7.1.18

Textbook: Measure Theory and Integration.
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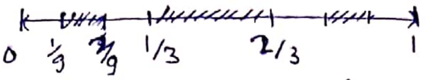
Cantor Set:



Step 1 $[0, \frac{1}{3}]$ $[\frac{2}{3}, 1]$

Step 2 $[0, \frac{1}{9}]$ $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, $[\frac{8}{9}, 1]$

Step n: 2^n intervals

$J_{n,1}, J_{n2} \ldots, J_{n,2^n}$

Let $J_n = \bigcup_{i=1}^{2^n} J_{n,i}$

Cantor set defined as (or Cantor Ternary Set)

$$C = \bigcap_{n=1}^{\infty} J_n$$

Remark:-

Cantor set consists of the points x which can be given an expansion to the base 3 with the form $x = 0.x_1x_2x_3\ldots$ with $x_n = 0$ or 2 $\forall n \geq 1$

Proof to show uncountable:

Suppose C is countable, say $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$

For each $n \geq 1$ if $x_n^{(n)} = 0$ Let $x_n = 2$ and if $x_n^{(n)} = 2$ then let $x_n = 0$. $x = 0.x_1x_2\ldots \in C$

but $x \neq x^{(n)}$ $\forall n \geq 1$

$\therefore$ C cannot be countable. $\therefore$ C is an uncountable set.

Def : The Lebesgue outer measure or an outer measure of a subset A of $\mathbb{R}$ is defined as

$$m^*(A) = \inf\left(\sum_{n\geq 1} l(I_n)\right)$$

Where infimum is taken over all finite or countable collection of intervals $\{I_n\}_{n\geq 1}$ such that

$$A \subseteq \bigcup_{n\geq 1} I_n \quad , \text{where } I_n = [a_n, b_n)$$

Theorem. Let $A \subseteq \mathbb{R}$

(i) $m^*(A) \geq 0$

(ii) $m^*(\phi) = 0$

(iii) If $A \subseteq B$, then

$$m^*(A) \leq m^*(B)$$

(iv) $m^*(\{x\}) = 0 \quad \forall x \in \mathbb{R}$

Proof: (i), (ii), (iii) follows from def^n. $\boxed{\text{Ex}}$.

(iv) Let $J_n = [x, x+\frac{1}{n})$ $\forall n \geq 1$

The $\{x\} \subseteq J_n$

$l(J_n) = \frac{1}{n}$

By (iii) $m^*(\{x\}) \leq m^*(J_n)$

$= l(J_n)$ $\forall n \geq 1$

$= \frac{1}{n}$.

$\therefore$ $m^*(\{x\}) \leq 0$

By (i) $m^*(\{x\}) = 0$.

Remark:- $m^*(I) = l(I)$ for any finite interval $I \subset \mathbb{R}$

Proposition :- Let $A \subseteq \mathbb{R}$ be $x \in \mathbb{R}$. Then $m^*(A+x) = m^*(A)$

wher $A+x = \{a+x \mid \forall a \in A\}$.

> Note: Let a set be S, $l = \inf(S)$ $\varepsilon > 0$
>
> Then $l+\varepsilon$ is not a lower bound.
>
> $\therefore \exists x \in S$ such that $x \leq l+\varepsilon$

Let $\varepsilon > 0$

Then $\exists$ a collection $\{I_n\}_{n\geq 1}$ such that $A \subseteq \bigcup_{n\geq 1} I_n$ &

$m^*(A) + \varepsilon \geq \sum_{n\geq 1} l(I_n)$ (by using inf. property)

We have $A + x \subseteq \bigcup_{n\geq 1} (I_n + x)$

$m^*(A+x) \leq \sum_{n\geq 1} l(I_n + x) = \sum_{n\geq 1} l(I_n)$

$\leq m^*(A) + \varepsilon$

$\therefore$ $m^*(A+x) \leq m^*(A)$

We have $A = (A+x) - x$

$m^*(A) = m^*((A+x)-x) \leq m^*(A+x)$ $\therefore$ $m^*(A) = m^*(A+x)$

$\boxed{\text{EOC}}$

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Thm: The outer measure of an interval is equal to its length.

Proof: Case 1.

Suppose $I = [a, b]$. For each $\varepsilon > 0$, there exists $\{I_n\}$ intervals such that $I \subset \bigcup_{n\geq 1} I_n$ and

$$m^*(I) + \varepsilon \geq \sum_{n\geq 1} \ell(I_n)$$

Let $I_n = [a_n, b_n) \quad \forall\, n \geq 1$

Let $I_n' = (a_n - \frac{\varepsilon}{2^n}, b_n) \quad \forall n \geq 1$ open interval

We have $I \subseteq \bigcup_{n\geq 1} I_n'$

Recall: Heine – Borel Theorem.

If A is closed and bounded set in $\mathbb{R}$ and & $A \subseteq \bigcup_{\alpha \in J} G_\alpha$, where G_α are open sets. then there exists a finite subcollection $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ such that $A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

Continuing,

$\therefore$ By using Heine Borel theorem, there exists a finite subcollection $J_1, J_2, \ldots, J_N$, of $\{I_n'\}_{n\geq 1}$ such that $I \subseteq \bigcup_{i=1}^{N} J_i$. Say $J_i = (c_i, d_i)$

We can assume that no J_i is contained in another i.e.

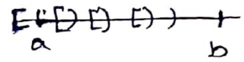
$c_1 \leq c_2 < c_3 < \ldots < c_N$

Now, $d_N - c_1 = \sum_{k=1}^{N}(d_k - c_k) - \underbrace{\sum_{k=1}^{N}(d_k - c_{k+1})}_{\geq 0}$

Note $d_k \geq c_{k+1}$ $\left\{ \begin{array}{l} \text{otherwise they cannot cover } I. \\ \text{ie } J_i \text{ are intersecting} \end{array} \right\}$

$$\leq \sum_{k=1}^{N}(d_k - c_k)$$

$$= \sum_{1}^{N} l(J_k)$$



$$d_N - c_1 \leq \sum_{k=1}^{N} l(J_k)$$

We have

$$\begin{aligned}
m^*(I) &\geq \sum_{n\geq 1} l(I_n) - \varepsilon \\
&\geq \sum_{n\geq 1}\left(l(I_n') - \frac{\varepsilon}{2^n}\right) - \varepsilon \\
&\geq \sum_{n\geq 1} l(I_n') - \varepsilon \sum_{n\geq 1}\frac{1}{2^n} - \varepsilon \\
&\geq \sum_{n\geq 1} l(I_n') - 2\varepsilon \\
&\geq \sum_{n=1}^{N} l(J_n) - 2\varepsilon \\
&\geq d_N - c_1 - 2\varepsilon \\
&\geq b - a - 2\varepsilon \\
&= l(I) - 2\varepsilon
\end{aligned}$$

$$\underline{m^*(I) \geq l(I) - 2\varepsilon} \quad (1)$$

$m^*(I) = m^*([a,b]) \leq m^*([a, b+\varepsilon])$ $\quad \{\because [a,b] \subseteq [a, b+\varepsilon)\}$

$$\leq b - a + \varepsilon$$

$$\underline{m^*(I) \leq l(I) + \varepsilon} \quad (11)$$

(1) & (11) $\Rightarrow$ $m^*(I) = l(I)$

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Case 2 - Suppose $I = (a, b]$

Let $\varepsilon > 0$ & $\varepsilon < b - a$

Let $I' = [a+\varepsilon, b]$

$I' \subseteq I$

$$m^*(I') \leq m^*(I)$$

$$\| $$

$$l(I') = b - a - \varepsilon = l(I) - \varepsilon$$

$$\therefore m^*(I) \geq l(I) - \varepsilon \quad \text{(iii)} \quad \forall \varepsilon > 0$$

Let $I'' = [a, b+\varepsilon)$

$I \subseteq I''$

$$\therefore m^*(I) \leq m^*(I'') = b + \varepsilon - a = l(I) + \varepsilon$$

$$m^*(I) \leq l(I) + \varepsilon \quad \forall \varepsilon > 0 \quad \text{(iv)}$$

From (iii) & (iv)

$$m^*(I) = l(I)$$

Similarly for (a, b)

Case 3 :- Suppose I is an infinite interval

Say $I = (-\infty, a]$

For any $M > 0$ $\exists\, k$ s.t. $I_k = [k, k+M) \subseteq I$

$$\therefore m^*(I_k) \leq m^*(I)$$

$$m^*(I_k) = k+M-k = M.$$

$$m^*(I) \geq M \quad \forall M > 0$$

$$m^*(I) = +\infty$$

<u>Theorem</u> :- For any sequence $\{E_i\}_{i \geq 1}$ of subsets of $\mathbb{R}$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

Proof: Let $\varepsilon > 0$

For each i, there exists a sequence of intervals

$\{I_{i,n}\}_{n \geq 1}$ such that $E_i \subseteq \bigcup_{n=1}^{\infty} I_{i,n}$

$\{I_{i,n}\}_{n \geq 1} = [\,,\,)$

& $m^*(E_i) + \frac{\varepsilon}{2^i} \geq \sum_{n=1}^{\infty} l(I_{i,n})$ (*)

$$\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty}\left(\bigcup_{n=1}^{\infty} I_{i,n}\right)$$

Thus $\{I_{i,n}\}_{i,n \geq 1}$ is a countable collection of intervals

that covers $\bigcup_{i=1}^{\infty} E_i$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} l(I_{i,n}) \quad [\text{from def}^n]$$

$$\leq \sum_{i=1}^{\infty}\left(m^*(E_i) + \frac{\varepsilon}{2^i}\right) \quad [\text{by } (*)]$$

$$\doteq \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon$$

$$\therefore \quad m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon \qquad \forall \varepsilon > 0$$

$$\Rightarrow \quad m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

Proposition :-

For any $E \subseteq \mathbb{R}$ any $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}$ such that $E \subseteq U$ & $m^*(U) \leq m^*(E) + \varepsilon$

Proof :- $\varepsilon > 0$ There exists seq. of intervals $\{I_n\}_{n \geq 1}$ s.t. $E \subseteq \bigcup_{n=1}^{\infty} I_n$ & $m^*(E) + \frac{\varepsilon}{2} \geq \sum_{n=1}^{\infty} l(I_n)$

Let $I_n = [a_n, b_n) \quad \forall n \geq 1$.

Let $I_n' = \left(a_n - \frac{\varepsilon}{2^{n+1}}, b_n\right) \quad \forall n \geq 1$ open

$$I_n \subseteq I_n' \qquad \forall n \geq 1$$

hence $\quad E \subseteq \bigcup_{n=1}^{\infty} I_n'$

Let $\quad U = \bigcup_{n=1}^{\infty} I_n'$

U is an open set & $E \subseteq U$

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$$m^*(E) \leq m^*(U) = m^*\left(\bigcup_{n=1}^{\infty} I_n'\right)$$

$$\leq \sum_{n=1}^{\infty} m^*(I_n') \quad \text{(by Countable Subadditivity property)}$$

$$= \sum_{n=1}^{\infty} l(I_n')$$

$$= \sum_{n=1}^{\infty} \left(b_n - a_n + \frac{\varepsilon}{2^{n+1}}\right)$$

$$= \sum_{n=1}^{\infty} l(I_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} l(I_n) + \frac{\varepsilon}{2} \leq m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= m^*(E) + \varepsilon$$

Thus $m^*(E) \leq m^*(U) \leq m^*(E) + \varepsilon$

Proposition

In the definition of outer measure, $m^*(E)$

$= \inf\left(\sum l(I_n)\right)$, $E \subseteq \bigcup_{n \geq 1} I_n$

We stipulate : (i) $I_n = (a_n, b_n)$
(ii) $I_n = [a_n, b_n)$
(iii) $I_n = (a_n, b_n]$
(iv) $I_n = [a_n, b_n]$
(v) mixture of above

For different n, of various types of intervals their inf. is the same.

~~Proof: Denote m_0^*~~

Proof :- Denote m_o^* the outer measure obtained from (i)

We denote m_m^* the outer measure obtained from (v)

Let $E \subseteq \mathbb{R}$

$$m_o^*(E) = \inf_{\substack{E \subseteq \bigcup_{n=1}^{\infty} O_n \\ O_n \text{ open}}} \left(\sum_{n=1}^{\infty} l(O_n) \right)$$

& $$m_m^*(E) = \inf_{\substack{E \subseteq \bigcup_{n=1}^{\infty} I_n \\ I_n \text{ any interval}}} \left(\sum_{n=1}^{\infty} l(I_n) \right)$$

By def^n of infimum $\quad m_m^*(E) \leq m_o^*(E) \quad$ --(i)

Let $\varepsilon > 0$ there exists $\{I_n\}_{n \geq 1}$ intervals of type (v)

Such that $E \subset \bigcup_{n=1}^{\infty} I_n$ and

$$m_m^*(E) + \varepsilon \geq \sum_{n=1}^{\infty} l(I_n)$$

$$\geq \sum_{n=1}^{\infty} (1+\varepsilon)^{-1} l(I_n')$$

But $E \subseteq \bigcup_{n=1}^{\infty} I_n'$

$$m_o^*(E) \leq \sum_{n=1}^{\infty} l(I_n')$$

Thus $m_m^*(E) + \varepsilon \geq (1+\varepsilon)^{-1} m^*(E)$

$\Rightarrow m_o^*(E) \leq (1+\varepsilon)\left(m_m^*(E) + \varepsilon\right)$

$\Rightarrow m_o^*(E) \leq m^*{}_m(E) + \varepsilon\left(m_m^*(E) + 1 + \varepsilon\right)$

$\forall \varepsilon > 0$

Thus $m_o^*(E) \leq m_m^*(E)$ --(ii)

Thus $m_o^*(E) = m_m^*(E)$ (i) and (ii)

Definition :-

Let X be any non empty set. let $\mathcal{F}$ be a ~~me~~ family of subsets of X. ~~$\mathcal{F}$ be a family of sub~~ $\mathcal{F}$ is said to be a σ algebra if it satisfies :-

(i) $X \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

(iii) If $E_1, E_2, \ldots \in \mathcal{F}$ $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$

Eg (i) $\{\phi, X\} \to \sigma$ algebra

(ii) $P(X) \to \sigma$ algebra

Definition :-

Let $E \subseteq \mathbb{R}$. We say E is said to be Lebesgue measurable or simply measurable if for each subset $A \subseteq \mathbb{R}$, the following holds :-

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

Remark

$$A = (A \cap E) \cup (A \cap E^c)$$

$$m^*(A) = m^*((A \cap E) \cup (A \cap E^c))$$

$$\leq m^*(A \cap E) + m^*(A \cap E^c)$$

by sub additive property of m^*

In defn of measurable set we can take only one inequality

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \quad \forall A \subseteq \mathbb{R}$$

$\boxed{\text{EOC}}$

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Recall.

A subset $E \subseteq \mathbb{R}$ is said to be measurable if $\forall A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad (*)$$

Eg ① Suppose $E \subseteq \mathbb{R}$ and $m^*(E) = 0$. Then E is measurable.

Proof: For $A \subseteq \mathbb{R}$, $A \cap E \subseteq E$ $m^*(A \cap E) \leq m^*(E) = 0$

$\therefore m^*(A \cap E) = 0$

& $A \cap E^c \subseteq A$

$\therefore m^*(A \cap E^c) \leq m^*(A)$

$\therefore m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A)$

The other inequality is always true.

$\therefore m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

$\therefore E$ is measurable.

② Let $\mathcal{M}$ be the set of all measurable subsets of $\mathbb{R}$.

Thm: $\mathcal{M}$ is a σ algebra.

(i) $\mathbb{R} \in \mathcal{M}$. For $A \subseteq \mathbb{R}$ $A \cap \mathbb{R} = A$, $A \cap \mathbb{R}^c = \phi$

$$\therefore m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^c) = m^*(A) + \underbrace{m^*(\phi)}_{=0}$$

$$= m^*(A)$$

$\therefore (*)$ holds for $E = \mathbb{R}$.

$\therefore \mathbb{R}$ is measurable, $\therefore \mathbb{R} \in \mathcal{M}$

(ii) Let $E \in \mathcal{M}$. To show $E^c \in \mathcal{M}$

For $A \subseteq \mathbb{R}$, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

$$= m^*(A \cap (E^c)^c) + m^*(A \cap E^c)$$

$\Rightarrow E^c \in \mathcal{M}$

iii) Let $E_i \in \mathcal{M} \quad \forall i \geq 1$

To show $\bigcup_{1}^{\infty} E_i \in \mathcal{M}$

Enough to show, for any $A \subseteq \mathbb{R}$

$$m^*(A) \geq m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)^c\right)$$

Given $E_1 \in \mathcal{M}$.

$$\therefore m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$$

Let $B = A \cap E_1^c$

Since E_2 is measurable.

$$m^*(B) = m^*(B \cap E_2) + m^*(B \cap E_2^c)$$

$$\therefore m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$$

$$= m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap (E_1 \cup E_2)^c)$$

Continuing, for any $n \geq 2$

$$m^*(A) = m^*(A \cap E_1) + m^*\left(A \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right) + m^*\left(A \cap \left(\bigcup_{j=1}^{n} E_j\right)^c\right) \quad \boxed{a}$$

We have $\bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i=1}^{\infty} E_i$

or, $\left(\bigcup_{i=1}^{n} E_i\right)^c \supseteq \left(\bigcup_{i=1}^{\infty} E_i\right)^c$, or, $\left(\bigcup_{i=1}^{n} E_i\right)^c \cap A \supseteq \left(\bigcup_{i=1}^{\infty} E_i\right)^c \cap A$

or, $m\left(\left(\bigcup_{1}^{n} E_i\right)^c \cap A\right) \geq m\left(\left(\bigcup_{1}^{\infty} E_i\right)^c \cap A\right)$

$E_1 \cap (E_2 \ldots$

$E_1 \cap E_2 \ldots$

$$m^*(A) \geq m^*(A \cap E_1) + m^*\left(A \cap E_i \cap \left(\bigcap_{j<i} E_j\right)^c\right) + m^*\left(\left(\bigcup_{i=1}^{\infty} E_i\right)^c \cap \ldots\right)$$

because $\geq m^*\left(A \cap \left(\bigcup_{1}^{\infty} E_i\right)\right) + m^*\left(A \cap \left(\bigcup_{1}^{\infty} E_i\right)^c\right)$

$$m^*\left(\bigcup_{1}^{\infty}\left(E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) \cap A\right) \leq \sum_{i=1}^{\infty} m^*\left(E_i \cap \left(\bigcup_{j<i} E_j\right)^c \cap A\right) \quad (+)$$

[count. subadd.]

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$$m^*(A) \geq m^*(A\cap E_1) + m^*\left(\bigcup_{i}^{\infty}(A\cap E_i \cap (\bigcup_{j<i} E_j)^c\right) + m^*((\bigcup_{1}^{\infty} E_i)^c \cap A)$$

$$= m^*\left(\underbrace{\bigcup_{i=1}^{\infty}(A\cap E_i \cap (\bigcup_{j<i} E_j)^c}_{\bigcup_{i=1}^{\infty} A\cap E_i}\right) + m^*((\bigcup_{1}^{\infty} E_i)^c \cap A)$$

$$m^*(A) \geq m^*(\bigcup_{i=1}^{\infty} A\cap E_i) + m^*(A\cap (\bigcup_{1}^{\infty} E_i)^c)$$

$$= m^*(A\cap (\bigcup_{1}^{\infty} E_j)) + m^*(A\cap(\bigcup_{1}^{\infty} E_j)^c)$$

Other inequality holds obviously.

$\therefore \bigcup_{1}^{\infty} E_i$ is measurable

$\Rightarrow \bigcup_{1}^{\infty} E_i \in \mathcal{M}$

(i)(ii)(iii) $\Rightarrow \mathcal{M}$ is a σ algebra

Theorem: Suppose $\{E_i\}_{i\geq 1}$ is a family of measurable subsets of $\mathbb{R}$ and they are disjoint. Then $m^*(\bigcup_{1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(E_i)$

Take $A = \bigcup_{i=1}^{\infty} E_i$ in the inequality (+) of prev. proof.

Then, $m^*(\bigcup_{1}^{\infty} E_i) \geq m^*((\bigcup_{1}^{\infty} E_i)\cap E_1) + \sum_{2}^{\infty} m^*(\bigcup_{j=1}^{\infty} E_j \cap E_i \cap (\bigcup_{j<i} E_j)^c)$

$$+ \underbrace{m^*((\bigcup_{1}^{\infty} E_i) \cap (\bigcup_{j<i} E_j))}_{\phi}$$

$$m^*(\bigcup_{1}^{\infty} E_i) \geq m^*(E_1) + \sum_{2}^{\infty} m^*(E_i)$$

$$\geq \sum_{i=1}^{\infty} m^*(E_i)$$

Subadditivity is obvious

$\therefore \sum m^*(E_i) = m^*(\bigcup_{1}^{\infty} E_i)$

$\boxed{\text{EOC}}$

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Suppose S is a σ algebra of all subsets of a set X.

Then if $E_1, E_2, \dots \in S \Rightarrow \bigcap_{1}^{\infty} E_i \in S$

Proof: Given $E_1, E_2, \dots \in S$

$\Rightarrow E_1^c, E_2^c \dots \in S$

$\Rightarrow \bigcup_{1}^{\infty} E_i^c \in S$

$\Rightarrow ((\bigcap_{1}^{\infty} E_i)^c)^c \in S$

$\Rightarrow \bigcap_{1}^{\infty} E_i \in S.$

Corollary: Let $\mathcal{M}$ denote the σ algebra of measurable subsets of $\mathbb{R}$. Then for $E_i \in \mathcal{M}$ $\forall\, i \geq 1$, we have

$\bigcap_{i=1}^{\infty} E_i \in \mathcal{M}$.

Proposition: Suppose $F \in \mathcal{M}$ & $m^*(\underbrace{(F \setminus G) \cup (G \setminus F)}_{F \Delta G}) = 0$ for a subset $G \in \mathbb{R}$. Then $G \in \mathcal{M}$, ie G is also measurable.

~~Proof~~:- Rem Suppose E is measurable & $F \subset E$. Then F is measurable. (Wrong!)

Proof:- To show

$$m^*(A) \geq m^*(A \cap F) + m^*(A \cap F^c)$$

Since E is measurable,

$$m^*(E) \geq m^*(A \cap E) + m^*(A \cap E^c)$$
$$\geq m^*(A \cap F) + m^*(A \cap E^c)$$

We know $\mathbb{R}$ is measurable, but every subset of $\mathbb{R}$ need not be measurable because $\exists$ non measurable sets. [Remark is false]

Proof:

$F \Delta G = (F \setminus G) \cup (G \setminus F)$

& $F \setminus G \subseteq F \Delta G$, $G \setminus F \subseteq F \Delta G$

$\Rightarrow m^*(F \setminus G) \leq \underbrace{m^*(F \Delta G)}_{=0}$ & $m^*(G \setminus F) \leq \underbrace{m^*(F \Delta G)}_{=0}$

$\therefore m^*(F \setminus G) = m^*(G \setminus F) = 0$.

$F \setminus G$, $G \setminus F$ are measurable

i.e. $F \setminus G, G \setminus F \in \mathcal{M}$. Since $\mathcal{M}$ is a σ algebra . . .

Thm. Every interval is measurable.

Let us prove for $[a, \infty)$

Note

$[a,b] = \bigcap_{n=1}^{\infty} [a, b+\frac{1}{n})$

$= [a, \infty) \cap (-\infty, b]$

$(a, b) = \dots$

$[a, b) = \dots$

$(a, b] = \dots$

To show:

For $A \subset \mathbb{R}$

$$m^*(A) \geq m^*(A \cap [a,\infty)) + m^*(A \cap (-\infty, a))$$

Let $A_1 = A \cap (-\infty, a)$

& $A_2 = A \cap [a, \infty)$

Given $\epsilon > 0$, there exists $\{I_n\}_{n \geq 1}$ intervals such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ & $m^*(A) + \epsilon \geq \sum_{n=1}^{\infty} l(I_n)$

Let $I_n' = I_n \cap (-\infty, a)$

$I_n'' = I_n \cap [a, \infty)$

Since $A \subseteq \bigcup_{n=1}^{\infty} I_n$ therefore

$$A \cap (-\infty, a) \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right) \cap (-\infty, a)$$

$$\& \quad A \cap [a, \infty) \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right) \cap [a, \infty)$$

$$\Rightarrow A_1 \subseteq \bigcup_{n=1}^{\infty} (I_n \cap (-\infty, a))'$$

$$\& \ A_2 \subseteq \bigcup_{n=1}^{\infty} (I_n \cap [a, \infty))$$

$$A_1 \subseteq \bigcup_{n=1}^{\infty} I_n'$$

$$\& \ A_2 \subseteq \bigcup_{n=1}^{\infty} I_n''$$

$$\Rightarrow m^*(A_1) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n'\right) \leq \sum_{n=1}^{\infty} m^*(I_n')$$

$$\& \ m^*(A_2) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n''\right) \leq \sum_{n=1}^{\infty} m^*(I_n'')$$

$$m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} \left(m^*(I_n') + m^*(I_n'')\right)$$

$$= \sum_{n=1}^{\infty} \left(\underbrace{l(I_n') + l(I_n'')}_{\text{disjoint, union to } I_n.}\right)$$

$$= \sum_{n=1}^{\infty} l(I_n)$$

$$m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} l(I_n)$$

$$\leq m^*(A) + \epsilon$$

True for any $\epsilon > 0$

$$m^*(A_1) + m^*(A_2) \leq m^*(A)$$

$\therefore [a, \infty)$ is measurable.

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Definition ..

Let $\mathcal{M}$ = set of all measurable subsets of $\mathbb{R}$.

Define a map $m : \mathcal{M} \to \mathbb{R} \cup \{+\infty, -\infty\}$

as $m(E) = m^*(E) \quad \forall\, E \in \mathcal{M}$

Known as the Lebesgue measure of E

m is countabley additive.

$\boxed{\text{EOC}}$

6.1.19

Let $\mathcal{A}$ be a family of subsets of a set X.

Thm Then there exists a smallest σ algebra containing $\mathcal{A}$. This σ algebra is known as the σ algebra generated by $\mathcal{A}$.

Proof This σ algebra is precisely, the intersection of all σ-algebras (on subsets of X) containing $\mathcal{A}$.

Say $\{S_\alpha\}_{\alpha \in J}$ be the family of all σ algebras containing $\mathcal{A}$. Then $\bigcap_{\alpha \in J} S_\alpha$ is the smallest σ algebra containing $\mathcal{A}$.

Def^{n}: Let B be the σ algebra generated by the intervals of the form $[a, b)$ $a, b \in \mathbb{R}$.

ie, $A = \{[a, b) \mid a, b \in \mathbb{R}\}$

$\mathcal{B}$ = σ algebra generated by A.

Then the members of $\mathcal{B}$ are called Borel Sets.

Theorem:

(i) Every Borel set is measurable ie, $\mathcal{B} \subseteq \mathcal{M}$

(ii) $\mathcal{B}$ is generated by each of the following classes: open intervals, open sets, G_δ sets, F_σ sets.

Proof (i) Note $[a, b) \in \mathcal{M}$ $\forall\, a, b \in \mathbb{R}$

$\Rightarrow \mathcal{B} \subseteq \mathcal{M}$ ($\because \mathcal{B}$ is the smallest σ algebra containing $[a, b)$ $\forall\, a, b \in \mathbb{R}$)

(ii) Let $\mathcal{B}_1$ be the σ algebra generated by open intervals. Since any open interval $(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b) \in \mathcal{B}$

$\therefore \mathcal{B}_1 \subseteq \mathcal{B}$

Also $[a, b) = \bigcap_{1}^{\infty} (a - \frac{1}{n}, b) \in \mathcal{B}_1$ $\Rightarrow \mathcal{B} \subseteq \mathcal{B}_1$ $\mathcal{B} = \mathcal{B}_1$

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Any open set is the countable union of open intervals

This implies $B_1 = B_2$, where $B_2 = \sigma$ alg. generated by open sets

Def^n

i) G_δ defined as countable intersection of open sets.

(ii) F_σ defined as countable union of closed sets.

Let $B_3 = \sigma$ algebra generated by G_δ sets in $\mathbb{R}$.

Any G_δ set $A = \bigcap_{n=1}^{\infty} O_n$ O is open. $\in B_2$

$B_3 \subseteq B_2$

Let U be any open set in $\mathbb{R}$

Let $O_n = O$ $\forall n \geq 1$

$\bigcap_{n=1}^{\infty} O_n = U$ is a G_δ set.

$B_2 \subseteq B_3$

$\therefore B_2 = B_3$

$\boxed{F_\sigma \text{ sets - } \quad \text{Exercise}}$

Remark: $\mathcal{B} \subsetneq \mathcal{M}$

$\therefore \exists$ a measurable set which is not Borel set

Proposition. For any subset $A \subseteq \mathbb{R}$, $\exists$ a measurable set E containing A & $m^*(A) = m(E)$

Proof. For $\varepsilon = \frac{1}{n}$. There exists an open set O_n s.t.

$m^*(O_n) \leq m^*(A) + \varepsilon = m^*(A) + \frac{1}{n}$

Let $E = \bigcap_{n=1}^{\infty} O_n$ E is G_δ set

E is measurable.

& $m^*(E) = m^*(\bigcap_1 O_n) \leq m^*(O_n) \leq m^*(A) + \frac{1}{n} \quad \forall n \geq 1$

$\therefore \; m^*(E) \leq m^*(A)$.

On the other hand

$$A \subseteq O_n \quad \forall n$$

$$\Rightarrow A \subseteq \bigcap_1^\infty O_n = E$$

$$\Rightarrow m^*(A) \leq m^*(E) = m(E)$$

$$\therefore m^*(A) = m^*(E) = m(E)$$

Let $\{E_i\}_{i \geq 1}$ be a sequence of subsets of X.

Def: $\limsup (E_i) = \bigcap_{n=1}^{\infty} \left(\bigcup_{i \geq n} E_i \right)$

$\liminf (E_i) = \bigcup_{n=1}^{\infty} \left(\bigcap_{i \geq n} E_i \right)$

$E_N \cap E_{N+1} \cdots$

Remark. $\liminf (E_i) \subseteq \limsup (E_i)$

Let $x \in \liminf (E_i)$

~~$\Rightarrow \exists N$ s.t. $x \in \bigcap_{i \geq N} E_i$~~

~~$\Rightarrow x \in E_i \quad \forall i \geq N$.~~

~~$\Rightarrow x \in \bigcup_{i \geq N} E_i \quad \forall$~~

$\Rightarrow$

for $n \geq 1$ $(E_n \cap E_{n+1} \cap \cdots) \subseteq \limsup (E_i)$

$$\Rightarrow \bigcup_{n \geq 1} (E_n \cap E_{n+1} \cap \cdots) \subseteq \limsup (E_i)$$

$$\Rightarrow \liminf (E_i) \subseteq \limsup (E_i)$$

If $\liminf(E_i) = \limsup(E_i)$

Then this set is defined as $\lim(E_i)$

Examples :-

① Suppose $E_1 \subset E_2 \subset \dots$

$$\lim \cancel{\sup}\inf (E_i) = \bigcup_{n=1}^{\infty}\left(\bigcap_{i\geq n} E_i\right)$$

$$= \bigcup_{n=1}^{\infty}(E_n \cap E_{n+1} \cap \dots) = \bigcup_{n=1}^{\infty} E_n.$$

$$\limsup(E_i) = \bigcap_{n=1}^{\infty}\left(\bigcup_{i\geq n} E_i\right) \subseteq \bigcup_{i\geq n} E_i \subseteq \bigcup_{1}^{\infty} E_i$$

$\bigcup_{i\geq n} E_i$ // ~~lim sup~~.

$\bigcup_{1}^{\infty} E_i$ = lim inf.

$\therefore \limsup(E_i) = \liminf(E_i)$

$\lim(E_i) = \bigcup_{n=1} E_n$.

② Suppose.

$E_1 \supset E_2 \supset \dots$

$$\limsup = \bigcap_{1}^{\infty}(E_n \cup E_{n+1} \cup \dots)$$

$$= \bigcap_{n=1}^{\infty} E_n \subseteq \bigcap_{i\geq n} E_i \quad \forall n.$$

$$\liminf = \bigcup_{n=1}^{\infty}\left(\bigcap_{i\geq n} E_i\right)$$

$$\Rightarrow \bigcap_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty}\left(\bigcap_{i\geq n} E_i\right)$$

$\bigcap_{n=1}^{\infty} E_n$ = $\limsup.(E_i)$ $\qquad$ $\bigcup_{n=1}^{\infty}\left(\bigcap_{i\geq n} E_i\right)$ = $\liminf(E_i)$

$$\therefore \lim(E_i) = \bigcap_{n=1}^{\infty} E_i$$

Theorem

Every nonempty open subset U of $\mathbb{R}$ is the union of disjoint open intervals, at most countable in number.

Proof: Define an ~~injection~~ relation $\sim$ on $\mathbb{R}$ as follows.

$a, b \in U$, $a \sim b$ if.

the closed interval $\underset{a \le b}{[a,b]}$ or $\underset{b \le a}{[b,a]}$ lives in U.

To show $\sim$ is an equivalence relation.

(a) $a \sim a$ $\quad (\because \{a\} = [a,a] \subseteq U)$

(b) Suppose $a \sim b$. To show $b \sim a$.

Suppose $a < b$ then $[a,b] \subseteq U$.

(c) Suppose $a \sim b$ and $b \sim c$ and hence

$[b,a] = \phi \subseteq U$.

To show $a \sim c$.

Say $a \le b$ & $b \le c$.

$[a,b]$, $[b, c] \subseteq U \Rightarrow [a,b] \cup [b,c] \subseteq U$.

$$[a,b] \cup [b,c] = [a,c]$$

$U =$ Disjoint union of equivalence classes of $\sim$.

For $a \in U$.

Let $C(a)$ denote the equivalence class containing a

Then $C(a)$ is an interval. For $b \in C(a)$ we have

$a \sim b \Rightarrow [a,b]$ or $[b,a] \subseteq U$

$\Rightarrow$ all elts b/w a or b also in $C(a)$

$\therefore$ $C(a)$ is an interval.

To show that $C(\alpha)$ is open.

For $k \in C(a) \subseteq U$ there is $\varepsilon > 0$

Such that $(k-\varepsilon, k+\varepsilon) \subseteq U$, U open.

but $(k-\varepsilon, k+\varepsilon) \subseteq C(a)$ for ε suff. small.

$\therefore$ $C(a)$ is open.

$$\therefore \quad U = \bigcup_{a} C(\alpha)$$

Recall: Lindelöf Theorem

Suppose $\mathcal{F} = \{ I_\alpha \mid \alpha \in A \}$ is a collection of open intervals. $\exists$ a subcollection, say $\{ I_i \mid i = 1, 2, \dots \}$ at most countable number. such that

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{i=1}^{\infty} I_i$$

$\therefore$ By Lindelöf Thm.

$\exists$ at most countable $a_1, a_2, \dots,$

Such that $U = \bigcup_{i=1}^{\infty} C(\alpha_i)$

[EOC]

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17.1.19

MEP

Theorem:

Let $\{E_j\}_{j\geq 1}$ be a sequence of measurable subsets of $\mathbb{R}$. Then (i) If $E_1 \subseteq E_2 \subseteq \ldots$ Then $m(\lim E_i) = \lim m(E_i)$

(ii) If $E_1 \supseteq E_2 \supseteq \ldots$ them $m(\lim(E_i)) = \lim(m(E_i))$

Proof (i) We have $\lim E_i = \bigcup_1^\infty E_i \in \mathcal{M}$.
is measurable.

Let $F_1 = E_1$.

$F_2 = E_2 \setminus F_1$.

$F_3 = E_3 \setminus F_2$.

$\vdots$

$F_i = E_i \setminus F_{i-1}$.

Then $\bigcup_{i=1}^\infty E_i = \bigcup_{i=1}^\infty F_i$

$$m(\lim(E_i)) = m\left(\bigcup_1^\infty E_i\right).$$

$$= m\left(\bigcup_1^\infty F_i\right) = m^*\left(\bigcup_1^\infty F_i\right)$$

$$= \sum^\infty m^*(F_i) \quad \text{[additive property]}$$

$$= \lim_{n\to\infty}\left(\sum_1^n m^*(F_i)\right)$$

$$= \lim_{n\to\infty}\left(m^*\left(\bigcup_{i=1}^n F_i\right)\right).$$

$$= \lim\left(m^*\left(\bigcup_1^n E_i\right)\right)$$

$$= \lim m^*(E_n)$$

$$= \lim m(E_n)$$

$$\therefore m(\lim(E_i)) = \lim(m(E_i))$$

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(ii) Assume $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$

Look at $E_1 \setminus E_2 \subseteq E_1 \setminus E_3 \subseteq E_1 \setminus E_4$.

where $E_1 \setminus E_2$ are measurable.

by (i)

$$m(\lim(E_1 \setminus E_i)) = \lim(m(E_1 \setminus E_i))$$

Since $E_1, E_1 \setminus E_i, E_i$ are measurable.

therefore $m(E_1 \setminus E_i) = m(E_1) - m(E_i)$

$$E_1 = E_i \cup (E_1 \setminus E_i)$$

$$m(\lim(E_1 \setminus E_i)) = \lim(m(E_1) - m(E_i)).$$

$$= m(E_1) - \lim m(E_i).$$

$$\lim(E_1 \setminus E_i) = \bigcup_{i=1}^{\infty}(E_1 \setminus E_i)$$

$$= E_1 \setminus \bigcap_1^{\infty} E_i = E_1 \setminus \lim E_i.$$

$$m(\lim(E_1 \setminus E_i)) = m(E_1 \setminus \lim E_i)$$

$$= m(E_1) - m(\lim E_i).$$

From $*$.

$$m(E_1) - m(\lim E_i)$$
$$= m(E_1) - \lim(m(E_i))$$

So, $m(\lim E_i) = \lim(m(E_i))$ $\quad m(E_1) < \infty$.

Proposition

1) Every non-empty set in $\mathbb{R}$ has +ve measur

2) suppose $Q = \{q_1, q_2, \dots, \}$.

& $G = \bigcup_{n=1}^{\infty}\left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$.

Then for any closed set $F \subseteq \mathbb{R}$ we have $m(G \Delta F) > 0$.

where $G \Delta F = (G \setminus F) \cup (F \setminus G)$.

Proof :- 1) Since every non empty open set of $\mathbb{R}$ is a countable disjoint union of open intervals, ① follows.

2) If $m(G \setminus F) > 0$, then nothing to prove.

Assume $m(G \setminus F) = 0$

$G \setminus F = G \cap F^c$ which is open.

Then, by ② if $G \setminus F \neq \phi$

$m(G \setminus F) > 0$.

$\Rightarrow G \setminus F = \phi$.

$\Rightarrow G \subseteq F$.

But $\mathbb{Q} \subseteq G$ (by defn of G).

$\Rightarrow \mathbb{Q} \subseteq G \subseteq F$.

$\Rightarrow \overline{\mathbb{Q}} \subseteq \overline{G} \subseteq \overline{F}$

$\overline{\mathbb{Q}} = \mathbb{R}$.

$\Rightarrow \overline{F} = \mathbb{R}$

$\overline{F} = F$.

$\Rightarrow F = \mathbb{R}$.

$m(F) = m(\mathbb{R}) = \infty$

$$m(G) \leq \sum_{1}^{\infty} \left(\left(q_n + \frac{1}{n^2}\right) - \left(q_n - \frac{1}{n^2}\right) \right)$$

$$= \sum \frac{2}{n^2} < \infty.$$

$\therefore m(F \setminus G) = m(F) - m(G)$

$= \infty > 0$.

$\therefore m(G \Delta F) > 0$.

(1) Let $k > 0$ & $A \subseteq \mathbb{R}$

Denote $kA = \{x \in \mathbb{R} \mid k^{-1}x \in A\}$.

To show that

i) $m^*(kA) = k\, m^*(A)$.

ii) A is measurable $\iff$ kA is measurable.

Solⁿ :-

$$m^*(kA) = \inf\left\{ \sum l(I_n) \,\Big|\, kA \subseteq \bigcup_1^{\infty} I_n \right\}.$$

Suppose $I_n = [a_n, b_n)$

$$kA \subseteq \bigcup_{n=1}^{\infty} [a_n, b_n)$$

$$\Rightarrow A \subseteq \bigcup_{n=1}^{\infty} [k^{-1}a_n, k^{-1}b_n)$$

$$\therefore m^*(kA) = \inf\left\{ \sum (b_n - a_n) \,\Big|\, A \subseteq \bigcup_{n=1}^{\infty} [k^{-1}a_n, k^{-1}b_n) \right\}.$$

$$= k \inf\left\{ \sum (k^{-1}b_n - k^{-1}a_n) \,\Big|\, A \subseteq \bigcup_1^{\infty} [k^{-1}a_n, k^{-1}b_n) \right\}$$

$$= k \inf\left\{ \sum l([k^{-1}a_n, k^{-1}b_n)) \,\Big|\, A \subseteq \bigcup_1^{\infty} [k^{-1}a, k^{-1}b) \right\}.$$

$\boxed{\text{EoC}}$

Thanks <3

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MTI.

Proposition

There exists an uncountable set whose measure is zero.

Proof

Look at the cantor set $C = \bigcap_{n=1}^{\infty} P_n$

where $P_n = \bigcup_{k=1}^{n} J_{n,k}$, where each $J_{n,k}$ is an interval.

Note C is uncountable.

Each $J_{n,k}$ is an interval. Thus each P_n is measurable & hence C is measurable and hence C is measurable

$$C \in \mathcal{M} \qquad (C \in \mathcal{M})$$

Look at $m(C^c) = m([0,1] \setminus C)$

$$= m([0,1]) - m(C)$$
$$= 1 - m(C)$$

$$C^c = \left(\bigcap_1^{\infty} P_n\right)^c = \bigcup_{n=1}^{\infty} P_n^c = \bigcup_{n=1}^{\infty}\left(\bigcup_{k=1}^{2^{n-1}} I_{n,k}\right)$$

$I_{n,k}$ is interval
$\ell(I_{n,k}) = \frac{1}{3^n}$

But $\bigcup_{n=1}^{\infty} P_n^c = \bigcup_{k=1}^{2^{n-1}} I_{n,k}$ $\quad \ell(I_{n,k}) = \frac{1}{3^n}$

$$m(C^c) = m\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^{n-1}} I_{n,k}\right) = \sum_{n=1}^{\infty} m\left(\bigcup_{k=1}^{2^{n-1}} I_{n,k}\right)$$
$$= \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} m(I_{n,k})$$
$$= \sum_1^{\infty}\sum_1^{2^{n-1}} \ell(I_{n,k})$$
$$= \sum_1^{\infty} \frac{2^{n-1}}{3^n} = 1.$$

$m(C^c) = 1.$

$\therefore m(C) = 0$

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Theorem

Let $E \subseteq \mathbb{R}$. Then the following are equivalent.

(i) E is measurable.

(ii) Given $\varepsilon > 0$, $\exists$ an open set $U \subseteq \mathbb{R}$ such that $U \supseteq E$ & $m^*(U \setminus E) < \varepsilon$

(iii) There exists a G_δ set $G \subseteq \mathbb{R}$ such that $G \supseteq E$ & $m^*(G \setminus E) = 0$ $(\because m^*(G) = m^*(E))$

(ii)' Given $\varepsilon > 0$, $\exists$ a closed set $F \subseteq \mathbb{R}$ such that $F \subseteq E$ & $m^*(E \setminus F) < \varepsilon$

(iii)' There exists an F_σ $F \subseteq E$ such that $m^*(E \setminus F)$

~~If a measure satisfies~~

(A non negative countably additive set function satisfying above equivalent conditions is said to be a regular measure)

Proof

(i) $\Rightarrow$ (ii)

Assume E is measurable

Case 1. Suppose $m(E) < \infty$

Then given $\varepsilon > 0$, we have shown that there exists U open in $\mathbb{R}$ such that $E \subseteq U$ & $m^*(U \setminus E) < \varepsilon$

Case 2. Suppose $m(E) = \infty$

Case 2 Suppose $m(E) = \infty$

Write $\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$ where I_n's are open intervals

$\Rightarrow E = \bigcup_{n=1}^{\infty} I_n \cap E$

& Let $E_n = I_n \cap E \quad \forall n \geq 1$

$m(E_n) < \infty \quad (\because E_n \subseteq I_n \;\&\; l(I_n) < \infty)$

Note E_n is measurable.

Let $\varepsilon > 0$

There exists open set $U_n \subseteq \mathbb{R}$.

such than $U_n \supseteq E_n$ & $m^*(U_n \setminus E_n) < \frac{\varepsilon}{2^n}$

Let $U = \bigcup_{n=1}^{\infty} U_n$. The U is open & $U = \bigcup_{n=1}^{\infty} U_n \supseteq \bigcup_{n=1}^{\infty} E_n$

$= \bigcup_{1}^{\infty} I_n \cap E = E$

$\Rightarrow U \supseteq E$

Look at $U \setminus E = \bigcup_{1}^{\infty} U_n \setminus \bigcup_{1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} (U_n \setminus E_n)$

$$m^*(U \setminus E) \leq m^*\left(\bigcup_{n=1}^{\infty} (U_n \setminus E_n)\right)$$

$$\leq \sum_{1}^{\infty} m^*(U_n \setminus E_n)$$

$$\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon\left(\frac{1/2}{1-1/2}\right) = \varepsilon$$

Proved That (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (iii)

To show - There exist a G_δ-set G such that $G \supseteq E$ & $m^*(G \setminus E) = 0$

By (ii) For each $n > 0$, $\exists$ open $U_n \subseteq \mathbb{R}$ such that $U_n \supseteq E$ & $m^*(U_n \setminus E) < \frac{1}{n}$

Let $G = \bigcap_{1}^{\infty} U_n$

G is a G_δ set & $G \supseteq E$

now $m^*(G \setminus E) \leq m^*(U_n \setminus E)$

$(\because G \setminus E \subseteq U_n \setminus E \ \forall n)$

$m^*(G \setminus E) < \frac{1}{n} \quad \forall n \geq 1$

$\Rightarrow m^*(G \setminus E) = 0.$

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(iii) ⇒ (i) Assume $\exists$ G_δ set $G \subset \mathbb{R}$ such that $G \supseteq E$ & $m^*(G \setminus E) = 0$

To show E is measurable.

We know $G \setminus E$ is measurable because $m^*(G \setminus E) = 0$

Also every G_δ set is measurable hence G is measurable.

Now $E = G \setminus (G \setminus E) \in \mathcal{M}$ $\therefore$ E is measurable.

(i) ⇒ (ii)' Assume E is measurable.

To show: Given $\varepsilon > 0$ $\exists$ F closed in $\mathbb{R}$ s.t. $F \subseteq E$ & $m^*(E \setminus F) < \varepsilon$

Let $\varepsilon > 0$.

We have E^c also measurable.

$\therefore$ $\exists$ U open in $\mathbb{R}$. such that $m^*(U \setminus E^c) < \varepsilon$ & $U \supseteq E^c$ (by (ii))

Now $U \setminus E^c = E \setminus U^c$

[Since $U \cap E = E \cap U$].

Then $m^*(E \setminus U^c) < \varepsilon$ & $U^c \subseteq E$

Let $F = U^c$

Then F is a closed set & $F \subseteq E$ & $m^*(E \setminus F) < \varepsilon$

(ii)' − (iii)'

Assume (ii)'

To show, $\exists$ a F_σ set $F \subseteq E$ & $m^*(E \setminus F) = 0$.

For each n, $\exists$ a closed set F_n such that $F_n \subseteq E$ & $m^*(E \setminus F_n) < \frac{1}{n}$ (By (ii)').

Let $F = \bigcup_1^\infty F_n$ $\Rightarrow F$ is F_σ

$F \subseteq E$

$$\begin{aligned} \& \ m^*(E \setminus F) &= m^*(E \setminus \bigcup_1^\infty F_n) \\ &= m^*(E \cap (\bigcup_1^\infty F_n)^c) \\ &= m^*(E \cap (\bigcap_1^\infty F_n^c)) \\ &\leq m^*(E \cap F_n^c) \quad \forall n \\ &= m^*(E \setminus F_n) \\ &\leq \frac{1}{n} \quad \forall n \geq 1 \end{aligned}$$

$\Rightarrow m^*(E \setminus F) = 0$

(iii)' $\Rightarrow$ (i)

There exists an F_σ set $F \subseteq \mathbb{R}$ such that $F \subseteq E$ & $m^*(E \setminus F) = 0$

$\Rightarrow$ $E \setminus F$ is measurable

& F is measurable.

$E = (E \setminus F) \cup F$

$\Rightarrow$ E is measurable.

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Corollary:

Lebesgue measure is a regular measure.

Definition

Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$

where $E \subseteq \mathbb{R}$ measurable be a function. Then f is said to be Lebesgue measurable function or simply a measurable function.

If for each $\alpha \in \mathbb{R}$, the set $\{x \in E \mid f(x) > \alpha\}$ is measurable.

Examples:

① $f : \mathbb{R} \to \mathbb{R}$

$f(x) = c \quad \forall x \quad (c \in \mathbb{R})$

$\alpha \in \mathbb{R}$

$\{x \in \mathbb{R} \mid f(x) > \alpha\} = \mathbb{R}$ or ϕ

$\Rightarrow F$ is measurable.

② Let $E \subseteq \mathbb{R}$ be measurable

Define $\chi_E : \mathbb{R} \longrightarrow [0,1]$.

$$\chi_E(x) = \begin{cases} 0, & x \notin E \\ 1, & x \in E \end{cases}$$

For $\alpha \in \mathbb{R}$

$\{x \in \mathbb{R} \mid \chi_E(x) > \alpha\} = A_\alpha$ (say)

$$A_\alpha = \begin{cases} \mathbb{R} & \text{if } \alpha < 0 \\ \phi & \text{if } \alpha \geq 1 \\ E & \text{if } \alpha = 0 \\ E & 0 < \alpha < 1 \end{cases}$$

In each value of α A_α is measurable.

$\therefore \chi_E$ is a measurable f^n

Thm. Every continuous function is measurable.

Proof: $\alpha \in \mathbb{R}$ & $f : E \to \mathbb{R}$ is continuous.

$\{x \in E \mid f(x) > \alpha\} = f^{-1}((\alpha, \infty))$

$\Rightarrow$ pre image of $\underbrace{(\alpha, \infty)}_{\text{open set}}$

$\Rightarrow$ open set in $\mathbb{R}$.

($\because f$ is continuous)

$\Rightarrow U$ is measurable & $\therefore F$ is measurable

Thm

Let $f : E \to \mathbb{R}$ be a function & $E \subseteq \mathbb{R}$ measurable

Then the following statements are equivalent.

(i) f is measurable

(ii) For any $\alpha \in \mathbb{R}$ $\{x \mid f(x) \geq \alpha\}$ is ~~uncountable~~ measurble

(iii) For any $\{x \mid f(x) < \alpha\}$ is measurable

(iv) For any $\alpha \in \mathbb{R}$ $\{x \mid f(x) \leq \alpha\}$ is measurable.

(v) For any $\alpha \in \mathbb{R}$ $\{x \mid f(x) = \alpha\}$

Proof:

(i) $\Rightarrow$ (ii) $\alpha \in \mathbb{R}$

$$\{x \mid f(x) \geq \alpha\} = \bigcap^{\infty} \underbrace{\{x \mid f(x) > \alpha - \tfrac{1}{n}\}}_{\text{measurable}}$$

$\therefore$ measurable.

(ii) $\Rightarrow$ (iii) $\alpha \in \mathbb{R}$

$\{x \mid f(x) < \alpha\} = \{x \mid f(x) \geq \alpha\}^c$

by (ii) $\{x \mid f(x) \geq \alpha\} \subseteq \mathcal{M}$

$\therefore$ (iii) follows.

(iii) $\Rightarrow$ (iv)

Let $\alpha \in \mathbb{R}$

$$\{x \mid f(x) \le \alpha\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \mid f(x) < \alpha + \tfrac{1}{n}\}}_{\in \mathcal{M} \text{ (iii)}}$$

$\therefore$ $\{x \mid f(x) \le \alpha\}$ is measurable.

$$\{x \mid f(x) > \alpha\} = \{x \mid f(x) \le \alpha\}^c$$

Proposition

Let $f, g : E \to \mathbb{R}$ be measurable functions & $c \in \mathbb{R}$ then $f+c$, $f \pm g$, cf, fg are measurable

Proof. $\alpha \in \mathbb{R}$.

(i) To show $f+c$ is measurable.

$$\{x \in E / (f+c)(x) > \alpha\}$$

$$= \{x \in E \mid f(x) + c > \alpha\}$$

$$= \{x \in E \mid f(x) > \alpha - c\}$$

is measurable, As f is measurable.

(ii) To show cf is measurable.

$$\{x \mid (cf)(x) > \alpha\} = \{x \mid c f(x) > \alpha\}$$

$$\cancel{= \{x \mid f(x) < \tfrac{\alpha}{c}\}}$$

$$= \begin{cases} \phi \text{ or } E, & c = 0 \\ \{x \mid f(x) > \frac{\alpha}{c}\} & c > 0 \\ \{x \mid f(x) < \frac{\alpha}{c}\} & c < 0 \end{cases}$$

In all cases, set is measurable.

<u>To show</u>: $f+g$ is measurable

Let $\alpha \in \mathbb{R}$

$\{x \in E \mid (f+g)(x) > \alpha\}$

$= \{x \in E \mid f(x) + g(x) > \alpha\}$

$= \{x \in E \mid f(x) > \alpha - g(x)\}$

Let $\{r_1, r_2, \ldots\}$ be the set of all ~~natural~~ rational numbers lies b/w

$f(x)$ & $\alpha - g(x)$

i.e. $f(x) > r_i > \alpha - g(x) \quad \forall i \geq 1$

$\{x \in E \mid f(x) > \alpha - g(x)\}$

$= \bigcup_{i=1}^{\infty} \{x \in E \mid f(x) > r_i\} \cap \{x \in E \mid r_i > \alpha - g(x)\}$

This set is measurable because

~~f(g)~~ f, g are measurable

$f+g$ is measurable.

$f - g = f + (-g)$ "

$f(g) = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$

To show f^2 is also measurable.

$\{x \mid f^2(x) > \alpha\}$

$= \{x \mid [f(x)]^2 > \alpha\}$

$= \begin{cases} \phi & \text{if } \alpha < 0 \\ \_\_\_\_ & \alpha \geq 0. \end{cases}$

Say $\alpha \geq 0$

$= \{x \mid f(x) > \sqrt{\alpha}\} \cup \{x \mid f(x) < -\sqrt{\alpha}\}$

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Theorem.

Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functions defined on a measurable set $E \subseteq \mathbb{R}$.

Then

(i) $\sup\limits_{1\leq i\leq n}(f_i)$ is a measurable f^n. $\forall n\geq 1$

(ii) $\inf\limits_{1\leq i\leq n}(f_i)$ is measurable fn. $\forall n\geq 1$

(iii) $\sup\limits_{n}(f_n)$ is measurable

(iv) $\inf\limits_{n}(f_n)$ is measurable.

(v) $\lim\sup(f_n)$ " "

(vi) $\lim\inf(f_n)$ " "

Proof:

(i) Let $\alpha \in \mathbb{R}$

Consider $\{x\in E \mid \sup\limits_{1\leq i\leq n}(f_i) < \alpha\}$

$= \{x\in E \mid \sup(f_i(x)) < \alpha\}$

$= \{x\in E \mid f_i(x) < \alpha \ \forall i, 1\leq i\leq n\}$

$= \bigcap\limits_{i=1}^{n}\{x\in E \mid f_i(x) < \alpha\}$ | Each $f_i \in \mathcal{M}$

$\in \mathcal{M}$

$\therefore \sup_{1 \leq i \leq n} \{f_i\}$ is measurable.

ii) Prove using greater than α as above.

OR

$$\inf_n (f_n) = -\sup_n (-f_n)$$

$\nwarrow$ is measurable. by (i)

(iii) $\{x \in E \mid \sup_n (f_n)(x) < \alpha\}$

$= \{x \in E \mid f_n(x) < \alpha \quad \forall \alpha\}$

$= \bigcap_1^\infty \{x \in E \mid f_n(x) < \alpha\} \subset \mathcal{M}$

$\bigcap \mathcal{M}$

$\therefore \sup(f_n)$ is a measurable fn.

iv) Use $\inf_n (f_n) = -\sup_n (-f_n)$

(v) Recall.

$$\lim\sup (f_n) = \inf_{n \geq 1} \left(\sup_{i \geq n} (f_i)\right)$$

By (iii) $\sup(f_i)$ is measurable $\forall n \geq 1$

& by (iv) $\inf_{n \geq 1} \left(\sup_{i \geq n} (f_i)\right)$ is measurable.

(vi) $\lim\inf (f_n) = -\lim\sup(-f_n)$

$f_n \to f$ p.w except on a set

is $m(\{x / f_n(x) \nrightarrow f(x)\}) = 0$

$\frac{f}{g}$ $g(x) \neq 0$ $\forall x \in E$

$\{x \mid \frac{1}{g}(x) > x\}$

$= \{x \mid \frac{1}{g(x)} > \alpha\}$

$= \{x \mid \frac{1}{\alpha} > g(x)\}$ if $\alpha > 0$

$= \{x \mid \frac{1}{\alpha} < g(x)\}$ if $\alpha < 0$

$\{x \mid \frac{1}{g(x)} > 0\} = \phi$ or E or subset of E

Let $E \subseteq [0,1]$ be a non measurable set

$g : [0,1] \to \mathbb{R}$

$$g(x) = \begin{cases} 1 & x \in E \\ -1 & x \in [0,1] \setminus E \end{cases}$$

g is not measurable.

$x^3 = x^9$

$x^3(x^6 - 1) = 0$

$x = 0$

or $x = e^{i\pi/6}$

$\{x \mid g^{-1}(x) > \alpha\}$ $\alpha \in F$

$\subseteq \{x \mid g(g^{-1}(x)) \geq g(\alpha)\}$

$\cup \{x \mid g(g^{-1}(x)) \leq g(\alpha)\}$

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f, g measurable & $g(x) \neq 0 \ \forall x \in E$ on E where E is measurable then f/g is measurable.

Proof: Let $\alpha \in \mathbb{R}$

$$\{x \in E / \frac{f(x)}{g(x)} > \alpha\} = \{x \in E / f(x) > \alpha g(x)\}$$

Let $\{r_1, r_2, \dots\}$ be the set of all rational numbers in E

$$= \bigcup_{i=1}^{\infty} \{x \in E \mid f(x) > r_i > \alpha g(x)\}$$

$$= \bigcup_{i=1}^{\infty} \{x \in E \mid f(x) > r_i\} \cap \{x \in E \mid \alpha g(x) < r_i\}$$

measurable

$\therefore f/g$ is measurable.

Defn

We say a function is defined on a measurable or Borel set is said to be Borel measurable function if for all $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) > \alpha\}$ is a Borel set.

Theorem:- Let $f : E \to \mathbb{R}$ is a function (measurable set)

The following are equivalent:

① f is Borel measurable

② for all $\alpha \in \mathbb{R}$, $\{x \in E / f(x) \geq \alpha\}$ ~~in Borel set~~ is Borel measurable (Borel set)

3. For all $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) < \alpha\}$ is Borel measurable (Borel set)

4. For all $\alpha \in \mathbb{R}$ $\{x \in E \mid f(x) \leq \alpha\}$ is Borel measurable. (Borel set)

Proof: Ex

Theorem

Let $f, g : E \to \mathbb{R}$ be Borel measurable functions where E is a measurable set. Let $c \in \mathbb{R}$. Then $c.f.$ $f \pm g$, fg, $f+c$. are all Borel measurable functions.

Proof Ex

Theorem: Let $f_n : E \to \mathbb{R}$ be a sequence of Borel measurable Then $\sup\limits_{1 \leq i \leq n} (f_i)$, $\inf\limits_{1 \leq i \leq n} (f_i)$, $\sup\limits_{n} (f_n)$, $\inf\limits_{n} (f_n)$, $\limsup\limits_{n} (f_n)$, & $\liminf\limits_{n} (f_n)$ are Borel measurable function.

Proof. Ex.

Defn:- If a property holds except on a set of measure zero, we say that it holds "almost everywhere" simply written at a.e.

Ex:-

$A = [0, 1]$

$B = [0, 1] \cup \{2\}$

$B \setminus A = \{2\}$

$m(B \setminus A) = 0$

$A \setminus B = \emptyset$

$m(A \setminus B) = 0$

$f(B) \to B$, $g : B \to B$

$$f(x) = \begin{cases} \sin x & \text{if } x \neq 2 \\ 3 & \text{if } x = 2. \end{cases}$$

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 2 \\ 2 & \text{if } x = 2 \end{cases}$$

$\Rightarrow f = g$ a.e.

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$\{x \in B \mid f(x) \neq g(x)\} = \{2\}$

which has measure zero $\Rightarrow f = g$ a.e.

Proposition: Let $f : E \to \mathbb{R}$ be measurable & let $g : E \to \mathbb{R}$ be any map such that $f = g$ a.e. Then g is also measurable.

Proof: Recall: Suppose F is measurable & $m^*(F \Delta G) = 0$ for some set G. Then G is measurable. Let $\alpha \in \mathbb{R}$.

Let $F = \{x \in E \mid f(x) > \alpha\}$
$G = \{x \in E \mid g(x) > \alpha\}$

To show. G is measurable set.

$$F \Delta G = (F \setminus G) \cup (G \setminus F)$$
$$= \{x \in F \mid f(x) > \alpha \ \& \ g(x) \leq \alpha\} \cup \{x \in E \mid g(x) > \alpha \ \& \ f(x) \leq \alpha\}$$
$$\subseteq \{x \in E \mid f(x) \neq g(x)\}$$

By assumption $m(\{x \in E \mid f(x) \neq g(x)\}) = 0$

$m^*(F \Delta G) = 0 \Rightarrow G$ is measurable.

Corollary. Let $f_n : E \to \mathbb{R}$ be a measurable functions & $\{f_n\}$ converges to f a.e. Then f is measurable.

Proof: $f_n \to f$ a.e.

i.e. $\{x \in E \mid f_n \not\to f(x)\}$ has measure zero

$$f = \limsup (f_n) \text{ a.e.}$$

We know $\limsup f_n$ is measurable.

$\Rightarrow f$ is measurable (by above Prop).

$\boxed{\text{EOC}}$

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Let $f: E \to \mathbb{R}$ be a measurable f^n

① $f^+ = \max\{f, 0\}$

② $f^- = -\min\{f, 0\}$ are measurable.

Proposition

The set of points to which a sequence of measurable functions $\{f_n\}$ converges, is measurable

i.e $\{x \in E \mid f_n(x) \text{ converges}\}$ is a measurable set.

Proof

$\{x \in E \mid f_n(x) \text{ converges}\}$

$= \{x \in E \mid \lim_{n \to \infty} f_n(x) \text{ exists}\}$

$= \{x \in E \mid \limsup_n f_n(x) - \liminf_n f_n(x) = 0\}$

Since $\limsup_n f_n$ and $\liminf_n f_n$ are measurable $\Rightarrow$ the above set is also measurable.

Defn

Let f be a measurable function defined on a measurable set E, then the essential supremum of f is defined as

$$\text{ess sup}(f) = \inf\{\alpha \in \mathbb{R} \mid f \leq \alpha \text{ a.e}\}$$

where $f \leq \alpha$ a.e.

② $\chi_{[0,1]}$

$$\text{ess sup}(\chi_{[0,1]}) = \inf\{\alpha \mid \chi_{[0,1]} \leq \alpha \text{ ae}\} = 1$$

Suppose $\alpha_0 < 1$

$\quad = \inf\{\alpha \mid \chi_{[0,1]} \leq \alpha \text{ ae}\}$

$$m(\{x \in \mathbb{R} \mid \chi_{[0,1]}(x) > \alpha_0\})$$

$$= m([0,1]) = 1 \neq 0$$

Prop^n

Let $f : E \to \mathbb{R}$ ~~[illegible]~~ be measurable.

$f \leq \text{ess sup}(f)$ a.e.

Proof

If $\text{ess. sup}(f) = +\infty$ Then we are done.

Suppose $\text{ess sup}(f) = -\infty$

For each $n \in \mathbb{Z}$, $f \leq n$ a.e. (by def)

& we have nothing to prove.

Assume $\text{ess sup}(f)$ is a finite number.

Let $E_n = \{x \in E \mid f(x) > \frac{1}{n} + \text{ess sup}(f)\}$

& $A = \{x \in E \mid f(x) > \text{ess sup}(f)\}$

Clearly $E_n \subseteq A \quad \forall n \geq 1$.

$\Rightarrow \bigcup_{n=1}^{\infty} E_n \subseteq A$.

$f(x) >$ ess sup

$\Rightarrow \quad \exists n$ s.t. $f(x) > \frac{1}{n} +$ ess sup(f)

$\Rightarrow x \in E_n \; \exists n.$

$\therefore A \subseteq \bigcup_{n=1}^{\infty} E_n$

$\Rightarrow A = \bigcup_{n=1}^{\infty} E_n$

By defn ess sup$(f_n) = \inf\{\alpha \mid f \leq \alpha \text{ ae}\}$

$m^*(E_n) = m^*(\{x \in E \mid f(x) \geq \frac{1}{n} +$ ess sup$(f)\})$

To show $m^*(E_n) = 0$

ess sup$(f) \leq \alpha =$ ess sup$(f) + \frac{1}{n}$ (Let)

The $f \leq \alpha$ a.e. by defn

$\Rightarrow m(\{x \in E \mid f(x) > \alpha\}$

$\Rightarrow m(E_n) = 0.$

$m^*(A) = m^*(\bigcup_1^{\infty} E_n) \leq \sum_1^{\infty} m^*(E_n) = 0.$

$\therefore m^*(A) = 0.$

$f \leq$ ess sup f a.e.

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Prop^n Let $f, g : E \to \mathbb{R}$ is a measurable fns.
Then ess sup $(f+g) \leq$ ess sup (f) + ess sup (g) a.e

Proof From above

$$f+g \leq \text{ess sup}(f) + \text{ess sup}(g) \text{ a.e.}$$

$$\Rightarrow \text{ess}(f+g) \leq \quad '' \quad '' \quad \text{a.e.}$$

Example

Let $f = \chi_{[-1,0)} - \chi_{[0,1]}$

& $g = -f$.

Then

ess. sup $(f+g) = 0$

ess sup $(f) = 1$

ess sup $(g) = 1$

ess sup (f) + ess sup $(g) = 2 \ngtr 0$.

Def^n Let $f : E \to \mathbb{R}$ be a measurable f^n. Then the essential infimum of f is defined as.

~~ess sup(~~

$$\text{ess inf}(f) = \sup\{\alpha \in \mathbb{R} \mid f \geq \alpha \text{ ae}\}$$

Remark ess sup $(f) = -$ ess inf$(-f)$

Pf: $\text{ess sup}(f) = \inf\{\alpha \in \mathbb{R} \mid f \leq \alpha \text{ ae}\}$

$= \inf\{\alpha \in \mathbb{R} \mid -f \geq -\alpha \text{ ae}\}$

$= -\sup\{-\alpha \mid -f \geq -\alpha \text{ ae}\}$

$= -\sup\{\beta \in \mathbb{R} \mid -f \geq \beta \text{ a.e}\}$

$= \cancel{\text{ess}} - \text{ess sup}(-f)$

Defn

~~Let~~

We say a measurable function $f: E \to \mathbb{R}$ is essentially bounded, if

$$\text{ess sup}(|f|) < \infty$$

Proposition

Let $f: E \to \mathbb{R}$ be a measurable f^n & B be a Borel set. Then $f^{-1}(B)$ is measurable.

Proof: We have for any $A_i \subseteq \mathbb{R}$ $\forall i \geq 1$

(i) $f^{-1}\left(\bigcup_1^\infty A_i\right) = \bigcup_1^\infty f^{-1}(A_i)$

& (ii) $f^{-1}(A_i^c) = (f^{-1}(A_i))^c$

To prove

Let $x \in$ LHS.

$\Rightarrow f(x) \in \bigcup_1^\infty A_i$

$\Rightarrow f(x) \in A_i \quad \exists i$

$\Rightarrow x \in f^{-1}(A_i)$

$\Rightarrow x \in$ RHS.

Let $x \in$ RHS

$\Rightarrow x \in f^{-1}(A_i) \ \exists i$

$\Rightarrow f(x) \in A_i \subseteq \bigcup_{j=1}^\infty A_j$

$\Rightarrow x \in f^{-1}\left(\bigcup_1^\infty A_j\right)$

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(ii) $x \in LHS$

$\Rightarrow f(x) \in A_i^c$

$\Rightarrow f(x) \notin A_i$

$\Rightarrow x \notin f^{-1}(A_i)$

$\Rightarrow x \in (f^{-1}(A_i))^c = RHS.$

Let $x \in RHS.$

$x \notin f^{-1}(A_i)$

$\Rightarrow f(x) \notin A_i$

$\Rightarrow f(x) \in A_i^c$

$\Rightarrow x \in f^{-1}(A^c) = LHS$

Then look at the class $\mathcal{C}$ = sets whose preimage under f is measurable.

$A_i \in \mathcal{C} \quad \forall i \geq 1.$

Then $\cup A_i$, $A_i^c \in \mathcal{C}$

$\therefore \mathbb{R}, \phi \in \mathcal{C}$

$(f^{-1}(\mathbb{R}) = E$ measurable$)$

$\mathcal{C}$ is a σ algebra.

Also all intervals belong to $\mathcal{C}$

$\Rightarrow \mathcal{B} \subseteq \mathcal{C}$

↳ Borel σ algebra

$\Rightarrow f^{-1}(B)$ is measurable $\forall$ Borel in B.

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Theorem:-

Let $E \subseteq \mathbb{R}$ is a ~~measurable~~ measurable set. Then for any $y \in \mathbb{R}$, The set $E+y = \{x+y \mid x \in E\}$ is measurable & $m(E) = m(E+y)$

Proof

Given $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}$ $U \supseteq E$ & $m(U \setminus E) < \varepsilon$ (E is measurable)

Let $y \in \mathbb{R}$

Look at $U+y \subseteq \mathbb{R}$ which is open set

& $U+y \supseteq E+y$

Now, $(U+y) \setminus (E+y) = (U \setminus E) + y$

$\Rightarrow m^*((U \setminus E)+y) = m^*(U \setminus E) = m(U \setminus E) < \varepsilon$

By one of equivalent condition we get $E+y$ is measurable.

$$m(E+y) = m^*(E+y)$$
$$= m^*(E)$$
$$= m(E).$$

ash. as

Theorem: $\exists$ a none measurable set.

Proof

Let $x, y \in [0,1]$

~~Define~~ $x \sim y$ if $y - x \in \mathbb{Q} \cap [-1,1]$

$\sim$ is an equivalence relation

$x, y \in [0,1]$ • $x \sim x$ $[x - x = 0 \in \mathbb{Q} \cap [-1,1]]$

• $x \sim y \Rightarrow y - x \in \mathbb{Q} \cap [-1,1]$

$\Rightarrow x - y \in \mathbb{Q} \cap [-1,1]$

$\Rightarrow y \sim x$

• $x \sim y$, $y \sim z$

$\Rightarrow y - x, z - y \in \mathbb{Q} \cap [-1,1]$

$\Rightarrow \cancel{y} - x + z - \cancel{y} \in \mathbb{Q} \cap [-1,1]$

$\Rightarrow z - x \in \mathbb{Q} \cap [-1,1]$

$\Rightarrow x \sim z$

$\therefore \sim$ is an equivalence ~~relat~~ relation

$[0,1] = \bigcup_\alpha E_\alpha$

where E_α is an equivalence relation $\sim$

Suppose $x, y \in E_\alpha \Rightarrow x \sim y \Rightarrow y \sim x \in \mathbb{Q} \cap [-1,1]$

$\Rightarrow E_\alpha$ is countable for some $x_\alpha \in E_\alpha$.

$\quad\;\; \|$

$y \in [x_\alpha]$ $\qquad y \sim x_\alpha$

Thus each E_α is a countable set.

Since $[0,1]$ is uncountable & every E_α is countable therefore there are uncountably many ~~equat~~ equivalence classes.

Recall – Axiom of choice:

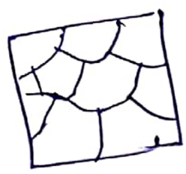
Let X be a non empty set. Suppose $\{E_\alpha \mid \alpha \in I\}$ is a non empty collection of nonempty disjoint subsets of X.

Then there exists a set $V \subseteq X$ containing just one element from each E_α.

By axiom of choice.

There exists a set $V \subseteq [0,1]$ containing exactly one element x_α from each E_α

in $V = \{x_\alpha \mid x_\alpha \in E_\alpha\} \subseteq [0,1]$



Claim :

V is not measurable.

Pf : Let $\mathbb{Q} \cap [-1,1] = \{r_1, r_2, \dots\}$

Denote $V_n = V + r_n \quad \forall n \geq 1$,

To show $V_n \cap V_m = \phi \quad \forall n \neq m$

Suppose $x \in V_n \cap V_m$

$\Rightarrow x = v_1 + r_n$ for some $v_1, v_2 \in V$

$= v_2 + r_m$

$v_1 - v_2 = r_m - r_n \in \mathbb{Q} \cap [-1,1]$

$\Rightarrow v_1 \sim v_2$

$\Rightarrow v_1 = v_2$ (By Def of V)

~~$\Rightarrow v_1 = $~~

$\Rightarrow r_m = r_n$

$\Rightarrow m = n$.

Thum if $m \neq n$, $\Rightarrow V_n \cap V_m = \phi$

~~Proof~~

Now $[0,1] \subseteq \bigcup^{\infty} V_n$.

$[0,1] \quad \mathbb{Q} = \bigcup E_\alpha$

$= \bigcup_\alpha [x_\alpha]$

$x \in [0,1]$

$\Rightarrow x \in E_\alpha$ from α

$x \sim x_\alpha$

$x = x_\alpha + r_n$ for some $n \in V_n$.

Then $\forall x \in [0,1]$

Then $[0,1] \subseteq \bigcup_1^\infty V_n$

Also $\bigcup_1^\infty V_n \subseteq [-1,2]$ [By Axiom of Choice?]

Suppose V is measurable. Then by above thm each $V_n = V + \gamma_n$ is measurable & $m(V_n) = m(V)$ $\forall n \geq 1$

Look at $[0,1] \subseteq \bigcup_{n=1}^{\infty} V_n \subseteq [-1, 2]$

$$\Rightarrow m([0,1]) \leq m\left(\bigcup_1^\infty V_n\right) < m([1,2])$$

$$= \sum_1^\infty m(V_n$$

$$\Rightarrow 1 \leq \underbrace{\sum_1^\infty m(V)}_{= 0 \text{ or } +\infty} \leq 3.$$

The possible value for $\sum_1^\infty m(V)$ either 0 or ∞

which gives a contradiction

V is not measurable.