

LIMIT (Cont.)

REMARK: Since $(x, y) \rightarrow (x_0, y_0)$ in the two dimensional plane, there are infinite number of paths joining (x, y) to (x_0, y_0) . Since the limit, if exists, is unique, the limit should be the same along all the paths. Thus, the limit cannot be obtained by approaching the point P along a particular path and finding the limit of $f(x, y)$. If the limit is dependent on a path, then the limit does not exist.

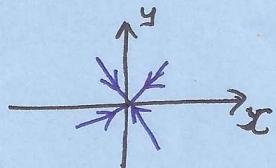
EXAMPLE :

- $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=mx}} \frac{x^2y}{x^4+y^2}$

- $\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=mx^2}} \frac{x^2y}{x^4+y^2}$

Along $y = mx$:

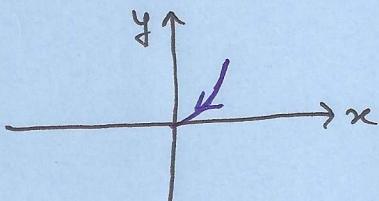
$$\lim_{x \rightarrow 0} \frac{x^2 \cdot mx}{x^4 + m^2x^2} = \lim_{x \rightarrow 0} \left(\frac{mx}{x^2 + m^2} \right) = 0$$



Along $y = mx^2$:

$$\lim_{x \rightarrow 0} \frac{x^2 \cdot mx^2}{x^4 + m^2x^4} = \frac{m}{1+m^2}$$

In particular, $y = x^2$, we get $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4+y^2} = \frac{1}{2}$



Hence limit does not exist in this case.

Ex. show that the limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$$

does not exist.

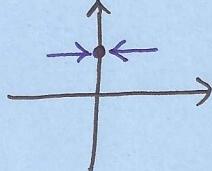
Sol. Consider $y = mx$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

The limit depends on path. Hence the limit does not exist.

Ex. Show that the limit $\lim_{(x,y) \rightarrow (0,1)} \tan^{-1}\left(\frac{y}{x}\right)$ does not exist.

Sol. Let us fix $y=1$ and calculate



$$\lim_{x \rightarrow 0^-} \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$$

} \Rightarrow limit does not exist.

$$\lim_{x \rightarrow 0^+} \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(+\infty) = \frac{\pi}{2}$$

Working with limits: Suppose, we have

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y) = L_1 \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0, y_0)} g(x,y) = L_2$$

then

i) $\lim_{(x,y) \rightarrow (x_0, y_0)} [K f(x,y)] = K L_1$ for any real constant K .

ii) $\lim_{(x,y) \rightarrow (x_0, y_0)} [f(x,y) \pm g(x,y)] = L_1 \pm L_2$

iii) $\lim_{(x,y) \rightarrow (x_0, y_0)} [f(x,y) g(x,y)] = L_1 \cdot L_2$

iv) $\lim_{(x,y) \rightarrow (x_0, y_0)} \left[\frac{f(x,y)}{g(x,y)} \right] = \frac{L_1}{L_2} \quad \text{provided, } L_2 \neq 0.$

CONTINUITY:

A function $z = f(x,y)$ is said to be continuous at a point (x_0, y_0) if

i) $f(x,y)$ is defined at the point (x_0, y_0)

ii) $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y)$ exists

iii) $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y) = f(x_0, y_0)$

OR

A function $f(x,y)$ is said to be continuous at (x_0, y_0) if for a given $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x,y) - f(x_0, y_0)| < \epsilon \text{ whenever } \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

If $f(x_0, y_0)$ is defined and $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y) = L$ exists but $f(x_0, y_0) \neq L$, then the point (x_0, y_0) is called a point of removable discontinuity.

If a function $f(x,y)$ is continuous at every point in a domain D , then it is said to be continuous in D .

EXAMPLE: Show that the following functions are continuous

$$\text{i) } f(x,y) = \begin{cases} \frac{2x^4+3y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\text{ii) } f(x,y) = \begin{cases} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & ; (x,y) \neq (0,0) \\ \frac{1}{2} & (x,y) = (0,0) \end{cases}$$

Sol: i) Change of coordinate system:

$$x = r\cos\theta \quad y = r\sin\theta$$

Note that $r = \sqrt{x^2+y^2} \neq 0$ since $(x,y) \neq (0,0)$.

Consider $|f(x,y) - f(0,0)| = \left| \frac{2r^4\cos^4\theta + 3r^4\sin^4\theta}{r^2(\cos^2\theta + \sin^2\theta)} \right| = |2r^2\cos^4\theta + 3r^2\sin^4\theta| \leq 2r^2\cos^4\theta + 3r^2\sin^4\theta < 5r^2 < 5\delta^2 < \epsilon$

$$\Rightarrow \text{Choose } \delta < \sqrt{\frac{\epsilon}{5}}$$

$$\Rightarrow |f(x,y) - f(0,0)| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y) = f(0,0) = 0$$

Hence $f(x,y)$ is continuous at $(0,0)$.

Alternative Approach: Change of coordinate $x = r\cos\theta, y = r\sin\theta$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^4+3y^4}{x^2+y^2} = \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{or } r \rightarrow 0}} \frac{2r^4\cos^4\theta + 3r^4\sin^4\theta}{r^2} = 0 \quad (\text{No dependency on } \theta) \text{ If it depends on } \theta \text{ then limit does not exist.}$$

$$\text{iii) } f(x,y) = \begin{cases} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & (x,y) \neq (0,0) \\ y_2 & (x,y) = (0,0) \end{cases}$$

Choose $x+2y = t$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} = \lim_{t \rightarrow 0} \frac{\sin^{-1}(t)}{\tan^{-1}(2t)} = \lim_{t \rightarrow 0} \frac{\frac{1}{\sqrt{1-t^2}}}{\frac{2}{1+4t^2}} = \frac{1}{2}$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \frac{1}{2} = f(0,0)$$

\Rightarrow The function is continuous.

Ex. Discuss the continuity of the functions

$$\text{i) } f(x,y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Choosing the path $y=mx$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x-y)^2}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{(x-mx)^2}{x^2+m^2x^2} = \lim_{x \rightarrow 0} \frac{(1-m)^2}{1+m^2} = \frac{(1-m)^2}{1+m^2}$$

The limit depends on the path. Therefore the limit does not exist and the function is not continuous at $(0,0)$.

ii)

$$f(x,y) = \begin{cases} \frac{\sin \sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin \sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{t \rightarrow 0} \frac{\sin \sqrt{t^2}}{\sqrt{t^2}} = 1 \neq f(0,0)$$

The function is discontinuous at $(0,0)$.

Note that the point $(0,0)$ is a point of removable discontinuity.

iii)

$$f(x,y) = \begin{cases} \frac{e^{xy}}{x^2+1} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{e^{xy}}{x^2+1} = \frac{1}{0+1} = 1 \neq f(0,0)$$

The function is not continuous at $(0,0)$

(iv)

$$f(x,y) = \begin{cases} \frac{x^4 y^4}{(x^2+y^2)^3} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

Choosing the path $y^2 = mx$

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{x \rightarrow 0} \left[\frac{x^4 y^4}{(x^2+y^2)^3} \right]_{y^2=mx} = \lim_{x \rightarrow 0} \frac{x^4 \cdot m^2 x^2}{(x^2+m^2 x^2)^3}$$

$$= \frac{m^2}{(1+m^2)^3}$$

The limit depends on the path and therefore does not exist.

The function is discontinuous at $(0,0)$.

v)

$$f(x,y) = \begin{cases} \frac{x^2+y^2}{\tan xy} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

S1:

Take path $y = mx$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{\tan xy} = \lim_{x \rightarrow 0} \frac{x^2+m^2x^2}{\tan(mx^2)} \quad (m \neq 0)$$

$$= \lim_{x \rightarrow 0} \frac{(1+m^2)}{m \cdot \frac{\tan(mx^2)}{mx^2}} = \frac{1+m^2}{m}$$

$$\text{as } \lim_{x \rightarrow 0} \frac{\tan(mx^2)}{mx^2} = 1.$$

The limit depends on the path, hence it does not exist.

The function is discontinuous at $(0,0)$.

Do not follow as:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{\tan xy} = \underbrace{\lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{xy}}_{\text{does not exist } (y=mx)} \cdot \underbrace{\lim_{(x,y) \rightarrow (0,0)} \frac{1}{\frac{\tan(xy)}{xy}}} _{=1} \Rightarrow \text{limit does not exist.}$$

$\lim(fg) = \lim f \cdot \lim g$ is valid when both limits $\lim f$ & $\lim g$ exist!

Consider for example:

$$\lim_{(x,y) \rightarrow (0,0)} \left(\frac{x^2+y^2}{1+xy} \right), \text{ clearly this limit exists and is equal to zero.}$$

However, if we rewrite as

$$\underbrace{\lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{xy}}_{\text{does not exist}} \cdot \underbrace{\lim_{(x,y) \rightarrow (0,0)} \left(\frac{xy}{1+xy} \right)} _{=0}$$

\Rightarrow limit does not exist.

REMARK: Changing to polar coordinate (subst. $x = r\cos\theta$, $y = r\sin\theta$) and investigating the limit of the resulting expression as $r \rightarrow 0$ is often very useful. for example:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2+y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3\theta}{r^2} = \lim_{r \rightarrow 0} r \cos^3\theta = 0.$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2+y^2} = 0 \quad (\text{useful in finding limit})$$

Also, Note that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2+y^4} = \lim_{r \rightarrow 0} \frac{r^2 \cos^2\theta}{r^2} = \cos^2\theta \quad (\text{depends on } \theta)$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2+y^4} \text{ does not exist.}$$

Shifting to polar coordinates does not always help, however, and may even tempt us to false conclusions.

For example:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r \rightarrow 0} \frac{r^2 \cos^2\theta * r \sin\theta}{r^4 \cos^4\theta + r^2 \sin^2\theta} = \lim_{r \rightarrow 0} \frac{r \cos^2\theta \sin\theta}{r^2 \cos^4\theta + \sin^2\theta}$$

If we hold θ constant then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4+y^2} = 0, \text{ BUT this is not the limit, we should not}$$

fix θ . Taking the path $r \sin\theta = r^2 \cos^2\theta$ ($y = x^2$):

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r \rightarrow 0} \frac{x \cos^2\theta \sin\theta * r^2 \cos^2\theta}{r^2 \cos^4\theta + r^2 \cos^4\theta} = \frac{1}{2}$$

\Rightarrow The limit does not exist.