Date 29/10/2019

Lecture 11

On multiplying eq^n (1) by $\frac{\partial \bar{x}^i}{\partial x^k}$ & summing over the index i from 1 to N, we obtain for transformation from $\bar{x}^i$ to x^i system,

$$\frac{\partial \bar{x}^i}{\partial x^k} \bar{A}_i = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^j}{\partial \bar{x}^i} A_j$$

$$= \frac{\partial x^j}{\partial x^k} A_j = \boxed{\delta^j_k A_j}$$

$$= A_k$$

i.e. $\boxed{A_k = \frac{\partial \bar{x}^i}{\partial x^k} \bar{A}_i}$ → (2)

(1) $\frac{\partial x^i}{\partial x^j} = \delta^i_j$, (2) $\delta^i_i = N$,

(3) $\delta^i_j A^{jk} = A^{ik}$, (4) $\delta^i_j \delta^j_k = \delta^i_k$

(5) $\delta^i_j \delta^j_i = N$.

②

2) Contravariant vector (or, Tensor of first order or rank 1).

A set of N-functions A^i or, $A^1, A^2, \ldots, A^N$ of N co-ordinates x^i or, $x^1, x^2, \ldots, x^N$ (ie, $A^i = A^i(x^i)$, $i = 1, 2, \ldots, N$)

are said to be the components of a contravariant vector or, contravariant tensor of the first order (or, rank 1) if they transform from x^i to $\bar{x}^i$ or, $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N$ co-ordinates, according to the

~~following~~ eqⁿs

$$\bar{A}^{i} = \sum_{j=1}^{N} \frac{\partial \bar{x}^{i}}{\partial x^{j}} A^{j}, \quad i = 1, 2, \ldots, N$$

or, by summation convention, we express

$$\bar{A}^{i} = \frac{\partial \bar{x}^{i}}{\partial x^{j}} A^{j} \longrightarrow (3)$$

On multiplying eqⁿ (3) by $\frac{\partial x^{k}}{\partial \bar{x}^{i}}$ & summing over the index i from 1 to N, we obtain for transformation from $\bar{x}^{i}$ to x^{i} system

4

$$\frac{\partial x^k}{\partial \bar{x}^i}\bar{A}^i = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^j}A^j$$

$$= \frac{\partial x^k}{\partial x^j}A^j$$

$$= \delta^k_j A^j$$

$$= A^k$$

ie, $$A^k = \frac{\partial x^k}{\partial \bar{x}^i}\bar{A}^i \rightarrow (4)$$

Note :- Superscript is used for contravariant tensor & subscript is used for covariant vector.

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EY / HW Prove that the transformation of

~~(i)~~ Covariant vector,

~~(ii)~~ Contravariant vector,

HW ~~(iii)~~ Mixed tensor A_j^i

possess the group property

i.e., forms a group.

Sol^n:- Suppose A^i be a contravariant vector. We consider co-ordinate transformations:

$$x'^i = x'^i(x^k), \quad x''^i = x''^i(x^k)$$

i.e., $x^i \longrightarrow x'^i \longrightarrow x''^i$

(i) $\longrightarrow$ (ii) $\longrightarrow$ (iii)

$A^i \longrightarrow A'^i \longrightarrow A''^i$

In the case of

$$x^i \longrightarrow x'^i,$$ we have

by transformation law

$$A'^j = \frac{\partial x'^j}{\partial x^i} A^i \longrightarrow (1)$$

& for the transformation

$$x'^i \longrightarrow x''^i$$

$$A''^i = \frac{\partial x''^i}{\partial x'^j} A'^j$$

[by eq^n (1)]

$$= \frac{\partial x''^i}{\partial x'^j} \cdot \frac{\partial x'^j}{\partial x^i} A^i$$

$$\Rightarrow A''^i = \frac{\partial x''^i}{\partial x^i} A^i \longrightarrow (2)$$

which proves that if we make the transformation from (i) to (iii), we get the same law of transformation. This property is expressed by stating that transformation of a contravariant vector form a group.

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Invariants or, Scalar

Any function I of the N-co-ordinates x^i, $(i=1,2,\dots,N)$, is called an invariant or a 'scalar' if $I = \bar{I}$, the change of co-ordinate transformation from x^i to $\bar{x}^i$ system, where $\bar{I}$ is the value of I in new co-ordinate system $\bar{x}^i$.

e.g. H.W. The quantity

$$\delta^i_i = \delta^1_1 + \delta^2_2 + \dots + \delta^N_N$$
$$= \underbrace{1+1+\dots+1}_{} = N$$

Ex2/ Show that if A^i & B_i are the components of a contravariant & covariant vectors resply, then their product $A^i B_i$ is an invariant.

Soln:- We form the components of A^i & B_i of a contravariant & covariant vectors resply in x^i co-ordinate system. We can form their product. On the change of x^i to $\bar{x}^i$ co-ordinate system,

~~functions A~~ functions A^i & B_i transform to $\bar{A}^i$ & $\bar{B}_i$, so that their product $A^i B_i$ transforms to $\bar{A}^i \bar{B}_i$.

Then

$$\bar{A}^i \bar{B}_i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \cdot \frac{\partial x^k}{\partial \bar{x}^i} B_k$$

$$= \frac{\partial \bar{x}^i}{\partial x^j} \cdot \frac{\partial x^k}{\partial \bar{x}^i} \cdot A^j B_k$$

$$= \frac{\partial x^k}{\partial x^j} A^j B_k = \delta^k_j A^j B_k$$

$$= A^k B_k$$

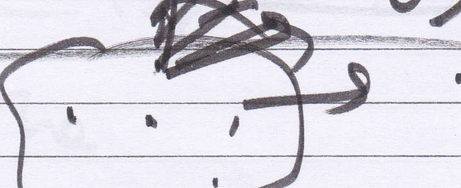
i.e., $\bar{A}^i \bar{B}_i = A^i B_i$ (as dummy suffix k can be replaced by i)

Thus the product

$A^i B_i$ is an <u>invariant</u>.

Note:- 1) An invariant or a scalar is also <u>called a tensor of rank 0</u> (or, order 0)

2) <u>Tensor field</u>:



Defn:- If to each point of a region in N-dim space, there corresponds a definite tensor, it is said that a 'tensor field' has been defined.

This is a 'vector field' or a 'scalar field', according as the tensor is of rank one or rank zero.

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§/ Covariant, Contravariant & Mixed tensors

(Higher order Tensors of rank or order 2)

1) Covariant Tensors :-

Suppose the N^2 – quantities A_{ij} in a co-ordinate system $(x^1, x^2, \ldots, x^N)$ are changed to N^2 other quantities $\bar{A}_{ij}$ in another co-ordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations

$$\bar{A}_{ij} = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A_{kl} \quad (i,j = 1 - N)$$

or, in the summation convention,

$$\overline{A_{ij}} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A_{kl} \rightarrow (1)$$

then the N^2 - quantities A_{kl} are called "Covariant components" of a tensor of the second rank or order.

b) Contravariant Tensors

Suppose the N^2 - quantities A^{ij} in a co-ordinate system $(x^1, x^2, \dots, x^N)$ are changed to N^2 - other quantities

$\bar{A}^{ij}$ in another co-ordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation eq^ns

$$\bar{A}^{ij} = \sum_{k=1}^{N} \sum_{j=1}^{N} \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial \bar{x}^j}{\partial x^l} A^{kl}$$

or, in the summation convention → (2)

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial \bar{x}^j}{\partial x^l} A^{kl}$$

→ (3)

then the N^2-quantities A^{kl} are called contravariant component of a tensor of rank 2 (or, order 2).

3) Mixed tensors

Suppose the N^2 - quantities A^i_j in a co-ordinate system $(x^1, x^2, \ldots, x^N)$ are changed to N^2 - other quantities $\bar{A}^i_j$ in another co-ordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation eqns

$$\bar{A}^i_j = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial x^l}{\partial \bar{x}^j} A^k_l \quad \rightarrow (4)$$

or, using the Einstein's summation convention,

17

$$\bar{A}^{i}_{j} = \frac{\partial \bar{x}^{i}}{\partial x^{k}} \cdot \frac{\partial x^{l}}{\partial \bar{x}^{j}} A^{k}_{l} ,$$

then the N^2 — quantities $\to$ (5)
A^{k}_{l} are called the components of a 'mixed tensor' of second rank (or, order 2).

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§/ Higher order Tensors :-

A set of N^{k+l} functions $A^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_l}$ of the N co-ordinates $(x^1, x^2, \ldots, x^N)$ are said to be the components of a 'mixed tensor' of $(k+l)^{th}$ order, (or, (k,l) order) contravariant of k^{th} order & covariant of l^{th} order, if they transform according to the eq^n, using summation convention

$$\bar{A}^{u_1 u_2 \cdots u_k}_{m_1 m_2 \cdots m_l} = \frac{\partial \bar{x}^{u_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{u_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{u_k}}{\partial x^{i_k}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{m_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{m_2}} \cdots \frac{\partial x^{j_l}}{\partial \bar{x}^{m_l}} \cdot A^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_l} \qquad (5)'$$

where

$\bar{A}^{u_1 u_2 \cdots u_k}_{m_1 m_2 \cdots m_l}$ are the new

N^{k+l} functions in the

$(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ co-ordinate

system.

19

Thus, $A^{i_1 i_2}_{j_1 j_2}$, $B^{u_1 u_2 u_3}_{s_1 s_2 s_3 s_4}$, A^{ijk}

B_{ijkl} etc. are higher order tensors of order $(2+2)=4$, $(3+4)=7$, $(3+0)=3$ & $(0+3)=3$ resply

or, of order $(2,2)$, $(3,4)$, $(3,0)$, $(0,3)$

Note 1 – The order of indices in a tensor is important. The tensor A^{ij} is not necessarily the same as the tensor A^{ji}.

Symmetric & Skew-symmetric ~~(Anti-symmetric)~~ tensor

Def^n^:- A tensor is called 'symmetric' w.r.t two contravariant or, two covariant indices if its components remain unaltered (unchanged) upon the interchange of the indices.

Thus, $A^{ij} = A^{ji}$, $A_{ij} = A_{ji}$ for two indices, & if

$$A^{ijk}_{lm} = A^{jik}_{lm},$$

the tensor is symmetric in the indices i & j.

If a tensor is symmetric w.r.t any two contravariant and any two covariant indices, it is called 'symmetric'.

Def^n:- A tensor is called skew-symmetric w.r.t two contravariant or, two covariant indices, if its components alters (change) in sign upon the interchange of the indices.

Thus, $A^{ij} = -A^{ji}$, $A_{ij} = -A_{ji}$

for two indices & i, j

$A^{ijk}_{lm} = -A^{jik}_{lm}$, then

tensor is skew-symmetric in indices i & j.

If a tensor is skew-symmetric w.r.t any two contravariant & any two covariant indices, it is called skew-symmetric

Q) ~~A^{i}_{j}~~

Note :- We cannot usually define symmetry w.r.t 2 indices, one contravariant & the other covariant because this symmetry may not be preserved after the co-ordinate transformation.

HW Ex.

However, kronecker delta (δ^i_j), a mixed tensor which possess symmetry w.r.t 2 indices (i.e, $\bar{\delta}^i_j = \delta^i_j$)

yyy

Ex1/ If a tensor is symmetric w.r.t two indices in any co-ordinate system, it remains symmetric w.r.t indices in any other co-ordinate system.

Solⁿ:- ~~For the~~ contravariant tensor (A^{ij}) which is symmetric in 2 indices i & j,

w.a.l.o.g

ie, $A^{ij} = A^{ji}$.

Under the transformation of co-ordinates, x^i to $\bar{x}^i$ & summation convention

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial \bar{x}^j}{\partial x^l} \underbrace{A^{kl}}$$

$$= \frac{\partial \bar{x}^j}{\partial x^l} \cdot \frac{\partial \bar{x}^i}{\partial x^k} \cdot A^{lk}$$

$[\text{as } A^{lk} = A^{kl}]$,

$$= \bar{A}^{ji}$$

i.e., $\bar{A}^{ij}$ is symmetric.

HW ✓ Covariant tensor.

check for mixed tensor

26

A_{ij}

N^2.

A_{11} A_{12} A_{13} . . A_{NN}

A_{21} A_{22} A_{23} . . A_{2N}

A_{N1} A_{N2} A_{N3} . . A_{NN}

$$\frac{N(N+1)}{2}$$

$$= N + \left(\frac{N^2 - N}{2}\right)$$