Date ___ / ___ / ____

Saathi

ode

Only One Independent Variable $x \in [a,b]$
$t \subset \mathbb{R}$

Evolutionary differentials : i.e wrt 't'.

General form:

$f(x)$

$f(t, y, y' \ldots y^{n-1}, y^n) = 0$.

t - independent.

If linear in terms of differentials ⇒

Linear — Non-Linear

IVP — BVP

$f(t, y, y' \ldots y^n) \longrightarrow y = f(t, c_i)$ — $i = 1$ to n
I.C.

∴ eq^n solved in an interval given initial cond^n.

TI - Dirichlet
TII - Neumann
} BVP

TIII - Robin

— linear $a_0 y(t$

b_0

③ Condn of existance & soln & ODE.

Date ___/___/___

Vectorisation of IVP

soln of IVP:

1. Picards method of successive Approximn.
2. Taylor series method.
3. Euler (forward) method.
4. Backward Euler method.
5. Modified — o —
6. A Class of Runge-Kutta method

Single step methods. // Approximate Method

Numerical | Non-numerical

Multistep Method

Date

1 Picard's Method of Successive Approxⁿ

IVP . $\frac{dy}{dt} = f(t, y)$ $t \in [t_0, b)$

$y(t_0) = y_0$

$$\int_{t_0}^{b} \frac{dy}{dt} = \int_{t_0}^{b} f(t, y)$$

$$y - y_0 = \int_{t_0}^{b} f(t, y)\, dt$$

$$y(t) = y_0 + \int_{t_0}^{b} f(t, y)\, dt$$

$$y_{n+1}(t) = y_0 + \int_{t_0}^{b} f(t, y_n)\, dt$$

$y_1(t)$: Approxⁿ to y at 1st level.

$n=0$ $\quad y_1(t) = y_0 + \int_{t_0}^{b} f(t, y_0)\, dt$. now its know hence Integrable

$n=n$ $\quad y_{n+1}(t) = y_0 + \int_{t_0}^{b} f(t, y$

sequence of Approxⁿ gener

Page No.

Date __/__/__

$\{y_{n+1}\}_0^\infty \rightarrow y(t)$

what's the cond^n that converges to $y(t)$.

we have solved
$t \in [t_0, b)$

take some pt from this interval
& see if they are close enough

II Taylor Series Method.

$$\frac{dy}{dt} = f(t,y) \xrightarrow[\text{①}]{} y(t)$$

s.t $y(t_0) = y_0$

$$y(t) = y_0 + t\,y'(t_0) + \frac{t^2}{2}y''(t_0) \ldots\ldots$$

t_0+h

$t=h$

Value of derivatives from eq^n ①

Date
Ex

IVP

$$y' = x^2 + y^2 \qquad y(0) = 1$$

construct y_1, y_2, y_3

$$y_{n+1} = y_0 + \int_0^x (x^2 + y_n^2)\, dx$$

$n = 0$

$$y_1 = 1 + \int_0^x (x^2 + 1)\, dx$$

$$y_1 = \frac{x^3}{3} + x + 1$$

$n = 1$

$$y_2 = y_1 + \int_0^x \left[(x^2) + \left(1 + x + \frac{x^3}{3} \right)^2 \right] dx$$

$\vdots$

so on so forth.

Date / /

Numerical solⁿ

$\frac{dy}{dt} = f(t, y)$

st $y(t_0) = y_0$) $t \in [t_0, b]$ (a)

step sizes.

S-I: ~~/~~ discritisation of domain:
dividing the interval into non-overlapping

t_0 — b

$a = t_0 < t_1 < t_2 \ldots < t_n = b$

data set $S: \{t_0, t_1, \ldots t_n\}$ Nodal Pts

t_0, t_N - Boundary Pts

$t_0, \ldots t_{N-1}$ - Interval Nodal Pts

∴ ~~we~~ do not have the value at each pt, just at nodal pts.

Bt, further refining helps.

$t_i - t_{i-1} = h \quad \forall i$

$\forall i$ then equally partitioned

⊗ It's posible that have different sub domains diff. partition

S-II:

When ever I replace differentials By finite differences → system of linear Algebraic eq.

$$y_{t_{i+1}} = y(t_i + h) = y(t_i) + h y'(t_i) + \frac{h^2}{2} y''(t_i) \ldots$$

$$y_{t_{i+1}} - y(t_i) = h y'(t) + \frac{h^2}{2} y''(t) \ldots$$

$$y_{t_{i+1}} - y(t_i) \simeq y'(t_{i+1}) \simeq y'(t_i) + O(h).$$

Date / /

I Forward euler Method

$$y_{i+1} = y_i + h f_i$$

$$\left\{ \begin{array}{l} y_1 = y_0 + h f_0 \\ \vdots \\ y_N = y_{N-1} + h f_{N-1} \end{array} \right\}$$ System of eqn.

(i) Explicit Method

(ii) Single step Method

(iii) $O(h)$ 1st Order Approxm.

h: step size.

II Backward euler method.

$$t_{i+1} = y(t_i - h)$$
$$= y(t_i) - h y'(t_i) + \frac{h^2}{2} y''(t_i) \ldots - \frac{h^3}{\lfloor 3} y'''(t_i) \ldots$$

$$\frac{y(t_{i-1}) - y(t_i)}{h} = -y'(t_i) + O(h)$$

$$\boxed{y_i - y_{i-1} = h f_i - O(h)}$$

$$y_i - y_{i-1} = h f_i . \qquad i = 1, 2 \ldots, N$$

$$\boxed{y_i = y_{i-1} + h f_i}$$ Backward difference Method.

Date ___/___/___

Saathi

(i) Implicit Method.

(ii) S.S.M.

(iii) $O(h)$

$$\left.\frac{dy}{dt}\right|_{t_i} \simeq \frac{y_{i+1} - y_{i-1}}{2h} \quad \delta \quad \text{Centred}$$

$$\frac{y_{i+1} - y_i}{h} \quad \Delta$$

$$\frac{y_i - y_{i-1}}{h} \quad \nabla$$

$$y_{i+1} = y_i + h\, y_i'(\,) + \frac{h^2}{2} \cdot y_i''(\,) + \frac{h^3}{L3} y_i''' \ldots$$

$$y_{i-1} = y_i - h\, y_i'(\,) + \frac{h^2}{2} y_i''(\,) \ldots$$

$$\boxed{\delta = y_i'(\,) + O(h^2)}$$

10/1/17

$\frac{dy}{dt} = f(t, y) \qquad t \in [a, b)$ —— ①

$y(a) = y_0.$

Mid point method

$$\left.\frac{dy}{dt}\right|_{t_i} = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2)$$

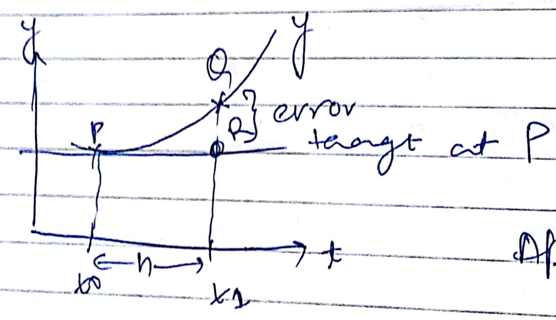
Numerical Method :

(1) FEM :

$$y_{i+1} = y_i + h f_i$$

(2) BEM $\quad y_{i+1} = y_i + h f_{i+1}$

(3) MidPoint : $\quad y_{i+1} = y_{i-1} + 2h f_i.$



Approx R
Actual Q.

$y_1 = y_0 + h f_0.$

$$y(t_1) = y(t_0) + h . f(t_0, y_0)$$
$$= y(t_0) + h . y'(t_0)$$

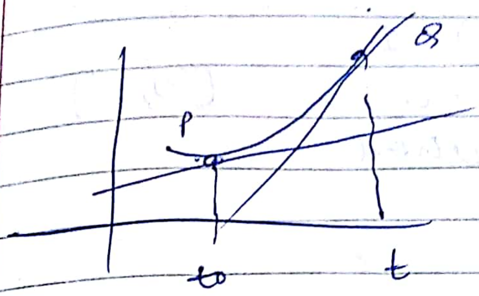
Date ___ / ___ / ___

$R = y_E(x_1) - y_1$

<u>Backward M.</u>

$$y_1 = y_0 + hf_1 \quad \ldots$$

$O(h)$
1st Order
~~Approx~~ soln



$$y_1 = y_0 + h f(t_0, y_0)$$

$$y_1 = y_0 + h y'(t_0)$$

$\boxed{\text{modificn}}$

$$y_1 = y_0 + h \frac{1}{2}(f_0 + f_1).$$

$$y_1 = y_0 + \frac{h}{2}[f(t_0, y_0) + f(t_1, y_1)]$$

Set up an iterative mechanism:

$$y_1 = y_0 + \frac{h}{2}[f(t_0, y_0) + f(t_1, y_1^{(s)})]$$

$s = 0, 1, 2 \ldots$

$$|y_1^{(s+1)} - y_1^{(s)}| < \varepsilon \text{ — error tollerence}$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(t_0, y_0) + f_1(t_1, y_1^{(0)})]$$

$$y_1^{(0)} = y_0 + hf_0$$

modifier

$$\boxed{y_{j+1} = y_j + \frac{h}{2}[f_j + f_{j+1}].}$$

Page No.

Date

$$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f_1(t_1, y_1^{(1)})\right)$$

$\underline{O(h^2)}$

Q $\frac{dy}{dx} = 2x + y, \quad y(0) = 1 \quad [0, 1]$

$h = 0.2$

Using forward ~~eler~~ euler method.

$\frac{dy}{dx} - y = 2x$

$\cancel{e^{-2xy}}$

e^{-y}

0   0.2   0.4   0.6   0.8   1

x_0 x_1 x_5

$y_1 = y_0 + hf_0$

$f = 2x + y$

$y_1 = 1 + 0.2(0) \quad (0,0)$

$0.2(1) = \underline{1.2} = y_1$

$y_1 = 1$

$y_2 = 1 + 0.2\left(2(0.2) + 1\right)$

$\cancel{y_2 = 1.6}$

1.4×0.2

1.28

$y_2 = 1.2 + (0.2)\left(2(0.2) + 1.2\right)$

0.4

$(0.2)(1.6)$

$1.2 + 0.32$

$y_2 = 1.52$

Date ___ / ___ / ___

$y_3 = \cancel{0.4} + 0.2(2(0.4) + 1.52)$

$1.52 \qquad\qquad 0.8$

$0.2(1.32)$

0.264

$\cancel{y_3 = 0.664}$

$y_3 = 1.784$ ✓

$\boxed{\text{decreasing step size} - \uparrow \text{accuracy}}$

$y_1 = 1 + 0.\cancel{2}(2(0) + 1) = 1.1$

$y_2 = 1.1 + 0.1(2(0.1) + 1.1)$.

1.3

$y_2 = 1.23$

(1) $h = 0.1$
(2) $h = 0.005$
(3) y : exact

Program

Q: $y' + \frac{1}{2}y = 0 \qquad y(0) = 1. \qquad [0, 20]$

$h =$

$\frac{dy}{y} = -\frac{x}{2}$ $\qquad$ ~~log~~ $\qquad y = e^{-x/2}$

(i) $h = 1$ $\qquad$ (ii) $h = 4$ $\qquad$ (iii) $h = 4.2$

Date

$y' = -2ty^2 \qquad y(0) = 1$

$h = 0.2 \qquad [0, 0.4]$

0   0.2   0.4

$y_1 = y_0 + h f_1$

$y_1 = 1 + h(-2t y_1^2)$

$y_1 = 1 - 2(0.4) y_1^2$

$0.08 y_1^2 + y_1 - 1 = 0$

$$y_1 = \frac{-1 \pm \sqrt{1 + 0.32}}{0.16}$$

$$y_1 = \frac{-1 \pm \sqrt{1.32}}{0.16}$$

0.931 ✓

~~-13.42~~ not physically possible.

Date 11/1/17

- In FEN: we consider One slope.
  $t_j = y_j'$ $\quad - O(h)$.

In modified EM: $\quad O(h^2)$

2 slopes used

⋮

* explicit method with 2 slopes:

$$K_1 = h(f(t_j, y_j))$$
$$K_2 = h(f(t_j+h, y_j+K_1))$$

$\boxed{y_{j+1} \cong y_{j+K_1}}$ approximate

$$y_{j+1} = y_j + \frac{K_1}{2} + \frac{K_2}{2}.$$

$$y_{j+1} = y_j + w_1 K_1 + w_2 K_2$$

w_i = weight of K_i in the method.

Runge-Kutta method of order 4:

$$y_{j+1} = y_j + w_1 K_1 + w_2 K_2 + w_3 K_3 + w_4 K_4.$$

Date

4th Order Range Kutta method

$$y_{j+1} = y_j + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$K_1 = h f(t_j, y_j)$

$K_2 = h \cdot f(t_j + h/2, y_j + \frac{K_1}{2})$

$K_3 = h \cdot f(t_j + h/2, y_j + \frac{K_2}{2})$

$K_4 = h \cdot f(t_j + h, y_j + K_3)$

#. $y' = 2x + y$ $\quad y(0) = 1 \quad y(0.2)$

$y(0.4)$

$h = 0.2$

explicit 2 slope method

$$y_{j+1} = y_j + \frac{1}{2}(K_1 + K_2)$$

$K_1 = h f(t_j, y_j)$

$K_2 = h f(t_{j+1}, y_1 + K_1)$

$y_1 = y_0 + \frac{1}{2}$

$K_1 = h f(t_0, y_0)$

$= h(\cancel{0}2(0) + 1)$.

$K_1 = 0.2$

$K_2 = 0.2(\underline{0.2}, 0.\cancel{2}4)$

$0.2(3(0.2) \cancel{0}1.6$

$K_2 = \cancel{0.12}\ 0.32$

Date

$y(0.2) = 1 + \frac{1}{2}(0.2 + 0.32)$

$\frac{0.52}{2} = 0.26$

$y(0.2) = 1.26$ //

$y(0.4) = y(0.2) + \frac{1}{2}(k_1 + k_2)$

$k_1 = h\,f(0.2, 1.26)$

$k_1 = 0.2\,(2(0.2) + 1.26)$

1.66

$k_1 = \underline{0.332}$

$k_2 = 0.2\,(0.4, 1.592)$

Date

Order of a method :

$y' = f$

$y(t_{j+1}) = y(t_j) + h y'(t_j) + \frac{h^2}{2} y''(t_j) + \dots$

Local truncation error :

Order = $\frac{\text{local } e}{h}$ for any One step method

$O(h^\alpha)$

- this is a measure of accuracy, It gives an indication of how rapidly the accuracy can be improved with refinement of grid spacing h.

1st Order

Error = E1

step size = h. $\Longrightarrow$ refine $h \to h/2$

$h \longrightarrow E1$

$h/2 \to E1/2$ error

Date ___/___/___

Saathi

2^{nd} order method:

$h : E_2$

$\frac{h}{2} : \frac{E_2}{4}$.

Q: $y' = 2t + y$ $\quad y(0) = 1$ $\quad h = 0.2$ $\quad y(0.2)$

$y_{j+1} = y_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

$k_1 = h f(t_j, y_j)$ $\quad k_2 = h(f(t_j + h/2, y_j + \frac{k_1}{2}))$.

$k_3 = h(f(t_j + h/2, (y_j + \frac{k_2}{2})))$

$k_4 = h(f(t_j + h, y_j + k_3))$

$y_1 = 1 + \cancel{0.2}\{$

$k_1 = 0.2(2(0.2) + 1)$

$k_1 = 0.2$

$k_2 = 0.2(2(0.1) + 0.1)$

$k_2 = 0.06 + 0.2 = 0.26$.

$k_3 = 0.2(2(0.1) + \cancel{0.03})$

$\cancel{0.2}3 \quad 1 + 0.13$

$0\cancel{.046} \quad (1.33)\,0.2\,0.266$

$k_4 = 0.2(2(0.2) + \cancel{0.046})$

$1 + 0.266)$

$= 0.2(0.4 + 1.266)$

$0.2(1.666)$

$y = 0.3332$

Date

$y_1 = 1 + \frac{1}{6}\left(0.2 + 2(0.26) \cancel{+} 2(0.226) + 0.333\right)$

$0.52 + 0.452 + \cancel{0.33}$

$0.72 \qquad 0.7852$

$y_1 = 1 + \frac{1.5052}{6}$

$6 \overline{)\,15052\,(}\, 25086$

12

30

30 52

48

40

36

$y_1 = 1.25086$

$y_1 = \boxed{1.2642}$

$y_{exact} =$

$y' - y = 2t$

$y = Ae^{t}$

$\frac{1}{D-1} 2t$

$1 - ($

program: (1) FEM Matlab

(2) BEM

(3) modified EM

(4) 4th order

for $y' = -t^2 \sin(t+y)$

$y(0) = 0$

$[0, 1]$ $h = 0.1$

Problems

* Picards

Q. Compute first few terms using Picards Iteration with $y_0(x)=0$ for the Initial value problem: $y' = xy + 2x - x^3$ with $y(0)=0$ & show they converge to soln. $y(x)=x^2$

Sol: Picards iteration formulae

$$y_n(x) = \underline{y_0(x)} + \int_{x_0}^{x} f(x, y_{n-1}(x))\, dx$$

$$f(x,y) = \overset{\text{always}}{xy} + 2x - x^3 \qquad x_0 = 0 \quad y_0 = 0$$

first Iteration putting $(n=1)$

$$y_1(x) = y_0 + \int_0^x (x_0 y_0 + 2x_0 - x_0^3)\, dx$$

$$\int_0^x (2x - x^3)\, dx$$

$$y_1(x) = x^2 - \frac{x^4}{4}$$

$n=2$

$$y_2(x) = y_0(x) + \int_0^x (x(y_1) + 2x - x^3)\, dx$$

$$0 \qquad \left(x\left(x^2 - \frac{x^4}{4}\right) + 2x - x^3 \right)$$

$$y_2(x) = \cancel{x^2} \cancel{-\frac{x^4}{4}} + \int_0^x \left(2x - \frac{x^5}{4}\right) dx$$

$$x^2 - \frac{x^6}{}$$

Date __/__/__

$$y_3(x) = y_0 + \int_0^x f(x, y_2(x))\,dx$$

$$y_3(x) = \int_0^x (x y_2 + 2x - x^3)\,dx$$

$$= x^3 - \frac{x^7}{24} + 2x - \cancel{x^3}$$

$$= \int_0^x \left(2x - \frac{x^7}{24}\right) dx$$

$$y_3(x) = x^2 - \frac{x^8}{8 \cdot 24}$$

From these values we can conclude that the solution $y = x^2$.

$$y' - xy = 2x - x^3$$

$$\frac{dy}{dx} - xy$$

$$\cancel{e^{-xy}}$$

$$e^0 \frac{dy}{dx} - e^{-\frac{x^2}{2}}$$

$$\frac{d\, y e^{-x^2/2}}{dx} = (2x - x^3)\, e^{-x^2/2}$$

$$e^{-x^2/2} y = \cancel{\left(x^2 - \frac{x^4}{4}\right) e^{-x}}$$

$$e^{-x^2/2} y = \int (2x - x^3)\, e^{-x^2/2}\, dx.$$

Saathi

② Use taylor series to find a numerical solution at $x = 0.5$ of ODE $y'' = xy$ when $x = 0$ given that $y' = 1$ & $y = 1$ (use $h = 0.5$)

$$y(x_0 + h) = y(x_0) + h y'(x_0) + \frac{h^2}{\lfloor 2} y''(x_0) + \frac{y'''(x_0)}{\lfloor 3} h^3 + \frac{h^4}{\lfloor 4} y^{iv}(x_0) \ldots$$

// truncate upto 4th orders (Just like that)

$$y(x_0+h) = y(x_0) + h y'(x_0) + \frac{h^2}{2} y''(x_0) + \frac{y'''(x_0) h^3}{\lfloor 3} + \frac{h^4}{\lfloor 4} y^{iv}(x_0)$$

given $y(0) = 1$ & $y'(0) = 1$

$\cancel{y(x_0)} = 1$ $\quad y''(0) = x_0 y_0 = 1 \times 0 = 0$.

$\cancel{y'(}$

$$y'''(x_0) = x y'(0) + \frac{y(0)}{1}$$

$$y'''(x_0) = 1$$

$$y^{iv} = x y''(0) + \frac{y'(0)}{2} + y'(0)$$

$$y^{iv}(0) = 2.$$

Date

$y(x_0+h) = 1 + 0.5 \times 1 + \frac{(0.5)^2}{2} \times (0) + \frac{(0.5)^3}{\lfloor 3}(2)$
(0.5)

$+ \frac{(0.5)^4}{\lfloor 4} \times (2)$

$\boxed{y(0.5) = 1.526}$

(5) Solve the ODE $y' = 2y + 3e^x$ with $y(0) = 0$ using Taylor series method of order (2) to approximate y for $x = 0.1, 0.2$

$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{\lfloor 2} y''(x_0)$ truncated

$y(x_0) = 0$

$y'(x_0) = 3 = 2(0) + 3e^0$.

$y'' = \cancel{2y_0' + 3.6}\ .2y_0' + 3e^{x_0} = 9$.

$y(0.1) = 0 + (0.1)(3) + \frac{(0.1)^2}{2} 9$.

$y(0.1) = 0.3 + \frac{0.09}{2} = 0.045 + 0.3$
$= 0.345$.

$y(0.2) = (0.2)(3) + \frac{(0.2)^2}{2} \cdot 9$.

$(0.02)9$ $= 0.18 + 0.6$
0.78.

Date

(4) Solve $y' = x - y^2$ $y(0) = 1$ by euler's method for $x = 0.2$ to 0.6 with $h = 0.2$.

(5)

$y_{n+1} = y_n + h f(x_n, y_n)$

here $f(x, y) = x - y^2$

$y =$ $y_1(0.2) = y_0 + (0.2) f(x_0, y_0)$

$= 1 + 0.2\ (0 - 1)$

$= 0.8$

$y_2(0.4) = y_1(0.2) + 0.2\left((0.2) - (0.8)^2\right)$

$= 0.8 + 0.2\ (0.2 - 0.64)$

$-(0.44)\,0.2$

$= 0.8 - 0.088$

$y_2(0.4) = \cancel{0.008}\ 0.712$

$\begin{array}{r} 800 \\ 088 \\ \hline 712 \end{array}$

$y_3(0.6) = y_2(0.4) + 0.2\left((0.4) - \cancel{(0.08)}^2\right)$

$(0.712)^2)$

$y_3(0.6) = \cancel{0.448}\ 0.6906.$

Sauthi

Date

(5) Given $y' = y - x$ where $y(0) = 2$.

find $y(0.1)$ & $y(0.2)$ euler method.

$h = 0.1$

soln.

$y_{n+1} = y_n + h\, f(x_n, y_n)$

$y_1 = y_0 + 0.1 (f(0, y_0))$

$= 2 + 0.1 (2 - 0)$

$y_1 = 2.2$

$y(0.2) = \underline{y(0.1)} + 0.1 (2.2 - 0.1)$

$0.1 (2.1)$

$y(0.2) = 2.2 + 0.21$

$= \underline{2.41}$

(6) given

$\frac{du}{dx} = x + u^2 \quad u(0) = 1$

Date ___/___/___

Any general single step Method for IVP :

$$\frac{dy}{dt} = f(t, y)$$

such that $y(t_0) = y_0$ $t \in [t_0, b]$

$y_{j+1} = y_j + h\phi(t_j, y_j, h)$ $\phi \to$ increment fn

$y_{j+1} = y_j + hf_j$ — euler method

Local Truncation Error

↳ Denoted by T_{j+1}

$$T_{j+1} = \underbrace{y(t_{j+1})}_{\text{exact soln}} - \underbrace{y_{j+1}}_{\text{numerical soln}}$$

$$T_{j+1} = y(t_{j+1}) - [y_j + h\underbrace{f_j}_{y'_j}]$$

O(LTE)

$$T_{j+1} = y(t_{j+1}) - (y_j + hy'_j)$$

$$= \left[y(t_j) + hy'(t_j) + \frac{h^2}{2} y''(t_j) \cdots\right] - [y_j + hy'(t_j)]$$

$$O(LTE) = O(h^2)$$

1Red 3 Brown - Liner Algebra

Date ___/___/______

Order of Numerical Method
(Single step Method)

as
$O(h^p)$

$$\left|\frac{1}{h} \cdot O(LTE)\right| = O(h^p)$$

FEM = $O(h)$ = BEM
~~ME~~ Mod. EM = $O(h^2)$

O(LT.E) / 4th Order R.K = $O(h^5)$

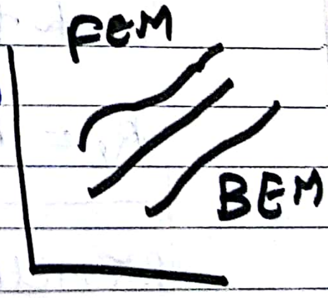
$y_{j+1} = y_j + hf$ —— FEM

$T_{j+1} = y(t_{j+1}) - \{y_j + hy_j)$

$= \frac{h^2}{2!} y_j''(\xi) \qquad t_j < \xi < t_{j+1}$

$\max_{[t_0, b]} |y''(\cdot)| = M$

$$\boxed{T \leq \frac{h^2}{2} M}$$



2 ways to increase accuracy
- h refinement.
- p refinement.

: Backward - less than exact.

Page No.

Class of Runga Kutta Method.

R-K methods use weighted avg. of slopes on the given interval $[x_j, x_{j+1}]$ instead of simple

general R-K Method with ν-slope

$y_{j+1} = y_j + h$ {weights avg. of slopes of given interval}

$$\left\{ y_{j+1} = y_j + [w_1 k_1 + w_2 k_2 \dots + w_\nu k_\nu] \right)$$

since I have defined : $\boxed{k_j = hf(x_j, y_j)}$

where

w_i — weights to be determined

Date

Explicit 2nd Order RK method.

$y_{j+1} = y_j + w_1 k_1 + w_2 k_2$

$k_1 = h(f(x_j, y_j))$

$k_2 = h(f(x_j + c_2 h, y_j + a_{21} k_1))$.

need to find : w_1, w_2, c_2, a_{21}

* Make y_{j+1} close to $y(x_{j+1})$ upto 2nd order

$= y(t_{j+1}) - y_{j+1}$

$= y(t_{j+1}) - \{ y_j + w_1 h . f(x_j, y_j) + w_2 h \, f(x_j + c_2 h, y_j + a_{21} k_2) \}$

$y(x_{j+1}) = y(x_j) + y'(x_j) h + \frac{h^2}{2} y''(x_j) + \frac{h^3}{\lfloor 3} y'''(x_j) \ldots$

$y' = f$

$y'' = \frac{\partial f}{\partial x}' = f_x + f_y y'$.

Page No.

Date / /

Book

1) Numerical methods using MATLAB
Pearson Publicn
Jokin Mathew

2) Numerical Methods for Scientific & engineering computn:
M. K. Jain Iyengnar.

Implicit methods with ν slops.

compact form:

$$y_{j+1} = y_j + \sum_{j=1}^{\nu} w_j K_j$$

where,

$$K_j = h f\left(x_j + c_j h,\ y_j + \sum_{m=1}^{\nu} a_{jm} k_m\right)$$

where $c_j = \sum_{i=1}^{\nu} a_{ji}$, $j = 1, 2 \ldots \nu$

and $a_{j,i}$ $i \leq j <$

w_i' are arbitary parameters :

no. of unknown parameters are :
$\nu(\nu+1)$

Saathi

Date ___/___/____

$y = 1$

$$y_{j+1} = y_j + w_1 k_1$$

$$k_1 = hf(x_j + c_1 k_1,\ y_j + a_{11} k_1)$$

$$T_{j+1} = y(x_{j+1}) - y_{j+1} \qquad y' = f.$$

comparision of h & h^2 terms:

$$w_1 = 1, \quad w_1 c_1 = 1/2 \qquad w_1 a_{11} = 1/2$$

$$c_1 = 1/2, \qquad a_{11} = 1/2.$$

$$y_j = y_j + k_1, \quad k_1 = h\ f\left(x_j + \frac{h}{2},\ y_j + \frac{1}{2} k_1\right)$$

Date / /

Higher Order IVP

$y y'' + 2y'^2 + 3y = e^x$ $[0,1]$

$y(0) = r_1$
$y'(0) = r_2$

$f(x, y, y', y'') = 0$.

* $y'' + 3y' + 2y = 0$

put $y = u$.
$y' = u' = v$
$y'' = v' = -3y' - 2y$

$u' = v$
$v' = -3u' - 2u$.

$$x = \begin{bmatrix} u \\ v \end{bmatrix}' = F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$u' = g_1(x, u, v) = v$
$v' = g_2(x, u, v) = -3u' - 2u$
$= -3v - 2u$.

$u' = v$
$v' = -3v - 2u$ $\Big\}$ given 2nd order into 2 1st order eqn

$\underline{w}' = \underline{F}(x, w)$

$$y' = f(x, y)$$

$$x : [a, b)$$

FEM : $y_{j+1} = y_j + h f_j()$

$$u_{j+1} = u_j + h g_{1j}$$
$$v_{j+1} = v_j + h g_{2j}$$

<u>Taylor series Method</u> :

$$y_{j+1} = y_j + h y_j' + \frac{h^2}{2} y_j'' + \dots + \frac{h^p}{p!} u_j^p$$

$$j = 0, 1, 2 \dots$$

where $u_j^{(k)} = \begin{cases} u_{1j}^{k} \\ u_{2j}^{k} \\ \vdots \\ u_{nj}^{k} \end{cases}$

Newton Raphson Method

$R^{n+1} = R^n - \frac{F(R^n)}{F'(R^n)}$

24/1/18 Date

Saathi

ex: $y' = -2xy^2$, $y(0)=1$ $[0, 0.4]$

$h = 0.2$

Using the I.R.K with one slope

y_0 — y_1 — y_2

0   0.2   0.4

$$y_{j+1} = y_j + K_1$$

$$K_1 = h f\left(x_j + \frac{h}{2}, y_j + \frac{K_1}{2}\right)$$

$j = 0, 1.$

$j = 0.$

$$y_1 = y_0 + K_1 = 1 + K_1$$

$$K_1 = 0.2 [\quad]$$

$$= 0.2\left[-2 \times \left(x_0 + \frac{0.2}{2}\right)\left(1 + \frac{K_1}{2}\right)^2\right]$$

K_1 — can be calculated.

$F(K_1) = 0$

$$K^{n+1} = K^n - \frac{F(K^n)}{F'(K^n)}$$

$$F(K_1) = K_1 + 0.2[2x_j + 0.2]\left[y_1 + \frac{1}{2}K_1\right]^2$$

$$F'(K_1) = 1 + (0.2)\left(y_j + \frac{K_1}{2}\right)(2x_j + 0.2).$$

Page No.

$$K_1^{(n+1)} = K_1^{(n)} - \frac{F(K_1^{n})}{F'(K_1^{n})} \quad n = 0, 1, 2 \ldots$$

$j = 0$ Assume:

$$K_1^{(n)} = h f(x_j, y_j)$$

$$K_1^{(0)} = 0$$

$$K_1^{(1)} = -0.0384615$$

$$K_1^{(2)} = -0.03847567$$

$$y(0.2) = y(0) + K_1$$

$$= 1 + (-0.03847567)$$

$$= 0.96152$$

Slly;

$$y(0.4) = y(0.2) + K_1$$

$$K_1 = 0.2\left[-2x\left(x_1 + \frac{R}{2}\right)\left(y_1 + \frac{K_1}{2}\right)^2\right.$$

$$y(0.4) = 0.86179013$$

$$y_{j+1} = y_j + \left(\frac{K_1 + K_2}{2}\right)$$

$$K_1 = h f\left(x_j + \frac{3-\sqrt{3}}{6}h,\ y_j + \frac{K_1}{4} + \frac{3-2\sqrt{3}}{12}K_2\right)$$

$$K_2 = h f\left(x_j + \frac{3+\sqrt{3}}{6}h,\ y_j + \frac{3+2\sqrt{3}}{12}K_1 + \frac{K_2}{4}\right)$$

solve $y' = -2xy^2$, $y(0) = 1$

$h = 0.1$ $[0, 0.4]$

Using implicit R.K with 2 slopes.

$\boxed{\text{example}}$ $u''' + 2u'' + u' - u = \cancel{\cos t}\ \cos(t)$.

$0 \le t \le 1$

$u(0) = 0$ $u'(0) = 1$ $u''(0) = 2$.

2nd order taylor series method $h = 1.0$.

soln $u = v_1$ $u' = v_1' = v_2$ $u'' = v_1'' = v_2' = v_3$

$$v' = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}' = \begin{Bmatrix} v_2 \\ v_3 \\ \cos(t) + v_1 - v_2 - 2v_3 \end{Bmatrix}$$

happy B day

4:30 / 4:30 Date

$$\text{Ily} \quad v''(0) = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$$

$$v(1) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$$

$$V(1) = \begin{bmatrix} 2 \\ 1 \\ 3/2 \end{bmatrix} = \begin{bmatrix} v_1(1) \\ v_2(1) \\ v_3(1) \end{bmatrix} = \begin{bmatrix} u(1) \\ u'(1) \\ u''(1) \end{bmatrix}$$

$$\cancel{u'' + 2u' + u' - u = \cos(t)}$$

Q: $u' = -3u + 2v$ $\qquad u(0) = 0$

$v' = 3u - 4v$ $\qquad v(0) = 0.5$

$h = 0.2$

$[0, 0.4]$

$u' = f(x, u, v)$

$v' = g(x, u, v)$

4th order R.K

soln. $u_{j+1} = u_j + \frac{1}{4}(k_1 + 2k_2 + 2k_3 + k_4)$

$v_{j+1} = v_j + \frac{1}{4}(L_1 + 2L_2 + 2L_3 + L_4)$

each k_i — have 3 arguments. $x_i \; v_i \; u_i$

Date ___/___/___

$K_1 = h\,(f\,(x_j, u_j, v_j))$

$L_1 = h g\,(x_j, u_j, v_j)$

$K_2 = h f\left(x_j + \frac{h}{2}, u_j + \frac{K_1}{2}, v_j + \frac{L_1}{2}\right)$.

$L_2 = h\, g\left(x_j + \frac{h}{2}, u_j + \frac{K_1}{2}, v_j + \frac{L_1}{2}\right)$

$K_3 = h \cdot f\left(x_j + \frac{h}{2}, u_j + \frac{K_2}{2}, v_j + \frac{L_2}{2}\right.$.

$K_1 = 0.2$, $\quad L_1 = -0.4$

$K_2 = 0.06 \quad L_2 = -0.18$

$K_3 = 0.646 \quad L_3 = -0.31$

$K_4 = -0.0116 \quad L_4 = -0.0644$

$u(0.2) = 0 + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right) = 0.1001$

Ily.

$v(0.2) = 0.5 + \frac{1}{6}\left(L_1 + 2L_2 + 2L_3 + L_4\right)$

$= 0.2593)$

IVP: $y' = f(x,y)$ $\quad y(x_0) = y_0$
$y_n(x) = ?$

linearized equation of $y' = f(x,y)$
called the test eq^n. $\quad y(x_0) = y_0$
Derived as $\quad [x_0, b]$

$$y' = f(x,y)$$

$\bar{x} \in [x_0, b]$
$(\bar{x}, \bar{y})$

expand $f(x,y)$ around $(\bar{x}, \bar{y})$ in T-series.

$$y' = f(\bar{x},\bar{y}) + (x-\bar{x}) \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y}}$$

$$+ (y-\bar{y}) \left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y}}$$

Call $\quad \lambda = \left.\frac{\partial f}{\partial y}\right|_{(\bar{x},\bar{y})}$

& $\quad c = f(\bar{x},\bar{y}) + (x-\bar{x}) f_x \Big|_{\bar{x},\bar{y}} - \bar{y} \left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y}}$

$$\bar{y} = \lambda y + c.$$

Date

choose

$$u = y + \frac{c}{\lambda}$$

$$y' = u' - \frac{1}{\lambda} f_x \Big|_p$$

$$u' = \lambda u + \frac{1}{\lambda} f_x \Big|_{(\bar{x}, \bar{y})}$$

or $\quad u' = \lambda u + m$

take $\quad w = u + \frac{m}{\lambda}$

$$w' = \lambda w \qquad w(x_0) = A$$

$$w(x) = A e^{\lambda x}$$

$$y'' = f(x, y, y') \qquad (\bar{x}, \bar{y}, \bar{y}')$$

$$w'' + bw' + cw = 0$$

$$\boxed{b = -f_{y'}\big/_p, \quad c = -f_y\big/_p}$$

* Numerical soln that we have derived:

(i) Consistency of the method

(ii) Stability of num. method

(iii) Convergence of num. method

(i) & (ii) $\Longrightarrow$ (iii)

$\Longleftarrow$

For a Consistant numerical Method : the Truncation error $\to 0$ as $h \to 0$

→ The method should be of order 1. ←

order of Local Truncation error.

$u(t_j)$ → exact solution of D.E at $t = t_j$

L.T.E

u_j → exact solution of the diffrence eq.

Rounding errors.

$\bar{U_j}$ → numerical solution of difference eqn.

Replacement of derivative By diffrences But

↳ not exact replacement after truncation.

$$|u(t_j) - \bar{U_j}| \to 0$$

I want exact soln to be as close to numerical soln as possible.

$$|u(t_j) - U_j + U_j - \bar{U_j}|$$

$$\le \big|\underbrace{U(t_j) - U_j}_{L.T.E}\big| + \underbrace{|U_j - \bar{U_j}|}_{\text{Rounding error}}$$

L.T.E

Date __/__/__

convergent :

A numerical method is said to convergent if as more grid points are taken (step size decreased) the numerical soln moves close to the exact solution in absence of rounding errors.

i.e. $u_j \longrightarrow u(t_j)$ as $h \to 0$

stability :

A numerical method is said to be stable if the effect of any single fixed round off error is bounded, independent of no. of mesh points

Stability of Single step Method

$u' = \lambda u$ $\quad u(t_0) = u_0$

$\lambda \to$ real / complex.

exact soln : $u(t_{j+1}) = e^{\lambda h} \cdot u(t_j)$

$$u(t) = U_0 \cdot e^{\lambda(t - t_0)} \Bigg|_{\text{at } t = t_{j+1},\ t = t_j} \quad u' = \lambda u,\ u(t_0) = U_0.$$

$u(\ )\ e^{\lambda h}$

- $e^{\lambda h} = \infty$ series
If we approximate

$e^{\lambda h} = 1 + \lambda h$ — (1st Order Approx)

$u(t_{j+1}) = (1+\lambda h)u(t_j)$

$u(t_{j+1}) = u(t_j) + \lambda h u(t_j)$

$u(t_{j+1}) = u(t_j) + h f(t_j, u_j)$.

$u(t_{j+1}) = u(t_j)$

$e^{\lambda h} = 1 + \lambda h + \frac{(\lambda h)^2}{2}$ — (2nd Order.)

$E(\lambda h) =$ as an Approximation to $e^{\lambda h}$.

$\boxed{u_{j+1} = E(\lambda h)\, u_j}$ Numerical Method.

Define Error $\epsilon_j = u(t_j) - u_j$. difference b/w exact & numerical solution

$\epsilon_{j+1} = u(t_{j+1}) - u_{j+1}$
$= e^{\lambda h} u(t_j) - E(\lambda h) u_j$

or

$\epsilon_{j+1} = e^{\lambda h} u(t_j) - E(\lambda h)[u(t_j) - \epsilon_j]$

$\epsilon_{j+1} = \underbrace{[e^{\lambda h} - E(\lambda h)]}_{L.T.E} u(t_j) + \underbrace{E(\lambda h)\epsilon_j}_{\text{propagation error.}}$

∴ propagation error must be controlled. else, unstable solⁿ, since ϵ keep on increasing away from pt.

$$\epsilon_{j+2} = \left(E^2(\lambda h) - e^{2\lambda h}\right) u(t_j) + E^2(\lambda h)\epsilon_j$$

$t_j \rightarrow$ point where error started to come

∴ now how the error propagates is what we are trying to understand.

So, the Single Step Method, when applied to test eqⁿ $u' = \lambda u$, $u(t_0) = u_0$

is (1) Absolutely stable if $|E(\lambda h)| \leq 1$ $\lambda < 0$

(2) Relatively stable if.

$|E(\lambda h)| < e^{\lambda h}$ $\lambda > 0$

(3) Periodically stable if

$|E(\lambda h)| = 1$ $\lambda \rightarrow$ pure imaginary

(4) Assymptotically stable if $u_j \rightarrow 0$ as $j \rightarrow \infty$.

31/01/18

Date ___ / ___ / ___

FEM.

$$u_{j+1} = u_j + h f_j$$

$$u' = f$$

Test Eqn

$$u' = \lambda u = f$$

$$u_{j+1} = u_j \underbrace{(1+\lambda h)}_{E(\lambda h)}$$

for stability - $|E(\lambda h)| < 1$

$$|1+\lambda h| < 1$$

$$-1 < 1+\lambda h < 1$$

$$-2 < \lambda h < 0 \qquad \lambda h = \bar{h}$$

$$\boxed{-2 < \bar{h} < 0}$$

$$0 < h < \frac{2}{|\lambda|}$$

Explicit Method
- Conditionally stable.

Saathi

Scanned by CamScanner

Date ___ / ___ / ___

Saathi

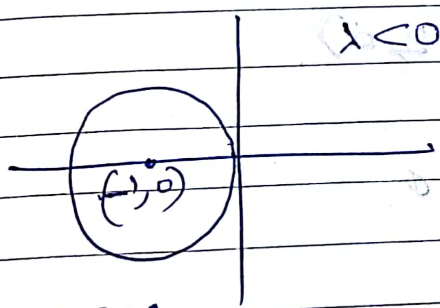
$y_{j+1} = (1-\lambda h)^{-1} y_j$

$\left|\frac{1}{(1-\lambda h)}\right| < 1$

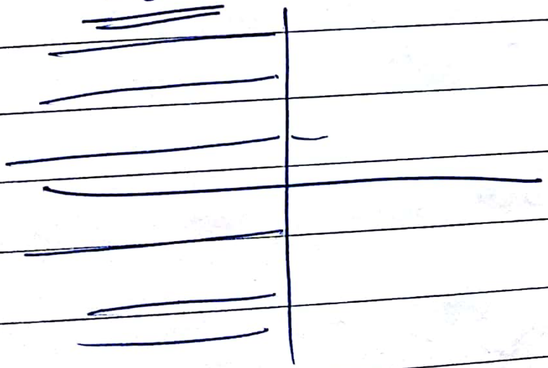
λ is Complex :

$(x+1)^2 + h^2 \leq 1$

$\lambda < 0$



BEM



Page No.

Usually explicit methods be conditionally stable where as implicit methods are unconditionally stable.

Stability of 2nd Order R-K Method

$\lambda < 0$

$$E(\lambda h) = \left| 1 + \lambda h + \frac{(\lambda h)^2}{2!} \right| < 1$$

$$-2 < (1+\lambda h)^2 + 1 < 2.$$

$$\lambda h \in (-2, 0)$$

modified euler method.

y_1

$$\lambda = \lambda_R + i\lambda_I$$

$$\nabla = \frac{A}{B} e^{t(\beta - \alpha)}$$

Date ___/___/___

$$A = \sqrt{\left(1+\frac{\lambda_R h}{2}\right)^2 + \left(\frac{\lambda_I^2 h^2}{4}\right)}$$

$$B = \sqrt{\left(1-\frac{\lambda_R h}{2}\right)^2 + \frac{\lambda_I^2 h^2}{4}}$$

$$\frac{A}{B} < 1$$

$\lim_{h \to \infty} \nabla$: y_0 $t - y_0$

4th Order RK.

Nt. Imp

Page No.

31/1/18

Multi step Method.

R- step Method.

$$u_{j+1} = a_1 u_j + a_2 u_{j+1} \dots \dots + a_R U_{j-R+1}$$
$$+ h(b_0 y'_{j+1} + b_1 u'_j + b_2 U'_{j-1} + b_R U'_{j-R+1}) - ①$$

$y' = f(t, y)$
$y(t_0) = y_0$

②
$$U_{j+1} = \sum_{i=1}^{R} a_i u_{j-i+1} + h \sum_{i=0}^{R} b_i U'_{j-i+1} \quad - ①$$

If $b_0 = 0$ — then explicit multi step Method

linear MSM. | predictor

If $b_0 \neq 0$, — implicit linear MSM
 — collector

$$\rho(E)\, u_{j-R+1} - h\sigma(E)\, U'_{j-K+1} = 0.$$

$$P(\xi) = \xi^R - a_1 \xi^{R-1} - a_2 \xi^{R-2} \dots - a_R.$$

$E f_j = f_{j+1}$

$\sigma(z) = b_0 z^{n} + b_1 z^{n-1} \qquad + b_n$

b_n

Date

$y' = x^2(1+y)$

$y(1) = 1$ $y(1.1) = (1.233)$

$y(1.2) = 1.548$ $y(1.3) = 1.979$.

find (1.4) using AB & AM

PC set. (4th order).

$h = 0.1$

$f = x^2(1+y)$ $f_i = f(x_i, y_i)$

$x_0 = 1$	1.1	1.2	1.3	1.4
$j-3$	$j-2$	$j-1$	j	$j+1$

x		y		f
x_{-3}	1	y_{-3}	1	2
x_{-2}	1.1	y_{-2}	1.233	2.702
x_{-1}	1.2	y_{-1}	1.548	3.669
x_0	1.3	y_0	1.979	5.035
x_1	1.4	y_1	?	5.

$y_1^p = 2.57245$ Predictor formula

$f(x_1, y_1^p) = f_1 = 7.002$.

Scanned by CamScanner

Date

$y_1^{c} = 2.5742$ $\quad E = 7.005432$

$y_1^{c_2} = 2.5751$ $\quad E =$

$\boxed{|y_1^{c_{n+1}} - y_1^{c_n}|} < \varepsilon$ (tollerance)

Used only once.

P — C — C — C.

f: Keepup

• Milne & Milne Simpson P–C Methods

Solve $\frac{dy}{dx} = f(x, y)$ $\quad y(x_0) = y_0.$

find $y(x_0 + 4h) = y_4$?

Consider newton forward interpol^n for $f(x,y)$

P: $y_4 = y_0 + \frac{4}{3}h[2f_1 - f_2 + 2f_3]$

$$\boxed{y_{j+1} = y_{j-3} + \frac{4}{3}h[2f_{j-2} - f_{j-1} + 2f_j]}$$

C: derived using simpson's $\frac{1}{3}$ rule

$$y_4 = y_2 + \int_{x_2}^{x_4} f(x, y).$$

$$y_4 = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$\boxed{y_{j+1} = y_{j-1} + \frac{h}{3}[f_{j-1} + 4f_j + f_{j+1}]}$$

ex: solve $y' = x - y^2$ $y(0) = 0$
$h = 0.2$

Date __/__/__

Q. given

$y'' + xy' + y = 0 \quad | \; y(0) = 1 \quad y'(0) = 0$

find $y(0.1) \,|\, y(0.2) \,|\, y(0.3)$ using taylor series method. Cal. $y(0.4)$ using milne PC.

sol^n

$y' = z = f(x, y, z)$

$z' = -(xz + y) = g(x, y, z)$

$y'' + xy' + y = 0$ — ①

diff ① w r t x n times.

$y_{n+2} + xy_{n+1} + ny_n + y_n = 0$

$y_{n+2} = -xy_{n+1} - (n+1)y_n$

$y_{(n+2)}(x=0) = -(n+1)y_n$

$y(2) = -(y_0) = -1$

$y_3 = -2y_1(0) = 0.$

$y_4 = -(3)(-1) = 3.$

⋮

Date ___

$y(x) = y(x_0) + x\,y'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{\lfloor 3} y'''(0)$ ...... (3)

$y(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{-15x^6}{6!}$ ...... (4)

$y' = -x + \frac{x^3}{2} - \frac{1}{8}x^5 + \dots$

	$z(0) = 0$
$y(0.1) = 0.995$	$z(0.1) = -0.0995$
$y(0.2) = 0.9802$	$z(0.2) = -0.196$
$y(0.3) = 0.956$	$z(0.3) = -0.2868$

milne Predictor Corrector :

$z(0.4) = z(0) + \frac{4}{3} h [2g_1 - g_2 + 2g_3]$

$= -0.3692$

$y(0.4) = y(0) + \frac{4}{3} h [2f_1 - f_2 + 2f_3]$

$= 0.9231$

$g = -(xz + y) \quad f = z.$

$E: \quad g(0.4) = -0.7754$

$f(0.4) = -0.3692$

$z(0.4) = y(0.2) + \frac{h}{3} [f_2 + 4f_3 + f_4]$

$= 0.9232$

Scanned by CamScanner

$T_{j+1} = \rho(E) - h\sigma(E) = 0.$

Problem

③ Solve the IVP :

$$y' = \frac{1}{4} y\left(1 - \frac{y}{20}\right) \qquad y(0) = 1$$

Using Adams-Bashforth method of order 2. Approximate $y(1)$

$h = 0.2$

Soln:

$$y_{i+1} = y_i + \frac{h}{2}\left[3(f(i)) - f_{i-1}\right)$$

$$f(t_i, y_i) = \frac{y}{4}\left(1 - \frac{y}{20}\right)$$

starting value can be computed using any single step method

$$y_1 = y_0 + hf(t_0, y_0) \qquad \text{(euler)}$$

$$= 1.048$$

$$y_2 = y_1 + \frac{h}{2}\left[3f_1 - f_0\right]$$

$$= 1.048 + \frac{0.2}{2}\left[3(0.248) - 0.238\right]$$

$$= 1.099$$

$$y_3 = y_2 + \frac{h}{2}\left[3f_2 - f_1\right)$$

$$= 1.099 + \frac{0.2}{2}\left[3 \times 0.26 - 0.248\right]$$

$$= 1.152.$$

②

⇒

fro

$y_4 = y_3 + \frac{h}{2}\left[3f_3 - f_2\right]$

$= 1.207$ $y_5 = 1.264$

(2) Use the Adams - Bashforth method of order 3 to obtain an approximate ______ to soln of the IVP

$$y' = 2t - y \qquad y(0) = 1$$

with $N=5$, at $t=1$ compare with exact value $y(t) = e^{-t} + 2t - 2$.

⇒ Adams - Bashforth of order 3 given by

$$y_{i+1} = y_i + \frac{h}{12}\left[23 f(t_i, x_i) - 16 f(t_{i-1}, x_{i-1}) + 5 f(t_{i-2}, x_{i-2})\right]$$

for $i = 2, 3, 4$.

here $f(t,y) = 2t - y$ $y_0 = -1$ $t_0 = 0$.

$h = \frac{1-0}{N} = \frac{1}{5} = 0.2$

from exact soln:

$y_1 = -0.7813$ $y_2 = -0.5297$

$y_3 = y_2 + \frac{0.2}{12}\left(23 f_2 - 16 f_1 + 5 f_0\right).$

$y_3 = -0.5297 + \frac{0.2}{12}\left[23(1.33) - 16 \times 1.181 + 5 \times 1\right].$

$y_5 = \underline{0.367}$

exact		
	0.6	−0.2512
	0.8	0.0493
	1.0	0.3679

(3) Use the Adams bash forth of order 4 to approximate soln of IIVP.

$$y' = 2 + \sqrt{y - 2t + 3} \qquad y(0) = 1$$

On the interval $[0, 1.5]$ with $\underline{h = 0.25}$

$$y = 1 + 4t + \frac{t^2}{4}.$$

soln: Adam's bashforth of Order 4 method is given by

From exact solution:

$$y_1 = 2.016$$
$$y_2 = 3.062$$
$$y_3 = 4.14$$

$$y_4 = 4.14 + \frac{0.25}{24}\left[55\times(4.375) - 59\,(4.245) + 37\,(4.125) - 9\times 4\right]$$

$= 5.252$	1.25	1.5
exact y_4 : 5.250	6.3906	7.5625.

(4). Solve following IVP using Adams-Bashforth-Moulton predictor-corrector method of Order 4.

$$y' = e^{-t} - y \qquad y(0) = 1$$

In $[0, 1.2]$ $h = 0.2$ Compare with exact

$$y(t) = e^{-t}(t+1)$$

soln

(explicit) P : $$y_{i+1}^{p} = y_i + \frac{h}{24}\left[55 f_i - 59 f_{i-1} + 37_{i-2} - 9 f_{i-3}\right]$$

implicit C : $$y_{i+1}^{c} = y_i + \frac{h}{24}\left[9 f(t_{i+1}, y_{i+1}^{p}) + 19 f_i - 5 f_{i-1} + f_{i-2}\right]$$

$i = 3, 4, 5.$

Date

starting values from exac[t]

given
$y_1 = 0.9825$
$y_2 = 0.9384$
$y_3 = 0.8781.$

$y^P = y_3 + \frac{0.2}{24}\left[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right]$

$y_4^P = 0.8085.$

$y_4^c = y_3 + \frac{0.2}{24}\left[9 f(x_4, y_4^P) + 19 f \ldots \ldots\right]$

$= 0.8088.$

$y_5^P = 0.7356$

$y_5^c = y \ldots$

$= 0.7358.$

(5) Use milne - Simpson predictor - corrector method

$x = 2$

$\frac{dy}{dx} = y \sin x$

$y(0) = 1$
$h = 0.5$

$f(x,y) = y \sin x$

$x_0 = 0 \mid y_0 = 1 \mid h = 0.5$

$y^P_{i+1} = y_{i-3} + \frac{4h}{3}\left[2f_{i-2} - f_{i-1} + 2f_i\right]$

$y^c_{i+1} = y_{i-1} + \frac{h}{3}\left[f_{i-1} + 4f_i + f_{i+1}\right]$

Scanned by CamScanner

Saathi

Euler — starting

$y_1 = 1$ $\quad y_2 = 1.24$ $\quad y_3 = 1.762$

$y_4^p = y_0 + \frac{4(0.5)}{3} \cdot [2 \times 0.479 - 1.042 + 2.4 \sin(1.5) \times \ldots]$

$y_4^c = y_2 + \frac{0.5}{3} [1.043 + 4 \times 1.7576 + 3 [3.287 \times \sin(2)]]$

$= 3.084$

Saathi

Date

Q3. $y' = 2t + y$ $y(0) = 1$ $h = 0.2$ $y(0.2)$

4th order RK Method.

$y_{j+1} = y_j + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$K_1 = h f(x_0, y_0)$

$0.2(1) = 0.2$

$K_2 = h f(0.1, 1 + 0.1)$

$0.2(2(0.1) + 1.1) = 0.2(1.3) = \underline{0.26}$

$K_3 = 0.2 f(0.1, 1 + 0.13)$

$= 0.2 (2(0.1) + 1.13) = 0.2(1.33) = \underline{0.266}$

$K_4 = 0.2 f(0.2, 1 + 0.266)$

$0.2(2(0.2) + 1.266) = 0.2(1.666)$

$= \boxed{0.3332}$

$y_{0.2} = 1 + \frac{1}{6}(0.2 + 2(0.26) + 2(0.266) + 0.332)$

$(0.2 + 0.52 \quad 0.536 + 0.332)$

$y(0.2) = 1.588$

Date

Q1.

soln:

Q2.

Date ___/___/___

Q1. $\frac{dy}{dx} = 2x+y$ | $y(0)=1$ | $h=0.2$

forward euler.

$\frac{dy}{dx} = f = 2x+y$

soln: $y_1 = 1 + (0.2)(1) = 1.2$

$y_2 = 1.2 + 0.2(0.4+1.2) = 1.2 + 0.2(1.6)$
$= 1.2 + 0.32$
$= 1.52$.

$y_3 =$ :: soon.

Q2. $y' = 2x+y$ $y(0)=1$ $h=0.2$.

explicit 2 slope method:

soln: $y_{j+1} = y_j + \frac{1}{2}(K_1+K_2)$ | $K_1 = hf(t_j, y_j)$
$K_2 = hf(t_{j+1}, y_j+K_1)$.

$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$

$K_1 = hf(0, 1) = 0.2(1) = 0.2$
$K_2 = 0.2f(0.2, 0.2+1) = 0.2f(0.2, 1.2)$
$= 0.2(2(0.2)+1.2) = 0.2(1.6)$
$K_2 = 0.32$

$y_1 = 1 + \frac{1}{2}(0.2+0.32) = 1.26$.

taking $e^h = \xi \Rightarrow h = \log \xi$

as $h \to 0 \qquad \xi \to 1$

then we write (2) in power of $(\xi - 1)$

$$\log(\xi) = \log(\xi - 1 + 1)$$
$$= \left[\xi - 1 - \frac{1}{2}(\xi-1)^2 \dots \right]$$

and $$h^{p+1} = [\log(\xi)]^{p+1}$$
$$= \left(\{\xi-1\}^{p+1} + O\{\xi-1\}^{p+2} \right)$$

in (2) we set

$$P(\xi) - \log(\xi)\sigma(\xi) = C_{p+1}\{\xi-1\}^{p+1} + O(\xi-1)^{p+2} \quad (3)$$

$\Rightarrow$

$\therefore \quad P(\xi) - \log(\xi)\sigma(\xi) = 0$

Scanned by CamScanner

Date

1. Adams - Bashforth methods:

$\rho(\xi) = \xi^{k-1}(\xi - 1)$

k^{th} degree

Find the explicit

$k=2$ $\quad \rho(\xi) = \xi(\xi-1) = \xi^2 - \xi$

$$\sigma(\xi) = \frac{\rho(\xi)}{\log(\xi)} = \frac{(\xi-1)^2 + \xi - 1}{(\xi-1) - \frac{1}{2}(\xi-1)^2 + \cdots}$$

$$= \left[1 - \frac{1}{2}(\xi-1) \cdots \right]^{-1}$$

$$= 1 + \frac{1}{2}(\xi-1) + (\xi-1) + \frac{1}{2}(\xi-1)^2 \cdots$$

$$= 1 + \frac{3}{2}(\xi-1) + O(\xi-1)^2$$

$\rho(\xi) = \xi^2 - \xi$

$\sigma(\xi) = \frac{3}{2}\xi - \frac{1}{2}$

The AB explicit Method:

$$\rho(E)\, u_{j-k+1} - h\,\sigma(E)\left(u'_{j-k+1}\right) = 0$$

$$(E^2 - E)u_{j-1} - h\left(\frac{3}{2}E - \frac{1}{2}I\right)u'_{j-1} = 0$$

$$(u_{j+1} - u_j) - \frac{h}{2}\left(3u_j' - u'_{j-1}\right) = 0$$

or $\boxed{u_{j+1} = u_j + \frac{h}{2}[3f_j - f_{j-1}]}$ show that $p=2$.

Adams - Morton method :

$\rho(\xi) = \xi^{k-1}(\xi-1)$

$\sigma(\xi) = k^{th}$ dep.

$k=2$;

$\sigma(\xi) = 1 + \frac{3}{2}(\xi-1) + \frac{5}{12}(\xi-1)^2$.

$u_{j+1} = u_j + \frac{h}{12}[5f_{j+1} + 8f_j - f_{j-1}]$. $\boxed{p=3}$

- Order calcn in multistep diff from single step method.

Nystrom method (Milne's).

$\rho(\xi) = \xi^{k-2}(\xi^2-1)$.

explicit method.

$\sigma(\xi) = (k-1)^{th}$

imp.

$\sigma(\xi) - k^{th}$.

└ Milne's Simpson.

$k=3$;

$u_{j+1} = u_{j-1} + \frac{h}{3}[7u_j' - 2u_{j-1}' + u_{j-2}']$

Date

Milne simpson method:

K=2

$$U_{j+1} = U_{j-1} + \frac{h}{3}\left[U'_{j+1} + 4U'_j + U'_{j-1}\right]$$

K=3

$[p=4]$

<u>Numerical diffrentiation formulae</u>

given

$$\sigma(\xi) = \xi^K$$

Fixed $\rho(\xi)$ of ~~deg~~ deg. K

K=2

$$\sigma(\xi) = \xi^2 = 1 + 2(\xi-1) + (\xi-1)^2$$

$$\sigma(\xi).\log(\xi) = \rho(\xi)$$

$$= (\xi-1) + \frac{3}{2}(\xi-1)^2 + O(\xi-1)^3$$

$$\rho(\xi) = \frac{3}{2}\xi^2 - 2\xi + \frac{1}{2}$$

$$\rho(E)\,U_{j-K+1} - h\,\sigma(E)\,U'_{j-K+1} = 0$$

$$\left[\frac{3}{2}E^2 - 2E + \frac{1}{2}\right]U_{j-1} = h\,E^2\,U'_{j-1}$$

$$hU'_{j+1} = \frac{3}{2}U_{j+1} - 2U_j + \frac{1}{2}U_{j-1}$$

K

1

2

3

4

Linear multi step method

LMSM

K	a_0	a_1	a_2	a_3	a_4
1	1	-1			
2	$3/2$	-2	$1/2$		
3	$11/6$	-3	$5/2$	$-1/3$	
4	$25/12$	-4	3	$-4/3$	$1/4$

<u>Excercise</u>

given $\sigma(\xi) = \dfrac{(25\xi^2 - 16\xi + 5)}{12}$.

find $\rho(\xi)$ — explicit

$$\rho(\xi) = \log \xi \cdot \sigma(\xi)$$

$$u_{j+} = u_j + \frac{h}{12}\left(23u_j' - 16u'_{j-1} + 5u'_{j-2}\right) \quad p = 3.$$

$\rho(\xi) = (\xi - 1)(\xi - \lambda)$ $\quad -1 \le \lambda < 1$

$\sigma(\xi)$ – implicit

when $\lambda = 0$ | $\lambda = -1$