchy's residue thm}$$

$$\Rightarrow \int_{C_{z_0}} d\psi = -2\pi m \quad \text{—(2)}$$

(2) indicates the value of fluid crossing unit thickness of C_{z_0} per unit time

$$\Rightarrow \int_{C_{\zeta_0}} d\psi' = -2\pi m$$

$\leftarrow$ indicates

$\Rightarrow$ the volume of the fluid crossing unit thickness of C_{ζ_0} per unit time

$\therefore$ The conformal mapping preserves the strength of the simple source.

5. Kutta - Joukowskii theorem :-

When a cylinder of any shape is placed in a uniform stream of speed U, the resultant ~~thrust~~ thrust on the cylinder is a lift of magnitude $K\rho U$ per unit length & at right angles to the stream. K is the circulation around cylinder.

Proof :- Let there be a fixed cylinder of some form in the finite region of the plane, its cross-section containing origin. The disturbance of the stream caused by the cylinder can be represented at a great distance in the form:

$$w = Az + B/z^2 + \dots\dots, \text{ where } A, B \text{ depend on } U \,\&\, K.$$

Let the direction of the stream makes an angle α with x-axis. Then the complex potential due to uniform stream velocity 'U' is

$$w_2 = Ue^{-i\alpha} z \quad \text{— (2)}$$

And due to circulation,

$$w_3 = \frac{ik}{2\pi} \log z \quad \text{— (3)}$$

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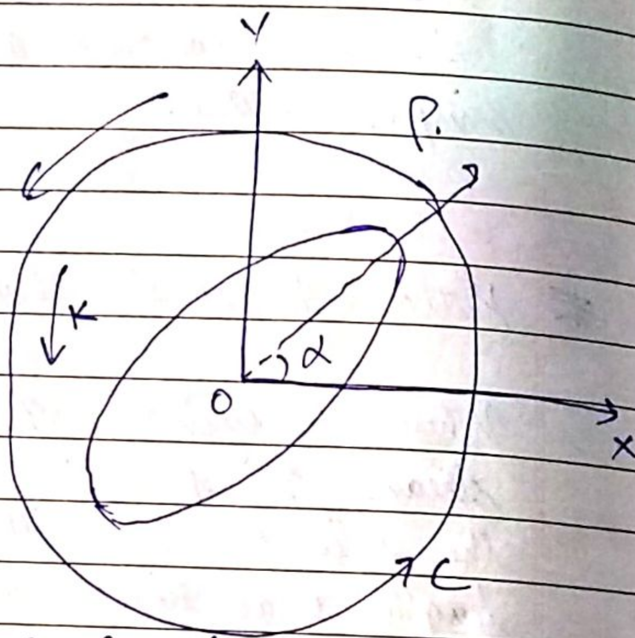
$\therefore$ The complete complex potential is

$$w = w_1 + w_2 + w_3$$
$$= Ue^{-i\alpha z} + \frac{iK}{2\pi}\log z + Az + \frac{B}{z^2} + \dots \quad -(4)$$

By Blasius Theorem :-

The force exerted on the cylinder is



$$X - iY = \tfrac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2$$

$$= \tfrac{1}{2} i\rho \int_C \left[-i\alpha Ue^{-i\alpha z} + \frac{iK}{2\pi z} + A - \frac{B}{z^2} - \frac{C}{z^3}\right]^2$$

$$= \tfrac{1}{2} i\rho \int_C \left[-i\alpha Ue^{-i\alpha z}\right] dz + \cancel{\tfrac{1}{2}i\rho\int_C}\left[\frac{iK}{2\pi z} + A - \frac{B}{z^2} - \frac{C}{z^3}\dots\right]^2 dz$$

$$= \tfrac{1}{2} i\rho \int_C \left[\left(-i\alpha Ue^{-i\alpha z}\right)^2 dz + \left(\frac{iK}{2\pi z} + A - \frac{B}{z^2} - \frac{C}{z^3}\right)^2 dz + 2\left(-i\alpha Ue^{-i\alpha z}\right)\left(\frac{iK}{2\pi z} + A - \frac{B}{z^2}\dots\right) dz\right]$$

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$$\boxed{dw/dz = Ue^{-i\alpha} + \frac{iK}{2\pi z} - \dots}$$

$X - iY = \tfrac{1}{2} i\rho \cdot 2\pi i$ (Sum of the residues at $z=0$)

Since $z=0$ is the pole of dw/dz inside C, therefore the residue at $z=0 = \frac{iK\,Ue^{-i\alpha}}{\pi}$

$$X - iY = \tfrac{1}{2} \cdot i\rho \cdot 2\pi i \cdot \frac{iK\,Ue^{-i\alpha}}{\pi}$$

$$= -i\rho KUe^{-i\alpha}$$

$$\Rightarrow X - iY = -i\rho KU(\cos\alpha - i\sin\alpha)$$

$$X = -KU\rho\sin\alpha$$

$$Y = \rho KU\cos\alpha$$

Resultant lift $= \sqrt{X^2 + Y^2}$

$$= \sqrt{(\rho K U)^2 (\cos^2\alpha + \sin^2\alpha)}$$

$$= \rho K U$$

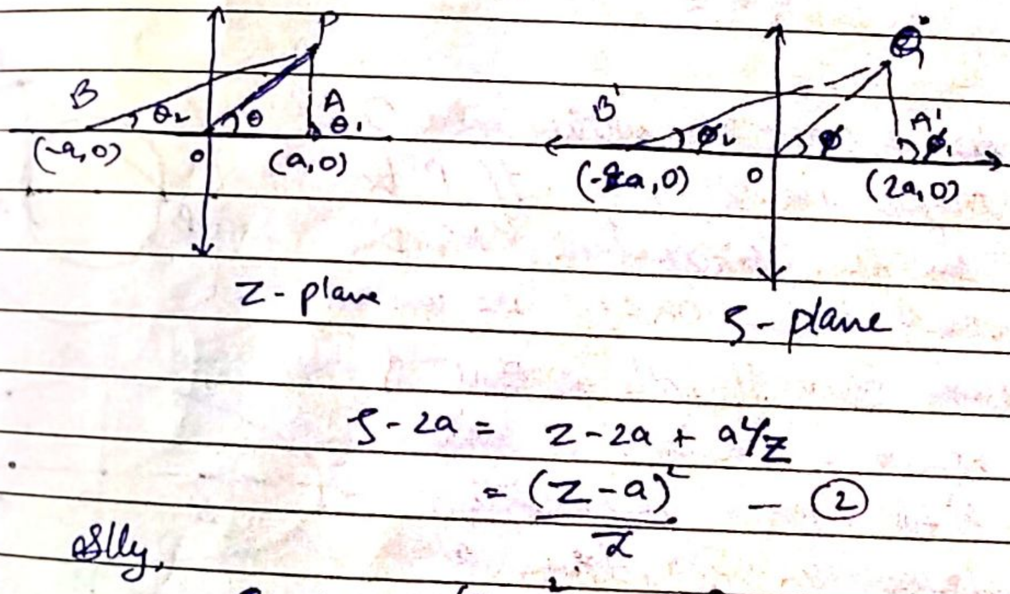
which always acts right angle to cylinder.

Joukowskii Transformation, Joukowskii aerofoils & Hypothesis

Consider the transformation, :-

$$\zeta = z + a^2/z \quad -(1)$$

Now, let A & B be the points at $z = -a, a$. These points are getting mapped to A' & B' resp. i.e. $\zeta = 2a$ & $\zeta = -2a$



$$\zeta - 2a = z - 2a + a^2/z$$

$$= \frac{(z-a)^2}{z} \quad -(2)$$

Slly,

$$\zeta + 2a = \frac{(z+a)^2}{z} \quad -(3)$$

So from (3) :-

$$A'Q e^{i\phi} = \frac{(AP e^{i\theta_1})^2}{OP e^{i\theta}}$$

$$\Rightarrow A'Q = \frac{AP^2}{OP} \ \& \ e^{i\phi_1} = e^{i(2\theta_1 - \theta)} \Rightarrow \phi_1 = 2\theta_1 - \theta \quad -(4)$$

Again from (2) :-

$$B'Q = BP^2/OP \quad \& \quad \phi_2 = 2\theta_2 - \theta \quad -(5)$$

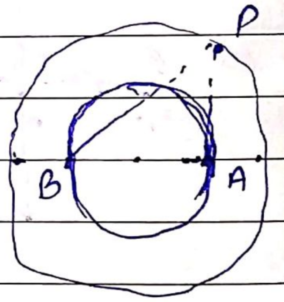
$$\therefore A'QB' = \phi_1 - \phi_2 = 2(\theta_1 - \theta_2) = 2\angle APB \quad -(6)$$

From (4) ; (5) :-

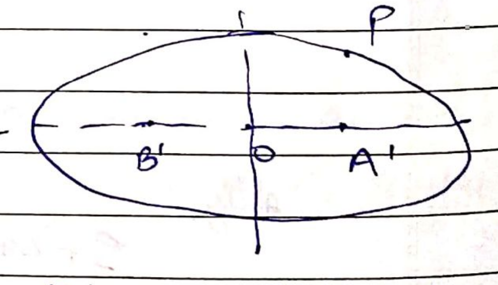
$$A'Q + B'Q = \frac{AP^2 + BP^2}{OP} = \frac{2(OP^2 + OA^2)}{OP},$$

since OA is median of ΔAPB,
i.e. $AP^2 + BP^2 = 2(OP^2 + OA^2)$

When $|z|$ is very large, then
$\xi = z$



Consider a circle C with centre O at origin in z-plane. If P be a pt on this circle then, $A'Q + B'Q$ = constant because OA & OP are constant from (vi). It follows that Q will describe an ellipse with foci A' & B'. Hence Joukowskii transformation (i) transforms a circle in z-plane to an ellipse in ξ-plane. Also the angle is getting doubled via the transformation. i.e if P be a point on the smaller circle with $\angle APB = \pi/2$ then $\angle A'QB' = \pi$ i.e. Q lies on A'B'!



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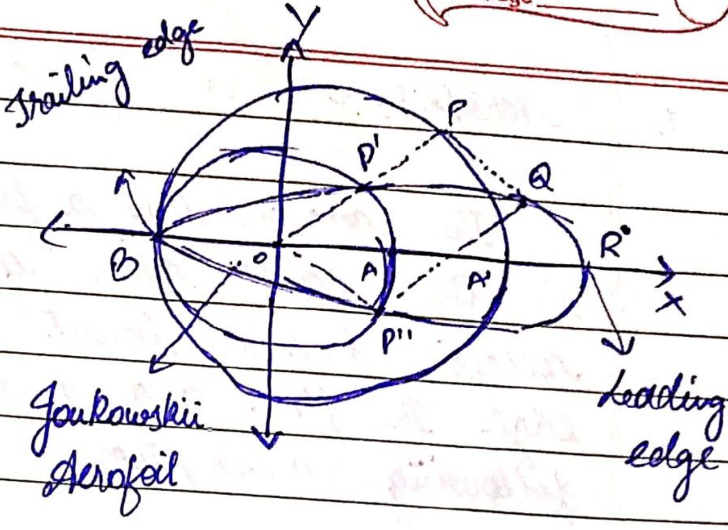


Let P be the pt. "Z" on the bigger circle & P' be it's inverse point w.r.t circle AB & P'' be the reflection of P' on x-axis, then P'' be the pt. a^2/z. Then we draw a parallelogram with OP, OP' as adjacent sides & OQ be the diagonal s.t Q be the pt. $\zeta = z + a^2/z$. The locus of the point Q is a fish shaped contour which touches the line BA on both sides & such ~~colour~~ contour is known as Joukowskii's aerofoils where B is the trailing edge & R is the leading edge.

Also,

$$\frac{d\zeta}{dz} = 1 - a^2/z^2$$

So, $d\zeta/dz = 0 \Rightarrow z = \pm a$

And $z = a$ is mapped $\zeta = 2a$ &
$\zeta = -2a$

But we need to know

$\frac{d\zeta}{dz}$ at B so either $\frac{d\zeta}{dz} = 0$ (or) $\frac{d\zeta}{dz} = \infty$

If $\vec{q}$ be the velocity at B for the circle & $\vec{q}'$ be the velocity at B' of the aerofoil, then

$$|q'| = |dw/d\zeta| = |dw/dz| \cdot |dz/d\zeta| = q\,|dz/d\zeta|$$

If $d\zeta/dz = 0$ then $dz/d\zeta = \infty \Rightarrow \vec{q}' = \infty$ at B $\rightarrow$ not suitable

To avoid infinite velocity at trailing edge. $\frac{d\zeta}{dz} = 0$ at trailing edge B.

The velocity at B is taken as 0, i.e taken as stagnation pt. of flow in Z-plane. This is Joukowskill Hypothesis.

7. __Aerofoils :-__

The aerofoil has a fish type profile. It is employed in the construction of modern airplanes. Such an aerofoil has a blunt leading edge & a sharp leading edge. The flow around the aerofoil depends on the following assumptions.

(i) The air behaves as an incompressible fluid

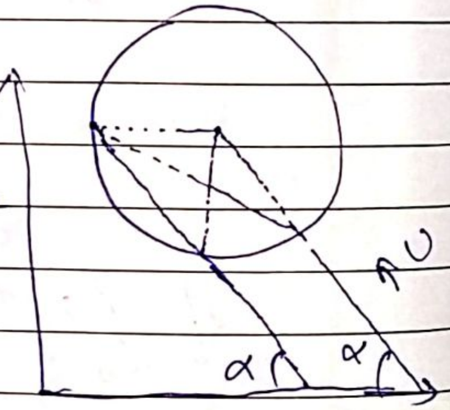
(ii) The aerofoil is a cylinder whose cross-section is a curve of fish type.

(iii) The flow is 2D irrotation cyclic motion

__Flow past a circle :-__

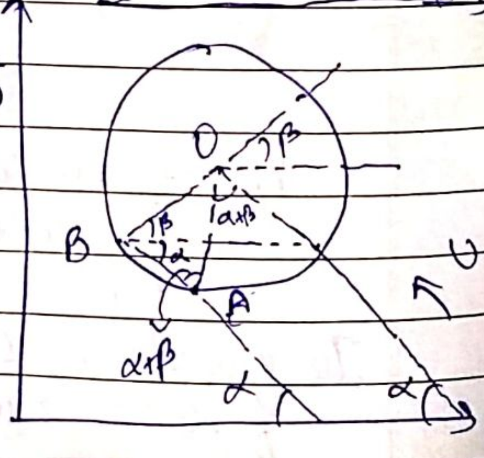
Let U be the velocity of the stream at infinity. Its direction making an angle α with the $-ve$ ~~dir~~ x-axis, K be the circulation around the ~~cylinder~~ circle whose center is at z_0 & radius b. Then,

$$w = Ue^{i\alpha}(z-z_0) + \frac{Ub^2e^{-i\alpha}}{z-z_0} + \frac{ik}{2\pi}\log(z-z_0), \quad b \text{ is the volume}$$



To find the stagnation point

$$\frac{dw}{dz} = 0 \Rightarrow Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{(z-z_0)^2} + \frac{ik}{2\pi(z-z_0)} = 0 \quad -(1)$$



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Taking the stagnation pt. as $z = z_0 + be^{i(\pi+\beta)}$
$= z_0 - be^{i\beta}$.

Then, ① reduces to,

$$U\left(e^{i\alpha} - e^{-i(\alpha+2\beta)}\right) - \frac{ik}{2\pi b} e^{-i\beta} = 0$$

$$\Rightarrow \quad K = 4\pi b U \sin(\alpha+\beta)$$

$$\Rightarrow K \leq 4\pi b U \quad \text{since } \sin(\alpha+\beta) \leq 1 \quad \text{—②}$$

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* Flow around a circle :-

$$w = Ue^{i\alpha}(z-z_0) + Ub^2e^{-i\alpha}/_{z-z_0} + {ik}/_{2\pi} \log(z-z_0)$$

$$\Rightarrow dw/dz = Ue^{i\alpha} - Ub^2e^{-i\alpha}/_{(z-z_0)^2} + {ik}/_{2\pi} \log(z-z_0)$$

For stagnation points, $dw/dz = 0$. Let us take the stagnation point as :-

$$z = z_0 + be^{i(\pi+\beta)} = z_0 - be^{i\beta}$$

Eq. 1 reduces to,

$$U\left[e^{i\alpha} - e^{-i(\alpha+2\beta)}\right] - {ik}/_{2\pi b}\, e^{-i\beta} = 0$$

$$\Rightarrow 2\pi b U\left[e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}\right] = ik$$

$$\Rightarrow 2\pi b U \, 2i \sin(\alpha+\beta) = ik$$

$$\Rightarrow k = 4\pi b U \sin(\alpha+\beta)$$

$$\Rightarrow k < 4\pi b U, \quad \text{since } b, U > 0$$

The stagnation points are given by $z - z_0 = be^{i(\pi+\beta)} = t$

From ①,

$$Ue^{i\alpha} - 1/_{t^2}\, U^2 b^2 e^{-i\alpha} + \frac{ik}{2\pi t} = 0$$

$$\Rightarrow t = be^{-i\alpha}\left[-i\sin(\alpha+\beta) \pm \cos(\alpha+\beta)\right], \quad \text{putting } k = \frac{4\pi b U}{\sin(\alpha+\beta)}$$

Blasius
Exam point of view

$= be^{-i\alpha} e^{-i(\alpha+\beta)}$ or, $-be^{+i\beta}$

$= be^{-i(2\alpha+\beta)}$ or $-be^{i\beta}$

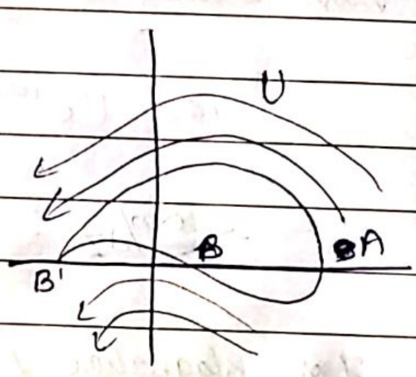
$\therefore$ the points $t_1 = be^{-i(2\alpha+\beta)}$

$= be^{-i[2\pi-(2\alpha+\beta)]}$

& $t_2 = -be^{+i\beta} = be^{i(\pi+\beta)}$

are points B & A, respectively where $dw/dz = 0$. i.e, A & B are the stagnation pts.

9. Flow past an aerofoil :-



From Joukowski transformation,

(1) $\zeta = z + a^2/z$ transforms a circle into an aerofoil. Thus the stagnation point P will be transformed into trailing edge B' and the other stagnation point A will be transformed to A''. B' will lie on the -ve side of ζ-axis s.t $OB' = -2a$. Let U be the velocity of the stream at infinity. its direction making an angle α with ζ-axis. We take the origins of both z & ζ planes to coincide then B & B' will lie on the left side of origin O.

By Blasius Theorem,

$$X - iY = \frac{1}{2} i\rho \int_{C_\zeta} \left(\frac{dw}{d\zeta}\right)^2 d\zeta \quad \text{—} (2)$$

From (1) :-

$$\frac{dw}{d\zeta} = \frac{dw}{dz} \cdot \frac{dz}{d\zeta} = \frac{dw/dz}{d\zeta/dz} = \frac{dw/dz}{1-a^2/z^2}$$

$$\left(\frac{dw}{d\zeta}\right)^2 = \frac{dw}{d\zeta} \cdot \frac{dw}{d\zeta} = \frac{(dw/dz)^2}{\ldots} \quad \text{—} (3)$$

Cauchy Residual Theorem

Exam point of view

Now,

$$w = Ue^{i\alpha}(z-z_0) + Ub^2e^{-i\alpha}/z + iK/2\pi \log(z-z_0)$$

$$dw/dz = Ue^{i\alpha} - Ub^2e^{-i\alpha}/(z-z_0)^2 + iK/2\pi(z-z_0)$$

$$1/(1-a^2/z^2)\,(dw/dz)^2 = (1-a^2/z^2)^{-1}\left[Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{z^2}(1-z_0/z)^{-2} + \frac{iK}{2\pi z}(1-z_0/z)^{-1}\right]^2$$

$$= (1+2a^2/z^2+\dots)\left[Ue^{-i\alpha} - \frac{Ub^2e^{-i\alpha}}{z^2}(1+2z_0/z+\dots) + \frac{iK}{2\pi z}(1+z_0/z+\dots)\right]^2 \quad -(4)$$

By, (2)

$$X - iY = i\rho/2 \int_{C_2} \frac{(dw/dz)^2}{(1-a^2/z^2)}\,dz$$

$$= i\rho/2 \cdot 2\pi i\left[\text{sum of the residues of } \frac{(dw/dz)^2}{(1-a^2/z^2)}\right]$$

$$= \cancel{i\rho/2 \cdot 2\pi i}$$

$$= -\pi\rho \text{ coeff. of } 1/z \text{ in RHS of } (4)$$

$$= -\pi\rho\, 2Ue^{i\alpha}\, iK/2\pi = -i\rho K e^{i\alpha} U$$

$X = \rho K U \sin\alpha \quad Y = \rho K U \cos\alpha \Rightarrow F = \rho K U$

Lift acting on the aerofoil $= \rho K U = 4\pi b U^2 \sin(\alpha+\beta)$

10. Moment of the force acting on aerofoil

$$(dw/d\zeta)^2 \times \zeta \, d\zeta = \frac{z+a^4/z}{1-a^4/z^2} \times \left(\frac{dw}{dz}\right)^2 \times dz$$

$$\frac{z+a^4/z}{1-a^4/z^2}\left(dw/dz\right)^2 = \left(z+a^4/z\right)\left(1+a^4/z^2+\cdots\right)\left[Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{z^2\left(1+\frac{2z_0}{z}\cdots\right)} + \frac{ik}{2\pi z}\left(1+\frac{z_0}{z}\cdots\right)\right]$$

$$= \left(z+\frac{2a^2}{z}+\cdots\right)\left(\cdots\cdots\right)$$

$\Rightarrow$ coeff. of $\frac{1}{z} = -2U^2b^2 - \frac{K^2}{4\pi^2} + \frac{iUe^{i\alpha}z_0}{\pi} + 2a^2U^2e^{2i\alpha}$

$$M = \text{Real part of } \left\{ -\frac{\rho}{2} \times \text{coeff. of } 1/z \times 2\pi i \right\}$$

$$= Re\left[-\frac{\rho}{2}\left\{ -2b^2U^2 - \frac{K^2}{4\pi^2} + \frac{iUe^{i\alpha}z_0}{\pi} + 2a^2U^2e^{2i\alpha} \right\} \times 2\pi i \right]$$

Taking $z_0 = ce^{i\lambda}, c>0$ then, $M = Re\left[-\frac{\rho}{2}\left\{ -2b^2U^2 - \frac{K^2}{4\pi^2} + \frac{iUe^{i\alpha}ce^{i\lambda}}{\pi} + 2a^2U^2e^{2i\alpha}\right\} 2\pi i\right]$

$$M = 2\pi\rho U^2\left[2bc\sin(\alpha+\beta)\cos(\alpha+\lambda) + a^2\sin 2\alpha\right]$$

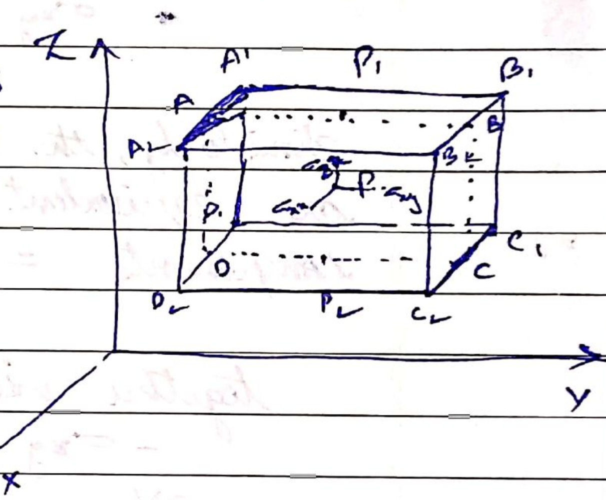
This is the amount at z_0. The moment at the trailing edge is $aY + M$

Required moment at B

$$= a\rho K U \cos\alpha + 2\pi\rho U^2 \left[2bc \sin(\alpha+\beta)\cos(\alpha+\eta) + a^2 \sin 2\alpha \right]$$

$$= 4a\pi b U^2 \rho \cos\alpha + 2\pi\rho U^2 \left[2bc \sin(\alpha+\beta)\cos(\alpha+\eta) + a^2 \sin 2\alpha \right]$$

Navier-Stokes Equation



With $P(x,y,z)$ as center and edges of length δx, δy, δz parallel to coordinates axes, we construct an elementary rectangular parallelopiped. Let us consider the fluid motion is viscous. We assume that the fluid element is moving & its mass is $\rho\,\delta x\,\delta y\,\delta z$. Let the coordinate of P_2 & P_1 be $(x+\frac{\delta x}{2}, y, z)$ & $(x-\frac{\delta x}{2}, y, z)$ respectively. At P the force components $\parallel$ to OX, OY, OZ on the rectangular surface $ABCD$ of area $\delta y\,\delta z$ through P & having $\hat{i}$ as the unit normal are $= (\sigma_{xx}\,\delta y\,\delta z,\ \sigma_{xy}\,\delta y\,\delta z,\ \sigma_{xz}\,\delta y\,\delta z)$ — ①

At the point $P_2(x+\delta x/2, y, z)$ the components of the force are $A_2B_2C_2D_2$ is

$$= \left[\left(\sigma_{xx} + \frac{\delta x}{2}\frac{\partial\sigma_{xx}}{\partial x}\right)\delta y\,\delta z,\ \left(\sigma_{xy} + \frac{\delta x}{2}\frac{\partial\sigma_{xy}}{\partial x}\right)\delta y\,\delta z,\ \left(\sigma_{zx} + \frac{\delta x}{2}\frac{\partial\sigma_{zx}}{\partial x}\right)\delta y\,\delta z \right] \quad \text{— ②}$$

Slly force on P_1 in $A_1B_1C_1D_1$,

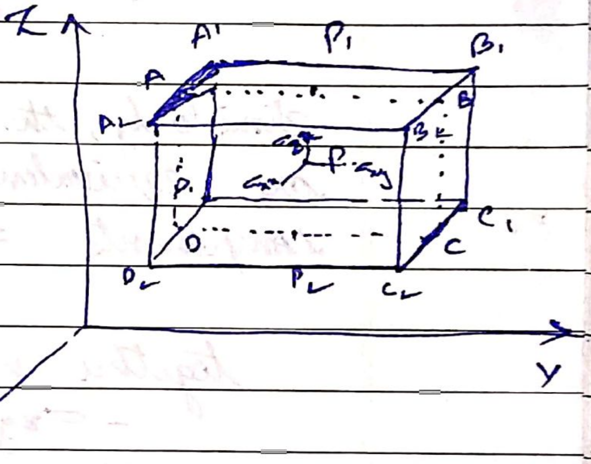
$$= \left[-\left(\sigma_{xx} - \frac{\delta x}{2}\frac{\partial\sigma_{xx}}{\partial x}\right)\delta y\,\delta z,\ -\left(\sigma_{xy} - \frac{\delta x}{2}\frac{\partial\sigma_{xy}}{\partial x}\right)\delta y\,\delta z,\ -\left(\sigma_{xz} - \frac{\delta x}{2}\frac{\partial\sigma_{xz}}{\partial x}\right)\delta y\,\delta z \right] \quad \text{— ③}$$

Required moment at B

$$= a\rho K U \cos\alpha + 2\pi\rho U^2 \left[2bc \sin(\alpha+\beta)\cos(\alpha+\eta) + a^2 \sin 2\alpha \right]$$

$$= 4a\pi b U^2 \rho \cos\alpha + 2\pi\rho U^2 \left[2bc \sin(\alpha+\beta)\cos(\alpha+\eta) + a^2 \sin 2\alpha \right]$$

Navier-Stokes Equation



With $P(x,y,z)$ as center and edges of length $\delta x, \delta y, \delta z$ parallel to coordinates axes, we construct an elementary rectangular parallelopiped. Let us consider the fluid motion is viscous. We assume that the fluid element is moving & its mass is $\rho\,\delta x\,\delta y\,\delta z$. Let the coordinate of P_2 & P_1 be $(x+\frac{\delta x}{2}, y, z)$ & $(x-\frac{\delta x}{2}, y, z)$ respectively. At P the force components $\parallel$ to OX, OY, OZ on the rectangular surface ABCD of area $\delta y\,\delta z$ through P & having $\hat{i}$ as the unit normal are $= (\sigma_{xx}\,\delta y\,\delta z,\ \sigma_{xy}\,\delta y\,\delta z,\ \sigma_{xz}\,\delta y\,\delta z)$ — ①

At the point $P_2(x+\delta x/2, y, z)$ the components of the force are $A_2B_2C_2D_2$ is

$$= \left[\left(\sigma_{xx} + \frac{\delta x}{2}\frac{\partial \sigma_{xx}}{\partial x}\right)\delta y\,\delta z,\ \left(\sigma_{xy} + \frac{\delta x}{2}\frac{\partial \sigma_{xy}}{\partial x}\right)\delta y\,\delta z,\ \left(\sigma_{zx} + \frac{\delta x}{2}\frac{\partial \sigma_{zx}}{\partial x}\right)\delta y\,\delta z \right] \quad — ②$$

Slly force on P_1 in $A_1B_1C_1D_1$,

$$= \left[-\left(\sigma_{xx} - \frac{\delta x}{2}\frac{\partial \sigma_{xx}}{\partial x}\right)\delta y\,\delta z,\ -\left(\sigma_{xy} - \frac{\delta x}{2}\frac{\partial \sigma_{xy}}{\partial x}\right)\delta y\,\delta z,\ -\left(\sigma_{xz} - \frac{\delta x}{2}\frac{\partial \sigma_{xz}}{\partial x}\right)\delta y\,\delta z \right] \quad — ③$$

Hence the forces on // planes $A_2B_2C_2D_2$ & $A_1B_1C_1D_1$ passing through P_2 & P_1 are equivalent to a single force at P with components

$$= \left[\frac{\partial \sigma_{xx}}{\partial x} \delta x \delta y \delta z, \frac{\partial \sigma_{xy}}{\partial x} \delta x \delta y \delta z, \frac{\partial \sigma_{zx}}{\partial x} \delta x \delta y \delta z \right] \quad -(4)$$

together with couple whose moment are

$$\left. \begin{array}{l} -\sigma_{xz} \delta x \delta y \delta z \text{ about } OY \\ \sigma_{xy} \delta x \delta y \delta z \text{ about } OZ \end{array} \right\} -(5)$$

Similarly, the forces parallel on ~~to the~~ planes ~~each other and~~ $\perp^r$ to z-axis are equivalent to a single force at P with components $= \left[\frac{\partial \sigma_{zx}}{\partial z}, \frac{\partial \sigma_{zy}}{\partial z}, \frac{\partial \sigma_{zz}}{\partial z} \right] \delta x \delta y \delta z$

together with couple of moment,
$-\sigma_{zy} \delta x \delta y \delta z$ about OX & $\sigma_{zx} \delta x \delta y \delta z$ about OY.

And forces on parallel planes $\perp^r$ to y-axis are equivalent to single force at P
$$= \left[\frac{\partial \sigma_{yx}}{\partial y}, \frac{\partial \sigma_{yy}}{\partial y}, \frac{\partial \sigma_{zy}}{\partial y} \right] \delta x \delta y \delta z$$

with couple of moment,
$-\sigma_{yx} \delta x \delta y \delta z$ about OZ & $\sigma_{yz} \delta x \delta y \delta z$ about OX

Thus the surface forces on all 6 faces of rectangular //piped are equivalent to a single force at P having compon.

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}, \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}, \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right.$$

with couple

$$= [(\sigma_{yz} - \sigma_{zy}) \delta x \delta y \delta z, (\sigma_{zx} - \sigma_{xz}) \delta x \delta y \delta z, (\sigma_{xy} - \sigma_{yx}) \delta x \delta y \delta z]$$

Let $\vec{q} = U\hat{i} + V\hat{j} + W\hat{k}$ be the velocity at P &
$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$ be the body force P. (per unit mass) The total force is given by

$$= \text{Body force} + \text{surface force}$$

$$= \vec{F} + \vec{S}$$

Total force along i-direction

$$= \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \right) \delta x \delta y \delta z$$
$$+ \rho F_x \delta x \delta y \delta z$$

By Newton's 2nd Law :-

$$\rho \delta x \delta y \delta z \frac{DU}{Dt} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho F_x \right) \delta x \delta y \delta z$$

$$\Rightarrow \rho \frac{DU}{Dt} = \rho F_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}$$

Slly, $$\rho \frac{DV}{Dt} = \rho F_y + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}$$

$$\rho \frac{DW}{Dt} = \rho F_z + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

From constitutive law of Newtonian Fluid, by stoke's law of fluid motion

$$\sigma_{xx} = 2\mu \, \partial u/\partial x - 2\mu/3 \, \vec{\nabla}\cdot\vec{q} - p$$

$$\sigma_{yy} = 2\mu \, \partial v/\partial y - 2\mu/3 \, \vec{\nabla}\cdot\vec{q} - p$$

$$\sigma_{zz} = 2\mu \, \partial w/\partial z - 2\mu/3 \, \vec{\nabla}\cdot\vec{q} - p$$

$$\sigma_{xy} = \sigma_{yx} = \mu\left(\partial u/\partial y + \partial v/\partial x\right)$$

$$\sigma_{yz} = \sigma_{zy} = \mu\left(\partial v/\partial z + \partial w/\partial y\right)$$

$$\sigma_{zx} = \sigma_{xz} = \mu\left(\partial w/\partial x + \partial u/\partial z\right)$$

$$\rho\frac{DU}{Dt} = \rho F_x + 2\mu\left(\partial^2 u/\partial x^2\right) + 2\mu/3 \frac{\partial}{\partial x}\left(\vec{\nabla}\cdot\vec{q}\right) - \frac{\partial p}{\partial x}\Big\}$$
$$+ \mu\left(\partial^2 u/\partial y^2 + \partial^2 v/\partial y\partial x\right) + \mu\left(\partial^2 w/\partial z\partial x + \partial^2 u/\partial z^2\right)$$

$$\rho\frac{Dq_i}{Dt} = \rho F_{x_i} - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left[\mu\left\{2\,\partial q_i/\partial x_i - 2/3\,\vec{\nabla}\cdot\vec{q}\right\}\right]$$
$$+ \partial/\partial x_j\left[\mu\left(\cancel{\partial u/\partial y + \partial v/\partial x}\right)\right] + \frac{\partial}{\partial x_k}\left[\mu\left(\cancel{\partial u/\partial z + \partial w/\partial x}\right)\right]$$
$$\partial q_i/\partial x_j + \frac{\partial q_j}{\partial x_i} \qquad \partial q_k/\partial x_i + \partial q_i/\partial x_k$$

$$\cancel{\rho\frac{Dv}{Dt} = \rho F_y - \partial p/\partial y}$$

$$\rho\frac{D\vec{q}}{Dt} =$$

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DISSIPATION OF ENERGY

It is that energy which is dissipated in a viscous fluid in motion on account of internal friction

To determine the rate of dissipation of energy of a fluid due to dissipation :-

Suppose we follow a particle of a viscous ~~incomp~~ incompressible fluid of density ρ & volume δV, such that its mass is $\rho \delta V$. It moves with velocity $\vec{q}$ at any time 't'. Then KE $T = \frac{1}{2}\rho \delta V \vec{q}^{\,2}$. Hence the rate of change of energy as the particle moves with time is given by,

$$\frac{D}{Dt}\left(\frac{1}{2}\rho \delta V \vec{q}^{\,2}\right) = \cancel{\frac{1}{2}\frac{D}{Dt}(\rho \delta V)\vec{q}^{\,2}}^{\,0} + \frac{1}{2}\rho \delta V \frac{D}{Dt}(\vec{q}^{\,2})$$

$$= \rho \delta V \, \vec{q} \, \frac{D\vec{q}}{Dt} \quad -(1)$$

Let total volume of the fluid be V & S be the total surface area.

$$\frac{DT}{Dt} = \frac{d}{dt}\int_V \frac{1}{2}\rho \vec{q}^{\,2} \, dV$$

$$= \rho \int_V \vec{q} \, \frac{d\vec{q}}{dt} \, dV \quad -(2)$$

From Navier-Stokes Equation for incompressible fluid.

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{\nabla p}{\rho} + \nu \nabla^2 \vec{q} \quad -(3), \quad \nu = \frac{\mu}{\rho} \text{ Kinematics of viscosity}$$

From ②, ③ :-

$$\frac{dT}{dt} = \rho \int_V \vec{q}\left[\vec{F} - \frac{\nabla p}{\rho} + \nu \nabla^2 \vec{q}\right] dV$$

$$= \int_V \vec{q}(\rho \vec{F})\, dV - \cancel{\int_V \vec{q}\cdot\nabla p\, dV} + \int_V \vec{q}\rho\nu \nabla^2 \vec{q}\, dV - \int_S p\, \vec{q}\cdot\vec{n}\, dS \quad -④$$

$$dV = -\left[-\int_V \overset{0}{\nabla\cdot\vec{q}}\, p\, dV + \int_S p\,\vec{q}\cdot\vec{n}\, dS\right]$$

The ~~dissipation~~ first term on RHS of ④ represents rate at which the external force $\vec{F}$ is doing work throughout the mass of the fluid while the second term represents the rate at which pressure is doing work at the boundary. For ideal fluid, the work done by the force $\vec{F}$ in the volume V and work by the pressure at the boundary are same.

$$\frac{dT}{dt} = D = \rho \int_V \frac{\mu}{\rho}\vec{q}\cdot\nabla^2\vec{q}\, dV = \int_V \mu\,\vec{q}\cdot\nabla^2\vec{q}\, dV$$

where D is the dissipation of energy. Now if the flow is rotational s.t $\vec{\Omega}$ represents the vorticity. Then $\vec{\nabla}\times\vec{q} = \vec{\Omega}$

We know $\vec{\nabla}\times(\vec{\nabla}\times\vec{q}) = \nabla(\underbrace{\vec{\nabla}\cdot\vec{q}}_{0}) - \nabla^2\vec{q} = -\nabla^2\vec{q}$

$$\Rightarrow \vec{\nabla}\times\vec{\Omega} = -\nabla^2\vec{q}$$

$$\Rightarrow \vec{q}\cdot(\nabla^2\vec{q}) = -\vec{q}\cdot(\vec{\nabla}\times\vec{\Omega})$$

Again,

$$\vec{\nabla}\cdot(\vec{q}\times\vec{\Omega}) = \vec{\Omega}\cdot(\vec{\nabla}\times\vec{q}) - \vec{q}\cdot(\vec{\nabla}\times\vec{\Omega})$$

$$= \vec{\Omega}\cdot\vec{\Omega} - \vec{q}(\vec{\nabla}\times\vec{q})$$

$$\Rightarrow \vec{\nabla}(\vec{q}\times\vec{\Omega}) = |\vec{\Omega}|^2 - \vec{q}(\vec{\nabla}\times\vec{\Omega})$$

$$\Rightarrow -\vec{q}(\vec{\nabla}\times\vec{\Omega}) = \vec{\nabla}(\vec{q}\times\vec{\Omega}) - |\vec{\Omega}|^2$$

This given,

$$\vec{q}(\nabla^2\vec{q}) = \vec{\nabla}(\vec{q}\times\vec{\Omega}) - |\vec{\Omega}|^2$$

This gives;

$$D = \int_V \mu[\vec{\nabla}\cdot(\vec{q}\times\vec{\Omega}) - |\vec{\Omega}|^2]\,dV$$

$$\cancel{= \int_V \vec{\nabla}(\vec{q}\cdot\vec{\Omega})}$$

$$= \mu\int_S (\vec{q}\times\vec{\Omega})\cdot\hat{n}\,dS - \mu\int_V |\vec{\Omega}|^2\,dV$$

(no flux at boundary)

If We assume no-slip condition, i.e. $\vec{q}\cdot\vec{n} = 0$, then first term will vanish,

$$D = -\int_V \mu|\vec{\Omega}|^2\,dV$$

$$|D| = \mu\int_V ((\xi^2+\eta^2+\zeta^2)\,dV$$

$$\Omega = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \xi\hat{i} + \eta\hat{j} + \zeta\hat{k}$$

$$|\Omega|^2 = \xi^2+\eta^2+\zeta^2$$