VECTOR SPACE: A nonempty set V of elements $a, b, \ldots$ is called a vector space and these elements are called vectors if in V the following algebraic operations are defined as follows:

Addition: $a, b \in V \Rightarrow a+b \in V$

Scalar multiplication: $\lambda \in \mathbb{R}$, and $a \in V \Rightarrow \lambda a \in V$

(These two properties are called closure properties)

and these vectors satisfy the following axioms:

(a1) commutativity $a+b = b+a \quad \forall\, a, b \in V$

(a2) Associativity $(a+b)+c = a+(b+c) \quad \forall\, a, b, c \in V$

(a3) There exists a zero vector such that

$$a + 0 = a \qquad \forall\, a \in V$$

(a4) $\forall\, a \in V,\ \exists\ (-a) \in V$ with $a + (-a) = 0$

(a5) Distributivity: $(\lambda+\mu)a = \lambda a + \mu a \quad \forall\, \lambda, \mu \in \mathbb{R}$ & $\forall\, a \in V$

(a6) Distributivity $\lambda(a+b) = \lambda a + \lambda b \quad \forall\, \lambda \in \mathbb{R}$ & $a, b \in V$

(a7) Associativity $\lambda(\mu a) = (\lambda\mu) a$

(a8) $\forall\, a \in V \quad 1.a = a$

↑ Vector space (over $\mathbb{R}$)!

Examples:

1. Row vector space $\mathbb{R}_n$

$$\mathbb{R}_n := \{(a_1, a_2, \cdots a_n) \mid a_i \in \mathbb{R} \quad \forall\ i = 1, \cdots n\}$$

with

$$(a_1, a_2, \cdots a_n) + (b_1, b_2, \cdots ; b_n) := (a_1+b_1, a_2+b_2, \cdots, a_n+b_n)$$

& $$\lambda(a_1, a_2, \cdots a_n) = (\lambda a_1, \lambda a_2, \cdots, \lambda a_n).$$

2. Column vector space $\mathbb{R}^n$.

$$\mathbb{R}^n := \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \mid a_i \in \mathbb{R}, \ \forall\ i = 1, 2, \cdots n \right\}$$

with $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{pmatrix}$ and $\lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}$

3. Vector spaces of matrices from $\mathbb{R}^{m \times n}$.

— ......

Linear Combination of vectors $v_1, v_2, \cdots ; v_n$:

is an expression $\lambda_1 v_1 + \lambda_2 v_2 + \cdots \lambda_n v_n$

where $\lambda_i \in \mathbb{R}$.

Linear independence of vectors: Let V be a vector space.

A finite set $\{v_1, v_2, \cdots v_n\}$ of the elements of V is said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$, not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

These vectors are called linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

Ex: $v_1 = (1,-1,0)$ $v_2 = (0,1,-1)$ $v_3 = (0,0,1)$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow (\alpha_1, -\alpha_1+\alpha_2, -\alpha_2+\alpha_3) = 0$$

$$\Rightarrow \alpha_1 = 0 = \alpha_2 = \alpha_3$$

$\Rightarrow$ The given set of vectors is linearly independent.

Ex: $v_1 = (0,0)$ $v_2 = (1,2)$

We know

$$\alpha(0,0) + 0.(1,2) = (0,0) \text{ for } \alpha \neq 0$$

$\Rightarrow$ vectors are linearly dependent.

If 0 is one of the vectors $v_1, v_2, \dots v_m$ then vectors must be linearly dependent.

Basis & Dimension of a vector space:

Dimension: maximum number of linearly independent vectors in V

Basis: Set of these linearly independent vectors is called basis.

Note that every vector in V can be uniquely written as a linear combination of the basis vectors.

SPAN of vectors $(v_1, v_2 \dots, v_n)$: The set of all linear combinations of given $v_1, v_2, \dots v_n$.

Note that a span is a vector space.

SPANNING SETS: Let V be a vector space over $\mathbb{R}$. Vectors $v_1, v_2 \dots v_n$ in V are said to span V or to form a spanning set of V if every v in V is a linear combination of the vectors $v_1, v_2, \dots v_n$, that is, if there exists scalars $\alpha_1, \alpha_2 \dots \alpha_n$ in $\mathbb{R}$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n.$$

Row space of a matrix: The span of the row vectors of a matrix.

similarly column space can be defined.

Rank of a matrix: = max. number of linearly independent row vectors of a matrix

= max. number of linearly independent column vectors of a matrix

= dimension of row or column space.

Ex: $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

Note that $R_3 = -\left(\frac{R_2}{2} - 6R_1\right)$

$\Rightarrow$ Rank of this matrix is 2.

REMARK: • Rank does not change under elementary row operations.

• The rank of a matrix is the number of non zero rows in its reduced row echelon form.

• The rank of an $m \times n$ matrix can not be greater than n or m. That is, $\text{Rank}(A) \leq \text{Min}(m,n)$.

Ex: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -2 & -1 & 1 \\ 2 & 4 & -4 & 0 & 3 \\ -1 & -2 & 3 & 3 & 4 \\ 3 & 6 & -7 & 1 & 1 \end{bmatrix}$

Using Gauss elimination technique

$$A \sim \begin{bmatrix} \boxed{1} & 2 & -2 & -1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 5 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{pivot rows} \\ \leftarrow \text{zero row} \end{array}$$

$\boxed{\text{RANK of } A = 3}$

Example: Let us take $V = \mathbb{R}^3$.

- The vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ form a spanning set of $\mathbb{R}^3$.

$\Rightarrow$ Take any $v \in \mathbb{R}^3$ then

$$\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = v_1\begin{pmatrix}1\\0\\0\end{pmatrix} + v_2\begin{pmatrix}0\\1\\0\end{pmatrix} + v_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

- We claim that the vectors

$$\underline{\omega_1} = \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad \underline{\omega_2} = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad \text{and} \quad \underline{\omega_3} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

form a spanning set of $\mathbb{R}^3$.

$\Rightarrow$ Take any $\underline{v} = (v_1, v_2, v_3)^T \in \mathbb{R}^n$ then we seek scalar $\alpha_1, \alpha_2, \alpha_3$ such that $\underline{v} = \alpha_1\underline{\omega_1} + \alpha_2\underline{\omega_2} + \alpha_3\underline{\omega_3}$; that is

$$\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \alpha_1\begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\1\\0\end{pmatrix} + \alpha_3\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} v_1 &= \alpha_1 + \alpha_2 + \alpha_3 \\ v_2 &= \alpha_1 + \alpha_2 \\ v_3 &= \alpha_1 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha_1 &= v_3 \\ \alpha_2 &= v_2 - v_3 \\ \alpha_3 &= v_1 - v_2 \end{aligned}$$

$$\Rightarrow \underline{v} = v_3\underline{\omega_1} + (v_2 - v_3)\underline{\omega_2} + (v_1 - v_2)\underline{\omega_3}$$

- One can also show that

$$\underline{u_1} = \begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \underline{u_2} = \begin{pmatrix}1\\3\\5\end{pmatrix} \qquad \underline{u_3} = \begin{pmatrix}1\\5\\9\end{pmatrix}$$

do not span $\mathbb{R}^3$.

SUBSPACES:

Def: Let V be a vector space (over $\mathbb{R}$) and let W be a subset of V. Then W is a subspace of V if W is itself a vector space (over $\mathbb{R}$) with respect to the same operations as in V.

CRITERIA FOR IDENTIFYING SUBSPACES:

for every $u, v \in W$ & $\lambda \in \mathbb{R}$ the following closure properties should hold:

$$u + v \in W$$

$$\& \quad \lambda u \in W.$$

TRIVIAL SUBSPACES of V:
- The set $\{0\}$
- The whole space V itself.

Ex: Let U consist of all vectors from $\mathbb{R}^3$ whose entries are equal; that is

$$U = \{(a, b, c) : a = b = c\}.$$

VERIFY above criteria.

Ex: Let S be a subset of a vector space V. Then $\mathrm{Span}(S)$ is a subspace of V that contain S.

Rank in terms of determinant:

Submatrix: Suppose A is any matrix of order $m \times n$ then a matrix obtained by leaving some rows and columns from A is called a submatrix of A.

Rank: An $m \times n$ matrix A has rank $r \geq 1$ iff A has $r \times r$ submatrix with non zero determinant, whereas the determinant of every square submatrix with $(r+1)$ or more rows is zero.

In particular, if A is square $n \times n$, it has rank n iff

$\det A \neq 0$.

Ex:

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$$

Clearly $|A| = 0$ (first two columns are identical)

Every 2×2 submatrices has zero determinant.

Therefore the rank is ~~zero~~.

Remark: Rank of a zero matrix is ~~zero~~ one.

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

$|A| = 0 \Rightarrow \text{Rank}(A) < 3$.

Also, $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \neq 0$

$\Rightarrow \text{Rank}(A) = 2$.

Ex:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|A| = 1 \neq 0 \Rightarrow \text{Rank}(A) = 3.$

Q: Rank of A matrix of order $m \times n$ whose every element is unity.

A: 1.

PROPERTIES:

(i) A is nonsingular if $\text{rank}(A) = n$ $(|A| \neq 0)$ & singular if $\text{rank}(A) < n$. $(|A| = 0)$

(ii) If $B \sim A$ then $\text{rank}(B) = \text{rank}(A)$ $(A, B \in \mathbb{R}^{m \times n})$

(iii) If $\text{rank}(A) = n$ and $AB = AC$ then $B = C$

(A^{-1} exists, Premult. by A^{-1} to get $B = C$)

(iv) If $\text{rank}(A) = n$ then $AB = 0 \Rightarrow B = 0$. Hence if $AB = 0$ But $A \neq 0$, $B \neq 0$, then $\rho(A) < n$ and $\rho(B) < n$.

Solution of system of equations using rank concept:

Ex: $$[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 0 \\ 2 & 5 & 1 & 1 & 2 \\ 3 & 7 & 2 & 2 & \beta \\ -1 & 0 & 1 & \alpha & 16 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 0 \\ 0 & 1 & 3 & 3 & 2 \\ 0 & 0 & 2 & 2 & \beta-2 \\ 0 & 0 & 0 & \alpha-1 & 3\beta+6 \end{array}\right]$$

(i) Unique solution $\Leftrightarrow$ $\alpha \neq 1$

What does that mean ?

$$\boxed{\text{Rank}(A) = 4 = \text{Rank}([A|b]) = \text{number of unknowns}}$$

(ii) No solution: $\alpha = 1$ & $\beta \neq -2$

$$\boxed{\text{Rank}(A) = 3 \neq \text{Rank}([A|b]) = 4}$$

(iii) Infinitely many solutions:

$\alpha = 1$ & $\beta = -2$.

$$\boxed{\text{Rank}(A) = 3 = \text{Rank}([A|b]) < \text{No. of unknowns.}}$$

No. of Free parameter (variable) $= 1 = 4-3$

$= \text{No. of unknowns} - \text{rank}$

$= n - r$

Consistency of a system of linear equations: (Fundamental theorem):

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ & $[A|b] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$

The system of equations $Ax = b$ is:

(i) inconsistent i.e. there is no solution if $\text{rank}(A) \neq \text{rank}([A|b])$

(ii) Consistent and there is a unique solution if

$$\text{rank}(A) = \text{rank}([A|b]) = \text{no. of unknowns } (n)$$

(iii) Consistent and there are infinitely many solutions if

$$\text{rank}(A) = \text{rank}([A|b]) < n.$$

Giving arbitary values to $(n-r)$ of unknowns we may express the other unknowns in terms of these.

For the system of linear homogeneous equations $Ax = b$

(i) If $\text{rank}(A) = n$ (number of unknowns), the equations have only a trivial solution $x_1 = x_2 = \cdots = x_n = 0$. (unique solution)

(ii) If $\text{rank}(A) < n$, the equations have $(n-r)$ linearly independent solutions. (infinitely many solutions)

Note:
- A homogeneous linear system is always consistent (It has atleast trivial solution)
- A homogeneous linear system with fewer equations than unknowns always has nontrivial solutions.

Solution of the system of equations $Ax = b$

$$\boxed{Ax = b}$$

$$\boxed{\rho(A) \neq \rho([A \mid b])}$$

Inconsistent
(no solution)

$$\boxed{\rho(A) = \rho([A \mid b])}$$ consistent

$$\boxed{\rho(A) = \rho([A \mid b]) < \text{no.of unknowns}}$$

Infinite solutions

$$\boxed{\rho(A) = \rho([A \mid b]) = \text{no.of unknowns}}$$

Unique solution