EXAMPLE: (CONTINUOUS, PARTIAL DERIVATIVES EXIST BUT NOT DIFFERENTIABLE)

$$f(x,y) = \begin{cases} \dfrac{x^3+2y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

i) CONTINUITY:

$$\lim_{\substack{x\to 0 \\ y \to 0}} \frac{x^3+2y^3}{x^2+y^2}$$

(Necessary for differentiability)

Changing to polar coordinates:

$$\lim_{r\to 0} \frac{r^3\cos^3\theta + 2r^3\sin^3\theta}{r^2}$$

$$= \lim_{r\to 0} r\left(\cos^3\theta + 2\sin^3\theta\right) = 0 = f(0,0)$$

ALTERNATIVE:

$$|f(x,y) - 0| = \left| \frac{r^3\cos^3\theta + 2r^3\sin^3\theta}{r^2} \right|$$

(subst. $x = r\cos\theta$, $y = r\sin\theta$)

$$\leq r|\cos^3\theta| + 2r|\sin^3\theta|$$

$$< 3r \overset{!}{<} \varepsilon$$

Choose $\delta < \dfrac{\varepsilon}{3}$ then

$$|f(x,y) - 0| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta$$

$\Rightarrow$ $f(x,y)$ is continuous at $(0,0)$.

ii) Existence of partial derivatives: (Necessary for differentiability)

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^3}{\Delta x^3} = 1.$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2\Delta y^3}{\Delta y^3} = 2.$$

iii) Differentiability:

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} \neq 0$$

as.

$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0)$$

$$= \frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2 + \Delta y^2}$$

$$dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y$$

$$= \Delta x + 2\Delta y$$

Now:

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho}$$

$$= \lim_{\Delta\rho \to 0} \left[\frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2 + \Delta y^2} - (\Delta x + 2\Delta y) \right] \frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$= \lim_{\Delta\rho \to 0} \frac{-\Delta x \Delta y^2 - 2\Delta x^2 \Delta y}{(\Delta x^2 + \Delta y^2)^{3/2}}$$

Along the path $\Delta y = m \Delta x$

$$\Rightarrow \lim_{\Delta x \to 0} - \frac{m^2 - 2m}{(1+m^2)^{3/2}}$$

limit depends on the path.

$\Rightarrow$ The given function is not differentiable.

EXAMPLE: (FUNCTION IS DIFFERENTIAL BUT f_x & f_y are not continuous)

$$f(x,y) = \begin{cases} (x^2+y^2)\cos\left[\frac{1}{\sqrt{x^2+y^2}}\right] & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

i) Continuity: $\lim\limits_{\substack{x\to 0 \\ y\to 0}} (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) = 0 = f(0,0)$

ii) Existence of partial derivatives:

$$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \Delta x \cos\left(\frac{1}{|\Delta x|}\right) = 0$$

$$f_y(0,0) = \lim_{\Delta y\to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \Delta y \cos\left(\frac{1}{|\Delta y|}\right) = 0$$

iii) Differentiability:

$dz = z_x \Delta x + z_y \Delta y = 0$

$$\lim_{\Delta\rho\to 0}\left(\frac{\Delta z - dz}{\Delta \rho}\right) = \lim_{\Delta\rho\to 0} \frac{(\Delta x^2 + \Delta y^2)}{\sqrt{\Delta x^2 + \Delta y^2}} \cos\left(\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\right)$$

$$= \lim_{\substack{\Delta x\to 0 \\ \Delta y\to 0}} \sqrt{\Delta x^2 + \Delta y^2} \cos\left(\frac{1}{\Delta x^2 + \Delta y^2}\right) = 0$$

Hence the function is differentiable.

iv) Continuity of f_x & f_y.

At $(x,y) \neq (0,0)$: $f_x(x,y) = -(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)\cdot\left(-\frac{1}{2}\frac{1\cdot 2x}{(x^2+y^2)^{3/2}}\right)$

$$+ 2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{x}{\sqrt{x^2+y^2}} \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + 2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

Along x-axis: $\lim\limits_{x\to 0} f_x(x,y) = \lim\limits_{x\to 0}\left[\frac{x}{|x|}\sin\left(\frac{1}{|x|}\right) + 2x\cos\left(\frac{1}{|x|}\right)\right] \neq 0$

Hence f_x is not continuous at $(0,0)$. Similarly one can show that f_y is not continuous.

This example shows that continuity of partial 1st order derivatives is not a necessary condition for differentiabity. A function can be differentiable without having first order partial derivatives continuous.

Ex. For the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y(x-y)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Find $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$.

Sol:

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$$

where

$$f_y(\Delta x, 0) = \lim_{\Delta y \to 0} \frac{f(\Delta x, \Delta y) - f(\Delta x, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta x^2 \cancel{\Delta y} (\Delta x - \Delta y)}{(\Delta x^2 + \Delta y^2)\cancel{\Delta y}} = \Delta x$$

Hence:

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

because

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = 0$$

Now:

$$\left.\frac{\partial^2 f}{\partial y\,\partial x}\right|_{(0,0)} = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y}$$

where. $$f_x(0,\Delta y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, \Delta y) - f(0,\Delta y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 \Delta y(\Delta x - \Delta y)}{\Delta x(\Delta x^2 + \Delta y^2)} = 0$$

& $$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$$

Hence

$$\left.\frac{\partial^2 f}{\partial y\,\partial x}\right|_{(0,0)} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

Ex: Test the continuity and existence of f_x & f_y at the origin of the followin function:

$$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

Sol. limit along $x = y$: $\lim_{x \to 0} f(x,y) = 0$

Since $f(0,0) = 1$, f is not continuous at $(0,0)$.

$$\left.f_x\right|_{(0,0)} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1-1}{\Delta x} = 0$$

$$\left.f_y\right|_{(0,0)} = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{1-1}{\Delta y} = 0.$$

$\Rightarrow$ First order partial derivatives exist at $(0,0)$

Ex. Test the differentiability of the following function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

at the origin.

Sol: Clearly the function is continuous at the origin as

$$\lim_{x\to 0, y\to 0} f(x,y) = \lim_{r\to 0} r\cos\theta\sin\theta = 0$$

Existence of partial derivatives:

$$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0}{\Delta x} = 0$$

Similarly: $$f_y(0,0) = \lim_{\Delta y\to 0} \frac{f(0,\Delta y)-f(0,0)}{\Delta y} = 0$$

So, $$df = \Delta x . f_x + \Delta y\, f_y = 0$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta f - df}{\sqrt{\Delta x^2+\Delta y^2}} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x\,\Delta y}{\Delta x^2+\Delta y^2}$$

Along the path $\Delta y = m\,\Delta x$:

$$= \lim_{\Delta x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$ limit does not exist.

$\Rightarrow$ f is not differentiable.

Ex. Find the total differential and the total increment of the function $z = xy$ at the point $(2,3)$ for $\Delta x = 0.1$, $\Delta y = 0.2$.

Sol.
$$\Delta z = (x+\Delta x)(y+\Delta y) - xy$$
$$= x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y$$

& $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = y\,dx + x\,dy = y\Delta x + x\Delta y$$

Consequently:
$$dz = 3.(0.1) + 2(0.2)$$
$$= 0.3 + 0.4 = 0.7$$

$$\Delta z = 0.7 + 0.1 \times 0.2$$
$$= 0.7 + 0.02 = 0.72$$

Q. Let
$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Discuss the continuity of f_{yx} at $(0,0)$.

Sol:
$$f_x = \frac{(x^2+y^2)\,2xy^2 - x^2y^2(2x)}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2}$$

$$f_{yx} = \frac{8x^3y^3}{(x^2+y^2)^3}$$

Along the path $y = mx$:
$$\lim_{x\to 0} f_{yx} = \lim_{x\to 0} \frac{8x^3m^3x^3}{(x^2+m^2x^2)^3} = \frac{8m^3}{(1+m^2)^3}$$

$\Rightarrow$ f_{yx} is not continuous at $(0,0)$.

COMPOSITE FUNCTIONS:

consider
$$z = f(x,y) \quad -(1)$$
and let

$$\left.\begin{array}{l} x = \varphi(t) \\ y = \psi(t) \end{array}\right\} (2) \qquad \text{or} \qquad \left.\begin{array}{l} x = \Phi(u,v) \\ y = \psi(u,v) \end{array}\right\} (2')$$

The equations (1 & 2) or (1 & 2') are said to define z as composite function of t or (u & v).

Differentiation of composite functions (Chain Rule)

Let $z = f(x,y)$ posses continuous partial derivatives (or differentiabl

and let $x = \varphi(t)$, $y = \psi(t)$ posses continuous derivatives (differentia

Then,
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Proof: Let $z = f(x,y)$, $x = \varphi(t)$, $y = \psi(t)$ be a composite function of t. Assuming z to be differentiable and also $\varphi(t)$ & $\psi(t)$ are differentiable function of t.

$$\Rightarrow \Delta z = z_x \Delta x + z_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\Rightarrow \frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\cdot\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\cdot\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta y}$$

Taking limit $\Delta t \to 0$; $(\Delta x \to 0, \Delta y \to 0)$

$$\Rightarrow \boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

For the case $x = \varphi(u,v)$ & $y = \psi(u,v)$, we have

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$

&

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$

Example: $z = xy$; $x = \cos t$ $\quad y = \sin t$

Find $\frac{dz}{dt}$.

Sol.
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$= y\cdot(-\sin t) + x\cos t$$
$$= -\sin^2 t + \cos^2 t$$
$$= \cos 2t$$

Ex. Let z be a function of x & y. Prove that if

$x = e^u + e^{-v}$ $\qquad y = e^{-u} + e^{v}$

Then $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Sol.
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u + \frac{\partial z}{\partial y}\cdot(-e^{-u})$$
$$= \frac{\partial z}{\partial x}e^u - \frac{\partial z}{\partial y}e^{-u} \quad \text{—(I)}$$

Similarly:
$$\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x}\cdot e^{-v} + \frac{\partial z}{\partial y}e^{v} \quad \text{—(II)}$$

(I)$-$(II) $\Rightarrow$
$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} + e^{v})$$
$$= \frac{\partial z}{\partial x}x - y\frac{\partial z}{\partial y}.$$

Derivative of a function defined implicitely:

ONE VARIABLE:

Let the function y of x be defined by

$$F(x,y) = 0$$

and let

$$z \equiv F(x,y) = 0$$

$$\Rightarrow \frac{dz}{dx} \equiv \frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y} \quad \text{if } \frac{\partial F}{\partial y} \neq 0$$

TWO INDEPENDENT VARIABLE

$$F(x,y,z) = 0$$

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} \quad \text{if } \frac{\partial F}{\partial z} \neq 0$$

$$\text{and } \frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z} \quad \text{if } \frac{\partial F}{\partial z} \neq 0$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of $x^2+y^2+z^2-c=0$

Sol:

$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} = -\frac{2x}{2z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z} = -\frac{2y}{2z} = -\frac{y}{z}$$

OR Directly from $x^2+y^2+z^2-c=0$

$$\Rightarrow 2x + 2z\cdot\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}$$

& Diff. w.r.t y $\Rightarrow 2y + 2z\cdot\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$.

HARMONIC FUNCTIONS

If a function of two variables $f(x,y)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

then we say that $f(x,y)$ is an harmonic function.

Example: i) $f(x,y) = x^3y - y^3x$

$$\frac{\partial f}{\partial x} = 3x^2y - y^3 \qquad \frac{\partial f}{\partial y} = x^3 - 3y^2x$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy \qquad \frac{\partial^2 f}{\partial y^2} = -6xy$$

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

ii) $f(x,y) = e^{-x}\cos y$

$$\frac{\partial f}{\partial x} = -e^{-x}\cos y \qquad \frac{\partial^2 f}{\partial x^2} = e^{-x}\cos y$$

$$\frac{\partial f}{\partial y} = -e^{-x}\sin y \qquad \frac{\partial^2 f}{\partial y^2} = -e^{-x}\cos y$$

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

HOMOGENEOUS FUNCTION

We say an expression in (x, y) is homogeneous of order n, if it can be expressed as

$$x^n f\left(\frac{y}{x}\right)$$

Examples:

i) $f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$

$$= x^n \underbrace{\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \dots + a_n\left(\frac{y}{x}\right)^n\right]}_{=: g\left(\frac{y}{x}\right)}$$

$\Rightarrow f(x,y)$ is a homo. func. of order n.

ii) $f(x,y) = \dfrac{\sqrt{y} + \sqrt{x}}{y+x} = \dfrac{\sqrt{x}}{x}\left[\dfrac{\sqrt{\left(\frac{y}{x}\right)} + 1}{\frac{y}{x} + 1}\right]$

$$= x^{-\frac{1}{2}} g(y/x)$$

$\Rightarrow f(x,y)$ is a homo. func. of order $-\frac{1}{2}$.

ALTERNATIVE DEF. A function $f(x,y)$ is said to be homogeneous of degree n if it satisfies

$$f(tx, ty) = t^n f(x,y).$$

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS:

If $z = f(x,y)$ be a homogeneous function of x & y of order n, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad \forall\, x, y \in D.$$

D: Domain of the function f.

PROOF: Given $z = f(x,y) = x^n g\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = nx^{n-1} g\left(\frac{y}{x}\right) + x^n \cdot \left(-\frac{y}{x^2}\right) g'\left(\frac{y}{x}\right)$$

$$= nx^{n-1} g\left(\frac{y}{x}\right) - x^{n-2} \cdot y\, g'\left(\frac{y}{x}\right) \quad \text{— (1)}$$

$$\frac{\partial z}{\partial y} = x^n \cdot g'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad \text{— (2)}$$

from (1) and (2)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n g\left(\frac{y}{x}\right) - yx^{n-1} g'\left(\frac{y}{x}\right) + yx^{n-1} g'\left(\frac{y}{x}\right)$$

$$= nz$$

□

Theorem: If $z = f(x,y)$ is a homogeneous function of x & y of degree n. Then

$$\boxed{x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z}$$

Example: If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, $x \neq y$, Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2x$$

Sol: Let $z = \tan u = \frac{x^3+y^3}{x-y} = x^2\left[\frac{1+(y/x)^3}{1-y/x}\right]$

Clearly z is a homogeneous of degree 2.

$$\Rightarrow \quad x \cdot \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$$

Subst. $z = \tan u$ gives,

$$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2 \cdot \tan u$$

$$\Rightarrow \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cdot \sin u \cdot \cos u \, .$$

$$= \sin 2u .$$

Ex: If $u = z e^{ax+by}$ where z is a homogeneous function in x & y of degree n.

Show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = (ax+by+n) u$

Sol: Since z is a homogeneous function of degree n,

we have $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$ (Euler's theorem)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x\left[\frac{\partial z}{\partial x} \cdot e^{ax+by} + z \cdot e^{ax+by} \cdot a\right] + y\left[\frac{\partial z}{\partial y} e^{ax+by} + z \cdot e^{ax+by} \cdot b\right]$$

$$= e^{ax+by}\left[x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}\right] + z \cdot \left[ax e^{ax+by} + by e^{ax+by}\right]$$

$$= (nz + azx + bzy) e^{ax+by}$$

$$= (n + ax + by) u :$$

Ex. Let $z = xy\, f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ where f & g are continuous and 2 times differentiable functions. Then, evaluate

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$$

Sol: Let $z = u_1 + u_2$ where $u_1 = \underbrace{xy\, f\left(\frac{y}{x}\right)}_{\text{homo fun. of deg. 2}}$ & $u_2 = \underbrace{g\left(\frac{y}{x}\right)}_{\text{homo. func. of deg. 0}}$

Applying Euler's theorem on u_1 & u_2 we get

$$x^2 \frac{\partial^2 u_1}{\partial x^2} + 2xy \frac{\partial^2 u_1}{\partial x \partial y} + y^2 \frac{\partial^2 u_1}{\partial y^2} = 2(2-1)\cdot u_1 \quad \text{—} (1)$$

& $$x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = 0 \quad \text{—} (2)$$

Adding (1) & (2):

$$x^2 \cdot \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 2 \cdot u_1$$

$$= 2 \cdot xy\, f\left(\frac{y}{x}\right)$$

Ex. If $z = y + f\left(\frac{x}{y}\right)$ where f is cont & differentiable function.

Find the value of $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$.

Sol. Let $z = u_1 + u_2$ where $u_1 = y$ & $u_2 = f\left(\frac{x}{y}\right)$

Now. $x \frac{\partial u_1}{\partial x} + y \frac{\partial u_1}{\partial y} = y$ (Either by Euler's theorem or direct result

& $x \frac{\partial u_2}{\partial x} + y \frac{\partial u_2}{\partial y} = 0$

Adding the above two, we get: $\boxed{x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = y}$