Recall

<u>BCT</u> $\{f_n\}$ meas.

$|f_n| \le M \ \forall n$

$\{f_n\}$ are supported on a set E with $m(E) < \infty$, $f_n \to f$ pt.wise as $n \to \infty$.

<u>Conclusion</u> $\to$ f is meas, bounded.

& $\int f_n \to \int f$. ✓

<u>Corollary</u>: Suppose f is measurable with $f \geq 0$, f is bdd. & f is supported on E, $m(E) < \infty$.

Suppose $\int f = 0$

Then $f = 0$ a.e.

<u>Proof</u>:– T.S. $m\{x : f(x) > 0\} = 0$

For each $K \geq 1$, take

$$E_k = \left\{ x \in E \mid f(x) \geq \frac{1}{k} \right\}$$

Clearly, $\frac{1}{k} \chi_{E_k}(x) \leq f(x)$ ✓

$$\Rightarrow \frac{1}{k} \int \chi_{E_k}(x)\, dx \leq \int f(x)\, dx$$

$$\Rightarrow \frac{1}{k} m(E_k) \leq \underset{0}{\overset{\shortparallel}{\int f}}$$

Then, $m(E_k) = 0 \quad \forall\, k \geq 1.$

But $\{x : f(x) > 0\} = \bigcup_{k=1}^{\infty} E_k$

$$0 \leq m\{x : f(x) > 0\} = m\left(\bigcup_{k=1}^{\infty} E_k\right)$$

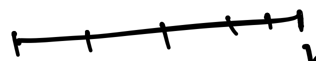
$$\leq \sum_{k=1}^{\infty} m(E_k)$$

$$= 0$$

$$\Rightarrow m\{x : f(x) > 0\} = 0$$

$$\Rightarrow f = 0 \ \underline{\text{a.e.}}$$ ✓✓

Riemann Integration



f continuous. on $[a, b]$

$$\int_a^b f(x)\,dx$$

$$\overline{\int_a^b} f(x)\,dx \quad , \quad \underline{\int_a^b} f(x)\,dx$$

↑ upper R- integral   Lower R-integr

$$U(P, f) = \sum_{i=0}^{n-1} M_i \,(x_{i+1} - x_i)$$

$$M_i = \sup_{x \in [x_i, x_{i+1}]} f(x)$$

$$L(P, f) = \sum_{i=0}^{n-1} m_i \,(x_{i+1} - x_i)$$

$$m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$$

$$\overline{\int_a^b} f(x)\,dx = \inf_P U(P, f) \quad \text{— } (*)$$

$$\int_a^b f(x)\,dx = \sup_P L(P,f) \quad (*)$$

$$(**) = (*) = \int_a^b f(x)\,dx$$

then, f is Riemann integrable

Dirichlet function is NOT Riemann integrable ✓

Fundamental theorem of integral Calculus

Thm :- $F : [a,b] \to \mathbb{R}$ continuous
Let F is differentiable (a,b)
s.t. $F' = f$ where $f : [a,b] \to \mathbb{R}$
is Riemann integrable, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

✓

Example $F : [0,1] \longrightarrow \mathbb{R}$ by

$$F(x) = \begin{cases} x^2 \sin\frac{1}{x} \; ; & 0 < x \leq 1 \\ 0 \; , & x = 0 \end{cases}$$

Is F continuous?

$$\lim_{x \to 0} F(x)$$

$$= \lim_{x \to 0} x^2 \sin\frac{1}{x}$$

$$0 \leq \left| x^2 \sin\frac{1}{x} \right| \leq x^2 \; , \quad x \neq 0$$

$$\lim_{x \to 0} \left| x^2 \sin\frac{1}{x} \right| = 0 \checkmark$$

$$\Rightarrow \left| \lim_{x \to 0} x^2 \sin\frac{1}{x} \right| = 0$$

$$\Rightarrow \lim_{x \to 0} x^2 \sin\frac{1}{x} = 0 \checkmark$$

$f(x) = x^2 \sin\frac{1}{x}$ is cont. at $x = 0$ ✓

$$F'(x) = \begin{cases} -\cos\frac{1}{x} + 2x \sin\frac{1}{x} \, , & 0 < x \leq 1 \\ 0 \, , & x = 0 \end{cases}$$

is F' bounded ?

Yes → F' is discontinuous at $x = 0$

So, F' is Riemann integrable.

$$\int_0^1 F'(x)\,dx = F(1) - F(0)$$
$$= \sin 1$$

$$\int_0^1 \left(-\cos\frac{1}{x} + 2x\sin\frac{1}{x}\right)dx = \underline{\sin 1} \quad \text{Ans}$$

(3) Fundamental theorem of integral Calculus (Version-II)

Statement :- Suppose $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ & continuous at a. Then,

$$\lim_{h \to 0+} \frac{1}{h} \int_a^{a+h} f(x)\, dx = f(a)$$

$$\boxed{\ell\,[a, a+h] = h}$$

Proof

$$\cancel{K = \frac{1}{h} \int_a^{a+h} f(x)\, dx}$$

$$\frac{1}{h} \int_a^{a+h} f(x)\, dx - f(a) \leftarrow$$

$$= \frac{1}{h} \left[\int_a^{a+h} \left(f(x) - f(a)\right) dx \right]$$

Since f is cont. at $x = a$ $\Rightarrow$

$$|f(x) - f(a)| < \varepsilon \text{ for } a \le x < a + \delta.$$

It follows that if $0 < h < \delta$,

$$\left| \frac{1}{h} \int_a^{a+h} (f(x) - f(a))\, dx \right|$$

$$\le \frac{1}{h} \sup_{a \le x \le a+h} |f(x) - f(a)| \cdot h$$

$$\leq \quad \varepsilon.$$

This proves the result.

<u>Example</u> $\quad f : \mathbb{R} \to \mathbb{R}$ (Sign function)

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

Then

$$\lim_{h \to 0+} \frac{1}{h} \int_0^h f(x)\, dx = 1$$

$$\lim_{h \to 0+} \frac{1}{h} \int_{-h}^0 f(x)\, dx = -1$$

neither limit is $f(0)$

$$\lim_{h \to 0+} \frac{1}{2h} \int_{-h}^{h} f(x)\, dx = 0 = f(0)$$

Theorem Suppose $f : [a,b] \to \mathbb{R}$ is integrable & $F : [a,b] \to \mathbb{R}$ is defined by $F(x) = \int_a^x f(t)\,dt$.

Then, F is continuous on $[a,b]$. ✓

If f is cont. at $a \le \boxed{c} \le b$, then F is diff at c & $F'(c) = f(c)$.

Proof $|f| \le M$ ✓

$$|F(x+h) - F(x)|$$

$$= \left| \int_x^{x+h} f(t)\,dt \right|$$

$$\le M|h|$$

As $h \to 0$, $\lim F(x+h) = F(x)$

$\Rightarrow$ F is cont. on $[a,b]$

To show
F is diff. at the pt. c

$$\frac{F(c+h)-F(c)}{h} = \frac{1}{h}\int_c^{c+h} f(t)\,dt$$

$$F'(c) = \lim_{h\to 0} \frac{F(c+h)-F(c)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\int_c^{c+h} f(t)\,dt$$

$$= f(c), \quad \text{(by previous Thm)}$$

<u>Remark</u> continuity assumption at $x=c$ for f is needed.

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Here, F is cont but not differentiable at 0.

$$F'(0^+) = 1$$
$$F'(0^-) = 0.$$

⊜ Integrals & sequence of functions

$\{f_n\} \xrightarrow{pt.} f$

$\{f_n\} \overset{unif}{\rightrightarrows} f.$

You know that unif. convergence & convergence in sup-norm is same

$$\|f\| = \sup\{|f(x)|, x \in A\}$$

Verify that this is indeed a norm? ✓

Theorem :- Suppose $\{f_n\} : [a,b] \to \mathbb{R}$ is Riemann integrable for each $n \in \mathbb{N}$ & $f_n \rightrightarrows f$ uniformly. Then

$f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ & $\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$

Proof :- Since $\{f_n\}$'s are bounded then f will also be bounded.

$$|f_n(x) - f(x)| < 1 \qquad \varepsilon = 1$$

$$\forall x \in A, \quad n \geq N_0$$

$$\big|\, |f| - |f_n| \,\big| \leq |f_n - f| < 1$$

$$n \geq N_0$$

$$|f| \leq |f - f_n| + |f_n|$$

$$< 1 + M.$$

Let $\varepsilon > 0$, $f_n \to f$ uniformly.

$\exists\ N \in \mathbb{N}$ s.t. if $n > N$, then

$$-\frac{\varepsilon}{b-a} < f - f_n < \frac{\varepsilon}{b-a}$$

$$\Rightarrow f_n - \frac{\varepsilon}{b-a} < f < f_n + \frac{\varepsilon}{b-a}$$

We know

$$\boxed{L\left(f_n - \frac{\varepsilon}{b-a}\right) \le L(f)} \text{ and } ✓$$

$$U(f) \le U\left(f_n + \frac{\varepsilon}{b-a}\right) ✓$$

Given that f_n is integrable. ✓

We know : upper integrals are greater than lower intergral

$$\underbrace{\int_a^b \left(f_n - \frac{\varepsilon}{b-a}\right) dx \le L(f)} \le U(f) \le \underbrace{\int_a^b \left(f_n + \frac{\varepsilon}{b-a}\right)}$$

$$\Rightarrow \int f_n - \varepsilon \le L(f) \le U(f) \le \int_a^b f_n + \varepsilon$$

$$\left(\because \text{Since } f_n \text{ is integrable, } \int_a^b \left(f_n - \frac{\varepsilon}{b-a}\right) dx = L\left(f_n - \frac{\varepsilon}{b-a}\right) \right)$$

$$\Rightarrow 0 \le U(f) - L(f) \le 2\varepsilon$$

Since $\varepsilon > 0$, is arbitrary

$$U(f) = L(f).$$

$\Rightarrow$ f is Riemann integrable.

$$\left| \int_a^b f_n - \int_a^b f \right|$$

$$= \left| \int_a^b (f_n - f) \right|$$

$$\leq \int_a^b |f_n - f|$$

$$\leq \|f_n - f\| \cdot (b-a)$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

$$\Rightarrow \int_a^b f_n \longrightarrow \int_a^b f. \checkmark$$

Problem :-

$$\lim_{n\to\infty} \int_0^1 \frac{n + \cos x}{n e^x + \sin x}\, dx.$$

with justification.

Ans : $f_n(x) = \dfrac{n + \cos x}{n e^x + \sin x}$

Claim, if, $f(x) = e^{-x}$, then

$f_n \longrightarrow f$ unif. on $[0,1]$

$$\left| \frac{n+\cos x}{ne^{x}+\sin x} - e^{-x} \right|$$

$$= \left| \frac{\cos x - e^{-x}\sin x}{ne^{x}+\sin x} \right| \leq \frac{2}{n} \quad \forall x \in [0,1]$$

$$|ne^{x}+\sin x| \geq n.$$

$$|\cos x - e^{-x}\sin x| \leq 2$$

$$\lim_{n\to\infty} \frac{n+\cos x}{ne^{x}+\sin x} = \frac{1}{e^{x}}$$

& this convergence is uniform

$$\lim_{n\to\infty} \int_0^1 \frac{n+\cos x}{ne^{x}+\sin x}$$

$$= \int_0^1 e^{-x}\,dx$$

$$= 1-\frac{1}{e}. \quad \square$$

Example

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

has r.o.c. $R > 0$.

$|x| < R$, in every compact intervals inside $|x| < R$, the above power series converges uniformly.

Term-by-term integration is possible

now

$$\int_a^x f(t)\,dx, \quad |x| < R,$$

$$= a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \cdots\cdots.$$

For instance

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots\cdots + x^n + \cdots \quad |x| < 1$$

$$\ln\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \quad |x| < 1$$

$x = \frac{1}{2}$,

$$\boxed{\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n 2^n}}$$

To Prove

" f is Riemann integrable iff set of discontinuity for f has Lebesgue measure zero"