Fundamental Thm of Abelian gp

Lecture 20

Fundamental Thm of finitely generated abelian groups:

Defn: A gp G is finitely generated if there is a finite subset A of G s.t $G = \langle A \rangle$.

Thm. Let G be a finitely generated abelian gp. Then

(1) $G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s}$

for some integers $r, n_1, n_2, \ldots, n_s$ satisfying the following condn-s

(a) $r \geq 0$ and $n_j \geq 2$ for all j.

(b) $n_{i+1} \mid n_i$ for $1 \leq i \leq s-1$.

(2) The expression in (1) is unique i.e

if $G \cong \mathbb{Z}^t \times \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_u}$

where $t, m_1, m_2, \ldots, m_u$ satisfies condn.s (a) & (b) then $t = r$, $u = s$ and $m_i = n_i$ $\forall i$.

The int r is called the free rank of G and the integers $n_1, n_2, \ldots, n_s$ are called invariant factors of G. and it is called invariant factor decomposition of G.

Observe that a f.g abelian gp is finite iff its free rank is zero. and the order of the gp is the product of the invariant factors.

If G is a finite abelian group with invariant factors $n_1, n_2, \ldots, n_s$ where $n_{i+1} \mid n_i$, $1 \le i \le s-1$, then G is said to be of type $(n_1, n_2, \ldots, n_s)$

The above thm gives an effective way of listing all finite abelian groups of a given order.

For this one must find all finite sequences of integers $n_1, n_2, \ldots, n_s$ s.t (1) $n_i \geq 2$, (2) $n_{i+1} \mid n_i$ and (3) $n_1 n_2 \cdots n_s = n$.

Want to find all abelian gps of order n. If p is a prime divisor of n then by (3) $p \mid n_i$ for some i^o.

Thus by (2) $p \mid n_j \quad \forall j \leq i^{0}$.

$\therefore$ It follows that every prime divisor of n must divide the first invariant factor n_1.

<u>Cor</u>. If n is product of distinct primes then there is only one abelian gp of order n i.e. cyclic g/p of order n, $\mathbb{Z}_n$.

<u>Example</u> Suppose $n = 180 = 2^2 . 3^2 . 5$

By above observation $2.3.5 \mid n_1$.

So the possible values for n_1 are

$n_1 = 2^2 . 3^2 . 5$; $n_1 = 2^2 . 3 . 5$,

$n_1 = 2 . 3^2 . 5$; $n_1 = 2 . 3 . 5$.

If $n_1 = 2^2 . 3 . 5$ then $n_2 = 3$.

If $n_1 = 2 . 3^2 . 5$ then $n_2 = 2$.

If $n_1 = 2 . 3 . 5$ then n_2 are

2 or 3 or 6.

But n_2 can not be 2 or 3.

Hence $n_2 = 6$.

$$\boxed{n = n_1 n_2 \cdots n_s}$$

Invariant factors	Abelian Group
$2^2 . 3^2 . 5$	$\mathbb{Z}_{180}$
$2^2 . 3 . 5 \, ; \, 3$	$\mathbb{Z}_{60} \times \mathbb{Z}_3$
$2 . 3^2 . 5 \, ; \, 2$	$\mathbb{Z}_{90} \times \mathbb{Z}_2$
$2 . 3 . 5 \, ; \, 6$	$\mathbb{Z}_{30} \times \mathbb{Z}_6$

Thm Let G be an abelian gp of order $n > 1$ and let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Then

(1) $G \cong A_1 \times A_2 \times \cdots \times A_k$ where

$$|A_i| = p_i^{\alpha_i}$$

(2) For each $A \in \{A_1, \ldots, A_k\}$ with $|A| = p^{\alpha}$

$$A \cong \mathbb{Z}_{p^{\beta_1}} \times \mathbb{Z}_{p^{\beta_2}} \times \cdots \times \mathbb{Z}_{p^{\beta_t}}$$

with $\beta_1 \geq \beta_2 \cdots \geq \beta_t \geq 1$

and $\beta_1 + \cdots + \beta_t = \alpha$.

(3) The decomposition in (1) & (2) is unique i.e if $G \cong B_1 \times \cdots \times B_m$ with $|B_i| = p_i^{\alpha_i}$ then $B_i \cong A_i$ and have same invariant factors.

Defn The integers $p_i^{\beta_i}$ are called the elementry divisors of G. This decomposition is known as elementry divisor decomposition of G.

The subgps A_i are actully Sylow p_i-subgp. (note that they are normal since G is abelian and hence unique) This is also known as primary decomposition Thm for finite abelian gps.

Example. let G be a gp of order $n = 1800$

$$1800 = 2^3 \cdot 3^2 \cdot 5^2.$$

$$G \cong A_1 \times A_2 \times A_3$$

$|A_1| = 2^3$

$|A_2| = 3^2$

$|A_3| = 5^2.$

order of $A_i = p_i^{\beta_i}$	partition of β	abelian gps
2^3	3; $2+1$; $1+1+1$;	$\mathbb{Z}_8$; $\mathbb{Z}_4 \times \mathbb{Z}_2$; $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
3^2	2; $1+1$	$\mathbb{Z}_9$; $\mathbb{Z}_3 \times \mathbb{Z}_3$
5^2	2; $1+1$	$\mathbb{Z}_{25}$; $\mathbb{Z}_5 \times \mathbb{Z}_5$

Total 12 nonisomorphic abelian gps of order 1800 are possible.

$\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$; $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_5$;

$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$; $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$;

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Obtaining elementary divisors from invariant factors:

Fact: $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ iff $\gcd(m,n) = 1$

If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_n^{\alpha_n}}$

Suppose G is an abelian gp of type $(n_1, n_2, \ldots, n_s)$

i.e $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \mathbb{Z}_{n_s}$.

for $n_i = p_1^{\beta_{i1}} \times \cdots \times p_t^{\beta_{ik}}$ where $\beta_{ij} \geq 0$.

$\therefore$ The elementary divisors of G are precisely the integers $p_j^{\beta_{ij}}$.

$1 \leq j \leq k$; $1 \leq i \leq s$ s.t $\beta_{ij} \neq 0$.

Example G is of type $(30, 30, 2)$.

$$G \cong \mathbb{Z}_{30} \times \mathbb{Z}_{30} \times \mathbb{Z}_2.$$

$$30 = 2 \cdot 3 \cdot 5 \quad ; \quad \mathbb{Z}_{30} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$

$$G \cong \underline{\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_2}$$

elementary factor decomposition