Lecture 7

Proposition:– Suppose $F \in \mathcal{M}$. Let $G \subseteq \mathbb{R}$ such that $m^*(F \Delta G) = 0$. Then $G \in \mathcal{M}$.

Proposition:– Let $\{E_i\}_{i \in \mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{R}$. Then $\bigcap_{i=1}^{\infty} E_i$ is also measurable.

i.e. if $E_i \in \mathcal{M}$ $\forall i$, then $\bigcap_{i=1}^{\infty} E_i \in \mathcal{M}$.

proof:–

Consider

$$\left(\bigcap_{i=1}^{\infty} E_i\right)^c = \bigcup_{i=1}^{\infty} E_i^c \in \mathcal{M}$$

because each $E_i^c \in \mathcal{M}$ & $\mathcal{M}$ is a σ-algebra

Thus $\bigcap_{i=1}^{\infty} E_i \in \mathcal{M}$.

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proof of the above proposition:–

Given $m^*(F \Delta G) = 0$. & $F \in \mathcal{M}$.

Then $F \Delta G \in \mathcal{M}$ & $F \in \mathcal{M}$.

To show: $G \in \mathcal{M}$.

$$F \Delta G = (F \setminus G) \cup (G \setminus F)$$

$$\Rightarrow \quad m^*(F\setminus G) \le m^*(F\Delta G) = 0$$

$$\Rightarrow \quad m^*(F\setminus G) = 0$$

$$\text{||ly} \quad m^*(G\setminus F) = 0.$$

$$\therefore \quad F\setminus G,\ G\setminus F \in \mathcal{M}.$$

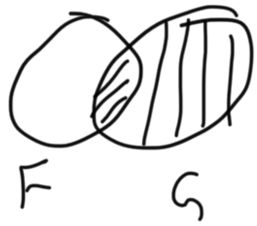
Now $\quad F\cap G = F\setminus(F\setminus G) \quad$ (check it)

$$= F\cap(F\setminus G)^c$$

$$(F\cap G)^c = \underbrace{F^c \cup (F\setminus G)}_{\text{measurable}} \quad \in \mathcal{M}$$

$\Rightarrow \quad F\cap G$ is measurable.

$$G = \underbrace{(F\cap G)}_{\in \mathcal{M}} \cup \underbrace{(G\setminus F)}_{\in \mathcal{M}} \quad \in \mathcal{M}$$



$$\therefore \ G \in \mathcal{M}.$$

Theorem:- Let $\{E_i\}$ be a sequence of disjoint measurable sets. Then $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$.

proof:- Recall that for any $A \subseteq \mathbb{R}$,

$$m^*(A) \ge m^*(A\cap E_1) + \sum_{i=2}^{\infty} m^*\left(A\cap E_i \cap \left(\bigcup_{1\le j<i} E_j\right)^c\right)$$

$$+m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right)$$

Take $A = \bigcup_{i=1}^{\infty} E_i$.

Then

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(\left(\bigcup_{j=1}^{\infty} E_i\right) \cap E_1\right) + \sum_{i=2}^{\infty} m^*\left(\left(\bigcup_{j=1}^{\infty} E_j\right) \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right)$$

$$+ m^*\Big(\underbrace{\left(\bigcup_{i=1}^{\infty} E_i\right) \cap \left(\bigcup_{i=1}^{\infty} E_i\right)^c}_{\phi}\Big)$$

We have

$$\left(\bigcup_{i=1}^{\infty} E_i\right) \cap E_1 = E_1$$

& $$\left(\bigcup_{j=1}^{\infty} E_j\right) \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c = E_i \cap \left(\bigcup_{j<i} E_j\right)^c$$

$$= E_i \cap \left(E_2^c \cap E_3^c \cap \cdots \cap E_{i-1}^c\right)$$

We have $E_2 \cap E_i = \phi$, $i > 2$

$$E_2^c \cup E_i^c = \mathbb{R}$$

$$\left(E_2^c \cup E_i^c\right) \cap E_i = \mathbb{R} \cap E_i = E_i$$

$\Rightarrow$ $E_2^c \cap E_i = E_i$

Repeat this argument, we get $E_i \cap \left(\bigcup_{j<i} E_j\right)^c = E_i$.

$$\therefore \; m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m^*(E_1) + \sum_{i=2}^{\infty} m^*(E_i) + 0.$$

$$\Rightarrow \; m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(E_i).$$

But we already proved the reverse inequality.

Thus $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$.

Def:- Define a map $m : \mathcal{M} \longrightarrow \mathbb{R}$, as

$$m(E) := m^*(E) \quad \forall E \in \mathcal{M}.$$

$m(E)$ is called the Lebesgue measure of E

Theorem:- Every interval is measurable.

proof:- Let $I = [a, \infty)$, $a \in \mathbb{R}$.

To show:- $I \in \mathcal{M}$ i.e, for any $A \subseteq \mathbb{R}$,

$$m^*(A) \geq m^*(A \cap I) + m^*(A \cap I^c)$$

Let $A \subseteq \mathbb{R}$. & $A_1 = A \cap (-\infty, a) = A \cap I^c$

$A_2 = A \cap [a, \infty) = A \cap I$.

Given $\varepsilon > 0$. There exists $\{I_n\}$ intervals such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n \quad \& \quad m^*(A)+\varepsilon \geq \sum_{n=1}^{\infty} l(I_n).$$

$$\boxed{I_n = I_n' \,\dot{\cup}\, I_n''}$$

intervals

Write $I_n' = I_n \cap (-\infty, a) = I_n \cap I^c$

$I_n'' = I_n \cap [a, \infty) = I_n \cap I.$

Then $l(I_n) = l(I_n') + l(I_n'')$.

Since $A \subseteq \bigcup_{n=1}^{\infty} I_n$, we have $A_1 \subseteq \bigcup_{n=1}^{\infty} I_n'$ &

$A_2 \subseteq \bigcup_{n=1}^{\infty} I_n''$.

Now

$$\begin{aligned} m^*(A_1)+m^*(A_2) &\leq \sum_{n=1}^{\infty} l(I_n') + \sum_{n=1}^{\infty} l(I_n'') \\ &= \sum_{n=1}^{\infty} \big(l(I_n') + l(I_n'')\big) \\ &= \sum_{n=1}^{\infty} l(I_n) \\ &\leq m^*(A)+\varepsilon \end{aligned}$$

Thus $m^*(A_1)+m^*(A_2) \leq m^*(A)+\varepsilon$

True for any $\varepsilon > 0$.

$\Rightarrow \quad m^*(A) \geq m^*(\underset{A\cap I^c}{\underset{\parallel}{A_1}}) + m^*(\underset{A\cap I}{\underset{\parallel}{A_2}})$ as required.

$\Rightarrow \quad \underset{[a,\infty)}{\underset{\parallel}{I}}$ is measurable.

$$[a,\infty)^c = (-\infty, a) \in \mathcal{M}.$$

$$(a,b) = (-\infty, b) \cap (a, \infty) \in \mathcal{M}$$

$$\vdots$$

<u>Theorem</u>:- Let $\mathcal{A}$ be a class of subsets of a metric space (X,d). Then there exists a smallest σ-algebra $\mathcal{S}$ containing $\mathcal{A}$.

We say that <u>$\mathcal{S}$ is the σ-algebra generated by $\mathcal{A}$</u>.

<u>proof</u>:- Let $\{\mathcal{S}_\alpha\}_\alpha$ be any collection of σ-algebras of subsets of X. Then by def. of σ-algebra,

$\bigcap_\alpha \mathcal{S}_\alpha$ is also a σ-algebra.

Now take the intersection of all σ-algebras of subsets of X which contains $\mathcal{A}$.

This is the σ-algebra generated by $\mathcal{A}$.
