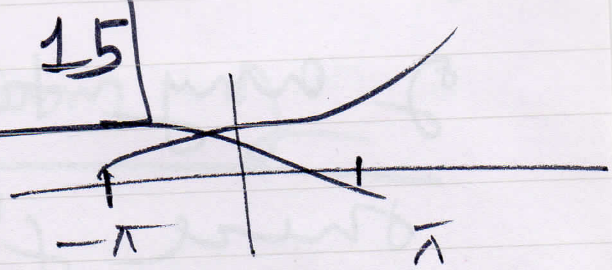
Date 05/09/2017

Lecture 15



$f: [-\pi, \pi] \to \mathbb{R}$

Orthogonality of the Trigonometric system

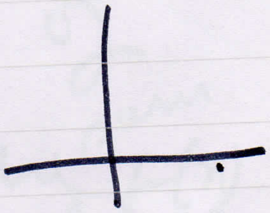
The trigonometric system

$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots$

is orthogonal on the interval

$-\pi \leq x \leq \pi$

(hence on any interval of length 2π, because of periodicity).



By defn, this means that the integral of the product of any two different of these f^ns over that interval is zero,

i.e., for any integers $m, n, n \neq m$, we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad (m \neq n)$$

$$\& \int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = 0, \quad m \neq n.$$

$f \cdot g$

$\langle f, g \rangle = \int_a^b f \bar{g} \, dx$

& for any integers $m \& n$ (including $m = n$), we have

$$\int_{-\pi}^{\pi} \cos mx \cdot \sin nx \, dx = 0$$

which is the most important property of the trigonometric system.

Convergence & Sum of Fourier Series

$f(x) \rightarrow$ periodic f^n of period 2π

(7) & (8).

$f(x)$ is cont. or piece-wise cont.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

If the Fourier series of $f(x)$ does not have the sum $f(x)$

or does not converge, then we still write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with a tilde '~', which indicates that the trigonometric series on the R·H·S has the Fourier coefficient of $f(x)$ as its co-efficient, so it is the Fourier series of $f(x)$

The class of fns that can be represented by F·S is surprisingly large & general.

Th-1 **Sufficient condn** for convergence of a F·S.

(Dirichlet's Theorem)

(Representation by a Fourier Series)

If a periodic function $f(x)$ with period 2π is piece-wise continuous in the interval $-\pi \leq x \leq \pi$ & has a left-hand derivative & right-hand derivative at each point of that interval, then the Fourier series (8) of $f(x)$ [with Fourier co-efficients (7)] is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous & the sum of the series is the average of the left-hand & right hand limits of $f(x)$ at x_0.

proof:- We prove the convergence for a continuous f^n $f(x)$ having continuous first & 2nd derivatives.

Integrating the eq^n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx, \quad n = 1, 2, \dots$$

by parts, we get

$$a_n = \left[\frac{f(x)\sin nx}{n\pi}\right]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx \, dx$$

$$\Rightarrow a_n = 0 - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx) \, dx.$$

Another integration by parts gives

$$a_n = \left[\frac{f'(x)\cos nx}{n^2\pi}\right]_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx\,dx$$

$= 0$, because $f'(x)$ is periodic & cont.

$$\left[\cancel{\frac{f'(\pi)\cos n\pi}{n^2\pi}} - \cancel{\frac{f'(-\pi)\cos(-n\pi)}{n^2\pi}}\right]$$

$$- \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx\,dx.$$

$$\left[\begin{aligned} f'(-\pi) &= f'(-\pi+2\pi) \\ &= f'(\pi). \end{aligned}\right]$$

Since $f''(x)$ is continuous in the interval of integration $-\pi \le x \le \pi$, it is bounded there in.

EX

$$|f''(x)| < M$$

for an appropriate constant M.

Furthermore, $|\cos nx| \leq 1$.

It follows that

$$|a_n| = \left| -\frac{1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \cos nx \, dx \right|$$

$$< \frac{1}{n^2\pi} \int_{-\pi}^{\pi} |f''(x)| \underbrace{|\cos nx|}_{\leq 1} dx$$

$$< \frac{1}{n^2\pi} \int_{-\pi}^{\pi} M \, dx$$

$$= \frac{2M}{n^2}$$

$$\Rightarrow |a_n| < \frac{2M}{n^2}, \forall n \in \mathbb{N}$$

Slly, $|b_n| < \frac{2M}{n^2}$, $\forall n \in \mathbb{N}$

Hence, the absolute value of each term of the F.S (Fourier series) of $f(x)$ is atmost equal to the the corresponding term of the series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$|f(x)| = \left| a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right|$$

$$\leq |a_0| + \sum_{n=1}^{\infty} \left(|a_n \cos nx| + |b_n \sin nx| \right)$$

$$\leq |a_0| + \sum_{n=1}^{\infty} \left(|a_n| + |b_n| \right)$$

$$|f(x)| < |a_0| + 2M\left(1+1+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\cdots\infty\right)$$

$\Rightarrow |f(x)|$ is convergent

$\Rightarrow f(x)$ is convergent (how?).

Infinite series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

p-test

$(p>1)$

(EX

Hence, the Fourier series (F-S) converges.

(EX Show the proof for piece-wise cont. f^n $f(x)$.

See EX)

$$f(x) = \begin{cases} -k, & \text{if } -\pi < x < 0 \\ k, & \text{if } 0 < x < \pi \end{cases}$$

It has a discontinuity at $x=0$.

L.H.L $= -k$ $= \lim_{x \to 0^-} f(x)$ (how? EX)

& R.H.L $= k$. $= \lim_{x \to 0^+} f(x)$

Hence, the average of these two limits is

$$\frac{(-k)+k}{2} = 0.$$

as F.S is

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$$

does indeed converge to this value when

$x = 0$ because then all its terms are 0.

Slly, for the other jumps

$(x = \pm \pi)$.

Find out
(EX).

— X —

$x = 0$ because then all its terms are 0.

Slly, for the other jumps

$(x = \pm\pi)$.

Find out (EX).

— x —