Sylow's Theorem

Lecture 17

<u>Propn</u>: Let p and q be distinct primes and $q < p$ and let G be a gp of order pq. Then

(1) If $q \nmid p-1$ then $G \cong \mathbb{Z}_{pq}$

(2) If $q \mid p-1$ and G is not cyclic then $G \cong \langle x, y \mid |x| = p, |y| = q, yxy^{-1} = x^s$ where $s \not\equiv 1 \pmod{p}$ & $s^q \equiv 1 \pmod{p}$

<u>pf</u>: n_p = # of Sylow p-subgps of G

n_q = # of Sylow q-subgp of G.

$n_p \mid q$ and $n_p \equiv 1 \pmod{p} \Rightarrow n_p = 1$.

$n_q \mid p$ and $n_q \equiv 1 \pmod q$

$\Rightarrow n_q = 1$ or p.

<u>Case 1</u>. Suppose $q \nmid p-1$. Then $n_q = 1$.

Thus $\exists$ only one Sylow p-subgp and only one Sylow q-subgp.

Let P be a Sylow p-subgp then $P = \langle x \rangle$ s.t $|x| = p$ and let Q be a Sylow q-subgp then $Q = \langle y \rangle$ s.t $|y| = q$.

$P \cap Q = \{1\}$ and $|PQ| = pq$.

Both P & Q are normal subgps of G.

$\therefore G = PQ \cong P \times Q \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/pq\mathbb{Z}$

Case 2. Suppose $q \mid p-1$. $n_q = p$.

i.e $\exists$ p many Sylow q-subgps.

Let P be a Sylow p-subgp s.t

$$P = \langle x \rangle = \{1, x, \ldots, x^{p-1}\}.$$

and Q be a Sylow q-subgp s.t

$$Q = \langle y \rangle \qquad |y| = q.$$

P is normal subgp of G.

$yxy^{-1} \in P$ i.e $yxy^{-1} = x^r$

for some $1 \le r \le p-1$.

If $r = 1$. then $yx = xy$.

then G is abelian we get $G \cong \mathbb{Z}_{pq}$.

$$yxy^{-1} = x^r$$

$$\Rightarrow y^2xy^{-2} = x^{r^2}$$

Similarly we get

$$x = y^q x y^{-q} = x^{r^q} \qquad [\because |y| = q]$$

$$\Rightarrow x^{r^q - 1} = 1$$

$$\Rightarrow p \mid r^q - 1. \qquad [\because |x| = p]$$

$$\Rightarrow r^q \equiv 1 \pmod p$$

$$G \cong \langle x, y \mid |x| = p,\ |y| = q,\ yxy^{-1} = x^r \text{ when } r^q \equiv 1 \pmod p$$

Cor. Let p be an odd prime and G be a gp of order $2p$.

Then either $G \cong \mathbb{Z}_{2p}$ or $G \cong D_p$.

Pf: $2 \mid p-1$.

$$G \cong \langle x, y \mid |x| = p, |y| = 2 \text{ and } yxy^{-1} = x^r \text{ where } r^2 \equiv 1 \pmod p \rangle.$$

$\Rightarrow p \mid r^2 - 1 = (r-1)(r+1)$.

If $p \mid r-1$ then $r \equiv 1 \pmod p$.

In this case $yxy^{-1} = x$

$\Rightarrow G$ is abelian.

$\therefore G \cong \mathbb{Z}_{2p}$.

If $p \mid r+1$ then $r \equiv -1 \pmod p$.

$$y x y^{-1} = x^{-1}$$

$$\therefore\ G \cong \langle x, y \mid |x| = p,\ |y| = 2,\ y x y^{-1} = x^{-1} \rangle$$

$$\cong D_p.$$

Classify group of order 12:

$$12 = 2^2 \times 3.$$

There are five different isomorphic classes of gp of order 12.

(1) $\mathbb{Z}_{12}$ - cyclic gp of order 12

(2) $A_4 :=$ gp of even permutations of S_4.

(3) $D_6 = \langle x, y \mid |x| = 6, |y| = 2,$
$yxy^{-1} = x^{-1} \rangle.$

(4) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$

(5) $G = \langle x, y \mid |x| = 4, |y| = 3,$
$xy = y^2 x \rangle.$

Pf: $n_3 = \#$ of Sylow 3-subgp.

$n_3 \mid 4$ and $n_3 \equiv 1 \pmod 3$

$\Rightarrow n_3 = 1, 4.$

$n_2 = \#$ of Sylow 2-subgp.

$n_2 \mid 3$ and $n_2 \equiv 1 \pmod 2$

$\Rightarrow n_2 = 1, 3$

Let H be a Sylow 2-subgp.

$|H| = 4$. $H \simeq$ $\mathbb{Z}/4\mathbb{Z}$ (cyclic gp)

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (V_4 - Klien 4gp).

and let K be a Sylow 3-subgp.

$|K| = 3$ $\quad K \simeq \mathbb{Z}/3\mathbb{Z}$

$H \cap K = \{1\}$ and $|HK| = 12 \Rightarrow G = HK$.

<u>Lemma</u> Either H or K is a normal subgp of G.

<u>Pf</u>: Let K be a Sylow 3-subgp which is not normal. Then $n_3 = 4$.

Let K_1 & K_2 are two Sylow 3-subgps

$\Rightarrow K_1 \cap K_2 = \{1\}$.

Any two Sylow 3-subgps intersect trivially.

$\therefore$ # of elts of order 3 in $G = 8$.

Hence there is only one Sylow 2-subgp in G $\therefore$ $H \trianglelefteq G$.

Case 1 H, K both are normal in G.

$H \cap K = \{1\}$ and $G = HK$.

$\therefore$ $G \cong H \times K$.

$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \boxed{\mathbb{Z}/12\mathbb{Z}}$

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \boxed{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}}$

<u>case 2</u> let $H \trianglelefteq G$ and $K \ntrianglelefteq G$.

Hence $\exists$ 4 Sylow 3-subgps of G say K_1, K_2, K_3, K_4.

In this case G has 8 elts of order 3.

Consider $S = \{K_1, K_2, K_3, K_4\}$.

Now consider the gp action

$$G \times S \longrightarrow S$$
$$(g, K_i) \longmapsto g K_i g^{-1}$$

which can be represented as

$$\phi : G \longrightarrow S_4$$
$$\phi(g) = \sigma_g \quad \text{where } \sigma_g(K_i) = gK_ig^{-1}$$

ϕ is a gp homo.

$$\ker \phi = \{ g \in G \mid \sigma_g = \mathrm{Id} \}$$

$$= \{ g \in G \mid g K_i g^{-1} = K_i \ \forall i \}.$$

Since all Sylow 3-subgps are conjugate to each other so S has only one orbit.

$$|S| = |O(K_i)| = [G : stb(K_i)] = 4.$$

$$\Rightarrow \frac{|G|}{|stb(K_i)|} = 4.$$

$$\Rightarrow |stb(K_i)| = 3.$$

Observe that $K_i \subseteq stb(K_i)$ and $|K_i| = 3 \Rightarrow K_i = stb(K_i)$.

Note that

$$\ker \phi = \{ g \in G \mid g k_i g^{-1} = k_i \ \forall i \}$$

$$= \bigcap_{i=1}^{4} (\mathrm{stb}(k_i))$$

$$= \bigcap_{i=1}^{4} k_i$$

$$= \{1\}.$$

$$\phi : G \longrightarrow S_4.$$

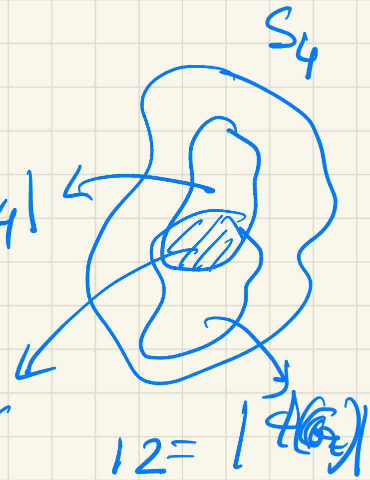
$$\therefore \quad G \cong \phi(G) \subset S_4.$$

Thus if $x \in G$ is an elt of order 3 then $\phi(x)$ is an elt of order 3 in S_4. The 3-cycles in S_4 are the elts of order 3 in S_4.

Since G contains 8 elts of order 3 and there are precisely 8 elts of S_4 of order 3 is in A_4.

$$|\phi(G) \cap A_4| \geq 9$$

$\Rightarrow \phi(G) = A_4$. $\quad 12 = |A_4|$



$\therefore G \cong A_4$.

<u>Case 3</u>: $K \triangleleft G$ and $H \ntriangleleft G$.

Let $K = \langle x \rangle \Rightarrow |x| = 3$

and H — $\mathbb{Z}/4\mathbb{Z} = \langle y \rangle$ with $|y| = 4$
— $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Subcase: Let $H = \langle y \rangle$ s.t. $|y| = 4$.

then $yxy^{-1} \in K = \{1, x, x^2\}$.

If $yxy^{-1} = x$ then G is abelian

$\Downarrow$

$H \lhd G$ which is a contradiction.

If $yxy^{-1} = x^2 \Rightarrow yx = x^2 y$.

$$\therefore \boxed{G \cong \langle x, y \mid |x| = 3, |y| = 4, yx = x^2 y \rangle}.$$

Subcase: $H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$$= \{1, u, v, uv \mid u^2 = 1, v^2 = 1, uv = vu\}$$

Since $K \trianglelefteq G$ $\left[x^3=1 \Rightarrow x^2=x^{-1}\right]$

$\therefore (*)\; uxu^{-1} = x^a, \quad vxv^{-1} = x^b$

where a, b are 1 or -1.

$uxu^{-1} \in K = \{1, x, \underset{\substack{\| \\ x^{-1}}}{x^2}\}$

and $(uv)(x)(uv)^{-1} = x^{ab}$ (check!)

If $a=b=1$ then G is abelian which is a contradiction ($\because H \ntrianglelefteq G$).

Assume $a=1$, $b=-1$ then $ux = xu$ from $(*)$

$\Rightarrow |xu| = 6$.

Let $z = xu$ then $|\langle z \rangle| = 6$.

and $v \notin \langle z \rangle$. ($\because |x| = 3$ check!)

and $vzv^{-1} = vxuv^{-1}$

$= vuxv^{-1}$ $[\because xu = ux]$

$= uvxv^{-1}$ $[\because uv = vu]$

$= ux^{-1}$ $[\because \underline{vxv^{-1} = x^{-1}}]$

$= u^{-1}x^{-1}$

$= (xu)^{-1} = z^{-1}$

$\therefore vzv^{-1} = z^{-1}$.

$G \cong \langle v, z \mid |z| = 6,\ |v| = 2$ and $vzv^{-1} = z^{-1} \rangle$

$\cong D_6$.