# Group Theory

Lecture 7

Let $G$ be a group and $H$ is a subgp of $G$. We define for a $g \in G$ [ $g+H = \{g+h \mid h \in H\}$ if written additively ]

$gH = \{gh \mid h \in H\}$ = a left coset of $H$.

$Hg = \{hg \mid h \in H\}$ = a right of $H$.

<u>Example</u> $G = \mathbb{Z}$, and $H = 5\mathbb{Z}$.

Let $1 \in \mathbb{Z}$.

$1+H = \{1+5n \mid n \in \mathbb{Z}\}$.

$2+H = \{2+5n \mid n \in \mathbb{Z}\}$.

$3+H = \{3+5n \mid n \in \mathbb{Z}\}$.

$4+H = \{4+5n \mid n \in \mathbb{Z}\}$

$5+H = \{5+5n \mid n \in \mathbb{Z}\} = 0+H$.

$6+H = 1+H$.

Example. $G = S_3$ and $H = \{(1), (12)\}$.

The left cosets of $H$ are

$$\begin{cases} (13)H = \{(13), (123)\}. \\ (23)H = \{(23), (132)\}. \\ (12)H = H. \end{cases}$$

$$(123)H = \{(123), (13)\} = (13)H.$$

$$(132)H = \{(132), (23)\} = (23)H.$$

Here we have 3 distinct left cosets.

$$H(13) = \{(13), (132)\}$$

Here left coset $\neq$ right coset.

**Lemma** Let $G$ be a finite gp and $H$ is a subgp of $G$. Then cardinality of any left coset = cardinality of any righ coset $= |H|$.

**Pf:** $f : H \longrightarrow aH$

$$f(h) = ah$$

$f$ is onto.

Let $f(h_1) = f(h_2)$.

$$\Rightarrow ah_1 = ah_2$$

$$\Rightarrow h_1 = h_2.$$

$\therefore$ $f$ is injective.

Thus $f$ is a bijection $\therefore$ $|H| = |aH|$

$$= |Ha|$$

Question Relation between the order of a gp and order of a subgp.

Let us define an equivalance relation on $G$ as let $a, b \in G$. then $a \sim b$ if $a = bh$ for some $h \in H$.

check that $\sim$ is an equivalance relation. Note that all the elemts in the same left coset are equivalant. Thus the lefts cosets are equivalance classes.

Therefore $aH = bH$ or $aH \cap bH = \phi$.

and $G$ is a disjoint union of the left cosets.

Let $g_1, \dots, g_r \in G$ s.t

$g_1 H, \dots, g_r H$ are distinct

all the distinct left cosets of $H$.

Then $G = g_1 H \cup \dots \cup g_r H$.

Since $|g_i H| = |H|$.

we have $|G| = r|H|$.

$\Rightarrow |H| \mid |G|$.

$r$ = no. of distinct left cosets of $H$

= index of the subgp $H$ in $G$.

$= [G : H]$.

**Lagrange's Thm** Let $G$ be a finite group and $H$ is a subgp of $G$. Then order of $H$ divides $|G|$ and

$$\frac{|G|}{|H|} = \text{no. of left cosets of } H$$
$$= \text{no. of right cosets of } H$$
$$= [G:H].$$

**Remark** The converse of Lagrange's Thm is not true in general.

**Defn.** The number of left cosets of a subgp $H$ of a gp $G$ = index of $H$ in $G$ $= [G:H]$.

Example (1) Even if $G$ is infinite but $[G:H]$ can be finite.

Consider $G = \mathbb{Z}$ and $H = n\mathbb{Z}$.

Then $[G:H] = n$. The left cosets are $n\mathbb{Z}, 1+n\mathbb{Z}, 2+n\mathbb{Z}, \ldots, (n-1)+n\mathbb{Z}$.

(2) $G = S_n$. and $H = A_n$ = even perm

$[S_n : A_n] = 2$.

Cor Let $G$ be a finite gp and $a \in G$. then $|a| \,\Big|\, |G|$ and $a^{|G|} = 1$.

Ans. Let $H = \langle a \rangle$ be the cyclic subgp of $G$ gen by $a$.

Then $|H| = |a|$.

By Lagrange's Thm $|a| \,\Big|\, |G|$.

**Cor** Let $G$ be a gp of prime order then $G$ is cyclic gp.

**Pf**: Let $1 \neq a \in G$. Consider the cyclic subgp $H = \langle a \rangle$ of $G$.

Then by Lagrange's Thm $|H| \,\Big|\, |G|$.

But $|G| = p$ where $p$ is a prime no.

Hence $|H| = p$. Therefore $H = G$

i.e $G$ is a cyclic gp.

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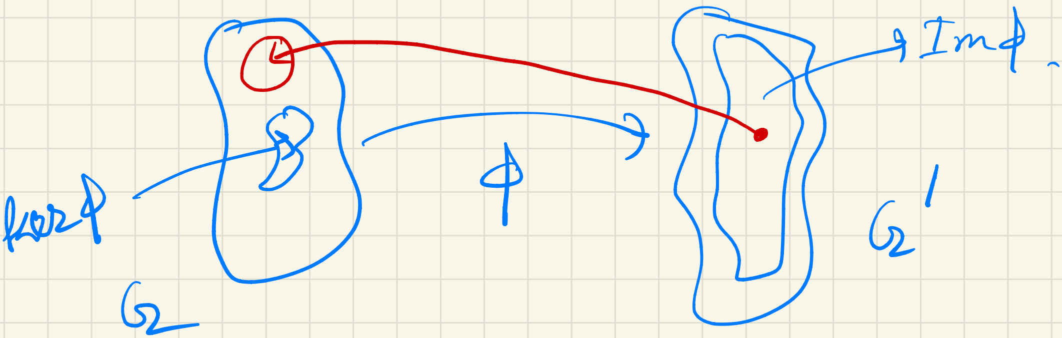
Is $gH$ a subgp of $G$?

No in general but if $g \in H$

then $gH = H$.

<u>Propn.</u> Let $\phi : G \to G'$ is a gp homomorphism of finite gps.

$$|G| = |\ker\phi| \, |\mathrm{Im}\,\phi|.$$




$H = \ker\phi$ is a subgp of $G$.

$gH$ is a left coset of $H$.

Let $gh_1$ & $gh_2 \in gH$ when $h_1, h_2 \in H$

$$\phi(gh_1) = \phi(g)\phi(h_1) = \phi(g)$$

$$\phi(gh_2) = \phi(g)\phi(h_2) = \phi(g).$$

The left cosets are the fibers of the map $\phi$.

We have seen that left cosets of $\ker \phi$ are the fibers of the map $\phi$ i.e There is a bijection betwen the no. of elts in the $\operatorname{im} \phi$ and the no. of left cosets of $\ker \phi$.

i.e $|\operatorname{Im} \phi| = [G : \ker \phi]$

$$= \frac{|G|}{|\ker \phi|}$$

$$\Rightarrow \quad |G| = |\ker \phi| \cdot |\operatorname{Im} \phi|.$$