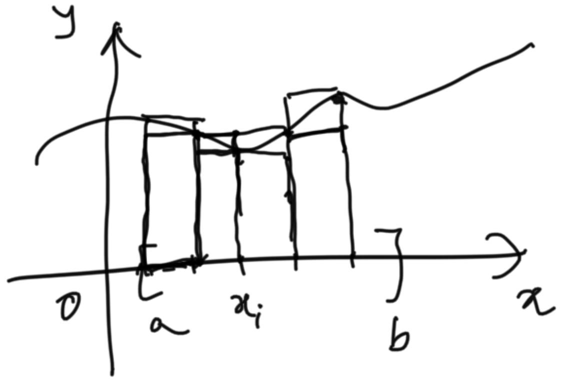
Lecture 25

Riemann integral vs Lebesgue integral.

Def:- Riemann integral :-

Let $f : \mathbb{R} \to \mathbb{R}$ be a function.



For any partition P: $a = x_0 < x_1 < \cdots < x_n = b$

of $[a,b]$, define

$$U(P,f) := \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]}(f) \cdot (x_i - x_{i-1}).$$

called Riemann upper sum.

$$L(P,f) := \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]}(f) \cdot (x_i - x_{i-1})$$

called Riemann lower sum.

The upper Riemann integral of f on $[a,b]$ is

$$\overline{\int_a^b} f(x)\,dx := \inf_{\substack{P \text{ partition} \\ \text{of } [a,b]}} \left(U(P,f) \right).$$

The lower Riemann integral of f on [a,b] is

$$\underline{\int_a^b} f(x)\,dx := \sup_P \left(L(P,f) \right).$$

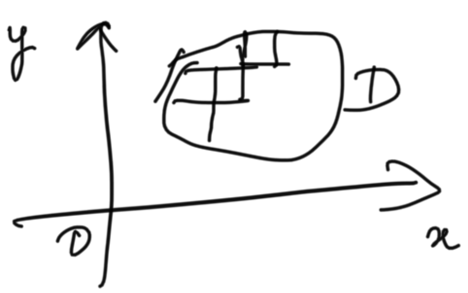
We say f is <u>Riemann integrable</u> on [a,b]

if the upper & <u>lower</u> Riemann integrals are equal. i.e. $\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$

& this common value is denoted by

$\int_a^b f(x)\,dx.$

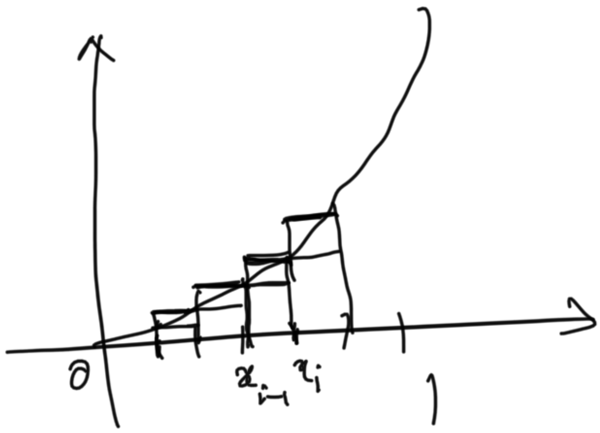
<u>Qn</u>: Can we define Riemann integral of a function $f: \mathbb{R}^d \to \mathbb{R}$?

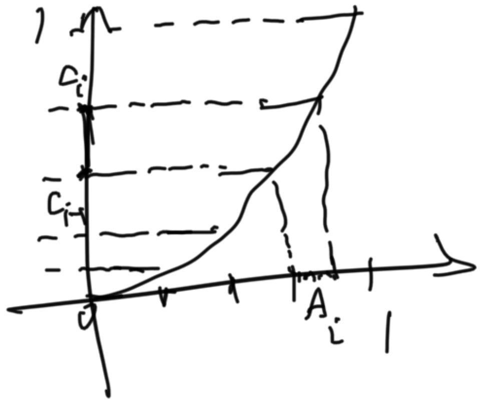


Example:- 1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

f is continuous on $[0,1]$.

$$\int_0^1 f(x)\, dx = ?$$





Let $P_n : 0 = x_0 < \frac{1}{n} < \frac{2}{n} < \cdots < 1 = b$

i.e., $x_i = \frac{i}{n}$ $\quad 0 \le i \le n$.

$$U(P_n, f) = \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]}(f) \cdot (x_i - x_{i-1}) \longrightarrow (*)$$

$$= \sum_{i=1}^{n} x_i^2 (x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \frac{i^2}{n^2}\left(\frac{1}{n}\right).$$

$$\therefore \int_0^1 f(x)\, dx = \inf_{P_n} U(P_n, f) = \lim_{n\to\infty} U(P_n, f)$$

*

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} x_i^2 (x_i - x_{i-1}) \right)$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{i^2}{n^3} \right)$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right).$$

$$= \frac{1}{3}.$$

Qn: How to Compute the Lebesgue integral of f on $[0,1]$. ?

First note that f is measurable ~~on~~ $[0,1]$

f is a bounded function supported on $[0,1]$.

on $[0,1]$

To find the $\int_{[0,1]} f$, suffices to find

$\mathcal{L}$

a sequence of simple functions $\{\varphi_n\}$

such that $\varphi_n(x) \to f(x)$ a.e on $[0,1]$.

$$\int_{[0,1]} f = \lim_{n\to\infty} \int_{[0,1]} \varphi_n .$$

consider $\varphi_n = \sum_{i=1}^{n} \underbrace{\sup_{[x_{i-1}, x_i]} (f)}_{= c_i \text{ (say)}} \chi_{A_i}. \quad \forall n \geq 1$

where $A_i = [x_{i-1}, x_i]$.

$= f^{-1}\left([c_{i-1}, c_i] \right)$

we have,

$\varphi_n(x) \to f(x)$ as $n \to \infty$.

$$\therefore \int_{\substack{[0,1] \\ \mathcal{L}}} f = \lim_{n \to \infty} \left(\int_{\substack{[0,1] \\ \mathcal{L}}} \varphi_n \right)$$

$$= \lim_{n \to \infty} \left(\int_{[0,1]} \left(\sum_{i=1}^{n} c_i \chi_{A_i} \right) \right)$$

$$= \lim_{n \to \infty} \left(\sum_{i=1}^{n} c_i \, m(A_i). \right) \qquad A_i = [x_{i-1}, x_i]$$

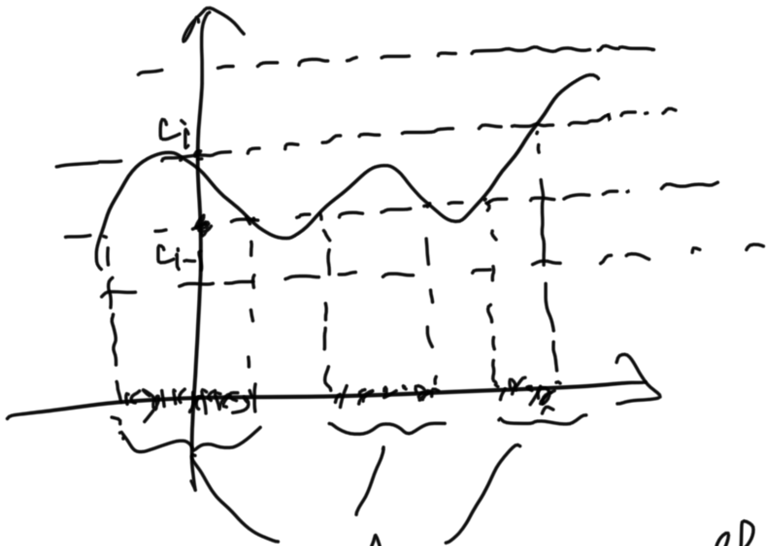
$$= \lim_{n \to \infty} \left(\sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]}(f) \cdot (x_i - x_{i-1}) \right) = \lim_{n \to \infty} U(P_n, f)$$

(by ⊛).

$$= \int_{\substack{0 \\ \mathcal{R}}}^{1} f(x)\, dx. = \frac{1}{3}.$$

The difference is that the Riemann subdivides the domain of a function, while the Lebesgue integral subdivides the range of that function.

The improvement from the Riemann integral to the Lebesgue integral is that the Lebesgue integral provides more generality than the Riemann integral does.



$y = f(x)$

A_i measurable. $\sum_i c_i \chi_{A_i}$

<u>Theorem</u>: If f is a bounded function defined on $[a,b]$ such f is Riemann integrable, then f is Lebesgue integrable &

$$\mathcal{R}\int_a^b f(x)\,dx = \mathcal{L}\int_{[a,b]} f.$$