Lecture 10

(ii) $\Rightarrow$ (iii): Assume (ii). Let $\varepsilon > 0$.

To show: There exists a G_δ-set G, $G \supseteq E$ & $m^*(G \setminus E) = 0$.

by (ii), there exists an open set $U_n \subseteq \mathbb{R}$ such that $U_n \supseteq E$ & $m^*(U_n \setminus E) \leq \frac{1}{n}$. $\forall n$

Let $G = \bigcap_{n=1}^{\infty} U_n$

Then G is a G_δ-set & $E \subseteq G$.

And $m^*(G \setminus E) \leq m^*(U_n \setminus E) \leq \frac{1}{n}$ $\forall n$

$\left(\because G \setminus E \subseteq U_n \setminus E\right)$

$\Rightarrow m^*(G \setminus E) \leq 0$.

$\Rightarrow m^*(G \setminus E) = 0$, as required.

(iii) $\Rightarrow$ (i): Assume (iii). To show: E is measurable.

we have $E = G \setminus (G \setminus E)$

We know that G is measurable

& $m^*(G \setminus E) = 0 \Rightarrow G \setminus E$ is measurable.

$E = G \setminus (G \setminus E) = G \cap (G \setminus E)^c$

is also measurable.

(i) $\Rightarrow$ (ii)*: Assume (i). ie, E is a measurable set.

To show: Given $\varepsilon > 0$, then exists a closed set $F \subseteq E$ such that $m^*(E \setminus F) \leq \varepsilon$.

Let $\varepsilon > 0$.

We have E^c is measurable.

Then by (ii), there exists an open set $U \supseteq E^c$ such that $m(U \setminus E^c) \leq \varepsilon$.

Now $U \setminus E^c = U \cap (E^c)^c$

$= U \cap E$

$= E \cap (U^c)^c$

$= E \setminus U^c$

Let $F = U^c$. F is a closed set.

Then we have $m(E \setminus F) \leq \varepsilon$, as required.

(ii)* $\Rightarrow$ (iii)*: Assume (ii)*. Then for each $n \in \mathbb{N}$,

Let F_n be a closed set, $F_n \subseteq E$ & $m^*(E \setminus F_n) \leq \frac{1}{n}$.

Let $F = \bigcup_{n=1}^{\infty} F_n$. Then F is an F_σ-set

& $F \subseteq E$.

Now $m^*(E \setminus F) \leq m^*(F \setminus F_n) \leq \frac{1}{n}$ $\forall n$

$(\because E \setminus F \subseteq E \setminus F_n)$

$\Rightarrow m^*(E \setminus F) \leq 0$.

Thus $m^*(E \setminus F) = 0$, as required.

$\underline{(iii)^* \Rightarrow (i)}$: Proof is analogue to $\underline{(iii) \Rightarrow (i)}$

EXERCISE.

Qn:- Suppose $E \subseteq \mathbb{R}$ such that $m^*(E) < \infty$. Then when E is measurable ?) Can we give a necessary & sufficient conditions to have that E is measurable. ?

Theorem:- Let $E \subseteq \mathbb{R}$ & $m^*(E) < \infty$. Then E is measurable if and only if

$\ast$ given $\varepsilon > 0$ there exists disjoint finite intervals $I_1, \dots, I_{n_0}$ such that

$$m^*\left(E \,\Delta\, \bigcup_{i=1}^{n_0} I_i\right) < \varepsilon.$$

We may stipulate that intervals I_i be open, closed or half-open.

proof:-

$\Longrightarrow$: Assume E is measurable.

Let $\varepsilon > 0$.

Then there exists an open set $U \supseteq E$ such that $m(U \setminus E) < \frac{\varepsilon}{2}$.

$$\Rightarrow m(U) < \infty \quad (\because m(E) < \infty)$$

$U = E \,\dot\cup\, (U \setminus E)$

$m(U) = m(E) + m(U \setminus E)$

But U is an open set, therefore

$$U = \dot{\bigcup_{i=1}^{\infty}} I_i,$$ disjoint union of open intervals I_i.

$$m(U) = m\left(\dot{\bigcup_{i=1}^{\infty}} I_i\right) = \sum_{i=1}^{n} m(I_i)$$

$$= \sum_{i=1}^{\infty} l(I_i) < \infty$$

$$\Rightarrow \quad \sum_{i=n_0+1}^{\infty} \ell(I_i) < \frac{\varepsilon}{2} \quad \text{for some } n_0 \in \mathbb{N}.$$

Let $V = \bigcup_{i=1}^{n_0} I_i \subseteq \bigcup_{i=1}^{\infty} I_i = U.$

$\therefore V \subseteq U.$

Then $E \Delta V = (E \setminus V) \cup (V \setminus E)$

$\qquad \subseteq (U \setminus V) \cup (U \setminus E)$

$$\therefore m^*(E \Delta V) \le m^*(U \setminus V) + m^*(U \setminus E) \quad \left(\because E \subseteq U,\ V \subseteq U\right)$$

$$= m^*\left(\bigcup_{i=n_0+1}^{\infty} I_i\right) + m^*(U \setminus E)$$

$$\lneq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $m^*\left(E \Delta \bigcup_{i=1}^{n_0} I_i\right) < \varepsilon$, as required.

If we wish the intervals to be, say, half open we first obtain open intervals $I_1, \dots, I_{n_0}$ as above. & then for each i, choose a half-open interval $J_i \subseteq I_i$ such that $m(I_i \setminus J_i) < \frac{\varepsilon}{n_0}$ $\forall\, i = 1, \dots, n$

Then $m\left(E \,\Delta\, \bigcup_{i=1}^{n_0} J_i\right) \leq m\left(E \Delta \bigcup_{i=1}^{n} I_i\right)$

$+ m\left(\left(\bigcup_{i=1}^{n_0} I_i\right) \Delta \left(\bigcup_{i=1}^{n_0} J_i\right)\right)$

(use some properties of Δ)

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

So the construction goes through for the intervals J_i.

Wly for any other type of intervals.

$\Leftarrow$: To show: E is measurable.

Assume (*). given $\varepsilon > 0$. there exists an open set $U \supseteq E$ such that $m^*(U) \leq m^*(E) + \varepsilon$.

To show E is measurable it is sufficient to show $m^*(U \setminus E)$ is sufficiently small. (by above Theorem)

We have (*). ie, there exists intervals $I_1, \ldots, I_{n_0}$ disjoint such that $m^*\left(E \Delta \bigcup_{i=1}^{n_0} I_i\right) < \varepsilon$.

Let $J = \bigcup_{i=1}^{n_0} I_i$. & $V = U \cap J$.

Now $U \Delta E \subseteq (U \Delta V) \cup (V \Delta E)$ (check it!)

$\Rightarrow m^*(U \Delta E) \leq m^*(U \Delta V) + m^*(V \Delta E)$

Since $V \subseteq J$, we have $V \setminus E \subseteq J \setminus E$ ✓

$\parallel^{ly}$ since $E \subseteq U$, we have $E \setminus V = E \setminus J$

Pf:- $a \in E \setminus V$

$\Rightarrow a \in \underset{\cup}{\overset{\cap\!|}{E}}$ & $a \notin V$

$\boxed{\begin{array}{l} V \subseteq U \\ E \setminus V \supseteq E \setminus U = \phi \end{array}}$

$\Rightarrow a \in U$ & $a \notin V = U \cap J$

$\Rightarrow a \notin J$

Thus $a \in E \setminus J$.

$\therefore E \setminus V \subseteq E \setminus J$.

Let $a \in E \setminus J \Rightarrow a \in \underset{V}{\overset{\cap\!|}{E}}$ & $a \notin J$

$\Rightarrow a \in V$ & $a \notin V$

$\Rightarrow a \in E \setminus V$.

$\therefore E \setminus J \subseteq E \setminus V$.

$\therefore E \setminus V = E \setminus J$.

Thus $V \Delta E \subseteq J \Delta E$

$\Rightarrow m^*(U \Delta E) \leq m^*(J \Delta E) < \varepsilon$.

But $E \subseteq V \cup (V \Delta E)$

$$\Rightarrow m^*(E) \leq m^*(V) + m^*(V \triangle E)$$
$$< m(V) + \varepsilon.$$

$$m(U \triangle V) = m\big((U \setminus V) \cup (\underbrace{V \setminus U}_{=\phi})\big)$$
$$= m(U \setminus V)$$
$$= m(U) - m(V)$$
$$\leq m^*(E) + \varepsilon - m(V)$$
$$< \varepsilon + \varepsilon = 2\varepsilon.$$

$$\therefore \quad m^*(U \setminus E) = m^*(U \triangle E) < 3\varepsilon.$$

(Check it!)

$\Rightarrow$ $m^*(U \setminus E)$ is sufficiently small as required.