Lecture 5

Theorem:- For any sequence of subsets $\{E_i\}$ of $\mathbb{R}$, we have $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$. $\quad \boxed{E_i \subseteq \mathbb{R}}$

proof:- Let $\varepsilon > 0$. Then for each i, there exists a sequence of intervals $\{I_{i,j}\}_{j=1,2,\ldots}$ such that $E_i \subseteq \bigcup_{j=1}^{\infty} I_{i,j}$ &

$$m^*(E_i) + \frac{\varepsilon}{2^i} \geq \sum_{j=1}^{\infty} l(I_{i,j})$$

$$\begin{array}{l} I_{1,1},\ I_{1,2},\ \cdots \\ I_{2,1},\ I_{2,2},\ \cdots \\ \vdots \end{array}$$

Now $\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty}\left(\bigcup_{j=1}^{\infty} I_{i,j}\right)$.

& $\{I_{i,j}\}_{i,j}$ is a countable class of finite intervals covering $\bigcup_{i=1}^{\infty} E_i$

$$\begin{aligned}
\therefore m^*\left(\bigcup_{i=1}^{\infty} E_i\right) &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} l(I_{i,j}) \\
&\leq \sum_{i=1}^{\infty}\left(m^*(E_i) + \frac{\varepsilon}{2^i}\right) \\
&= \sum_{i=1}^{\infty} m^*(E_i) + \sum_{i=1}^{\infty}\frac{\varepsilon}{2^i} \\
&= \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon.
\end{aligned}$$

Thus $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon$

True for any $\varepsilon > 0$.

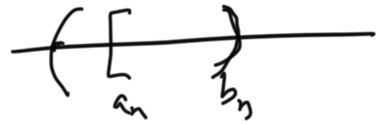
$$\therefore\ m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i).$$

<u>Proposition</u>:- For any $A \subseteq \mathbb{R}$, & $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}$ such that $U \supseteq A$ & $m^*(U) \leq m^*(A) + \varepsilon$.

<u>Proof</u>:- Let $\varepsilon > 0$. Then there exists intervals $\{I_n\}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ &

$$m^*(A) + \frac{\varepsilon}{2} \geq \sum_{n=1}^{\infty} \ell(I_n). \longrightarrow (*)$$

Let $I_n = [a_n, b_n)$



Set $I_n' = \left(a_n - \frac{\varepsilon}{2^{n+1}}, b_n\right)$

$I_n \subseteq I_n'$ $\forall n$. & I_n' are open.

Let $U = \bigcup_{n=1}^{\infty} I_n'$. Then U is an open set.

Also, $A \subseteq \bigcup_{n=1}^{\infty} I_n' = U$

Now $m^*(U) = m^*\left(\bigcup_{n=1}^{\infty} I_n'\right) \leq \sum^{\infty} \ell(I_n')$.

(by above Thm)

$$= \sum_{n=1}^{\infty}\left(b_n - a_n + \frac{\varepsilon}{2^{n+1}}\right)$$

$$= \sum_{n=1}^{\infty}(b_n - a_n) + \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}}.$$

$$= \sum_{n=1}^{\infty} l(I_n) + \frac{\varepsilon}{2}$$

$$\leq m^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

(by (*))

Thus $m^*(U) \leq m^*(A) + \varepsilon.$

<u>Remark</u> $m^*(A) \leq m^*(U)$.

<u>Remark</u> $m^*(U) = m^*(A)$ $\exists$ open set $U \supseteq A$.

<u>Proposition</u>:– Let $E \subseteq \mathbb{R}$. In the definition of outer measure $m^*(E) := \inf_{A \subseteq \cup I_n}\left(\sum_{n=1}^{\infty} l(I_n)\right)$

we stipulate (i) I_n open

(ii) $I_n = [a_n, b_n)$

(iii) $I_n = (a_n, b_n]$

(iv) I_n closed

or (v) mixture of all above,

for different values of n of the various types of intervals.

Then the same m^* is obtained.

proof:- Our definition of m^* in the case (ii).

Write the corresponding m^* as m_o^* in case (i), m_{oc}^* in case (iii), m_c^* in case (iv), m_m^* in case (iv).

We show that each equals to m_m^*.

Consider m_0^*, the proof is similar in other cases.

To show: For $E \subseteq \mathbb{R}$, $m_o^*(E) = m_m^*(E)$.

From the definition, we have

$$m_m^*(E) \leq m_o^*(E).$$

Let $\varepsilon > 0$. Then there exists $\{I_n\}$ intervals (mixture) such that $E \subseteq \bigcup_{n=1}^{\infty} I_n$ &

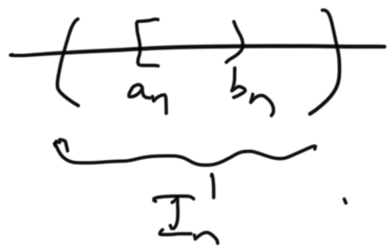
$$m_m^*(E) \geq \sum_{n=1}^{\infty} l(I_n) - \varepsilon.$$

Let I_n' be an open interval containing I_n with $l(I_n') = (1+\varepsilon)\, l(I_n)$.

Say for example, $I_n = [a_n, b_n]$ or $[a_n, b_n)$ or $(a_n, b_n]$, (a_n, b_n), choose

$$I_n' = \left(a_n - \frac{(b_n - a_n)}{2}\varepsilon,\ b_n + \frac{(b_n - a_n)}{2}\varepsilon\right).$$

We have $I_n \subseteq I_n' \quad \forall n$



$$\therefore \; m_m^*(E) + \varepsilon \geq \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} l(I_n')\,(1+\varepsilon)^{-1}.$$

$$\Rightarrow \left(m_m^*(E) + \varepsilon\right) \geq (1+\varepsilon)^{-1} \sum_{n=1}^{\infty} l(I_n'). \longrightarrow (*)$$

Also we have $E \subseteq \bigcup_{n=1}^{\infty} I_n'$ & I_n' open,

Therefore $\boxed{m_0^*(E) \leq \sum_{n=1}^{\infty} l(I_n')}$ (by def.)

$$\leq (1+\varepsilon)\, m_m^*(E) + \varepsilon(1+\varepsilon).$$
by $(*)$.

Since $\varepsilon > 0$ is arbitrary, then we get

$$m_0^*(E) \leq m_m^*(E).$$

Hence $m_0^*(E) = m_m^*(E)$.

Remaining proof : EXERCISE.

Definition:- Let $E \subseteq \mathbb{R}$. Then we say E is **Lebesgue measurable** or simply **measurable** if for each $A \subseteq \mathbb{R}$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Remark:- Since m^* is subadditive & $A = (A \cap E) \cup (A \cap E^c)$, therefore we have

$$m^*(A) \le m^*(A \cap E) + m^*(A \cap E^c).$$

Thus E is measurable $\Longleftrightarrow$

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

proposition:- Let $E \subseteq \mathbb{R}$ & $m^*(E) = 0$. Then E is measurable.

proof:- To show: for $A \subseteq \mathbb{R}$,

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

But

$$m^*(A \cap E) \le m^*(E) = 0.$$

$$\Rightarrow m^*(A \cap E) = 0.$$

Also $A \cap E^c \subseteq A$. $\Rightarrow m^*(A \cap E^c) \le m^*(A)$

$$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

//.

$\therefore$ E is measurable.

Examples:- $\{x\}$, $\mathbb{Q}$, any finite set are measurable

There exist a non-measurable subset of $\mathbb{R}$.

proof:- Later.