Linear Algebra

Lecture 17

Gram-Schmidt orthogonalization

<u>Definition</u>: Let V be an inner product space. A subset B of V is said to be an orthonormal basis if for

$B = \{v_1, \cdots, v_n\}$

$$\langle v_i, v_j \rangle = 0 \quad \text{for } i \neq j, \quad i,j = 1,2,\cdots,n$$

$$\|v_i\|^2 = \langle v_i, v_i \rangle = 1 \quad \forall\, i = 1,\cdots,n.$$

<u>Example</u>:

$\{e_1, e_2, \cdots, e_n\}$ is orthonormal basis of $\mathbb{F}^n$.

Example:

Is $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}$ orthonormal basis of $\mathbb{R}^2$??

Theorem: Let V be an inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal subset of V (none of these v_i's is zero). If $y \in \operatorname{span}(S)$

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$$

$\| a_i$

Proof:

Let $y = \sum_{i=1}^{k} a_i v_i$ where $a_1, \ldots, a_k \in F$.

$$\langle y, v_j \rangle = \left\langle \sum_{i=1}^{k} a_i v_i , v_j \right\rangle$$

$$= \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle$$

$$= a_j \langle v_j, v_j \rangle$$

$$= a_j \|v_j\|^2$$

$$\Rightarrow a_j = \frac{\langle y, v_j \rangle}{\|v_j\|^2}$$

$\boxtimes$

$\mathbb{R}^3$ $\{e_1, e_2, e_3\}$

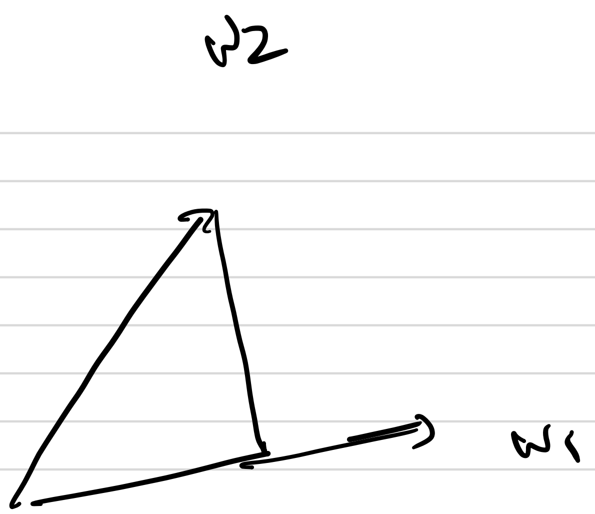
$y \in \mathbb{R}^3$

$\boxed{2}$

Corollary: If $B = \{v_1, \dots, u_k\}$ in previous theorem is orthonormal, $y \in \text{span}\{B)$

$$y = \sum_{i=1}^{k} \langle y, u_i \rangle v_i$$

Corollary: Let V be an inner product space and S be an orthogonal subset of V containing non-zero vectors, then S is linearly independent.



$$u_1 = \frac{w_1}{\|w_1\|}$$

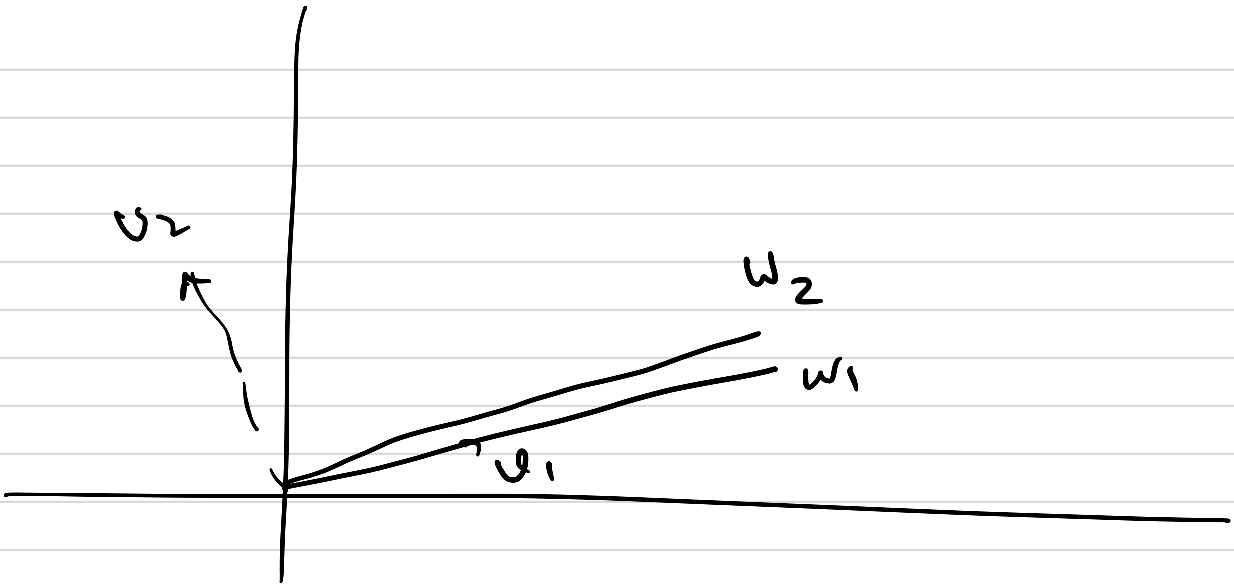
$$u_2 = w_2 - c_1 w_1$$

$$0 = \langle u_2, w_1 \rangle = \langle w_2 - c_1 w_1, w_1 \rangle$$

$$0 = \langle w_2, w_1 \rangle - c \langle w_1, w_1 \rangle$$

$$\Rightarrow c = \frac{\langle w_2, w_1 \rangle}{\|w_1\|^2}$$

$$u_2 = w_2 - \frac{\langle w_2, w_1 \rangle}{\|w_1\|^2} w_1$$



<u>Theorem</u>: Let V be an inner product space and $S = \{w_1, w_2, \dots, w_n\}$ be a linearly independent subset of V.
Define $S' = \{v_1, \dots, v_n\}$ where

$$v_1 = w_1$$

and
$$v_k = w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j$$

for $2 \leq k \leq n$.

Then S' is an orthogonal set of nonzero vectors such that $\mathrm{span}(S') = \mathrm{Span}(S)$.

Proof: By induction.

$k = 1, 2, \cdots, n$

If $n = 1$, trivially true.

Assume that the set

$\hat{S}_{k-1} = \{v_1, \cdots, v_{k-1}\}$ is with

the desired properties.

To show: $S'_k = \{v_1, \cdots, v_k\}$ also has the desired properties. where

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j$$

If $v_k = 0 \Rightarrow w_k \in \text{span}(\hat{S}_{k-1}) = \text{span}(S_{k-1})$

induction hypothesis.

Contradicts to the fact that $\{w_1, \cdots, w_n\}$ is a linearly independent set.

$\therefore v_k \neq 0$.

Now for $i = 1, 2, \cdots, k-1$

$$\langle v_k, v_i \rangle = \langle w_k - \sum_{j=1}^{n} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j, v_i \rangle$$

$$= \langle w_k, v_i \rangle - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} \langle v_i, v_j \rangle$$

$$= \langle w_k, v_i \rangle - \frac{\langle w_k, v_i \rangle}{\cancel{\|v_i\|^2}} \cancel{\langle v_i, v_i \rangle}$$

$$= 0$$

$\Rightarrow \{v_1, \dots, v_k\} = \text{span}\{S_k\}$ is an orthogonal set of non-zero vectors.

To show $\text{span}(S_k') = \text{span}(S_k)$

Consider

$$v_k = w_k - \underbrace{\sum_{i=1}^{k-1} \frac{\langle w_k, v_i \rangle}{\|v_i\|^2} v_i}_{\in \text{span}\{w_1, \dots, w_{k-1}\}}$$

$\Rightarrow \text{span}(S_k') \subseteq \text{span}(S_k)$

$\dim(\text{span}(S_k')) = k$ ($\because S_k'$ is an orthogonal set with non-zero vectors)

Also $\dim(\text{span}(S_k)) = k$

$\Rightarrow \text{span}(S_k') = \text{span}(S_k)$ $\blacksquare$

The process described in the theorem is called as Gram - Schmidt orthogonalization process.

Ex:

Let $\left\{ \underset{w_1}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{w_3}{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}} \right\}$ be a linearly independent set in $\mathbb{R}^4$.

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = v_1$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is an orthogonal set.
In order, orthonormal set which spans $\{w_1, w_2, w_3\}$, we divide these vectors by their respective norms.

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$$

QR decomposition.

Observe: $\mathbb{R}^n$

$\{w_1, \ldots, w_n\}$ given independent set.

$\{u_1, \ldots, u_n\}$ obtained orthogonal set by G.S. orthogonalization.

$$\begin{cases} u_1 = w_1 \\ u_k = w_k - \displaystyle\sum_{i=1}^{k-1} \frac{\langle w_k, u_i \rangle}{\|u_i\|^2} u_i \end{cases}$$

$k = 2, \ldots, n.$

$$\Downarrow$$

$$w_1 = u_1$$

$$w_k = u_k + \sum_{i=1}^{k-1} \underbrace{\frac{\langle w_k, u_i \rangle}{\|u_i\|^2}}_{= r_{ik}} u_i$$

$k = 2, \ldots, n$

$$\begin{bmatrix} | & | & & | \\ w_1 & w_2 & \cdots & w_n \\ | & | & & | \end{bmatrix}_{n \times n} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}_{n \times n} \begin{bmatrix} 1 & r_{12} & \cdots & r_{ik} \\ & 1 & \cdots & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{bmatrix}}_{A}{}_{n\times n} = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & & r_{ik} \\ & 1 & \\ 0 & & \ddots \\ & & 1 \end{bmatrix}}_{\tilde{R}}$$

$$= \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \overbrace{\begin{bmatrix} \|u_1\| & & \\ & \ddots & \\ & & \|v_n\| \end{bmatrix}}^{} R$$

$$= \underbrace{\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \|v_1\| & & r_{ik} \\ & \|v_2\| & \\ 0 & & \ddots \\ & & \|v_n\| \end{bmatrix}}_{R}$$

Check:

$$Q^T Q = Q Q^T = I_{n\times n}$$

System of linear eq$^{\underline{n}}$:-

$Ax = b$; $A \in \mathbb{R}^{n \times n}$; A is invertible
$b \in \mathbb{R}^n$

Collect the columns of A as $\{a_1, \cdots, a_n\}$.

Then $\text{span}\{a_1, \cdots, a_n\}$ = col span of A actually spans $\mathbb{R}^n$.

In particular $\{a_1, \cdots, a_n\}$ is a linearly independent set.

By G.S. $\{v_1, \cdots, v_n\}$ is orthogonal and $\text{span}\{v_1, \cdots, v_k\} = \text{span}\{a_1, \cdots, a_k\}$
$k = 1, 2, \cdots, n$.

Construct $\{q_1, \cdots, q_n\}$ from $\{v_1, \cdots, v_n\}$

$$q_i = \frac{v_i}{\|v_i\|} \qquad i = 1, 2, \cdots, n.$$

Then in the matrix notation,

$$A = QR$$

where $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \in \mathbb{R}^{n \times n}$

$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{n \times n}$

with $Q^T Q = Q Q^T = I_{n \times n}$

$R = \begin{bmatrix} \|v_1\| & & r_{ik} \\ & \|v_2\| & \\ 0 & & \ddots \quad \|v_n\| \end{bmatrix}$

upper triangular matrix

$$Ax = b$$

$\Rightarrow QRx = b \quad \Rightarrow \quad Rx = Q^T b$

Example:

$V = P(\mathbb{R})$: the real vecter space of all polynomials.

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

Consider $\mathbb{P}_2(\mathbb{R})$: subspace of all polynomials upto & including degree 2.

Let $\{1, x, x^2\}$ be standard basis of $P_2(\mathbb{R})$.

Compute the corresponding orthonormal basis using G-S. orthogonalization.

$$B = \{\underset{w_1}{\underset{\shortparallel}{1}}, \underset{w_2}{\underset{\shortparallel}{x}}, \underset{w_3}{\underset{\shortparallel}{x^2}}\}$$

$v_1 = w_1 = 1.$

(ii)

$w_2 = x$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$\|v_1\|^2 = \langle v_1, v_1 \rangle$$

$$= \int_{-1}^{1} 1 \, dt$$

$$= 2$$

$$\langle w_2, v_1 \rangle = \int_{-1}^{1} t \, dt = 0$$

$$v_2 = x - 2 \cdot 0 = x$$

(iii) $w_3 = x^2$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$\|v_2\|^2 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

$$\langle w_3, v_1 \rangle = \int_{-1}^{1} t^2 \, dt = \frac{2}{3}$$

$$\langle w_3, v_2 \rangle = \int_{-1}^{1} t^2 \cdot t \, dt = 0$$

$$v_3 = x^2 - \frac{1}{3} - 0$$

$$v_3 = x^2 - \frac{1}{3}$$

$\left\{1, x, x^2 - \frac{1}{3}\right\}$ is orthogonal basis.

$\downarrow$ normalizing

$$\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \sqrt{\frac{5}{8}}\,(3x^2 - 1)\right\}$$

is orthonormal basis of $\mathbb{P}_2(\mathbb{R})$.

$\blacksquare$

Theorem: Let V be a finite dimensional non-zero inner product space. Then V has an orthonormal basis β. Further $\beta = \{v_1, \dots, v_n\}$ and for any $x \in V$

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$$

Example:

Express polynomial $f(x) = 1 + 2x + 3x^2$ as a linear combination of

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \sqrt{\frac{5}{8}}\,(3x^2 - 1) \right\}$$

Corollary: Let V be a finite dimensional inner product space with orthonormal basis $\beta = \{v_1, \dots, v_n\}$. Let T be a linear operator on V. Let A be a matrix representation of T w.r.t. β.

then $A_{ij} = \langle T(v_j), v_i \rangle$ $\blacksquare$

<u>Definition</u>: Let B be an orthonormal subset (possibly infinite) of an inner product space V. Let $x \in V$. Then we define <u>Fourier Coefficients</u> of x relative to B as $\langle x, y \rangle$ for $y \in B$.

<u>Example</u>:

Let $S = \{e^{int} : n \text{ is an integer}\}$ is an orthonormal set of $C[0,2\pi]$

Continuous complex functions on $[0,2\pi]$.

$f, g \in C[0,2\pi]$

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, \overline{g(t)}\, dt$$

Q: Can you compute the Fourier coefficients of $f(t) = t$ relative to S.

for $n \neq 0$

$$\langle f, e^{int} \rangle = \frac{1}{2\pi} \int_0^{2\pi} t \, \overline{e^{int}} \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} t e^{-int} \, dt$$

$$= -\frac{1}{in}.$$

$n = 0$

$$\langle f, 1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} t \, dt = \pi$$