Lecture 18

Theorem:- Let $E \subseteq \mathbb{R}^d$. Then the following are equivalent for any $\varepsilon > 0$.

(i) There exists an open set $U \supseteq E$ such that $m^*(U \setminus E) \leq \varepsilon$.
(i.e, E is measurable).

(ii) There exists a closed set $F \subseteq E$ such that $m^*(E \setminus F) \leq \varepsilon$.

(iii) If $m^*(E) < \infty$, then there exists a compact set K with $K \subseteq E$ & $m^*(E \setminus K) \leq \varepsilon$.

(iv) If $m^*(E) < \infty$, then there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes Q_j such that $m^*(E \Delta F) \leq \varepsilon$.

Def:- Let $E \subseteq \mathbb{R}^d$ be a measurable set. Then a function $f: E \to \mathbb{R} \cup \{\pm \infty\}$ is said to be a measurable function, if for all $\alpha \in \mathbb{R}$,

$$\{ \underline{x} \in E \mid f(x) < \alpha \} \text{ is measurable.}$$

Theorem:-

① f is measurable $\iff$ for all $\alpha \in \mathbb{R}$,

$$\{x \in E \mid f(x) > \alpha\} \text{ is measurable}$$

② f is measurable $\iff$ for all $\alpha \in \mathbb{R}$,

$$\{x \in E \mid f(x) \leq \alpha\} \text{ is measurable.}$$

④ f is measurable $\iff$ for all $\alpha \in \mathbb{R}$

$$\{x \in E \mid f(x) \geq \alpha\} \text{ is measurable.}$$

Proposition:-

① $f : E \to \mathbb{R}$, $E \subseteq \mathbb{R}^d$ measurable. Then f is measurable $\iff$ $f^{-1}(U)$ is measurable, for all $U \subseteq \mathbb{R}^d$ open.

② Let $f_n : E \to \mathbb{R}$ be a sequence of measurable functions defined on a measurable set $E \subseteq \mathbb{R}^d$. Then

$\sup(f_n)(x)$, $\inf(f_n)(x)$, $\limsup f_n(x)$, $\liminf(f_n)(x)$ are measurable functions.

(3) Let $\{f_n\}$ be a sequence of measurable functions & $f_n(x) \to f(x)$ $\forall x \in E$

i.e., $f_n \to f$ pointwise. Then f is measurable

(4) f, g are measurable functions. Then $f \pm g$, fg are measurable, if f, g are finite valued

i.e. $f, g : E \to \mathbb{R}$.

Examples:— (1) Every constant function is measurable.

$f : \mathbb{R}^d \to \mathbb{R}$, $f(x) = c$ $\forall x$.

(2) Every continuous function is measurable.

(3) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 - 2y^2$ measurable.

$g(x,y) = e^x + 2e^y - 2y^2$ measurable.

becaus f, g are continuous.

Def:- A step function is a finite sum

$$f = \sum_{k=1}^{N} a_k \chi_{R_k}, \quad \text{where } R_k \text{ are rectangles } \forall\, k = 1, \ldots, N.$$

χ_{R_k} = the characteristic function of R_k.

$$\chi_{R_k}(x) = \begin{cases} 1 & \text{if } x \in R_k \\ 0 & \text{otherwise.} \end{cases}$$

$$f(x) = \sum_{k=1}^{N} a_k \chi_{R_k}(x), \quad \text{where } a_k \text{ are constants.}$$

Def:- A simple function is a finite sum

$$f = \sum_{k=1}^{N} a_k \chi_{E_k}$$

where each E_k is a measurable set of finite measure & a_k are constants.

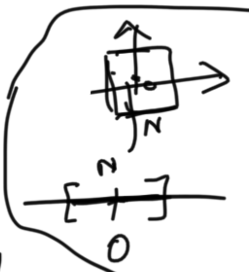
Theorem:- Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\varphi_k\}_{k=1}^{\infty}$ that converges pointwise to f, namely $\varphi_k(x) \le \varphi_{k+1}(x)$ $\forall x, \forall k$

& $\lim_{k\to\infty} \varphi_k(x) = f(x)$ $\forall x$.

$\boxed{\varphi_k(x) \to f(x) \ \forall x.}$

<u>proof:-</u> For $N \ge 1$,

let Q_N = the cube centered at the origin & of side length N. in $\mathbb{R}^d$.



$Q_1 \subseteq Q_2 \subseteq \ldots$

Define
$$F_N(x) = \begin{cases} f(x) & \text{if } x \in Q_N \ \& \ f(x) \le N \\ N, & \text{if } x \in Q_N \ \& \ f(x) > N \\ 0 & \text{otherwise} \end{cases}$$

As $N \to \infty$, $F_N(x) \to f(x)$ $\forall x$.

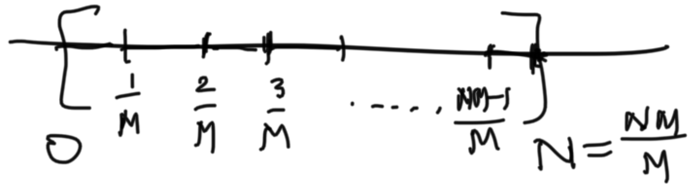
Now range of $F_N(x)$ is $[0, N]$ $\left(f(x) \ge 0 \ \forall x \right)$

Now partition the range $[0, N]$ as follows:

For fixed $N, M \ge 1$, we define

$$E_{l,M} \doteq \left\{ \underline{x \in Q_N} \,\middle|\, \frac{l}{M} \lneq F_N(x) \leq \frac{l+1}{M} \right\}$$ disjoint. & measurable ∵ F_N measurable

for $0 \leq l \lneq NM$.



Consider the function

$$\boxed{F_{N,M}(x) = \sum_{l=0}^{NM-1} \frac{l}{M} \chi_{E_{l,M}}}$$

These functions are simple functions $\forall N, M$.

$$F_{N,M}(x) \leq F_N(x) \quad \forall x.$$

On the set $E_{l,M}$, $F_{N,M}(x)$ takes the values is $\frac{l}{M}$ only

but on the set $E_{l,M}$, $F_N(x)$, values is at least $\frac{l}{M}$ & at most $\frac{l+1}{M}$

$$\therefore \quad F_{N,M}(x) \leq F_N(x) \quad \forall x \in E_{l,M}$$

$$\forall l = 0, \dots, NM-1.$$

$$\boxed{0 \leq F_N(x) - F_{N,M}(x) \leq \frac{1}{M} \quad \forall x.} \quad \forall N, M \geq 1$$

Now choose $N = M = 2^k$ with $k \geq 1$.

Let $\varphi_k = F_{2^k, 2^k}$ $\forall$ $k \geq 1$.

<u>Claim:</u> $\varphi_k(x) \longrightarrow f(x)$ as $k \to \infty$.