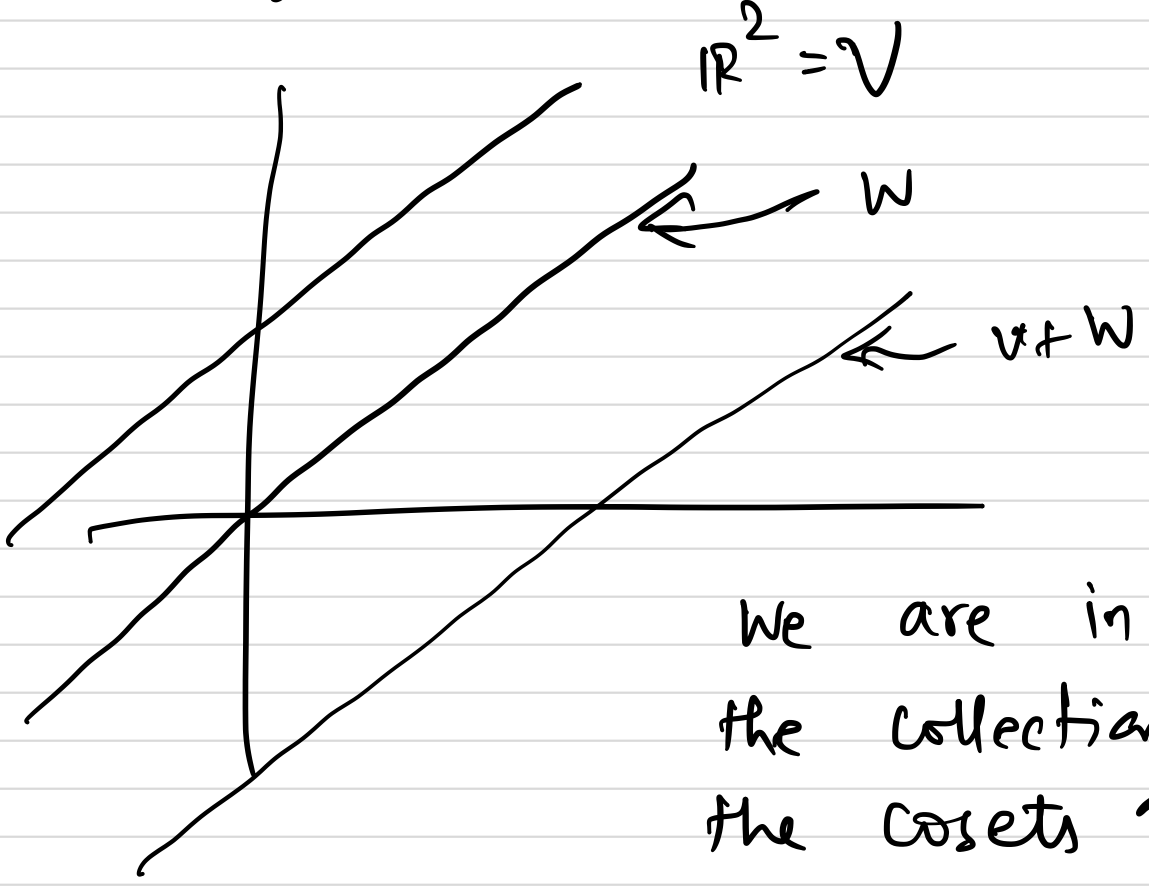
Linear Algebra

Lecture - 11

Quotient Spaces:

Definition: Let W be a vector subspace of a vector space V over a field $\mathbb{F}$. Then the set $v+W = \{v+w \mid w \in W\}$ for some $v \in V$ is called a co-set of W in V.



We are interested in the collection of all the cosets of W in V.

Question: Does this collection have any interesting algebraic structure??

Theorem: Let W be a vector subspace of a vector space V over $\mathbb{F}$. Let $v_1 + W$ and $v_2 + W$ be two cosets of W in V. Then exactly one of the following statements is true.

(1) $(v_1 + W) \cap (v_2 + W) = \phi$

(2) $v_1 + W = v_2 + W$

Moreover, $v_1 + W = v_2 + W \iff v_1 - v_2 \in W$

Proof:

Let $v_1 + W$ and $v_2 + W$ have some non-empty intersection.

Let $v \in V$ be such that

$$v \in (v_1 + W) \cap (v_2 + W)$$

$$\Rightarrow \quad v = v_1 + w_1 \quad \text{for some } w_1 \in W$$

$$v = v_2 + w_2 \quad \text{for some } w_2 \in W$$

$$\Rightarrow v_1 - v_2 = w_1 - w_2 \in W$$

Let $u = v_1 - v_2 \in W$ ☆

Let $x \in v_1 + W$

Then $\exists$ $w \in W$ such that

$$x = v_1 + w$$

$$= v_1 - v_2 + v_2 + w$$

$$= v_2 + u + w \quad \in W$$

$$\in v_2 + W$$

$\Rightarrow$ $v_1 + W \subseteq v_2 + W$

Let $y \in v_2 + W$

$\exists$ some $w \in W$ s.t.

$$y = v_2 + w$$

$$= v_2 + v_1 - v_1 + w$$

$$= v_1 + (v_2 - v_1) + w \quad \in W$$

$$\in v_1 + W$$

$\Rightarrow$ $v_2 + W \subseteq v_1 + W$

$\blacksquare$

By V/W we mean the collection of all cosets of W in V.

$$V/W = \{ v + W \mid v \in V \}$$

<u>Definition</u>: Let $\xi \in V/W$ be a coset. If $\xi = v + W$, for some $v \in V$, then v is called a representative of ξ in V/W.

Notice that $\xi = v + W$ and $\xi = v_1 + W$ then from the theorem,

$$v - v_1 \in W.$$

Thus any two representatives of a coset of W in V differ by an element of W.

In particular, $x + W = W$ if and only if $x \in W$.

Addition of two cosets.

Let $\xi_1, \xi_2 \in V/W$.

$$\xi_1 = u_1 + W$$

$$\xi_2 = u_2 + W$$

where u_1 & u_2 are representatives of ξ_1 & ξ_2 respectively.

Then define $\xi_1 + \xi_2 = (u_1 + W) + (u_2 + W)$

$$= (u_1 + u_2) + W$$

Question: Is this addition well-defined??

Since we know that ξ_1 may have more than one representative in V/W, we consider,

$$\xi_1 = x_1 + W$$

$$\xi_2 = x_2 + W$$

then $\xi_1 + \xi_2 = (x_1 + x_2) + W$

Then is $(x_1+x_2)+W = (v_1+v_2)+W$??

iff $(x_1+x_2)-(v_1+v_2) \in W$

iff $\underbrace{(x_1-v_1)}_{\in W} + \underbrace{(x_2-v_2)}_{\in W} \in W$ is true.

because $\xi_1 = x_1+W = v_1+W$

$\xi_2 = x_2+W = v_2+W$

This proves that addition of two cosets is well-defined.

Similarly for $\alpha \in \mathbb{F}$, define scalar multiplication on V/W as

for $\xi_1 \in V/W$

$$\alpha\xi_1 = \alpha(x_1+W) = \alpha x_1 + W$$

Note that for another representative v_1 of ξ_1, $\alpha\xi_1 = \alpha v_1 + W$

$\alpha v_1 + W = \alpha x_1 + W \Leftrightarrow \alpha(v_1 - x_1) \in W$

Thus scalar multiplication in V/W is also well-defined.

Theorem: Let V be a vector space over $\mathbb{F}$ and let W be a subspace of V. Then V/W with addition & scalar multiplication as defined above is a vector space over $\mathbb{F}$.

Proof:

For $\xi_1 = v_1 + W$, $\xi_2 = v_2 + W$ & $\xi_3 = v_3 + W \in V/W$

$$\xi_1 + (\xi_2 + \xi_3) = (\xi_1 + \xi_2) + \xi_3$$

$$\|$$

$$v_1 + W + (v_2 + v_3) + W$$

$$= v_1 + (v_2 + v_3) + W$$

$$= (v_1 + v_2) + v_3 + W$$

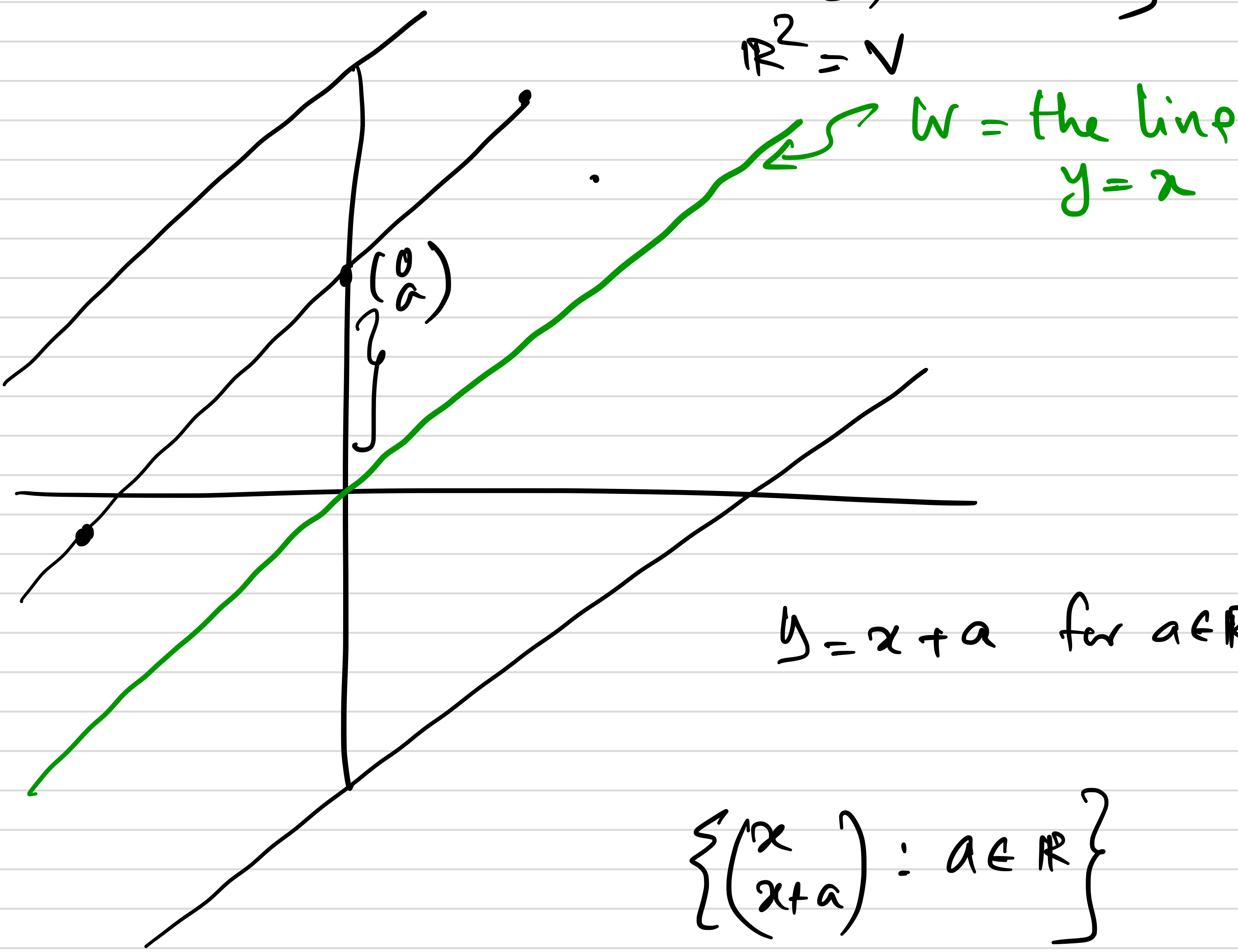
$$= (\xi_1 + \xi_2) + \xi_3$$

Note: zero vector in V/W is W.

Further, additive inverse of $\xi_1 = v_1 + W$ in V/W is $-\xi_1 = -v_1 + W$.

The remaining axioms can be very easily proved. ∎

<u>Ex:</u> $V = \mathbb{R}^2$, $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x = y \right\}$



$y = x + a$ for $a \in \mathbb{R}$.

$$\left\{ \begin{pmatrix} x \\ x+a \end{pmatrix} : a \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 0 \\ a \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} : \begin{matrix} x = y \\ a \in \mathbb{R} \end{matrix} \right\}$$

$$= \begin{pmatrix} 0 \\ a \end{pmatrix} + W$$

Let $\begin{pmatrix} p \\ q \end{pmatrix}$ be another representative of $\begin{pmatrix} 0 \\ a \end{pmatrix} + W$.

Then $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix} \in W$

$$\Leftrightarrow \quad p = q - a$$

V/W = collection of all lines parallel to W (the line $y = x$ in this case)

$$= \left\{ \begin{pmatrix} 0 \\ a \end{pmatrix} + W \;\middle|\; a \in \mathbb{R} \right\}$$

<u>Theorem</u>: Let V be a finite dimensional vector space over $\mathbb{F}$ and W be a subspace of V.

Then $\dim(V/W) = \dim(V) - \dim(W)$

<u>Proof:</u>

Let $\{w_1, w_2, \dots, w_p\}$ be a basis for W. Then extend this basis as $\{w_1, \dots, w_p, v_1, \dots, v_q\}$ as a basis for V.

Now we will prove that $\{v_1 + W, v_2 + W, \dots, v_q + W\}$ is a basis for V/W.

Let $x + W \in V/W$

Since x is a representative,

$x \in V$, then $x = \sum_{i=1}^{p} \alpha_i w_i + \sum_{j=1}^{q} \beta_j v_j$

$$x + W = \sum_{i=1}^{p} \alpha_i w_i + \sum_{j=1}^{q} \beta_j v_j + W$$

$$= \sum_{j=1}^{q} \beta_j v_j + W$$

$$= \sum_{j=1}^{q} \beta_j (v_j + W)$$

$$V/W = \text{span}\{v_1+W, v_2+W, \ldots, v_q+W\}$$

Now we will prove that

$\{v_1+W, v_2+W, \ldots, v_q+W\}$ is

linearly independent.

For scalars $\alpha_1, \ldots, \alpha_q \in \mathbb{F}$ let

$$\alpha_1(v_1+W) + \alpha_2(v_2+W) + \cdots + \alpha_q(v_q+W) = W$$

this is the zero vector in V/W.

$$\iff (\alpha_1 v_1 + \cdots + \alpha_q v_q) + W = W$$

$$\iff \alpha_1 v_1 + \cdots + \alpha_q v_q \in W$$

$\iff \exists$. scalars $\beta_1, \beta_2, \ldots, \beta_p$ such that

$$\alpha_1 v_1 + \cdots + \alpha_q v_q = \beta_1 w_1 + \cdots + \beta_p w_p$$

$$\iff \alpha_1 v_1 + \cdots + \alpha_q v_q - \beta_1 w_1 - \cdots - \beta_p w_p = 0$$

is the zero vector in V.

$\Leftrightarrow \alpha_1 = \cdots = \alpha_q = \beta_1 = \cdots = \beta_p = 0$

the zero in the field of scalars $\mathbb{F}$.

$\Rightarrow \{v_1 + W, v_2 + W, \ldots, v_q + W\}$ is linearly independent.

$$\Rightarrow \dim(V/W) = q = (p+q) - p$$
$$= \dim(V) - \dim(W)$$

$\boxtimes$

Remember: This theorem (specially the proof) is extremely similar to the proof of rank-nullity theorem.

We will come back to this when we study isomorphisms.

If $T: V \to V$ is a linear transformation,

$$\dim(V/N(T)) = \dim(R(T))$$

Coming back to linear transformations.

[Matrix representations & linear transformations.]

<u>Definition</u>: Let V be a finite-dimensional vector space. An ordered basis for V is a basis for with specific order.

For example in $\mathbb{F}^3$

$\{e_1, e_2, e_3\}$ is a basis and so is

$\{e_2, e_1, e_3\}$.

But $\{e_1, e_2, e_3\} \neq \{e_2, e_1, e_3\}$ as ordered bases.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Definition: Let $\{u_1, u_2, \ldots, u_n\}$ be an ordered basis of V. For $x \in V$, there exist scalars $a_1, a_2, \ldots, a_n \in \mathbb{F}$ such that $x = \sum_{i=1}^{n} a_i u_i$.

Then we define coordinate vector of x relative to $\{u_1, \ldots, u_n\}$ as

$$x = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Ex: Let $V = P_3(\mathbb{R})$. Let $\{1, x, x^2, x^3\}$ be an ordered basis for V. Then any $p(x) \in P_3(\mathbb{R})$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$p(x) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad p(x) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix}$$

$$p(x) = 2x^3 + x^2 - x$$

Matrix representation of a linear transformation.

Let V and W be finite dimensional vector spaces over $\mathbb{F}$. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be ordered bases for V & W respectively. Let $T : V \to W$ be a linear map.

We just need to understand the action of T on a basis for V.

There exists scalars $a_{ij} \in \mathbb{F}$

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \qquad \text{for } 1 \leq j \leq n.$$

<u>Definition</u>: Using the notation stated above, we call the matrix $A = [a_{ij}]$ with size $m \times n$, the matrix representation of T in the ordered bases of V & W.

Example:

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation defined by

$$T(a_1, a_2) = (a_1 + 3a_2,\ 0,\ a_1 - 4a_2).$$

The ordered basis for $\mathbb{R}^2$ is $\{e_1, e_2\}$ and ordered basis for $\mathbb{R}^3$ is $\{e_1, e_2, e_3\}$.

$$T(e_1) = T(1, 0) = (1, 0, 1)$$

$$= \underset{a_{11}}{1} \cdot \overset{w_1}{e_1} + \underset{a_{21}}{0} \cdot \overset{w_2}{e_2} + \underset{a_{31}}{1} \cdot \overset{w_3}{e_3}$$

$$T(e_2) = T(0, 1) = (3, 0, -4)$$

$$= \underset{a_{12}}{3} \cdot \overset{w_1}{e_1} + \underset{a_{22}}{0} \cdot \overset{w_2}{e_2} \underset{a_{32}}{-4}\, \overset{w_3}{e_3}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 1 & -4 \end{bmatrix}$$

is the matrix representation of T.

Example:

Let $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ with ordered basis $\{1, x, x^2, x^3\}$

and $T(p(x)) = \frac{d}{dx} p(x)$.

$$T(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$$

$$T(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$$

$$T(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$$

$$T(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is the matrix representation of T with the ordered basis $\{1, x, x^2, x^3\}$.

Example:

$$T : \mathbb{P}_2(\mathbb{R}) \longrightarrow \mathbb{P}_3(\mathbb{R})$$

$$T(p(x)) = 2p'(x) + \int_0^x 3p(t)\,dt$$

with $\{1, x, x^2\}$ as ordered basis for $\mathbb{P}_2(\mathbb{R})$ and $\{1, x, x^2, x^3\}$ as ordered basis for $\mathbb{P}_3(\mathbb{R})$. Obtain the matrix representation of T.

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T : \underset{n}{V} \longrightarrow \underset{m}{W}$$

$$A = \begin{bmatrix} \quad \end{bmatrix}_{m \times n}$$