Ring Theory

Lecture 24

__Homomorphism Thm__: Let $f: R \to S$ be a surjective ring homo. Then $\exists$ a bijection between

$$\left\{\begin{matrix}\text{all ideals of } R \\ \text{containing } K\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Set of all} \\ \text{ideals of } S\end{matrix}\right\}$$

where $K = \ker f$.

$$J \rightsquigarrow f(J)$$

$$f^{-1}(I) \leftsquigarrow I$$

__Pf__: __1st Step__: Let J be an ideal of R

__WTS__ $f(J)$ is an ideal of S.

Let $a, b \in f(J)$ then __WTS__ $a+b \in f(J)$.

$\exists\, x, y \in J$ s.t $f(x) = a$ & $f(y) = b$

and $x+y \in J$. $f(x+y) \in f(J)$.

$f(x+y) = f(x) + f(y) = a+b \in f(J)$.

Let $s \in S$. and $a \in f(J)$.

WTS $sa \in f(J)$.

Since f is surjective ring homo

$\exists\ s' \in R$ s.t $f(s') = s$.

and $\exists\ x \in J$ s.t $f(x) = a$.

Then $s'x \in J$.

$$f(s'x) = f(s') f(x) = sa \in f(J).$$

$\therefore f(J)$ is an ideal.

<u>2nd Step</u>: Let I be an ideal of S.

WTS $f^{-1}(I) := \{a \in R \mid f(a) \in I\}$.

is an ideal of R.

Let $x, y \in f^{-1}(I)$

then $f(x), f(y) \in I$.

$\Rightarrow \quad f(x) + f(y) \in I$.

$\Rightarrow \quad f(x+y) \in I$

$\Rightarrow \quad x+y \in f^{-1}(I)$.

Let $a \in R$. and $x \in f^{-1}(I)$

WTS $\quad ax \in f^{-1}(I)$.

$$f(ax) = f(a) f(x) \in I.$$

$\Rightarrow \quad ax \in f^{-1}(I)$.

$\therefore \quad f^{-1}(I)$ is an ideal of of R.

Note that $f^{-1}(I) \supseteq \ker f$

Let $x \in \ker f$ then $f(x) = 0 \in I$
$\Rightarrow x \in f^{-1}(I)$.

3rd Step: Let J be an ideal s.t. $J \supseteq K$.

WTS $f^{-1}(f(J)) = J$ and $f(f^{-1}(I)) = I$.

It is clear $J \subseteq f^{-1}(f(J))$.

WTS $f^{-1}(f(J)) \subseteq J$.

Let $a \in f^{-1}(f(J))$

$\Rightarrow f(a) \in f(J)$.

$\Rightarrow f(a) = f(y)$ for some $y \in J$.

$\Rightarrow f(a-y) = 0$.

$\Rightarrow a-y \in \ker f \subseteq J$.

$\Rightarrow a \in J$. $[\because y \in J]$.

$\therefore f^{-1}(f(J)) = J$.

Note that $f(f^{-1}(I)) \subseteq I$.

<u>WTS</u> $I \subseteq f(f^{-1}(I))$.

Let $a \in I$.

Since f is surjective $\exists\, x \in R$ s.t

$$f(x) = a \in I.$$

$$\Rightarrow \quad x \in f^{-1}(I).$$

$$\therefore \quad a \in f(f^{-1}(I)).$$

$$\therefore \quad f(f^{-1}(I)) = I.$$

<u>Remark</u>. Let R be a ring and I be an ideal of R. Consider the ring homo $R \xrightarrow{\varphi} R/I$

$$\varphi(r) = r + I$$

The ideals of R/I will be of the form J/I where J is an ideal of $R \supseteq I$.

$\mathbb{Z}$. is a ring but not a field.

$\mathbb{Z} \subseteq \mathbb{Q}$ is a field.

Consider the set $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$

$$= \left\{ (a,b) \,\middle|\, \begin{array}{l} a, b \in \mathbb{Z} \\ \& \ b \neq 0 \end{array} \right\}.$$

Define an equivalance relation on the above set as follows:

$(a, b) \sim (c, d)$ if $ad - bc = 0$.

[check that it is an equivalance relation].

Then $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ is disjoint union of equivalance classes.

Let me denote equivalace class of (a, b) by a/b.

$$\frac{a}{b} = \frac{c}{d} \qquad ad - bc = 0.$$

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$$

$\mathbb{Q}$ is nothing but the equivalence classes : $\mathbb{Z} \times \mathbb{Z} \setminus \{0\} / \sim$

Define '+' & '·' on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\} / \sim$.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

check that these operation are well-defined.

With respect to '+' & '·' $\mathbb{Q}$ forms a ring with $\frac{0}{1}$ and multiplicative identity $\frac{1}{1}$.

WTS Every non-zero elt of $\mathbb{Q}$ has an inverse.

Let $\frac{a}{b} \neq \frac{0}{1}$ i.e $a \neq 0$.

then $\frac{b}{a}$ is the inverse of $\frac{a}{b}$.

$\therefore$ Every nonzero elt has an inverse.

$$\phi : \mathbb{Z} \longrightarrow \mathbb{Q}$$

$$\phi(r) = \frac{r}{1}.$$

$$\ker \phi = \left\{ r \in \mathbb{Z} \;\middle|\; \phi(r) = \frac{0}{1} \right\}$$

$$= \left\{ r \in \mathbb{Z} \;\middle|\; \frac{r}{1} = \frac{0}{1} \right\}.$$

$$= (0).$$

$\therefore \phi$ is inj ring homo.

We identify $\mathbb{Z}$ with $\phi(\mathbb{Z}) = \left\{ \frac{r}{1} \mid r \in \mathbb{Z} \right\} \subseteq \mathbb{Q}$.

<u>Thm</u>. Let R be an int domain. Then $\exists$ an injective ring homo $R \to F$ where F is a field.