Book :

Fluid Mechanics by Kundu and Cohen

Chapter 1 : Basic concepts in fluid mechanics

15/7/19

(1) Fluid < Incompressible / Compressible

(i) Liquids are usually incompressible, their volumes don't change with change in pressure.

(ii) Gases are compressible, their volumes change when pressure changes.

(2) Continuum hypothesis

→ We assume that the fluid is uniformly / macroscopically distributed in a region.

(3) Isotropy

→ A fluid is said to be isotropic w.r.t. some properties (say pressure, velocity, density etc) if those properties remain unchanged in all directions.

→ If these properties change, the fluid is said to be anisotropic.

(4) Density :

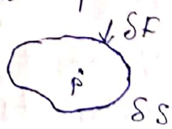
→ The density of a fluid is mass per unit volume. Mathematically,

$$\rho = \lim_{\delta V \to 0} \frac{\delta M}{\delta V}$$

(5) Pressure :

→ Pressure of fluid at a point P is defined as force per unit area at that point P.

$$P = \lim_{\delta S \to 0} \frac{\delta F}{\delta S}$$



(6) Temperature

(7) Thermal conductivity

→ It is given by Fourier's law as;

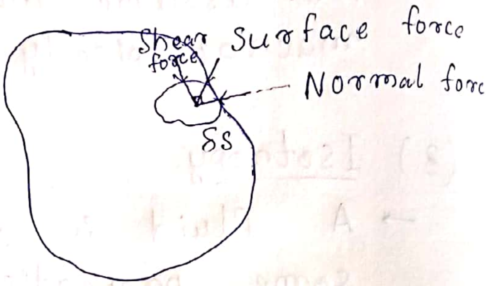
$$q_n \propto \frac{\partial T}{\partial n}$$

$$\Rightarrow q_n = -k \frac{\partial T}{\partial n}$$, when

q_n is the conductive heat flow per unit area, k is the thermal conductivity

Viscous and Inviscous fluid

Classification of forces :



→ An infinitesimal fluid element is acted upon by two types of forces,

(a) body force

(b) surface force

→ Body force is proportional to the mass of the body and surface force is proportional to the surface area of the body

→ Surface force ⇗ Normal force
⇘ Shear force / Shear stress.

(a) A fluid is said to be viscous if both normal and shear forces are present.

(b) A fluid is inviscous / inviscid / non-viscous / frictionless / perfect / ideal if it ~~is~~ doesnot exert any shear stress.

(8) Viscosity

→ Viscosity of a fluid is that property which exhibits a certain resistance to the alteration of form.

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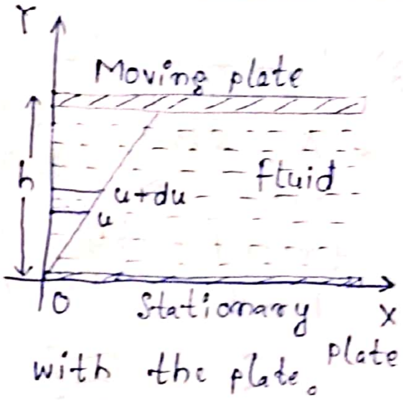

→ The upper plate moves with velocity u in x-direction whereas the lower plate is stationary.

→ The fluid at $y=0$ is at rest and at $y=h$, it is in motion and it is moving with the plate.

Then the shear stress τ is given by:

$$\tau = \mu \frac{du}{dy}$$, where μ is a constant

of proportionality and it is called as viscosity of the fluid.

→ For ideal fluid, shear stress must be zero, which means $\mu = 0$. (Does not exist in nature)

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Two types of forces exist on the fluid element.

(1) Body force: It is distributed over the entire mass or volume of the element. It is expressed as the force per unit mass of the element.
e.g., Gravitational force, electromagnetic force.

(2) Surface force: Force exerted on the fluid element by its surroundings through direct contact with the surface is called surface force.

Surface force < Normal force, Shear force.

(1) Normal force — along normal to the surface area.

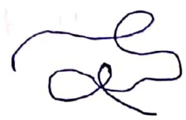
(2) Shear force — along the plane of the surface area.

$$\tau = \mu \frac{du}{dy}.$$

(3) Laminar flow / Streamline flow

→ A flow in which each fluid particle traces out a definite curve and the curves traced by any two different particles don't intersect

(4) Turbulent flow



. Reynold's number ≥ 200

(5) Steady and ~~unsteady~~ unsteady flow

→ If we have a fluid flow in which properties such as pressure, velocity etc are independent of time t, i.e., $\frac{dP}{dt} = 0$, then such flows are called steady flows.

(6) Rotational and irrotational flows

→ A flow in which the fluid is rotating about ~~about~~ an axis.

\# Lagrangian Approach

\# Initially at $t = t_0$, let the fluid particle is at P_0.

The current position is:

$$\left.\begin{aligned} x &= f(x_0, y_0, t_0, t), \\ y &= g(x_0, y_0, t_0, t), \\ z &= h(x_0, y_0, t_0, t). \end{aligned}\right\} \text{Lagrangian approach}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

$$\frac{d\vec{r}}{dt} = \Sigma \frac{dx}{dt}\hat{i}$$

Eulerian Approach

→ We fix a point P in space and look how a fluid particle is behaving at that point.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\frac{d\vec{r}}{dt}, \quad \frac{d^2\vec{r}}{dt^2}$$

Relationship between Lagrangian and Eulerian approach.

(1) Lagrange → Euler

→ Suppose $\phi(x_0, y_0, z_0, t)$ be a physical quantity in Lagrangian description.

$$\phi = \phi(x_0, y_0, z_0, t)$$

Since Lagrangian description gives the current position as $x = f_1(x_0, y_0, z_0, t)$

and $y = f_2(x_0, y_0, z_0, t)$

$z = f_3(x_0, y_0, z_0, t)$

$$\Rightarrow x_0 = g_1(x, y, z, t)$$

$$y_0 = g_2(x, y, z, t)$$

$$z_0 = g_3(x, y, z, t)$$

$$\therefore \phi = \phi(x_0, y_0, z_0, t)$$

$$= \phi\left(g_1(x,y,z,t), g_2(x,y,z,t), g_3(x,y,z,t), t\right)$$

(2) Euler → Lagrange

– Suppose $\psi(x, y, z, t)$ is a physical quantity associated with the flow. $\psi = \psi(x, y, z, t)$.

Eulerian description ⇒ we can write

$$\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = \frac{d\vec{r}}{dt} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$= F_1(x, y, z, t)\hat{i} + F_2(x, y, z, t)\hat{j} + F_3(x, y, z, t)$$

$$\frac{dx}{dt} = u = F_1(x, y, z, t), \quad \frac{dy}{dt} = v = F_2(x, y, z, t), \quad \frac{dz}{dt} = w = F_3(x, y, z, t)$$

$$\Downarrow \quad x(t) - x(t_0) = \int_{t_0}^{t} F_1(x, y, z, t)\,dt$$

Similarly $\quad y(t) - y(t_0) = \int_{t_0}^{t} F_2 \, dt$

and $\quad z(t) - z(t_0) = \int_{t_0}^{t} F_3 \, dt$

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Exercise 1. The velocity components for a 2-dimensional fluid system can be given by Eulerian description

$$u = 2x + 2y + 3t$$

$$v = x + y + t/2$$

Find the displacement of fluid in Lagrangian form.

Solution. $\dfrac{dx}{dt} = 2x + 2y + 3t$

$$\frac{dy}{dt} = x + y + \frac{1}{2}t$$

$$D = \frac{d}{dt}.$$

$$(D-2)x = 2y + 3t \qquad -(4)$$

$$(D-1)y = x + \frac{1}{2}t \qquad -(5)$$

(5) × (D − 2)

$$(D^2 - 3D + 2)y - (D-2)x = \frac{1}{2} - t \qquad -(6)$$

(4) + (6)

$$\Rightarrow (D^2 - 3D + 2)y = 2y + \frac{1}{2} + 2t$$

$$\Rightarrow (D^2 - 3D)y = 2t + \frac{1}{2}$$

$$\therefore \; y(t) = C_1 + C_2 e^{3t} - \frac{1}{3}t^2 - \frac{7}{18}t$$

$$\Rightarrow \frac{dy}{dt} = 3C_2 e^{3t} - \frac{2}{3}t - \frac{7}{18}$$

Now, $\frac{dy}{dt} = x + y + \frac{t}{2}$

$$\Rightarrow x = \frac{dy}{dt} - y - \frac{1}{2}t$$

$$= 2c_2 e^{3t} - c_1 + \frac{t^2}{3} - \frac{7t}{9} - \frac{7}{18}$$

Initially, $t_0 = 0$, $\cancel{x_0 = x_0, y_0 = y}$ $x(t_0) = x_0$, $y(t_0) = y_0$.

$$y_0 = c_1 + c_2$$

$$x_0 = c_1 + 2c_2 - \frac{7}{18}$$

$$\Rightarrow c_1 = \frac{2y_0 - x_0}{3} - \frac{7}{54}$$

$$c_2 = \frac{x_0 + y_0}{3} + \frac{7}{54}$$

$$\therefore x = \frac{2y_0 - x_0}{3} - \frac{7}{54} + 2\left(\frac{x_0 + y_0}{3} + \frac{7}{54}\right) + e^{3t} + \frac{t^2}{3} - \frac{7t}{9} - \frac{7}{18}$$

$$= f(x_0, y_0, t)$$

$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$\cancel{\# \ \vec{\nabla}(f\vec{g}) = \vec{\nabla}f \times \vec{g} + f(\vec{\nabla} \times \vec{g})}$

Gauss divergence theorem

→ Let S be a closed surface bounding a volume V, and $\hat{n}$ be the unit outward normal to S, then

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \operatorname{div} \vec{F} \, dv$$

Stokes's theorem

→ Let S be an open surface bounded by a closed curve $\hat{C}$, then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, ds$$

Green's theorem

→ Let ϕ and ψ be continuously differentiable functions, then

$$\int_C \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds$$

$$= \int_S \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi \right) dx\,dy$$

Cartesian system

→ $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

Spherical polar system

$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta,$

$r \geq 0, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \theta \leq \pi$

Cylindrical system

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$r \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad z \in \mathbb{R}$

Material, local and convective derivative

→ Suppose a fluid particle moves from $P(x, y, z)$ to $Q(x + \delta x, y + \delta y, z + \delta z)$ in time $t + \delta t$. Further let $f(x, y, z)$ be some fluid property associated with the flow. Let the total change of the fluid property from p to Q be δf.

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial t}\delta t$$

$$\Rightarrow \frac{\delta f}{\delta t} = \frac{\partial f}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}\cdot\frac{\delta y}{\delta t} + \frac{\partial f}{\partial z}\cdot\frac{\delta z}{\delta t} + \frac{\partial f}{\partial t}$$

lim $\delta t \to 0$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial f}{\partial t}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) + \frac{\partial f}{\partial t}$$

$$= \nabla f\cdot\vec{q} + \frac{\partial f}{\partial t}$$

$$= \left(\frac{\partial}{\partial t} + q\cdot\vec{\nabla}\right)f = \frac{df}{dt}$$

$$\therefore \underbrace{\frac{Df}{Dt}}_{\text{Material derivative}} = \frac{df}{dt} = \Big(\underbrace{\frac{\partial}{\partial t}}_{\text{Local derivative}} + \underbrace{\cancel{\vec{\nabla}\cdot\vec{q}}\ \vec{q}\cdot\vec{\nabla}}_{\text{Convective term}}\Big)f$$

→ $\frac{D}{Dt}$ (or $\frac{d}{dt}$) is called material derivative and it is spoken as of differentiation following the motion of the fluid.

→ $\frac{\partial}{\partial t}$ is the local derivative and it is associated with time variation at a fixed point. ~~$\vec{q}\cdot\vec{\nabla}$~~

→ $\vec{q}\cdot\vec{\nabla}$ is called convective derivative and it is associated with change in the fluid property f, due to the motion of the fluid.

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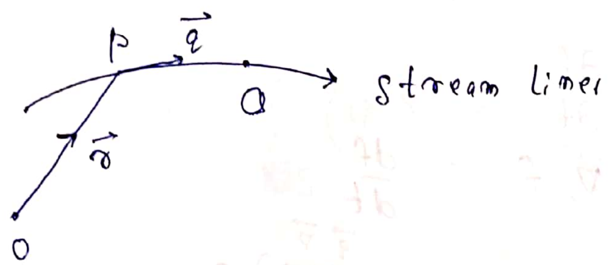
§ Line of flow

→ A line of flow is a line whose direction coincides with the direction of the resultant velocity of the fluid.

§ Streamlines ~~flow~~

→ A streamline is a continuous line of flow drawn in the fluid so that the tangent at every point of it at any instant of time coincides with direction of motion of fluid.



→ Consider ds. as an element of the streamline passing through the point P, s.t. $\vec{OP} = \vec{r}$. Let $\vec{q}$ be the fluid velocity at that point.

→ The direction of the tangent and the direction of velocity are same, i.e., parallel, i.e.,

$$d\vec{r} \times \vec{q} = \vec{0}.$$

$$\Rightarrow (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) \times (u\hat{i} + v\hat{j} + w\hat{k}) = \vec{0}$$

$$\Rightarrow (w\,dy - v\,dz)\,\hat{i} + (u\,dz - w\,dx)\,\hat{j} + (v\,dx - u\,dy)\,\hat{k} = \vec{0}$$

$$\Rightarrow w\,dy - v\,dz = 0 \qquad \Rightarrow \frac{dy}{v} = \frac{dz}{w}$$

$$u\,dz - w\,dx = 0 \qquad \Rightarrow \frac{dx}{u} = \frac{dz}{w}$$

$$v\,dx - u\,dy = 0 \qquad \Rightarrow \frac{dx}{u} = \frac{dy}{v}$$

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$\therefore \dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w}$

↳ streamline of the fluid at any point P.

e.g., The velocity vector $\vec{q}$ is given by
$\vec{q} = x\hat{i} - y\hat{j}$
Determine the equation of streamline.

Sol→ Here, $\vec{q} = x\hat{i} - y\hat{j} + 0.\hat{k}$

$\therefore \vec{q} \times d\vec{r} = \vec{0}$

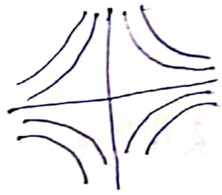
$\Rightarrow (x\hat{i} - y\hat{j}) \times (dx\,\hat{i} + dy\,\hat{j}) = \vec{0}$

$\Rightarrow (x\,dy + y\,dx)\hat{k} = \vec{0}$

$\Rightarrow x\,dy + y\,dx = 0$

$\Rightarrow d(xy) = 0$

$\Rightarrow xy = \text{constant}$



e.g., The velocity of the fluid is given by $\vec{q} = 2x\hat{i} - y\hat{j} - z\hat{k}$. Determine the equation of the streamline passing through the point $(1,1,1)$.

Sol→ $u = 2x,\; v = -y,\; w = -z$

$\Rightarrow \dfrac{dx}{2x} = \dfrac{dy}{-y} = \dfrac{dz}{-z}$

$\Rightarrow \log x + 2\log y = c_1$

$\Rightarrow xy^2 = c_1 \quad \Rightarrow xy^2 = 1$

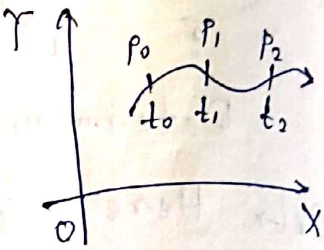
$\log y - \log z = c_2$

$\Rightarrow \dfrac{y}{z} = c_2 \Rightarrow z = \dfrac{y}{c_2} \quad \Rightarrow z = y$

$\therefore \left.\begin{array}{l} xy^2 = 1 \\ z = y \end{array}\right\}$

§ Path lines

→ A curve described in space by moving a fluid particle is known as path line or trajectory, i.e., path line is a line traced by the fluid particle.



→ The path line is obtained by

$$\vec{q} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

e.g., 3 The velocity field at a point P in a fluid is given by $\vec{q} = (x/t, y, 0)$. Obtain the path lines.

Solution. Here $u = \frac{x}{t}$, $v = y$, $w = 0$.

∴ The equation of path line is:

$$\frac{dx}{dt} = \frac{x}{t}, \quad \frac{dy}{dt} = y$$

$$\therefore x = c_1 t, \quad y = c_2 e^{t}.$$

Let at $t = t_0$, $x = x_0$, $y = y_0$

$$\therefore x_0 = c_1 t_0 \Rightarrow c_1 = \frac{x_0}{t_0}$$

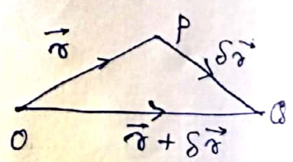
$$y_0 = c_2 e^{t_0} \Rightarrow c_2 = y_0 e^{-t_0}$$

$$\therefore x = \frac{x_0}{t_0} t, \quad y = y_0 e^{(t-t_0)}$$

is the required equation of the path lines.

§ Velocity and acceleration of a fluid particle

→ Let P be the position of the fluid particle at time t and Q be the position at $t + \delta t$,



s.t. $\vec{OP} = \vec{r}$, $\vec{OQ} = \vec{r} + \delta\vec{r} \Rightarrow \vec{PQ} = \delta\vec{r}$

→ The velocity $\vec{q}$ at P is given by!

$$\vec{q} = \lim_{\delta t \to 0} \frac{\vec{r} + \delta\vec{r} - \vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

$$\therefore \vec{q} = \frac{d\vec{r}}{dt}$$

Acceleration of this fluid particle, $\vec{a}$

$$= \frac{d}{dt}(\vec{q}) = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

$$= \frac{d^2\vec{r}}{dt^2}$$

→ $$\frac{d}{dt}(\vec{q}) = \frac{\partial \vec{q}}{dt} + u\frac{\partial \vec{q}}{\partial x} + v\frac{\partial \vec{q}}{\partial y} + w\frac{\partial \vec{q}}{\partial z}$$

e.g., $$\vec{q} = \underbrace{(Ax^2yt)}_{u}\hat{i} + \underbrace{(Bxy^2t)}_{v}\hat{j} + \underbrace{(Cxyz)}_{w}\hat{k}$$

$$\Rightarrow \frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + u\frac{\partial \vec{q}}{\partial x} + v\frac{\partial \vec{q}}{\partial y} + v\frac{\partial \vec{q}}{\partial z}$$

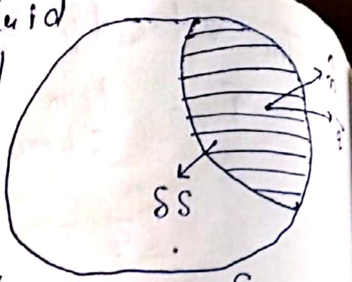
$$= (Ax^2y)\hat{i} + (Bxy^2)\hat{j} \cancel{+ (Cxy)}$$
$$+ \{(2Axyt)\hat{i} + (By^2t)\hat{j} + (Cyz)\hat{k}\}(Ax^2yt)$$
$$+ \{(Ax^2t)\hat{i} + (2Bxyt)\hat{j} + (Cxz)\hat{k}\}(Bxy^2t)$$
$$+ (Cxy\hat{k})(Cxyz)$$

§ Equation of continuity or conservation of mass

→ By continuity / conservation of mass, we mean that the fluid always remains a continuum, i.e., a continuously distributed matter. When a region of fluid contains no source or sink, then the amount of fluid within the region is conserved in accordance with principles of conservation of mass.

∴ Fluid in − fluid out + Source − Sink
= accumulation.

→ Consider an infinitesimal fluid element δs of volume δV and density ρ which is situated at a point whose position vector is $\vec{r}$. Let $\vec{q}$ be the fluid velocity at the element δs, then the normal component of $\vec{q}$ measured outward from the volume V is $\vec{q}\cdot\hat{n}$, where $\hat{n}$ is the unit outward drawn normal and V is the volume of the fluid in the closed region S fixed in space.



→ The mass of the fluid element $= \rho\,\delta v$.
Throughout the motion, the mass of any fluid element must be conserved. hence the mass of any fluid element remains unchanged as it moves about.

This means

$$\frac{D}{Dt}(\rho\,\delta v) = 0 \qquad \text{--- (1)}$$

→ Rate of mass flow across δs (per unit mass) $= \rho(\hat{n}\cdot\vec{q})\,\delta s$

∴ Total rate of mass flow (into V) through V is

$$-\int_S \rho(\hat{n}\cdot\vec{q})\,ds$$

$$= -\left(\int \vec{\nabla}\cdot(\rho\vec{q})\,dv\right)$$

Also, the rate increase of mass within V $= \frac{\partial}{\partial t}\left[\int_V \rho\,dv\right] = \int_V \frac{\partial\rho}{\partial t}\,dv$

~~Now~~. Now, by principle of conservation of mass,

$$\int \frac{\partial \rho}{\partial t} \cdot dv = -\int_V \vec{\nabla} \cdot (\rho \vec{q}) dv$$

$$\Rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{q}) \right] dv = 0 \quad \text{--- (ii)}$$

(ii) is valid for ~~any~~ arbitrary V provided it lies entirely in the fluid.
Hence, the integrand must be zero.

$$\therefore \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{q}) = 0 \quad \text{--- (iii)}$$

This equation is called equation of conservation of mass or equation of continuity.
It is free from any source or sink.

$$\rightarrow \frac{\partial \rho}{\partial t} + \rho(\vec{\nabla} \cdot \vec{q}) + (\vec{\nabla} \rho) \cdot \vec{q} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \vec{q} \cdot \vec{\nabla} \right) \rho + \rho(\vec{\nabla} \cdot \vec{q}) = 0$$

→ For incompressible fluid, ρ is unchanged.

$$\therefore \frac{D\rho}{Dt} = 0$$

$$\therefore \vec{\nabla} \cdot \vec{q} = 0.$$

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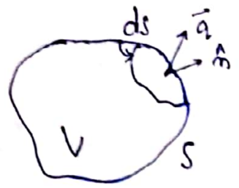
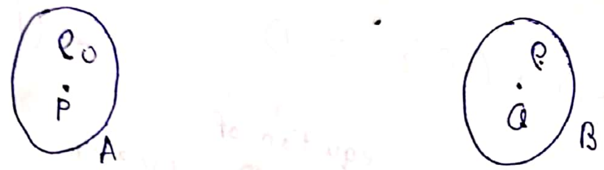
Equation of continuity



→ $\dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{q}) = 0$

→ $\vec{\nabla}\cdot(\vec{q}) = 0$ for incompressible fluids.

Equation of continuity by Lagrange's method



Let A be the fluid region occupied by a fluid at time $t = 0$, and B is the region occupied by the same fluid at time t. Let ρ_0 and ρ be the densities of the fluid at points P and Q ~~who~~ respectively whose co-ordinates are given by (a, b, c) and (x, y, z) respectively.

→ Mass of the fluid at P at $t = 0$ is

$\rho_0\, \delta a\, \delta b\, \delta c$

→ Mass of the fluid at Q at time t

$= \rho\, \delta x\, \delta y\, \delta z$

Now, the total mass ~~of the~~ should be the same in both the regions, i.e.,

$$\iiint_A \rho_0\, \delta a\, \delta b\, \delta c = \iiint_B \rho\, \delta x\, \delta y\, \delta z$$

$$\cancel{\Rightarrow \iiint_A \rho_0\, \delta a\, \delta b\, \delta c = \iiint_B \rho\, \delta x\, \delta y\, \delta z}$$

$$\Rightarrow \iiint_A \rho_0 \, da\,db\,dc = \iiint_B \rho \, dx\,dy\,dz$$

$$\Rightarrow \iiint_A \rho_0 \, da\,db\,dc = \iiint_A \rho \left|\frac{\partial(x,y,z)}{\partial(a,b,c)}\right| da\,db\,dc$$

$$\Rightarrow \iiint_A (\rho_0 - \rho|J|)\, da\,db\,dc = 0,$$

$$J = \frac{\partial(x,y,z)}{\partial(a,b,c)}$$

→ Since A is arbitrary, we obtain,

$$\rho_0 - \rho J = 0$$

$$\Rightarrow \rho_0 = \rho J.$$

This is the required equation of continuity in Lagrangian form.

Lagrangian → Eulerian (Equation of continuity)

→ The equation of continuity in Lagrangian form is:

$$\rho_0 = \rho J, \quad \text{where } J = \frac{\partial(x,y,z)}{\partial(a,b,c)}$$

→ Differentiating w.r.t. t gives:

$$\cancel{\frac{\partial \rho_0}{\partial t} = \rho\frac{\partial J}{\partial t} + J\frac{\partial \rho}{\partial t}}$$

$$\frac{d\rho_0}{dt} = \rho\frac{dJ}{dt} + J\frac{d\rho}{dt} \Rightarrow \rho\frac{dJ}{dt} + J\frac{d\rho}{dt} = 0 \qquad \text{—(i)}$$

→ Now, we change the variable from Lagrangian to Eulerian form.

$$\frac{\partial u}{\partial a} = \frac{\partial}{\partial a}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right),$$

$$\frac{\partial v}{\partial a} = \frac{d}{dt}\left(\frac{\partial y}{\partial a}\right), \quad \frac{\partial w}{\partial a} = \frac{d}{dt}\left(\frac{\partial z}{\partial a}\right)$$

Similarly, ... —(ii)

Again, $\frac{d\rho}{dt} = \left(\frac{\partial}{\partial t} + \vec{q}\cdot\nabla\right)\rho$

$$= \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \quad \text{— (iii)}$$

Since, $J = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix}$

→ We differentiate the Jacobian w.r.t. t,

$$\frac{dJ}{dt} = \begin{vmatrix} \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) & \cdots \\ \frac{d}{dt}\left(\frac{\partial y}{\partial a}\right) & \cdots \\ \frac{d}{dt}\left(\frac{\partial z}{\partial a}\right) & \cdots \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{d}{dt}\left(\frac{\partial y}{\partial a}\right) & \cdots \\ \frac{\partial x}{\partial b} & \frac{d}{dt}\left(\frac{\partial y}{\partial b}\right) & \cdots \\ \frac{\partial x}{\partial c} & \frac{d}{dt}\left(\frac{\partial y}{\partial c}\right) & \cdots \end{vmatrix} + \begin{vmatrix} \cdots \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \vdots & \frac{\partial v}{\partial a} & \vdots \\ \vdots & \frac{\partial v}{\partial b} & \vdots \\ \vdots & \frac{\partial v}{\partial c} & \vdots \end{vmatrix} + \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix}$$

$$\Rightarrow \frac{dJ}{dt} = \frac{\partial(u,y,z)}{\partial(a,b,c)} + \frac{\partial(x,v,z)}{\partial(a,b,c)} + \frac{\partial(x,y,w)}{\partial(a,b,c)}$$

$$= J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

But, $\frac{\partial u}{\partial a} = \left[\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial a}\right]$

Again, $\frac{\partial(u,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \frac{\partial u}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix}$

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$$= \frac{\partial u}{\partial x} \cdot \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} = \frac{\partial u}{\partial x} \cdot J$$

$$\frac{\partial(x, v, z)}{\partial(a, b, c)} = \frac{\partial v}{\partial y} \cdot \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} = \frac{\partial v}{\partial y} \cdot J$$

and $\frac{\partial(x, y, w)}{\partial(a, b, c)} = \frac{\partial w}{\partial z} \cdot J.$

from $-(i)$, $\rho \frac{dJ}{dt} + J \frac{d\rho}{dt} = 0$

$$\Rightarrow \rho J \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + J \frac{d\rho}{dt} = 0$$

$$\Rightarrow \frac{d\rho}{dt} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

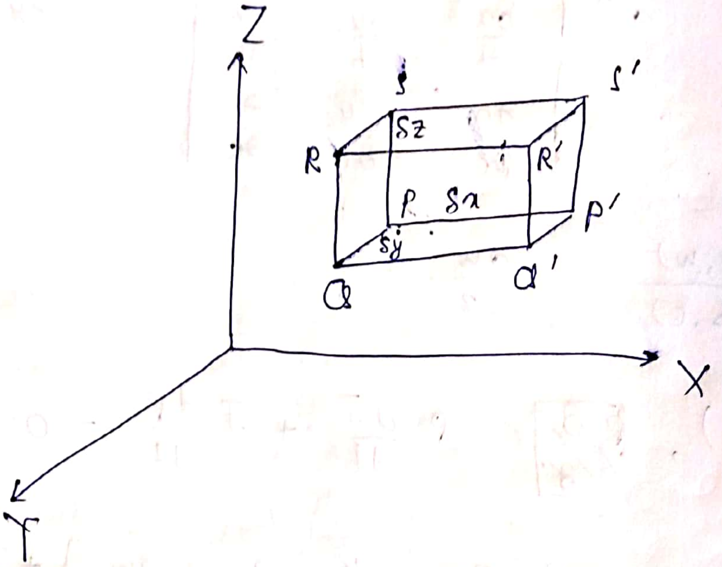
$$\Rightarrow \frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{q} = 0, \quad \vec{q}(u, v, w).$$

This is the required Eulerian equation of continuity.

Equation of continuity –

(1) Cartesian co-ordinate system
(2) Spherical co-ordinate system
(3.) Cylindrical co-ordinate system
(4) Curvilinear co-ordinate system

(1.)



- Consider ρ to be the density of the fluid at $P(x, y, z)$ and $\vec{q}\,(u, v, w)$ be the fluid velocity. Consider a small rectangular parallelopiped in the fluid with length, breadth and height as δx, δy, and δz respectively.

- Mass of the fluid that passes through

$$PQRS = \rho(\delta y\, \delta z)\, u \quad \text{(per unit mass)}$$
$$= f(x, y, z)$$

- Mass of the fluid that passes through the opposite side $= f(x + \delta x, y, z)$

$$= f(x) + (\delta x) f'(x) + \frac{(\delta x)^2}{\lfloor 2} f''(x) + \dots \text{; where}$$

$f'(x) = \frac{\partial}{\partial x} f$, $f''(x) = \frac{\partial^2}{\partial x^2} f$.

→ Mass of excess of fluid within the region in x-direction

$$= -\{ f(x+\delta x, y, z) - f(x,y,z) \}$$

$$= -\{ f(x,y,z) + \delta x\, f'(x) + \frac{(\delta x)^2}{\lfloor 2} f''(x) + \ldots - f(x,y,z) \}$$

$$= -\delta x \frac{\partial f}{\partial x} .$$

$$= -\delta x \frac{\partial}{\partial x} (\rho\, \delta y\, \delta z) u$$

$$= -\frac{\partial}{\partial x} (\rho u\, \delta x\, \delta y\, \delta z)$$

Similarly, mass of excess of fluid within the region in y-direction is :

$$- \frac{\partial}{\partial y} \{ (\rho v)\, \delta x\, \delta y\, \delta z \}$$

and similarly mass of excess of fluid within the region along z-axis is :

$$- \frac{\partial}{\partial z} (\rho w\, \delta x\, \delta y\, \delta z)$$

∴ The excess of fluid along all directions is :

$$- \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] \delta x\, \delta y\, \delta z$$

∴ The total mass inside the parallelopiped is

$$\rho (\delta x\, \delta y\, \delta z)$$

∴ The rate of change / increase of mass inside the parallelopiped

$$= \frac{\partial}{\partial t} (\rho\, \delta x\, \delta y\, \delta z)$$

By conservation law,
rate of mass accumulation = rate of mass in − rate of mass out

$$\Rightarrow \frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x\,\delta y\,\delta z$$

$$\Rightarrow \left[\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{q})\right](\delta x\,\delta y\,\delta z) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{q}) = 0$$

(2) Continuity equation is spherical polar co-ordinate system

$$\rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho q_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho q_\phi) = 0$$

$$\rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{q}) = 0$$

$\rightarrow$ $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

(3) Equation of continuity in cylindrical polar system

$$\rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0$$

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Velocity potential

→ $\vec{q} = -\nabla\phi \Rightarrow \nabla^2\phi = 0$. (for incompressible fluids)

→ If $\vec{q}$ is the fluid velocity at any ~~istant~~ instant of time, then the equations of the streamlines are given by:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad - (1)$$

→ These curves from (1) will intersect the surfaces $u\,dx + v\,dy + w\,dz = 0$ orthogonally.

→ Consider a scalar function $\phi(x, y, z, t)$ at any instant of time t s.t.

$$u\,dx + v\,dy + w\,dz = -d\phi$$

$$= -\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$\Rightarrow u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}, \quad w = -\frac{\partial\phi}{\partial z}$$

$$\Rightarrow u\hat{i} + v\hat{j} + w\hat{k} = -\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial y}\hat{k}\right)$$

$$= -\vec{\nabla}\phi$$

$$\Rightarrow \vec{\nabla}\cdot\vec{q} = -\nabla^2\phi = 0$$

(for incompressible fluids)

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\# $\vec{q} = -\nabla\phi$; ϕ is the potential function.

\# $\vec{q}$, if taken as the velocity of the fluid, the flow is irrotational if $\vec{\nabla}\times\vec{q} = \vec{0}$.

\# <u>Proposition</u> : The necessary and sufficient condition for $\vec{q} = -\nabla\phi$ is $\vec{\nabla}\times\vec{q} = \vec{0}$.

~~Solution~~

Proof. $\vec{q} = -\nabla\phi$,

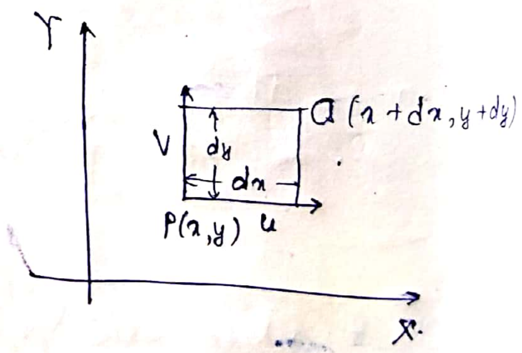
$$\begin{aligned} \text{L.H.S.} &= \vec{\nabla}\times\vec{q} \\ &= \vec{\nabla}\times(-\vec{\nabla}\phi) \\ &= -\{\vec{\nabla}\times(\vec{\nabla}\phi)\} \\ &= \vec{0} \end{aligned}$$

\# <u>Rotational flow</u>:

→ Consider a two dimensional flow in x-y plane and at every point $P(x,y)$ the velocity has two components u and v, i.e., $\vec{q} = (u, v)$. Let Q be the displaced position of P and at any time t. Then the velocity components at Q will be

$$u + \frac{\partial u}{\partial y}dy$$

$$v + \frac{\partial v}{\partial x}dx$$



The velocities will make the fluid rotate about an axis ⊥ to xy - plane. The rotation ω_z about z - axis of the fluid element is determined as:

$$\omega_z = \frac{v + \frac{\partial v}{\partial x}dx - v}{dx} = \frac{\partial v}{\partial x} \rightarrow \text{anticlockwise}$$

$$\omega_z = \frac{u - (u + \frac{\partial u}{\partial y}dy)}{dy} = -\frac{\partial u}{\partial y} \leftarrow \text{clockwise}$$

→ The rotation ω_z is given by mean angular velocity.

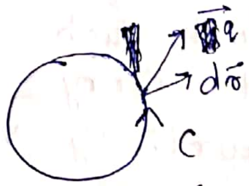
$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

→ $\omega_z \gtrless 0$ → rotational flow

$= 0$ → irrotational flow.

~~Vorticity~~

Circulation :



→ Circulation and vorticity are two primary measures of rotation of a fluid.

$$\oint_C \vec{F} \cdot d\vec{r}$$

→ Circulation is a scalar integral quantity which measures the rotation of a fluid particle along a closed curve.

→ The circulation, C, about a closed curve/contour in a fluid is given by:

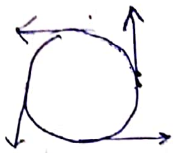
$$C = \oint_C \vec{q} \cdot d\vec{r} > 0 \quad (\text{if anticlockwise})$$

Vorticity

→ The vorticity is a type of pseudovector field ~~field~~ that describes the local spinning motion of the fluid element/continuum near some point. (The tendency of fluid to rotate).

→ The vorticity vector field $\vec{\omega}$ is defined as

$$\vec{\omega} = \vec{\nabla} \times \vec{q}$$



Stokes' theorem states that the circulation about any closed curve/~~[illegible]~~ contour is equal to the integral of the normal component of vorticity over the area enclosed by the closed contour.

Vortex lines

→ A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of vorticity vector $\vec{\omega}$.

→ Let $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ be the vorticity vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point P on a vortex line.

→ Since the vorticity is in the same direction as the tangent at P.

→ $\vec{\omega} \times d\vec{r} = \vec{0}$

$$(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) \times (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = \vec{0}$$

$$\Rightarrow \omega_2\, dz = \omega_3\, dy, \quad \omega_3\, dx = \omega_1\, dz, \quad \omega_1\, dy = \omega_2\, dx$$

$$\Rightarrow \frac{dx}{\omega_1} = \frac{dy}{\omega_2} = \frac{dz}{\omega_3}$$ (equation of vortex line)

e.g., (1) Determine whether the motion

$$\vec{q} = \frac{A(x\hat{j} - y\hat{i})}{x^2+y^2}, \quad A = \text{constant}.$$

is of incompressible fluid, or not.

If it is incompressible then determine streamlines.

Also check whether it is irrotational.

If yes, then find the potential function.

Sol → $\nabla \cdot \vec{q}$

$$= A\left\{-\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right\}$$

$$= 0$$

$$= A\left\{\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(+\frac{y}{x^2+y^2}\right)\right\}$$

$$= A\left\{\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + (+1)\left\{\frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}\right\}\right\}$$

$$= A\left\{\frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right\}$$

∴ The fluid is incompressible.

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{0}, \quad z = C_1$$

$$\Rightarrow \frac{dx}{-\frac{yA}{x^2+y^2}} = \frac{dy}{+\frac{xA}{x^2+y^2}} \Rightarrow x\,dx + y\,dy = 0$$

$\Rightarrow x^2+y^2 = C_2$

⇒ The streamline is given by:

$x^2+y^2 = C_2 \quad z = C_1$

$\nabla \times \vec{q} = \vec{0}$ for a fluid to be irrotational.

$$\nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \left\{\frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right\}\hat{k}$$

$$= \vec{0}$$

$\vec{q} = -\vec{\nabla}\phi$

$$\Rightarrow A\left\{-\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}\right\} = -\frac{\partial\phi}{\partial x}\hat{i} - \frac{\partial\phi}{\partial y}dy\,\hat{j}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = A\frac{y}{x^2+y^2}, \quad \frac{\partial\phi}{\partial y} = -A\frac{x}{x^2+y^2}$$

$$d\phi = A\left\{\frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy\right\}$$

$$= \cancel{A\,d\tan^{-1}\frac{x}{y}} \quad A\,d\left(\tan^{-1}\frac{x}{y}\right)$$

$$\Rightarrow \phi = A\tan^{-1}\frac{x}{y} + B$$

e.g., Determine whether the velocity potential $\phi(x,y,z) = \frac{1}{2}a(x^2+y^2-2z^2)$ satisfies Laplace equation. Also determine streamlines.

Sol → $\nabla^2\phi = \frac{1}{2}\{2+2-4\} = 0$.

$$\vec{q} = -\vec{\nabla}\phi$$

$$= -\frac{1}{2}a\{2x\hat{i} + 2y\hat{j} - 4z\hat{k}\}$$

$$= -a\{x\hat{i} + y\hat{j} - 2z\hat{k}\}$$

$$\frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2za}$$

$$x\,dy - y\,dx = 0 \Rightarrow \frac{dx}{x} - \frac{dy}{y} = 0$$

$$\Rightarrow \ln\frac{x}{y} = \ln c_1 \Rightarrow \frac{x}{y} = c_1$$

$$\frac{2dy}{y} - \frac{dz}{z} = 0 \Rightarrow \ln y^2 - \ln z = \ln c_2$$

$$\Rightarrow y^2 = z c_2.$$

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e.g., Find the vorticity components of a fluid particle whose velocity is given by:

$$\vec{q} = (k_1 x^2 y t)\hat{i} + (k_2 y^2 z t)\hat{j} + (k_3 z t^2)\hat{k}$$

Sol → $\vec{\omega} = \vec{\nabla} \times \vec{q}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ k_1 x^2 y t & k_2 y^2 z t & k_3 z t^2 \end{vmatrix}$$

$$= \hat{i}(-k_2 y^2 t) + \hat{j}(0) + \hat{k}(-k_1 x^2 t) \neq \vec{0}$$

The vortex lines are given by:

$$\vec{\omega} \times d\vec{r} = \vec{0}$$

$$\Rightarrow \frac{dx}{\omega_1} = \frac{dy}{\omega_2} = \frac{dz}{\omega_3}$$

$$\Rightarrow \frac{dx}{-k_2 y^2 t} = \frac{dy}{0} = \frac{dz}{-k_1 x^2 t}$$

$$dy = 0 \Rightarrow y = c_1 \quad \text{— (1)}$$

$$\Rightarrow (k_1 x^2 t)\, dx = k_2 y^2 t\, dz$$

$$\Rightarrow k_1 \frac{x^3}{3} = k_2 y^2 z + c_2 \quad \text{— (2)}$$

e.g., Determine the vortex lines of the flow whose velocity is given by:

$$\vec{q} = \hat{i}(Az - By) + \hat{j}(Bx - Cz) + \hat{k}(Cy - Ax)$$

A, B, C are non-zero constants.

Sol → $\vec{\nabla} \times \vec{q} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Az - By & Bx - Cz & Cy - Ax \end{vmatrix}$$

$$= \hat{i}(C + C) + \hat{j}(A + A) + \hat{k}(B + B)$$

$$= 2C\hat{i} + 2A\hat{j} + 2B\hat{k} \neq \vec{0}$$

for vortex lines, we have:

$$\frac{dx}{\omega_1} = \frac{dy}{\omega_2} = \frac{dz}{\omega_3}$$

$$\Rightarrow \frac{dx}{2C} = \frac{dy}{2A} = \frac{dz}{2B}$$

$$\Rightarrow Ax - Cy = c_1$$

and $By - Az = c_2$ constitute the vortex lines.

Streak lines

→ A streak line is defined as the locus of different particles passing through a fixed point.

→ A streak line is a line on which lie all those fluid elements that at some earlier instant passed through a particular point in space.

→ Consider a fluid particle $P(x_0, y_0, z_0)$ passing through a fixed point $\vec{r}_1\ (x_1, y_1, z_1)$ in the course of time.

By Lagrange description,

$$x_1 = f_1(x_0, y_0, z_0, t)$$

$$y_1 = f_2(x_0, y_0, z_0, t)$$

$$z_1 = f_3(x_0, y_0, z_0, t).$$

Solving for x_0, y_0, z_0, we have:

$$x_0 = F_1(x_1, y_1, z_1, t),$$

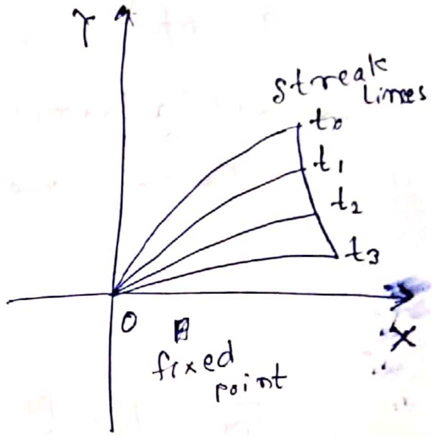
$$y_0 = F_2(x_1, y_1, z_1, t),$$

$$z_0 = F_3(x_1, y_1, z_1, t).$$

Since the streakline is the locus of the positions of the particles which have passed through the fixed point (x_1, y_1, z_1). Therefore the streaklines at any instant of time,

$x = G_1(x_0, y_0, z_0, t)$
$y = G_2(x_0, y_0, z_0, t)$
$z = G_3(x_0, y_0, z_0, t)$

$x = G_1(F_1, F_2, F_3, t)$
$y = G_2(F_1, F_2, F_3, t)$
$z = G_3(F_1, F_2, F_3, t)$



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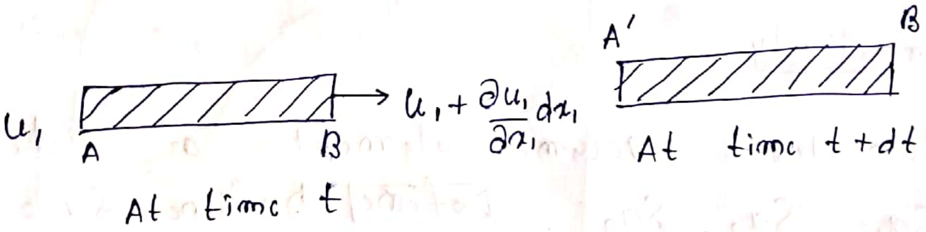
Linear strain rule

→ A study of dynamics of fluid involves determination of forces on a fluid element which depends upon the amount and nature of its deformation.



$$A'B' = AB + BB' - AA'$$

The deformation of the fluid is similar to deformation in solid, where one defines the normal strain as the change in length per unit length of the linear fluid element and the shear strain as the change as of angle 90°.

→ We consider first the normal strain rate of a fluid element in x-direction which is given by,

$$\frac{1}{\delta x_1} \frac{D}{Dt}(\delta x_1) = \frac{1}{dt} \frac{A'B' - AB}{AB}$$

$$= \frac{1}{dt} \cdot \frac{1}{\delta x_1} \left[\delta x_1 + \frac{\partial u_1}{\partial x_1} \delta x_1 dt - \delta x_1 \right]$$

$$= \frac{\partial u_1}{\partial x_1}$$

Similarly, the normal strain along x_2-direction,

$$\frac{1}{\delta x_2} \frac{D}{Dt}(\delta x_2) = \frac{\partial u_2}{\partial x_2}$$

→ The general formula for normal ~~direction~~ strain along x_α

$$= \frac{\partial u_\alpha}{\partial x_\alpha}, \quad \alpha = 1, 2, 3.$$

∴ The total normal strain $= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$

$= \frac{\partial u_\beta}{\partial x_\beta}$, where β is the dummy summation convention.

<u>Alternatively</u>

→ Consider a volume element of sides $\delta x_1, \delta x_2, \delta x_3$. ~~Define~~ Define $\delta V = \delta x_1 \delta x_2 \delta x_3$. Then the volume strain rate per unit volume is:

$$\frac{1}{\delta x_1 \delta x_2 \delta x_3} \cdot \frac{D}{Dt}(\delta x_1 \delta x_2 \delta x_3)$$

$$= \frac{1}{\delta x_1 \delta x_2 \delta x_3} \cdot \left(\frac{D}{Dt}\delta x_1\right) \delta x_2 \delta x_3$$

$$+ \frac{1}{\delta x_1 \delta x_2 \delta x_3} \cdot \left(\frac{D}{Dt}\delta x_2\right) \delta x_1 \delta x_3$$

$$+ \frac{1}{\delta x_1 \delta x_2 \delta x_3} \cdot \left(\frac{D}{Dt}\delta x_3\right) \delta x_1 \delta x_2$$

$$= \frac{1}{\delta x_1} \cdot \left(\frac{D}{Dt}\delta x_1\right) + \frac{1}{\delta x_2}\left(\frac{D}{Dt}\delta x_2\right) + \frac{1}{\delta x_3}\left(\frac{D}{Dt}\delta x_3\right)$$

$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}, \; i = 1, 2, 3$$

In addition to undergoing normal strain rate, a fluid element may also simply deform in shape. The shear strain rate of an element is defined as the rate of decrease/increase of the angle formed by two mutually ⊥ lines on the element.

The deformation of the fluid element due to strain rate tensor is denoted by:

$$\varepsilon_{ij} = e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right); \; i, j = 1, 2, 3$$

$i = 1 \Rightarrow \varepsilon_{ii} \rightarrow$ principal stresses.

6/08/19

→ We used $\frac{D}{Dt}$ because it signifies that a specific fluid particle is followed, so that the volume of the particle is $\propto \frac{1}{\rho}$, i.e., $dv \propto \frac{1}{\rho}$.

$$\rightarrow \frac{1}{\rho}\frac{D\rho}{Dt} = \sum \frac{\partial u_i}{\partial x_i} \qquad \text{—— (1)}$$

→ An another type of continuity equation, which means that the fluid flow has no void in it. The density of the fluid doesn't change appreciably / drastically throughout the flow under several conditions, the most important one is that the flow speed should be small compared to the speed of sound in that medium. This assumption is called Boussinesq approximation.

From (1), $\frac{1}{\rho}\frac{D\rho}{Dt} = 0 \Rightarrow \sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad -(2)$$

$$u(x, y) = \frac{\partial \psi}{\partial y}, \quad v(x, y) = -\frac{\partial \psi}{\partial x} \quad -(3)$$

$\psi(x, y, t) \rightarrow$ streamfunction.

// Streamlines of the flow are given by:

$$\frac{dx}{u} = \frac{dy}{v}$$

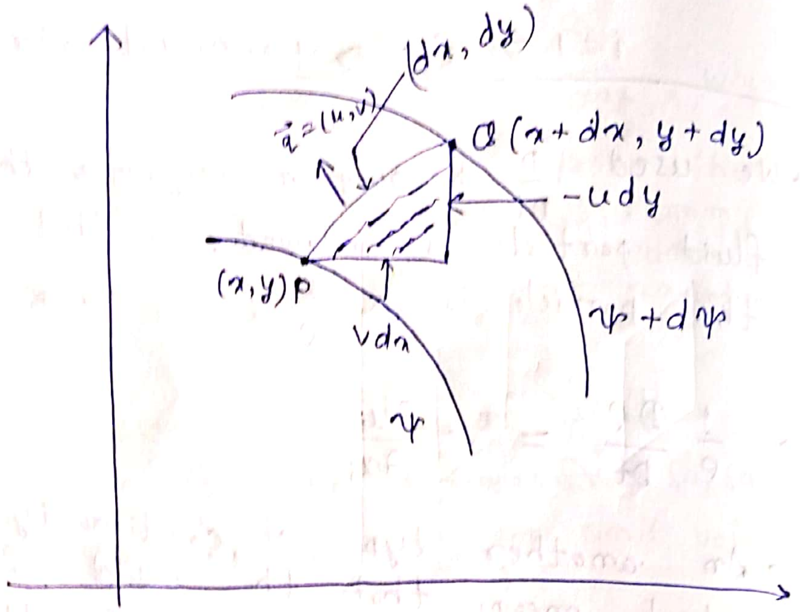
$$\Rightarrow v\,dx = u\,dy$$

$$\Rightarrow \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$\Rightarrow d\psi = 0$$

$$\Rightarrow \psi = \text{constant}$$

$$\therefore \psi(x, y, t) = C.$$



→ Consider an arbitrary fluid element (line element) $(dx, dy) = d\vec{x}$, in the flow. The volume rate of flow across such element will be:

$$v\,dx + (-u\,dy) = v\,dx - u\,dy$$
$$= -\frac{\partial \psi}{\partial x} dx - \frac{\partial \psi}{\partial y} dy$$
$$= -d\psi$$

→ This shows that the volume flow rate between a pair of streamlines is numerically equal to difference of streamlines $d\psi$. The $-$ve sign of ψ is s.t. facing the direction of motion, ψ increases to the left.

Conservation laws

(1) → Gauss divergence theorem

$\cdot \vec{F}(\vec{x}, t) = \vec{F}(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$

$$\iiint_V \operatorname{div} \vec{F}\, dv = \iint_S \vec{F} \cdot \hat{n}\, ds$$

(2.) Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a differentiable function.

$$\frac{d}{dx} \int_c^d f(x, y)\, dy = \int_c^d \frac{\partial f}{\partial x}\, dy$$

$$\frac{d}{dx} \int_{c(x)}^{d(x)} f(x, y)\, dy = \int_{c(x)}^{d(x)} \frac{\partial}{\partial x} f(x, y)\, dy$$

$$+ f(x, d(x))\, d'(x) - f(x, c(x))\, c'(x).$$

(Leibnitz rule of differentiation)

Generalisation:

$$\frac{d}{dt} \int_{V(t)} \vec{F}(\vec{x}, t)\, dv = \iint_{S(t)} (d\vec{S} \cdot \vec{V_s}) \vec{F} + \int_{V(t)} \frac{\partial \vec{F}}{\partial t}\, dV, \quad \text{—(1)}$$

where $\vec{V_s}$ is the velocity of the boundary and $S(t)$ is the surface of $V(t)$.

→ For a material volume $V(t)$, the surface of the fluid moves with the velocity $\vec{u}$, then (1) reduces to:

$$\frac{D}{D(t)} \int_{V(t)} \vec{F}(\vec{x},t)\,dv = \int_V \frac{\partial F}{\partial t}\,dv + \int_S (d\vec{s}\cdot\vec{u})\vec{F} \quad -(2)$$

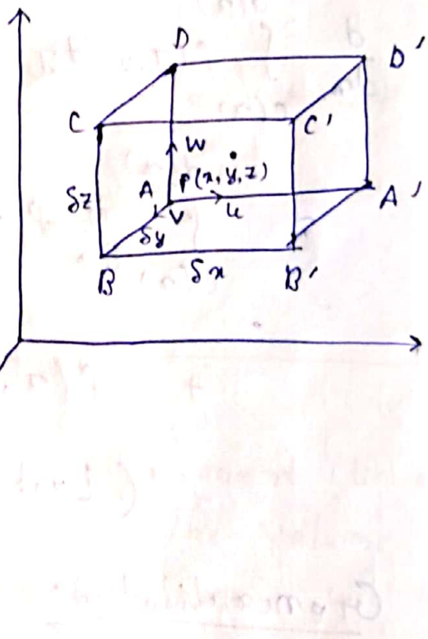
→ This is called Reynolds transportation rule.

→ For fixed volume,

$$\frac{d}{dt} \int_{V(t)} F(x,t)\,dv = \int_V \frac{\partial F}{\partial t}\,dv$$

Euler's equation of motion

→ Let p be the pressure and ρ be the density at point $P(x,y,z)$ in an inviscid (perfect fluid).



→ We consider an elementary fluid element $\delta x \delta y \delta z$.

→ Let $\vec{q} = (u,v,w)$ be the fluid velocity and (X,Y,Z) be the components of the external force per unit mass at a time t at P. Then if $p = f(x,y,z)$. We have:

Force on the plane through P parallel to side ABCD $= p\,\delta y\,\delta z$

and Force on the face ABCD $= f(x-\frac{1}{2}\delta x, y, z)\,\delta y\,\delta z$

$$= \left\{ f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \left(\frac{\delta x}{2}\right)^2 \frac{\partial^2 f}{\partial x^2}\frac{1}{\underline{|2}} - \cdots \right\} \delta y\,\delta z$$

$$= \left\{ f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} \right\} \delta y\,\delta z. \quad -(1)$$

Similarly, force on the face $A'B'C'D'$.

$$= f(x + \tfrac{1}{2}\delta x, y, z)$$

$$= \left(f + \tfrac{1}{2}\delta x \frac{\partial f}{\partial x}\right)\delta y \delta z \quad -(2)$$

$\therefore$ The net force in x-direction due to forces on $ABCD$ and $A'B'C'D'$

$$= \left\{f - \tfrac{1}{2}\delta x \frac{\partial f}{\partial x}\right\}\delta y \delta z - \left\{f + \tfrac{1}{2}\delta x \frac{\partial f}{\partial x}\right\}\delta y \delta z$$

$$= -\frac{\partial f}{\partial x}\delta x \delta y \delta z \quad -(3)$$

The mass of the fluid element is $\rho\,\delta x \delta y \delta z$.
Hence the external force acting on the element along x-axis is $X\rho\,\delta x \delta y \delta z$.
We know that $\frac{Du}{Dt}$ is the total acceleration of the element in x-direction.

→ By Newton's 2nd law of motion,

F_{ext} = mass × acceleration

(sum of components of ext force)

$$\Rightarrow \rho\,\delta x \delta y \delta z \frac{Du}{Dt} = X\rho\,\delta x \delta y \delta z - \frac{\partial f}{\partial x}\delta x \delta y \delta z$$

$$\Rightarrow \rho \frac{Du}{Dt} = \rho X - \frac{\partial f}{\partial x} \quad (4)$$

→ Proceeding similarly,

$$\rho \frac{Dv}{Dt} = \rho Y - \frac{\partial f}{\partial y} \quad (5)$$

$$\rho \frac{Dw}{Dt} = \rho Z - \frac{\partial f}{\partial z} \quad (6)$$

$$\rightarrow \quad \frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial f}{\partial x}$$

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial f}{\partial y}$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial f}{\partial z}$$

$$\therefore \frac{D}{Dt}(u\hat{i} + v\hat{j} + w\hat{k}) = X\hat{i} + Y\hat{j} + Z\hat{k} - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right)$$

$$\Rightarrow \frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}(\vec{\nabla} p)$$

$$\Rightarrow \frac{D\vec{q}}{Dt} + \frac{1}{\rho}(\vec{\nabla} p) = \vec{F} \quad \text{(Euler's equation)}$$

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Equation of perfect fluid (Euler's equation) 08/08/19

→ $\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p$ (for inviscous fluid)

We know:

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p \quad -(I)$$

$$\vec{\nabla}(\vec{q}\cdot\vec{q}) = 2\left[\vec{q}\times(\vec{\nabla}\times\vec{q}) + (\vec{q}\cdot\vec{\nabla})\vec{q}\right]$$

$$\Rightarrow (\vec{q}\cdot\vec{\nabla})\vec{q} = \frac{1}{2}\vec{\nabla}|\vec{q}|^2 - \vec{q}\times(\vec{\nabla}\times\vec{q})$$

$$= \frac{1}{2}\vec{\nabla}(\vec{q}\cdot\vec{q}) - \vec{q}\times(\vec{\nabla}\times\vec{q}) \quad -(II)$$

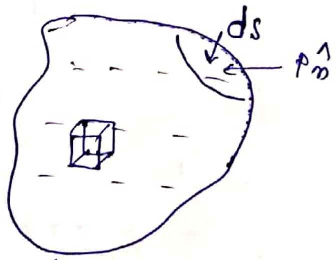
From (I) and (II)

$$\frac{\partial \vec{q}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}(\vec{q}\cdot\vec{q})\right) - \vec{q}\times(\vec{\nabla}\times\vec{q}) = \vec{F} - \frac{1}{\rho}\vec{\nabla}p$$

$$\Rightarrow \frac{\partial \vec{q}}{\partial t} - \vec{q}\times(\vec{\nabla}\times\vec{q}) = \vec{F} - \frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\left(\frac{1}{2}\vec{q}\cdot\vec{q}\right) \quad -(III)$$

→ Euler's equation is no longer true for viscous fluid.

Euler's equation of motion of an inviscous fluid (in vector form / method)



→ We consider an arbitrary closed surface S drawn in the region occupied by an incompressible fluid and it is moving with it, so that it contains some fluid particles at every instant.

→ By Newton's second law,
external force acting on the mass of the fluid = rate of change of linear momentum $-(I)$

→ Now the mass of the fluid under consideration experiences the following forces:
(I) Normal pressure thrusts on the boundary
(II) The external forces $\vec{F}$, per unit mass.

(For viscous fluid, there is one more force, namely viscous force).

→ Let ρ be the density of the fluid particle at P whose volume is δv. Then the mass of the fluid particle is $\rho\delta v$.

Then the mass of the entire region whose volume is V is

$$M = \int_V \rho\, dv$$

→ So the momentum $\vec{M}$ of the volume V is given by $\vec{M} = \vec{q}\int_V \rho\, dv$

$$\therefore \frac{D\vec{M}}{Dt} = \frac{D}{Dt}\vec{q}\int_V \rho\, dv$$

$$= \frac{D\vec{q}}{Dt}\int_V \rho\, dv + \vec{q}\int_V \frac{D}{Dt}\rho\, dv$$

$$= \frac{D\vec{q}}{Dt}\int_V \rho\, dv \quad \text{—(II)} \quad \left(\because \frac{D\rho}{Dt} = 0 \text{ for incompressible fluid}\right)$$

→ If $\vec{F}$ is the external force per unit mass acting on the fluid particle P, then the total force on the region S, $\vec{F}_{total} = \int_V \vec{F}\rho\, dv$ —(III)

Finally, if p be the pressure at a point of surface element δs, then the total pressure on the region S,

$$P_{total} = \int_S p \cdot (-\hat{n})\, ds$$

$$= -\int_V \vec{\nabla} p\, dv \quad (\text{check})$$

$$\int_V \frac{D\vec{q}}{Dt} \rho\, dv = \int_V \vec{F} \rho\, dv - \int_V \vec{\nabla} p\, dv$$

$$\Rightarrow \int_V \left(\frac{D\vec{q}}{Dt} - \vec{F} + \frac{1}{\rho} \vec{\nabla} p \right) dv = 0, \quad \rho \neq 0$$

This is true for arbitrary volume,

$$\Rightarrow \frac{D\vec{q}}{Dt} - \vec{F} + \frac{1}{\rho} \vec{\nabla} p = \vec{0}.$$

This is the required Euler's equation.

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Equation of perfect fluid (Euler's equation)

→ $\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla} p$

We know :

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla} p \quad \text{— (i)}$$

$$\vec{\nabla}(\vec{q}\cdot\vec{q}) = 2\left[\vec{q}\times(\vec{\nabla}\times\vec{q}) + (\vec{q}\cdot\vec{\nabla})\vec{q}\right]$$

$$\Rightarrow (\vec{q}\cdot\vec{\nabla})\vec{q} = \frac{1}{2}\left[\vec{\nabla}(\vec{q}\cdot\vec{q})\right] - \vec{q}\times(\vec{\nabla}\times\vec{q}) \quad \text{— (ii)}$$

→ from relation (i) and (ii),

$$\frac{\partial \vec{q}}{\partial t} + \vec{\nabla}\left(\frac{\vec{q}\cdot\vec{q}}{2}\right) + (\vec{\nabla}\times\vec{q})\times\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla} p$$

We denote $\vec{\nabla}\times\vec{q}$, the vorticity vector by $\vec{\omega}$.

$$\therefore \frac{\partial \vec{q}}{\partial t} + \vec{\nabla}\left(\frac{\vec{q}\cdot\vec{q}}{2}\right) + \vec{\omega}\times\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla} p \quad \text{— (iii)}$$

This equation is known as Lamb's Hydrodynamical equation. This relation remains invariant under co-ordinate transformation.

→ Suppose the force field is conservative.
In a conservative ~~field~~ force field, the work done by the force $\vec{F}$ is independent of the path.

– If $\vec{F} = X\hat{i} + Y\hat{j} + Z\hat{k}$ and $\vec{F}$ is conservative, then $\exists\ \phi$ s.t. $\vec{F} = -\vec{\nabla}\phi$, ϕ is a scalar. (iv)

– From (iii) and (iv), we have :

$$\frac{\partial \vec{q}}{\partial t} + \vec{\nabla}\left(\frac{\vec{q}\cdot\vec{q}}{2}\right) + \vec{\omega}\times\vec{q} + \vec{\nabla}\phi + \frac{1}{\rho}\vec{\nabla} p = \vec{0}$$

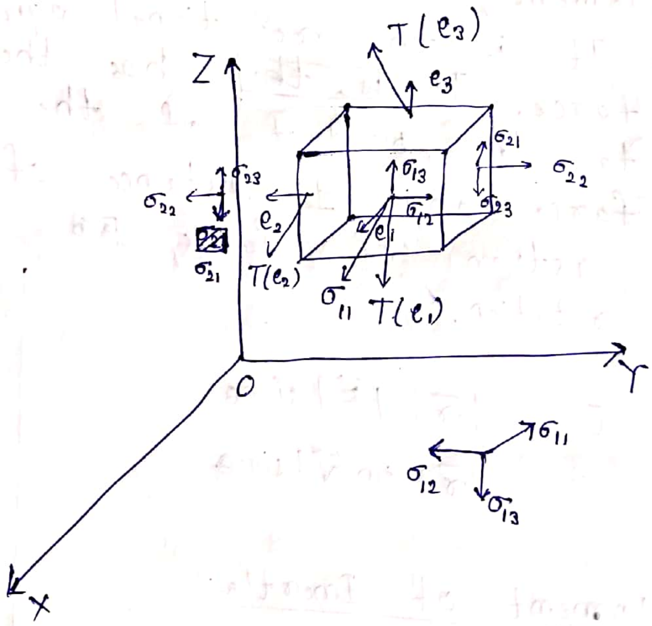
Stress at a point (Cauchy stress tensor)

→ In continuum mechanics, the Cauchy stress tensor, or simply a stress tensor is a second order tensor consisting of nine components, σ_{ij}, where $1 \le i, j \le 3$.

→ This tensor completely defines the stress at a point inside a material in the deformed state, placement or configuration. This relates a unit length ~~vector~~ direction vector $\hat{n}$ to the stress vector $T^{(n)}$.

$$T^{(n)} = \hat{n} \cdot \vec{\sigma}, \quad T_j^{(n)} = \sigma_{ij} n_i, \text{ where } j = 1, 2, 3.$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



Cauchy postulate

→ According to the Cauchy postulate, the stress vector $T^{(n)}$ remains unchanged for all ~~surfaces~~ surfaces passing through a point P and having the same normal $\hat{n}$. This means stress vector is a function of $\hat{n}$ only, and not influenced by the curvature of the internal surface.

Cauchy's fundamental lemma:

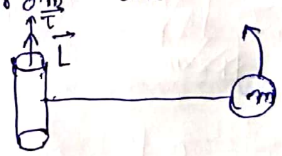
→ The stress vectors acting on the opposite sides of the same surface are equal in magnitude and opposite in direction,

i.e., $T^{(-n)} = -T^{(n)}$.

Torque and angular momentum

→ Moment / Moment of a force / turning effect:

- It is the rotational equivalent of the linear force. It ~~is the~~ has the magnitude equal to the product of the magnitude of the force and $\perp$ distance of the line of action of force from the axis of rotation.



$$\tau = |\vec{r}|\,|\vec{F}|\sin\theta$$

$$L = |\vec{r}|\, m|\vec{v}|\sin\theta$$

Torque $\vec{\tau} = \vec{r} \times \vec{F}$

$= \vec{r} \times m\dfrac{d\vec{v}}{dt}$

$= m\left(\vec{r} \times \dfrac{d\vec{v}}{dt}\right)$

$\vec{L} = \vec{r} \times \vec{p}$

$= \vec{r} \times m\vec{v}$

$= m(\vec{r} \times \vec{v})$

$\dfrac{d\vec{L}}{dt} = m\left\{\vec{r} \times \dfrac{d\vec{v}}{dt} + \dfrac{d\vec{r}}{dt} \times \vec{v}\right\}$

$= \vec{\tau}$.

Moment of Inertia

- Moment of inertia is defined as the ratio of angular momentum L of a system to its angular ~~moment~~ velocity ω.

i.e., $I = \dfrac{L}{\omega}$.

Principle of conservation of angular momentum 19/02/19

Lévi - Civita System

→ This symbol is used quite often in tensor analysis, vector analysis,

$$\varepsilon_{i_1, i_2 i_3 \dots i_n} = (-1)^{p} \varepsilon_{123\dots n}$$, where

$i_1, i_2, \dots, i_n$ are distinct and in ordered fashion $1, 2, 3, \dots, n$ and $\varepsilon_{123\dots n} = 1$.

→ In two dimensions,

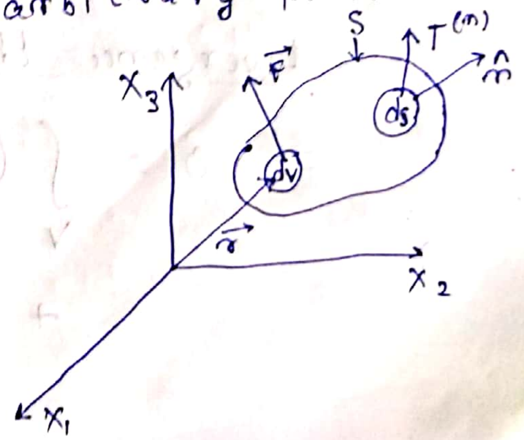
$$\varepsilon_{ij} = 1 \qquad (i,j) = (1,2)$$
$$= -1 \qquad (i,j) = (2,1)$$
$$= 0 \qquad , i = j$$

→ In 3-D,

$$\varepsilon_{ijk} = 1 \text{ , if } (i,j,k) = (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2)$$
$$= -1 \text{ if } (3,2,1), (1,3,2) \text{ or } (2,1,3)$$
$$= 0 \text{ if } (1,1,2), (2,2,3), \dots$$

Principle of conservation of angular momentum

→ Statement: According to the principle of conservation of angular momentum, the moment of a force acting on a body with respect to an arbitrary point is zero



→ Consider a continuum body of volume V and surface area S. Let $\vec{F}$ be the body force and $T^{(n)}$ be the stress / surface force per unit area.

→ $\vec{r} \times (T^{(n)} + \vec{F}) = \vec{0}$

→ $\vec{r} \times T^{(n)} + \vec{r} \times \vec{F} = \vec{0}$

$$\Rightarrow \int_S (\vec{r} \times T^{(n)}) ds + \int_V (\vec{r} \times \vec{F}) dv = \vec{0},$$

where $\vec{r} = x_j e_j$ is the position vector.

$\vec{a} \times \vec{b}$

$= \varepsilon_{ijk} a_j b_k$

$i = 1$

$\Rightarrow \varepsilon_{ijk} a_j b_k$

$= \varepsilon_{123} a_2 b_3$

$+ \varepsilon_{132} a_3 b_2$

$$\rightarrow 0 = \int_S \varepsilon_{ijk} x_j \sigma_{mk} n_m ds + \int_V \varepsilon_{ijk} x_j F_k dv \quad \because T_k^{(n)} = \sigma_{mk} n_m$$

$m = 1, 2, 3$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

n_m = mth component of normal.

<u>Divergence theorem</u>

$$\Rightarrow 0 = \int_V (\varepsilon_{ijk} x_j \sigma_{mk})_{,m} dv + \int_V \varepsilon_{ijk} F_k x_j dv = 0$$

$\{\varepsilon_{ijk}\, x_j\, (\sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3)\}$ $\cancel{= F_k^{(n)}}$

20/08/19

$$\Rightarrow \int_V \left[\varepsilon_{ijk}\, x_{j,m}\, \sigma_{mk} + \varepsilon_{ijk}\, x_j .\, \sigma_{mk,m}\right] dV$$

$$+ \int_V \varepsilon_{ijk}\, x_j\, F_k\, dV = 0$$

(m - co-ordinate system is (x_1, x_2, x_3)

$\frac{\partial}{\partial x_1}(x_j)$.

$\frac{\partial}{\partial x_m}(x_j) = x_{j,m}$
$= \delta_{jm}$

$= 1,\ j = m$

$(x_j)_{,m} = x_{j,m}$ $\quad = 0,\ j \neq m$)

$(\sigma_{mk})_{,m} = \sigma_{mk,m}$

$\frac{\partial}{\partial x}(f g) = \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x}$

$\nabla f = \boxed{\cancel{f}}\ m\ (f)_{,x}$
or $\overline{f_x}$ or ∇f

$$\Rightarrow \int_V \varepsilon_{ijk}\, x_{j,m}\, \sigma_{mk}\, dv + \int_V \left(\varepsilon_{ijk}\, (\sigma_{mk,m} + F_k) x_j\right) d$$
$$= 0 \quad -(2)$$

→ By Cauchy's law of motion,

$$\sigma_{mk,m} + F_k = 0 \quad \text{——} \quad (3)$$

from (2) and (3), we have!

$$\int_V \varepsilon_{ijk}\, x_{j,m}\, \sigma_{mk}\, dV = 0 .$$

$$\Rightarrow \int_V \varepsilon_{ijk}\, \sigma_{jk}\, dV = 0 .$$

$\therefore V$ is arbitrary, $\varepsilon_{ijk}\,\sigma_{jk} = 0$

$$\Rightarrow \varepsilon_{1jk}(\sigma_{jk})\hat{e}_1 + \varepsilon_{2jk}(\sigma_{jk})\hat{e}_2 + \varepsilon_{3jk}(\sigma_{jk})\hat{e}_3 = \vec{0}$$

~~Similarly~~

$$\Rightarrow (\sigma_{23} - \sigma_{32})\hat{e}_1 + (\sigma_{31} - \sigma_{13})\hat{e}_2 + (\sigma_{12} - \sigma_{21})\hat{e}_3 = \vec{0}$$

$$\Rightarrow \sigma_{23} = \sigma_{32},\quad \sigma_{31} = \sigma_{13},\quad \sigma_{12} = \sigma_{21}.$$

($\varepsilon_{1jk}\,\sigma_{jk} = 0$

$\varepsilon_{1jk}\,\sigma_{jk} = 0$

$\Rightarrow \varepsilon_{123}\,\sigma_{23} + \varepsilon_{132}\,\sigma_{32} = 0$

$\Rightarrow \sigma_{23} - \sigma_{32} = 0$

$\Rightarrow \sigma_{23} = \sigma_{32}$)

Similarly $\sigma_{31} = \sigma_{13}$, $\sigma_{12} = \sigma_{21}$

$\therefore \sigma_{ij} = \sigma_{ji}$ $\forall\, i, j \le 3$.

Hence, σ is symmetric.

Cauchy's first law of motion :

→ According to the principle of conservation of linear momentum, if a continuum body is in static equilibrium, it can be demonstrated that the components of Cauchy stress tensor in every material point in the body satisfy the equilibrium equation,

$$\sigma_{ji,j} + f_i = 0$$

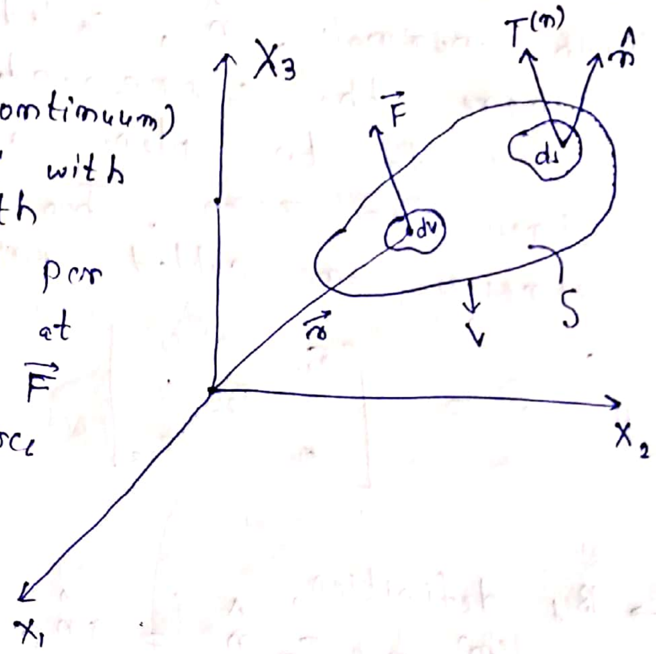
i.e., $\sigma_{ji,j} = -f_i$.

In particular; for a hydrostatic fluid in equilibrium, we have:

$$\sigma_{ij} = -p\,\delta_{ij}.$$

proof.

Consider a body (continuum) occupying volume V with surface area S, with surface forces $T^{(n)}$ per unit area acting at every point and $\vec{F}$ be the body force per unit volume.



→ If the body is in equilibrium, then the sum of external forces must be zero.

$$\therefore \int_S T_i^{(n)} \, ds + \int_V F_i \, dv = 0.$$

$$\Rightarrow \int_S \sigma_{ji} n_j \, ds + \int_V F_i \, dv = 0$$

$$\Rightarrow \int_V (\sigma_{ji})_{,j} \, dV + \int_V F_i \, dv = 0$$

$$\Rightarrow \int_V (\sigma_{ji,j} + F_i) \, dv = 0$$

$\therefore$ V is arbitrary, we have!

$$\sigma_{ji,j} + F_i = 0.$$

$T^{(n)} = \sigma n$

$T_i^{(n)} = \sigma_{ji} n_j$

$i = 1, 2, 3.$

e.g.,

$T^{(n)} = \begin{pmatrix} f \\ g \end{pmatrix}$

$\sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$\hat{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T^{(n)} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$f = 1$, $g = 2$

$f = \sigma_{11} n_1 + \sigma_{12} n_2$

Principal stress, stress invariants and deviatoric stress tensor

→ At every point in a stressed body, there are at least three planes, with normals, $\hat{n}$, called principal directions, where the corresponding stress vector is $\perp$ to the plane and the shear stresses are zero. Then the three stresses are called principal stresses.

$$\sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

→ By definition,

$$T^{(n)} = \sigma_n \hat{n} = \lambda \hat{n}.$$

Also, $T_i^{(n)} = \sigma_{ij} n_j$ and $n_i = \delta_{ij} n_j$.

$$\sigma_{ij} n_j = \cancel{\delta_{ij} n_j} \lambda n_i = \lambda \delta_{ij} n_j$$

$$\Rightarrow (\sigma_{ij} - \lambda \delta_{ij}) n_j = 0 \quad \text{— (1)}$$

(1) is a homogeneous equation in n_j.
This equation has non-trivial solution.

$$\det(\sigma_{ij} - \lambda \delta_{ij}) = 0.$$

$$\Rightarrow \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0$, where

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}, \quad = tr([\sigma_{ij}])$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 = \frac{1}{2}\left[(tr[\sigma_{ij}])^2 - tr[\sigma_{ij}]^2\right]$$

$$I_3 = \det[\sigma_{ij}] \quad \boxed{\cancel{\sigma_{11}\sigma_{22}\sigma_{33}}}$$

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$$\left.\begin{aligned} \sigma_{11}' &= \max\{\lambda_1, \lambda_2, \lambda_3\} \\ \sigma_{33}' &= \min\{\lambda_1, \lambda_2, \lambda_3\} \\ \sigma_{22}' &= I_1 - \sigma_{11}' - \sigma_{33}' \end{aligned}\right\} \rightarrow \text{principal stresses}$$

$$\sigma = \begin{bmatrix} \sigma_{11}' & 0 & 0 \\ 0 & \sigma_{22}' & 0 \\ 0 & 0 & \sigma_{33}' \end{bmatrix}$$

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Stress Deviative tensor

- The stress tensor σ_{ij} can be expressed as the sum ~~of~~ of two other stress tensors:

(1) mean hydrostatic stress tensor or volumetric stress tensor or mean normal stress tensor, $\pi \delta_{ij}$ which tends to change the volume of the stressed body, and

(2) a deviatoric component called stress derivative tensor, S_{ij}, which tends to distort it. So,

$$\sigma_{ij} = \pi \delta_{ij} + S_{ij}, \text{ where}$$

π is mean stress given by!

$$\pi = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

$$= \frac{1}{3}\sigma_{KK} = \frac{1}{3} I_1$$

e.g., At a point P, the stress tensor is given by

$$\sigma = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix} \text{ and}$$

the normal is!

$$\hat{n} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}.$$

Then find the stress vector at the point P.

Soln $\vec{T}^{(n)} = \sigma . \vec{n}$

$$= \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -10/3 \end{bmatrix}$$

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$$\vec{T}^{(n)} = \begin{bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{bmatrix} = \begin{bmatrix} 4 \\ -10/3 \\ 0 \end{bmatrix}$$

or $\vec{T}^{(n)} = 4\hat{i} - \frac{10}{3}\hat{j} + 0\hat{k}$

e.g., Find out the principal stresses of

$$\sigma = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Sol→ The principal stresses are

$$|\sigma - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2-4) + 1(\lambda+2) + 1(\lambda+2) = 0$$

$$\Rightarrow (\lambda+2)\left((\lambda-2)(3-\lambda) + 2\right) = 0$$

$$\Rightarrow (\lambda+2)(\lambda^2 - 5\lambda + 4) = 0$$

$$\Rightarrow (\lambda+2)(\lambda-4)(\lambda-1) = 0$$

$$\Rightarrow \lambda = 1, -2, 4.$$

$\cancel{\sigma_{11} = (-2),}$ $\cancel{\lambda_{11} = 4,}$

$$\sigma_{11} = 4,$$

$$\sigma_{33} = -2, \quad \sigma_{22} = 1$$

$$(\sigma - \lambda I)\vec{n} = \vec{0} \qquad (n_1, n_2, n_3).$$

Bernoulli's equation

→ When the velocity exists (so that the flow / motion of the fluid may be irrotational) and external forces are given in the form of a potential function, then the equations of motion can always be integrated.

~~→ for any ϕ, $\vec{F}$~~

~~→ $\vec{F} = -\vec{\nabla}\phi$~~

~~$\Rightarrow f_1 = -\frac{\partial \phi}{\partial x}, \; f_2 = -\frac{\partial \phi}{\partial y}, \; f_3 = -\frac{\partial \phi}{\partial z}$.~~

→ $\vec{q} = -\vec{\nabla}\phi$

$$\Rightarrow u = -\frac{\partial \phi}{\partial x}, \; v = -\frac{\partial \phi}{\partial y}, \; w = -\frac{\partial \phi}{\partial z}. \qquad \text{— (1)}$$

~~F~~.

→ $\vec{F} = -\vec{\nabla} V$

$$\Rightarrow f_1 = -\frac{\partial V}{\partial x}, \; f_2 = -\frac{\partial V}{\partial y}, \; f_3 = -\frac{\partial V}{\partial z}. \qquad \text{— (2)}$$

→ $$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right) = -\frac{\partial^2 \phi}{\partial y \partial x}$$

$$= -\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right)$$

$$= \frac{\partial v}{\partial x}.$$

→ $$\frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \; \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}. \quad \boxed{\cancel{[(1), (2)]}}. \qquad \text{— (3)}$$

→ From Euler's equation, we have:

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p$$

$$\Rightarrow \left\{ \frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} \right) = \vec{F} - \frac{1}{\rho}\vec{\nabla}p$$

$$\Rightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad -\boxed{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}, \quad -\boxed{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}. \quad -\boxed{3}$$

Now, we use (1), (2), (3) to get!

$$-\frac{\partial^2\phi}{\partial t\,\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$-\frac{\partial^2\phi}{\partial t\,\partial y} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$-\frac{\partial^2\phi}{\partial t\,\partial z} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}.$$

$$\therefore -\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right) + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right) + u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right) + u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Therefore,

$$-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right) = X - \frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right) = Y - \frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right) = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}.$$

$$\Rightarrow -\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right)dx + \frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2)dx = \underset{\left(-\frac{\partial V}{\partial x}\right)}{\overset{\shortparallel}{X}}\,dx - \frac{1}{\rho}\frac{\partial p}{\partial x}\cdot\partial x \quad -(i)$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)dy + \frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2+w^2)dy = \underset{\left(-\frac{\partial V}{\partial y}\right)}{\overset{\shortparallel}{Y}}\,dy - \frac{1}{\rho}\frac{\partial p}{\partial y}\partial y \quad -(ii)$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)dz + \frac{1}{2}\frac{\partial}{\partial z}(u^2+v^2+w^2)dz = \underset{\left(-\frac{\partial V}{\partial z}\right)}{\overset{\shortparallel}{Z}}\,dz - \frac{1}{\rho}\frac{\partial p}{\partial z}\partial z \quad -(iii)$$

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(i) + (ii) + (iii)

$$\Rightarrow -d\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}d(u^2+v^2+w^2)$$

$$= -dV - \frac{1}{\rho}dp$$

$$\Rightarrow \frac{\partial \phi}{\partial t} - \frac{u^2+v^2+w^2}{2} = V + \int \frac{dp}{\rho} + \psi(t)$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{u^2+v^2+w^2}{2} + V + \int \frac{dp}{\rho} = \psi(t)$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{\vec{q}^{\,2}}{2} + V + \int \frac{dp}{\rho} = \psi(t).$$

(general)

This is Bernoulli's equation of motion for an irrotational fluid in conservative force field.

Special case: ρ = constant

$$\rightarrow -\frac{\partial \phi}{\partial t} + \frac{\vec{q}^{\,2}}{2} + V + \frac{p}{\rho} = \psi(t).$$

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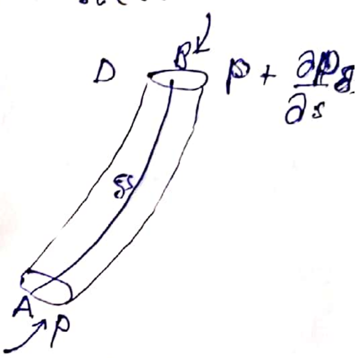
Bernoulli's theorem for steady flow with no ~~[illegible]~~ velocity potential and conservative force field

→ **Statement:** When a motion is steady potential doesn't exist, and the velocity potential doesn't exist, then

$$\frac{\vec{q}^2}{2} + V + \int \frac{dP}{\rho} = \text{constant},$$

where P is the pressure, $\vec{q}$ is the velocity, V is the force potential from which the external forces are derivable.

Proof. Let us consider a streamline AB in the fluid. Let δs be an element of this ~~streamli~~ streamline and CD be a small cylinder of cross-sectional area a and δs as its axis.



If $\vec{q}$ be the fluid velocity and S be the component of the external force per unit mass in the direction of streamline, then by Newton's law,

$$\rho a \cdot \delta s \frac{Dq}{Dt} = \rho a \,\delta s \cdot S + Pa - \left(P + \frac{\partial P}{\partial s} ds\right)a.$$

$$\Rightarrow \rho a \,\delta s \frac{Dq}{Dt} = \rho a \,\delta s \cdot S + Pa - Pa - a\frac{\partial P}{\partial s} ds$$

$$\Rightarrow \rho \frac{Dq}{Dt} = \rho S - \frac{\partial P}{\partial s}$$

$$\Rightarrow \frac{Dq}{Dt} = S - \frac{1}{\rho}\frac{\partial P}{\partial s}.$$

$$\Rightarrow \underset{(=0)}{\frac{\partial q}{\partial t}} + q\frac{\partial q}{\partial s} = S - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

$$\Rightarrow q\frac{\partial q}{\partial s} = S - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

$$\Rightarrow q\,\partial q = S\,\partial s - \frac{1}{\rho}\partial p$$

$$\Rightarrow \int q\,\partial q = \int S\,\partial s - \int \frac{1}{\rho}\partial p$$

$$\Rightarrow \frac{1}{2}\int \partial(q^2) = -V - \int\frac{\partial p}{\rho} = \text{constant} \qquad \left(S = -\frac{\partial V}{\partial s}\right)$$

$$\Rightarrow \frac{1}{2}q^2 + V + \frac{\partial p}{\rho} = \text{constant}.$$

$$\boxed{\begin{aligned} S &= g\frac{\partial h}{\partial x} \\ &= \frac{\partial}{\partial x}(gh) \\ V &= gh \end{aligned}}$$

02/09/19

1. Cartesian co-ordinates

→ $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ in 2D

$\equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ in 3D

2. Polar co-ordinates in 2D

→ Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a function.

$u(x,y) = u(r\cos\theta, r\sin\theta)$, $x = r\cos\theta$, $y = r\sin\theta$ is the substitution,

where $r > 0$, $0 \le \theta < 2\pi$.

$r^2 = x^2 + y^2$

$2r\frac{\partial r}{\partial x} = 2x$

$r\frac{\partial r}{\partial x} = x$

$r\frac{\partial r}{\partial y} = y$

$\theta = \cot^{-1}\frac{x}{y}$

$\frac{\partial\theta}{\partial x} = -\frac{1}{1+\frac{x^2}{y^2}}\,\frac{1}{y} = -\frac{y}{x^2+y^2}$

$\frac{\partial\theta}{\partial y} = -\frac{y^2}{x^2+y^2}\cdot -\frac{x}{y^2} = +\frac{x}{x^2+y^2}$

$\cancel{r = \frac{x}{\cos\theta}}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\times\frac{x}{r} - \frac{\partial u}{\partial\theta}\cdot\frac{y}{r^2}$$

$$= \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{y}{r} + \frac{\partial u}{\partial\theta}\cdot\frac{x}{r^2}$$

$$= \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = (\sin\theta + \cos\theta)\frac{\partial u}{\partial r} + \frac{1}{r}(\cos\theta - \sin\theta)\frac{\partial u}{\partial\theta}.$$

3. Spherical polar co-ordinates

→ $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$.

→ $u(x,y,z) = u_1(r,\theta,\phi)$.

$r > 0$, $0 \le \theta < 2\pi$, $-\pi \le \theta < \pi$.

→ $x^2 + y^2 + z^2 = r^2$

→ $\phi = \tan^{-1}\frac{x}{y}$,

→ $\theta = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right)$.

(4) Cylindrical co-ordinate system

→ $x = r\cos\theta,\quad y = r\sin\theta,\quad z = z.$

$u(x,y,z) = u_1(r, \theta, z).$

$\theta = \tan^{-1}\left(\frac{y}{x}\right),\quad r = x^2+y^2,\quad z = z.$

The energy equation

Statement: The rate of change of total energy (kinetic, potential, intrinsic) of any portion of a compressible ~~fluid~~ inviscid fluid as it moves about is equal to the rate at which work is being done by pressure on the boundary provided the forces are conservative. The potential due to ~~external~~ extraneous forces is supposed to be independent of time.

3/08/19

Definitions :

(1) Internal energy :

→ In thermodynamics, the internal energy of a system is the total energy contained within the system. It is the energy necessary to create/prepare the system in a given state.

(2.) Potential energy :

→ In physics the potential energy is the energy held by an object because of its position relative to other objects, stresses within itself, or other factors.

(3) Kinetic energy :

→ The energy due to the motion of the body $= \frac{1}{2} m v^2$.

$$\left(\frac{d}{dt} (K.E. + P.E. + E.) = \text{Power} \right)$$

Principal of energy :

→ Let us consider any arbitrary closed surface/region S occupied by an inviscid fluid and let V be the volume of the fluid with ~~S~~ S. Let ρ be the density of the fluid P within S with volume dv surrounding P. Let $\vec{q}(\vec{r}, t)$ be the velocity of the particle at P. Then the Euler's equation is:

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho} \vec{\nabla} P \quad \text{--- (1)}$$

→ Let the external forces be conservative so that $\exists$ a ~~particle~~ potential Ω independent of time s.t.

$$\vec{F} = -\vec{\nabla}\Omega \; , \; \frac{\partial \Omega}{\partial t} = 0 \quad \text{--- (2)}$$

Now taking dot product both side in (1), we have:

$$\vec{q}\cdot\frac{d\vec{q}}{dt} = (-\vec{\nabla}\Omega\cdot\vec{q}) - \frac{1}{\rho}\vec{\nabla}p\cdot\vec{q}$$

$$\text{or } \rho\vec{q}\cdot\frac{d\vec{q}}{dt} = -\vec{q}\cdot\vec{\nabla}p - \rho\vec{\nabla}\Omega\cdot\vec{q} \quad -(3)$$

For Ω, $\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial t} + \vec{q}\cdot(\vec{\nabla}\Omega)$

$$= \vec{q}\cdot(\vec{\nabla}\Omega) \quad -(4)$$

From (3) and (4), we have:

$$\rho\vec{q}\cdot\frac{d\vec{q}}{dt} + \rho\frac{d\Omega}{dt} = -\vec{q}\cdot\vec{\nabla}p$$

$$\Rightarrow \frac{\rho}{2}\frac{d}{dt}(\vec{q}^{\,2}) + \rho\frac{d\Omega}{dt} = -\vec{q}\cdot\vec{\nabla}p$$

$$\Rightarrow \int_V \rho\frac{d}{dt}\left(\frac{1}{2}\vec{q}^{\,2}\right)dv + \int_V \frac{d}{dt}(\rho\Omega)dv = -\int_V \vec{q}\cdot\vec{\nabla}p\,dv$$

$$\boxed{\cancel{\Rightarrow \frac{1}{2}\int\frac{d}{dt}}}$$

$$\Rightarrow \frac{d}{dt}\int\frac{1}{2}\rho\vec{q}^{\,2}dv + \frac{d}{dt}\int_V\rho\Omega\,dV = -\int_V\vec{q}\cdot\vec{\nabla}p\,dv \quad -(5)$$

→ Let us ~~demod~~ denote K.E. by T, P.E. by $\mathbb{P}$, Internal energy $= \int_V \rho E dv$, where E is the internal energy per unit mass.

$$\because \vec{\nabla}\cdot(p\vec{q}) = \vec{q}\cdot\nabla\vec{p} + p\vec{\nabla}\cdot\vec{q}$$

$$\Rightarrow \vec{q}\cdot\nabla\vec{p} = \vec{\nabla}\cdot(p\vec{q}) - p(\vec{\nabla}\cdot\vec{q})$$

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$\therefore\ -\int \vec{q}\cdot\vec{\nabla}p\,dV$

$$= -\int_V [\vec{\nabla}\cdot(p\vec{q}) - p(\vec{\nabla}\cdot\vec{q})]\,dV$$

$$= -\int_V \vec{\nabla}\cdot(p\vec{q})\,dV + \int_V p(\vec{\nabla}\cdot\vec{q})\,dV$$

$$= -\int_S (p\vec{q})\cdot\hat{n}\,ds + \int_V p(\vec{\nabla}\cdot\vec{q})\,dV \quad \text{---}(6)$$

where $\hat{n}$ is the unit normal and ds is the surface element.

Now, we try to show that $\int_V p(\vec{\nabla}\cdot\vec{q})\,dV$

$$= -\frac{dI}{dt} \quad (7)$$

Since, E is the internal energy, E is defined as the work done by the unit mass of the fluid,

$$E = \text{Force} \times dl$$

$$= \frac{\text{Force}}{\text{Area}} \times dV$$

$$= \int_V p\,dV \quad (8)$$

→ For E as the work done per unit mass,

$V\rho = 1$.

$\Rightarrow V = \frac{1}{\rho}$. $\Rightarrow dV = -\frac{1}{\rho^2}\,d\rho$

→ By (8), $E = \int_V^{V_0} p\,dV$

$$= -\int_{\rho}^{\rho_0} p\,\frac{d\rho}{\rho^2}$$

$$\frac{dE}{d\rho} = -\frac{d}{d\rho}\int_{\rho}^{\rho_0} \frac{p\,d\rho}{\rho^2}$$

$$\cancel{\Rightarrow \frac{dE}{d\rho} = p(\rho_0)\frac{d\rho_0}{d\rho} - p(\rho)\frac{d\rho}{d\rho} + \int_{\rho}^{\rho_0}\left(\frac{d}{d\rho}p\right)d\rho}$$

$$\cancel{\Rightarrow \frac{dE}{d\rho} = -p(\rho)}$$

$$\cancel{\Rightarrow \frac{dE}{dt} = \frac{dE}{d\rho}\cdot\frac{d\rho}{dt} = -p(\rho)\frac{d\rho}{dt}}$$

$$= +\frac{p}{\rho^2}$$

$$\Rightarrow \frac{dE}{dt} = \frac{dE}{d\rho}\cdot\frac{d\rho}{dt} = \frac{p}{\rho^2}\frac{d\rho}{dt}.$$

$$\Rightarrow \frac{dE}{dt}\cdot\rho\,dV = \frac{p}{\rho^2}\frac{d\rho}{dt}\cdot\rho\,dV$$

$$= \frac{p}{\rho}\frac{d\rho}{dt}dV \quad \text{--- (9)}$$

But; $\frac{d}{dt}(E\rho\,dV) = \frac{dE}{dt}\rho\,dV + E\underbrace{\frac{d}{dt}(\rho\,dV)}_{=0}$ --- (10)

By (9) and (10),

$$\frac{d}{dt}(E\rho\,dV) = \frac{p}{\rho}\frac{d\rho}{dt}dV$$

$$\Rightarrow \frac{d}{dt}(E\rho\,dV) = \frac{p}{\rho}(-\rho\,\vec{\nabla}\cdot\vec{q})\,dV$$

$$\Rightarrow \int_V \frac{d}{dt}(E\rho\,dV) = -\int_V p\,\vec{\nabla}\cdot\vec{q}\,dV$$

$$\Rightarrow \frac{d}{dt}\int_V E\rho\,dV = -\int_V p(\vec{\nabla}\cdot\vec{q})\,dV$$

$$\Rightarrow \frac{dI}{dt} = -\int_V p(\vec{\nabla}\cdot\vec{q})\,dV$$

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from (5),

$$\frac{dT}{dt} + \frac{dP}{dt} + \frac{dI}{dt} = \int_S p(\vec{q}\cdot\vec{n})\,ds$$

→ Rate of work done by the fluid pressure on an element ds of S is

$$P\,ds\;\hat{n}\cdot\vec{q}$$

$\Rightarrow$ work done (total), $W = \int_S p(\vec{n}\cdot\vec{q})\,ds$ —(11)

$$\therefore \frac{dT}{dt} + \frac{dP}{dt} + \frac{dI}{dt} = W$$

$$\Rightarrow \frac{d}{dt}(I+P+T) = W.$$

→ for incompressible fluid,

$$\frac{d}{dt}(T+P) = W.$$

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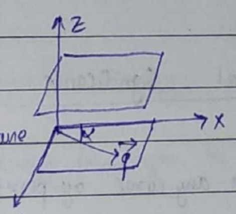
Note: For incompressible fluid, $\frac{d}{dt}\left(\quad T + P \right) = \omega$

5/9/19
Thursday

Two dimensional Flow



Suppose the plane under consideration is xy-plane & let P be any arbitrary point in that plane. Draw a st. line Q to P and let R be another pt. on the plane ∥ to xy-plane and lies on PQ. If $\vec{q}$ be the velocity of the fluid in XY-plane which makes an angle α, then $\vec{q}$ be the velocity in the ∥ plane and of same magnitude making an angle α with the x-axis.

Stream function / Current function.

let $\vec{q} = (u, v)$ be the fluid velocity. Then the eq^n of the Stream function / current function / lines of flow -

$$\frac{dx}{u} = \frac{dy}{v}$$

$$\Rightarrow v\,dx - u\,dy = 0 \qquad (1)$$

From eq^n of continuity,

$$\nabla \cdot \vec{q} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \qquad (11)$$

$\Rightarrow$ (11) is the exactness condition for (1)

$\Rightarrow \exists\ \psi$ st $\quad d\psi = v\,dx - u\,dy = 0$

$$\Rightarrow \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = v\,dx - u\,dy$$

$M\,dx + N\,dy$

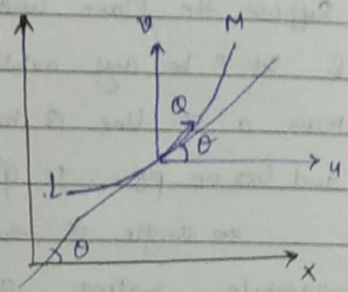
is exact if

$\frac{\partial M}{\partial y} = \frac{\partial}{}$

$$\boxed{u = \frac{-\partial \psi}{\partial y} \; ; \; v = \frac{-\partial \psi}{\partial x}.}$$

we can find ψ.

Physical Significance on Stream function in 2D flow.



let LM be any curve xy-plane and let ψ_1 and ψ_2 be the two stream functions at L and M. let P be any point on LM st $LP = s$ and $LQ = s + ds$ where Q is another pt on LM. let θ be the angle between x-axis and the target at P. let $\vec{q} = (u, v)$ be the fluid velocity.

Now, velocity component along the normal $= \vec{q} \cdot \hat{n}$

$$= (u, v) \cdot (-\sin\theta, \cos\theta)$$

$$= -u\sin\theta + v\cos\theta. \quad \text{(i)}$$

The flux across $PQ = \vec{q} \cdot \vec{n}\, ds$

$$= (v\cos\theta - u\sin\theta)\, \delta s.$$

$\Rightarrow$ total flux across LM (from right to left)

$$= \int_{LM} (v\cos\theta - u\sin\theta)\, ds. \quad \text{(ii)}$$

$$= \int_{LM} \left(v\frac{\partial x}{\partial s} - u\frac{\partial y}{\partial s} \right) ds \qquad u = \frac{\partial \psi}{\partial y} \quad v = \frac{\partial \psi}{\partial x}$$

$$= \int_{LM} \left(\frac{\partial \psi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial s} \right) ds$$

$$= \int_{LM} \left(\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \right) = \int_{LM} d\psi = \psi_2 - \psi_1$$

Thus a property of a string function is that the difference of its values at two pts represent the flow across any line joining the pt.

Remark 1) Since the velocity normal to δs contributes to the flux across δs, therefore

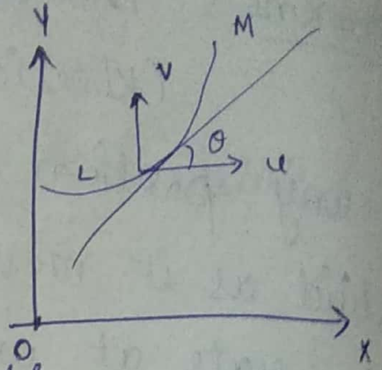
flux across δs = δs × normal velocity

$\Rightarrow \psi + \delta\psi - \psi = \delta s \times$ (velocity from right to left)

$\Rightarrow$ (velocity from right to left) = $\dfrac{\delta\psi}{\delta s}$

• Stream functions:

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \quad \text{—— (I)}$$



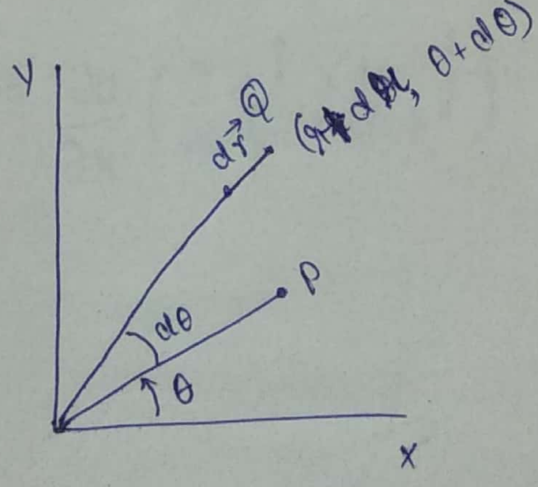
Total flux across LM (from right to left) $= \psi_2 - \psi_1$ —— (II)

Velocity from right to left across

$$\delta s = \frac{\partial \psi}{\partial s} \quad \text{—— (III)}$$

• Stream function / velocity component in Polar co-ordinate:

Let the stream function ψ is given in terms of polar- coordinate system (r, θ) and velocity components $\vec{q} = (q_r, q_\theta)$



$$\tan d\theta = p/r$$

$$\Rightarrow p = r\,d\theta.$$

q_r = velocity from right to left across $r\,d\theta$

$$= \lim_{d\theta \to 0} \frac{\partial \psi}{r\,d\theta} = \frac{1}{r}\frac{d\psi}{d\theta}$$

q_θ = velocity from left to right across dr

$$= \lim_{\delta r \to 0} \frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial r}$$

$$\therefore \boxed{q_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad q_\theta = \frac{\partial \psi}{\partial r}}$$

- Complex potential:

$w = f(z)$, $z = x + iy$

$= \phi(x,y) + i\,\psi(x,y)$

$$\frac{df}{dz} = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \qquad \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

from compressibility :

$$\vec{\nabla} \cdot \vec{q} = 0. \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

i.e. $u = -\frac{\partial \psi}{\partial y}$, $v = \frac{\partial \psi}{\partial x}$ — (I)

Irrotational :

$$\vec{q} = -\vec{\nabla}\phi \Rightarrow u = -\frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \quad \text{— (II)}$$

(I) & (II) $\Rightarrow \frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

$\therefore$ $\boxed{\text{complex potential} = \text{potential func}^n + i\,(\text{Stream func.})}$

- Spin- component of ψ :

The spin / rotation in a flow is given by :

$$\Omega = \frac{1}{2} \text{ curl } \vec{q} = \frac{1}{2} (\vec{\nabla} \times \vec{q})$$

In a 2D flow, $\vec{q} = (u, v, 0)$

$$\text{curl } \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & 0 \end{vmatrix}$$

$$= \hat{i} \frac{\partial v}{\partial z} - \hat{j} \frac{\partial u}{\partial z} + \hat{k} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$\therefore \text{ curl } \vec{q} = \hat{k} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Thus, spin of flow $= \frac{1}{2}$ curl $\vec{q}$

$$= \frac{1}{2} \hat{k} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

If flow is

curl $\vec{q}$ = (

Eg: Find o

$\vec{q}$

Sol] ϕ =

$\frac{\partial u}{\partial z}$

u

If flow is irrotational, then -

$$\text{curl } \vec{q} = 0 \Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \text{ ; subs. } \boxed{u = -\frac{\partial \psi}{\partial y}, \; v = \frac{\partial \psi}{\partial x}}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Laplace eq^n.

Eg: Find out ~~complex potential~~ velocity of the flow.

$\vec{q}$ $w = ikz$

Sol] $\phi = 0$, $\psi = kz$.

$\begin{pmatrix} u \\ v \end{pmatrix}$

$$\frac{\partial w}{\partial z} = ik = -u + iv.$$

$u = 0$, $v = k$

(*) Let $\omega = f(z)$ $\qquad z = x + iy.$

$= \phi + i\psi$ $\qquad \frac{\partial x}{\partial z} = 1.$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

for 2D flow: $u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}$

$$\frac{\partial \omega}{\partial z} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial z} + i \frac{\partial \psi}{\partial x} \cdot \frac{\partial x}{\partial z}$$

$$\cancel{= u \frac{\partial x}{\partial z} (-1+i)}$$

$$\cancel{= u(-1+i)}$$

$$\frac{ik}{-1+i}$$

$$\frac{\partial w}{\partial z} = (-u \mp iv) \cdot 1.$$

Ex. 2] $\omega = -k e^{-i\alpha} z.$

$$\frac{\partial \omega}{\partial z} = \cancel{0} - k e^{-i\alpha} = -u + iv.$$

$$= -k(\cos\alpha - i\sin\alpha) = -u + iv.$$

$u = -k\cos\alpha, \quad v = k\sin\alpha.$