Date
02/09/2019

Lecture 1

Subject Name :- Mathematical Methods

Subject code :- MA30107
(3-0-0·3)

Syllabus :- Hypergeometric equation and functions, properties of hyper-geometric functions, Bessel equation & its solution, Bessel function, Modified Bessel function, Generating function for Bessel function,

Recurrence relations betn Bessel functions,

orthogonality of Bessel functions.

Definition of Tensors,

summation convention,

Kronecker Delta, covariant & contravariant & mixed tensors.

Fundamental operations with tensors,

the line element & metric tensor, length of a vector, Christoffel's symbols, the co-variant derivative, tensor form of gradient, divergence & curl.

Examples from continuum mechanics, elasticity, plasticity, fluids.

Books :-

1. Advanced Engineering Mathematics (Wiley) (10th edition) by Erwin Kreyszig.

2. Special functions for scientists & Engineers by W. W. Bell (Dover publication)

—

Bessel's equation & its solution

The d.eqn of the form

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \longrightarrow (1)$$

is called Bessel's equation of order n, n being a non-negative constant.

We now solve (1) in series by using the well-known method of Frobenius.

[Note:- Here, $x=0$ regular singular point of the eqn & Frobenius series soln exists for the eqn.]

Let the series solution

(1) be

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0 \longrightarrow (2)$$

$$y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$$

$$\therefore y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2}$$

Substituting for y, y', y'' in eqn (1), we get

$$x^2\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m-2}$$

$$+\, x\sum_{m=0}^{\infty} c_m (k+m)\, x^{k+m-1}$$

$$+\,(x^2-n^2)\sum_{m=0}^{\infty} c_m\, x^{k+m} = 0$$

$$\text{or, } \sum_{m=0}^{\infty} c_m \left[(k+m)(k+m-1)+(k+m)-n^2\right] x^{k+m}$$

$$= (k+m+n)(k+m-n)$$

$$+\sum_{m=0}^{\infty} c_m\, x^{k+m+2} = 0$$

or, ~~$\sum c_m (k+m+n)(k+$~~

$$\begin{aligned} &(k+m)(k+m-1)+(k+m)-n^2 \\ &= (k+m)^2 - \cancel{(k+m)} + \cancel{(k+m)} - n^2 \end{aligned}$$

$$\text{or, } \sum_{m=0}^{\infty} c_m (k+m+n)(k+m-n)\, x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0 \longrightarrow (3)$$

Equating to zero the smallest power of x, namely x^k eq^n (3) gives the <u>indicial equation</u>

$$c_0 (k+n)(k-n) = 0$$

$$\Rightarrow (k+n)(k-n) = 0, \text{ as } c_0 \neq 0.$$

Its roots are $k = n, -n$.

[$n-(-n) = 2n$; n is not an integer

Next equating to zero the co-efficient of x^{k+1} in (3) gives

$$c_1 (k+1+n)(k+1-n) = 0$$

$\Rightarrow$ $c_1 = 0$, for $k = n$ & $k = -n$.

Finally, equating to zero the coefficient of x^{k+m}, in (3), gives

$$c_m (k+m+n)(k+m-n) + c_{m-2} = 0$$

$$\Rightarrow c_m = \frac{1}{(k+m+n)(n-k-m)} c_{m-2} \rightarrow (4)$$

Putting $n = 3, 5, 7, \dots$ in (4)

& using $c_1 = 0$, we get

$$c_1 = c_3 = c_5 = c_7 = \dots = 0 \longrightarrow (5)$$

Putting $n = 2, 4, 6, \dots$ in (4), we get

$$c_2 = \frac{1}{(k+2+n)(n-k-2)} c_0,$$

$$c_4 = \frac{1}{(k+4+n)(n-k-4)} c_2$$

$$c_4 = \frac{1}{(k+4+n)(n-k-4)(k+2+n)(n-k-2)} c_0$$

& so on.

Putting these values in 2

$$Y = c_0 x^k \times \left[1 + \frac{x^2}{(n+k+2)(n-k-2)} + \frac{x^4}{(n+k+2)(n-k-2)(n+k+4)(n-k-4)} + \cdots\right]$$

Replacing k by n & $(-n)$ & also replacing c_0 by a & b in the above eqn

$$Y = ax^n\left[1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4\cdot 8\,(1+n)(2+n)} - \cdots\right] \rightarrow 6$$

12

$$\therefore y = b x^{-n}\left[1 - \frac{x^2}{4(1-n)} + \frac{x^4}{4.8(1-n)(2-n)} - \cdots\right] \rightarrow 7$$

The particular soln of 1 obtained from 6 above by taking the arbitrary constant $a = \frac{1}{\left\{2^n \Gamma(n+1)\right\}}$ is called the Bessel fn of the first kind of order n.

It will be denoted by

$J_n(x)$.

Thus, we have

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4 \cdot 8 (n+1)(n+2)} - \cdots \right] \to (8)$$

$$\text{or, } J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r! \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n} \to (9)$$

Replacing 'b' by $\left\{ \frac{1}{2^{-n} \Gamma(-n+1)} \right\}$ in (7)

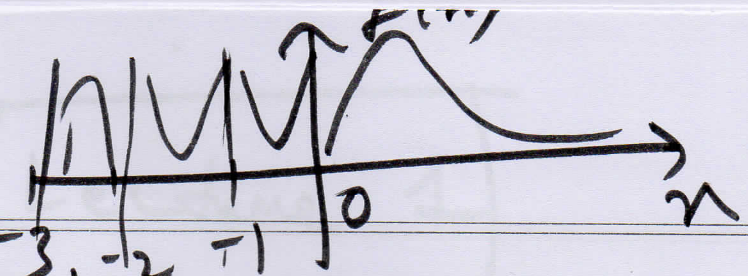
& proceeding as above gives

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \rightarrow (10)$$

Thus, the general solution of Bessel eq^n (1) when n is not an integer is

$$\boxed{y = A J_n(x) + B J_{-n}(x)}$$

where A & B are arbitrary constants.



Note :- Let n be non-integral

$\Gamma(n) = \infty$, if n is zero or, a negative integer

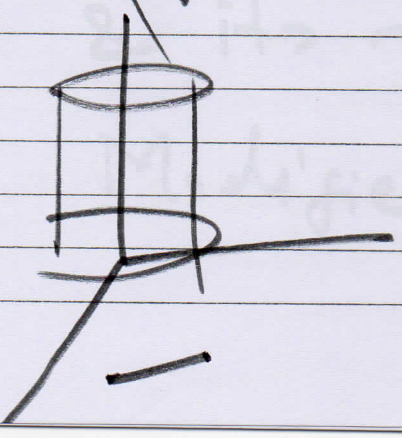
& $\Gamma(n)$ is finite otherwise.

Since n is not an integer

.

Note :- p.d.eqn

(r, θ, z) — cylindrical symmetry



eg, rectangular membrane in pd.eqn.