# Line Integrals:

## Definitions:

**Smooth curves:** Let $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ denote the position vector of a point $P(x,y,z)$ in three dimensional space.

If $\bar{r}(t)$ posses a continuous first order derivative for all values of t under consideration then the curve is known as smooth.

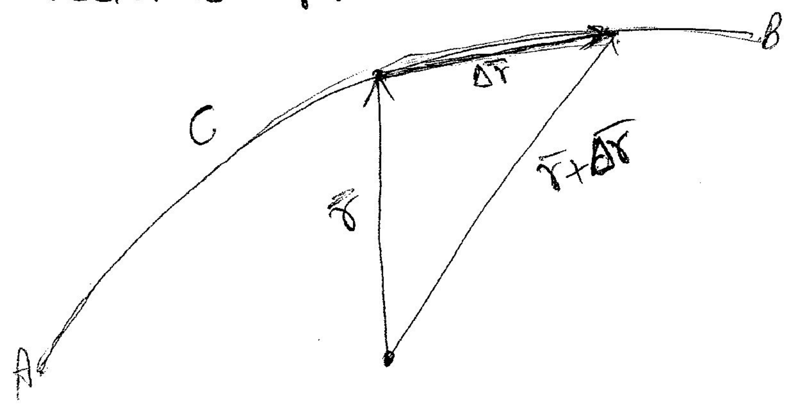
piecewise smooth if it is madeup of a finite number of smooth curves.

**Simple Closed curve:** A closed smooth curve which does not intersect itself anywhere is known as simple closed curve.

**Smooth surfaces:** A surface $\bar{r} = \bar{f}(u,v)$ is said to be smooth if $\bar{f}(u,v)$ possesses continuous first order partial derivatives.

## Line integrals: (work done by a force).

Let a force $\bar{F}$ act upon a particle which is displaced along a given curve C in space from the point P whose position vector is $\bar{r}$.



first divide the curve C into a large number of small pieces.

Consider the work done when the particle moves from the position $\bar{r}$ to $\bar{r}+\Delta\bar{r}$.

On this small section of the curve C the work done is $\bar{F}\cdot\Delta\bar{r}$

Total work done $W \simeq \sum_{i=1}^{N} \bar{F}_i \cdot \Delta\bar{r}_i$

The line integral is defined as:

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \lim_{N\to\infty} \sum_{i=1}^{N} \bar{F}_i \cdot \Delta\bar{r}_i}$$

Evaluation of the line integral:

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t)) \cdot \frac{d\bar{r}}{dt}\, dt}$$

In component form: $\bar{F}(r) = \hat{i}\, F_1(x,y,z) + \hat{j}\, F_2(x,y,z) + \hat{k}\, F_3(x,y,z)$

$d\bar{r} = \hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz$. Then

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \int_C F_1\, dx + F_2\, dy + F_3\, dz}$$

**Example:** Find the work done by $\bar{F} = (y-x^2)\bar{i} + (z-y^2)\bar{j} + (x-z^2)\bar{k}$ over the curve $\bar{r}(t) = t\,\hat{i} + t^2\hat{j} + t^3\bar{k}$, $0 \le t \le 1$ from $(0,0,0)$ to $(1,1,1)$.

**Solution:**

$$\frac{d\bar{r}}{dt} = \hat{i} + 2t\,\hat{j} + 3t^2\,\bar{k}$$

$$\bar{F}(\bar{r}t) = (t^2 - t^2)\,\hat{i} + (t^3 - t^4)\,\hat{j} + (t - t^6)\,\bar{k}$$

$$= (t^3 - t^4)\,\hat{j} + (t - t^6)\,\bar{k}$$

$$\bar{F} \cdot \frac{d\bar{r}}{dt} = 2t(t^3 - t^4) + 3t^2(t - t^6)$$

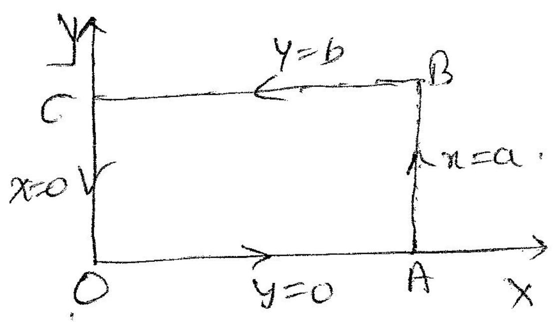
$$= 2t^4 - 2t^5 + 3t^3 - 3t^8$$

$$\int_C \bar{F} \cdot d\bar{r} = \int_{t=0}^{1} (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt$$

$$= \frac{29}{60}. \qquad \text{Ans.}$$

**Example: 2:** Evaluate $\int_C \bar{F} \cdot d\bar{r}$ $\quad \bar{F} = (x^2+y^2)\,\hat{i} - 2xy\,\hat{j}$

C: rectangle in $xy$ plane bounded by ~~$y=0, y=b, x=0, x=a$~~
$y=0, x=a, y=b, x=0$




$$\int_C \vec{F} \cdot d\bar{r} = \int_C \left( (x^2+y^2)\,\hat{i} - 2xy\,\hat{j} \right) \cdot \left( dx\,\hat{i} + dy\,\hat{j} \right).$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \left[(x^2+y^2)\,dx - 2xy\,dy\right].$$

Along OA : $y=0$, $dy=0$ & $x$ varies from 0 to $a$.

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_0^a x^2\,dx = \frac{a^3}{3}.$$

Along AB : $\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^b -2a\cdot y\,dy = -ab^2.$

Along BC : $\int_{BC} \bar{F}\cdot d\bar{r} = \int_a^0 (x^2+b^2)\,dx$

$$= -\left[\frac{a^3}{3} + ab^2\right].$$

Along DO: $\int_{CO} \bar{F}\cdot d\bar{r} = \int_b^0 0\cdot dy = 0$

$$\Rightarrow \int_C \bar{F}\cdot d\bar{r} = -2ab^2$$

Ans.

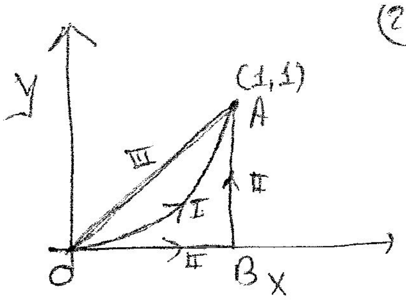
<u>Example:</u> If $\bar{F} = y\hat{i} - x\hat{j}$ Evaluate $\int_C \bar{F}\cdot d\bar{r}$ from $(0,0)$ to $(1,1)$ along the following ~~that~~ path:

i) the parabola $y = x^2$

ii) the straight line $(0,0)$ to $(1,0)$ and then to $(1,1)$

iii) the straight line joining $(0,0)$ to $(1,1)$.

Solution:




$$\bar{r} = x\hat{i} + y\hat{j}$$

$$d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (y\hat{i} - x\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C y\,dx - x\,dy.$$

i) parabola $y = x^2 \Rightarrow dy = 2x\,dx$. $x$ varies from 0 to 1.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 x^2\,dx - x\cdot 2x\,dx$$

$$= \int_0^1 -x^2\,dx = -\frac{1}{3}.$$

ii) along OB & then BA:

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OB} (y\,dx - x\,dy) + \int_{BA} (y\,dx - x\,dy)$$

along OB, $y = 0$ $dy = 0$

along BA $x = 1$ $dx = 0$,

$$\int_C \bar{F}\cdot d\bar{r} = \int_{y=0}^{1} -dy = -1.$$

iii) Along straight line $\bar{O}A$. along $\bar{O}A$ $y = x$.
$dy = dx$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (x\,dx - x\,dx) = 0.$$

Ans.

Example: If $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$ evaluate $\int_c \bar{F}\cdot d\bar{r}$ where $c$ is the straight line joining $(0,0,0)$ to $(1,1,1)$.

Solution: Equation of the line:

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \text{ (parameter)}$$

$$x=t \quad y=t \quad z=t$$

$$\bar{r} = t\hat{i} + t\hat{j} + t\bar{k}$$

$$\frac{d\bar{r}}{dt} = (\hat{i}+\hat{j}+\bar{k})$$

$$F = (3t^2+6t)\hat{i} - 14t^2\hat{j} + 20t^3\bar{k}$$

$$\int_c \bar{F}\cdot d\bar{r} = \int_0^1 [(3t^2+6t)\hat{i} - 14t^2 j + 20t^3\bar{k}]\cdot[\hat{i}+\hat{j}+\bar{k}]\,dt$$

$$= \int_0^1 [3t^2+6t - 14t^2 + 20t^3]\,dt$$

$$= \int_0^1 [-11t^2 + 6t + 20t^3]\,dt$$

$$= \frac{-11}{3} + \frac{6}{2}{}^{3} + \frac{20}{4}{}^{5}$$

$$= \frac{13}{3}. \quad \text{Ans}$$

Example: Find the total work done in moving a particle in a force field $\bar{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x=t^2+1$, $y=2t^2$ $z=t^3$ from $t=1$ to $2$.

Solution:

$$\int_c \bar{F}\cdot d\bar{r} = \int_c \cancel{3xy}\; 3(t^2+1)(2t^2)(2t) - 5t^3(4t) + 10(t^2+1)3t^2\,dt$$

$$= 303 \quad \text{Ans.}$$

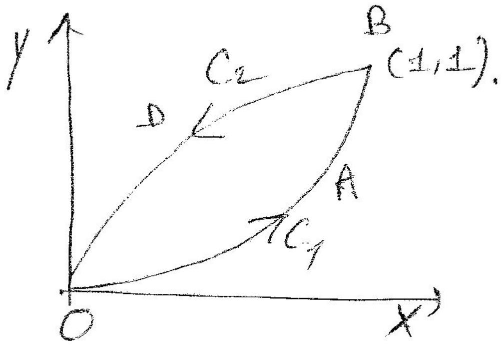
Note: The integral around a closed curve, $\oint$, is called Circulation integral.

Example: Find the circulation of F around the curve C where $\bar{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$ and C in the curve $y = x^2$ for $(0,0)$ to $(1,1)$ and the curve $y^2 = x$ from $(1,1)$ to $(0,0)$.

Solution: $\bar{r} = xi + yj$

$d\bar{r} = i\,dx + j\,dy$

$F \cdot d\bar{r} = (2x+y^2)dx + (3y-4x)dy$




$\int_C \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r}$

Along OAB: $y = x^2$ $dy = 2x\,dx$.

$\int_{C_1} F\cdot d\bar{r} = \int_{C_1} (2x+x^4)dx + (3.x^2 - 4x)\cdot 2x\,dx$.

$= \int_0^1 (x^4 + 6x^3 - 8x^2 + 2x)\,dx$

$= \left[\frac{x^5}{5} + 6\frac{x^4}{4} - 8\cdot\frac{x^3}{3} + 2\frac{x^2}{2}\right]_0^1$

$= \frac{1}{30}$.

Along $C_2$: $x = y^2$ $dx = 2y\,dy$. $y$ varies from 0 to 1.

$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_{y=1}^{0} (2y^2+y^2)\,2y\,dy + (3y - 4.y^2).dy$.

$= -\int_0^1 6y^3 - 4y^2 + 3y$

$= -5/3$.

$\therefore \int_C \bar{F}\cdot d\bar{r} = \frac{1}{30} - \frac{5}{3} = -\frac{49}{30}$ Ans.

## Evaluation of the line integral:

$$\int_C \bar{F} \cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t)) \cdot \frac{d\bar{r}}{dt}\, dt$$

OR

$$\int_C \bar{F} \cdot d\bar{r} = \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$

Path independence:

Let $\bar{F}$ be a vector field defined on an open region $D$ and suppose that for any two points $A$ and $B$ in $D$ the integral

$\int_A^B \bar{F}\cdot d\bar{r}$ is the same over all paths from $A$ to $B$.

Then the integral $\int \bar{F}\cdot d\bar{r}$ is path independent in $D$ and

Fundamental theorem of line integral:

1. Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be a vector field whose components are continuous throughout an open connected region $D$ in space. Then there exists a differentiable function $f$ such that

$$\bar{F} = \nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k} \quad \left(\text{or } F \text{ is conservative on } D\right).$$

if and only if for all points $A$ and $B$ in $D$ the value of $\int_A^B \bar{F}\cdot d\bar{r}$ is independent of the path joining $A$ to $B$ in $D$.

Proof: $\bar{F} = \nabla f \Rightarrow$ path independence.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$= \overline{\nabla f}\cdot\frac{d\bar{r}}{dt} = \bar{F}\cdot\frac{d\bar{r}}{dt}$$

Therefore

$$\int_C F\cdot d\bar{r} = \int_{t=a}^{b} F\cdot\frac{d\bar{r}}{dt}\,dt = \int_a^b \frac{df}{dt}\cdot dt = f\Big|_a^b = f(b)-f(a)$$

2. If the integral is independent of the path from $A$ to $B$ its value is

$$\int_A^B \bar{F}\cdot d\bar{r} = f(B) - f(A)$$

**Theorem 2:** The following statements are equivalent:

1. $\int \bar{F}\cdot d\bar{r} = 0$ around every closed loop in $D$.

2. The field $F$ in conservative on $D$.

## Component test for conservative field:

Let $\bar{F} = F_1(x,y,z)\,i + F_2(x,y,z)\,j + F_3(x,y,z)\,k$ be a vector field whose components have continuous first partial derivatives. Then, $\bar{F}$ is conservative iff

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

**Proof:**

$$F = F_1\bar{i} + F_2\bar{j} + F_3\bar{k} = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\Rightarrow \quad \frac{\partial F_3}{\partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$$

$$= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}\cdot F_2$$

Similarly others relation can be prooved. □

**Example:** Show that $F = (e^x \cos y + yz)\bar{i} + (xz - e^x \sin y)\bar{j} + (xy + z)\bar{k}$ is conservative and find a potential function for it.

**Solution:**

$$F_1 = e^x \cos y + yz \qquad F_2 = (xz - e^x \sin y)$$

$$F_3 = (xy + z)$$

$$\frac{\partial F_3}{\partial y} = x = \frac{\partial F_2}{\partial z}$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = -e^x \sin y + z$$

$$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} = y.$$

$\Rightarrow$ $\bar{F}$ is conservative that is $\bar{F} = \nabla \varphi$.

$$\Rightarrow \frac{\partial \varphi}{\partial x} = e^x \cos y + yz \quad (1) \qquad \frac{\partial \varphi}{\partial y} = xz - e^x \sin y. \quad (2)$$

$$\frac{\partial \varphi}{\partial z} = xy + z \quad (3)$$

$$(1) \Rightarrow \varphi = e^x \cos y + xyz + h(y,z)$$

$$(2) \Rightarrow -e^x \sin y + xz + \frac{\partial h}{\partial y} = xz - e^x \sin y$$

$$\Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow h = h(z).$$

$$(3) \qquad xy + \frac{dh}{dz} = xy + z$$

$$\Rightarrow \frac{dh}{dz} = z \qquad \Rightarrow h(z) = \frac{z^2}{2} + C.$$

$$\Rightarrow \varphi = e^x \cos y + xyz + \frac{z^2}{2} + C.$$

<u>Example:</u> Show that $\bar{F} = (2x-3)\bar{i} - z\bar{j} + \cos z\,\bar{k}$ is not conservative.

<u>Solution:</u> $\frac{\partial F_3}{\partial y} = 0 \neq \frac{\partial F_2}{\partial z} = -1.$

$\Rightarrow$ $\bar{F}$ is not conservative. $\square$.

## Green's theorem in the plane:

(Transformation between double integrals and line integrals)

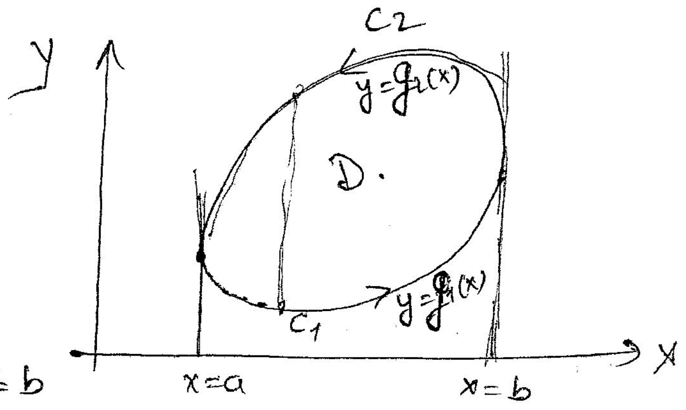
If $D$ is a region in $\mathbb{R}^2$, $\partial D$ is its boundary, oriented in the positive sence, and $F_1$ & $F_2$ are $C^1$ functions of $x$ and $y$ then

$$\boxed{\oint_{\partial D} F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy.} \quad (1)$$

Note that equation (1) can be written as

$$\boxed{\oint_{\partial D} \bar{F}\cdot d\bar{r} = \iint_D (\operatorname{curl} \bar{F})\cdot \bar{K}\, dx\,dy}$$

Proof: Let $C$ be a smooth simple closed curve in the $xy$ plane with the property that lines parallel to axes cut it in no more than two points:




$C_1$: $y = g_1(x)$ $\quad a \le x \le b$

$C_2$: $y = g_2(x)$ $\quad b \ge x \ge a$. $\qquad C = C_1 + C_2$

Integrate $\frac{\partial F_1}{\partial y}$ with respect to $y$ from $y = g_1(x)$ to $y = g_2(x)$

$$\int_{g_1(x)}^{g_2(x)} \frac{\partial F_1}{\partial y}\, dy = F_1(x,y)\Big|_{g_1(x)}^{g_2(x)} = F_1(x, g_2(x)) - F_1(x, g_1(x)).$$

Now integrate with respect to $x$ from $a$ to $b$:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial F_1}{\partial y}\, dy\, dx = \int_a^b \left[F_1(x, g_2(x)) - F_1(x, g_1(x))\right] dx$$

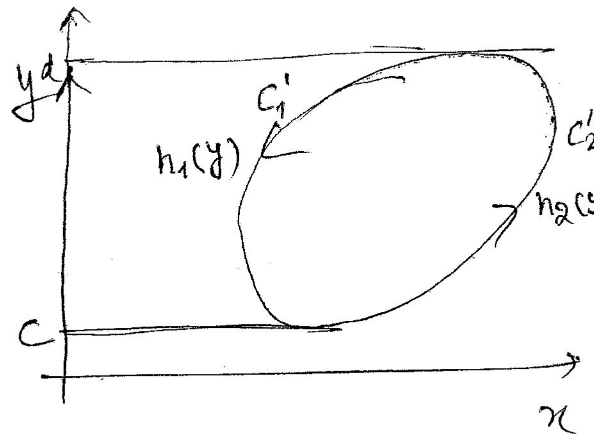
$$= -\int_b^a F_1(x, g_2(x))\,dx - \int_a^b F_1(x, g_1(x))\,dx$$

$$= -\int_{C_2} F_1\,dx - \int_{C_1} F_1\,dx$$

$$= -\oint_{C \text{ OR } \partial D} F_1\,dx .$$

$$\Rightarrow \oint_C F_1\,dx = \iint_D \left(-\frac{\partial F_1}{\partial y}\right) dx\,dy. \quad \text{— (1)}$$

Now we derive the other half by integrating $\frac{\partial F_2}{\partial x}$ first with respect to $x$ and then with respect to $y$.




$C_1'$ : $x = h_1(y)$ $d \geq y \geq c$

$C_2'$ : $x = h_2(y)$ $c \leq y \leq d$.

$$\int_{h_1(y)}^{h_2(y)} \frac{\partial F_2}{\partial x}\,dx = F_2(h_2(y), y) - F_2(h_1(y), y)$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial F_2}{\partial x}\,dx\,dy = \int_c^d F_2(h_2(y), y)\,dy - \int_c^d F_2(h_1(y), y)\,dy$$

$$= \oint_C F_2\,dy . \quad \text{— (2)}$$

combining equation (1) & (2).

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C F_1\,dx + F_2\,dy.$$

$\square$

Example: Verify Green's theorem for the vector field:

$$\bar{F}(x,y) = (x-y)\hat{i} + x\hat{j}$$

The region $R$ is bounded by the circle

$C$: $r(t) = \cos t\, i + \sin t\, j \quad 0 \leq t \leq 2\pi$

Solution: $F_1 = (x-y) \quad F_2 = x.$

$x = \cos t \quad y = \sin t$

$dx = -\sin t\, dt \quad dy = \cos t\, dt$

$\frac{\partial F_1}{\partial y} = -1 \quad \frac{\partial F_2}{\partial x} = 1.$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = 2 \iint_R dx\,dy. = 2\pi$$

$$\oint_C F_1 dx + F_2 dy = \oint_C \bar{F} \cdot d\bar{r} = \int_{t=0}^{2\pi} \left[(\cos t - \sin t)i + \cos t\, j\right] \cdot \left[-\sin t\, i + \cos t\, j\right] dt$$

$$= \int_0^{2\pi} \left[-\cos t \sin t + \sin^2 t + \cos^2 t.\right] dt$$

$$= 2\pi - \frac{1}{2}\int_0^{2\pi} \sin 2t\, dt$$

$$= 2\pi + \frac{1}{2}\left[\cos 2t\right]_0^{2\pi}$$

$$= 2\pi$$

Verified.

Example: Evaluate the integral

$$\oint_C xy\,dy - y^2\,dx$$ using Green's theorem.

Here C is the square cut from the first quadrant by the lines $x=1$ and $y=1$.

Solution:

$$\oint_C F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \int_0^1\int_0^1 \{y - (-2y)\}\,dx\,dy$$

$$= \int_0^1\int_0^1 3y\,dx\,dy$$

$$= \frac{3}{2}.$$ ANS.

Example: Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C x\,dy - y\,dx$.

Solution:

Using Green's theorem:

$$\underbrace{\oint_C x\,dy - y\,dx}_{F_2\,dy + (-F_1)\,dx} = \iint_R \underbrace{[1-(-1)]}_{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}\,dx\,dy = 2A.$$

$$\Rightarrow \boxed{A = \frac{1}{2}\oint x\,dy - y\,dx.}$$

Example: Find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$

Solution:

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_{\theta=0}^{2\pi} (a\cos\theta)(b\cos\theta)\,d\theta - (b\sin\theta)(-a\sin\theta)\,d\theta$$

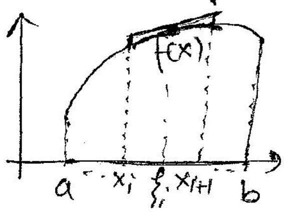
$$= \frac{1}{2}\int_0^{2\pi} ab\,(\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \pi ab.$$ Ans.

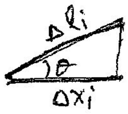
# Surface integrals:

## I) Surface area:

Recall evaluation of arc lenght




$$L = \int_C dl = \int_a^b \underbrace{\sqrt{1+f_x^{2}}}_{\frac{1}{\cos\theta}} \cdot dx$$




$$\frac{\Delta x_i}{\Delta l_i} = |\cos\theta|$$

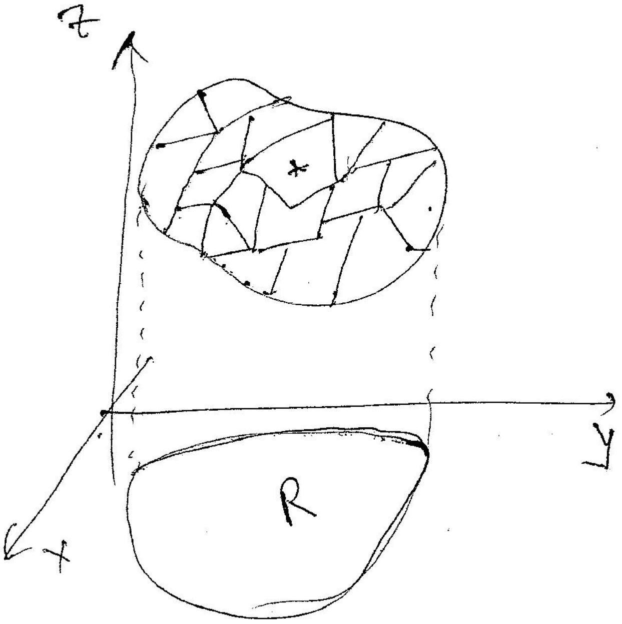
or. $\boxed{\Delta l_i = \frac{1}{|\cos\theta|} \cdot \Delta x_i}$

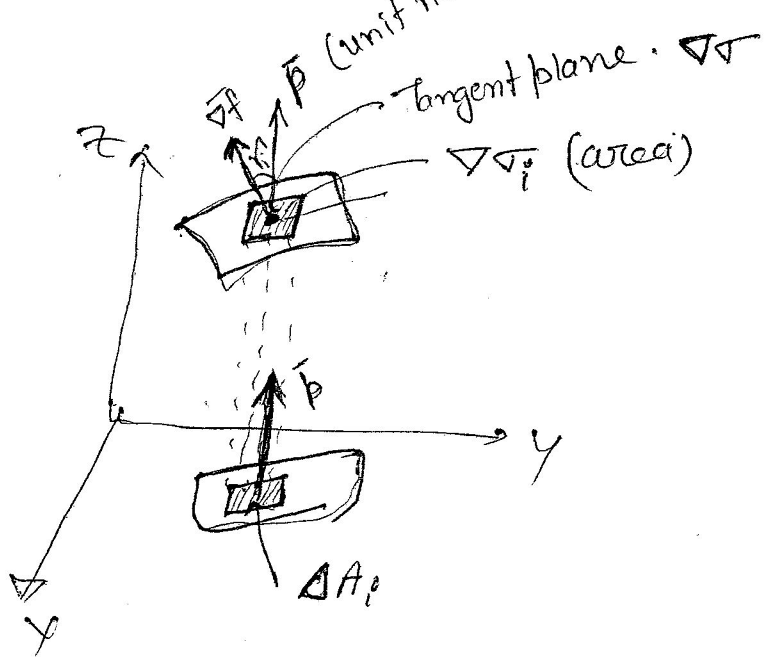
$$L = \int_C dl = \int_a^b \frac{1}{|\cos\theta|} dx$$

Arc length differential

$$dl = \frac{1}{|\cos\theta|} dx$$

Similarity for surface area:




$$\frac{\Delta A_i}{\Delta \sigma_i} = |\cos\gamma_i|$$

$$\Delta \sigma_i = \frac{1}{|\cos\gamma_i|} \Delta A_i$$

see also (391)

Surface area $= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta\sigma_i = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{|\cos\gamma_i|} \Delta A_i$

$$= \iint_R \frac{1}{|\cos\gamma|}\, dA.$$

where $R$ is the projection of the surface on the $xy$, $yz$ or $zx$ plane.

Also Note that

$$|\nabla f \cdot \bar{p}| = |\nabla f| \cdot |\bar{p}|\, |\cos\gamma|$$

$$\Rightarrow \frac{1}{|\cos\gamma|} = \frac{|\nabla f|}{|\nabla f \cdot \bar{p}|}$$

§: The area of the surface $f(x,y,z)=c$ over a closed and bounded plane region $R$ is

$$\text{Surface area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \bar{p}|}\, dA = \iint_S d\sigma$$

where $p$ is a unit normal to $R$ and $\nabla f \cdot \bar{p} \neq 0$

Example: Find the area of the cap cut from the hemisphere $x^2+y^2+z^2=2$, $z\geq 0$ by the cylinder $x^2+y^2=1$.

Solution: Projection of the surface $f(x,y,z)=c$ ie $x^2+y^2+z^2=2$ onto the $xy$ plane: $x^2+y^2\leq 1$.

$f(x,y,z) = x^2+y^2+z^2-2=0$

$\nabla f = 2x\bar{i}+2y\bar{j}+2z\bar{k}$

$|\nabla f| = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{2}$

Extra work.

Earlier formula to calculate surface area of $z = g(x,y)$ was:

$$S = \iint_R \sqrt{1 + z_x^2 + z_y^2} \, dx\,dy.$$

In the vector form we got.

$$S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \bar{p}|} \cdot \underbrace{dA}_{dx\,dy}$$

Let $f = z - g(x,y) = 0$.

$$\nabla f = (-g_x, -g_y, 1)$$

$$|\nabla f| = \sqrt{z_x^2 + z_y^2 + 1}$$

$$|\nabla f \cdot \bar{p}| = 1.$$

So. $S = \iint_R \sqrt{1 + z_x^2 + z_y^2} \cdot dx\,dy.$

The vector $\bar{p} = \bar{k}$ is normal to the surface.

$$|\nabla f \cdot p| = |2z| = 2z.$$

$$\text{surface area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \bar{k}|} \cdot dA$$

$$= \iint_A \frac{2\sqrt{2}}{2z} \cdot dA = \sqrt{2} \iint_R \frac{dA}{z}.$$

$$= \sqrt{2} \iint_R \frac{dA}{\sqrt{2-(x^2+y^2)}}$$

$$= \sqrt{2} \cdot \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{r\,dr\,d\theta}{\sqrt{2-r^2}}$$

$$= \sqrt{2} \int_0^{2\pi} \left[-(2-r^2)^{\frac{1}{2}}\right]_{r=0}^{1} d\theta$$

$$= \sqrt{2} \int_0^{2\pi} (\sqrt{2}-1)\,d\theta$$

$$= 2\pi\,(2-\sqrt{2}).$$ Ans.

Surface integrals: Integrating a function over a surface using the idea just developed for calculating surface area.

Suppose, for example, we have electrical charge distributed over a surface $f(x,y,z)=c$. Let the function $g(x,y,z)$ gives the charge per unit area (charge density) at each point on S.

$$\text{Total charge on } S = \iint_S g(x,y,z)\, d\sigma$$

$$= \iint_R g(x,y,z) \frac{dA}{|\cos \gamma|}$$

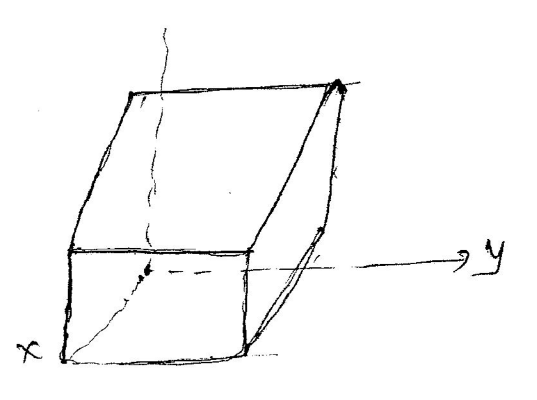
$$= \underbrace{\iint_R g(x,y,z) \frac{|\nabla f|}{|\nabla f \cdot \bar{p}|}\, dA}_{\text{surface integral of } g \text{ over } S.}$$

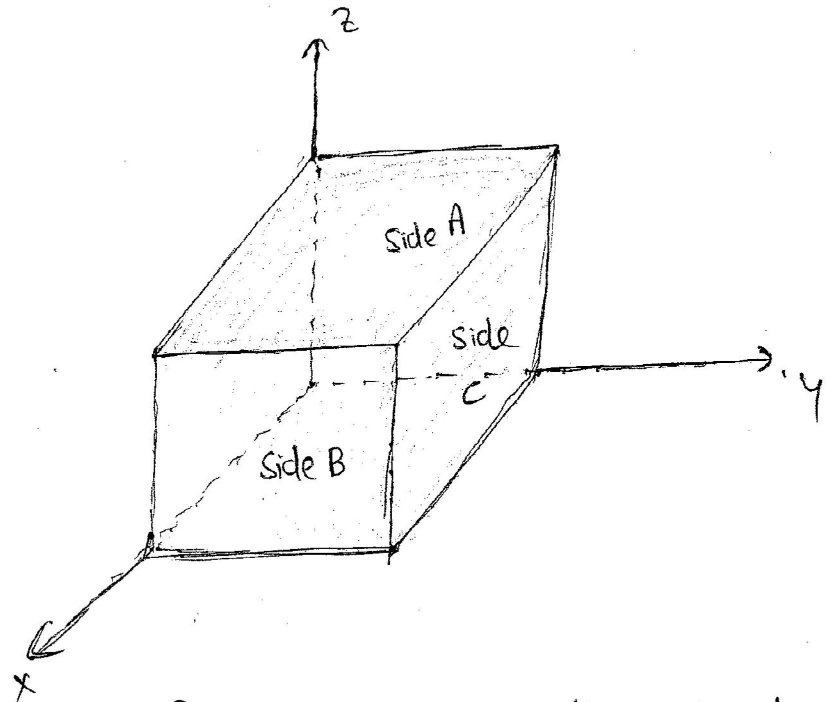
Another application: (i) If $g$ gives the mass density of a thin shell of material modelled by S, the integral gives the mass of the shell.

(ii) If $g=1$ then the integral will simply gives the total area of the surface.

Example: Integrate $g(x,y,z)=xyz$ over the surface of the cube cut from the first octant by the planes $x=1$, $y=1$, and $z=1$.

Solution:






Since $xyz = 0$ on the sides that lie in the coordinate planes, the integral over the surface of the cube reduces to

$$\iint_{\substack{\text{Cube}\\\text{Surface}}} xyz\, d\sigma = \iint_{\text{Side}A} xyz\, d\sigma + \iint_{\text{side }B} xyz\, d\sigma + \iint_{\text{side}C} xyz\, d\sigma .$$

Side A is the surface $f(x,y,z) = z = 1$ over the region $R_{xy}: 0 \le x \le 1,\ 0 \le y \le 1$ in the $xy$ plane. For this surface and region:

$$p = \bar{k}, \quad \nabla f = \bar{k} \qquad |\nabla f| = 1$$

$$|\nabla f \cdot p| = 1.$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot p|} dA = dx\,dy$$

~~fxcept~~ $xyz = xy(1) = xy.$

$$\iint_{\text{side}A} xyz\, d\sigma = \int_0^1\int_0^1 xy\, dx\, dy = \frac{1}{4}.$$

Symmetry tells us that the integral of $xyz$, over side B and C are also $1/4$. Hence.

$$\iint_{\text{cube surface}} xyz\, d\sigma = 3 \cdot \frac{1}{4} = \frac{3}{4}. \qquad \text{Ans.}$$

Orientable surface: S is an orientable surface if it has two sides which may be painted in two different colours.

Example of nonorientable surface is Möbius strip.

Mathematically, a smooth surface S is said to be orientable if there exists a continuous unit normal vector field $\bar{n}$ defined at each point $(x, y, z)$ on the surface.

Flux of a vector field $\bar{F}$ through a surface S.

The flux of a vector field $\bar{F}$ across an oriented surface S in the direction of $\bar{n}$ is given by the formula

$$\text{Flux} = \iint_S F \cdot \bar{n} \, d\sigma.$$

Here $\bar{n}$ is a unit normal ~~vector field~~ to the surface.

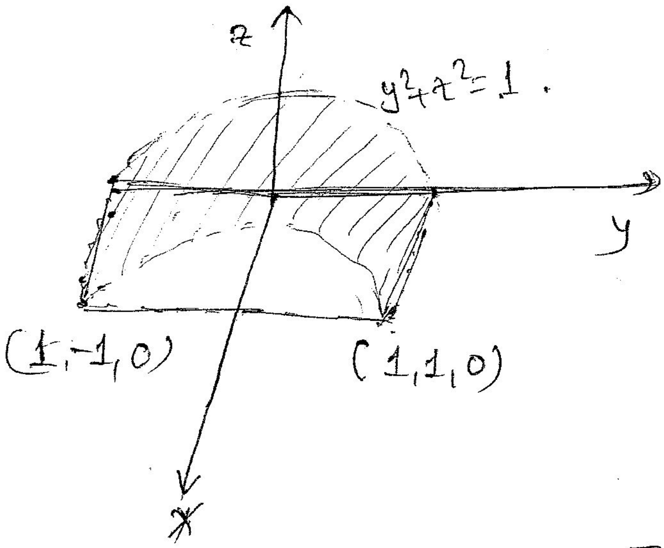
If S is a part of a level surface $g(x, y, z) = c$ then $\bar{n}$ may be taken to be one of the two fields $\bar{n} = \pm \frac{\nabla g}{|\nabla g|}$

$$\text{Flux} = \iint_S \bar{F} \cdot \bar{n} \, d\sigma$$

$$= \iint_R \left( \bar{F} \cdot \frac{\pm \nabla g}{|\nabla g|} \right) \frac{|\nabla g|}{|\nabla g \cdot p|} \cdot dA$$

$$= \iint_R \bar{F} \cdot \frac{\pm \nabla g}{|\nabla g \cdot p|} \cdot dA.$$

Example: Find the flux of $\bar{F} = yz\bar{j} + z^2\bar{k}$ outward through the surface S cut from the cylinder $y^2+z^2=1$, $z \geq 0$ by the planes $x=0$ and $x=1$.




surface. $g(x,y,z) = c$

$\Leftrightarrow y^2+z^2=1$.

$$\bar{n} = \frac{\nabla g}{|\nabla g|} = \frac{2y\bar{j} + 2z\bar{k}}{\sqrt{4(y^2+z^2)}}$$

$$= y\bar{j} + z\bar{k}$$

$\bar{p} = \bar{k}$ we have:

$$d\sigma = \frac{|\nabla g|\, dA}{|\nabla g \cdot \bar{k}|} = \frac{2}{|2z|} dA = \frac{2}{2z} dA$$

$$= \frac{1}{z} \cdot dA$$

Also. $\bar{F} \cdot \bar{n} = y^2z + z^3 = z$

Flux through S is:

$$\iint_S F \cdot n \, d\sigma = \iint_{R_{xy}} \cancel{z} \cdot \frac{1}{\cancel{z}} \cdot dA = \iint_{R_{xy}} dA = 2.$$

Ans.

Ex. Evaluate the surface integral

$$\iint_S \bar{F}\cdot\bar{n}\, d\sigma \quad \text{where } \bar{F} = 6z\hat{i} + 6\hat{j} + 3y\hat{k}$$

and S is the portion of the plane $2x+3y+4z=12$

which is in the first octant

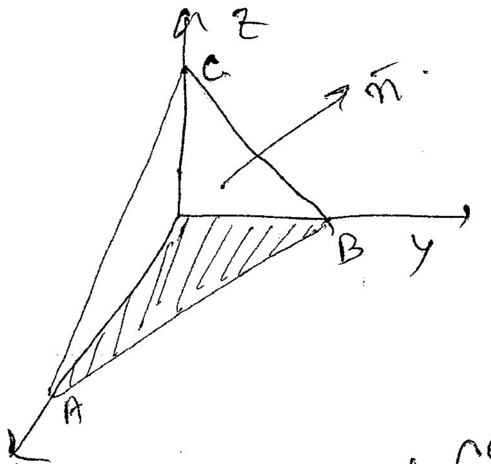
Sol: Let $g(x,y,z) = 2x+3y+4z$

$$\nabla g = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

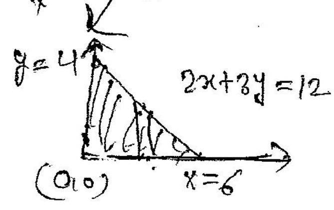
$$\bar{n} = \frac{\nabla g}{|\nabla g|} = \frac{1}{\sqrt{29}}(2\hat{i}+3\hat{j}+4\hat{k})$$

$$d\sigma = \frac{|\nabla g|}{|\nabla g\cdot \bar{p}|}dA = \frac{\sqrt{29}}{4}dA. \qquad (\bar{p} = \hat{k})$$

We have considered projection of S on the x-y plane. The Projection R is bounded by $x=0$, $y=0$ and $2x+3y=12$.




$$\iint_S \bar{F}\cdot\bar{n}\, d\sigma = \iint_R \frac{1}{\sqrt{29}}\cdot(12z+18+12y)\frac{\sqrt{29}}{4}dA$$




$$= \frac{1}{4}\iint_R (54-6x+3y)\, dA$$

$$= \frac{1}{4}\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3}(54-6x+3y)\,dy\,dx$$

$$= \cdots = 138.$$

Ans.

Example: Evaluate the surface integral $\iint_S F.\bar{n}\, d\sigma$ where $F = z^2 i + xy j - y^2 k$ and $S$ is the portion of the surface of the cylinder $x^2+y^2=36$ $0 \le z \le 4$ included in the first octant.

Solution: Let $f(x,y,z) = x^2+y^2-36 = 0$ then

$$\nabla f = 2x i + 2y j$$

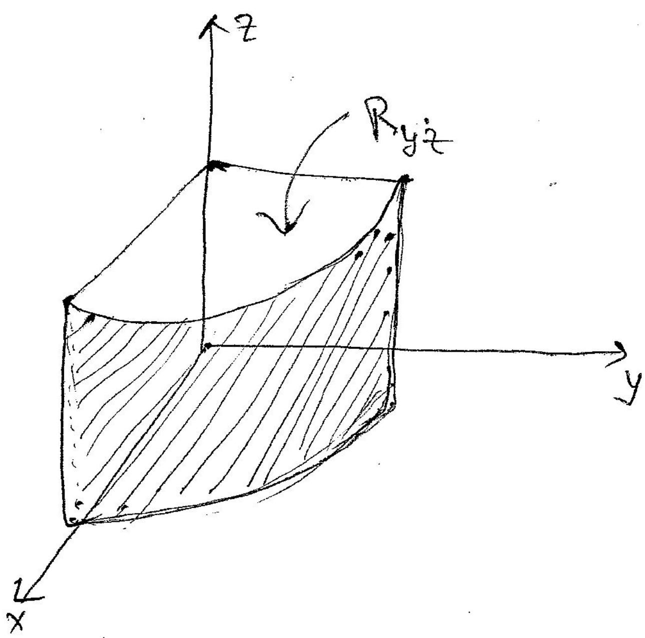
$$\bar{n} = \frac{\bar{\nabla} f}{|\nabla f|} = \frac{2x i + 2y j}{\sqrt{4 \cdot 36}} = \frac{1}{6}(x i + y j)$$

$$d\sigma = \frac{|\bar{\nabla} f|}{|\nabla f \cdot p|} dA$$

$$\bar{p} = (1, 0, 0)$$

$$d\sigma = \frac{\sqrt{4 \cdot 36}}{|2x|} dA$$

$$= \frac{6}{x} dA.$$




Therefore: $\iint_S F.\bar{n}\, d\sigma =$

$$\iint_{R_{yz}} \frac{1}{6}(xz^2 + xy^2) \frac{6}{x} dA$$

$$= \int_{z=0}^{4} \int_{y=0}^{6} \frac{x(y^2+z^2)}{x} dy\, dz$$

$$= \int_0^4 \left[\frac{y^3}{3} + z^2 y\right]_0^6 dz = \int_0^4 (72 + 6z^2)\, dz$$

$$= 72 \cdot 4 + \frac{6}{3} \cdot 64 = 288 + 128$$

$$= 416$$ Ans

# Parameterized surface:

Parametric equation of a surface:

$$x = f(u,v) \quad y = g(u,v) \quad z = h(u,v).$$

In vector form:

$$\bar{r}(u,v) = f(u,v)\,\hat{i} + g(u,v)\,\hat{j} + h(u,v)\,\hat{k}.$$

Example: Parametrization of the cone:

$$z = \sqrt{x^2+y^2} \qquad 0 \le z \le 1.$$

Cylindrical coordinate provides

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \sqrt{x^2+y^2} = r$$

$$\bar{r}(r,\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + r\,\hat{k}$$

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

Ex: Parametrization of sphere: $x^2+y^2+z^2 = a^2$.

$$r(\theta,\varphi) = a\sin\theta\cos\varphi\,\bar{i} + a\sin\theta\sin\varphi\,\bar{j} + a\cos\theta\,\hat{k}$$

$$0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi$$

Ex: Parametrization of the cylinder:

$$x^2 + (y-3)^2 = 9 \qquad 0 \le z \le 5.$$

In cylindrical coordinate system: $x = r\cos\theta \quad y = r\sin\theta$

then: $x^2 + (y-3)^2 = 9 \Rightarrow r^2 - 6r\sin\theta = 0$

$$\Rightarrow r = 6\sin\theta$$

$x = 6\sin\theta\cos\theta = 3\sin 2\theta$ ; $y = 6\sin\theta\sin\theta = 6\sin^2\theta$

$z = z$.

$$\bar{r}(\theta,z) = (3\sin 2\theta)\,\bar{i} + 6\sin^2\theta\,\bar{j} + z\,\bar{k} \qquad 0 \le \theta \le \pi$$
$$0 \le z \le 5$$

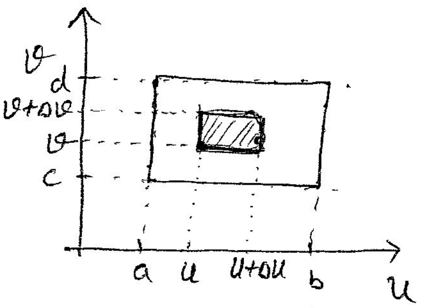
**Def.** A parametrized surface $\bar{r}(u,v) = f(u,v)\bar{i} + g(u,v)\bar{j} + h(u,v)\bar{k}$ is smooth if

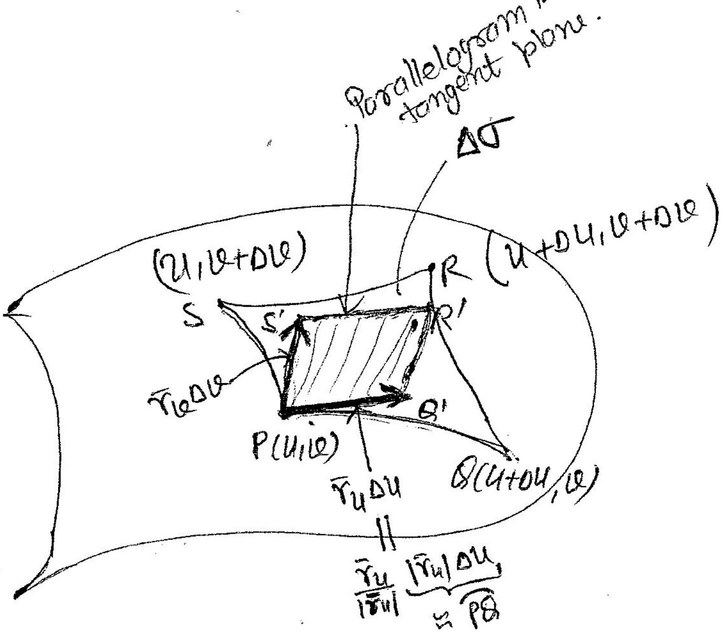
$$\bar{r}_u = \frac{\partial r}{\partial u} = \frac{\partial f}{\partial u}\bar{i} + \frac{\partial g}{\partial u}\bar{j} + \frac{\partial h}{\partial u}\bar{k}. \quad \text{and}$$

$$\bar{r}_v = \frac{\partial r}{\partial v} = \frac{\partial f}{\partial v}\bar{i} + \frac{\partial g}{\partial v}\bar{j} + \frac{\partial h}{\partial v}\bar{k}.$$

are continuous and $\bar{r}_u \times \bar{r}_v$ is never zero on the parameter domain.

**Surface Area:**




$$\Delta\sigma \approx \left| \frac{\partial \bar{r}}{\partial u}\Delta u \times \frac{\partial \bar{r}}{\partial v}\Delta v \right| = |\bar{r}_u \times \bar{r}_v|\, \Delta u\, \Delta v$$

The area of the smooth surface

$$\bar{r}(u,v) = f(u,v)\bar{i} + g(u,v)\bar{j} + h(u,v)\bar{k} \qquad a \le u \le b,\ c \le v \le d$$

is

$$A = \int_c^d \int_a^b |\bar{r}_u \times \bar{r}_v|\, du\, dv = \iint_S d\sigma$$

Surface area differential:

$$\boxed{d\sigma = |\bar{r}_u \times \bar{r}_v|\, du\, dv}$$

**Note:** Equation of normal : $\bar{r}_u \times \bar{r}_v$

Unit normal: $\dfrac{\bar{r}_u \times \bar{r}_v}{|\bar{r}_u \times \bar{r}_v|}$.

Example: Find the surface area of the cone

$z=\sqrt{x^2+y^2}$ $\quad 0\le z\le 1. \quad 0\le\theta\le 2\pi$

$\bar{r}(r,\theta) = r\cos\theta\,\bar{i} + r\sin\theta\,\bar{j} + r\,\bar{k}$

$$\bar{r}_r \times \bar{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$|r_r \times r_\theta| = \sqrt{2}\, r.$

$$\text{Area} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \sqrt{2}\, r .\, dr\, d\theta \quad = \pi\sqrt{2}. \quad \text{Ans.}$$

Surface integral:

If $S$ is a smooth surface defined parametrically as

$\bar{r}(u,v) = f(u,v)\bar{i} + g(u,v)\bar{j} + h(u,v)\bar{k}$, $a\le u\le b$, $c\le v\le d$,

and $G(x,y,z)$ is a continuous function defined on $S$, then the integral of $G$ over $S$ is

$$\iint_S G(x,y,z)\, d\sigma = \int_c^d\int_a^b G\big(f(u,v), g(u,v), h(u,v)\big)\, |\bar{r}_u \times \bar{r}_v|\, du\, dv.$$

Example: Integrate $G(x,y,z) = x^2$ over the cone

$z = \sqrt{x^2+y^2}$ $\quad 0\le z\le 1.$

$$\iint_S x^2\, d\sigma = \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)\, \sqrt{2}\, r .\, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1}{4} .\, \cos^2\theta .\, d\theta$$

$$= \frac{\pi\sqrt{2}}{4}$$

Ans-

Example: Find the flux of $F = yzi + xj - z^2k$ outward through the parabolic cylinder $y = x^2$ $0 \le x \le 1$ $0 \le z \le 4$.

Solution:

$$\bar{r}(x,z) = x\hat{i} + x^2\hat{j} + z\bar{k}.$$

$$\bar{r}_x \times \bar{r}_z = \begin{vmatrix} i & j & k \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= i\,2x - j$$

$$\bar{n} = \bar{r}_x\times\bar{r}_z / |\bar{r}_x \times \bar{r}_z| = \frac{i2x - j}{\sqrt{4x^2+1}}$$

On the surface $y = x^2$; the vector field:

$$\bar{F} = x^2z\,\bar{i} + x\bar{j} - z^2\bar{k}.$$

$$\bar{F}\cdot\bar{n} = \frac{1}{\sqrt{4x^2+1}}(2x^3z - x)$$

$$\text{Flux}: = \iint_S \bar{F}\cdot\bar{n}\,d\sigma = \int_{z=0}^{4}\int_{x=0}^{1} \frac{2x^3z - x}{\sqrt{4x^2+1}}\,|\bar{r}_x\times\bar{r}_z|\,dx\,dz$$

$$= \int_0^4\int_0^1 (2x^3z - x)\,dx\,dz$$

$$= \int_0^4 \left[2z\cdot\frac{1}{4} - \frac{1}{2}\right] dz$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot 16 - \frac{1}{2}\cdot 4$$

$$= 4 - 2$$

$$= 2.$$ Ans

## Stokes Theorem:

Recall Green's theorem in the plane:

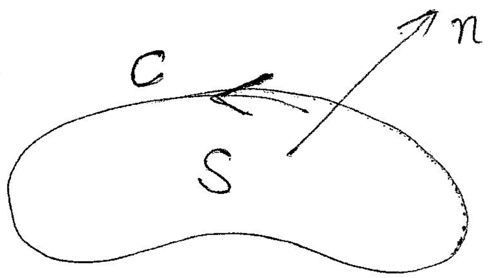
$$\oint_C \bar{F} \cdot d\bar{r} = \iint_D (\text{curl } \bar{F}) \cdot \bar{K} \, dx \, dy .$$

### Stokes Theorem:

* Let $C$ be a closed curve which forms the boundary of a surface $S$. Then for a continuously differentiable vector field $\bar{F}$, Stokes's theorem states that

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n} \, ds$$

where the direction of the line integral around $C$ and the normal $\bar{n}$ are oriented in a right-handed sense.




* If $\nabla \times \bar{F} = 0$ ($\bar{F}$ is irrotational, or $\bar{F}$ is conservative) then Stokes's theorem tells us that

$$\oint_C \bar{F} \cdot d\bar{r} = 0 .$$

Example: Verify Stokes's theorem for the hemisphere $S$: $x^2+y^2+z^2=9$, $z \geq 0$, its boundary $C: x^2+y^2=9, z=0$. and the field $\bar{F} = y\hat{i} - x\hat{j}$.

Solution: Parametric equation of the curve.



$$\bar{r}(\theta) = 3\cos\theta\, i + 3\sin\theta\, j$$

$$\frac{d\bar{r}}{d\theta} = -3\sin\theta\, i + 3\cos\theta\, j$$

$$\bar{F} = 3\sin\theta\, \hat{i} - 3\cos\theta\, \hat{j}$$

$$F \cdot \frac{d\bar{r}}{d\theta} = -9\sin^2\theta - 9\cos^2\theta = -9$$

$$\int_C \bar{F} \cdot d\bar{r} = \int_{\theta=0}^{2\pi} F \cdot \frac{d\bar{r}}{d\theta}\, d\theta$$

$$= \int_0^{2\pi} -9 \cdot d\theta = -18\pi.$$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = i(0) - j(0) + k(-1-1)$$

$$= -2\hat{k}$$

$$\bar{n} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4 \cdot 9}} = \frac{1}{3}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\iint_S (\nabla \times F) \cdot \hat{n}\, ds = \iint_{x^2+y^2 \le 9} -\frac{2z}{3} \cdot \underbrace{\frac{|\nabla(x^2+y^2+z^2)|}{|\nabla(x^2+y^2+z^2) \cdot k|}}\, dx\, dy$$

$$= \iint_{x^2+y^2 \le 9} -\frac{\cancel{2z}}{3} \cdot \frac{6}{\cancel{2z}}\, dx\, dy$$

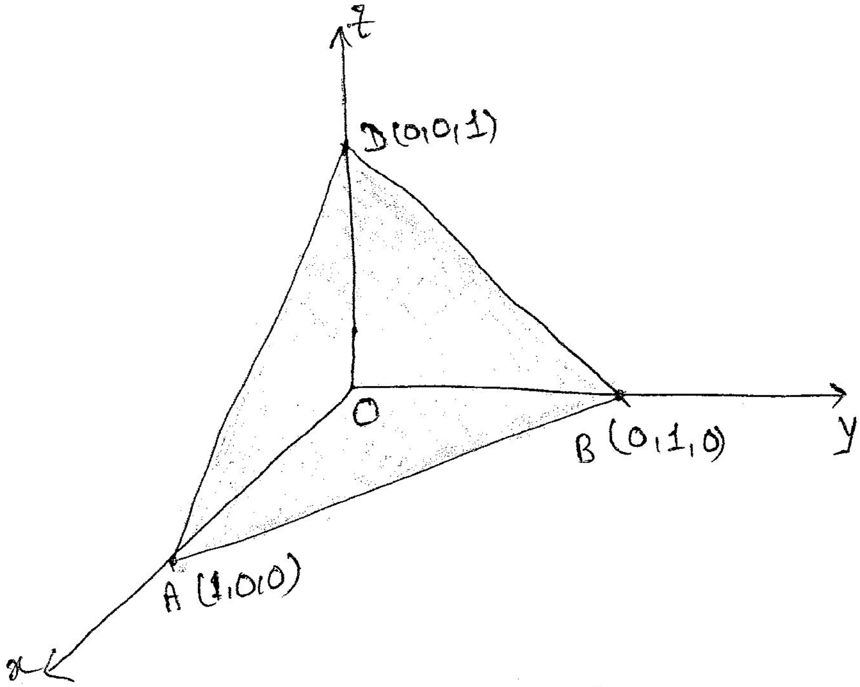
$$= -2 \iint_{x^2+y^2 \le 9} dx\, dy$$

$$= -18\pi$$

Ans.

**Example:** Verify Stoke's theorem for the function $\bar{F} = x\hat{i} + z^2\hat{j} + y^2\hat{k}$ over the plane surface $x+y+z=1$ lying in the first quadrant.

**Solution:**




Stoke's theorem: $\oint_C \bar{F} \cdot d\bar{r} = \iint_S \text{Curl} \bar{F} \cdot \bar{n} \, ds$.

S: triangle ABD.

C: lines AB, BD and DA.

$$\oint_C \bar{F} \cdot d\bar{r} = \oint_C (x\hat{i} + z^2\hat{j} + y^2\hat{k}) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_{AB} x\,dx + z^2 dy + y^2 dz + \int_{BD} x\,dx + z^2 dy + y^2 dz + \int_{DA} x\,dy + z^2 dy + y^2 dz$$

Equation of the line AB:

$$\frac{x-1}{0-1} = \frac{y-0}{1-0} = \frac{z-0}{0-0} = t.$$

$x = 1-t \quad y = t \quad z = 0.$

$$\int_{AB} x\,dx + z^2 dy + y^2 dz = \int_{t=0}^{1} (1-t)(-dt) = \frac{(1-t)^2}{2} = -\frac{1}{2}.$$

Similarly we have:

$$\int_{BD}(xdx + z^2dy + y^2dz) = 0 \quad \& \quad \int_{DA} xdx + z^2dy + y^2dz = \frac{1}{2}.$$

$$\Rightarrow \int_C \bar{F}\cdot d\bar{r} = 0.$$

Projecting $S$ on the $x$-$y$ plane let $R$ be its projection.

$R$ is bounded by the $x$-axis, $y$axis and straight line $AB$. Equation of $AB$: $x+y=1$.

Given surface $f = x+y+z = 1$

$$\nabla f = \hat{i}+\hat{j}+\hat{k}$$

$$\hat{n} = \text{Unit normal to the surface} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\frac{|\nabla f|}{|\nabla f\cdot k|} = \frac{\sqrt{3}}{|1|} = \sqrt{3}$$

$$\text{Curl } \bar{F}\cdot\hat{n} = (2(y-z)\hat{i})\cdot\left(\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})\right)$$

$$= \frac{2}{\sqrt{3}}\cdot(y-z) = \frac{2}{\sqrt{3}}(y-(1-x-y))$$

$$= \frac{2}{\sqrt{3}}(2y+x-1).$$

$$\iint_S \text{curl}\bar{F}\cdot\bar{n}\, ds = \iint_{R_{xy}} \frac{2}{\sqrt{3}}(2y+x-1)\cdot\sqrt{3}\cdot dxdy$$

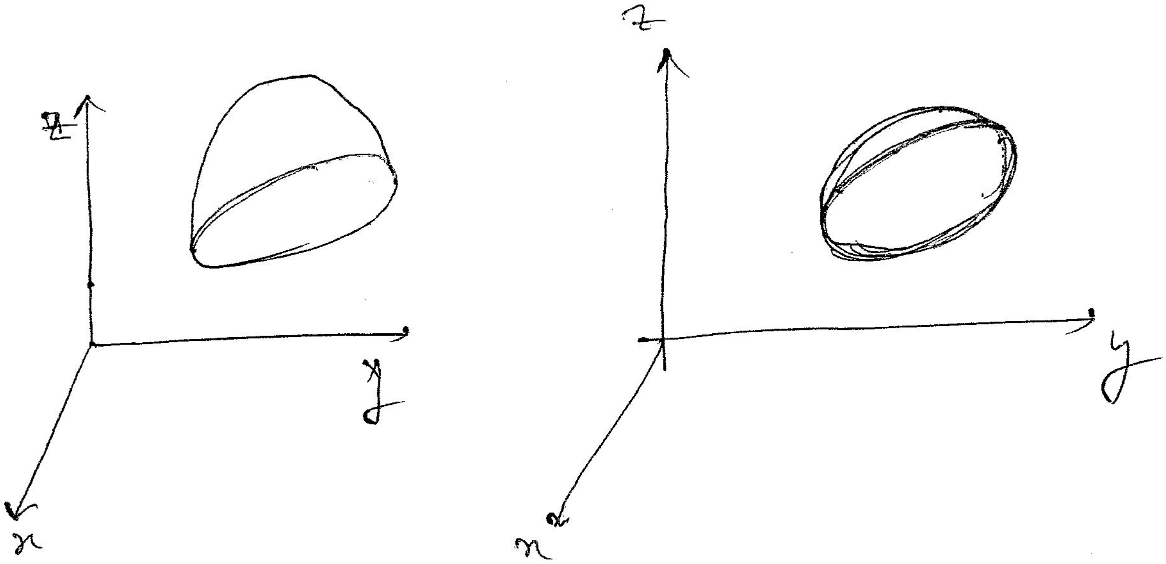
$$= 2\cdot\int_0^1\int_0^{1-x}(2y+x-1)\cdot dydx.$$

$$= 0$$

verified.

Remarks: (1) Green's theorem and Stoke's theorem in plane are just the same.

(2) Stokes's theorem states that the value of the surface integral is same for any surface as long as the bounding curve is the same curve C.




(3) Physical interpretation of curl:

Let $C_r$ be a small circle with centre at $P^*(x^*, y^*, z^*)$. Then by Stokes's theorem:


$$\oint_{C_r} \bar{V} \cdot d\bar{r} = \iint_{S_r} \text{curl } V \cdot \bar{n} \, dA$$

Here $\bar{V}$ is the velocity of a field. Let P be any arbitrary point on $S_r$. We approximate curl $\bar{V}(P)$ by curl $V(P^*)$

Then.

$$\oint_{C_r} \bar{V} \cdot d\bar{r} \approx \iint_{S_r} \text{curl } V(P^*) \cdot \bar{n}(P^*) \, dA$$

$$= \text{curl } \bar{V}(P^*) \cdot \bar{n}(P^*) A_r$$

Let $r \to 0$

Then.

$$\boxed{\text{curl } \bar{V}(P^*) \cdot \bar{n}(P^*) = \lim_{r \to 0} \frac{1}{A_r} \oint_{C_r} \bar{V} \cdot d\bar{r}}$$

Circulation density.

## SUMMARY

Green's Theorem: (in the plane)

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_R (\text{curl } \bar{F})\cdot \bar{k}\ dx\,dy$$

SURFACE INTEGRAL: ( Surface is defined by $f(x,y,z)=c$)

1) Surface area $= \iint_S d\sigma = \iint_R \dfrac{|\nabla f|}{|\nabla f\cdot p|}\, dA$

2) Surface integral of $g(x,y,z)$ over S:

$$= \iint_S g(x,y,z)\,d\sigma = \iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f\cdot p|}\, dA$$

3) Flux of $\bar{F}$ across S in the direction of $\bar{n}$:

$$= \iint_S \bar{F}\cdot\bar{n}\ d\sigma = \iint_R \bar{F}\cdot\bar{n}\ \frac{|\nabla f|}{|\nabla f\cdot p|}\ dA$$

Surface Integral ( surface is given in parametrized form

$$\bar{r}(u,v) = f(u,v)\,\bar{i} + g(u,v)\,\bar{j} + h(u,v)\,\bar{k}$$

$$a \le u \le b \qquad c \le v \le d.$$

1) surface area: $= \iint_S d\sigma = \int_{v=c}^{d}\int_{u=a}^{b} |\bar{r}_u \times \bar{r}_v|\,du\,dv$

$$\boxed{d\sigma = |\bar{r}_u \times \bar{r}_v|\ du\,dv}$$

2) Equation of the normal to the surface $= \bar{r}_u \times \bar{r}_v$

Unit normal $\bar{n} = \dfrac{\bar{r}_u \times \bar{r}_v}{|\bar{r}_u \times \bar{r}_v|}$

## The Divergence theorem:

The flux of a vector field $F = F_1 i + F_2 j + F_3 k$ across a closed oriented surface $S$ in the direction of the surface's outward unit normal field $\bar{n}$ equals the integral of $\nabla \cdot \bar{F}$ over the region $D$ enclosed by the surface.

$$\iint_S F \cdot \bar{n} \, d\sigma = \iiint_D \nabla \cdot F \, dV$$

Intutively it states that sum of all sources minus the sum of all sinks gives the net flow of a region.

Example: Verify divergence theorem for the field $F = xi + yj + zk$ over the sphere $x^2+y^2+z^2 = a^2$.

Solution:

$$\bar{n} = \frac{xi + yj + zk}{a}$$

$$F \cdot \bar{n} = \frac{1}{a} \cdot (x^2+y^2+z^2) = a.$$

$$\iint_S F \cdot \bar{n} \, d\sigma = \iint_S a \, d\sigma = a \iint d\sigma = a \cdot (4\pi a^2)$$
$$= 4\pi a^3.$$

$$\text{Div} F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3.$$

So. $$\iiint_D \text{Div} F \, dv = 3 \iiint dV = 3 \cdot \frac{4}{3}\pi a^3 = 4\pi a^3.$$

Ans.

Example: Find the flux of $F = xy i + yz j + xz k$ outward through the surface of the cube from the first octant by the planes $x=1$, $y=1$ and $z=1$.

Solution: $\nabla \cdot F = y + z + x.$

$$\text{Flux} = \iint_{\partial D} F \cdot \bar{n} \, d\sigma = \iiint_D \nabla \cdot F \, dV$$

$$= \int_0^1 \int_0^1 \int_0^1 (x+y+z) \, dx\,dy\,dz = \frac{3}{2}.$$

Ans

Example: If V is the volume enclosed by a closed surface S and $\bar{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$

Show that $\iint_S F \cdot \bar{n}\, ds = 6V$.

Solution: $\operatorname{div} F = \frac{\partial}{\partial x} \cdot x + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z) = 6$.

By Gauss's divergence theorem:

$$\iint_S F \cdot \bar{n}\, ds = \iiint_V \operatorname{div} F\, dv = 6 \iiint_V dv = 6V$$

Example: Evaluate

$$\iint_S \left((x^3 - yz)\hat{i} - 2x^2y\hat{j} + 2\hat{k}\right) \cdot \hat{n}\, ds$$ where S

denotes the surface of the cube bounded by the planes

$x = 0,\ x = a,\ y = a,\ z = a,\ z = a.$
$y = a,$

Solution: $\operatorname{div} \bar{F} = 3x^2 - 2x^2 + 0$

$= x^2$

By Gauss divergence theorem:

$$\iint_S \bar{F} \cdot \hat{n}\, ds = \iiint_V x^2\, dx\,dy\,dz$$

$$= \int_0^a \int_0^a \int_0^a x^2\, dx\, dy\, dz$$

$$= \frac{a^3}{3} \cdot a^2 = \frac{a^5}{3}. \quad \text{Ans.}$$

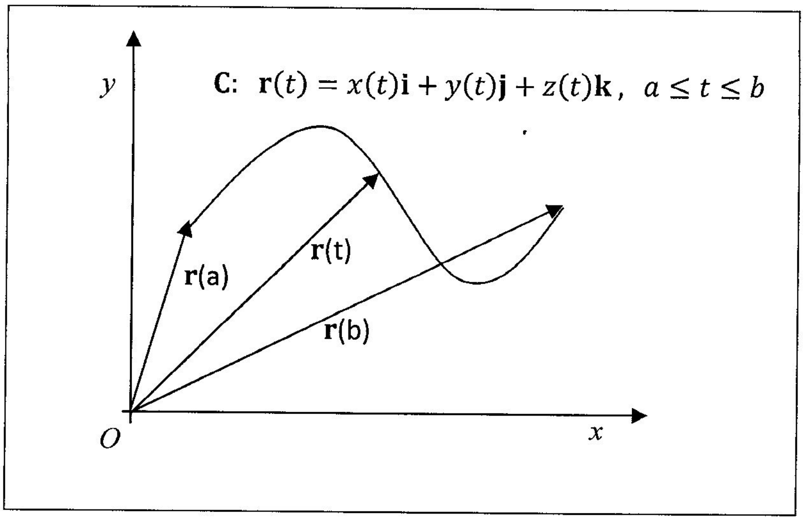
Example: If $\bar{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$, a, b, c are constant. Evaluate $\iint_S \bar{F} \cdot \bar{n}\, ds$ where S is the surface of unit sphere.

Solution: $\operatorname{div} \bar{F} = a + b + c$.

$$\iint_S F \cdot \hat{n}\, ds = \iiint_V \operatorname{Div} F\, dv = (a+b+c)\frac{4}{3}\pi = \frac{4\pi}{3}(a+b+c)$$

Ans.

**Line and Surface Integrals:**




**Line Integral of a scalar function:**

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t)) \left|\frac{d\mathbf{r}}{dt}\right| dt$$

**Line Integral of a vector function:** Let **t** be the unit tangent vector to the curve C.

$$\int_C \mathbf{F}(x,y,z)\cdot\mathbf{t}\,ds = \int_a^b \mathbf{F}(x(t),y(t),z(t))\cdot\frac{\frac{d\mathbf{r}}{dt}}{\left|\frac{d\mathbf{r}}{dt}\right|}\left|\frac{d\mathbf{r}}{dt}\right| dt = \int_a^b \mathbf{F}(x(t),y(t),z(t))\cdot\frac{d\mathbf{r}}{dt}dt$$

**Special Case:** Arc length of a curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \ a \le t \le b$

$$\int_C ds = \int_a^b \left|\frac{d\mathbf{r}}{dt}\right| dt$$

**Surface Integral of a scalar function:** Let $g(x, y, z) = 0$ be the equation of the surface.

$$\iint_S f(x,y,z)d\sigma = \iint_{R_{xy}} f(x,y,z)\frac{|\nabla g|}{|\nabla g\cdot \boldsymbol{p}|}dx\,dy$$

**Note:** Replace $z$ in terms of $x$ and $y$ from $g(x, y, z) = 0$ in the integrand of right hand side integral. Here $\boldsymbol{R_{xy}}$ is the projection of the surface S on the $xy$ plane. This can also be done by projecting S on the $yz$ or $xz$ planes.

**Surface Integral of a vector function (Flux):** Let $g(x, y, z) = 0$ be the equation of the surface.

$$\iint_S \mathbf{F}(x,y,z)\cdot\mathbf{n}\,d\sigma = \iint_R \mathbf{F}(x,y,z)\cdot\frac{\nabla g}{|\nabla g|}\frac{|\nabla g|}{|\nabla g\cdot\mathbf{p}|}dA$$

**Special Case:** Area of the surface $S$:

$$\iint_S d\sigma = \iint_{R_{xy}} \frac{|\nabla g|}{|\nabla g \cdot p|} dx\, dy$$

**If the surface is given in the parametric form** as $\mathbf{r}(u,\ v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k},\ a \leq u \leq b,\ c \leq v \leq d$ then the above surface integral can be evaluated as

$$\iint_S f(x,y,z) d\sigma = \int_a^b\!\!\int_c^d f(x,y,z)|\mathbf{r}_u \times \mathbf{r}_v| dv\, du$$

$$\iint_S \mathbf{F}(x,y,z) \cdot \mathbf{n}\, d\sigma = \int_a^b\!\!\int_c^d \mathbf{F}(x,y,z) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v| dv\, du$$

**Stokes Theorem:**

$$\oint_C \boldsymbol{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

**Green Theorem:**

$$\oint_C \boldsymbol{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dx\, dy = \iint_D \left(\frac{\partial \mathrm{F}_2}{\partial \mathrm{x}} - \frac{\partial \mathrm{F}_1}{\partial \mathrm{y}}\right) dx\, dy$$

**Gauss Divergence Theorem:**

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V \nabla \cdot \mathbf{F}\, dV$$