Date
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Lecture 26

Here, A, B, C may be functions of x & y & the type of (1) may be different in different parts of the XY-plane

Ex 1) Laplace eqn $u_{xx} + u_{yy} = 0$

$A = 1, C = 1$ is

$B = 0$. $\therefore AC - B^2 = 1 - 0 = 1 > 0$

Elliptic p.d. eqn

2) 1-dim heat eqn $u_t = c^2 u_{xx}$

$A = c^2$, $B = 0$, $C_1 = 0$ is parabolic p.d. eqn

$AC_1 - B^2 = 0 - 0 = 0$

1-dim

3) the wave eqn

$$u_{tt} = c^2 u_{xx}$$

$A = 1$, $C = -c^2$

$B = 0$

Hyperbolic p.d. eqn

$$AC - B^2 = -c^2 < 0 .$$

Solutions by use of Fourier transforms (F.T)

Q1) By taking the F.T w.r.t the variable x, show that

(a) $\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = (i\alpha)\,\mathcal{F}(u)$,

(b) $\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right) = -\alpha^2 \mathcal{F}(u)$,

(c) $\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\mathcal{F}(u)$.

Sol^n:- By def^n (Integrating by parts)

(i) $\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left(\frac{\partial u}{\partial x}\right)}_{2^{nd} f^n} \underbrace{e^{-i\alpha x}}_{1^{st} f^n} \, dx$

$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\left[e^{-i\alpha x} u \right]_{-\infty}^{\infty}}_{(=0)} + i\alpha \int_{-\infty}^{\infty} u \, e^{-i\alpha x} \, dx \right]$

$= i\alpha \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \, e^{-i\alpha x} \, dx$

$= i\alpha \, \mathcal{F}(u)$,

where, we suppose that $\boxed{u \to 0 \text{ as } x \to \pm\infty.}$

(b) $\mathcal{F}\left(\dfrac{\partial^2 v}{\partial x^2}\right)$.

Let $v = \dfrac{\partial w}{\partial x}$ in part (a), then

$$\mathcal{F}\left(\frac{\partial^2 w}{\partial x^2}\right) = i\alpha\,\mathcal{F}(v)$$

$$= \mathcal{F}\left(\frac{\partial v}{\partial x}\right) = i\alpha\,\underbrace{\mathcal{F}\left(\frac{\partial w}{\partial x}\right)}$$

$$= (i\alpha)^2\,\mathcal{F}(w)$$

$$\Rightarrow \mathcal{F}\left(\frac{\partial^2 w}{\partial x^2}\right) = -\alpha^2\,\mathcal{F}(w)$$

Then if we formally replace w by v, we have

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right) = -\alpha^2 \mathcal{F}(u);$$

provided that u & $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$.

In general, we can show that

$$\mathcal{F}\left(\frac{\partial^n u}{\partial x^n}\right) = (i\alpha)^n \mathcal{F}(u).$$

if $u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}} \to 0$

as $x \to \pm\infty$.

By def^n

(c) $$\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\alpha x}\, dx$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u\, e^{-i\alpha x}\, dx\right]$$

$$\therefore \mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\{\mathcal{F}(u)\}$$

Q1) Use Fourier Transforms

a) to solve the boundary value problem

1-dim Heat eq^n → Type parabolic p.d.eq^n

$$\frac{\partial u(x,t)}{\partial t} = k\,\frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = f(x),$$

$$|u(x,t)| < M, \text{ where}$$

$$-\infty < x < \infty, \ t > 0$$

b) Give a physical interpretation

Sol^n:- Taking the Fourier Transform w.r.t x on both sides of the given p.d. eq^n, we have

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = k\,\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\}.$$

$$\Rightarrow \underline{\frac{d}{dt}}\{\mathcal{F}(u)\} = -k\alpha^2 \mathcal{F}(u) \longrightarrow (1)$$

where we have written the total derivative since $\mathcal{F}(u)$ depends only on t & not on x.

Here,

$$\frac{dy}{dt} = c \cdot y$$

$$\Rightarrow \frac{dy}{y} = c\,dt$$

$\underline{\int}$ (both sides)

$$\ln y = ct + \mathrm{const}$$

$$\Rightarrow y = d e^{ct}$$

$$\therefore \mathcal{F}(u) = c\, e^{-k\alpha^2 t} \longrightarrow (2).$$

$$\therefore \mathcal{F}\{u(x,t)\} = c\, e^{-k\alpha^2 t} \longrightarrow (3)$$

Putting $t=0$ in (3), we see that

$$\boxed{\mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} = c}$$

so that (2) becomes

$$\tilde{F}(u) = \mathcal{F}\{f\}\, e^{-k\alpha^2 t} \longrightarrow (5)$$

We can now apply the convolution theorem (why?)

We know that (Do we?)

$$e^{-k\alpha^2 t} = \mathcal{F}\left\{\sqrt{2\pi}\sqrt{\frac{1}{4\pi k t}}\; e^{-\left(x^2/4kt\right)}\right\} \longrightarrow (6)$$

$$\therefore \mathcal{F}\{u(x,t)\} = \mathcal{F}\{f(x)\}\cdot\mathcal{F}\left\{\sqrt{2\pi}\sqrt{\frac{1}{4\pi k t}}\; e^{-x^2/4kt}\right\}$$

Hence, $u(x,t) = \frac{1}{\sqrt{2\pi}}\,\sqrt{2\pi}\left\{f(x) * \sqrt{\frac{1}{4\pi k t}}\; e^{-\left(x^2/4kt\right)}\right\}$

$$= \frac{1}{\cancel{\sqrt{2\pi}}}\int_{-\infty}^{\infty} f(\omega)\sqrt{\frac{1}{4\pi k t}}\; e^{-\left[\frac{(x-\omega)^2}{4kt}\right]}\, d\omega \longrightarrow (7)$$

If we now change variable from ω to z by

$$\frac{(x-\omega)^2}{4kt} = z^2$$

$$\Rightarrow \frac{x-\omega}{2\sqrt{kt}} = z.$$

$$\Rightarrow x-\omega = 2z\sqrt{kt}$$

$$\Rightarrow \omega = x - 2z\sqrt{kt}.$$

$$\therefore dz = -\frac{d\omega}{2\sqrt{kt}}.$$

Limits of $\int$

when $\omega \to -\infty$, $z \to \infty$

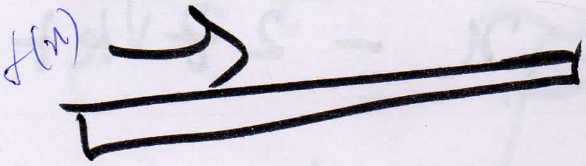
$\omega \to \infty$, $z \to -\infty$

Then (7) becomes,

$$u(x,t) = -\int_{z=\infty}^{-\infty} \frac{1}{2\sqrt{\pi}\sqrt{kt}} e^{-z^2} f(x-2z\sqrt{kt}) \cdot 2\sqrt{kt}\,dz$$

$$\therefore u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} f(x - 2z\sqrt{kt})\, dz$$

(b) The problem is that of determining the temperature in a thin infinite bar whose surface is insulated & whose initial temperature is $f(x)$.



We have to show that

To show.

$$\mathcal{F}\left\{\underbrace{\sqrt{\frac{1}{4\pi kt}}\, e^{-x^2/4kt}}\right\} = \boxed{e^{-k\omega^2 t}}$$

By defn

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\left(x^2/4kt\right)} \left[\cos \omega x - \underbrace{i \sin \omega x}_{(=0)}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\pi kt}} \int_{0}^{\infty} e^{-\left(x^2/4kt\right)} \cos(\omega x)\, dx$$

We know that

$$\left[\int_0^\infty e^{-\alpha x^2} \cos \beta x \, dx = \frac{1}{2}\sqrt{\pi/\alpha}\; e^{-\beta^2/4\alpha}\right]$$

$$= \frac{1}{\sqrt{\pi k t}} \cdot \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{4kt}} \cdot e^{-\frac{\omega^2}{4 \cdot 4kt}}$$

(Solve it?)

Here, $\alpha = \frac{1}{4kt}$, $\beta = \alpha$

$$= \frac{1}{\cancel{\sqrt{\pi k t}}} \cdot \frac{1}{\cancel{2}} \cancel{\sqrt{\pi}} \cdot \cancel{2}\cancel{\sqrt{kt}}\ e^{-\frac{\alpha^2}{4} \cdot 4kt}$$

$$= e^{-\alpha^2 k t}.$$

$$\therefore \mathcal{F}\left\{ \sqrt{2\pi} \cdot \sqrt{\frac{1}{4\pi k t}} \cdot e^{-x^2/4kt} \right\} = e^{-\alpha^2 k t}$$