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Higher Order BVP

$$y''' + A(x)y'' + B(x)y' + C(x)y = D(x)$$

$$y(0) = y_0, \quad y'(0) = y'_0, \quad y'(a) = y'_a \qquad x_0 = 0$$

$$x_i \qquad x_n = a$$

Put $p = \frac{dy}{dx}$ (1) $\Rightarrow \frac{d^2p}{dx^2} + A(x)\frac{dp}{dx} + B(x)\,p + C(x)\,y = D(x)$ (ii)

$$y(0) = y_0, \quad p(0) = y'_0, \quad p(a) = y'_a$$

(1) & (ii) can be discretized ~~such~~ they y_n doesn't app

$$y_i - y_{i-1} - \frac{h}{2}(P_i + P_{i-1}) = 0 \quad \text{—(i)}$$

$$\left(\frac{P_{i+1} - 2P_i + P_{i-1}}{h^2}\right) + A_i\left(\frac{P_{i+1} - P_{i-1}}{2h}\right) + B_i P_i + C_i y_i = D_i \rightarrow \text{(ii)}$$

$$i = 1, 2, \dots n-1.$$

y_0, p_0, p_n are prescribed.

let $X_i = \begin{bmatrix} y_i \\ P_i \end{bmatrix}$

$$y_i - y_{i-1} - \frac{h}{2}P_i - \frac{h}{2}P_{i-1} = 0.$$

$$\left(\frac{1}{h^2} - \frac{A_i}{2h}\right)P_{i-1} + \left(\frac{-2}{h^2} + B_i\right)P_i + \left(\frac{1}{h^2} + \frac{A_i}{2h}\right)P_{i+1}$$

$$+ C_i y_i = D_i$$

let $X_i = \begin{bmatrix} y_i \\ P_i \end{bmatrix}$

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$$\underbrace{\begin{bmatrix} -1 & -\frac{h}{2} \\ 0 & \left(\frac{2 - A_i}{2h^2}\right) \end{bmatrix}}_{A_i^*}\begin{bmatrix} y_{i-1} \\ P_{i-1} \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & -\frac{h}{2} \\ C_i & \left(\frac{-2}{h^2} + B_i\right) \end{bmatrix}}_{B_i^*}\begin{bmatrix} y_i \\ P_i \end{bmatrix}$$

Base form

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{1}{h^2} + \frac{A_i}{2h}\right) \end{bmatrix}}_{C_i^*}\begin{bmatrix} y_{i+1} \\ P_{i+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ D_i \end{bmatrix}}_{D_i^*}$$

$$A_i^* X_{i-1} + B_i^* X_i + C_i^* X_{i+1} = D_i^*$$

Q $y''' + 4y'' + y' - 6y = 1, \qquad 0 < x < 1.$

$$y(0) = 0, \quad y'(0) = 0, \quad y'(1) = 1.$$

let $y' = p$

$$p'' + 4p' + p - 6y = 1$$

discretising the eq^n we get,

$$y_i - y_{i-1} + \frac{h}{2}(P_i + P_{i-1}) = 0$$

$y(0) = 0 \quad p(0) = 0 \quad p(1) = 1.$

$$\left(\frac{P_{i+1} - 2P_i + P_{i-1}}{h^2}\right) + 4\left(\frac{P_{i+1} - P_{i-1}}{2h}\right) + P_i - 6y_i = 1.$$

$$\Rightarrow \left(\frac{1}{h^2} - \frac{2}{h}\right)P_{i-1} + \left(\frac{-2}{h^2} + 1\right)P_i + \left(\frac{1}{h^2} + \frac{2}{h}\right)P_{i+1} - 6y_i = 1.$$

let $X_i = \begin{bmatrix} y_i \\ P_i \end{bmatrix}$.

$\frac{-0.2}{2}$

$X_0 \checkmark$ $\quad X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad X_n = \begin{bmatrix} y_n \\ \\ \end{bmatrix}$

$$\underbrace{\begin{bmatrix} -1 & -\frac{h}{2} \\ 0 & \left(\frac{1}{h^2} - \frac{2}{h}\right) \end{bmatrix}}_{A_i} \begin{bmatrix} y_{i-1} \\ p_{i-1} \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & -\frac{h}{2} \\ -6 & 1 - \frac{2}{h^2} \end{bmatrix}}_{B_i} \begin{bmatrix} y_i \\ p_i \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{2}{h} \end{bmatrix}}_{C_i} \begin{bmatrix} y_{i+1} \\ p_{i+1} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{D_i}$$

$$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i$$

$\checkmark$

$A_1 X_0 + B_1 X_1 + C_1 X_2 = D_1$

$\Rightarrow \quad B_1 X_1 + C_1 X_2 = \underbrace{D_1 - A_1 X_0}_{D_1'}$

$D_1' = D_1 - A_1 X_0$

$A_i = \begin{bmatrix} -1 & -0.1 \\ 0 & 15 \end{bmatrix}$ $\quad$ & $\quad X_1 + C_1' X_2 = D_1''$, $\quad D_1'' = B_1^{-1} D_1'$

$C_1' = B_1^{-1} C_1$

$C_i = \begin{bmatrix} 0 & 0 \\ 0 & 35 \end{bmatrix}$, $\quad B_i = \begin{bmatrix} 1 & -0.1 \\ -6 & -49 \end{bmatrix}$ $\quad D_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$D_1' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 & -0.1 \\ 0 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$B_1^{-1} = \begin{bmatrix} \frac{245}{248} & \frac{-1}{496} \\ \frac{-15}{124} & \frac{-5}{248} \end{bmatrix} \qquad C_1' = B_1^{-1} C_1 = \begin{bmatrix} 0 & -35/496 \\ 0 & -175/248 \end{bmatrix}$$

$$D_1'' = \begin{bmatrix} -1/496 \\ -5/248 \end{bmatrix}$$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

$0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1$

$B_2' = A_2^{-1} B_2$

$C_2' = A_2^{-1} C_2$

$D_2' = A_2^{-1} D_2$

$X_1 + C_1' X_2 = D_1''$

$\Rightarrow \cancel{A_2 X_1} + A_2 C_1' X_2 = A_2 D_1''$

$\cancel{A_2} X_1 + B_2 X_2 + C_2 X_3 = D_2$

$A_2 X_1 + B_2 X_2 + C_2 X_3 = D_2$

$\Rightarrow \cancel{X_1 + B_2' X_2 + C_2' X_3 = D_2'}$

$(B_2 - A_2 C_1') X_2 + C_2 X_3 = D_2 - A_2 D_1''$

$X_2 + C_2' X_3 = D_2'$

$C_2' = (B_2 - A_2 C_1')^{-1} C_2$

$D_2' = (B_2 - A_2 C_1')^{-1} (D_2 - A_2 D_1'')$.

$\Rightarrow \quad X_1 + C_1' X_2 = D_1''$ — (1)

Now,

we have $C_2' = (B_2 - A_2 C_1')^{-1} C_2$

$$= \begin{bmatrix} 0.9637 & -6.049 \times 10^{-3} \\ -0.150519 & -0.025086 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.2117178 \\ 0 & -0.878027 \end{bmatrix}$$

$$D_2' = \begin{bmatrix} & \\ & \end{bmatrix} \times \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 & -0.1 \\ 0 & 15 \end{bmatrix} \begin{bmatrix} -1/496 \\ -5/248 \end{bmatrix} \right]$$

$$= \begin{bmatrix} & \\ & \end{bmatrix} \times \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4.032 \times 10^{-3} \\ -75/248 \end{bmatrix} \right]$$

$$= \begin{bmatrix} -0.01176 \\ -0.08206 \end{bmatrix}$$

$$B_i' = B_i - A_i C_{i-1}'$$
$$C_i' = (B_i')^{-1} C_i \,,\quad D_i' = (B_i')^{-1}(D_i - A_i D_{i-1}')$$

at $i = n-2$,
$$X_{n-2} + C_{n-2}' X_{n-1} = D_{n-2}'$$

Now, $\quad A_{n-1} X_{n-2} + B_{n-1} X_{n-1} + C_{n-1} X_n = D_{n-1}$

we don't know y_n so we can't directly do backsub
but;
$$y_n - y_{n-1} - \frac{h}{2}(P_n + P_{n-1}) = 0$$
$$\hookrightarrow y_n = y_{n-1} + \frac{h}{2}(P_n + P_{n-1})$$

$$\therefore X_n = \begin{bmatrix} y_{n-1} + \frac{h}{2}(P_n + P_{n-1}) \\ P_n \end{bmatrix}$$

So, we can change the final eqn for $i = n-1$

initially we had

$$\begin{bmatrix} -1 & -0.1 \\ 0 & 15 \end{bmatrix}\begin{bmatrix} y_{n-2} \\ P_{n-2} \end{bmatrix} + \begin{bmatrix} 1 & -0.1 \\ -6 & -49 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ P_{n-1} \end{bmatrix} + \cancel{\begin{bmatrix} 0 & 0 \\ 0 & 35 \end{bmatrix}\begin{bmatrix} \; \\ \; \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 35 \end{bmatrix}\begin{bmatrix} y_{n-1} + \frac{h}{2}(P_n + P_{n-1}) \\ P_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad -y_{n-2} - 0.1P_{n-2} + y_{n-1} - 0.1P_{n-1} + 0 + 0 = 0$$
$$0 + 15P_{n-2} - 6y_{n-1} - 49P_{n-1} + 0 + 35P_n = 0$$

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$$\therefore \text{we can take } X_n = \begin{bmatrix} 0 \\ P_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

so we will have,

$$\cancel{AX_{n-2} + AC_{n-2}' X_{n-1} = AD_{n-2}'}$$
$$-\; A_{n-1}\cancel{X_{n-2}} + A_{n-1}C' C_{n-2} X_{n-1} = -A D_{n-2}'$$
$$A_{n-1}\cancel{X_{n-2}} + B_{n-1} X_{n-1} = (D_{n-1} - C_{n-1} X_n)$$

$$(B_{n-1} - A_{n-1} C_{n-2}') X_{n-1} = (D_{n-1} - C_{n-1} X_n - A_{n-1} D_{n-2}')$$

$$\Rightarrow X_{n-1} = (B_{n-1} - A_{n-1} C_{n-2}')^{-1}(D_{n-1} - C_{n-1} X_n - A_{n-1} D_{n-2}').$$

we had done till $i = 2$, for $i = 3$ onwards
$$B_3' = B_3 - A_3 C_2'$$
$$C_3' = (B_3')^{-1} C_3 \quad \& \quad D_3' = (B_3')^{-1}(D_3 - A_3 D_2')$$

$$B_3' = \begin{bmatrix} 1 & -0.4 \\ -6 & -35.8296 \end{bmatrix} \qquad B_3^{-1} = \begin{bmatrix} 0.937292 & -0.01045 \\ -0.156958 & -0.0261597 \end{bmatrix}$$

$$C_3' = \begin{bmatrix} 0 & -0.365797 \\ 0 & -0.9155899 \end{bmatrix}, \quad D_3' = \begin{bmatrix} -0.0295049 \\ -0.0363909 \end{bmatrix}$$

$$X_4 = (B_4 - A_4 \cdot C_3')^{-1}(D_4 - C_3 X_5 - A_4 D_3').$$

$$\rightarrow \begin{bmatrix} y_4 \\ P_4 \end{bmatrix} = \begin{bmatrix} 0.913387 & -0.0144354 \\ -0.15539899 & -0.025899 \end{bmatrix}\begin{bmatrix} -0.033144 \\ -33.4541365 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} y_4 \\ P_4 \end{bmatrix} = \begin{bmatrix} 0.45265 \\ 0.87158 \end{bmatrix}$$

&

$$X_3 = D_3' - C_3' X_4$$

$$= \begin{bmatrix} 0.28931645 \\ 0.761618945 \end{bmatrix} = \begin{bmatrix} y_3 \\ P_3 \end{bmatrix}$$

$$X_2 = D_2' - C_2' X_3$$

$$= \begin{bmatrix} 0.149488 \\ 0.6366619 \end{bmatrix} = \begin{bmatrix} y_2 \\ P_2 \end{bmatrix}.$$

$$X_1 = D_1' - C_1' X_2$$

$$= \begin{bmatrix} 0.0429096 \\ 0.4290961 \end{bmatrix} = \begin{bmatrix} y_1 \\ P_1 \end{bmatrix}.$$

$$y_5 = y_4 + \frac{h}{2}(P_5 + P_4)$$

$$= 0.45265 + 0.1(1 + 0.87158)$$

$$= \underline{\underline{0.639808}}$$

Q. $y^{iv} = q/EI$. $\quad y(0) = y''(0) = y(L) = y''(L) = 0$

Let $p = y''$ — (*) $\quad p'' = \frac{q}{EI}$ — (**)

$y(0) = y(L) = 0$, $\quad p(0) = p(L) = 0$

$\Rightarrow$ Discretizing (*) & (**)

we get,

let $\frac{q}{EI} = 1$.

x_0 (0), x_1 (0.25), x_2 (0.5), x_3 (0.75), x_4 (1)

$$\left(\frac{P_{i+1} - 2P_i + P_{i-1}}{h^2}\right) = 1 \quad — (I)$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = P_i \quad — (II)$$

Let $X_i = \begin{bmatrix} y_i \\ P_i \end{bmatrix}$, So (I) & (II) can be written as

$$\underbrace{\begin{bmatrix} 0 & \frac{1}{h^2} \\ \frac{1}{h^2} & 0 \end{bmatrix}}_{A_i} \begin{bmatrix} y_{i-1} \\ P_{i-1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & \frac{-2}{h^2} \\ \frac{-2}{h^2} & -1 \end{bmatrix}}_{B_i} \begin{bmatrix} y_i \\ P_i \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & \frac{1}{h^2} \\ \frac{1}{h^2} & 0 \end{bmatrix}}_{C_i} \begin{bmatrix} y_{i+1} \\ P_{i+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{D_i}$$

we know, $X_0 = \begin{bmatrix} y_0 \\ P_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. & $X_4 = \begin{bmatrix} y_4 \\ P_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A_i = \begin{bmatrix} 0 & 16 \\ 16 & 0 \end{bmatrix} \quad B_i = \begin{bmatrix} 0 & -32 \\ -32 & -1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 0 & 16 \\ 16 & 0 \end{bmatrix}, \quad D_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A_1 X_0 + B_1 X_1 + C_1 X_2 = D_1$$

$\Rightarrow B_1 X_1 + C_1 X_2 = (D_1 - A_1 X_0)$ $\qquad \because X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow B_1 X_1 + C_1 X_2 = D_1$ $\qquad \therefore D_1 - A_1 X_0 = D_1$

$\Rightarrow X_1 + C_1' X_2 = D_1'$

$$C_1' = B_1^{-1} C_1, \quad D_1' = B_1^{-1} D_1$$

$$= \begin{bmatrix} -0.5 & 1/64 \\ 0 & -1/2 \end{bmatrix}, \quad D_1' = \begin{bmatrix} 1/1024 & -1/32 \\ -1/32 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/1024 \\ -1/32 \end{bmatrix}$$

Now, $B_i' = B_i - A_i C_{i-1}'$

$$C_2' = (B_2')^{-1} C_2 .$$

$$= \begin{bmatrix} -2/3 & 5/144 \\ 0 & -2/3 \end{bmatrix}$$

$$(B_2')^{-1} = \begin{bmatrix} \frac{5}{2304} & -1/24 \\ -1/24 & 0 \end{bmatrix}$$

$$D_2' = (B_2')^{-1}(D_2 - A_2 D_1') = \begin{bmatrix} \frac{5}{2304} & -\frac{1}{24} \\ -\frac{1}{24} & 0 \end{bmatrix} \begin{bmatrix} 1.5 \\ \frac{1}{64} \end{bmatrix} = \begin{bmatrix} 2.604166 \times 10^{-3} \\ -0.0625 \end{bmatrix}$$

$$(B_3')^{-1} = \begin{bmatrix} 0 & -64/3 \\ -64/3 & -14/9 \end{bmatrix} \qquad B_3' = B_3 - A_3 C_2'$$

$$(C_3)' = (B_3')^{-1} C_3 = \begin{bmatrix} -1024/3 & 0 \\ -224/9 & -1024/3 \end{bmatrix}$$

$$D_3' = (B_3')^{-1}(D_3 - A_2 D_2')$$

$$= \begin{bmatrix} 0 & -64/3 \\ -64/3 & -14/9 \end{bmatrix} \begin{bmatrix} 2 \\ \frac{0.375}{9} \end{bmatrix}$$

$$= \begin{bmatrix} -8/9 \\ -42.73148148 \end{bmatrix}$$

Now,

$$X_3 = (B_3')^{-1}(D_3 - A_2 D_2') \qquad \left(\because X_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right.$$

$$= \cancel{\begin{bmatrix} -8/9 \\ -42.73148 \end{bmatrix}} = \begin{bmatrix} y_3 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0.009765 \\ -0.09375 \end{bmatrix}$$

$$X_2 = D_2' - C_2' X_3$$

$$= \cancel{\begin{bmatrix} 3.995306 \\ -28.55 \end{bmatrix}} = \begin{bmatrix} y_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.01367107 \\ -0.125 \end{bmatrix}$$

$$X_1 = D_1' - C_1' X_2 = \begin{bmatrix} 0.009765627 \\ -0.09375 \end{bmatrix}$$

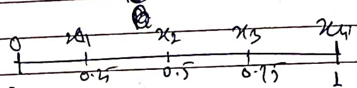
$$= \cancel{\begin{bmatrix} 2.194725 \\ -14.3 \end{bmatrix}}.$$

Q) $y^{iv} + 81y = 81x^2 \qquad y(0) = y(1) = y''(0) = y''(1) = 0$

Let $p = y''$, $\Rightarrow p'' + 81y = 81x^2$

Discretizing we get

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - p_i = 0$$



$$\frac{p_{i+1} - 2p_i + p_{i-1}}{h^2} + 81y_i = 81x_i^2$$

$$\Rightarrow \underbrace{\begin{bmatrix} \frac{1}{h^2} & 0 \\ 0 & \frac{1}{h^2} \end{bmatrix}}_{A_i^*} \begin{bmatrix} y_{i-1} \\ p_{i-1} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{-2}{h^2} & -1 \\ 81 & \frac{-2}{h^2} \end{bmatrix}}_{B_i^*} \begin{bmatrix} y_i \\ p_i \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{h^2} & 0 \\ 0 & \frac{1}{h^2} \end{bmatrix}}_{C_i^*} \begin{bmatrix} y_{i+1} \\ p_{i+1} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 \\ 81x_i^2 \end{bmatrix}}_{D_i^*}$$

$$X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad X_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we get

$$X_1 = \begin{bmatrix} 0.10082765 \\ -0.6115064 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0.16343614 \\ -1.4704651 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 0.13747923 \\ -1.784357 \end{bmatrix}$$

$x_0 \quad x_1 \quad x_2 \quad x_3$

$0 \quad \tfrac{1}{3} \quad \tfrac{2}{3} \quad 1$

Q. $y^{iv} - y''' + y = x^2 \quad | \; y(0) = y'(0) = 0, \; y(1) = 2, \; y'(1) = 0$

Ans.

$$y_i^{iv} = (y_i'')'' = \left(\frac{y_{i+1}'' - 2y_i'' + y_{i-1}''}{h^2} \right)$$

$$= \left(\frac{y_{i+2} - 2y_{i+1} + y_i - 2y_{i+1} + 4y_i - 2y_{i-1} + y_i - 2y_{i-1} + y_{i-2}}{h^4} \right)$$

$$= \left(\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} \right)$$

$$y_i''' = (y_i')'' = \frac{y_{i+1}' - 2y_i' + y_{i-1}'}{h^2}$$

$$= \frac{y_{i+2} - \cancel{y_i} - 2y_{i+1} + 2y_{i-1} + \cancel{y_i} - y_{i-2}}{2h^2}$$

$$= \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^2}$$

So, we have

$$\left(\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} \right) - \left(\frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^2} \right) + y_i = x_i^2$$

$y_0 = 0, \quad y_3 = 2, \quad \dfrac{y_1 - y_{-1}}{2h} = 0$

$\Rightarrow y_{-1} = y_1$

& $y'(1) = 0$

$\Rightarrow \dfrac{y_4 - y_2}{2h} = 0 \Rightarrow y_4 = y_2$

$y_3 = 2, \; y_0 = 0$

So we have

for $i = 1$.

$$\left(\frac{y_3 - 4y_2 + 6y_1 - 4y_0 + y_1}{h^4} \right) - \left(\frac{y_3 - 2y_2 + 2y_0 - y_1}{2h^2} \right) + y_1 = x_1^2$$

$$\Rightarrow \left(\frac{-4}{h^4} + \frac{1}{h^2} \right) y_2 + \left(\frac{7}{h^4} + \frac{1}{2h^2} + 1 \right) y_1 = x_1^2 - \frac{y_3}{h^4} + \cancel{\frac{4y_0}{h^4}}^{\,0} + \frac{y_3}{2h^2} + \cancel{\frac{y_0}{h^2}}^{\,0}$$

$$\Rightarrow -315 y_2 + 572.5 y_1 = \frac{-1376}{9} \quad \text{(i)}$$

for $i = 2$,

$$\left(\frac{y_2 - 4y_3 + 6y_2 - 4y_1 + \cancel{y_0}^{\,0}}{h^4} \right) - \left(\frac{y_2 - 2y_3 + 2y_1 - \cancel{y_0}^{\,0}}{2h^2} \right) + y_2 = \left(\frac{2}{3}\right)^2$$

$$\Rightarrow \left(\frac{7}{h^4} - \frac{1}{2h^2} + 1 \right) y_2 = -\left(\frac{-4}{h^4} + \frac{1}{h^2} \right) y_3 + \left(\frac{2}{3}\right)^2 + \left(\frac{-4}{h^4} - \frac{1}{h^2} \right) y_1$$

$$\Rightarrow 563.5 y_2 - 333 y_1 = \frac{5674}{9} \quad \text{(ii)}$$

$y_2 = 1.424, \; y_1 = 0.5165$ Ans

Spline interpolation.

we have,

$$P_k(x) = \frac{M_k}{h}\frac{(-x+x_{k+1})^3}{3!} + \frac{M_{k+1}}{h}\frac{(x-x_k)^3}{3!}$$

$$+ C_k(x-x_k) + D_k(x_{k+1}-x)$$

we have $P_k(x_k) = f_k$ (i) & $P_k(x_{k+1}) = f_{k+1}$ (ii)

from (i)

$$\Rightarrow \frac{M_k}{h}\frac{(-x_k+x_{k+1})^3}{3!} + \frac{M_{k+1}(x_k-x_k)^3}{3!\,h} + C_k(0)$$

$$+ D_k(x_{k+1}-x_k)$$

$$= f_k = y_k$$

$$\Rightarrow D_k = \frac{1}{h}\left(y_k - \frac{M_k h^2}{6}\right), \quad C_k = \frac{1}{h}\left(y_{k+1} - \frac{M_{k+1}h^2}{6}\right)$$

$\therefore$ P_k can be written as

$$P_k(x) = \frac{M_k}{6}\left[\frac{(x_{k+1}-x)^3}{h} - h(x_{k+1}-x)\right] + \frac{y_k(x_{k+1}-x)}{h}$$

$$+ \frac{M_{k+1}}{6}\left[\frac{(x-x_k)^3}{h} - h(x-x_k)\right] + \frac{y_{k+1}(x-x_k)}{h}$$

for $k = 0, 1, \dots, n-1$, $\therefore$ we need to find $M_0, M_1, \dots M_n$

Q

x	1	2	3	4
y	1.5	2.2	3.1	4.3

$k = 0 \quad 1 \quad 2 \quad 3$

(1) Find $y(1.2)$ & $y'(1)$ by spline interpolation

(2) Find $S(x) = \{P_k(x)\}$ $k = 0, 1, 2$ y

Ans Here we have $h = 1$

$M_0 = M_3 = 0$, we have to find M_1 & M_2

we have,

$$M_{k-1} + 4M_k + M_{k+1} = \frac{6}{h^2}(y_{k+1} - 2y_k + y_{k-1}).$$

for $k=1$,

$$M_0 + 4M_1 + M_2 = \frac{6}{5}$$

for $k=2$

$$M_1 + 4M_2 + M_3 = \frac{9}{5}$$

$$\Rightarrow M_1 = 1/5 \ \& \ M_2 = 2/5$$

So, we have,

$$P_0(x) = \frac{M_0}{6}\left[(x_1-x)^3 - (x_1-x)\right] + 1.5(x_1-x) + \frac{M_1}{6}\left[(x-x_0)^3 - (x-x_0)\right]$$

$$+ 2.2(x-x_0)$$

$$= 1.5(2-x) + \frac{1}{30}\left((x-1)^2 - (x-1)\right) + 2.2(x-1)$$

$$P_1(x) = \frac{1}{30}\left[(3-x)^3 - (3-x)\right] + 2.2(3-x) + \frac{1}{15}\left((x-2)^3 - (x-2)\right)$$

$$+ 3.1(x-2).$$

$P_2(x) = \frac{1}{15}\left((4-x)^3 - (4-x)\right) + 3.1(4-x)$

$+ 0(\quad) + 4.3(x-3)$

Now $y(1.2) = P_0(1.2) = 1.6346$

& $y'(1) = P_0'(1) = 2/3 = 0.6667$

Q $y'' - y = 0$, $y(0) = y(1) = 1$, $h = 1/2$

for $x = x_k$, we have

$x_0 \quad x_1 \quad x_2$; $0 \quad 0.5 \quad 1$

$y''_k = M_k$

$\Rightarrow M_k - y_k = 0 \Rightarrow M_k = y_k$

$\Rightarrow y_{k-1} + 4y_k + y_{k+1} = \frac{6}{h^2}(y_{k-1} - 2y_k + y_{k+1})$

$y_0 = 1 \quad y_2 = 1 \qquad k = 1$

$\Rightarrow y_0 + 4y_1 + y_2 = 6 \times 4(y_0 - 2y_1 + y_2)$

$\Rightarrow 1 + 4y_1 + 1 = 24(1 - 2y_1 + 1)$

$\Rightarrow 2 + 4y_1 = 24(2 - 2y_1)$

$\Rightarrow \cancel{2}(1 + 2y_1) = 24 \times \cancel{2}(1 - y_1)$

$\Rightarrow 1 + 2y_1 = 24 - 24y_1$

$26y_1 = 23 \Rightarrow y_1 = \frac{23}{26}$

& Abs ans ≤ 0.88

So, for in general, we have

$y'' + b(x)y = c(x)$

$M_k + b_k y_k = c_k$

$M_k = c_k - b_k y_k$

$\Rightarrow (c_{k-1} - b_{k-1}y_{k-1}) + 4(c_k - b_k y_k) + c_{k+1} - b_{k+1}y_{k+1}$

$= \frac{6}{h^2}(y_{k-1} - 2y_k + y_{k+1})$.

$\Rightarrow \underbrace{(c_{k-1} + 4c_k + c_{k+1})}_{D_k} = y_{k-1}\underbrace{\left(\frac{6}{h^2} + b_{k-1}\right)}_{A_k}$

$+ y_k\underbrace{\left(\frac{-12}{h^2} + 4b_k\right)}_{B_k} + y_{k+1}\underbrace{\left(\frac{6}{h^2} + b_{k+1}\right)}_{C_k}$

$\Rightarrow A_k y_{k-1} + B_k y_k + C_k y_{k+1} = D_k$

we have our tridiagonal system with which we can solve it to get y_k & M_k

Q $y'' + 2y' + y = 30x$, $y(0) = 0 = y(1)$, $h = 0.5$.

$x_0 \quad x_1 \quad x_2$; $0 \quad 0.5 \quad 1$

Ans $y_0 = y_2 = 0 \qquad n = 2$.

we have $\left(1 - \frac{h A_k}{3}\right)M_k - \frac{h}{6}M_{k+1}A_k = C_k - B_k y_k - \frac{A_k}{h}y_{k+1} + \frac{A_k}{h}y_k$

Here, $A_k = 2$, $B_k = 1$, $k = 0, 1$,

$C_k = 30x_k$

$$\frac{A_k M_{k-1} h}{6} + \left(1 + \frac{h A_k}{3}\right) M_k = C_k - B_k y_k - \frac{A_k y_k}{h} + \frac{A_k y_{k+1}}{h}$$

$k = 1, 2$

So, we have

$K=0$
$$\left(1 - \frac{1}{3}\right) M_0 - \frac{M_1}{6} = 30(0) - y_0 - 4y_1 + 4y_0$$

$$\Rightarrow \frac{2M_0}{3} - \frac{M_1}{6} = \cancel{3y_0}^{0} - 4y_1 \quad \text{— (I)}$$

$K=1$
$$\left(1 - \frac{1}{3}\right) M_1 - \frac{M_2}{6} = 30(0.5) - y_1 - 4y_2 + 4y_1$$

$$\Rightarrow \frac{2M_1}{3} - \frac{M_2}{6} = 15 + 3y_1 - \cancel{4y_2}^{0} \quad \text{— (II)}$$

$K=1$
$$\frac{M_0}{6} + \left(1 + \frac{1}{3}\right) M_1 = 30(0.5) - y_1 - 4y_1 + \cancel{4y_0}^{0}$$

$$\Rightarrow \frac{M_0}{6} + \frac{4M_1}{3} = 15 - 5y_1 \quad \text{— (III)}$$

$K=2$
$$\frac{M_1}{6} + \frac{4}{3} M_2 = 30 - \cancel{y_2}^{0} - \cancel{4y_2}^{0} + 4y_1$$

$$\Rightarrow \frac{M_1}{6} + \frac{4M_2}{3} = 30 + 4y_1 \quad \text{— (IV)}$$

we have the unkno variables M_0, M_1, M_2, y_1

Solving these we get

$M_0 = 16.487$ $\quad M_2 = 14.2562$

$M_1 = 16.8595$ $\quad y_1 = -2.0455$

The absolute soln

is $y = 30e^{-x}\left(e^{x}(x-2) + ex - 2x + 2\right)$

& $y(0.5) = -2.073$

so error is quite less.

Non-linear B.V.P.

Q $y'' + 2yy' = 4 + 4x^2$, $y(1) = 2$, $y(2) = 4$, $h = 0.25$

Ans Discretizing, we get

$$f_i = \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + y_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) - 4 - 4x_i^2$$

1, 1.25, 1.5, 1.75, 2

x_0, x_1, x_2, x_3, x_4 $\quad n = 4$

Now, consider $y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$

to $f_i\left(y_{i-1}^{(k+1)}, y_i^{(k+1)}, y_{i+1}^{(k+1)}\right) = \cancel{\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right)}\, f_i^{(k)}$

$$+ \left.\frac{\partial f_i}{\partial y_{i-1}}\right|^{(k)} \Delta y_{i-1} + \left.\frac{\partial f_i}{\partial y_i}\right|^{(k)} \Delta y_i + \left.\frac{\partial f_i}{\partial y_{i+1}}\right|^{(k)} \Delta y_{i+1}$$

$$= \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + y_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) - 4 - 4x_i^2 \quad -D_i$$

$$+ \underbrace{\left(\frac{1}{h^2} - \frac{y_i^{(k)}}{2h}\right)}_{A_i} \Delta y_{i-1} + \underbrace{\left(\frac{-2}{h^2} + \left(\frac{y_{i+1}^{(k)} - y_{i-1}^{(k)}}{2h}\right)\right)}_{B_i} \Delta y_i +$$

$$\underbrace{\left(\frac{1}{h^2} + \frac{y_i^{(k)}}{2h}\right)}_{C_i} \Delta y_{i+1} = 0$$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$0 \quad \pi/4 \quad \pi/2 \quad 3\pi/4 \quad \pi$

$\rightarrow \quad A_i \Delta y_{i-1} + B_i \Delta y_i + C_i \Delta y_{i+1} = D_i$

we can solve this using the tridiagonal system

& then $y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$

till the solution converges

the initial guess can be taken as

$y_0 = 2, \quad y_1 = 2.5, \; y_2 = 3, \; y_3 = 3.5, \; y_4 = 4$

Solving this by code we get

~~i~~	~~y~~	~~x~~	i	x	y
~~0~~			0	1	2
~~1~~			1	1.25	1.6854
~~2~~			2	1.5	2.00801
~~3~~			3	1.75	2.85116
~~4~~			4	2.	4.

Q $y'' - (y')^2 - y^2 + y + 1 = 0, \quad y(0) = 0.5, \; y(\pi) = -0.5$

Ans Discretizing by central diff scheme, we get

$$f_i = \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) - \left(\frac{y_{i+1} - y_{i-1}}{2h}\right)^2 - y_i^2 + y_i + 1 = 0$$

At the $(k+1)^{th}$ iteration, we have

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i, \quad k \geq 0, \; i = 1, 2, \ldots, n-1$$

So, $f_i^{(k+1)} = f_i + \left.\frac{\partial f_i}{\partial y_{i-1}}\right|^{(k)} \Delta y_{i-1} + \left.\frac{\partial f_i}{\partial y_i}\right|^{(k)} \Delta y_i + \left.\frac{\partial f_i}{\partial y_{i+1}}\right|^{(k)} \Delta y_{i+1}$

$-d_i$

a_i

$$f_i^{(k+1)} = \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) - \left(\frac{y_{i+1} - y_{i-1}}{2h}\right)^2 - y_i^2 + y_i + 1 + \left(\frac{1}{h^2} + 2\left(\frac{y_{i+1} - y_{i-1}}{4h^2}\right)\right)$$

$$\times \Delta y_{i-1} + \left(\frac{-2}{h^2} - 2y_i + 1\right)\Delta y_i + \left(\frac{1}{h^2} - 2\left(\frac{y_{i+1} - y_{i-1}}{4h^2}\right)\right)\Delta y_{i+1}$$

b_i

$= 0$

initial guess $y_i^0(x) = -\frac{x}{\pi} + 0.5 \quad \Big| \; h = \pi/4$

Solving this, we get,

i	x	y
0	0	0.5
1	$\pi/4$	-0.150979
2	$\pi/2$	-0.853574
3	$3\pi/4$	-1.01181
4	π	-0.5

Q $3yy'' + (y')^2 = 0, \quad y(0) = 0, \; y(1) = 1$.

Ans Discretizing the eq^n, we get,

$$3y_i\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \left(\frac{y_{i+1} - y_{i-1}}{2h}\right)^2 = 0$$

At the $(k+1)^{th}$ iteration, we have,

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$$

So, $f_i^{(k+1)} = f_i^{(k)} + \left.\frac{\partial f_i}{\partial y_{i-1}}\right|^{(k)} \Delta y_{i-1} + \left.\frac{\partial f_i}{\partial y_i}\right|^{(k)} \Delta y_i + \left.\frac{\partial f_i}{\partial y_{i+1}}\right|^{(k)} \Delta y_{i+1}$

0 — 1/3 — 2/3 — 1

$$f_i^{(k+1)} = \underbrace{3y_i\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right)+\left(\frac{y_{i+1}-y_{i-1}}{2h}\right)^2}_{-d_i} = 0$$

$$+\underbrace{\left(\frac{3y_i}{h^2}-\frac{(y_{i+1}-y_{i-1})}{2h^2}\right)}_{a_i}\Delta y_{i-1}+\underbrace{\left(\frac{3(y_{i+1}-4y_i+y_{i-1})}{h^2}\right)}_{b_i}\Delta y_i$$

$$+\underbrace{\left(\frac{3y_i}{h^2}+\frac{(y_{i+1}-y_{i-1})}{2h^2}\right)}_{c_i}$$

initial guess, $y_0=0,\ y_1=1/3\ \ y_2=2/3,\ y_3=1$.

final sol^n

i	x	y
0	0	0
1	1/3	0.426414
2	2/3	0.737918
3	1	1.

Q) $f''' + ff'' + 1 - (f')^2 = 0,\quad f(0)=0,\ f'(0)=0,\ f'(10)=1.$

Ans let $F = f' \rightarrow F'' + fF' + 1 - F^2 = 0$.

$f(0)=0,\ F(0)=0,\ F(10)=1.$

$g_a \rightarrow\quad f_i - f_{i-1} - \frac{h}{2}(F_i + F_{i-1}) = 0 \quad$ — (a)

$g_b \rightarrow\quad \left(\frac{F_{i+1}-2F_i+F_{i-1}}{h^2}\right) + f_i\left(\frac{F_{i+1}-F_{i-1}}{2h}\right) + 1 - F_i^2 = 0 \quad$ — (b)

Now, we have to solve (a) & (b) set of equation

At iteration $(k+1)$, we have

$$f_i^{(k+1)} = f_i^{(k)} + \Delta f_i$$

$$F_i^{(k+1)} = F_i^{(k)} + \Delta F_i$$

$$g_a^{(k+1)} = g_a^{(k)} + \left.\frac{\partial g_a}{\partial f_{i-1}}\right|^{(k)}\Delta f_{i-1} + \left.\frac{\partial g_a}{\partial f_i}\right|\Delta f_i + \left.\frac{\partial g_a}{\partial f_{i+1}}\right|\Delta f_{i+1}$$

$$+ \left.\frac{\partial g_a}{\partial F_{i-1}}\right|^{(k)}\Delta F_{i-1} + \left.\frac{\partial g_a}{\partial F_i}\right|\Delta F_i + \left.\frac{\partial g_a}{\partial F_{i+1}}\right|\Delta F_{i+1}$$

$$= \underbrace{\left(f_i - f_{i-1} - \frac{h}{2}(F_i + F_{i-1})\right)}_{-D_1} + \underbrace{(-1)}_{A_{11}}\Delta f_{i-1} + \underbrace{(1)}_{B_{11}}\Delta f_i + \underbrace{0}_{C_{11}}\times\Delta f_{i+1}$$

$$+ \underbrace{\left(-\frac{h}{2}\right)}_{A_{12}}\Delta F_{i-1} + \underbrace{\left(\frac{-h}{2}\right)}_{B_{12}}\Delta F_i + \underbrace{(0)}_{C_{12}}\Delta F_{i+1} = 0$$

if $X_i = \begin{pmatrix}\Delta f_i \\ \Delta F_i\end{pmatrix}$, the above eq^n can be written as

$$[A_{11}\quad A_{12}]\,[X_{i-1}] + [B_{11}\quad B_{12}]\,X_i + [C_{11}\quad C_{12}]\,X_{i+1} = D_1$$

Similarly for g_b.

$$g_b^{(k+1)} = \underbrace{\left(\frac{F_{i+1}-2F_i+F_{i-1}}{h^2}\right) + f_i\left(\frac{F_{i+1}-F_{i-1}}{2h}\right) + 1 - F_i^2}_{-D_2}$$

$$+ \underbrace{(0)}_{A_{21}}\Delta f_{i-1} + \underbrace{\left(\frac{F_{i+1}-F_{i-1}}{2h}\right)}_{B_{21}}\Delta f_i + \underbrace{0}_{C_{21}}\,\Delta f_{i+1}$$

$$+ \underbrace{\left(\frac{1}{h^2}-\frac{f_i}{2h}\right)}_{A_{22}}\Delta F_{i-1} + \underbrace{\left(\frac{-2}{h^2}-2F_i\right)}_{B_{22}}\Delta F_i + \underbrace{\left(\frac{1}{h^2}+\frac{f_i}{2h}\right)}_{C_{22}}\Delta F_{i+1}$$

So, let $A_i = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B_i = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

& $C_i = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ & $D_i = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$

Using this we have

$$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i \quad \text{for } i = 1, 2, \ldots n-1$$

we can solve this to get X_i, $i = 1, \ldots n-1$

$X_0 = \begin{bmatrix} \Delta f_0 \\ \Delta F_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $X_n = \begin{bmatrix} \Delta f_n \\ \Delta F_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

& then we can have

$Z_i = \begin{bmatrix} f_i \\ F_i \end{bmatrix}$ & $Z_i^{(k+1)} = Z_i^{(k)} + X_i^{(k)}$

$i = 1, 2, \ldots n-1$

till error is reduced

Solving this, we get,

i	x	f
0	0	0
1	1	0.3904
2	2	1.2709
3	3	2.2614
4	4	3.2616
5	5	4.2616
6	6	5.2616
7	7	6.2616
8	8	7.2616
9	9	8.2615

Q $F''' + (2F+4)F' = 0, \quad F(0)=0, \; F''(0) = -K, \; F'(\omega) = 0$

$K = 0.1, \quad \omega = 0.087$

Ans $F' = g, \Rightarrow g'' + (2F + 4)g = 0$

$F(0) = 0, \; g'(0) = -K, \; g(\omega) = 0$

$$g_0' = \left(\frac{-3g_0 + 4g_1 - g_2}{2h}\right) = -k \Rightarrow g_0 = \frac{(4g_1 - g_2 + 2hk)}{3}$$

$$\& K_a = F_i - F_{i-1} - \frac{h}{2}(g_i + g_{i-1}) = 0$$

$$\& K_b = \left(\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2}\right) + (2F_i + 4)g_i = 0$$

$i = 1, 2, \dots, n-1$

At $(k+1)^{th}$ iteration

$$F_i^{(k+1)} = F_i^{(k)} + \Delta F_i \quad \& \quad g_i^{(k+1)} = g_i^{(k)} + \Delta g_i$$

$$\Delta F_0 = \Delta F_n = \Delta g_0 = \Delta g_n = 0$$

$$K_a^{(k+1)} = F_i - F_{i-1} - \frac{h}{2}(g_i + g_{i-1}) + (-1)\Delta F_{i-1} + \Delta F_i + 0\,\Delta F_{i+1}$$

$$+ \left(-\frac{h}{2}\right)\Delta g_{i-1} + \left(-\frac{h}{2}\right)\Delta g_i + (0)\,\Delta g_{i+1} = 0$$

$$K_b^{(k+1)} = \left(\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2}\right) + (2F_i + 4)g_i + (0)\Delta F_{i-1} + 2g_i\,\Delta F_i + 0\,\Delta F_{i+1}$$

$$+ \left(\frac{1}{h^2}\right)\Delta g_{i-1} + \left(\frac{-2}{h^2} + (2F_i + 4)\right)\Delta g_i$$

$$+ \left(\frac{1}{h^2}\right)\Delta g_{i+1}$$

Converting it into tri diagonal f

$$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i$$

we can then solve it.

Quasilinearization

$y(a) \;\& \; y(b)$ are given

$$F(x, y, y', y'') = F(x, y^{(0)}, y'^{(0)}, y''^{(0)}) + (y - y^{(0)})\frac{\partial F}{\partial y}\bigg|^{(0)}$$

$$+ (y' - y'^{(0)})\frac{\partial F}{\partial y'}\bigg|^{(0)} + (y'' - y''^{(0)})\frac{\partial F}{\partial y''}\bigg|^{(0)} = 0$$

$$F(x, y^{(k)}, y'^{(k)}, y''^{(k)}) + (y^{(k+1)} - y^{(k)})\frac{\partial F}{\partial y}\bigg|^{(k)} + (y'^{(k+1)} - y'^{(k)})\frac{\partial F}{\partial y'}\bigg|^{(k)}$$

$$+ (y''^{(k+1)} - y''^{(k)})\frac{\partial F}{\partial y''}\bigg|^{(k)}$$

$y^{(k+1)}(a) = y_0, \quad y^{(k+1)}(b) = y_n, \quad k \geq 0$

Repeat till

$$\|y^{(k+1)} - y^{(k)}\| < \epsilon$$

Q $3yy'' + (y')^2 = 0 \qquad y(0) = 0, \; y(1) = 1$

~~$h = 0.25$, $n = 4$, $y_0^{(k)} = 0$, $y_4^{(k)} = 1$~~ $y_a = 0 \quad y_b = 1$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$0 \quad 0.25 \quad 0.5 \quad 0.75 \quad 1$

Now, the initial guess ~~is $y_0^{(0)} = 0$ $y_1^{(0)} = 0.25$ $y_2^{(0)} = 0.5$ $y_3^{(0)} = 0.75$~~

Now,

$$F(x, y^{(k)}, y'^{(k)}, y''^{(k)})$$

$$+ (y^{(k+1)} - y^{(k)})\,3y''^{(k)} + (y'^{(k+1)} - y'^{(k)})\,2y'^{(k)}$$

$$+ (y''^{(k+1)} - y''^{(k)}) \cdot 3 \cdot y^{(k)} = 0$$

$$\Rightarrow \cancel{3y^{(k)}y''^{(k)}} + \cancel{(y'^{(k)})^2} + 3y^{(k+1)}y''^{(k)} - 3 \cdot y^{(k)}y''^{(k)}$$

$$+ 2 \cdot y'^{(k)} \cdot y'^{(k+1)} - 2(y'^{(k)})^2 + 3y^{(k)}y''^{(k+1)}$$

$$- \cancel{3y^{(k)}y''^{(k)}}$$

$$\Rightarrow 3y^{(k+1)}y''^{(k)} - 3y^{(k)}y''^{(k)} + 2y'^{(k)}y'^{(k+1)} + 3y^{(k)}y''^{(k+1)}$$

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$F(x, y^{(k)}, y'^{(k)}, y''^{(k)})$

$$+ \left(y^{(k+1)} - y^{(k)}\right) \cdot \left(3y''^{(k)}\right) + \left(y'^{(k+1)} - y'^{(k)}\right) 2y'^{(k)}$$

$$+ \left(y''^{(k+1)} - y''^{(k)}\right) 3y^{(k)}$$

$$3y^{(k)} y''^{(k+1)} + 2y'^{(k)} y'^{(k+1)} + 3y''^{(k)} y^{(k+1)}$$

$$- \cancel{3y''^{(k)} y^{(k)}} - \cancel{2}(y'^{(k)})^2 - 3y^{(k)} y''^{(k)}$$

$$+ \cancel{3y^{(k)} y''^{(k)}} + \cancel{(y'^{(k)})^2} = 0$$

$$\Rightarrow 3y^{(k)} y''^{(k+1)} + 2y'^{(k)} \cdot y'^{(k+1)} + 3y''^{(k)} y^{(k+1)}$$

$$= 3y^{(k)} y''^{(k)} + (y'^{(k)})^2$$

this is a linear BVP.

For $k=0$, we have

$$3y^{(0)} y''^{(1)} + 2y'^{(0)} y'^{(1)} + 3y''^{(0)} y^{(1)} = 3y^{(0)} y''^{(0)} + (y'^{(0)})^2$$

So, $y^{(0)}$, $y'^{(0)}$ & $y''^{(0)}$ are known, so, these can be solved after using the initial values.

& then we can ~~solve~~ find out $y^{(k+1)}$, $y'^{(k+1)}$ & $y''^{(k+1)}$ and so on for values of k till we get $\|y^{(k+1)} - y^{(k)}\| \leq \varepsilon$, $a \leq x \leq b$

Q $f''' + f \cdot f'' + 1 - (f')^2 = 0$, $f(0) = f'(0) = 0$, $f'(10) = 1$, $0 < \eta < 10$.

Ans At the $(k+1)^{th}$ iteration, we have,

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$$F(x, y^{(k+1)}, y'^{(k+1)}, y''^{(k+1)}) = F(x, y^{(k)}, y'^{(k)}, y''^{(k)}) + \left(y^{(k+1)} - y^{(k)}\right) \cdot \frac{\partial F}{\partial y}\Big|^{(k)} + \left(y'^{(k+1)} - y'^{(k)}\right) \cdot \frac{\partial F}{\partial y'}\Big|^{(k)} + \left(y''^{(k+1)} - y''^{(k)}\right) \frac{\partial F}{\partial y''}\Big|^{(k)} + \left(y'''^{(k+1)} - y'''^{(k)}\right) \frac{\partial F}{\partial y'''}\Big|^{(k)} = 0$$

$$\Rightarrow \frac{\partial^3 f^{(k+1)}}{\partial \eta^3} \frac{\partial F}{\partial f'''}\Big|^{(k)} + \frac{\partial^2 f^{(k+1)}}{\partial \eta^2} \frac{\partial F}{\partial f''}\Big|^{(k)} + \frac{\partial f^{(k+1)}}{\partial \eta} \frac{\partial F}{\partial f'}\Big|^{(k)}$$

$$+ f^{(k+1)} \frac{\partial F}{\partial f}\Big|^{(k)} = - F(f'''^{(k)}, f''^{(k)}, f'^{(k)}, f^{(k)})$$

$$+ f'''^{(k)} \frac{\partial F}{\partial f'''}\Big|^{(k)} + f''^{(k)} \frac{\partial F}{\partial f''}\Big|^{(k)} + f'^{(k)} \frac{\partial F}{\partial f'}\Big|^{(k)} + f^{(k)} \frac{\partial F}{\partial f}\Big|^{(k)}.$$

Boundary Conditions:- $f^{(k+1)} = 0 = f'^{(k+1)}$, $f'^{(k+1)}(10) = 1$.

$f^{(0)}(\eta) \rightarrow$ this can be guessed.

let $p^{k+1} = f'^{(k+1)}$

$$\Rightarrow p''^{(k+1)} + p'^{(k+1)} f^{(k)} - 2p^{k+1} f'^{(k)} + f^{(k+1)} \cdot f''^{(k)}$$

$$= f''^{(k)} f^{(k)} - (f'^{(k)})^2 - 1$$

Now, after discretization we get

$$f_i^{(k+1)} - f_{i-1}^{(k+1)} - \frac{h}{2}\left(F_i^{(k+1)} + F_{i-1}^{(k+1)}\right) = 0 \quad -(i)$$

$$\frac{P_{i+1}^{(k+1)} - 2P_i^{(k+1)} + P_{i-1}^{(k)}}{h^2} + f^{(k)} \frac{\left(P_{i+1}^{(k+1)} - P_{i-1}^{(k+1)}\right)}{2h} - 2P_i^{(k+1)} f_i'^{(k)} + f_i^{(k+1)} f''^{(k)}$$

$$= f_i''^{(k)} f_i^{(k)} - (f''^{(k)})^2 - 1 \quad -(ii)$$

if we take $X_i = \begin{bmatrix} f_i^{(k+1)} \\ P_i^{(k+1)} \end{bmatrix}$

then, $A_i = \begin{bmatrix} -1 & -h/2 \\ 0 & \frac{1}{h^2} - \frac{f_i^{(k)}}{2h} \end{bmatrix}$ $B_i = \begin{bmatrix} 1 & -h/2 \\ f''^{(k)} & \frac{-2}{h^2} - 2f' \end{bmatrix}$

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$$C_i = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{f^{(x)}}{2h} \end{bmatrix} \quad \& \quad D_i = \begin{bmatrix} 0 \\ f''^{(x)} f^{(x)} - (f'^{(x)})^2 - 1 \end{bmatrix}$$

(i) & (ii)

This can be written as

$$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i$$

we can use our modification of the Thomas Algo. to solve this Block tridiagonal system.