Minimize $$z = 5x_1 + 7x_2$$

Subject to the constraints

$$\begin{aligned} 2x_1 + 3x_2 &\geq 42 \\ 3x_1 + 4x_2 &\geq 60 \\ x_1 + x_2 &\geq 18 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Ans: x_1=12, x_2=6 , Min z = 102

Introducing the surplus and artificial variables, R_1, R_2, the LPP is modified as follows:

Minimize $z = 5x_1 + 7x_2 + MR_1 + MR_2 + MR_3$

Subject to the constraints

$$
\begin{aligned}
2x_1 + 3x_2 - s_1 \quad + R_1 \quad &= 42 \\
3x_1 + 4x_2 \quad - s_2 \quad + R_2 &= 60 \\
x_1 + x_2 \quad - s_3 \quad + R_3 &= 18 \\
x_1, x_2, s_1, s_2, s_3, R_1, R_2, R_3 &\geq 0
\end{aligned}
$$

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
		-5+6M	-7+8M	-M	-M	-M	0	0	0	120M
z	1	~~-5~~	~~-7~~	~~0~~	~~0~~	~~0~~	~~-M~~	~~-M~~	~~-M~~	~~0~~
R1	0	2	$\boxed{3}$	-1	0	0	1	0	0	42
R2	0	3	4	0	-1	0	0	1	0	60
R3	0	1	1	0	0	-1	0	0	1	18
z	1	$-\frac{1}{3}+\frac{2M}{3}$	0	$-\frac{7}{3}+\frac{5M}{3}$	-M	-M	$\frac{7}{3}-\frac{8M}{3}$	0	0	98+8M
x2	0	2/3	1	-1/3	0	0	1/3	0	0	14
R2	0	1/3	0	4/3	-1	0	-4/3	1	0	4
R3	0	1/3	0	1/3	0	-1	-1/3	0	1	4

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
z	1	$-\frac{1}{3}+\frac{2M}{3}$	0	$-\frac{7}{3}+\frac{5M}{3}$	-M	-M	$\frac{7}{3}-\frac{8M}{3}$	0	0	98+8M
x2	0	2/3	1	-1/3	0	0	1/3	0	0	14
R2	0	1/3	0	$\boxed{4/3}$	-1	0	-4/3	1	0	4
R3	0	1/3	0	1/3	0	-1	-1/3	0	1	4
z	1	$\frac{1}{4}+\frac{M}{4}$	0	0	$-\frac{7}{4}+\frac{M}{4}$	-M	- M	$\frac{7}{4}-\frac{5M}{4}$	0	105+3M
x2	0	3/4	1	0	-1/4	0	0	1/4	0	15
s1	0	1/4	0	1	-3/4	0	-1	3/4	0	3
R3	0	1/4	0	0	1/4	-1	0	-1/4	1	3

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
z	1	$\frac{1}{4}+\frac{M}{4}$	0	0	$-\frac{7}{4}+\frac{M}{4}$	-M	-M	$\frac{7}{4}-\frac{5M}{4}$	0	105+3M
x2	0	3/4	1	0	-1/4	0	0	1/4	0	15
s1	0	1/4	0	1	-3/4	0	-1	3/4	0	3
R3	0	$\boxed{1/4}$	0	0	1/4	-1	0	-1/4	1	3
z	1	0	0	0	-2	1	-M	2-M	-1-M	102
x2	0	0	1	0	-1	3	0	1	-3	6
s1	0	0	0	1	-1	1	-1	1	-1	0
x1	0	1	0	0	1	-4	0	-1	4	12

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
z	1	0	0	0	-2	1	-M	2-M	-1-M	102
x2	0	0	1	0	-1	3	0	1	-3	6
s1	0	0	0	1	-1	$\boxed{1}$	-1	1	-1	0
x1	0	1	0	0	1	-4	0	-1	4	12
z	1	0	0	-1	-1	0	1-M	1-M	2-M	102
x2	0	0	1	-3	2	0	3	-2	0	6
s3	0	0	0	1	-1	1	-1	1	-1	0
x1	0	1	0	4	-3	0	-4	3	0	12

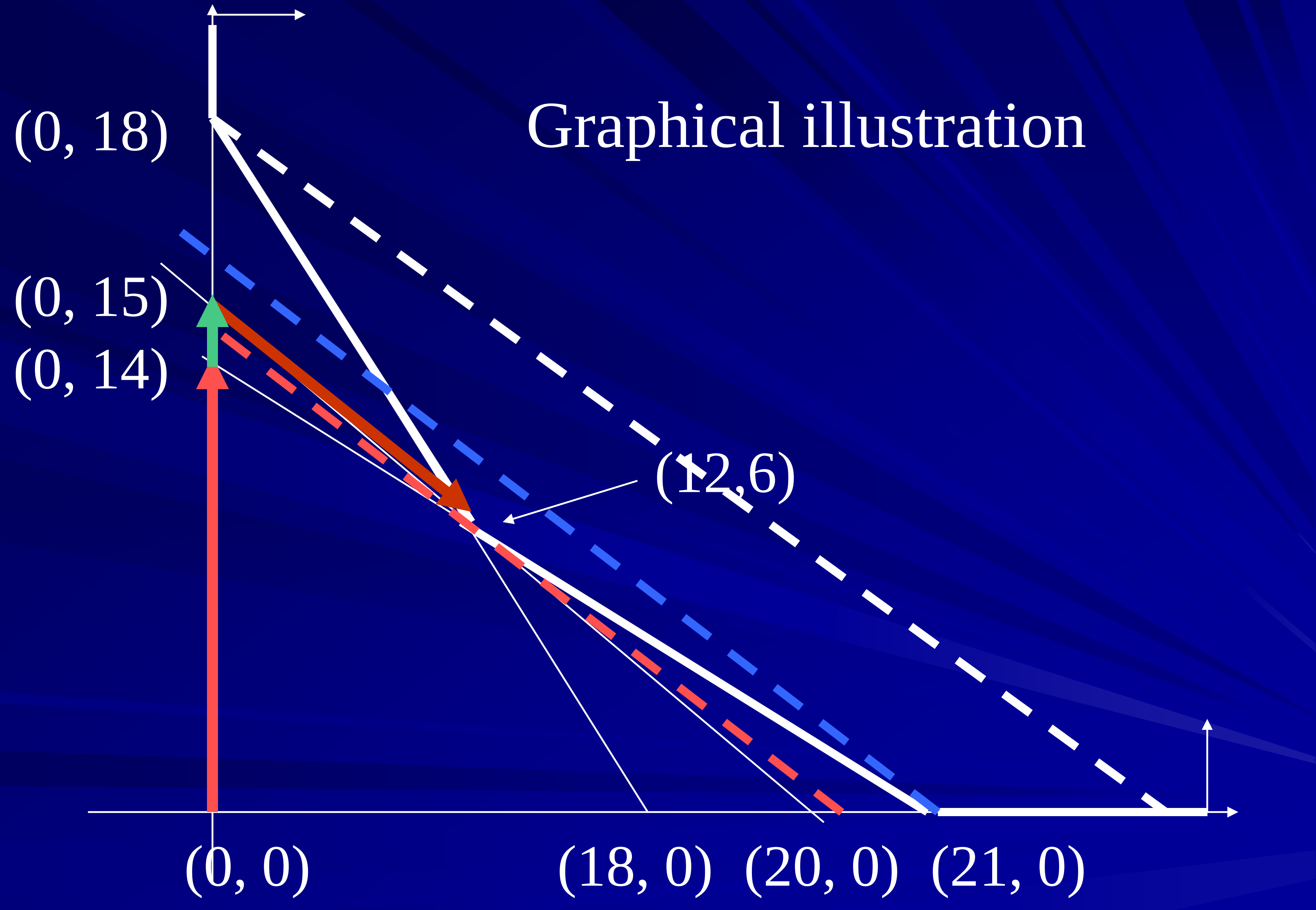
Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
z	1	0	0	-1	-1	0	1-M	1-M	2-M	102
x2	0	0	1	-3	2	0	3	-2	0	6
s3	0	0	0	1	-1	1	-1	1	-1	0
x1	0	1	0	4	-3	0	-4	3	0	12

This is the optimal tableau.

Optimal solution: $x_1=12$, $x_2=6$

Minimum $z = 102$

Graphical illustration



TWO PHASE SIMPLEX METHOD

The Big M method involves manipulation with small and large numbers and so is not suited to a computer. We now look at the Two-Phase method. As the name suggests, the method consists of two phases: In phase-I we minimise the sum of all the artificial variables subject to the same constraint equations. If the original problem had a feasible solution this new problem will give a solution with all artificial variables zero. Taking this as a starting BFS, we solve the original problem. We illustrate by an example.

Consider the LPP:

Minimize $z = 2x_1 + x_2$

Subject to the constraints

$$3x_1 + x_2 \geq 9$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

Introducing the surplus and artificial variables, R_1, R_2, the LPP is same as:

Minimize $\boxed{z = 2x_1 + x_2}$

Subject to the constraints

$$\boxed{\begin{aligned} 3x_1 + x_2 - s_1 \quad + R_1 \quad &= 9 \\ x_1 + x_2 \quad - s_2 \quad + R_2 &= 6 \\ x_1, x_2, s_1, s_2, R_1, R_2 &\geq 0 \end{aligned}}$$

Phase I:

Minimize $\boxed{r = R_1 + R_2}$

Subject to the constraints

$$\boxed{\begin{array}{l} 3x_1 + x_2 - s_1 \quad + R_1 \qquad = 9 \\ x_1 + x_2 \qquad - s_2 \quad + R_2 = 6 \\ x_1, x_2, s_1, s_2, R_1, R_2 \geq 0 \end{array}}$$

We now solve it by Simplex method.

Basic	r	x1	x2	s1	s2	R1	R2	Sol.
r	1	4 ~~0~~	2 ~~0~~	-1 ~~0~~	-1 ~~0~~	0 ~~-1~~	0 ~~-1~~	15 ~~0~~
R1	0	$\boxed{3}$	1	-1	0	1	0	9
R2	0	1	1	0	-1	0	1	6
r	1	0	2/3	1/3	-1	-4/3	0	3
x1	0	1	1/3	-1/3	0	1/3	0	3
R2	0	0	$\boxed{2/3}$	1/3	-1	-1/3	1	3
r	1	0	0	0	0	-1	-1	0
x1	0	1	0	-1/2	1/2	1/2	-1/2	3/2
x2	0	0	1	1/2	-3/2	-1/2	3/2	9/2

Note that Phase I has ended as min r =0.

Phase II:

Now we solve the original LPP with

the starting BFS: $x_1 = \frac{3}{2},\ x_2 = \frac{9}{2}$

Note that the starting Simplex tableau is same as the *last* tableau except for the first row which is our z-Row. Since R_1, R_2 have served their purpose (of giving a starting BFS), we **suppress their columns**.

Basic	z	x1	x2	s1	s2	R1	R2	Sol.
z	1	0 ~~-2~~	0 ~~-1~~	-1/2 ~~0~~	-1/2 ~~0~~			15/2 ~~0~~
x1	0	1	0	-1/2	1/2			3/2
x2	0	0	1	1/2	-3/2			9/2

This is the optimal Tableau.

Optimal solution: $x_1 = 3/2$, $x_2 = 9/2$

Min $z = 15/2$