Problems 3

(1) Let $E \subseteq \mathbb{R}$ be a measurable set such that $m(E) = 0$. Then show that E^c is dense in $\mathbb{R}$.

Sol:-

Hint: if I is an open interval in $\mathbb{R}$, then $m(I) > 0$.

Let I be any open interval.

To show: $E^c \cap I \neq \phi$.

Suppose $E^c \cap I = \phi$

$$\Rightarrow E \supseteq I.$$

$$\Rightarrow m(E) \geq m(I) > 0$$

$$\Rightarrow m(E) > 0$$

$$\Rightarrow \Leftarrow$$

$\therefore E^c \cap I \neq \phi$.

$\therefore E^c$ is dense in $\mathbb{R}$.

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(2) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a simple function defined by $\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$,

where $A_k = \{x \in \mathbb{R} \mid \varphi(x) = a_k\}$

Prove that φ is measurable if and only if A_k's are measurable.

Sol:- $\Rightarrow$: Assum φ is a measurable function.

To show: Each A_k is measurable.

A_k are measurable.

$\because$ $A_k = \{x \in \mathbb{R} \mid \varphi(x) \geq a_k\} \setminus \{x \in \mathbb{R} \mid \varphi(x) > a_k\}$.

$\Leftarrow$: Assum all A_k sets are measurable.

To show: φ is a measurable function,

To show: For each $\alpha \in \mathbb{R}$,

$$\{x \in \mathbb{R} \mid \varphi(x) > \alpha\} \text{ is measurable}$$

$$\{x \in \mathbb{R} \mid \varphi(x) > \alpha\} = \bigcup_{a_k > \alpha} A_k \text{ is measurable.}$$

$$\varphi = \sum_{k=1}^{n} a_k \chi_{A_k} \qquad \varphi(x) \in \{a_1, a_2, \ldots, a_n\}$$

$$\varphi(x) = \begin{cases} a_k & \text{if } x \in A_k \\ 0 & , \text{o.w} \end{cases}$$

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③ Consider the sequence $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n = n\,\chi_{[0,\frac{1}{n}]} \qquad \forall\, n \geq 1.$$

(i) Does $\{f_n\}$ converge pointwise a.e. on $\mathbb{R}$?

(ii) Does $\{f_n\}$ converges uniformly a.e on $\mathbb{R}$?

Sol:– Given $f_n = n\,\chi_{[0,\frac{1}{n}]} \qquad \forall\, n \geq 1.$

$$f_n(x) = \begin{cases} n & \text{if } x \in [0, \frac{1}{n}] \\ 0 & , \text{otherwise.} \end{cases}$$

(i) Suppose $x \in (1, \infty)$, $f_n(x) = 0 \quad \forall\, n \geq 1$

$$f_n(x) \to 0 \quad \text{as } n \to \infty$$
$$\forall\, x \in (1, \infty).$$

Suppose $x \in (-\infty, 0)$, $f_n(x) = 0, \ \forall\, n \geq 1.$

$\therefore f_n(x) \to 0$, as $n \to \infty$, $\forall x \in (-\infty, 0)$

Thus $f_n(x) \to 0$, $\forall x \in (-\infty, 0) \cup (1, \infty)$.

In fact, $f_n \to 0$ uniformly on $(-\infty, 0) \cup (1, \infty)$.

At $x = 1$: $f_1(1) = 1$, $f_n(1) = 0$ $\forall n \geq 2$

$\therefore f_n(1) \to 0$ as $n \to \infty$

Thus $\boxed{f_n \to 0 \text{ uniformly on } (-\infty, 0) \cup [1, \infty)}$

At $x = 0$: $f_n(0) = n$, $\forall n \geq 1$.

$\therefore \{f_n(0)\}$ is not Convergent

$\Rightarrow \{f_n(x)\}$ is not Convergent at $x = 0$, as $n \to \infty$.

For $0 < x < 1$:

Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$.

$\Rightarrow x \notin [0, \frac{1}{n_0}]$ (n_0 depends on x).

$f_n(x) = 0$, $\forall n \geq n_0$, ($\because [0, \frac{1}{n}] \subseteq [0, \frac{1}{n_0}]$, $n \geq n_0$)

$\therefore f_n(x) \longrightarrow 0$ as $n \to \infty$

for $x \in (0,1)$.

Thus $f_n \to 0$ pointwise on $\mathbb{R} \setminus \{0\}$.

$\Rightarrow f_n \to 0$ pointwise almost everywhere.

Let $N \in \mathbb{N}$, and Consider, $(\frac{1}{N}, 1)$

For $x \in (\frac{1}{N}, 1)$, &

for all $n \geq N$, $f_n(x) = 0$

$\left(\because \left(\frac{1}{n} \leq \frac{1}{N} \right) \text{ if } x \in (\frac{1}{N}, 1), \text{ then } x \notin [0, \frac{1}{n}] \right)$

Thus $f_n(x) \to 0$ as $n \to \infty$

$\forall x \in (\frac{1}{N}, 1)$.

$\therefore f_n \to 0$ uniformly on $\underbrace{(-\infty, 0) \cup [1, \infty) \cup (\frac{1}{N}, 1)}_{= E}$

$\forall N$,

$E^c = [0, \frac{1}{N}]$, $m(E^c) = \frac{1}{N}$

& $f_n \to 0$ uniformly on E.

Thus we verified the Littlewood 3rd principle (Egorov's thm).