Linear Algebra

Lecture 10

Linear Transformations

Rank-Nullity theorem:
Let V and W be finite dimensional vector spaces over $\mathbb{F}$ and $T: V \to W$ be a linear transformation. Then

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(V).$$

1-1 and onto linear transformation.

Thm: Let V and W be vector spaces over a field $\mathbb{F}$ and $T: V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$.

Proof: Let us assume T is one-to-one.

$x \in N(T)$.

$$\Rightarrow T(x) = \underset{\text{in } W}{0} = T(\underset{\text{in } V}{0})$$

$$\Rightarrow x = 0 \quad \text{using definition of one-to-one function.}$$

Conversely, let $N(T) = \{0\}$.

Assume that $T(x) = T(y)$

$$\Rightarrow T(x-y) = 0$$

$$\Rightarrow x - y = 0 \Rightarrow x = y$$

$$\Rightarrow T \text{ is one-to-one}$$

Ex:

Let $V = W = P(\mathbb{R})$ = real vector space of polynomials.

$T(f) = \frac{d}{dx} f(x)$

Here $N(T) = \mathbb{R}$ $\Rightarrow$ T is not one-to-one

Theorem: Let V and W be vectorspaces of equal and finite dimension. and let $T: V \to W$ be linear. Then the following statements are equivalent.

1) T is one-to-one
2) T is onto
3) $\text{rank}(T) = \dim(V)$

Proof: From rank-nullity theorem:

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

T is one-to-one

$\Leftrightarrow$ $N(T) = \{0\}$

$\Leftrightarrow$ $\dim N(T) = \text{nullity}(T) = 0$

$\Leftrightarrow$ $\text{rank}(T) = \dim(V)$

$\Leftrightarrow$ $\text{rank}(T) = \dim(W)$

$\Leftrightarrow$ $\dim(R(T)) = \dim(W)$

$\Leftrightarrow$ $R(T) = W$ $\Leftrightarrow$ T is onto. $\blacksquare$

Example: Let $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 - x_2 \\ 0 \end{pmatrix}$$

$$N(T) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , \ T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 - x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow N(T) = \{0\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$\Rightarrow$ T is one-to-one.

Clearly T is not onto.

Example: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \end{pmatrix}$$

$$N(T) = \left\{ \begin{pmatrix} t \\ t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\Rightarrow$ T is not one-to-one

Rank-Nullity theorem $\Rightarrow$ T is onto.

Note: Let $T: V \to W$ be linear.
If $\dim(V) < \dim(W)$, there does NOT exist a linear map T which is onto.
If $\dim(V) > \dim(W)$, there does NOT exist a linear map T which is one-to-one.

Exercise: Construct a linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ which is neither one-to-one nor onto.

Example: Let $T: P_2(\mathbb{R})$ to $P_3(\mathbb{R})$ be the linear transformation defined as

$$T(f) = 2f'(x) + \int_0^x 3f(t)\,dt$$

If $\{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$ then

$$\text{span}\{T(1), T(x), T(x^2)\} = R(T).$$

$T(1) = 3x$

$T(x) = 2 + \frac{3x^2}{2}$, $\quad T(x^2) = 4x + x^3$

for scalars a_1, a_2, a_3

$$a_1(4x + x^3) + a_2\left(2 + \frac{3x^2}{2}\right) + a_3(3x) = 0$$

$$a_1 x^3 + \frac{3}{2} a_2 x^2 + (3a_3 + 4a_1)x + 2a_2 = 0$$

$\Rightarrow$ ~~Rk~~ rank(T) = 3 $\Rightarrow$ nullity(T) = 0.

$\Rightarrow$ T is not onto but one-to-one.

<u>Theorem</u>: Let V and W be vector spaces over $\mathbb{F}$. Let $\{v_1, \cdots, v_n\}$ be a basis for V. For $w_1, w_2, \cdots, w_n \in W$, there exists exactly one linear transformation $T: V \to W$ such that

$$T(v_i) = w_i \qquad i = 1, 2, \cdots, n$$

<u>Proof:</u> Let $x \in V$. Then

$$x = \sum_{i=1}^{n} a_i v_i$$

where $a_1, a_2, \cdots, a_n$ are unique scalars.

Define $T: V \to W$ as

$$T(x) = \sum_{i=1}^{n} a_i w_i$$

To prove that T is linear.

take $u, v \in V$ and $d \in \mathbb{F}$.

$$u = \sum_{i=1}^{n} b_i v_i \quad ; \quad v = \sum_{i=1}^{n} a_i v_i$$

$$T(du + v) = dT(u) + T(v)$$

Check $T(v_i) = w_i$ $i = 1, 2, \dots, n$

To show: T is unique.

Suppose there is another linear transformation $\tilde{T}: V \to W$ such that $\tilde{T}(v_i) = w_i$ for $i = 1, 2, \dots, n$

Take $x \in V$, then

$$x = \sum_{i=1}^{n} a_i v_i$$

$$\tilde{T}(x) = \tilde{T}\left(\sum_{i=1}^{n} a_i v_i\right)$$

$$= \sum_{i=1}^{n} a_i \tilde{T}(v_i)$$

$$= \sum_{i=1}^{n} a_i w_i$$

$$= T(x)$$

$\Rightarrow \tilde{T} = T$

Corollary:

Quotient spaces:

Let V be a vector space over $\mathbb{F}$ and let W be a subspace of V.

Define a coset of W in V as

$$v + W = \{ v + w \mid w \in W \}$$

Example: Let $V = \mathbb{R}^2$ and W be

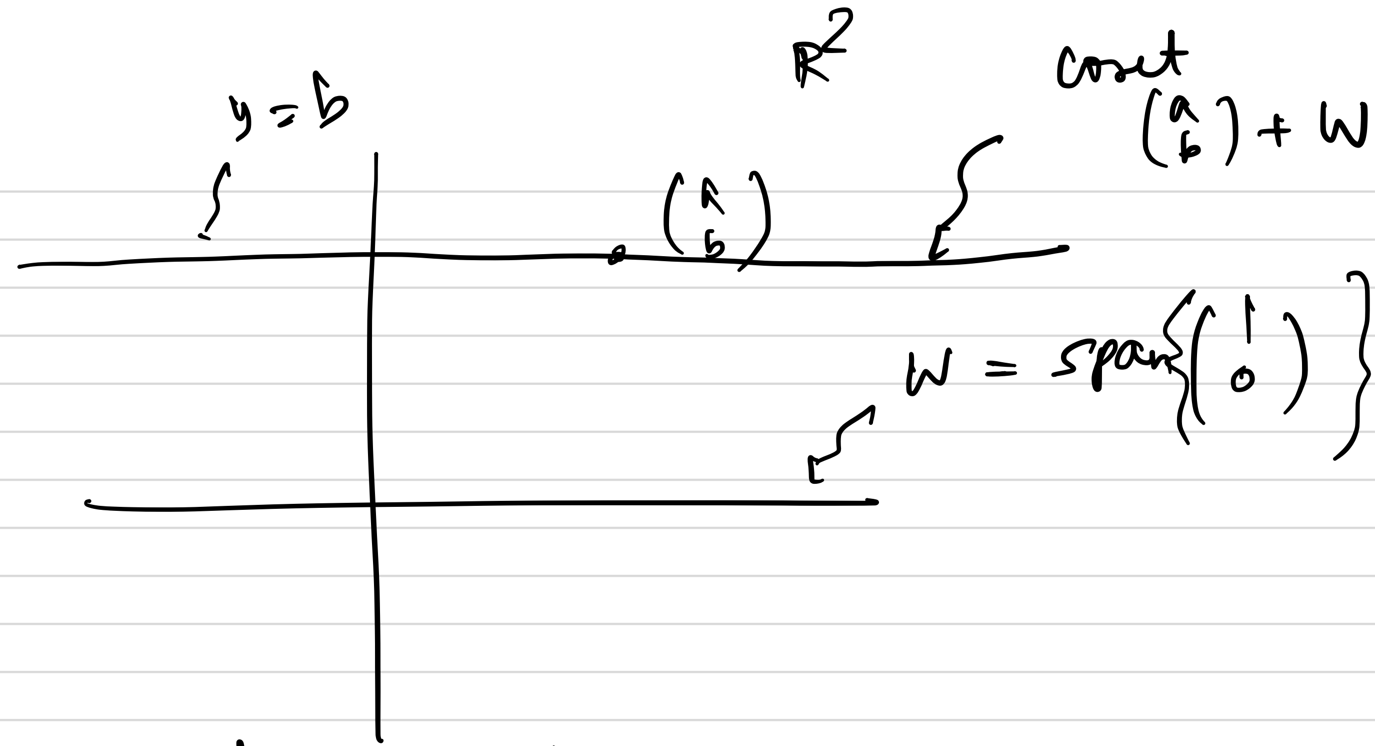
$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

For any $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} + W = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a+t \\ b \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} t' \\ b \end{pmatrix} : t' \in \mathbb{R} \right\}$$



By V/W, to be read as V mod W, we denote the collection of all cosets of W in V.

Then this set V/W is essentially a set of all lines of the type $y = b$.

Then we can define "addition" of two lines in V/W as

$$(y = b_1) + (y = b_2) = (y = b_1 + b_2)$$

and scalar multiplication as

$$\alpha(y = b_1) = (y = b\alpha_1)$$

In the notation of cosets introduced earlier,

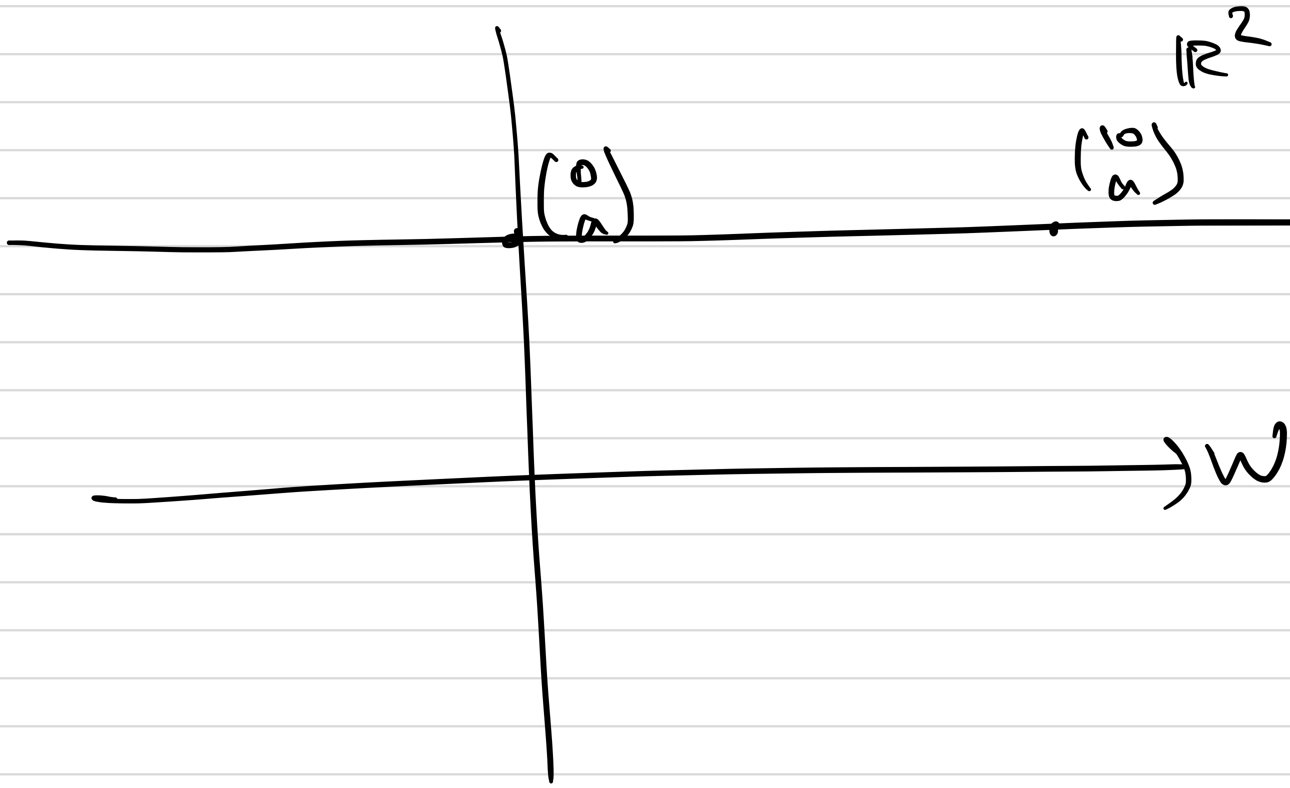
$$\big((\underset{10}{\cancel{0}}, a) + W\big) + \big((0, b) + W\big)$$

$$= (\underset{10}{\cancel{0}}, a+b) + W$$

... addition of cosets.

$$\alpha\big((0, a) + W\big) = (0, \alpha a) + W$$

... scalar multiplication for $\alpha \in \mathbb{F}$.



$$(0, a) + W = (10, a) + W$$