$i \downarrow \; n \rightarrow$	0	1	2	3	4
0	0	0	0	0	0
1	0.707	0.5	$\frac{1}{2\sqrt{2}}$	0.25	$\frac{1}{4\sqrt{2}}$
2	1	$\frac{1}{\sqrt{2}}$	0.5	$\frac{1}{2\sqrt{2}}$	0.25
3	$\frac{1}{\sqrt{2}}$	0.5	$\frac{1}{2\sqrt{2}}$	0.25	$\frac{1}{4\sqrt{2}}$
4	0	0	0	0	0

8|3|24

$u_t = \nu u_{xx} + f(x,t), \quad 0 < x < a, \quad t > 0.$

I.C: $u(x,0) = F(x)$

B.C: $u(0,t) = u_0, \quad u(a,t) = u_a, \quad t > 0$

Forward time marching, $u_i^n = u(x_i, t_n)$, $i = 0, 1, \ldots N$

$u_i^0 = F_i, \quad u_0^n = u_0, \quad u_N^n = u_a \rightarrow$ given. $\quad n = 0, 1, 2, \ldots$

I. Explicit method $\quad$ FTCS

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\delta x^2} + f_i^n$$

$$u_i^{n+1} = a_i u_{i-1}^n + b_i u_i^n + c_i u_{i+1}^n + d_i$$

$n = 0, 1, \ldots, \quad i = 1, 2, \ldots, N-1$

$u_i^1 \; \forall i, \quad u_i^2, \ldots, \quad \boxed{\gamma = \frac{\nu \delta t}{\delta x^2}} \leq \frac{1}{2}$

$\delta x < 1, \quad \delta x = 0.05,$

$\delta t \leq 0.5 (\delta x)^2 \cdot \nu \quad$ for stability

$\delta t \sim O((\delta(x))^2) \quad \delta t << 1 \quad 0 \rightarrow T$

Huge time steps are required to meet the stability criteria

Implicit Scheme:

Let u_i^n are known. We need to obtain u_i^{n+1}

$t_n \rightarrow t_{n+1}$

Satisfy the PDE at (x_i, t_{n+1})

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$$\left.\frac{\partial u}{\partial t}\right|_i^{n+1} = \nu \left.\frac{\partial^2 u}{\partial x^2}\right|_i^{n+1} + f(x_i, t_{n+1}).$$

BTCS : Backward Time Central Space

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \nu \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\delta x^2} + f(x_i, t_{n+1}) \quad (*)$$

$i = 1, 2, \dots, N-1$, for $n = 0, 1, \dots$

unknowns are with superscript $(n+1)$, which appear in both sides of the discretized eq^n. $(*)$

At any $n (= 0, \dots)$.

where

$$a_i u_{i-1}^{n+1} + b_i u_i^{n+1} + c_i u_{i+1}^{n+1} = d_i$$

$i = 1, 2, \dots, N-1$

$$X = \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \end{bmatrix}, \quad AX = d$$

$A \to (N-1) \times (N-1)$ tri-diagonal matrix with elements

$a_i = \gamma$, $b_i = -1-2\gamma$, $c_i = \gamma$

$$d_i = -f_i^{n+1}\delta t - u_i^n$$

$$d_1 = -f_1^{n+1} - u_1^n - \gamma u_0$$

$$d_{N-1} = -f_{N-1}^{n+1} - u_{N-1}^n - \gamma u_a$$

At any $n = 0, \dots$ solve the tri-diagonal system $\boxed{AX = d}$ to obtain $u_i^{n+1}\ \forall i$ $\longrightarrow$ unconditionally stable, i.e, stable for any choice of γ.

$$T.E = O(\delta t, \delta x^2)$$

T.E, the residue by which the exact solution of the PDE fails to satisfy the discrete eq^n (finite difference eq^n) | difference eq^n.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \qquad (\text{let } f(x,t) = 0.)$$

$$\frac{u_i^{n+1} - u_i^n}{\delta t} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\delta x^2} = 0$$

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Let $U(x,t)$ be the exact solution of the PDE

$$\text{Then } T.E = \frac{U_i^{n+1} - U_i^n}{\delta t} - \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\delta x^2}$$

Expand by Taylor series about (x_i, t_{n+1}).

Since $U(x,t)$ is a continuous differentiable function, which can be expressed in Taylor series,

$$\frac{1}{\delta t}\left[U_i^{n+1} - U(x_i, t_{n+1} - \delta t)\right] - \frac{1}{\delta x^2}\left[U(x_i + \delta x, t_{n+1}) - 2U(x_i, t_{n+1}) + U(x_i - \delta x, t_{n+1})\right] = T.E$$

$$\frac{\left[\cancel{U_i^{n+1}} - \cancel{U_i^{n+1}} + \delta t \left.\frac{\partial U}{\partial t}\right|_i^{n+1} - \frac{(\delta t)^2}{2}\left.\frac{\partial^2 U}{\partial t^2}\right|_i^{n+1} + \dots\right]}{\delta t}$$

$$- \frac{1}{\delta x^2}\left[\cancel{U_i^{n+1}} + \cancel{\delta x \left.\frac{\partial U}{\partial x}\right|_i^{n+1}} + \frac{\delta x^2}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_i^{n+1} + \cancel{\frac{\delta x^3}{3!}\left.\frac{\partial^3 U}{\partial x^3}\right|_i^{n+1}} + \frac{\delta x^4}{4!}\left.\frac{\partial^4 U}{\partial x^4}\right|_i^{n+1} + \dots\right.$$

$$- \cancel{2U_i^{n+1}} + \cancel{U_i^{n+1}} - \cancel{\delta x \left.\frac{\partial U}{\partial x}\right|_i^{n+1}} + \frac{\delta x^2}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_i^{n+1} - \cancel{\frac{\delta x^3}{3!}\frac{\partial^3 U}{\partial x^3}}$$

$$\left. + \frac{\delta x^4}{4!}\frac{\partial^4 U}{\partial x^4} + \dots\right] = T.E$$

$$T.E = \left(\left.\frac{\partial U}{\partial t}\right|_i^{n+1} - \left.\frac{\partial^2 U}{\partial x^2}\right|_i^{n+1}\right) - \frac{1}{2}\delta t \frac{\partial^2 U}{\partial t^2} - \frac{2\delta x^2}{4!}\frac{\partial^4 U}{\partial x^4} + O(\delta t^3, \delta x^4).$$

Since $U(x,t)$ is the exact solution of the PDE,

$$\left.\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)\right|_i^{n+1} = 0, \quad T.E = O(\delta t, \delta x^2),$$

$T.E \to 0$ as $\delta x, \delta t \to 0$.

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i.e, the scheme is consistent with the PDE.

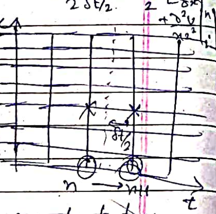
$$(n) \longrightarrow (n+1) \longrightarrow (n+2) \longrightarrow$$

$$(t=0) \longrightarrow (\delta t) \longrightarrow (2\delta t) \longrightarrow$$

$$\left.\frac{\partial u}{\partial t}\right|_i^{n+1/2} = \nu \left.\frac{\partial^2 u}{\partial x^2}\right|_i^{n+1/2}$$

$$\frac{u_i^{n+1} - u_i^n}{2\,\delta t/2} = \frac{\nu}{2}\left[\left.\frac{\partial^2 u}{\partial x^2}\right|_i^{n+1} + \left.\frac{\partial^2 u}{\partial x^2}\right|_i^{n}\right]$$

<u>Crank-Nicolson Scheme</u>



$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + f(x,t) \quad -(*)$$

Let u_i^n are known, we need to obtain u_i^{n+1} for all i, when $n = 0, 1, \ldots$

Integrate (*) w.r.t. t between t_n to t_{n+1}

$$\int_{t_n}^{t_{n+1}} \frac{\partial u}{\partial t}\, dt = \nu \int_{t_n}^{t_{n+1}} \frac{\partial^2 u}{\partial x^2}\, dt + \int_{t_n}^{t_{n+1}} f(x,t)\, dt$$

at $x = x_i$, we apply Trapezoidal rule to integrate the 2^{nd} derivative of u, i.e, U_{xx} w.r.t. t.

$$u_i^{n+1} - u_i^n = \nu \frac{\delta t}{2}\left[\left.\frac{\partial^2 u}{\partial x^2}\right|_i^{n+1} + \left.\frac{\partial^2 u}{\partial x^2}\right|_i^{n}\right] + F(x_i, t_{n+1})$$

$F \rightarrow$ is known, as f is known.

discretize the derivatives w.r.t. x through central differences.

$$u_i^{n+1} - u_i^n = \nu \frac{\delta t}{2}\left[\frac{U_{i+1}^{n+1} - 2u_i^{n+1} + U_{i-1}^{n+1}}{\delta x^2} + \frac{U_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{\delta x^2}\right] + F\cancel{(x_i, t_n)}$$

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$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \frac{\nu}{2}\left[\frac{U_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\delta x^2} + \frac{U_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{\delta x^2}\right] + O(\delta t^2, \delta x^2)_i^n$$

For simplicity, let $f(x,t) = 0$.

which is $O(\delta t^2, \delta x^2)$, i.e, second-order accurate in both time and space as well as implicit which implies, this scheme is unconditionally stable and 2^{nd} order accurate. if we vary $i = 1, 2, \ldots, N-1$

$$a_i U_{i-1}^{n+1} + b_i U_i^{n+1} + c_i U_{i+1}^{n+1} = d_i, \quad n \geq 0$$

$$\begin{cases} a_i = \\ b_i = \\ c_i = \\ d_i = \end{cases}$$

leading to a tri-diagonal system. $Ax = d$

(*) H.T. Expand by Taylor Series about $u(x_i, t_{n+1/2})$ and show that T.E is $O(\delta t^2, \delta x^2)$ and the Crank-Nicolson scheme is consistent.

Ex: $U_t = U_{xx}$

$$\frac{U_i^{n+1} - U_i^{n-1}}{2\delta t} - \frac{U_{i+1}^n - 2\{\theta U_i^{n+1} + (1-\theta) U_i^{n-1}\} + U_{i-1}^n}{(\delta x)^2} = 0$$

$0 < \theta < 1$. Find whether consistent if (a) $\delta t = r\delta x$ (b) $\delta t = r\,\delta x^2$. r is a constant and find θ for consistency.

$$T.E = \cancel{\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right)}\Bigg|_i^n \rightarrow 0 + \left\{\frac{\delta t^2}{6} U_{ttt} - \frac{h^2}{12} U_{xxxx} + (2\theta - 1) \times \frac{2\delta t}{\delta x^2} \cdot U_t + \frac{\delta t^2}{\delta x^2} U_{tt}\right\} + O\left(\frac{\delta t^3}{\delta x^2}, \delta t^4, \delta x^4\right).$$

(1) $\delta t = r\delta x$, $\theta \neq 1/2$, $\delta t, \delta x \rightarrow 0$, $\frac{\delta t}{\delta x} = r \neq 0$

the third term tends to infinity if $\theta \neq 1/2$

if $\theta = 1/2$, the discretized scheme is consistent with :-

Implicit Scheme → unconditionally stable..

$O(\delta t^2, \delta x^2)$.

$O(\delta t, \delta x^2)$..

$$\boxed{\partial u/\partial t - \frac{\partial^2 u}{\partial x^2} + \gamma^2 U_{tt} = 0} \rightarrow H.T$$

Q1 $U_t = U_{xx}$, $U(x,0) = \sin \pi x$, $0<x<1$, $U(0,t) = U(1,t) = 0$, $t>0$.

H.T & Lab

$\delta x = 1/4$, $\gamma = 1/2$

$\boxed{Ax = d.}$

$U_i^0 = \sin \pi x_i$, $U_0^n = U_4^n = 0$, $i = 1, 2, 3, 4$.

H.T → Lab Task for $\delta x = 0.25, 0.05, 0.01$. with $\gamma = 1/2$.

Solve by (a) Explicit (b) Implicit (c) Crank-Nicolson.

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Stability Analysis:

U_j^n be the numerical solution of the difference

$\bar{U}_j^n$ be the exact numerical solⁿ.

$U_j^n = \bar{U}_j^n + \epsilon_j^n$, ϵ_j^n as the error in

$U_t = \nu U_{xx} \rightarrow \frac{U_j^{n+1} - U_j^n}{\delta t} = \frac{\nu}{\delta x^2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$

$\bar{U}_j^n$ be the exact solⁿ of the finite diff. eqⁿ. and U_j^n be the solution obtained.

$U_j^n = \bar{U}_j^n + \epsilon_j^n$, $\epsilon_j^n \rightarrow$ round-off error.

Substitute into the linear difference eqⁿ (*) find that ϵ_j^n that satisfies the difference eqⁿ. (*).

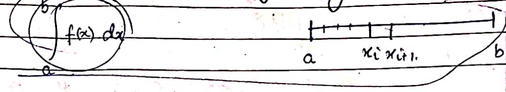
$$\frac{(\bar{U}_j^{n+1} + \epsilon_j^{n+1} - U_j^n - \epsilon_j^n)}{\delta t} = \frac{\nu}{\delta x^2}(\bar{U}_{j+1} + \epsilon_{j+1}^n + \cdots)$$

Since, $\bar{U}_j^n$ satisfies the difference eqⁿ exactly. what we find that ϵ_j^n satisfy the linear difference eqⁿ.

$$(\epsilon_j^{n+1} - \epsilon_j^n)/\delta t = \nu/\delta x^2 (\epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n)$$

Thus, the error propagates through the difference eqⁿ (*)

Von Neumann Stability Analysis.



we express the distribution of error in terms of a finite fourier series.

$$\epsilon(x,t) = \sum_{m=0}^{M}\left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L}\right)$$

$$= \sum^{M} A_m(t)\, e^{im\pi x/L}$$

which provide the distribution of error in an analytic form. A_m's are arbitrary complex coefficient.

$$\epsilon(x_j, t_n) = \epsilon_j^n = \sum^{M} A_m(t_n)\, e^{im\pi x_j/L}$$

Since, ϵ_j^n satisfies a linear equation. we may examine the behaviour of a single term of the series i.e,

$$A_m(t)\, e^{im\pi x/L}, \quad 0 < x < a$$

at (x_j, t_n), any term of the Fourier series is $\bar{A}_m^n e^{imx_j\pi/L}$, let $\bar{A}^n = \max_{0 \le m \le M} \{A_m^n\}$

$x_j = j\delta x$,

$$\epsilon_j^n = \bar{A}^n e^{im\pi j \delta x/L}$$

$$\theta = \frac{m\pi \delta x}{L} \quad \epsilon_j^n = \bar{A}^n e^{i\theta j}$$

$\epsilon_j^n = \bar{A}^n e^{i\theta j}$, $|\bar{A}^n|$ is the amplitude.

$$|\epsilon_j^n| \geq |\epsilon_j^{n+1}| \Rightarrow |\bar{A}^n| \geq |\bar{A}^{n+1}|$$

let ξ be the amplification factor defined as

$$\xi = \frac{\bar{A}^{n+1}}{\bar{A}^n}$$

For stability, $|\xi| \leq 1$, i.e., the error is either decaying or remaining constant.

Unstable, $|\xi| > 1$, error is growing.

Working Rule: error at (x_j, t_n) is expressed as

$$\epsilon_j^n = \bar{A}^n e^{i\theta j} \quad — (*)$$

ϵ_j^n satisfies the difference eq^n. Substitute $\epsilon_j^n = \bar{A}^n e^{i\theta j}$ in the difference eq^n and find $\xi = \frac{\bar{A}^{n+1}}{\bar{A}^n}$.

For stability, $|\xi| \leq 1$, $\forall\, \theta$.

Unstable > 1

Q. Stability of the explicit scheme.

$$\epsilon_j^{n+1} - \epsilon_j^n = r\left(\epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n\right)$$

$$\epsilon_j^n = A^n e^{i\theta j}$$

$$A^{n+1} - A^n = r\left(A^n e^{i\theta} - 2A^n + A^n e^{-i\theta}\right)$$

$$= r A^n \left(e^{i\theta} + e^{-i\theta} - 2\right)$$

$$\Rightarrow \xi - 1 = r(2\cos\theta - 2)$$

$$\Rightarrow \xi = 1 + 2r(\cos\theta - 1).$$

$$\xi = 1 - 4r \sin^2\theta/2$$

$$|\xi| \leq 1$$

$$\Rightarrow -1 \leq \xi \leq 1$$

$$-1 \leq 1 + 2r(\cos\theta - 1) \leq 1$$

$$-2 \leq 2r(\cos\theta - 1) \leq 0$$

$$\Rightarrow -1 \leq -2r\sin^2\theta/2 \leq 0$$

$$-2r\sin^2\theta/2 \geq -1 \quad \Rightarrow \cancel{2}r\sin^2\theta \geq \frac{1}{2}$$

$$r \geq \frac{1}{2\sin^2\theta}$$

& $-2r\sin^2\theta/2 \leq 0$ → true for all r

$$2r(1 - \cos\theta) \leq 1 \qquad 2r(1-\cos\theta) \leq 2$$

$$r \leq 1/2$$

$$1 - \cos\theta \leq 2 \Rightarrow r \leq 1/2$$

The explicit scheme is conditionally stable for $r = \frac{\nu\, dt}{(\delta x)^2} \leq \frac{1}{2}$

H.T. → Stability for implicit & crank nicolson scheme.