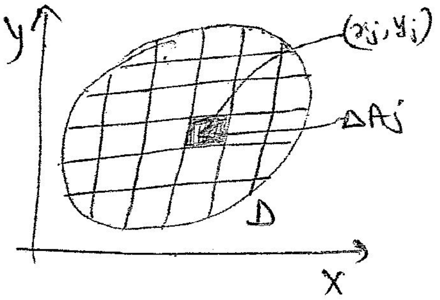
Multiple Integrals:

Double Integrals:

Let $f(x,y)$ be defined in a closed region D of the xy plane. Divide D into n subregions of area ΔA_j, $j = 1, 2, \ldots n$. Let (x_j, y_j) be some point of ΔA_j. Then consider



$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j, y_j)\, \Delta A_j$$

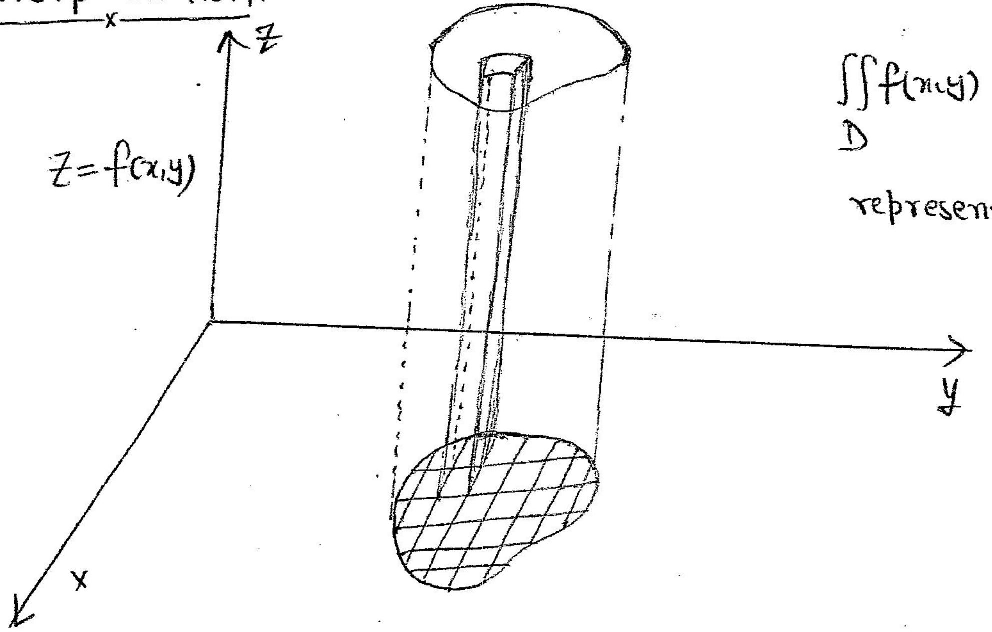
If this limit exists, then it is denoted by

$$\iint_D f(x,y)\, dA \quad \text{or} \quad \iint_D f(x,y)\, dx\, dy.$$

Note: It can be proved that the above limit exists if $f(x,y)$ is continuous or piecewise continuous in D.

Physical Interpretation:

1)



$\iint_D f(x,y)\, dA$ represents volume.

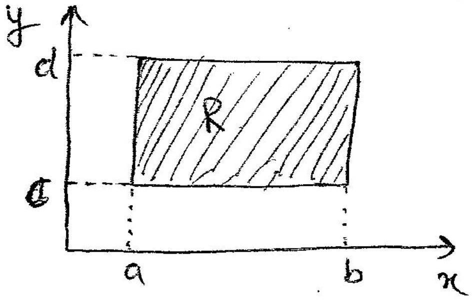
2) Area: $= \lim_{n\to\infty} \sum_{j=1}^{n} \Delta A_j$

$$= \iint_D dx\, dy \text{ or } \iint_D dA$$

Evaluation of double integrals:

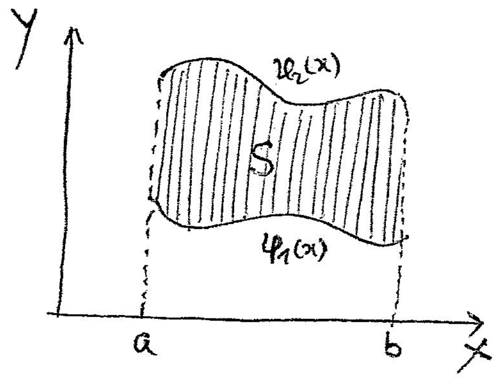
a) If $f(x,y)$ is continuous* on rectangular region $R: a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x,y)\, dA = \int_c^d \underbrace{\left\{\int_a^b f(x,y)\, dx\right\}}_{\varphi(y)} dy = \int_a^b \underbrace{\left\{\int_c^d f(x,y)\, dy\right\}}_{=:\,\psi(x)} dx .$$



*. or $f(x,y)$ is defined and bounded on R.

b)

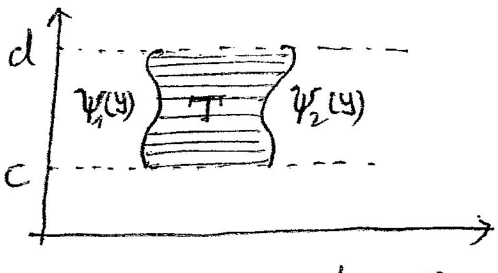


- $\psi_1(x)$ & $\psi_2(x)$ are continuous between a & b.
- $f(x,y)$ be defined and bounded on S

Then.

$$\iint_S f(x,y)\, dA = \int_a^b \underbrace{\int_{\psi_1(x)}^{\psi_2(x)} f(x,y)\, dy}_{} \, dx$$

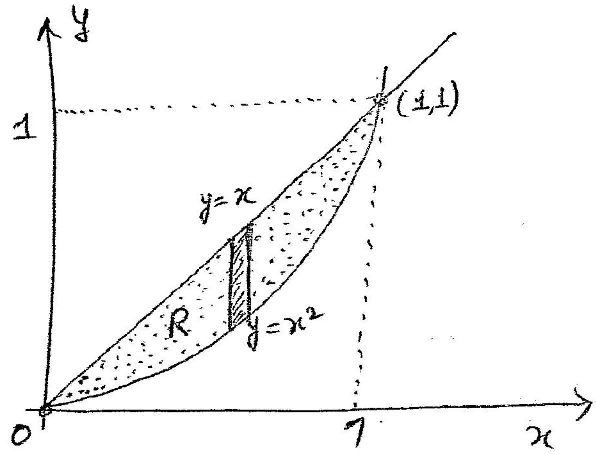
c)



$$\iint_T f(x,y)\, dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx\, dy$$

Example 1: Evaluate $\iint_R xy(x+y)\,\cancel{dx\,dy}\,dA$ where R is the region bounded by the line $y=x$ and the curve $y=x^2$

Solution:



$$I = \int_{x=0}^{1}\int_{x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_0^1 \left[\frac{y^2}{2}x^2 + x\cdot\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

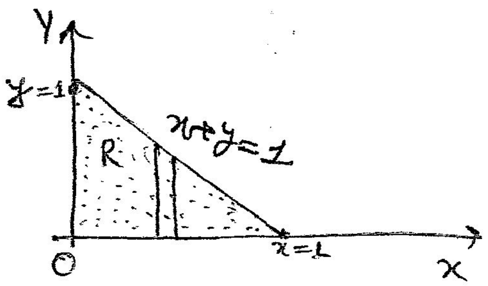
$$= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$= \frac{5}{6}\cdot\frac{1}{5} - \frac{1}{2}\cdot\frac{1}{7} - \frac{1}{3}\cdot\frac{1}{8} = \frac{3}{56}$$

OR

$$I = \int_{y=0}^{1}\int_{x=y}^{\sqrt{y}} xy(x+y)\,dx\,dy = \cdots = \frac{3}{56}.$$

Example 2: Evaluate $\iint_R e^{2x+3y}\,dx\,dy$ R is a triangle bounded by $x=0,\ y=0$ and $x+y=1$.



$$I = \int_{x=0}^{1}\int_{y=0}^{1-x} e^{2x+3y}\,dy\,dx$$

$$= \int_0^1 e^{2x}\cdot\left[\frac{e^{3y}}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 e^{2x}\,\frac{1}{3}\left\{e^{3-3x} - 1\right\} dx$$

$$= \frac{1}{3}\cdot\int_0^1 \left(e^{3-x} - e^{2x}\right) dx = \frac{1}{3}\left[-e^{3-x} - \frac{e^{2x}}{2}\right]_0^1$$

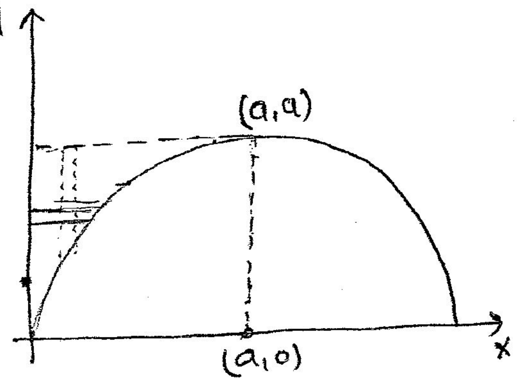
$$= -\frac{1}{3}\left[e^2 + \frac{e^2}{2} - e^3 - \frac{1}{2}\right] = -\frac{1}{3}\left[3\frac{e^2}{2} - e^3 - \frac{1}{2}\right].$$

Ans.

Example: Change the order of integration

$$\int_{y=0}^{a}\int_{x=0}^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy \quad \text{and evaluate.}$$

Solution:



x^2+a^2

$(x-a)^2+y^2=a^2$

$$\int_{y=0}^{a}\int_{x=0}^{a-\sqrt{a^2-y^2}} \cdots\, dx\,dy = \int_{x=0}^{a}\int_{y=\sqrt{a^2-(x-a)^2}}^{a} \frac{xy\log(x+a)}{(x-a)^2}\,dy\,dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\cdot\frac{1}{2}\cdot\left[a^2-\{a^2-(x-a)^2\}\right]\cdot dx$$

$$= \frac{1}{2}\int_0^a x\log(x+a)\,dx$$

$$= \frac{1}{2}\cdot\left[\left\{\frac{x^2}{2}\log(x+a)\right\}_0^a - \int_0^a \frac{x^2}{2}\cdot\frac{1}{(x+a)}\cdot dx\right]$$

$$= \frac{1}{2}\left[\left\{\frac{a^2}{2}\log(2a)\right\} - \frac{1}{2}\int_0^a\left[(x-a)+\frac{a^2}{x+a}\right]dx\right]$$

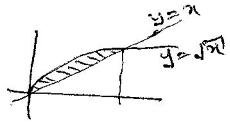
$$= \frac{1}{2}\left[\frac{a^2\log(2a)}{2} - \frac{1}{2}\left\{\frac{a^2}{2}-a^2+a^2\log(2a)-a^2\log a\right\}\right]$$

$$= \frac{a^2}{8}\left[1+2\log a\right].$$

Ans.

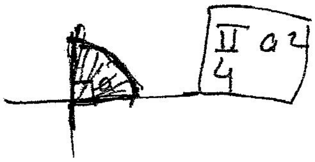
Q: Change the order of integration:

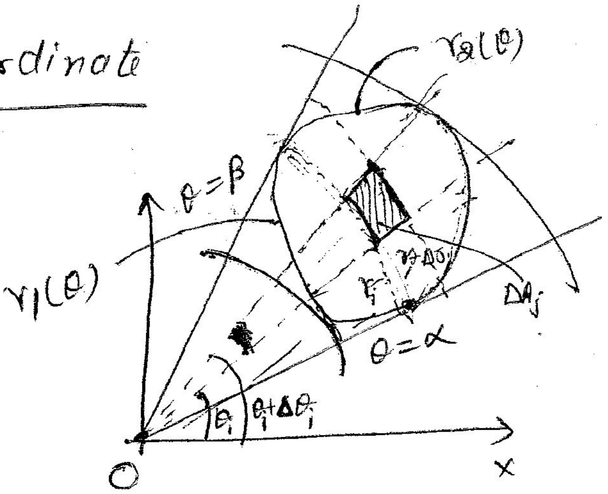
$$\int_0^1\int_x^{\sqrt{x}} f(x,y)\,dy\,dx$$



ANS: $\int_0^1\int_{y^2}^{y} f(x,y)\,dx\,dy$

Double integral in polar coordinate





πr^2

$$\Delta A_i = (r_i+\Delta r_i)^2 \frac{\Delta\theta_i}{2} - r_i^2 \frac{\Delta\theta_i}{2}$$

$$= \left(2r_i \Delta r_i + \Delta r_i^2\right) \frac{\Delta\theta_i}{2}$$

$$= \frac{(2r_i + \Delta r_i)}{2} \cdot \Delta r_i \Delta\theta_i$$

$$= \left(r_i + \frac{\Delta r_i}{2}\right) \Delta r_i \Delta\theta_i$$

$$= r_i^* \Delta r_i \Delta\theta_i \qquad r_i < r_i^* < r_i+\Delta r_i$$

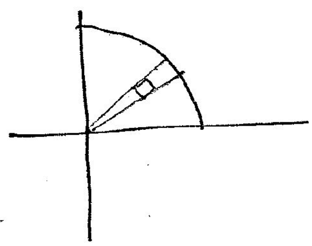
$$I = \lim_{n\to\infty} \sum_{j=1}^{n} f(r_j, \theta_j)\, \Delta A_j$$

$$I = \int_{\theta=\alpha}^{\beta} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Example: compute area of first quadrant of a circle:

$$A = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r\, dr\, d\theta$$

$$= \frac{a^2}{2} \cdot \pi/2 = \frac{\pi a^2}{4}$$



$\square$

Change of variables:

Analogous to the method of substitution in single variable.

$$\int_a^b f(x)\,dx \overset{x=g(t)}{=} \int_c^d f(g(t)) \cdot \underbrace{g'(t)}_{=\frac{dx}{dt}}\,dt$$

where $a = g(c)$ & $b = g(d)$.

We can change variables in two dimensional case.

Let the variables x, y in the double integral

$$\iint_R f(x,y)\,dx\,dy$$

be changed to new-variables u, v by means of relations

$$x = u(u,v) \quad , \quad y = \psi(u,v)$$

then the double integral is transformed into

$$\iint_{R'} f\{u(u,v), \psi(u,v)\}\,\underbrace{|J|}\,du\,dv$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ Jacobian.

R' is the region in uv plane which corresponds to the region R in the xy-plane.

Special Case: Cartesian to polar coordinate $x = r\cos\theta$ $y = r\sin\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

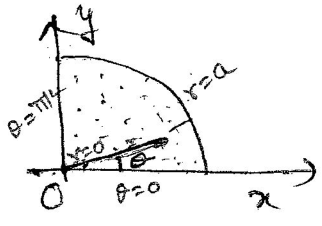
$$\Rightarrow \iint_S f(x,y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example 1: Volume of one octant of a sphere of radius a,

$$\iint_S \sqrt{a^2-x^2-y^2}\, dx\, dy$$

where S is the first quadrant of the circular disk $x^2+y^2 \le a^2$.

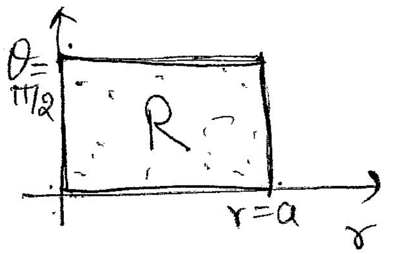
Change of variables:



$x = r\cos\theta \quad y = r\sin\theta$.

$|J| = r$.

$$\iint_S \sqrt{a^2-x^2-y^2}\cdot dx\,dy = \iint_R \sqrt{a^2-r^2}\cdot r\, dr\, d\theta$$



$$\iint_R \sqrt{a^2-r^2}\, r\, dr\, d\theta = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} \sqrt{a^2-r^2}\, r\, dr\, d\theta$$

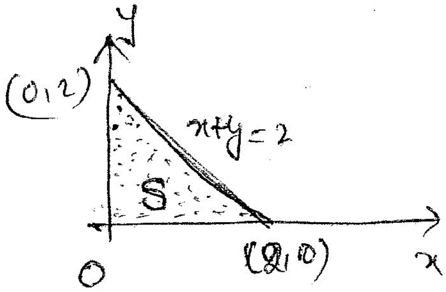
$$= \frac{\pi}{2}\cdot\left(-\frac{1}{2}\right)\frac{(a^2-r^2)^{3/2}}{3/2}\Bigg|_0^a$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}\right)(-a^3)\cdot\frac{2}{3}$$

$$= \frac{\pi}{6}a^3.$$

Example: 2:

$$\iint_S e^{(y-x)/(y+x)}\, dx\, dy$$



Change of variables

$$y-x=u \qquad\Rightarrow\qquad x=\frac{v-u}{2}$$
$$y+x=v$$
$$y=\frac{v+u}{2}.$$

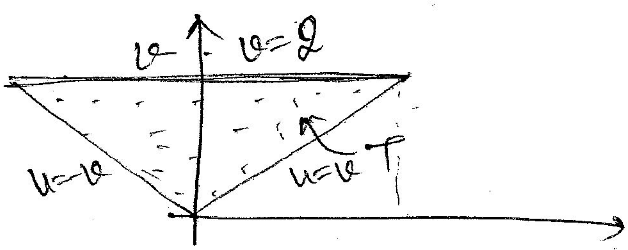
$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{vmatrix} = -\frac{1}{4}-\frac{1}{4}=-\frac{1}{2}.$$

domain in the uv plane:

line $x=0$ maps to $v=u$.

line $y=0$ maps to $v=-u$

line $x+y=2$ maps to $v=2$.



$$\iint_S e^{(y-x)/(y+x)}\, dx\, dy = \iint_T e^{u/v}\, \frac{1}{2}\, du\, dv$$

$$= \frac{1}{2}\int_{v=0}^{2}\int_{u=-v}^{v} e^{u/v}\, du\, dv.$$

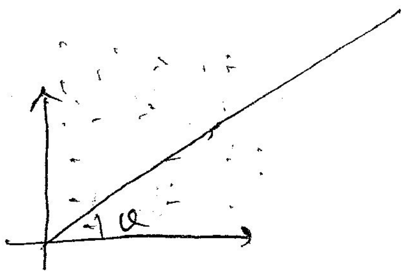
$$=\frac{1}{2}\int_0^2 v\left(e-\frac{1}{e}\right) dv$$

$$= e-\frac{1}{e}.$$

Ans.

Example: Change into polar coordinates and evaluate

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$$



$x = r\cos\theta \quad y = r\sin\theta$

$$\Rightarrow \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta$$

$$= \int_0^{\pi/2}\frac{1}{2}\,d\theta$$

$$= \frac{\pi}{4}.$$

Note: Let $I = \int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-y^2}dy$

$$\Rightarrow I^2 = \int_0^\infty e^{-x^2}dx \cdot \int_0^\infty e^{-y^2}dy$$

$$= \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \pi/4$$

$$\Rightarrow \boxed{I = \frac{\sqrt{\pi}}{2}}$$

$$\Rightarrow \boxed{\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}}$$

Ans.

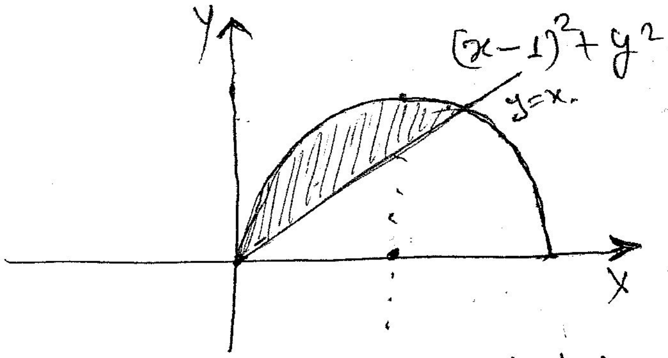
Example: Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$ by changing to polar co-ordinates.

Solution: The region of integration is bounded by

$y = x$, $y = \sqrt{2x-x^2}$, $x=0$ & $x=1$

$\Updownarrow$

$(x-1)^2 + y^2 = 1.$



Polar equation of the circle: $(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$

$\Rightarrow r^2 - 2r\cos\theta = 0$

$\Rightarrow \boxed{r = 2\cos\theta}$

$$\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\, dy dx$$

$$= \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2 \cdot r\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = \int_{\pi/4}^{\pi/2} 4\cos^4\theta\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (2\cos^2\theta)^2 d\theta = \int_{\pi/4}^{\pi/2} (1+\cos 2\theta)^2 d\theta$$

$$= \int_{\pi/4}^{\pi/2} [1 + \cos^2 2\theta + 2\cos 2\theta]\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[1 + \frac{1}{2}(1+\cos 4\theta) + 2\cos 2\theta\right] d\theta$$

$= \dots \frac{1}{8}(3\pi - 8)$ Ans

Example: Given that $x+y=u$, $y=uv$, show by change of the variables to u, v from x, y in the double integral

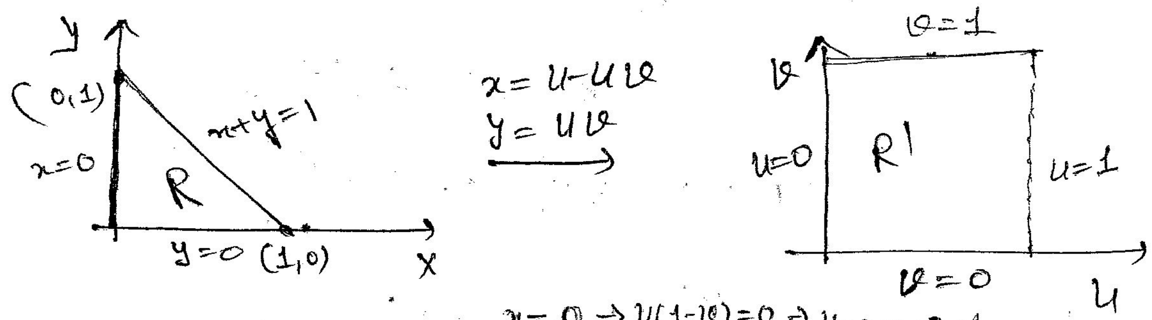
$$\iint \left[xy\{1-x-y\}\right]^{1/2} dx\,dy,$$ taken over the area enclosed by the lines $x=0$, $y=0$ and $x+y=1$ is $\frac{2\pi}{105}$.

Solution: $x+y=u \qquad y=uv$

$$x = u-uv = u(1-v)$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} (1-v) & -u \\ v & u \end{vmatrix}$$

$$= u-uv+uv = u.$$



$x=0 \Rightarrow u(1-v)=0 \Rightarrow u=0$ or $v=1$
$y=0 \Rightarrow uv=0 \Rightarrow u=0$ or $v=0$
$x+y=1 \Rightarrow u=1.$

$$\iint_R \left[xy[1-x-y]\right]^{1/2} dx\,dy = \iint_{R'} \sqrt{u(1-v)\cdot uv\,(1-u)}\cdot u\cdot du\,dv.$$

$$\Rightarrow \int_{v=0}^{1}\int_{u=0}^{1} u^2\sqrt{1-u}\cdot\sqrt{v(1-v)}\,du\,dv$$

$$= \int_{u=0}^{1} u^2\sqrt{1-u}\,du \int_{v=0}^{1} \sqrt{v(1-v)}\cdot dv.$$

$= \ldots$ Subst. $u=\sin^2\theta$, $v=\sin^2\beta$.

$$= \ldots \frac{2\pi}{105}$$

Example: Evaluate the integral

$$\iint_R \sqrt{x^2+y^2}\, dx\, dy$$ by changing to polar coordinates, where R is the region in the x-y plane bounded by the circles $x^2+y^2=4$ and $x^2+y^2=9$.

Solution: $x = r\cos\theta \quad y = r\sin\theta$.

$|J| = r$.

$$I = \int_0^{2\pi}\int_2^3 r\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^3}{3}\right]_2^3 d\theta$$

$$= \frac{(27-8)}{3}\cdot 2\pi$$

$$= \frac{19}{3}\cdot 2\pi$$

$$= \frac{38}{3}\cdot\pi$$

□