Solution Model for Mathematical Methods (MA31007)

Q1) Prove that (i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$,

(ii) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, (iii) $\left\{J_{1/2}(x)\right\}^2 + \left\{J_{-1/2}(x)\right\}^2 = \frac{2}{\pi x}$,

(2+2+1 = 5M)

Solⁿ :- By the defⁿ of $J_n(x)$, we have

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2\cdot 2\cdot (n+1)} + \frac{x^4}{2\cdot 4\cdot 2^2 (n+1)(n+2)} - \cdots\right] \longrightarrow (1)$$

(i) Replacing n by $(-1/2)$ in (1) & simplifying, we get

$$J_{1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\Gamma(1/2)} \left[1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \cdots\right]$$

$$= \boxed{\sqrt{\frac{2}{\pi x}} \cos x}, \text{ as } \Gamma(1/2) = \sqrt{\pi} \longrightarrow (2M)$$

(ii) Replacing n by $(1/2)$ in (1) & simplifying, we get

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\Gamma(3/2)} \left[1 - \frac{x^2}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots\right]$$

$$= \sqrt{\left(\frac{x}{2}\right)} \cdot \frac{1}{\frac{1}{2}\Gamma(1/2)} \cdot \frac{1}{x}\left[x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots\right]$$

$$\left[\because \Gamma(P+1) = P\,\Gamma(P)\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right] = \boxed{\sqrt{\frac{2}{\pi x}} \sin x} \longrightarrow (2M)$$

(iii) Squaring & adding the results of (i) & (ii), we get

$$\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \left(\frac{2}{\pi x}\right)\left(\sin^2 x + \cos^2 x\right)$$

$$= \left(\frac{2}{\pi x}\right) \longrightarrow (1M)$$

Q2) (a) Show that the Bessel function

$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ satisfies the Bessel eq^n of order $1/2$. (4M)

Sol^n :- Bessel eq^n of order $(1/2)$ is given by

$$x^2\left(\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx}\right) + \left(x^2 - \tfrac{1}{4}\right)y = 0 \longrightarrow (1).$$

$\longrightarrow$ (1M)

$$\text{Let } y = J_{1/2}(x) = \left(\frac{2}{\pi}\right)^{1/2} \times \left(x^{-1/2}\sin x\right) \longrightarrow (2)$$

$$\therefore \frac{dy}{dx} = \left(\frac{2}{\pi}\right)^{1/2} \times \left[x^{-1/2}\cos x + \left(-\tfrac{1}{2}\right)x^{-3/2}\sin x\right].$$

$\longrightarrow$ (1M)

$$\therefore \frac{d^2y}{dx^2} = \left(\frac{2}{\pi}\right)^{1/2} \times \left\{-\tfrac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x + \tfrac{3}{4}x^{-5/2}\sin x - \tfrac{1}{2}x^{-3/2}\cos x\right\}$$

$\longrightarrow$ (1M)

Substituting the above values of y, $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in (1), we get

$$x^2 \times \left(\frac{2}{\pi}\right)^{1/2}\left\{-x^{-3/2}\cos x - x^{-1/2}\sin x + \left(\frac{3}{4}\right)x^{-5/2}\sin x\right]$$

$$+ x \times \left(\frac{2}{\pi}\right)^{1/2}\left\{x^{-1/2}\cos x - \frac{1}{2}\times\left(x^{-3/2}\right)\sin x\right]$$

$$+\left(x^2 - \tfrac{1}{4}\right)\times\left(\frac{2}{\pi}\right)^{1/2}\times\left(x^{-1/2}\sin x\right) = 0.$$

$$\text{or, } \left(\frac{2}{\pi}\right)^{1/2}\left\{-x^{1/2}\cos x - x^{3/2}\sin x + \left(\tfrac{3}{4}\right)x^{-1/2}\sin x + x^{1/2}\cos x - \tfrac{1}{2}x^{-1/2}\sin x + x^{3/2}\sin x - \frac{1}{4}x^{-1/2}\sin x\right] = 0,$$

or, $0 = 0$, which is true. $\longrightarrow$ (1M)

Hence, $y = J_{1/2}(x)$ satisfies the Bessel eq^n (1) of order $1/2$.

2) b) Write the general solution for the following eq^n (1M)

$$z\frac{d^2y}{dz^2}+\frac{dy}{dz}+zy=0.$$

Sol^n:- Re-writing the given eq^n,

$$z\frac{d^2y}{dz^2}+\frac{dy}{dz}+zy=0 \quad (\times z)$$

or, $$z^2\frac{d^2y}{dz^2}+z\left(\frac{dy}{dz}\right)+z^2y=0,$$

which is a Bessel equation of order 0, which is an integer.

Its solution is ——

$$y=AJ_0(z)+BY_0(z). \rightarrow (1M)$$

Q3) Prove that

(i) $\frac{d}{dx}\{x^n J_n(x)\}=x^n J_{n-1}(x)$.

(ii) $\frac{d}{dx}\{x^{-n}J_n(x)\}=-x^{-n}J_{n+1}(x)$. (2+3=5M)

Sol^n:- (i) Using the def^n of $J_n(x)$, we have

$$\frac{d}{dx}\{x^n J_n(x)\}=\frac{d}{dx}\left[x^n\sum_{m=0}^{\infty}(-1)^m\frac{1}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n}\right]$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(n+m+1)}\frac{1}{2^{2m+n}}\frac{d}{dx}\left(x^{2m+2n}\right)$$

$$=\sum_{m=0}^{\infty}\left\{\frac{(-1)^m(2m+2n)\,x^{2m+2n-1}}{m!\,\Gamma(n+m+1)\,2^{2m+n}}\right\} \rightarrow (1M)$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m\,2\,\cancel{(m+n)}}{m!\,\cancel{(n+m)}\,\Gamma(n+m)}\cdot\frac{x^n\,x^{2m+n-1}}{2^{2m+n}} \quad \left[\because \Gamma(n+1)=n\Gamma(n)\right]$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n-\cancel{1}+r+\cancel{1})} \left(\frac{x}{2}\right)^{2r+n-1}.$$

$= x^n J_{n-1}(x)$, by the def^n of $J_{n-1}(x)$. $\rightarrow$ (1M)

(ii) Using the def^n of $J_n(x)$, we have

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \frac{d}{dx}\left\{x^{-n} \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n}} \cdot \frac{d}{dx}\left(x^{2r}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2\cancel{r}\, x^{2r-1}}{\cancel{r}(r-1)!\,\Gamma(n+r+1)\, 2^{2r+n}}$$

$\rightarrow$ (1M)

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r-1}}{(r-1)!\,\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n-1}}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r x^{2r-1}}{(r-1)!\,\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n-1}}$$

$\rightarrow$ (1M)

[since $(r-1)! = \infty$ when $r=0$, so the term corresponding to $r=0$ vanishes]

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} x^{2m+2-1}}{m!\,\Gamma(n+m+2)} \cdot \frac{x^n \cdot x^{-n}}{2^{2m+2+n-1}} \quad [r = 1+m]$$

(On changing the variable of summation to $m = r-1$ so that $r = m+1$. Then $m=0$ when $r=1$ & $m=\infty$ when $r=\infty$)

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+2)} \left(\frac{x}{2}\right)^{n+1+2m}$$

$$= -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+1+r+1)} \left(\frac{x}{2}\right)^{n+1+2r}$$

(On changing the variable of summation from m to r)

$= -x^{-n} J_{n+1}(x)$, by the def^n of $J_{n+1}(x)$. $\longrightarrow$ (1M)

Q4) Prove that

$$\exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right] = \sum_{n=-\infty}^{\infty} z^n J_n(x),$$

where $\exp\left[\frac{x}{2}\left(z-\frac{1}{z}\right)\right]$ is the generating f^n for $J_n(x)$.

(5M)

Sol^n:- We have,

$$\exp\left[\left(\frac{x}{2}\right)\left(z-\frac{1}{z}\right)\right] = e^{\left(\frac{xz}{2} - \frac{x}{2z}\right)} = e^{\frac{xz}{2} - \frac{x}{2z}}$$

$$= e^{\frac{xz}{2}} \cdot e^{-\frac{x}{2z}}$$

$$= \left[1 + \left(\frac{x}{2}\right)z + \left(\frac{x}{2}\right)^2 \frac{z^2}{2!} + \dots + \left(\frac{x}{2}\right)^n \frac{z^n}{n!} + \left(\frac{x}{2}\right)^{n+1} \frac{z^{n+1}}{(n+1)!} + \dots\right]$$

$$\times \left[1 - \left(\frac{x}{2}\right) z^{-1} + \left(\frac{x}{2}\right)^2 \frac{z^{-2}}{2!} + \dots + \left(\frac{x}{2}\right)^n \frac{(-1)^n z^{-n}}{n!} + \left(\frac{x}{2}\right)^{n+1} \frac{(-1)^{n+1} z^{-(n+1)}}{(n+1)!} + \dots\right] \longrightarrow (1)$$

$\longrightarrow$ (1M)

The co-efficient of z^n in the product (1) is obtained by multiplying the co-efficients of $z^n, z^{n+1}, z^{n+2}, \ldots$ in the first bracket with the co-efficient of $z^0, z^{-1}, z^{-2}, \ldots$ in the second bracket respectively.

$\therefore$ Co-efficient of z^n in product (1)

$$= \left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2} \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \frac{1}{(n+2)!\,2!} - \ldots$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} = J_n(x).$$

[$\because$ $(n+r)! = \Gamma(n+r+1)$, $(n+r)$ being positive integer].

(1M)

The coefficient of z^{-n} in the product (1) is obtained by multiplying the co-efficients of $z^{-n}, z^{-n-1}, z^{-n-2}, \ldots$ of the second bracket with the co-efficients of $z^0, z^1, z^2, \ldots$ in the first bracket resly.

$\therefore$ Co-efficient of z^{-n} in product (1)

$$= \left(\frac{x}{2}\right)^n \frac{(-1)^n}{n!} \cdots + \left(\frac{x}{2}\right)^{n+1} \cdot \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)$$

$$+ \left(\frac{x}{2}\right)^{n+2} \frac{(-1)^{n+2}}{(n+2)!\, 2!}\left(\frac{x}{2}\right)^2 + \cdots$$

$$= (-1)^n \left[\left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2} \cdot \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \cdot \frac{1}{(n+2)2!} - \cdots\right]$$

$= (-1)^n J_n(x)$, as before $\longrightarrow$ (1M)

Thus, the co-efficient of z^{-n}

$$= (-1)^n J_n(x)$$

$\Rightarrow J_n(x) = (-1)^n \times$ the co-efficient of z^{-n}.

Finally, in the product (1), the co-efficient of z^0 is obtained by multiplying the co-efficient of $z^0, z^1, z^2, \ldots$ in the first bracket with the co-efficients of $z^0, z^{-1}, z^{-2}, \ldots$ in the second bracket & is thus

$$= 1 - \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 \left(\frac{1}{2!}\right)^2 - \left(\frac{x}{2}\right)^6 \left(\frac{1}{3!}\right)^2 + \cdots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \cdots = J_0(x).$$

$\longrightarrow$ (1M)

We observe that the coefficients of

z^0, $(z - z^{-1})$, $(z^2 + z^{-2})$, ..., $[z^n + (-1)^n z^{-n}]$ ...

are $J_0(x)$, $J_1(x)$, $J_2(x)$, ..., $J_n(x)$, ... resp^ly.

Thus, (1) gives

$$\exp\left\{\left(\frac{x}{2}\right) \times (z - 1/z)\right\} = J_0(x) + (z - z^{-1}) J_1(x) + (z^2 + z^{-2}) J_2(x) + \cdots + [z^n + (-1)^n z^{-n}] J_n(x) + \cdots$$

$$= \sum_{n=-\infty}^{\infty} z^n J_n(x),$$

as $J_{-n}(x) \Rightarrow (-1)^n J_n(x)$. $\longrightarrow$ (1M)

Q5) a) Show that $J_n(x) = 0$ has no repeated roots except at $x = 0$;

b) Express $J_4(x)$ in terms of J_0 & J_1.

(2+3=5)

soln:- a) If possible, suppose $J_n(x) = 0$ has repeated roots, then at least two roots must be equal (say α), i.e., α is a double root of $J_n(x) = 0$. Then from the theory of eqn, we have

$J_n(\alpha) = 0$ & $J_n'(\alpha) = 0 \cdot \rightarrow (1)$

$\rightarrow$ (1M)

Now, using Recurrence relations III & IV,

$$J_{n-1}(x) = \left(\frac{n}{x}\right) J_n(x) + J_n'(x) \rightarrow (2)$$

& $$J_{n+1}(x) = \left(\frac{n}{x}\right) J_n(x) - J_n'(x) \rightarrow (3)$$

Replacing x by α in (2) & (3) & using (1), we get

$J_{n+1}(\alpha) = 0$ & $J_{n-1}(\alpha) = 0$, except

when $x = 0$.

Since two different power series have distinct sum functions, so

$J_{n+1}(\alpha) = 0 = J_{n-1}(\alpha)$ must be absurd.

Hence, $J_n(x) = 0$ has no repeated roots except at $x = 0$. $\rightarrow$ (1M)

Q5) b) Express $J_4(x)$ in terms of J_0 & J_1.

(3M)

Sol^n :- Recurrence relation VI is

$$J_{n+1}(x) = \left(\frac{2n}{x}\right) J_n(x) - J_{n-1}(x) \rightarrow (1)$$

Replacing n by 3 in (1), we get

$$J_4(x) = \left(6/x\right) J_3(x) - J_2(x) \rightarrow (2)$$

$\rightarrow$ (1M)

Now, replacing n by 2 in (1), we get

$$J_3(x) = \left(4/x\right) J_2(x) - J_1(x) \rightarrow (3)$$

Using (3), (2) becomes

$$J_4(x) = \frac{6}{x}\left[\frac{4}{x}J_2(x) - J_1(x)\right] - J_2(x)$$

or, $$J_4(x) = \left(\frac{24}{x^2} - 1\right)J_2(x) - \frac{6}{x}J_1(x) \rightarrow (4)$$

$\rightarrow$ (1M)

Next, replacing n by 1 in (1) gives

$$J_2(x) = \left(\frac{2}{x}\right)J_1(x) - J_0(x) \rightarrow (5)$$

Using (5), (4) becomes —

$$J_4(x) = \left(\frac{24}{x^2} - 1\right)\left[\frac{2}{x}J_1(x) - J_0(x)\right] - \frac{6}{x}J_1(x)$$

or, $$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) - \left(\frac{24}{x^2} - 1\right)J_0(x).$$

$\rightarrow$ (1M)

Q6) a) Show that

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\,d\phi,$$

where n is a positive integer. (3M)

Solⁿ:- We shall use the following results:-

$$\int_0^{\pi} \cos m\phi \cos n\phi \, d\phi = \int_0^{\pi} \sin m\phi \sin n\phi \, d\phi$$

$$= \begin{cases} \pi/2, & \text{when } m = n \\ 0, & \text{when } m \neq n \end{cases} \rightarrow (1)$$

$$\cos(x\sin\phi) = J_0 + 2J_2\cos(2\phi) + 2J_4\cos 4\phi + \cdots \rightarrow (2)$$

& $$\sin(x\sin\phi) = 2J_1\sin\phi + 2J_3\sin 3\phi + 2J_5\sin 5\phi + \cdots \rightarrow (3)$$

Multiplying both sides of (2) by $\cos(n\phi)$ & then integrating w.r.t 'ϕ' betn limits 0 to π & using (1), we have ——

$$\int_0^{\pi} \cos(x \sin\phi) \cos n\phi \, d\phi = \begin{cases} 0, & \text{if } n \text{ is odd} \quad \longrightarrow (4) \\ \pi J_n, & \text{if } n \text{ is even} \quad \longrightarrow (5) \end{cases}$$

Again, multiplying both sides of (3) by $\sin(n\phi)$ & then integrating w.r.t 'ϕ' betn limits 0 to π & using (1), we get

$$\int_0^{\pi} \sin(x \sin\phi) \sin n\phi \, d\phi = \begin{cases} \pi J_n, & \text{if } n \text{ is odd} \quad \longrightarrow (6) \\ 0, & \text{if } n \text{ is even} \quad \longrightarrow (7) \end{cases}$$

(1M)

Let n be odd. Adding (4) & (6), we get

$$\int_0^{\pi} [\cos(x \sin\phi) \cos n\phi + \sin(x \sin\phi) \sin n\phi] = \pi J_n$$

or, $$\int_0^{\pi} \cos(n\phi - x\sin\phi) \, d\phi = \pi J_n$$

or, $$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\phi - x\sin\phi) \, d\phi \quad \longrightarrow (8)$$

(1M)

Next, let n be even. Then adding (5) & (7) as before, we again get (8). Thus, (8) holds for each positive integer (even as well as odd). (1M)

Q7) Show that

$$\frac{d}{dx}F(\alpha,\beta;\gamma;x) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1;\gamma+1;x)$$

Hence, deduce that

$$\frac{d^n}{dx^n}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\beta+n;\gamma+n;x)$$

where $F(\alpha,\beta;\gamma;x)$ is the Hyper-geometric function.

[3+2=5]

Sol^n:– By def^n, we have —

$$F(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r\, r!}x^r \longrightarrow (1M)$$

Differentiating both sides w.r.t 'x', we have

$$\frac{d}{dx}F(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r\, r!}\cdot r\, x^{r-1}$$

$$= \sum_{r=1}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r\,(r-1)!}x^{r-1}$$ [$\because$ the term with $r=0$ vanishes.

$$= \sum_{m=0}^{\infty}\frac{(\alpha)_{m+1}(\beta)_{m+1}}{(\gamma)_{m+1}\, m!}x^m \longrightarrow (1M)$$

(taking m as the new variable of summation such that

[Taking $r=m+1$ i.e., $m=r-1$ so that when $r=1$, $m=0$ & $r=\infty$, $m=\infty$]

$$= \sum_{m=0}^{\infty}\frac{\alpha(\alpha+1)_m\,\beta(\beta+1)_m}{\gamma(\gamma+1)_m\, m!}x^m$$

$$= \frac{\alpha\beta}{\gamma}\sum_{m=0}^{\infty}\frac{(\alpha+1)_m(\beta+1)_m}{(\gamma+1)_m\, m!}x^m$$

(14)

$$= \frac{\alpha\beta}{\gamma} F\left(\alpha+1, \beta+1; \gamma+1; x\right)$$

$$\therefore \frac{d}{dx} F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1; \gamma+1; x) \quad \text{(1m)} \longrightarrow (1)$$

Deduction :- (i) For each positive integer, we must show that

$$\frac{d^n}{dx^n} F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} F\left(\alpha+n, \beta+n, \gamma+n, x\right) \longrightarrow (2)$$

Since $\alpha = (\alpha)_1$, $\beta = (\beta)_1$ & $\gamma = (\gamma)_1$, (1) shows that (2) is true for $n = 1$. We now assume that (2) is true for a particular value of n (say, $n=m$) so that

$$\frac{d^m}{dx^m} F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} F\left(\alpha+m, \beta+m; \gamma+m; x\right) \longrightarrow (3)$$

Differentiating both sides of (3) w.r.t 'x', we get

$$\frac{d^{m+1}}{dx^{m+1}} F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \frac{d}{dx} F\left(\alpha+m, \beta+m; \gamma+m; x\right)$$

$$= \frac{(\alpha)_m (\beta)_m (\alpha+m)(\beta+m)}{(\gamma)_m (\gamma+m)} F(\alpha+m+1, \beta+m+1; \gamma+m+1; x)$$

[Using ① for $\alpha+m, \beta+m, \gamma+m$, in place of α, β, γ resly] $\longrightarrow$ (1M)

$$\therefore \frac{d^{m+1}}{dx^{m+1}} F(\alpha, \beta; \gamma; x)$$

$$= \frac{(\alpha)_{m+1} (\beta)_{m+1}}{(\gamma)_{m+1}} F(\alpha+m+1, \beta+m+1; \gamma+m+1; x) \longrightarrow ④$$

eqn ④ shows that ② is true for $n = m+1$. Thus, if $\left[\because (\alpha+n)(\alpha)_n = (\alpha)_{n+1}\right]$ ② is true for $n = m$, then ② is also true for $n = m+1$.

Hence, by mathematical induction, ② is true for each positive integer. $\longrightarrow$ (1M)

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28) a) Evaluate $\lim\limits_{a \to \infty} {}_2F_1\left(1, a; 1; \frac{x}{a}\right)$, where ${}_2F_1\left(1, a; 1; \frac{x}{a}\right)$ is the Hyper-geometric function. (2M)

Sol^n:— By def^n,

$${}_2F_1(1, a; 1; x/a)$$

$$= 1 + \frac{1\cdot a}{1\cdot 1}\left(\frac{x}{a}\right) + \frac{1\cdot 2\, a(a+1)}{1\cdot 2\cdot 1\cdot 2}\left(\frac{x}{a}\right)^2 + \cdots \longrightarrow (1M)$$

$$= 1 + \frac{x}{1!} + \left(1+\frac{1}{a}\right)\frac{x^2}{2!} + \left(1+\frac{1}{a}\right)\left(1+\frac{2}{a}\right)\frac{x^3}{3!} + \cdots.$$

$$\therefore \lim_{a\to\infty} {}_2F_1(1, a; 1; x/a) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \boxed{e^x}. \longrightarrow (1M)$$

Q8) b) Find the solution of the following equation

$$x(1-x)\frac{d^2y}{dx^2} + \left(\frac{3}{2} - 2x\right)\frac{dy}{dx} + 2y = 0,$$

about $x = 0$.

(3 M)

Sol^n:— Rewriting the given eq^n, we have

$$x(1-x)y'' + (3/2 - 2x)y' + 2y = 0 \longrightarrow (1)$$

Comparing (1) with

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0,$$

we have

$$\gamma = 3/2, \quad \alpha+\beta+1 = 2 \ \ \& \ \ \alpha\beta = -2. \longrightarrow (1/2 M)$$

Solving, we, $\alpha = 2$, $\beta = -1$, $\gamma = 3/2$.

Here, γ is not an integer. The

$\therefore (\alpha, \beta, \gamma) = (2, -1, 3/2)$ or, $(-1, 2, 3/2)$

general solution of ① is $y = au + bv$,
where

$$u = {}_2F_1(\alpha, \beta; \gamma; x) = F(2, -1; 3/2; x)$$

$$= 1 + \frac{2\times(-1)}{1\times(3/2)} + \frac{2\times3\times(-1)\times0}{1\times2\times(3/2)\times(5/2)}x^2 + \cdots$$

$$= 1 - \frac{4x}{3}$$ → (1/2 M)

$$\& \; v = x^{1-\gamma}\,{}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x)$$

$$= x^{1-(3/2)}\,{}_2F_1(2+1-3/2, -1+1-3/2; 2-3/2; x)$$

$$= x^{-1/2}\,{}_2F_1(3/2, -3/2; 1/2; x)$$ → (1 M)

Hence, the general solution of ① is given by

$$y = a(1 - 4x/3) + b\,x^{-1/2}\,{}_2F_1(3/2, -3/2; 1/2; x)$$

→ (1 M)

Q9) a) Give definition of Symmetric & skew-symmetric tensors. Prove that a skew-symmetric tensor of second order has atmost $\frac{1}{2}N(N-1)$ independent components.

(3 M)

Sol^n:– A tensor is called symmetric w.r.t two contravariant or, two covariant indices if its components remain unaltered (unchanged) upon the interchange of the indices.

Thus, $A^{ij} = A^{ji}$, $A_{ij} = A_{ji}$ for two indices i & j.

& if $A^{ijk}_{lm} = A^{jik}_{lm}$, the tensor is symmetric in indices i & j. If a tensor is symmetric w.r.t any two contravariant & any two covariant indices, it is called 'symmetric'.

→ (1M)

A tensor is called skew-symmetric w.r.t two contravariant or, two covariant indices, if its components alters (change) in sign upon interchange of the indices.

Thus, $A^{ij} = -A^{ji}$, $A_{ij} = -A_{ji}$ for two indices

& if $A^{ijk}_{lm} = -A^{jik}_{lm}$, the tensor is skew-symmetric in indices i & j.

If a tensor is skew-symmetric w.r.t any two contra-variant & any two covariant indices, it is called 'skew-symmetric'

→ (1M)

Suppose A^{ij} is a second-order contravariant tensor that has N^2 — components in the space V_N. Then the components are as follows:

$$A^{11} \quad A^{12} \quad A^{13} \quad \ldots \quad A^{1N}$$
$$A^{21} \quad A^{22} \quad A^{23} \quad \ldots \quad A^{2N}$$
$$\text{——————————————}$$
$$A^{N1} \quad A^{N2} \quad A^{N3} \quad \ldots \quad A^{NN}$$

As A^{ij} is skew-symmetric, $A^{ij} = -A^{ji}$, for every $i \,\&\, j$,

ie, $A^{12} = -A^{21}$, $A^{32} = -A^{23}$ etc.

Taking $j = i$, we get $A^{ii} = -A^{ii}$

$$\Rightarrow 2A^{ii} = 0 \Rightarrow A^{ii} = 0.$$

$$\Rightarrow A^{11} = A^{22} = \ldots = A^{NN} = 0.$$

ie, number of components corresponding to a repeated suffix is zero. Then, no. of components corresponding to different suffixes is $N^2 - N$.

However, as A^{ij} is skew-symmetric, this number reduces to half,

ie, $\dfrac{N^2 - N}{2} = \dfrac{1}{2}N(N-1)$. → (1 M)

Thus, the total no. of independent components in the tensor A^{ij} is $0 + \frac{1}{2}N(N-1) = \frac{1}{2}N(N-1)$.

This can be proved for skew-symmetric covariant tensor A_{ij} of rank 2 (ie, $A_{ij} = -A_{ji}$).

Q9) b) Define Christoffel Symbols or Brackets of first & second kind. Show that

$$[i,j,m] = g_{lm}\left\{ {l \atop ij} \right\}$$

(2M)

Soln :- Christoffel symbols or brackets of first & second kinds, are defined as

$$[i,j,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)$$

& $\left\{ {l \atop ij} \right\} = g^{lk}[ij,k]$. (1/2 M)

Other symbols for $\left\{ {l \atop ij} \right\}$ are $\{ij,l\}$ or Γ^l_{ij}.

They are not tensors. (1/2 M)

we have,

$$\left\{ {l \atop ij} \right\} = g^{lk}[ij,k] \quad \text{(1)}$$

Inner multiplication of (1) by g_{lm}, gives

$$g_{lm}\left\{ {l \atop ij} \right\} = g_{lm}\, g^{lk}[ij,k]$$

$$= \delta^k_m [ij,k] = [ij,m]$$

Thus, $[ij,m] = g_{lm}\left\{ {l \atop ij} \right\}$ $\quad [\because g_{lm}\, g^{lk} = \delta^k_m]$

(1M)

(10) State & prove the Quotient law of tensors. (5M)

Sol:- **Quotient law :- (Statement)**

A set of N^P – functions of co-ordinates x^i $(i = 1, 2, \ldots, N)$ form the components of a tensor of order P, provided that an inner product of these functions with an arbitrary tensor is itself a tensor. → (1M)

proof:- It will suffice to consider the proof for the following particular case.

The set of N^3 – functions A^{ijk} form the components of a tensor of the type (character) by its indices if

$$A^{ijk} B^p_{ij} = C^{pk}, \rightarrow (1)$$

provided that B^p_{ij} is an arbitrary tensor & C^{pk} is a tensor. The transformed quantities, referred to a system of co-ordinates x^i to $\bar{x}^i$, $(i = 1, 2, \ldots, N)$ satisfy the equations → (1M)

$$\bar{A}^{ijk} \bar{B}^p_{ij} = \bar{C}^{pk} \rightarrow (2)$$

This becomes by transformation law,

$$\bar{A}^{ijk} \frac{\partial \bar{x}^p}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} B^l_{mn} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} C^{qr}$$

$$= \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} A^{ijr} B^q_{ij} \quad \left[\begin{array}{l}\text{As by (1)}, \\ C^{qr} = A^{ijr} B^q_{ij}\end{array}\right.$$

With a change of dummy suffixes (indices) [q by l, (i,j) by (m,n)] we get

$$\frac{\partial \bar{x}^p}{\partial x^l}\left[\bar{A}^{ijk}\frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j} - A^{mnr}\frac{\partial \bar{x}^k}{\partial x^r}\right]B^l_{mn} = 0 \rightarrow (3)$$

(IM)

On multiplying this eq^n (3) by $\frac{\partial x^s}{\partial \bar{x}^p}$ & summing over p from 1 to N, we obtain

$$\left[\bar{A}^{ijk}\frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j} - A^{mnr}\frac{\partial \bar{x}^k}{\partial x^r}\right]B^s_{mn} = 0 \rightarrow (4)$$

$$\left[\because \frac{\partial \bar{x}^p}{\partial x^l}\frac{\partial x^s}{\partial \bar{x}^p}B^l_{mn} = \frac{\partial x^s}{\partial x^l}B^l_{mn} = \delta^s_l B^l_{mn} = B^s_{mn}\right]$$

Since B^s_{mn} is an arbitrary tensor, we can arrange that only one of its components differs from zero. Now, each component of B^s_{mn} may be chosen in turn as that one which does not vanish. This shows that the expression in brackets in eq^n (4) is identically zero.

i.e, $$\bar{A}^{ijk}\frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j} = A^{mnr}\frac{\partial \bar{x}^k}{\partial x^r} \rightarrow (5)$$

(IM)

An inner multiplication of eq^n (5) by $\frac{\partial \bar{x}^s}{\partial x^m}\frac{\partial \bar{x}^t}{\partial x^n}$ gives the result

$$\bar{A}^{stk} = \frac{\partial \bar{x}^s}{\partial x^m} \frac{\partial \bar{x}^t}{\partial x^n} \frac{\partial \bar{x}^k}{\partial x^r} A^{mnr} \longrightarrow (6)$$

$$\left[\frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial \bar{x}^s}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^j} \frac{\partial \bar{x}^t}{\partial x^n} = \delta^s_i \, \delta^t_j \right]$$

Thus, A^{mnr} is a tensor of the third order & contravariant in all its indices, which proves the quotient law for 3 indices. $\longrightarrow$ (1M)

Slly, we can prove the theorem for N^{p+s} functions $A^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_s}$

— X —

Q6) b) Show that

$$\cos(x \sin\phi) = J_0 + 2\cos(2\phi)J_2 + 2\cos(4\phi)J_4 + \cdots$$

Soln:- We know that, (2M)

$$e^{\left(\frac{x}{2}\right)\left(z-\frac{1}{z}\right)} = J_0 + (z - z^{-1})J_1 + (z^2 + z^{-2})J_2 + (z^3 - z^{-3})J_3 + \cdots \longrightarrow (1)$$

Let $z = e^{i\phi}$, so that $z^n = e^{in\phi}$ & $z^{-n} = e^{-in\phi}$. Then (1) gives —

$$e^{\left(\frac{x}{2}\right)\left(e^{i\phi} - e^{-i\phi}\right)} = J_0 + \left(e^{i\phi} - e^{-i\phi}\right)J_1 + \left(e^{2i\phi} + e^{-2i\phi}\right)J_2 + \left(e^{3i\phi} - e^{-3i\phi}\right)J_3 + \cdots \longrightarrow (2)$$

since $\cos(n\phi) = \left(e^{ni\phi} + e^{-ni\phi}\right)/2$

& $\sin(n\phi) = \left(e^{ni\phi} - e^{-ni\phi}\right)/2i$,

eqn (2) gives —

$$e^{xi\sin\phi} = J_0 + 2i\sin\phi\, J_1 + 2\cos(2\phi)J_2 + 2i\sin(3\phi)J_3 + \cdots$$

or, $\cos(x\sin\phi) + i\sin(x\sin\phi)$ $\longrightarrow$ (1M)

$$= \left(J_0 + 2\cos(2\phi)J_2 + 2\cos(4\phi)J_4 + \cdots\right) + 2i\left(\sin\phi J_1 + \sin(3\phi)J_3 + \cdots\right) \longrightarrow (3)$$

Equating real parts in (3), we get

$$\cos(x\sin\phi) = J_0 + 2\cos(2\phi)\cdot J_2 + 2\cos(4\phi)\cdot J_4 + \cdots$$ $\longrightarrow$ (1M)