Tutorial 4 Discussion

Lecture 19

Q5. N be a normal subgp of G s.t $|N| = 5$ and $|G|$ is odd

WTS $N \subseteq Z(G)$.

Define a gp action

$$\phi : G \times N \longrightarrow N$$
$$(g, n) \longmapsto gng^{-1}$$

The corresponding permutation representation of the gp is

$$\psi : G \longrightarrow S_5$$
$$g \longmapsto \sigma_g$$

$\sigma_g : N \to N$

$\sigma_g(n) = gng^{-1}$

$\Downarrow$

σ_g is an automorphism of N

which is a gp homo.

$$\ker \psi = \{ g \in G \mid \sigma_g = Id \}$$
$$= \{ g \in G \mid gng^{-1} = n \quad \forall n \in N \}$$

what can I say about $\operatorname{Im}\psi$.

Note that $\operatorname{Im}\psi \subseteq \operatorname{Aut}(N)$.

Since $|N| = 5$, thus N is a cyclic gp of order 5.

$$|\operatorname{Aut}(N)| = 4$$

$$\psi : G \longrightarrow \operatorname{Im}\psi \subseteq \operatorname{Aut}(N)$$

$$G/\ker\psi \cong \operatorname{Im}\psi.$$

As order G is odd $\operatorname{Im}\psi$ has odd number of elts.

But $\operatorname{Im}\psi$ is a subgp of $\operatorname{Aut}(N)$ which has even order. Therefore

$|\operatorname{Im}\psi| = 1$. $\Rightarrow$ $G = \ker\psi$.

$\therefore$ $N \subset Z(G)$.

<u>Lemma</u>: If H is a subgp of S_n then either all its elements are even permutations or the number of even permutation is same as number of odd permutations.

<u>Pf</u>: Let H is not contained in A_n. Then $\exists$ at least one odd permutation in H. Let $\phi : H \longrightarrow \{1, -1\}$.

$$\phi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

$$\ker \phi = H \cap A_n.$$

$$H/\ker\phi \cong \{1, -1\}$$

$$\Rightarrow \quad \frac{|H|}{|H \cap A_n|} = 2.$$

Since $|G| = 2m$. $\Rightarrow \exists$ an elt $x \in G$ s.t $|x| = 2$.

$$\phi : G \longrightarrow S_{2m}. \qquad G \cong \phi(G).$$
$$g \longmapsto \sigma_g$$

why G will be having an odd permutation?

$\sigma_g : G \longrightarrow G$, $G = \{1, \ldots, 2m\}$

$\sigma_g(x) = gx$.

Then $\phi(x)$ will also be an elt of order 2 in S_{2m} which will be product transpositions.

Therefore $\phi(x)$ will be an odd permutation.

$\phi(G) \cap A_n$ is a proper subgp of $\phi(G)$ whose index is 2.

$m=3$ $\quad$ $2m=6$ $\quad$ $(12)(34)$

Hint $|GL_2(\mathbb{F}_3)| = 12.$

$$\cong$$

$$D_6 = \langle r, s \mid r^6 = 1, s^2 = 1, rs = sr^{-1} \rangle$$

Q8. No of Sylow p-subgps of S_p.

$$|S_p| = p! = p\,(p-1)!$$

$n_p \mid (p-1)!$ and $n_p \equiv 1 \pmod p.$

A Sylow p-subgp is of order p

An elt of order p is a p-cycle.

The no of p-cycles in S_p

is $p!/p = (p-1)!$

Each subgp of order p contains

$(p-1)$ elts of order p.

and any p-subgps intersect

trivially.

$\therefore$ The total no. of Sylow

p-subgp is $(p-1)!/_{p-1} = (p-2)!$.

$\therefore$ $n_p = (p-2)!$

$(p-1)! \equiv -1$ (mod p).

$(p-2)! \equiv 1 \pmod p$

$(p-1)! \equiv p-1 \pmod p$

Q14 $|GL_2(\mathbb{F}_p)| = p(p-1)^2(p+1)$.

$n_p \mid (p-1)^2(p+1)$ and $n_p \equiv 1 \pmod p$

$\therefore n_p = 1, p+1, (p-1)^2, (p+1)(p-1)^2$.

Let P be a Sylow p-subgp.

$|P| = p$. which will be cyclic.

$P = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$ is a Sylow p-subgp.

Let $N(P)$ be the normalizer of P in G.

$n_p = [G : N(P)]$

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N(P)$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 - \frac{ac}{\Delta} & a^2/\Delta \\ -c^2/\Delta & 1 + ac/\Delta \end{pmatrix}$$

where $\Delta = ad - bc \neq 0$.

then $c = 0$.

$$\therefore \; N(P) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{F}_p \right\}$$

$$|N(P)| = p(p-1)^2.$$

$$\therefore \; n_p = \frac{|G|}{|N(P)|} = p+1.$$