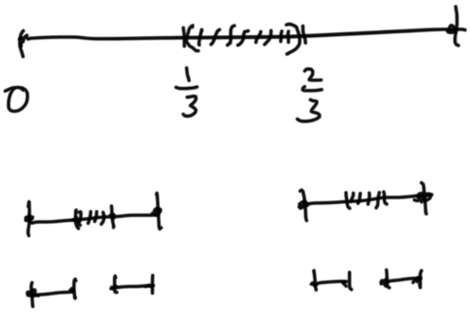
Lecture 3

The Cantor set



$[0,1]$

$[0,\frac{1}{3}]\cup[\frac{2}{3},1]$

$[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1]$.

$\vdots$

From the closed interval $[0,1]$, first remove $(\frac{1}{3},\frac{2}{3})$, then $(\frac{1}{9},\frac{2}{9})$, & $(\frac{7}{9}, 8/9)$ etc removing at each stage, the open interval containing the "middle thirds" of closed intervals left at the previous stage.

At the n^{th} stage we get the closed intervals $J_{n,1}, J_{n,2}, \ldots, J_{n,2^n}$ each of length $\frac{1}{3^n}$.

Let $P_n = \bigcup_{k=1}^{2^n} J_{n,k} \quad \forall n\geq 1.$

Then $P=\bigcap_{n=1}^{\infty} P_n$ is called the <u>Cantor set</u> or the <u>Cantor ternary set</u>.

$P_1 = [0,1]$

$P_2 = [0,\frac{1}{3}]\cup[\frac{2}{3},1]$

$P_3 = [0,\frac{1}{3^2}]\cup[\frac{2}{3^2},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{3^2}]\cup[\frac{8}{3^2},1]$

$\vdots$

$$P_n = \frac{P_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{P_{n-1}}{3}\right).$$

where $\frac{P_{n-1}}{3} = \left\{ \frac{s}{3} \,\middle|\, s \in P_{n-1} \right\}.$

$\frac{2}{3} + S = \left\{ \frac{2}{3} + s \,\middle|\, s \in S \right\}$

for any set S.

- $P \neq \phi$. $\because 0, 1 \in P$. $\boxed{P = \frac{P}{3} \cup \left(\frac{2}{3} + \frac{P}{3}\right)}$

<u>Proposition</u>:– Let $x \in P$. Then x has the ternary expansion, $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, where $a_i \in \{0, 2\}$ $\forall i \in \mathbb{N}$.

& Conversely.

<u>proof</u>:– Let $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \in P$. $x = 0.a_1 a_2 a_3 \dots$ ternary.

$$\longrightarrow \quad \frac{1}{3}x = \sum_{k=1}^{\infty} \frac{a_k}{3^{k+1}} = \sum_{k=1}^{\infty} \frac{b_k}{3^k}, \text{ where } \underline{b_1 = 0} \text{ \& } b_k = a_{k-1}, \forall k \geq 2$$

$$\frac{1}{3}x + \frac{2}{3} = \sum_{k=1}^{\infty} \frac{b_k}{3^k} + \frac{2}{3}.$$

$$= \sum_{k=1}^{\infty} \frac{c_k}{3^k}, \text{ where } \underline{c_1 = 2}, \text{ \& } c_k = a_{k-1} \ \forall k \geq 2.$$

Thus

$x \in P_1 \iff x$ has a ternary exp. where either $a_1 = 0$ or $a_1 = 2$

since it is of the form $\frac{1}{3}y$ or $\frac{1}{3}y + \frac{2}{3}$

for some $y \in [0,1]$.

Repeat this argument, we see that

$x \in P_n \iff$ its equals a ternary exp.

where $a_k = 0$ or 2

for $1 \le k \le n$.

<u>Proposition</u>:- Cantor set is uncountable.

<u>proof</u>:- By above proposition, P consists of those points x which can be given an expansion to the base 3 in the form

$$x = 0.x_1 x_2 x_3 \cdots \quad \text{with } x_n = 0 \text{ or } 2 \ \forall n.$$

$$= \sum_{i=1}^{\infty} \frac{x_i}{3^i}.$$

Suppose P is countable. & let $x^{(1)}, x^{(2)}, \ldots$ be an enumeration of P.

Then let $x = 0.x_1 x_2 x_3 \cdots$ be such that

if $x_n^{(n)} = 0$, then let $x_n = 2$

& if $x_n^{(n)} = 2$, then let $x_n = 0$.

Then $x \neq x^{(n)}$ $\forall n$. $\quad (x_n \neq x_n^{(n)})$

& $x \in P$.

$\therefore$ P is not countable.

$\Rightarrow$ P is uncountable.

Let $I \subset \mathbb{R}$ be an interval,

say $I = [a, b]$, $\quad$ length$(I) = b-a$.

length$([a,b]) = b-a$.

$2.2220\cdots = 2\times 3^0 + 2\cdot\frac{1}{3} + 2\cdot\frac{1}{3^2} + 2\frac{1}{3^3} + 0$

$P \subset [0,1] \quad > 1 \quad \therefore$

$0.1 = 0\times 3^0 + 1\times\frac{1}{3} = \frac{1}{3}$.

terny exp ?

$0.1 = 0.022\ldots$

length $= b-a$

length$([a,b]) = b-a$.

Qn: Can we have a general notion ~~for~~ the length of any subset of $\mathbb{R}$?

For $A \subseteq \mathbb{R}$, can we define a number assign to A so that if A is an interval then this number coincides with the length of the interval.

<u>Def</u>:- Let $A \subseteq \mathbb{R}$. Then the <u>Lebesgue outer measure</u> or simply <u>outermeasure</u> of A is defined

as $$m^*(A) := \inf \left(\sum_{n=1}^{\infty} l(I_n) \right)$$

where inf is taken over all finite or countable collections of intervals $\{I_n\}$, $I_n = [a_n, b_n)$, such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$.

$$m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) \;\middle|\; \begin{array}{l} \{I_n\} \text{ intervals, } I_n = [a_n, b_n) \\ \& \ A \subseteq \bigcup_{n=1}^{\infty} I_n \end{array} \right\}.$$

① If $I = [a, b)$, then $m^*(I) = b - a$.

<u>Pf</u>: $I_1 = I$, $I_n = \phi$, $\forall n \geq 2$.

$I \subseteq \bigcup_{n=1}^{\infty} I_n \qquad \sum l(I_n) = b - a;$

② $m^*(\mathbb{Q}) = ??$ we see this later.