Tutorial-2 Dissicion.

Tutorial 2

Q12 $\phi(x) = x^2$ $|G|$ is odd.

$\ker\phi = \{x \in G \mid x^2 = 1\}$.

then $|x| = 2$ or 1

let $1 \neq x \in \ker\phi$. then $|x| = 2$.

$H = \langle x \rangle$ then $|H| = 2$.

But $|H| \mid |G|$ which is a contradiction

Hence $\ker\phi = \{1\}$.

Q16 $K < H < G$. If $|G|$ is finite.

$$[G:K] = \frac{|G|}{|K|}$$

$$[G:H] \cdot [H:K] = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|}$$

$$= \frac{|G|}{|H|}.$$

But if $|G|$ is infinite then how to prove?

Ans. $K \subset H \subset G$.

$[H : K] = m < \infty$.

$$G = \bigcup_{i=1}^{n} g_i H \qquad H = \bigcup_{j=1}^{m} h_j K$$

$$G = \bigcup g_i H = \bigcup g_i \left(\bigcup h_j K \right)$$

$[G : H] = n < \infty$

$$= \bigcup_{i,j} g_i h_j K$$

WTS $g_i h_j K \cap g_t h_s K = \phi$.

we have $g_i H \cap g_t H = \phi$

$\Rightarrow g_i (h_j K) \cap g_t (h_s K) = \phi \quad \forall i \neq t$.

since $h_j K \cap h_s K = \phi$

$\Rightarrow g_i h_j K \cap g_i h_s K = \phi$.

A Dihedral group is the group of symmetries of a regular n-gon including both rotations & reflections.

A regular polygon with n-sides has $2n$ different symmetries: n- rotational symmetries and n- reflection symmetries.

$$D_n = \langle x, y \mid x^n = 1, \ y^2 = 1, \ yx = x^{-1}y \rangle$$

$$= \{1, x, x^2, \dots, x^{n-1}, y, xy, x^2y, \dots, x^{n-1}y\}$$

$$|D_n| = 2n$$

Q15

$$\mathrm{lcm}(m,n) \le \overset{\ell}{\overset{\shortparallel}{[G:H\cap K]}} \le mn.$$

$$[G:H]=m, \qquad [G:K]=n.$$

$$\begin{tikzcd}[column sep=small] & G \arrow[dl, dash] \arrow[dr, dash] & \\ H \arrow[dr, dash] & & K \arrow[dl, dash] \\ & H\cap K & \end{tikzcd}$$

$$\ell = [G:H\cap K] = [G:H][H:H\cap K]$$
$$= [G:K][K:H\cap K]$$

$$\ell = m\, p_1$$
$$= n \cdot p_2 .$$

$$\Rightarrow \quad \mathrm{lcm}(m,n) \le [G:H\cap K].$$

Ex: $[H:H\cap K] \le [G:K]$.

using the above inequality get the upper bound.

Q20. $G = \mathbb{Z}/(p^n-1)\mathbb{Z}^{\times}$

$= \{ \bar{a} \mid \gcd(a, p^n-1) = 1 \}$.

G is a gp wrt multiplication.

$\bar{p}^n = \bar{1}$ in G.

$(\mathbb{Z}/10\mathbb{Z})^{\times} = \{ \bar{a} \mid \gcd(a,10) = 1 \}$

$= \{ \bar{1}, \bar{3}, \bar{7}, \bar{9} \}$. form a gp wrt multipl

$(\mathbb{Z}/n\mathbb{Z})^{\times}$ is always a gp wrt $\cdot$

$\shortparallel$

$U(\mathbb{Z}/n\mathbb{Z})$

WTS $|\bar{p}| = n$.

Let $0 < k < n$ s.t. $\bar{p}^k = 1$.

$$\left(\mathbb{Z}/p^n - 1\right)^\times \rightsquigarrow \bar{p}^n - \bar{1} = \bar{0}$$

$$p^k < p^n - 1.$$

$$p^k - 1 = l\,(p^n - 1).$$

$$\left|\mathbb{Z}/(p^n-1)^\times\right| = \phi\,(p^n - 1).$$

$$\cancel{\alpha}\ |\bar{p}| = n \Rightarrow n \mid \phi(p^n - 1).$$