Lecture 6

Recall, A subset $E \subseteq \mathbb{R}$ is <u>measurable</u>

if for any $A \subseteq \mathbb{R}$,

$$\boxed{m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)}$$

$\leq$ ~~en~~ true always.

<u>Qn</u>:- What are all the measurable subsets of $\mathbb{R}$?

<u>Def</u>:- Let $\mathcal{Y}$ be a class of subsets of a metric space (X, d). Then $\mathcal{Y}$ is said to be a <u>σ-algebra</u> or a <u>σ-field</u>, if it satisfies the following conditions:

(i) $X \in \mathcal{Y}$

(ii) if $A \in \mathcal{Y}$, then $A^c \in \mathcal{Y}$.

& (iii) If $E_i \in \mathcal{Y}\ \forall i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{Y}$.

(i.e., $\mathcal{Y}$ is closed under countable union).

Let $\mathcal{M}$ = the class of all Lebesgue measurable subsets of $\mathbb{R}$.

<u>Theorem</u>:- $\mathcal{M}$ is a σ-algebra.

<u>proof</u>:-

(ii) follows directly from the def.

ii) $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.

For $A \subseteq \mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$
$$= m^*(A \cap (E^c)^c) + m^*(A \cap E^c)$$
$$= m^*(A \cap E^c) + m^*(A \cap (E^c)^c)$$

$\Rightarrow$ E^c is measurable. $\Rightarrow E^c \in \mathcal{M}$.

i) we know $m^*(\phi) = 0$. Then $\phi \in \mathcal{M}$.

Then by (ii) $\underset{\mathbb{R}}{\underset{\shortparallel}{\phi^c}} \in \mathcal{M}$

$\therefore \mathbb{R} \in \mathcal{M}$.

Remains to prove (iii).

Let $\{E_j\}_{j \in \mathbb{N}}$ be a collection of subsets of $\mathbb{R}$ such that $E_j \in \mathcal{M}$ $\forall j \in \mathbb{N}$.

To show: $\bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$.

i.e. $\bigcup_{j=1}^{\infty} E_j$ is measurable.

Let $A \subseteq \mathbb{R}$.

$E_1 \in \mathcal{M}$ i.e. E_1 is measurable

$$\Rightarrow m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c). \longrightarrow (S)$$

Now replace E_1 by E_2 & A by $A \cap E_1^c$,

Then we get

$$\boxed{m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)}$$

($\because E_2$ is measurable).

Now substitute this in (8).

Then we get

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + \underline{m^*(A \cap E_1^c \cap E_2^c)}.$$

Now repeat the above process then we get (after n steps)

$$m^*(A) = m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\Big(A \cap E_i \cap \big(\bigcap_{2 \le j < i} E_j^c\big)\Big)$$

$$+ m^*\Big(A \cap \big(\bigcap_{j=1}^{n} E_j^c\big)\Big)$$

$$\ge m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\Big(A \cap E_i \cap \big(\big(\bigcup_{2 \le j < i} E_j\big)^c\big)\Big)$$

$$+ m^*\Big(A \cap \underline{\big(\big(\bigcup_{j=1}^{\infty} E_j\big)^c\big)}\Big)$$

True for all $n \ge 2$

$$\Rightarrow m^*(A) \ge m^*(A \cap E_1) + \sum_{i=2}^{\infty} m^*\Big(A \cap E_i \cap \big(\bigcup_{j<i} E_j\big)^c\Big)$$

$$+ m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right)$$

$$\Rightarrow m^*(A) \geq m^*(A\cap E_1) + m^*\left(\underline{\bigcup_{i=2}^{\infty} \left(A\cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right)}\right)$$

$$+ m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right)$$

$$\left(\because m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} m^*(A_j)\right)$$

Now, $\boxed{\bigcup_{i=2}^{n} \left(A\cap E_i \cap \left(\bigcup_{2\leq j<i} E_j\right)^c\right) = \bigcup_{i=2}^{n} (E_i \cap A)}$

$\underline{n=3:}$ $\bigcup_{i=2}^{3} (A\cap E_i) \cap \left(\bigcup_{j<i} E_j\right)^c\Big) = \left(A\cap E_2 \cap (\phi)^c\right) \cup \left(A \cap E_3 \cap (E_2^c)\right)$

$$= (A\cap E_2 \cap \mathbb{R}) \cup (A\cap E_3 \cap E_2^c)$$

$$= (A\cap \underline{E_2}) \cup (A\cap E_3 \cap \underline{E_2}^c)$$

$$= (A\cap E_2) \cup (A\cap E_3)$$

$$\Rightarrow \bigcup_{j=2}^{\infty} \left(A\cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) = \bigcup_{i=2}^{\infty} (A\cap E_i).$$

$$\therefore m^*(A) \geq m^*(\underline{A\cap E_1}) + m^*\left(\underline{\bigcup_{i=2}^{\infty} (A\cap E_i)}\right) + m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right).$$

$$\geq m^*\left(\bigcup_{i=1}^{\infty}(A\cap E_i)\right) + m^*\left(A\cap\left(\bigcup_{j=1}^{\infty}E_j\right)^c\right)$$

$$\left(\because\ m^*(S\cup T) \leq m^*(S)+m^*(T)\right)$$

$$\Rightarrow m^*(A) \geq m^*\left(A\cap\left(\bigcup_{i=1}^{\infty}E_i\right)\right) + m^*\left(A\cap\left(\bigcup_{i=1}^{\infty}E_i\right)^c\right)$$

$\Rightarrow \bigcup_{i=1}^{\infty}E_i$ is measurable

$\Rightarrow \bigcup_{i=1}^{\infty}E_i \in \mathcal{M}$.

$\therefore$ $\mathcal{M}$ is a σ-algebra.