Lecture 8

Def:- The σ-algebra generated by the class of all the intervals of the form $[a,b)$, $a,b \in \mathbb{R}$ is called the <u>Borel σ-algebra</u>, & denote by $\mathcal{B}$.

The members of $\mathcal{B}$ are called <u>Borel sets</u> of $\mathbb{R}$.

Theorem:- ① Every Borel set is measurable

i.e, $\mathcal{B} \subseteq \mathcal{M}$.

② $\mathcal{B}$ is the σ-algebra generated by each of the following classes:

the open intervals, the open sets, the G_δ-sets, the F_σ-sets.

Proof:- (1) Since $[a,b) \in \mathcal{M}$ for any $a,b \in \mathbb{R}$,

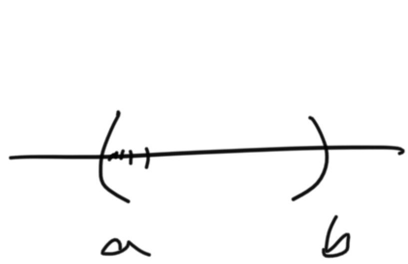
Therefore $\mathcal{B} \subseteq \mathcal{M}$.

$\boxed{[a,b)^c \in \mathcal{B} \quad \{[a,b) / \ldots\}}$

② Let $\mathcal{B}_1$ = the σ-algebra generated by the open intervals.

To show: $\mathcal{B}_1 = \mathcal{B}$.

Any open intvl, $(a,b) = \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, b)$

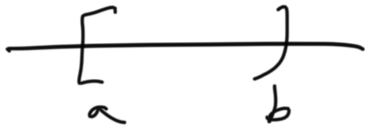


$\in \mathcal{B}$

Thus any open interval belongs to $\mathcal{B}$.

$\implies \mathcal{B}_1 \subseteq \mathcal{B}$.

Now $[a,b) = \underbrace{\bigcap_{n=1}^{\infty} \underbrace{(a-\tfrac{1}{n}, b)}_{\in \mathcal{B}_1}}_{\in \mathcal{B}_1}$



Thus $[a,b) \in \mathcal{B}_1 \ \forall a,b \in \mathbb{R}$

$\implies \mathcal{B} \subseteq \mathcal{B}_1$.

Thus $\mathcal{B} = \mathcal{B}_1$.

Remaining : EXERCISE. $\mathcal{B}_2$ = the σ-algebra gen by open sets.

Prove $\mathcal{B} = \mathcal{B}_2$.

⋮

Qn:- Does $\mathcal{B} = \mathcal{M}$?

Ans: NO! $(\mathcal{B} \subsetneq \mathcal{M})$

$\mathbb{Q} = \bigcup_{x \in \mathbb{Q}} [x,x] \in \mathcal{B}$.

Proposition:- Let $A \subseteq \mathbb{R}$. Then there exists a measurable set E such that $E \supseteq A$ &

$m^*(A) = m^*(E)$.

<u>proof:-</u> We already proved: Given $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}$ such that $A \subseteq U$ & $m^*(U) \leq m^*(A) + \varepsilon$.

Take $\varepsilon = \frac{1}{n}$. Then there exists open set U_n such that $A \subseteq U_n$ & $m^*(U_n) \leq m^*(A) + \frac{1}{n}$. $\forall n$.

Let $E = \bigcap_{n=1}^{\infty} U_n$. E is a G_δ-set.

$\therefore E$ is measurable. (in fact E is a Borel set)

& $m^*(E) \leq m^*(U_n)$

$\leq m^*(A) + \frac{1}{n} \quad \forall n.$

$\Rightarrow m^*(E) \leq m^*(A)$

Also $A \subseteq \bigcap_{n=1}^{\infty} U_n = E$

$\Rightarrow m^*(A) \leq m^*(E)$.

$\therefore m^*(A) = m^*(E)$.

Def:- For any sequence of sets $\{E_i\}$,

$$\limsup (E_i) := \bigcap_{n=1}^{\infty} \left(\bigcup_{i \geq n} E_i \right)$$

$$\bigcup_{i\geq 1} E_i \supseteq \bigcup_{i\geq 2} E_i \supseteq \cdots$$

$$\liminf (E_i) := \bigcup_{n=1}^{\infty} \left(\bigcap_{i \geq n} E_i \right)$$

$$\bigcap_{i\geq 1} E_i \subseteq \bigcap_{i\geq 2} E_i \subseteq \cdots$$

Remark:-

① $\limsup (E_i)$ = the set of points belonging to infinitely many of the sets E_i.

$\liminf (E_i)$ = the set of points belonging to all but finitely many of the sets E_i.

② $\liminf (E_i) \subseteq \limsup (E_i)$.

If they are equal, then we denote this set as $\lim (E_i)$.

Example:- ① Suppose $E_1 \subseteq E_2 \subseteq \cdots\cdots$

Then $\limsup (E_i) = \bigcup_{i=1}^{\infty} E_i$

$\liminf (E_i) = \bigcup_{i=1}^{\infty} E_i$

$\therefore \quad \lim(E_i) = \bigcup_{i=1}^{\infty} E_i.$

② Suppose $E_1 \supseteq E_2 \supseteq \cdots\cdots$ Then

$$\limsup(E_i) = \bigcap_{i=1}^{\infty} E_i = \liminf(E_i) \quad \text{(EXERCISE)}$$

$$\therefore \lim(E_i) = \bigcap_{i=1}^{\infty} E_i$$

<u>Theorem:</u>— Let $\{E_i\}$ be a sequence of measurable sets in $\mathbb{R}$. Then

(i) if $E_1 \subseteq E_2 \subseteq \cdots$, then $m(\lim E_i) = \lim(m(E_i))$

(ii) if $E_1 \supseteq E_2 \supseteq \cdots$, & $m(E_i) < \infty$ for all i, then

$$m(\lim E_i) = \lim m(E_i).$$

<u>proof:</u>— (i) Let $F_1 = E_1$ $\qquad \lim(E_i) = \bigcup_{i=1}^{\infty} E_i$

$F_i = E_i \setminus E_{i-1} \quad \forall i \geq 2.$

Then $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$ & F_i's are measurable

$(F_i = E_i \cap E_{i-1}^c)$.

$$\therefore m^*(\lim E_i) = m\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$= m\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \sum_{i=1}^{\infty} m(F_i).$$

$$= \lim_n \left(\sum_{i=1}^{n} m(F_i)\right)$$

$$= \lim_n \left(m\left(\bigcup_{i=1}^{n} F_i\right)\right)$$

$$= \lim_n \left(m\left(\bigcup_{i=1}^{n} E_i\right)\right)$$

$$= \lim_n \left(m(E_n)\right) \qquad E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n$$

(ii) Given $E_1 \supseteq E_2 \supseteq \cdots$

$$\Rightarrow \underbrace{E_1 \setminus E_2}_{= A_1} \subseteq \underbrace{E_1 \setminus E_3}_{= A_2} \subseteq \cdots \qquad (A_1 \subseteq A_2 \subseteq \cdots)$$

$\therefore$ By (i) $m\left(\lim(\underbrace{E_1 \setminus E_i}_{= A_i})\right) = \lim m(\underbrace{E_1 \setminus E_i}_{= A_i})$

$$= \lim\left(m(E_1) - m(E_i)\right)$$

$$= m(E_1) - \lim m(E_i)$$

Now $\lim(E_1 \setminus E_i) = \bigcup_{i=1}^{\infty}(E_1 \setminus E_i)$

$$= E_1 \setminus \bigcap_{i=1}^{\infty} E_i$$

$$= E_1 \setminus \lim(E_i)$$

Thus $m\left(E_1 \setminus \lim(E_j)\right) = m(E_1) - \lim m(E_j)$

$\Rightarrow m(E_1) - m\left(\lim(E_j)\right) =$ "

$\Rightarrow \quad m\left(\lim(E_j)\right) = \lim\left(m(E_j)\right).$
