Eigenvalue Eigenvectors:

Th1: λ is a characteristic root of a matrix A iff $\exists$ a nonzero vector x such that

$$Ax = \lambda x$$

A is a square matrix.

Th2: If x is characteristic vector of A corresponding to the characteristic value λ then kx is also a characteristic vector corresponding to the same characteristic value λ. Here k is any nonzero scalar.

Proof: $Ax = \lambda x$ since x is a char. vector corr. to char. value λ.

If k is any nonzero scalar then $kx \neq 0$.

Also, $A(kx) = kAx = k(\lambda x) = \lambda(kx)$

$\Rightarrow$ kx is a characteristic vector corr. to char. value λ.

Th3: If x is a characteristic vector of a matrix A then x cannot correspond to more than one char. value of A.

Proof: Let us assume

$$Ax = \lambda_1 x \quad \& \quad Ax = \lambda_2 x$$

$$\Rightarrow \lambda_1 x = \lambda_2 x$$

$$\Rightarrow (\lambda_1 - \lambda_2)x = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = 0 \quad \text{since } x \neq 0$$

$$\Rightarrow \lambda_1 = \lambda_2$$

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS:

Let λ be an eigenvalue of A and x be its corresponding eigenvector. Then:

(1) αA has eigenvalue $\alpha\lambda$ and corresponding eigenvector x.

$$Ax = \lambda x \Rightarrow (\alpha A)x = (\alpha\lambda)x$$

(2) A^m has eigenvalues λ^m and corresponding eigenvectors x for any positive integer m.

$$Ax = \lambda x$$

Premultiply by A

$$A^2x = \lambda Ax = \lambda^2 x$$

(3) $A-kI$ has eigenvalue $\lambda - k$ and corresponding vector x.

$$Ax=\lambda x \Rightarrow Ax - kIx = \lambda x - kx$$

$$\Rightarrow (A-kI)x = (\lambda - k)x$$

(4) A^{-1} (if it exists) has eigenvalue $1/\lambda$ and corr. eigen vector x.

$$Ax = \lambda x \Rightarrow A^{-1}Ax = A^{-1}\lambda x$$

$$\Rightarrow x = \lambda A^{-1}x \Rightarrow A^{-1}x = \left(\frac{1}{\lambda}\right)x.$$

(5) A and A^T have the same eigenvalues.

(6) For a real matrix A if $\alpha + i\beta$ is an eigenvalue then its conjugate $\alpha - i\beta$ is also an eigenvalue.

Theorem 4: The characteristic roots of a Hermitian matrix are real

Proof: Remember A is hermitian $\Rightarrow$ $A^* = A$.

Let λ be a characteristic root of A and x its eigenvector.

Then. $Ax = \lambda x$

Premultiply both side by x^*

$$x^* A x = x^* \lambda x = \lambda x^* x \quad \text{—} (1)$$

Taking conjugate transform both side:

$$(x^* A x)^* = (\lambda x^* x)^*$$

$$\Rightarrow x^* A^* (x^*)^* = \bar{\lambda}\, x^* (x^*)^* \qquad (x^*)^* = x$$

$$\Rightarrow x^* A^* x = \bar{\lambda} x^* x \qquad A^* = A.$$

$$\Rightarrow x^* A x = \bar{\lambda} x^* x \quad \text{—} (2)$$

$$(1) \& (2) \Rightarrow \quad \lambda x^* x = \bar{\lambda} x^* x$$

$$\Rightarrow (\lambda - \bar{\lambda}) x^* x = 0$$

$$\Rightarrow \quad \lambda - \bar{\lambda} = 0 \quad \text{since } x^* x \neq 0$$

$$\Rightarrow \lambda = \bar{\lambda}$$

$\Rightarrow$ λ is real. □

In a similar way we can prove

1) The characteristic roots of a real symmetric matrix are all real.

2) The characteristic roots of a skew-hermitian matrix are either purely imaginary or zero.

3) The characteristic roots of a real skew-symmetric matrix are either zero or purely imaginary.

Theorem 5: The characteristic roots of a Unitary matrix are of unit modulus.

Proof: Unitary matrix $A^*A = I$

$$Ax = \lambda x \quad \text{---}(1)$$

$$\Rightarrow (Ax)^* = (\lambda x)^* \Rightarrow x^* A^* = \bar{\lambda} x^* \quad \text{---}(2)$$

$$(1) \,\&\, (2) \Rightarrow (x^* A^*)(Ax) = (\bar{\lambda} x^*)(\lambda x)$$

$$\Rightarrow x^*(A^*A)x = \bar{\lambda}\lambda\, x^* x$$

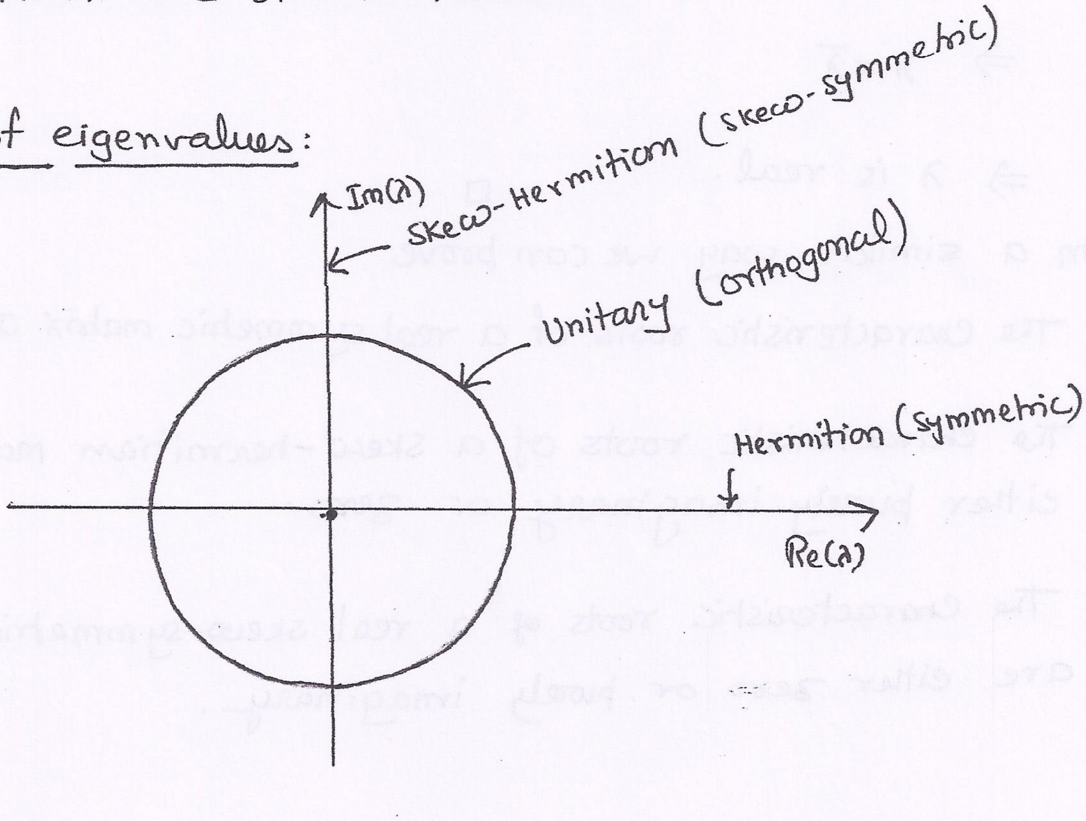
$$\Rightarrow x^* I x = \bar{\lambda}\lambda\, x^* x$$

$$\Rightarrow x^* x (1 - \bar{\lambda}\lambda) = 0$$

$$\Rightarrow 1 - \bar{\lambda}\lambda = 0 \Rightarrow \bar{\lambda}\lambda = 1 \Rightarrow |\lambda|^2 = 1.$$

Corollary: The characteristic roots of an orthogonal matrix are of unit modulus.

location of eigenvalues:



WORKING RULES FOR FINDING EIGEN VALUES and EIGEN VECTORS:

roots of Characteristic equation $|A-\lambda I| = 0$ are eigenvalues.

Eigen vector corresponding to eigenvalue λ_i can be obtained by solving:

$$Ax = \lambda_i x$$

$$\Leftrightarrow (A-\lambda_i I)x = 0.$$

Example: Determine the eigenvalues & eigenvectors of the matrix:

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

Characteristic equation:

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-6) = 0$$

$$\Rightarrow \lambda = 1, 6.$$

Eigen vector corresponding to $\lambda = 6$:

$$(A-\lambda I)x = 0 \Rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\Rightarrow \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

linearly independent solution $n - r = 2 - 1 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Eigenvector corresponding to $\lambda = 1$.

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

<u>Theorem</u>: Let $\lambda_1, \lambda_2, \cdots \lambda_k$ be distinct eigenvalues of $n \times n$ matrix. Then corresponding eigenvectors $x_1, x_2, \cdots, x_k$ form a linearly independent set.

<u>Algebraic</u> multiplicity of an eigenvalue is defined as the multiplicity of the corresponding root of the characteristic polynomial.

<u>Geometric</u> multiplicity of an eigenvalue is defined as the number of linearly independent eigenvectors corresponding to that eigenvalue.

$$\boxed{\text{Geometric multiplicity} \leq \text{algebraic multiplicity}}$$

Ex. Determine the characteristic roots and the corresponding characteristic vectors of the matrix:

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol: Characteristic equation:

$$|A-\lambda I| = 0$$

$$\Rightarrow (2-\lambda)(\lambda-2)(\lambda-8) = 0$$

$$\Rightarrow \lambda = 2, 2, 8.$$

algebraic multiplicity of 2 is 2
" " 8 is 1.

Eigenvector corresponding to $\lambda = 8$: $(A-\lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These equations possess $3-2=1$ linearly indepent solution.

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}. \qquad \alpha \neq 0, \ \alpha \in \mathbb{R}.$$

Eigenvector corresponding to $\lambda = 2$: $(A-\lambda I)x = 0$.

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These equations possess $3-1=2$ linearly independent solutions.

$x_3 = \alpha_2$
$x_2 = \alpha_1$

$$x_1 = \frac{1}{2}(\alpha_1 - \alpha_2) \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \underbrace{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}} + \alpha_2 \underbrace{\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}}$$

Ex: Determine the characteristic roots and the corresponding characteristic vectors of the matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Eigenvalues: $\lambda = 2, 2, 3.$

Note: Eigenvalues of a triangular matrix are its diagonal elements.

Eigenspace of $\lambda = 2$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_3 = 0, \quad x_1 = 0, \quad x_2 = \alpha$ free parameter

eigenvector. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

No of linearly independent vectors corresponding to $\lambda = 2$ is one.

Eigenspace of $\lambda = 3$:

$$\begin{bmatrix} -1 & 0 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 \text{ free variable.} \end{array}$$

eigenspace: $\alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

Algebraic multiblicity of $\lambda = 2$: 2

Geometric multiplicity of $\lambda = 2$: 1

Algebraic multiplicity of $\lambda = 3$: 1

Geometric multiplicity of $\lambda = 3$: 1