Date
22/10/2019

Lecture 9

(1)

$$\sin^{-1} x = x\, F\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; x^2\right)$$

Sol^n:- we have

$$F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha . \beta}{\gamma} \frac{x}{1!}$$

$$+ \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \cdots \longrightarrow (1)$$

Replacing α, β, γ, x by $\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}$ & x^2 resp. in (1), we get

$$F\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; x^2\right) = 1 + \frac{\left(\tfrac{1}{2}\right) . \left(\tfrac{1}{2}\right)}{\left(\tfrac{3}{2}\right)} . \frac{x^2}{1!}$$

$$+ \frac{\left(\tfrac{1}{2}\right) . \left(\tfrac{3}{2}\right) . \left(\tfrac{1}{2}\right) . \left(\tfrac{3}{2}\right)}{\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)} . \frac{x^4}{2!} + \cdots$$

②

$$\Rightarrow x\,F\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};x^2\right)$$

$$= x + 1^2 \cdot \frac{x^3}{3!} + 1^2 \cdot 3^2 \cdot \frac{x^5}{5!}$$

$$+ 1^2 \cdot 3^2 \cdot 5^2 \cdot \frac{x^7}{7!} + \cdots$$

$$= \sin^{-1} x$$

~~HW~~

EX/ Show that

$${}_2F_1(a,1;a;x) = (1-x)^{-1}$$

EX/ Show that

$$\frac{d}{dx}\left[{}_2F_1(a,b;c;x)\right]_{x=0} = \frac{ab}{c}$$

HW

EX/ Show that

$$1 - nx\, {}_2F_1(1-n, 1; 2; x) = (1-x)^n$$

where n is any natural no.

HW

EX/ Show that

$$\lim_{a,b \to \infty} {}_2F_1\left(a, b; \tfrac{1}{2}; \tfrac{x^2}{4ab}\right) = \cosh(x).$$

Note :- $\underbrace{m = n = 1}$ & $\underbrace{m = 2, n = 1}$

$\Rightarrow$ Confluent Hypergeometric function (or, Kummer f^n)

$\downarrow$ Hypergeometric f^n.

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We now consider the convergence of the hyper-geometric series:

EX (i) The confluent hyper-geometric series is convergent for all values of x. (i.e., Radius of convergence is ∞)

EX (ii) The hyper-geometric series is convergent if $|x| < 1$ & divergent if $|x| > 1$.

For $x = 1$, the series converges if $\beta > \alpha_1 + \alpha_2$, which for

$x=-1$, it converges

if $\beta > \alpha_1+\alpha_2-1$.

Ex

Th-/ $J_n(x) = \frac{e^{-ix}}{n!}\left(\frac{x}{2}\right)^n {}_1F_1(n+\frac{1}{2}; 2n+1; 2ix)$

—x—

Th-/ **Gauss Theorem**:

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$$

Ex / **Vandermonde's Theorem**

$$F(-n, \beta; \gamma; 1) = \frac{(\gamma-\beta)_n}{(\gamma)_n}$$

Ex / **Kummer's Theorem**

$$F(\alpha, \beta; \beta-\alpha+1; -1)$$

$$= \frac{\Gamma(\beta-\alpha+1)\,\Gamma(\beta/2+1)}{\Gamma(\beta+1)\,\Gamma(\beta/2-\alpha+1)}.$$

— x —

Tensors / Tensor Algebra

Syllabus :-

Books :- Tensor Calculus
— A concise introduction
by Barry Spain
(Dover)

History :-

tensor $\xrightarrow{\text{origin}}$ in the development of differential geometry by Gauss, Riemann, Christoffel & others.

Tensor calculus / Absolute Differential calculus.

But it was popularized & made a separate branch of mathematics by Ricci & his pupil Levi - Civita.

"The investigation of relations which remain valid when we move (change) from one co-ordinate system to another, is the main purpose of tensor calculus."

- Einstein found it as an excellent tool for the presentation of his General Theory of Relativity (G-R) theorem.

Preliminaries :-

1) Einstein's Summation convention

The expression

$$a_1 x^1 + a_2 x^2 + \dots + a_N x^N$$

$$= \sum_{i=1}^{N} a_i x^i$$

may be written as $a_i x^i$

$(i = 1, 2, \dots, N)$

$(i = 1 - N)$

(VN) — space

2) If a suffix occurs in a term, once in upper position & once in lower position (ie, $a_i^j x^i$) then that suffix (i) is a dummy / umbral / dextral suffix.

3) A suffix which is not repeated, is called a real or free suffix.

4) (mixed ~~tensor of rank 2~~) Kronecker delta δ_j^i has the form $\delta_j^i = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

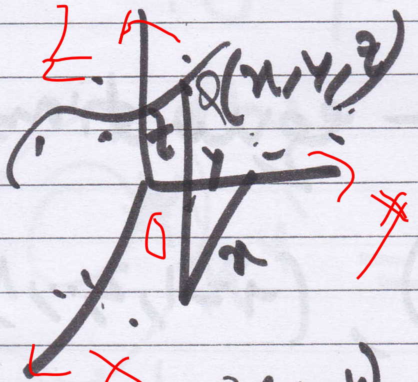
ie, $\delta_1^1 = \delta_2^2 = 1 = \delta_3^3 = \ldots$

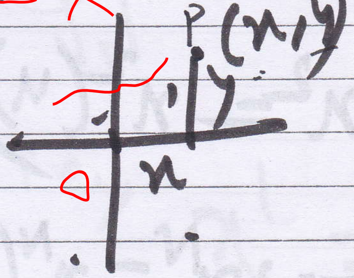
2 $\delta_2^1 = 0 = \delta_2^3 = \delta_2^4 = \ldots$

5) A^i_j $\rightarrow$ superscript (i)

$\rightarrow$ subscript (j)

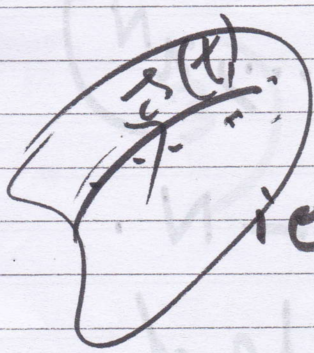
§/ N-Dimensional Space





We consider an ordered set of N real variables $(x^1, x^2, \dots, x^i, \dots, x^N)$ or, x^i, $i = 1 - N$. which are called the co-ordinates of a point. Then, all the points corresponding to all values of the co-ordinates are said to form an N-dimensional space (V_N).

Curve :- A curve in the N-dimensional space V_N is defined as the collection of points which satisfy the N-equations



$$x^i = x^i(u), \quad (i=1,2,\dots,N)$$

i.e., $x^1 = x^1(u)$, $x^2 = x^2(u)$, $\dots$, $x^i = x^i(u)$, $\dots$, $x^N = x^N(u)$.

where u is a parameter & $x^i(u)$ are N-functions of u which obey certain continuity eqns.

Subspace :- A subspace V_M of V_N is defined for $M < N$ as the collection of points which satisfy the N eqns

$$x^i = x^i(u^1, u^2, \ldots, u^M),$$

$$(i = 1, 2, \ldots, N)$$

i.e.,

$$x^1 = x^1(u^1, u^2, \ldots, u^M)$$

$$x^2 = x^2(u^1, u^2, \ldots, u^M)$$

$$\cdots\cdots\cdots\cdots$$

$$x^N = x^N(u^1, u^2, \ldots, u^M)$$

where there are M-parameters $u^1, u^2, \ldots, u^M$.

The $x^i(u^1, u^2, \dots, u^M)$ are N-functions of $u^1, u^2, \dots, u^M$, satisfying certain conditions of continuity.

- Hypersurface :-

The $M \times N$ matrix formed from the partial derivatives $\frac{\partial x^i}{\partial x^j}$ is assumed to be of rank M^*.

When $M^* = N - 1$, the subspace is called a hyper-surface

Note:-

1) A dummy suffix can be replaced by another dummy suffix, not used in that term.

e.g., $a_i^k x^i = a_1^k x^1 + a_2^k x^2 + \dots + a_N^k x^N$

& $a_j^k x^j = a_1^k x^1 + a_2^k x^2 + \dots + a_N^k x^N$

shown that $a_i^k x^i = a_j^k x^j$

(here i, j being dummy suffixes)

2) A real suffix (or, a free suffix) cannot be replaced by another real suffix.

e.g., $a_i^k x^i \neq a_i^l x^i$, k, l being real suffixes

3) The symbol δ^i_j, called <u>Kronecker delta</u> has the properties:

HW. ✓(i) $\frac{\partial x^i}{\partial x^j} = \delta^i_j$, (ii) $\delta^i_i = n$,

(iii) $\delta^i_j A^{jk} = A^{ik}$,

(iv) $\delta^i_j \delta^j_k = \delta^i_k$,

(v) $\delta^i_j \delta^j_i = n$,

(vi) $\delta_{ij} = \delta_{ji}$ for all i, j

(vii) $\delta^i_j A^j = A^i$,

where $\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Sol^n:- (ii) $\delta^i_i = n$.

By summation convention,

$$\delta^i_i = \delta^1_1 + \delta^2_2 + \dots + \delta^n_n$$

$$= \underbrace{1 + 1 + \dots + 1}_{n \text{ times}}$$

$$= n.$$

(iii) $\delta^i_j A^{jk} = A^{ik}$

Sol^n:- $\delta^1_j A^{jk} = \underbrace{\delta^1_1}_{=1} A^{1k} + \delta^1_2 A^{2k}$

$(\delta^1_j A^{jk}) \qquad + \delta^1_3 A^{3k} + \dots + \delta^1_n A^{nk}$

$$= A^{1k} + 0 + \dots + 0 = A^{1k}$$

Slly, for $i = 2, 3, \dots, n$

$\delta^2_j A^{jk} = A^{2k}$, $\delta^3_j A^{jk} = A^{3k}$, ...

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$$\therefore \delta^n_j A^{jk} = A^{nk}$$

Generalizing the above, we have

$$\boxed{\delta^i_j A^{jk} = A^{ik}}$$

iv) $\delta^i_j \delta^j_k = \delta^i_k$.

$$\text{L.H.S} = \delta^i_j \delta^j_k$$

$$= \delta^i_1 \delta^1_k + \delta^i_2 \delta^2_k + \delta^i_3 \delta^3_k + \cdots + \underbrace{\delta^i_i}\,\delta^i_k + \cdots + \delta^i_n \delta^n_k$$

$$= 0 \cdot \delta^1_k + 0 \cdot \delta^2_k + 0 \cdot \delta^3_k + \cdots + 1 \cdot \delta^i_k + \cdots + 0 \cdot \delta^n_k$$

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2) $\delta_j^i \delta_k^j = \delta_k^i$,

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v) $\delta_j^i \delta_i^j = n$.

Putting $k = i$ in iv), it follows.

$$\delta_j^i \delta_i^j = \delta_i^i = \delta_1^1 + \delta_2^2 + \cdots + \delta_n^n$$

$$= \underbrace{1 + 1 + \cdots + 1}_{n \text{ time}}$$

$$= n.$$

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vi) $\delta_{ij} = \delta_j^i = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$

& $\delta_j^i = \delta_{ij} = \delta^{ij} = \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$

$\therefore \delta_{ij} = \delta_{ji}$ ie, $\delta_j^i = \delta_i^j \ \forall\ i \,\&\, j$.

i.e., kronecker delta is symmetric in its two indices.

(vii) $\delta^i_j A^j = A^i$

sol^n: $\delta^i_j A^j = \delta^i_1 A^1 + \delta^i_2 A^2 + \cdots + \underbrace{\delta^i_i}\, A^i + \cdots + \delta^i_n A^n$

$= 0 \cdot A^1 + 0 \cdot A^2 + \cdots + 1 \cdot A^i + \cdots + 0 \cdot A^n$

i.e., $\boxed{\delta^i_j A^j = A^i}$.

4) Indicial convention:—

Latin indices, used as either subscript (a_i) or superscript (a^i), will take all values from ($i=$) 1 to N unless otherwise stated.

Thus, equations

$$\bar{x}^i = \phi^i(x^1, x^2, \dots, x^N)$$

have N equations for $i = 1$ to N.

(Sir William Rowan Hamilton) 22

Hamilton. coined the word "tensor"

→ 'Voigt' used the tensor in modern way in 20th century.

Applications:

"Tensor" ↓ Latin word 'tensus' → tension or stress → mechanical stress.

1) mechanics
2) elasticity
3) hydrodynamics
4) electro-magnetic theory
5) differential geometry
6) continuum mechanics
7) Fluid mechanics
etc.

Scalars, vectors, tensors

Defn:– A vector is the
1) mathematical representation of a physical entity (or, object) that may be characterized by size (or, magnitude) & direction (also sense).

e.g., velocity, acceln etc.

2) A scalar is the mathematical representation of a physical entity (or object) that may be characterized by magnitude only. e.g., speed, length, mass, etc.

Tensor :- A tensor is the mathematical representati[on] of a physical entity (or, object) that may be characterized by magnitude & multiple directions.

(A tensor is the generalizatio[n] of scalars & vectors)

Ex, scalar (x) vector (x_1, x_2, x_3) or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Tensor (Rank 2 or, Order 2)

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \text{ or } \begin{pmatrix} x^{11} & x^{12} & x^{13} \\ x^{21} & x^{22} & x^{23} \\ x^{31} & x^{32} & x^{33} \end{pmatrix}$$

$$(3 \times 3 = 3^2 = 9)$$

These are generally defined as an organized array of mathematical objects, such as numbers or functions.

Rank or Order

In generic terms, the rank of a tensor signifies (expresses) the complexity of its structure.

Tensors may be of different orders or ranks.

e.g., rank-0 tensor or, tensor of order 0 are called scalars., while rank-1 tensor or, tensor of order 1 are called vectors.

whereas, rank-2 or order 2 tensors may be called <u>dyads</u>, rank-3 or order 3 tensors are called <u>triads</u> & for higher rank tensors we use the term <u>polyads</u>.

—x—