

Q: Consider the system

$$x+2y+z=3$$
$$ay+5z=10$$
$$2x+7y+az=b$$

a) Find those values of a for which the system has a unique solution

b) Find those values of (a,b) for which the system has more than one solution.

S:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 2 & 7 & a & b \end{array}\right] \xrightarrow{R_3-2R_1} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 0 & 3 & a-2 & b-6 \end{array}\right]$$

$$\xrightarrow{aR_3-3R_2} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 0 & 0 & a(a-2)-15 & a(b-6)-30 \end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 0 & 0 & a^2-2a-15 & ab-6a-30 \end{array}\right]$$

The system has unique solution if

$$a^2-2a-15 \neq 0 \Rightarrow (a-5)(a+3) \neq 0$$

$$\Rightarrow a \neq 5 \text{ and } a \neq -3$$

The system has more than one solution if both $a^2-2a-15$ and $ab-6a-30$ are zero.

$\Rightarrow$ $a=5$ & $a=-3$. for left side to be zero.

& When $a=5$; $5b-30-30=0 \Rightarrow b=12$

When $a=-3$; $-3b+18-30=0 \Rightarrow b=-4$.

Thus $(5,12)$ and $(-3,-4)$

ITERATIVE METHODS:

$$x^{(k+1)} = G x^{(k)} + Hb$$

Jacobi iteration method:

$$G = -D^{-1}(L+U) \qquad H = D^{-1}$$

Gauss-Seidel iteration method:

$$G = -(L+D)^{-1}U \qquad H = (L+D)^{-1}.$$

<u>Theorem:</u> If A is strictly diagonally dominant by rows then the Jacobi and Gauss-Seidel methods converge for any initial guess. (Sufficient condition)

Theorem: The Gauss-Seidel and Jacobi iterations converge for every guess if and only if all the eigenvalues of the iteration matrix G have absolute value less that 1.

Assignment:
$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

Jacobi:
$$G = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}.$$
eigenvalues $-5.3190\times10^{-6} \pm i\, 9.2127\times10^{-6}$
1.0638×10^{-5}.

converges.

Gauss-Seidel:
$$G = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$
eigen values $0,\ 2, 2$

diverges.

VECTOR SPACES. A nonempty set V:

Closure properties: $v_1, v_2 \in V \Rightarrow v_1 + v_2 \in V$

$\lambda \in \mathbb{R}$ and $v \in V \Rightarrow \lambda v \in V$.

a1) $v_1 + v_2 = v_2 + v_1 \quad \forall\, v_1, v_2 \in V$

a2) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \quad \forall\, v_1, v_2, v_3 \in V$

a3) Existance of zero vector, denoted by 0, such that

$$v + 0 = 0 + v = v \quad \forall\, v \in V$$

a4) Existence of negative vector, denoted by $-v$ for each $v \in V$, such that

$$v + (-v) = (-v) + v = 0.$$

a5) $(\lambda + \mu) v = \lambda v + \mu v \quad \forall\, \lambda, \mu \in \mathbb{R}$ & $\forall\, v \in V$

a6) $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2 \quad \forall\, \lambda \in \mathbb{R}$ & $v_1, v_2 \in V$

a7) $\lambda(\mu v) = (\lambda \mu) v$

a8) $\forall\, v \in V \quad 1 \cdot v = v$

Q: Let V be the set of ordered pairs (a, b) of real numbers with addition in V and scalar multiplication on V defined by

(i) $(a, b) + (c, d) = (a+c, b+d)$; $k(a, b) = (ka, 0)$

(ii) $(a, b) + (c, d) = (a+d, b+c)$; $k(a, b) = (ka, kb)$

(iii) $(a, b) + (c, d) = (a+c, b+d)$; $k(a, b) = (a, b)$

(iv) $(a, b) + (c, d) = (0, 0)$; $k(a, b) = (ka, kb)$

(v) $(a, b) + (c, d) = (ac, bd)$; $k(a, b) = (ka, kb)$

S: (i) a8): $1 \cdot (a, b) = (a, 0) \neq (a, b)$ NOT A VECTOR SPACE

(ii) a2) Let (a_1, b_1) (a_2, b_2) $(a_3, b_3) \in V$

$$\left((a_1, b_1) + (a_2, b_2)\right) + (a_3, b_3) = (a_1+b_2, b_1+a_2) + (a_3, b_3)$$
$$= (a_1+b_2+b_3, b_1+a_2+a_3)$$

$$(a_1, b_1) + \left((a_2, b_2) + (a_3, b_3)\right) = (a_1, b_1) + (a_2+b_3, b_2+a_3)$$
$$= (a_1+b_2+a_3, b_1+a_2+b_3)$$

NOT A VECTOR SPACE.

(iii) a5) $(\lambda+\mu) \cdot (a, b) = (a, b)$

& $\lambda(a, b) + \mu(a, b) = (a, b) + (a, b) = (2a, 2b)$

$\Rightarrow (\lambda+\mu)(a, b) \neq \lambda(a, b) + \mu(a, b)$ NOT A VECTOR SPACE

(iv) a3) $(a, b) + (,) = (0, 0) \neq (a, b)$ NOT a vector space.

(v) Existence of zero vector. $(a, b) + \underbrace{(1, 1)}_{\text{zero vector.}} = (a, b)$

BUT: (a4): if we take $(0, 0) + \underbrace{(\cdot, \cdot)}_{\text{do not exist.}} = (1, 1)$ Not a vector space.

Q. Consider the subspaces $U = \{(a,b,c,d) : b-2c+d=0\}$ and $W = \{(a,b,c,d) : a=d,\ b=2c\}$ of $\mathbb{R}^4$.

Find a basis and the dimension of ...

(a) U, (b) W (c) $U \cap W$

S: a): U: $b-2c+d=0$

dimension $= n-r = 4-1 = 3.$

On the other hand, we can choose 3 free variables: $a=\alpha_1$
$d=\alpha_2$
$c=\alpha_3$
$b = 2\alpha_3 - \alpha_2$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

Basis: $= \{(1,0,0,0), (0,1,0,-1), (0,0,1,2)\}.$

b): W: $a-d=0$
$b-2c=0$

$\dim = n-r = 4-2 = 2.$

Choose free variables: $c=\alpha_1$
$\Rightarrow b = 2\alpha_1$
$d=\alpha_2$
$a=\alpha_2$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \alpha_1 \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Basis: $= \{(0,2,1,0), (1,0,0,1)\}.$

c) $U \cap W$:

$$\begin{aligned} b-2c+d &= 0 \\ a-d &= 0 \\ b-2c &= 0 \end{aligned} \iff \begin{bmatrix} 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

No of free variables $= 1 = (4-3)$ $\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$c=\alpha$, $d=0$; $b=2\alpha$. $a=0$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix};$$ Basis: $= \{(0,2,1,0)\}.$

Q: Find a basis and dimension of the solution space W of

$$x+2y-2z+2s-t=0$$
$$x+2y-z+3s-2t=0$$
$$2x+4y-7z+s+t=0$$

S: $$\begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 1 & 2 & -1 & 3 & -2 \\ 2 & 4 & -7 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 3 \end{bmatrix}$$

$$\begin{array}{c} \\ \sim \\ \\ \end{array} \begin{array}{ccccc} x & & z & & \\ \left[\begin{matrix} \boxed{1} & 2 & -2 & 2 & -1 \\ 0 & 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}\right] \\ & y & & s & t \end{array}$$

dim = 5 − 2 = 3.

$t=\alpha_1 \qquad s=\alpha_2 \qquad z=-\alpha_2+\alpha_1 \qquad y=\alpha_3$

$$x = -2\alpha_3+2(-\alpha_2+\alpha_1)-2\cdot\alpha_2+\alpha_1$$
$$= -2\alpha_3-4\alpha_2+3\alpha_1$$

$$\begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \alpha_1\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2\begin{bmatrix} -4 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Basis: $\{(3,0,1,0,1),(-4,0,-1,1,0),(-2,1,0,0,0)\}$.

Q: Whether the set $S=\{(3,0,1,0,1),(-4,0,-1,1,0),(-2,1,0,0,0),(1,1,1,0,1)\}$ form a basis of the solution space of

$$x+2y-2z+2s-t=0$$
$$x+2y-z+3s-2t=0$$
$$2x+4y-7z+s+t=0.$$

If not, find the basis of the solution space from S.

S. The set S does not form a basis of the solution space because dim (sol space) = 3. OR Check the linear dependency of S and conclude.

Basis: any three linearly independent vectors from S; that is

$(3,0,1,0,1)\ (-2,1,0,0,0)\ (1,1,1,0,1)$

(any two from 1, 2 & 3 will form basis)

LINEAR MAPPING:

$$F: X \to Y$$

$$F(x+y) = F(x) + F(y) \quad \forall\, x, y \in X$$

$$F(kx) = kF(x) \quad \forall\, x \in X \;\&\; k \in \mathbb{R}.$$

MATRIX REPRESENTATION OF A LINEAR OPERATOR:

Let $F: X \to Y$ be linear mapping.

Let $S = \{x_1, x_2, \ldots x_m\}$ and $S' = \{y_1, y_2, \ldots y_n\}$ are basis of X and Y.

Then.

$$F(x_1) = a_{11} y_1 + a_{12} y_2 + \cdots + a_{1n} y_n$$

$$F(x_2) = a_{21} y_1 + a_{22} y_2 + \cdots + a_{2n} y_n$$

$$\vdots$$

$$F(x_m) = a_{m1} y_1 + a_{m2} y_2 + \cdots + a_{mn} y_n$$

Then:

Matrix representation of F, denoted by $[F]_S^{S'} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$

PROPERTIES OF EIGENVALUES EIGENVECTORS: ....

DIAGONALIZATION: $P^{-1}AP = D$.

$A^{n\times n}$ matrix is diagonalizable iff it has n linearly independent eigenvectors.

Quadratic form: $q(x_1, x_2, \cdots x_n) = q(\underset{\sim}{x})_{\in \mathbb{R}^n} = \sum_{i \le j} a_{ij} x_i x_j = x^T A x$ ↑ sym. (unique)

A is called a) positive definite if the quadratic form x^TAx is positive definite that is $x^TAx > 0 \;\; \forall\, x \neq 0 \in \mathbb{R}^n$

b) positive semidefinite ....

PRINCIPAL Axes Theorem:

$$x^TAx \xrightarrow{x = Py} y^TDy$$

A is an $n\times n$ symmetric matrix

How would you relate eigenvalues to the definiteness of the matrix A?

Q: Let G be the linear operator on $\mathbb{R}^3$ defined by $G(x,y,z) = (2y+z,\ x-4y,\ 3x)$.
Find the matrix representation of G relative to the basis

$$S = \{w_1, w_2, w_3\} = \{(1,1,1),\ (1,1,0),\ (1,0,0)\}.$$

S:

$$G(w_1) = (3,-3,3)^T = a_{11}\begin{pmatrix}1\\1\\1\end{pmatrix}^T + a_{12}\begin{pmatrix}1\\1\\0\end{pmatrix}^T + a_{13}\begin{pmatrix}1\\0\\0\end{pmatrix}^T$$

$$G(w_2) = (2,-3,3)^T = a_{21}\begin{pmatrix}1\\1\\1\end{pmatrix}^T + a_{22}\begin{pmatrix}1\\1\\0\end{pmatrix}^T + a_{23}\begin{pmatrix}1\\0\\0\end{pmatrix}^T$$

$$G(w_3) = (0,1,3)^T = a_{31}\begin{pmatrix}1\\1\\1\end{pmatrix}^T + a_{32}\begin{pmatrix}1\\1\\0\end{pmatrix} + a_{33}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 3 & 2 & 0 \\ 1 & 1 & 0 & -3 & -3 & 1 \\ 1 & 0 & 0 & 3 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 3 & 2 & 0 \\ 0 & 0 & -1 & -6 & -5 & 1 \\ 0 & -1 & -1 & 0 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 3 & 2 & 0 \\ 0 & -1 & -1 & 0 & 1 & 3 \\ 0 & 0 & -1 & -6 & -5 & 1 \end{array}\right]$$

$\boxed{a_{13} = 6}$

$\boxed{a_{23} = 5}$

$\boxed{a_{33} = -1}$

$-a_{12} - a_{13} = 0 \Rightarrow \boxed{a_{12} = -a_{13} = -6.}$

$a_{11} + a_{12} + a_{13} = 3 \Rightarrow \boxed{a_{11} = 3+6-6 = 3}.$

$-a_{22} - a_{23} = \phi \Rightarrow \boxed{a_{22} = -5-1 = -6.}$

$+a_{21} + a_{22} + a_{23} = 2 \Rightarrow \boxed{a_{21} = 2+6-5 = 3}$

$-a_{32} - a_{33} = 3 \Rightarrow \boxed{a_{32} = +1-3 = -2}$

$a_{31} + a_{32} + a_{33} = 0 \Rightarrow \boxed{a_{31} = 3}$

$$[G] = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

ANS :

Q. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ be the matrix representation of the linear transformation T with respect to the ordered basis vectors $V_1 = [1,2]^T$, $V_2 = [3,4]^T$ in $\mathbb{R}^2$ and $W_1 = [-1,1,1]^T$, $W_2 = [1,-1,1]^T$, $W_3 = [1,1,-1]^T$ in $\mathbb{R}^3$. Then determine the linear transformation T. (JI-3.91)

S. We need to find $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix}$

IDEA: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow \alpha = \dfrac{-4x_1+3x_2}{2}$, $\beta = \dfrac{-x_2+2x_1}{2}$

Then. $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha T\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta T\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

What is $T\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ & $T\begin{pmatrix} 3 \\ 4 \end{pmatrix}$?

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$$

Then. $$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{-4x_1+3x_2}{2}\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \frac{-x_2+2x_1}{2}\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -6x_1+7x_2 \\ -2x_1+3x_2 \\ 2x_1-x_2 \end{pmatrix}$$

Ans.

Q: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -4 \end{bmatrix}$ be the matrix representation of the linear transformation with respect to the ordered basis vectors $v_1 = [1, -1, 1]^T$, $v_2 = [2, 3, -1]^T$, $v_3 = [1, 1, -1]^T$ in $\mathbb{R}^3$ and $w_1 = [1, 1]^T$, $w_2 = [2, 3]^T$ in $\mathbb{R}^2$. Then determine the linear transformation T.

S: Let $x \in \mathbb{R}^3$. Then it can be expressed as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \quad \alpha = \frac{x_1}{2} - \frac{x_2}{4} + \frac{x_3}{4}$$

$$\beta = \frac{x_2}{2} + \frac{x_3}{2}$$

$$\gamma = \frac{x_1}{2} - \frac{3}{4}x_2 - \frac{5}{4}x_3$$

$$T\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 4\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -7 \\ -11 \end{pmatrix}$$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha\begin{pmatrix} 5 \\ 7 \end{pmatrix} + \beta\begin{pmatrix} -4 \\ -7 \end{pmatrix} + \gamma\begin{pmatrix} -7 \\ -11 \end{pmatrix}$$

$$= \begin{pmatrix} 5\alpha - 4\beta - 7\gamma \\ 7\alpha - 7\beta - 11\gamma \end{pmatrix}$$

$$= \begin{pmatrix} 2x_2 - x_1 + 8x_3 \\ 3x_2 - 2x_1 + 12x_3 \end{pmatrix}$$

ANS

Q: Let $A=\begin{bmatrix}4 & 1 & -1\\ 2 & 5 & -2\\ 1 & 1 & 2\end{bmatrix}$. Is A diagonalizable?. If yes, find P such that $D = P^{-1}AP$ is diagonal.

S: $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix}4-\lambda & 1 & -1\\ 2 & 5-\lambda & -2\\ 1 & 1 & 2-\lambda\end{vmatrix} = 0 \Rightarrow (4-\lambda)[(5-\lambda)(2-\lambda)+2] - 1(4-2\lambda+2) - 1(2-5+\lambda) = 0$

$\Rightarrow (\lambda-3)^2(\lambda-5) = 0$

$\Rightarrow \lambda = 3, 3, 5.$

Eigenvector corresponding to $\lambda=3$:

$$\begin{pmatrix}1 & 1 & -1\\ 2 & 2 & -2\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

dim of solution space $= \cancel{3}-1 = \underline{\underline{2}}.$

let $x_3 = \alpha_1 \quad x_2 = \alpha_2$

$x_1 = -\alpha_2 + \alpha_1$

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \alpha_1\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$\Rightarrow$ A is diagonalizable.

Eigenvecto corresponding to $\lambda = 5$.

$$\begin{pmatrix}-1 & 1 & -1\\ 2 & 0 & -2\\ 1 & 1 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \sim \begin{pmatrix}-1 & 1 & -1\\ 0 & 2 & -4\\ 0 & 2 & -4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Chose. $x_3 = \alpha_1 \qquad x_2 = 2\alpha_1$

$x_1 = 2\alpha_1 - \alpha_1 = \alpha_1$

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \alpha_1\begin{pmatrix}1\\2\\1\end{pmatrix}.$$

Then.

$$P = \begin{bmatrix}1 & -1 & 1\\ 0 & 1 & 2\\ 1 & 0 & 1\end{bmatrix} \qquad P^{-1} = \begin{bmatrix}-0.5 & -0.5 & 1.5\\ -1 & 0 & 1\\ 0.5 & 0.5 & -0.5\end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{bmatrix} = D.$$

Q: Let 3, 3, 5 be the eigenvalues & $[1\ 0\ 1]^T, [-1, 1, 0]^T, [1\ 2\ 1]^T$ be the corresponding eigenvectors ~~of~~ a matrix A. Find A.

Q: Let a 4×4 matrix A have eigenvalues $1, -1, 2, -2$. Find the value of the determinant of the matrix $B = 2A + A^{-1} - I$.

S. determinant of A = product of eigenvalues.

$P^{-1}AP = D$

$|A| = |D| =$ product of eigenvalues.

eigenvalues of B ?

$$Bx = (2A + A^{-1} - I)x = 2\lambda x + \frac{1}{\lambda}x - x$$

$$= \left(2\lambda + \frac{1}{\lambda} - 1\right)x.$$

eigenvalues of B are.

$\left(2+\frac{1}{1}-1\right)$ $= 2$ $(-2-1-1)$ $= -4$ $\left(4+\frac{1}{2}-1\right)$ $7/2$ $\left(-4-\frac{1}{2}-1\right)$ $= -11/2$

determinant of B $= 2 \times -4 \times \frac{7}{\cancel{2}} \times \frac{-11}{\cancel{2}}$

$= 154$.

Q: Let a 3×3 matrix A have eigenvalues $1, 2, -1$. Find the value of the determinant of the matrix $B = A - A^{-1} + A^2$

S. $Bx = Ax - A^{-1}x + A^2x = \left(\lambda x - \frac{1}{\lambda}x + \lambda^2 x\right)$

$$Bx = \left(\lambda - \frac{1}{\lambda} + \lambda^2\right)x.$$

Eigenvalues of B are:

$(1-1+1)$ $= 1$ $\left(2-\frac{1}{2}+4\right)$ $= 11/2$ & $(-1+1+1)$ $= 1$

determinant of B $= 11/2$

Q: Check the definiteness of quadratic form corresponding to the matrix (13)

a) $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 1 & 4 \end{bmatrix}.$$

S. Find the eigenvalues of A:

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 5-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)[(5-\lambda)(4-\lambda)-1]$$
$$-2(8-2\lambda-3)+3(2-15+3\lambda)=0$$

$$\Rightarrow (1-\lambda)[(20-9\lambda+\lambda^2)-1]$$
$$-16+4\lambda+6+6-45+9\lambda=0$$

$$\Rightarrow 20-9\lambda+\lambda^2-1-20\lambda+9\lambda^2-\lambda^3+\lambda$$
$$-16+4\lambda+6+6-45+9\lambda=0$$

$$\Rightarrow -\lambda^3+10\lambda^2-15\lambda-30=0$$

one root is between 0 & −2 (neg root)
one root lies between 0 & 4 (positive root)

The quadratic form is ~~hee~~ indefinite.

b) $$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 7 \end{bmatrix}$$

All eigenvalues are positive so the matrix is ~~pos~~ or corresponding quadratic form is positive definite.

Improper Integrals:

Test integrals:

1) $\int_{a>0}^{\infty} \frac{1}{x^n}\, dx$ converges for $n > 1$.
& diverges for $n \leq 1$

2) $\int_a^b \frac{dx}{(x-a)^n}$ converges if $n < 1$
diverges for $n \geq 1$.

Comparison test I: $0 \leq f(x) \leq g(x)$ . . . .

(II) $\lim_{x \to \cdot\cdot} \frac{f(x)}{g(x)} = k$

i) $k \neq 0$ then $\int f(x)\, dx$ & $\int g(x)\, dx$ behave the same.

ii) $k = 0$ then $\int f(x)\, dx$ converges if $\int g(x)\, dx$ converges.

iii) $k = \infty$ then $\int f(x)\, dx$ diverges if $\int g(x)\, dx$ diverges.

Abel's test: 1) $\int f(x)\, dx$ converges ii) g is monotone & bounded

$\Rightarrow \int f(x) g(x)\, dx$ converges.

Dirichlet test: i) $\left| \int_a^b f(x)\, dx \right| \leq C \quad \forall\, b > a$

ii) g is monotone bounded & $\lim_{x \to \cdot\cdot} g(x) = 0$

$\Rightarrow \int f(x)\, g(x)\, dx$ converges.

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1) $\int_a^\infty (1-e^{-x}) \frac{\cos x}{x^2} \, dx \qquad a>0.$

ANS: convergent by Abel's test.

$(1-e^{-x})$ is monotone & bounded.

also $\int_a^\infty \frac{\cos x}{x^2} \, dx$ is Convergent.

2) $\int_a^\infty e^{-x} \frac{\sin x}{x^2} \, dx \qquad a>0.$

ANS: Convergent by Abel's test.

3) $\int_0^\infty \sin x^2 \, dx.$

ANS. Convergent:

$$\int_0^\infty \sin x^2 \, dx = \int_0^1 \sin x^2 dx + \underbrace{\int_1^\infty \sin x^2 dx}_{= \int_1^\infty 2x \cdot \sin x^2 \underbrace{\frac{1}{2x}}_{\text{monotone converging to zero.}} dx}$$

& $\int_1^\infty 2x \sin x^2 \, dx$ is bounded by 2.

by Dirichlet's test.

4). $\int_a^\infty \frac{\sin x}{\sqrt{x}} \, dx \qquad a>0.$

ANS: convergent by Dirichlet's test.

5) $\int_0^\infty \frac{\sin x}{x} \, dx$

ANS. Convergent $\int_0^\infty \frac{\sin x}{x} dx = \int_0^a \frac{\sin x}{x} dx + \int_a^\infty \frac{\sin x}{x} dx \rightarrow$ by Dirichlet test.

6] $\int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx$

Ans: Convergent by Abel's test

since e^{-ax} is monotone & bounded

& $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

7) $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$.

Ans. convergent

$g(x) = \frac{1}{x^{1/3}}$ then $\lim_{x \to 0} \frac{f(x)}{g(x)} = 1$.

& $\int g(x) \, dx$ is convergent

$\Rightarrow$ - -

8]. $\int_0^1 \frac{\sec x}{x} \, dx$

Ans. Divergent.

$g(x) = \frac{1}{x}$. & $\frac{f(x)}{g(x)} = \sec x \to \phi$ as $x \to 0$.

$\int \frac{1}{x} dx$ is divergent

$\Rightarrow$ - - -

9) $\int_0^{\pi/4} \frac{dx}{\sqrt{\tan x}} = \int_0^{\pi/4} \frac{\sqrt{\cos x}}{\sqrt{\sin x}} dx$.

Ans: $g(x) = \frac{1}{\sqrt{x}}$. $\lim_{x \to 0} \frac{f(x)}{g(x)} = 1 \Rightarrow$ convergent.

(10) $\int_0^{\pi} \frac{\sqrt{x}}{\sin x} dx$:

Ans: convergent

$$g(x) = \frac{1}{\sqrt{x}}$$

$$\frac{f(x)}{g(x)} = \frac{x}{\sin x} \to 1$$

$\int g(x)\, dx$ is convergent

(11) $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

Ans. Convergent (test integral) $1-x=t \Rightarrow \int_0^1 \frac{dt}{\sqrt{t}}$.

(12) $\int_1^{\infty} \frac{dx}{x}$

Ans Divergent

(13) $\int_1^2 \frac{\sqrt{x}}{\log x} \cdot dx$

Ans: $f(x) = \frac{\sqrt{x}}{\log x}$ $g(x)\ \frac{1}{x-1}$

$$\frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{\sqrt{x}\,(x-1)}{\log x} = \lim_{x\to 1} \frac{\frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}}{1/x} = \frac{\frac{3}{2} - \frac{1}{2}}{1} = 1.$$

Since $\int g(x)\, dx$ diverges

the $\int_1^2 \frac{\sqrt{x}}{\log x}$ diverges.

(14) $\int_0^1 \log\sqrt{x} \cdot dx$.

ANS: CONVERGENT.

$f(x) = -\log\sqrt{x} \; = \; \log\left(\frac{1}{\sqrt{x}}\right)$

Now $f(x) \geq 0$ in $(0, 1]$

$g(x) = \frac{1}{\sqrt{x}}$

$$\lim_{x\to 0+} \frac{\log\left(\frac{1}{\sqrt{x}}\right)}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0} \sqrt{x} \log \cancel{\left(\frac{1}{\sqrt{x}}\right)}$$

$$= \log\frac{1}{}$$

$$= \lim_{x\to 0} \frac{\sqrt{x} \cdot \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)}{\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)}$$

$$= 0.$$

Also $\int_0^1 \frac{1}{\sqrt{x}} dx$ converges.

hence. $\int_0^1 -\left[\log(\sqrt{x})\right]$ converses

and therefore

$\int_0^1 \log(\sqrt{x})\, dx$ converges.

Q: value of the integral $\int_{-1}^{1} \frac{1}{x^2} dx$ D.

ANS: Integral is ...

Question:

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}\, dx$$

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}\, dx = \underbrace{\int_0^1 \frac{\cos x}{\sqrt{x}}\, dx}_{\text{Convergent}} + \underbrace{\int_1^\infty \frac{\cos x}{\sqrt{x}}\, dx}_{\text{Convergent}}$$

Convergent because

$g(x) = \frac{1}{\sqrt{x}}$

$\lim \frac{f(x)}{g(x)} = \cos x \to 1.$

and $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ converges

hence $\int_0^1 \frac{\cos x}{\sqrt{x}}\, dx$ converges.

Convergent because $\int_1^\infty \cos x\, dx$ is uniformly bounded by 2.

and $\frac{1}{\sqrt{x}}$ is monotone and bounded converging to zero. Hence by Dirichlet test it converges.

$$\Rightarrow \int_0^\infty \frac{\cos x}{\sqrt{x}}\, dx \text{ converges.} \quad \square$$