# Lecture 19

Proof the claim:-

We know $F_N(x) \to f(x)$ as $N \to \infty$.

$\Rightarrow F_{2^k}(x) \to f(x)$ as $k \to \infty$

$\Rightarrow$ Given any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\left| F_{2^k}(x) - f(x) \right| < \frac{\varepsilon}{2}$

$\forall\, k \geq n_0$.

Now to show: $\varphi_k(x) \to f(x)$.

Let $\varepsilon > 0$.

$$\left| \varphi_k(x) - f(x) \right| = \left| F_{2^k, 2^k}(x) - f(x) \right|$$

$$= \left| F_{2^k,2^k}(x) - F_{2^k}(x) + F_{2^k}(x) - f(x) \right|$$

$$\leq \left| F_{2^k,2^k}(x) - F_{2^k}(x) \right| + \left| F_{2^k}(x) - f(x) \right|$$

$$\leq \underbrace{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}}_{=\varepsilon} \text{ (as below) } \forall\, n \geq \underbrace{\max\{n_0, n_1\}}_{n_2}$$

Recall that $\left| F_{2^k}(x) - F_{2^k, 2^k}(x) \right| \leq \frac{1}{2^k} \quad \forall x$.

As $k \to \infty$, then $\left| F_{2^k}(x) - F_{2^k, 2k}(x) \right| \to 0$

becau $\frac{1}{2k} \to 0$.

i.e., $\lim_{k\to\infty}\left(F_{2^k}(x) - F_{2^k,2^k}(x)\right) = 0.$

Given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $\left| F_{2^k}(x) - F_{2^k,2^k}(x) \right| < \frac{\varepsilon}{2}$

$\forall k \geq n_1.$

Thus $\varphi_k(x) \to f(x)$ as $k \to \infty$.

$\{\varphi_k\}$ is an increasing sequence because

if $x \in E_{l,2^k}$ for some $l$, $0 \leq l \lneq 2^k \cdot 2^k$.

$$\varphi_k(x) = F_{2^k,2^k}(x) = \frac{l}{2^k} \quad \left(\text{by def. of } F_{N,M}\right)$$

$$\varphi_{k+1}(x) \in \left\{ \underset{\overset{\shortparallel}{\frac{l}{2^k}}}{\frac{2l}{2^{k+1}}}, \frac{2l}{2^{k+1}} + \frac{1}{2^{k+1}} \right\} \quad (\text{Check it!})$$

Idea: Compare $E_{l,2^k}$, $E_{l,2^{k+1}}$.

Thus $\varphi_{k+1}(x) \geq \varphi_k(x)$ $\forall k$, $\forall x$

Thus $\{\varphi_k\}$ is increasing.

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Definition:- Let $f : E \to \mathbb{R}$ be any function.

Then $f^+ \doteq \max\{f, 0\}$

$f^- \doteq \max\{-f, 0\}$.

i.e, $f^+(x) = \max\{f(x), 0\} \quad \forall x \in E$

$f^-(x) = \max\{-f(x), 0\}$

Remark:- ① $f^+, f^-$ are non-negative functions.

$f^+(x) \geq 0, \; f^-(x) \geq 0 \quad \forall x \in E$.

② $f^+(x) = \dfrac{f(x) + |f(x)|}{2}$

$f^-(x) = \dfrac{-f(x) + |f(x)|}{2}$.

③ $\boxed{f = f^+ - f^-}$ i.e, $f(x) = f^+(x) - f^-(x)$.

$\forall x \in E$.

④ If $E \subseteq \mathbb{R}^d$ is measurable & $f$ is measurable, then $f^+, f^-$ are also measurable functions.

Theorem:- Suppose $f$ is a measurable function on $\mathbb{R}^d$. Then there exists a sequence of simple functions $\{\varphi_k\}_{k=1}^{\infty}$, that satisfies $|\varphi_k(x)| \leq |\varphi_{k+1}(x)|$ $\forall k, \forall x$

& $\lim_{k\to\infty} \varphi_k(x) = f(x)$, $\forall x$.

i.e., $\{|\varphi_k|\}$ is increasing & $\varphi_k \to f$ pointwise.

proof:- We have $f = f^+ - f^-$,

where $f^+, f^-$ are non-negative measurable functions.

Therefore by above theorem, there exist a sequence of non-negative simple functions $\{\varphi_k^{(1)}\}$, $\{\varphi_k^{(2)}\}$ such that $\{\varphi_k^{(1)}\}$, $\{\varphi_k^{(2)}\}$ are increasing &

$$\lim_{k\to\infty} \varphi_k^{(1)}(x) = f^+(x), \quad \lim_{k\to\infty} \varphi_k^{(2)}(x) = f^-(x)$$

$\forall x.$

Let $\varphi_k(x) = \varphi_k^{(1)}(x) - \varphi_k^{(2)}(x), \quad \forall x$

Then $\varphi_k(x) \to f^+(x) - f^-(x) = f(x). \quad \forall x.$

as $k \to \infty$

We have $g^+ = \dfrac{g + |g|}{2}, \quad g^- = \dfrac{-g + |g|}{2}.$

$g^+ + g^- = |g|$ for any function $g$.

$\boxed{\varphi_k^{(1)}(x) + \varphi_k^{(2)}(x) = |\varphi_k(x)|}$ is increasing

because $\{\varphi_k^{(1)}\}, \{\varphi_k^{(2)}\}$ are increasing $\cancel{x}$

$$|\varphi_k(x)| = \varphi_k^+(x) + \varphi_k^-(x)$$

$$= \left(\varphi_k^{(1)} - \varphi_k^{(2)}\right)^+(x) + \left(\varphi_k^{(1)} - \varphi_k^{(2)}\right)^-(x).$$

$= ?$