Date 06/11/2017

Lecture 27

~~*~~ Solve p.d.eqn using Laplace Transform Technique

13/11/2017 (Monday)

Class Test -10

Syllabus: Fourier series, Fourier integral, & Fourier transform.

Time: 1 hour.

Laplace Transforms w.r.t 't'

$t \longrightarrow s$ $\quad t : 0 - \infty$

$$\mathcal{L}\{\phi'\} = \mathcal{L}\left\{\frac{d\phi}{dt}\right\} = \int_0^\infty e^{-st}\frac{d\phi}{dt}\,dt$$

$$= s\,\overline{\phi}(s) - \phi(0)$$ where $\overline{\phi}(s) = \mathcal{L}\{\phi(t)\}$

$$\frac{\partial \phi}{\partial t} = k \frac{\partial^2 \phi}{\partial x^2} \rightarrow 1\text{ di}$$

Heat eq

turn it into the o. d. eqn

$$\mathcal{L}\left\{\frac{\partial \phi}{\partial t}\right\} = k\, \mathcal{L}\left\{\frac{\partial^2 \phi}{\partial x^2}\right\}$$

$$\Rightarrow s\bar{\phi}(s) - \phi(0) = k \frac{d^2 \bar{\phi}}{dx^2};$$

where

$$\mathcal{L}\left\{k \frac{\partial^2 \phi}{\partial x^2}\right\} = \int_0^\infty k \cdot \frac{\partial^2 \phi}{\partial x^2} e^{-st}\, dt$$

$$= k \frac{d^2}{dx^2} \Big\{ \underbrace{\int_0^\infty \phi\, e^{-st}\, dt}_{= \mathcal{L}\{\phi\}} \Big\}.$$

$$= k \frac{d^2}{dx^2} (\bar{\phi})$$

) Solve the heat conduction eq^n

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2},$$

in the region $\underline{\underline{t > 0,\ x > 0}}$ with

boundary conditions

✓ $\phi(x, 0) = 0,\ x > 0$ (initial cond^n)

$\phi(0, t) = 1,\ \underline{\underline{t > 0}}$

(temperature held constant at the origin) &

✓ $\lim\limits_{x \to \infty} \phi(x, t) = 0$

(the temperature a long way away from the origin remains at its initial value).

Sol^n:– Taking the Laplace Transform (in t of course) gives

$$\mathcal{L}\left\{\frac{\partial\phi}{\partial t}\right\} = \mathcal{L}\left\{\frac{\partial^2\phi}{\partial x^2}\right\}$$

$$\Rightarrow s\bar{\Phi} - \underbrace{\phi(0)}_{(=0)} = \frac{d^2\bar{\Phi}}{dx^2} \qquad \left[\phi(x,t) \xrightarrow{\mathcal{L}} \bar{\Phi}(x,s)\right]$$

$$\Rightarrow \frac{d^2\bar{\Phi}}{dx^2} = s\bar{\Phi}, \text{ since } \phi(x,0)=0 \qquad \left[\phi(0) = \phi(x,0) = 0\right]$$

This is an ordinary differential eq^n (o.d.eq^n) with constant co-efficients (not dependent on x, the presence of s does not matter as there is no s–derivative)

with sol^n

$$\bar{\phi}(x,s) = A e^{-\sqrt{s}x} + B e^{x\sqrt{s}}$$

$$\frac{d^2y}{dx^2} = sy$$

$\cancel{A} \cdot e^{mx}$, $m^2 - s = 0$

$\Rightarrow m = \pm\sqrt{s}$

$y = a e^{m_1 x} + b e^{m_2 x}$

$= a e^{\sqrt{s}x} + b e^{-\sqrt{s}x}$

$\longrightarrow (1)$

$$= A(s) e^{-x\sqrt{s}} + B(s) e^{x\sqrt{s}} \longrightarrow (1)$$

Now, since $\boxed{\lim_{x\to\infty} \phi(x,t) = 0}$

& $\bar{\phi}(x,s) = \mathcal{L}\{\phi(x,t)\}$

$$= \int_0^\infty e^{-st} \phi(x,t)\, dt$$

$$\therefore \lim_{x\to\infty} \bar{\phi}(x,s) = \int_0^\infty \left\{\lim_{x\to\infty} \phi(x,t)\right\} e^{-st}\, dt$$

$$= 0$$

$\Rightarrow \bar{\phi}(x,s) \to 0$ as $x \to \infty$.

$\therefore$ from eqn (1), we have:

$$B(s) = 0.$$

as for every fixed positive s, the fn $e^{x\sqrt{s}}$ increases as x increases

(but L.H.S $\to 0$, hence R.H.S should be bounded)

$\therefore \boxed{B = 0}$.

We can assume $s > 0$ (why?) since a Laplace transform generally exists for all $s > k$ (k fixed)

$$\therefore \overline{\Phi}(x, s) = A(s)\, e^{-x\sqrt{s}} \quad \rightarrow (2)$$

Let $x=0$ in eq^n (2), we get

$$\overline{\Phi}(0, s) = A(s).$$

& $\overline{\Phi}(0, s) = \mathcal{L}\{\Phi(0, t)\}$

$$= \int_0^\infty \underbrace{\Phi(0, t)}_{=1} \cdot e^{-st}\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

$$= -\left[\frac{e^{-st}}{s}\right]_0^\infty$$

$$= \frac{1}{s}, \; (s>0),$$

$\Rightarrow \boxed{A(s) = \frac{1}{s}}$ given that $\Phi(0, t) = 1, \; \underline{\underline{\forall t}}.$

$\therefore$ the soln for $\bar{\phi}(x,s)$ is

$$\bar{\phi}(x,s) = \frac{e^{-x\sqrt{s}}}{s}$$

Inverting this, we have

$$\phi(x,t) = \mathcal{L}^{-1}\left\{\bar{\phi}(x,s)\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s}\right\}$$

$$\Rightarrow \phi(x,t) = \boxed{\operatorname{erfc}\left(\tfrac{1}{2} x t^{-1/2}\right)}$$

We know that

$$\mathcal{L}^{-1}\left\{\frac{e^{-k\sqrt{s}}}{s}\right\} = \operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$$

Complementary error fn.

This cannot ~~solve by~~ be

Q1) zeperation of variables method.

or the soln $\phi(x,t) = \operatorname{erfc}\left(\tfrac{1}{2} x t^{-1/2}\right)$ is in no way expressible in variables zeperable form.

~~Another method~~

Let
$$\boxed{\xi = x\,t^{-1/2}}$$
& transform the eq^n

$$\phi_t = \phi_{xx}$$

where $\phi_t = \dfrac{\partial \phi}{\partial t}$, $\phi_{xx} = \dfrac{\partial^2 \phi}{\partial x^2}$

chng from $(x, t) \rightarrow$ into ξ-space.

This method is called using a similarity variable

Note

Uniqueness: - The heat conduction eq^n has a unique sol^n provided boundary condt^{ns} are given at $t=0$ (initial cond^n) together with values of ϕ (or, its derivative w.r.t x)

at $x=0$ &

another value ($x=a$, say)

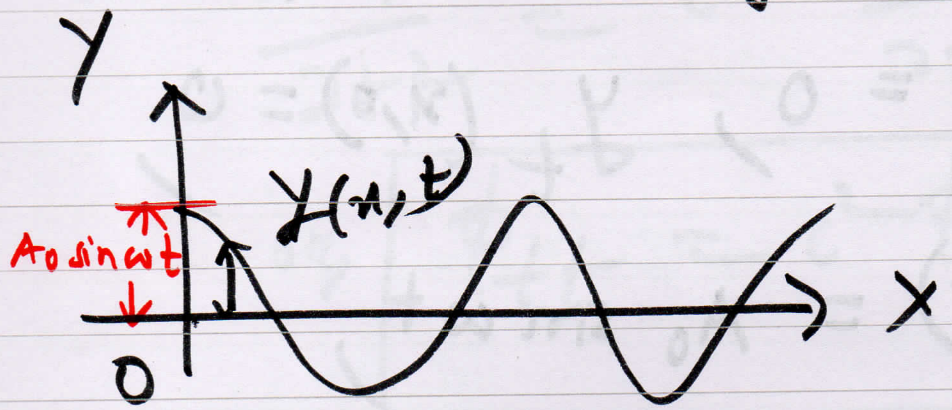
for all t

** problems that do not have a unique sol^n are called ill-posed

(non-linear problem)

(Q) Investigate the problem)

An infinitely long string having one end at $x=0$ is initially at rest on the X-axis. The end $x=0$ undergoes a periodic transverse displacement given by $A_0 \sin \omega t,\ t>0$.



Find the displacement of any point on the string at any time.

Sol^n:- If $y(x,t)$ is the transverse displacement of the string at any point x at any time t, then

The boundary value problem

is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2},$$

$$x > 0, \quad \underline{t > 0}$$

$$y(x,0) = 0, \quad y_t(x,0) = 0,$$

$$y(0,t) = A_0 \sin \omega t,$$

$$|y(x,t)| < M \quad \text{(ie, bounded)}$$

<u>Hint</u> :-

$$y(x,t) = \begin{cases} A_0 \sin \omega (t - x/a) & t > x/a \\ 0, & t < x/a. \end{cases}$$

L.T can be used to solve <u>second order wave eq</u>

L.T can be used to solve second order wave eq^n

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

or $\boxed{\phi_{tt} = c^2 \phi_{xx}}$

where c is a constant, called the celerity or wave speed, x is the displacement & t is time.

One kind of problem that can solved is that the cond^{n}s at time $t=0$ must be specified.

Let $\overline{\Phi} = \mathcal{L}\{\phi\}$.

$$\mathcal{L}\left\{\frac{\partial^2 \phi}{\partial t^2}\right\} = s^2\overline{\Phi} - s\phi(0) - \phi'(0)$$

$\therefore$ in our case

$$s^2\overline{\phi} - s\phi(0) - \phi'(0) = c^2 \frac{d^2(\overline{\phi})}{dx^2}.$$

The sol^n to this eq^n is written in terms of hyperbolic f^n since the (why?) x boundary $cond^{ns}$ are almost always given at two finite values of x.

generally, in an infinite half-space,

eg, the vibration of a very long beam fixed at one end (a cantilever) the complementary fn

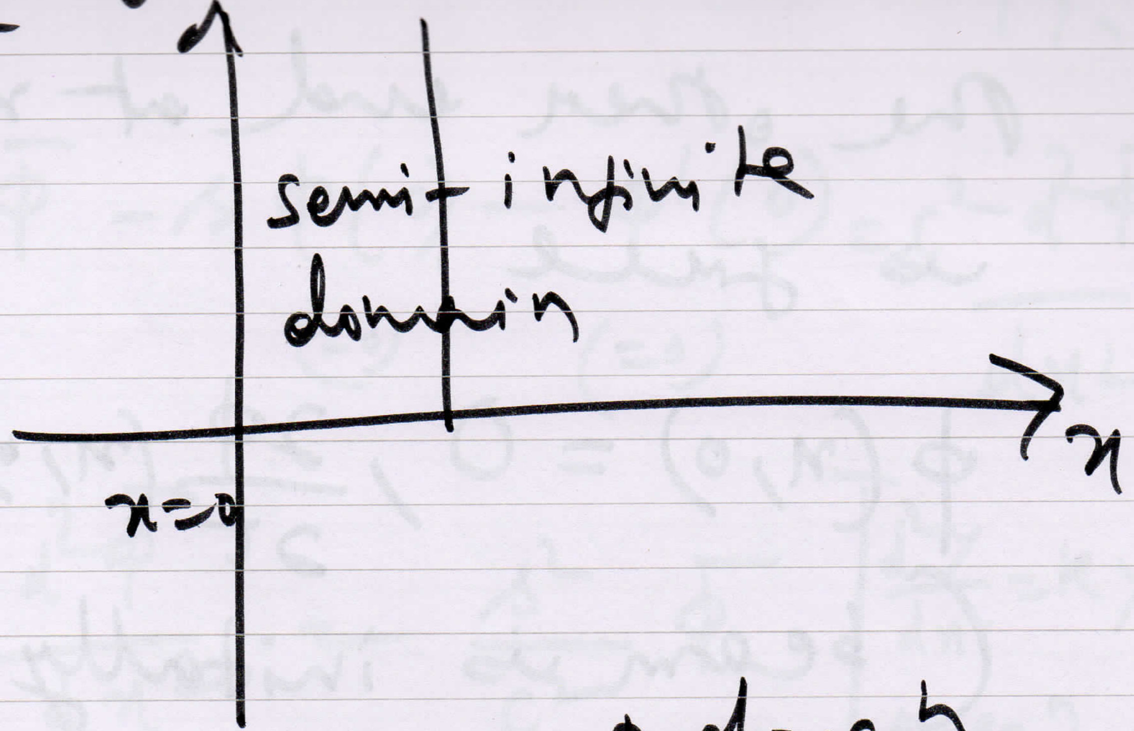
$$Ae^{sx/c} + Be^{-sx/c}$$

(same e^x)

EX/ A beam of length a initially at rest has one end fixed at $x=0$ with the other end free to move at $x=a$.

Assuming that the beam only moves longitudinally (in the x-direction) & is subject to a constant force ED along its length where E is the Young's modulus of the beam & D is the displacement per unit length, find the longitudinal displacement of the beam at any subsequent time. Find also the motion of the free end at $x=a$.

1)! -



The given p.d.eqn is

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}.$$

$$\mathcal{L}\left[\frac{\partial^2 \phi}{\partial t^2}\right] = c^2 \mathcal{L}\left(\frac{\partial^2 \phi}{\partial x^2}\right)$$

$\phi(x,t)$ is the reqd. displacement

& $\overline{\Phi}(x,s)$ is its Laplace Transform

Now, the beam is initially at rest, $x=0$ is fixed for all time

the other end at $x=a$
is free:

$$\phi(x,0)=0, \; \frac{\partial\phi}{\partial t}(x,0)=0$$

(beam is initally at rest)

together with

$$\phi(0,t)=0$$

(ie., $x=0$ fixed for all time)

&

$$\frac{\partial\phi(a,t)}{\partial x}=0, \; \forall t$$

(the end at $x=a$ is free)

$$s^2\bar{\Phi} - s\underbrace{\phi(0)}_{(=0)} - \underbrace{\phi'(0)}_{(=0)} = c^2 \frac{d^2\bar{\Phi}}{dx^2}$$

$$\Rightarrow \frac{d^2\bar{\Phi}}{dx^2} = \frac{s^2}{c^2}\bar{\Phi}$$

$$\frac{d^2y}{dx^2} = ky$$

$A \cdot e^{mx}$

$m^2 - (s/c)^2 = 0$

$\Rightarrow m = \pm s/c$

$$\Rightarrow \bar{\Phi}(x,s) = k_1 \cosh\left(\frac{sx}{c}\right) + k_2 \cdot \sinh\left(\frac{sx}{c}\right).$$

To find k_1, k_2, we use the x boundary cond^n.

If $\phi(0,t) = 0$, then

$$\bar{\Phi}(0,s) = \mathcal{L}\{\phi(0,t)\} = \int_0^\infty \underbrace{\phi(0,t)}_{(=0)} e^{-st}\, dt = 0.$$

from (2),

$$0 = k_1 \cdot 1 \Rightarrow k_1 = 0.$$

$$\therefore \bar{\phi}(x, s) = k_2 \sinh\left(\frac{xs}{c}\right),$$

$$\& \frac{d\bar{\phi}}{dx} = \frac{s k_2}{c} \cosh\left(\frac{sx}{c}\right).$$

$$\therefore \frac{d\bar{\phi}}{dx} = \frac{D}{s} \text{ at } x = a,$$

Hence,
$$\frac{D}{s} = \frac{s k_2}{c} \cosh\left(\frac{sa}{c}\right)$$

$$\Rightarrow k_2 = \frac{Dc}{s^2 \cosh\left(\frac{sa}{c}\right)}$$

Hence,
$$\bar{\phi}(x, s) = \left\{\frac{Dc \sinh\left(\frac{sx}{c}\right)}{s^2 \cosh\left(\frac{sa}{c}\right)}\right\}.$$

~~For~~ we know that,

$$\mathcal{L}^{-1}\left\{\frac{\sinh(sx)}{s^2\cosh(sa)}\right\}$$

$$= x + \frac{8a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\,.$$

$$\left[\sin\left(\frac{(2n-1)\pi x}{2a}\right).\cos\left(\frac{(2n-1)\pi t}{2a}\right)\right]$$

[Use Inverse L.T general formula]

$\therefore$ hence

$$\Rightarrow \phi(x,t) = \mathcal{L}^{-1}\left\{\bar{\Phi}(x,s)\right\}$$

$$= Dx + \frac{8a}{\pi^2}D\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\,.$$

$$\left[\sin\left(\frac{(2n-1)\pi x}{2a}\right).\cos\left(\frac{(2n-1)\pi ct}{2a}\right)\right]$$

Q3) (Solve using Fourier integral)

An infinite string is given an initial displacement $y(x,0) = f(x)$, & then released. Determine its displacement at any later time t.

The boundary value problem is

1-dim wave eqn $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ $\rightarrow (1)$

$y(x,0) = f(x)$, $y_t(x,0) = 0$,

$|y(x,t)| < M$ where $-\infty < x < \infty, t > 0.$ $\rightarrow (2)$

Letting $y = XT$ in eqn (1)

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x).\cos(\lambda a t)$$

By assuming that A & B are fns of λ & integrating as $x, t \to$ from $\lambda = 0$ to ∞, we then arrive at the possible soln

$$y(x,t) = \int_0^\infty [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x].\cos(\lambda a t)\,d\lambda \to (3)$$

Putting $t=0$ in (3), we see from the first boundary condn from (2), we get

$$f(x) = \int_0^\infty \left(A(\lambda)\cos(\lambda x) + B(\lambda)\sin\lambda x \right) d\lambda$$

Then it follows from Fourier integral theorem that

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\lambda v)\,dv$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\lambda v)\,dv \rightarrow (4)$$

where, we have changed the dummy variable from x to v.

Substituting (4) in (3), we get

$$y(x,t) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u)\Big[\cos\lambda x \cos\lambda u + \sin\lambda x \sin\lambda u\Big] \cos(\lambda a t)\, du\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u) \cos\lambda(x-u) \cdot \cos(\lambda a t)\, du\, d\lambda.$$

$$= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u) \Big[\cos\lambda(\underline{x+a}t-u) + \cos\lambda(\underline{x-a}t-u)\Big] du\, d\lambda$$

$$\Rightarrow y(x,t) = \frac{1}{2}\Big[f(x+at) + f(x-at)\Big]$$

which is the reqd. soln.

$$\boxed{\cos A \cos B = \frac{1}{2}\Big[\cos(A+B) + \cos(A-B)\Big]}$$

Again, F-I Thm

$$f(\underline{x}) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u) \cos\lambda(\underline{\underline{x}}-u)\, du\, d\lambda$$

(separation of variables Method) 26

Let $y = XT$

$y_{tt} = a^2 y_{xx}$

$\Rightarrow y(x,t) = X(x)\,T(t).$

$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

$y_x = X'T$
$y_{xx} = X''T$

$X T'' = a^2 X'' T$

$y = XT$
$y_t = XT'$
$y_{tt} = XT''$

$\Rightarrow \frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda^2.$

$\Rightarrow X'' + \lambda^2 X = 0$ | $T'' + (\lambda a)^2 T = 0$

$\therefore$ [$m^2 + \lambda^2 = 0$
$\Rightarrow m = \pm i\lambda$] | $\Rightarrow$ [$m^2 + (\lambda a)^2 = 0$]

$X = (a_1 \cos \lambda x + b_1 \sin \lambda x)$ | $T = a_2 \cos(\lambda a t) + b_2 \sin(\lambda a t)$

$\therefore y(x,t) = \underbrace{(a_1 \cos\lambda x + b_1 \sin\lambda x)}_{(=P)} \cdot \left(a_2 \cos(\lambda a t) + b_2 \sin \lambda a t\right)$

$\left.\frac{\partial y}{\partial t}\right|_{t=0} = P \cdot \left[-\lambda a a_2 \sin(\lambda a t) + b_2 \lambda a \cos(\lambda a t)\right]_{t=0}$

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)\cos(\lambda a t)$$

— X —

H.W

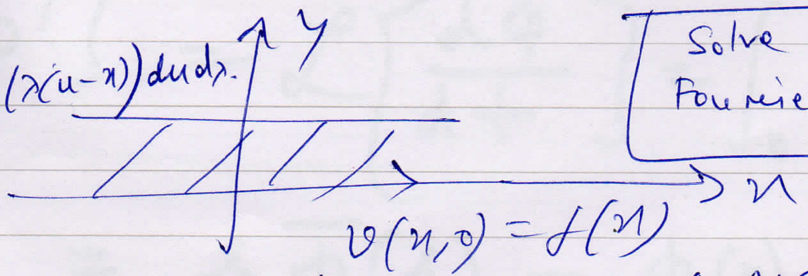
Q) Find a bounded sol^n to Laplace's eq^n $\nabla^2 v = 0$ for the half-plane $y > 0$ if v takes on the value $f(x)$ on the x-axis.

ie, solve $v(x,y)$ for

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

$$v(x,0) = f(x), \quad |v(x,y)| < M.$$

sol^n: $\boxed{v(x,y) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{u=-\infty}^{\infty} e^{-\lambda y} f(u)\cos(\lambda(u-x))\,du\,d\lambda.}$



Solve using Fourier integral.

Hint :- Let $v = XY$. ie, $v(x,t) = X(x)\,Y(t)$