Real Analysis

Th: Suppose $\{s_n\}, \{t_n\}$ are complex sequences and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then,

(a) $\lim_{n\to\infty} (s_n + t_n) = s+t$

(b) $\lim_{n\to\infty} cs_n = cs$,

$\lim_{n\to\infty} (c+s_n) = c+s$.

(c) $\lim_{n\to\infty} s_n t_n = st$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, Provided $s_n \neq 0 (n=1,2,\ldots)$ & $s \neq 0$

Proof: (a) $\epsilon > 0$, $\exists$ integers N_1, N_2 s.t.

$$n \geq N_1 \Rightarrow |s_n - s| < \epsilon/2$$

$$n \geq N_2 \Rightarrow |t_n - t| < \epsilon/2$$

let $N = \max(N_1, N_2)$.

$$|(s_n + t_n) - (s+t)| = |(s_n - s) + (t_n - t)|$$

$$\leq |s_n - s| + |t_n - t|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$$\underline{n \geq N}$$

(b) $|cs_n - cs| = |c(s_n - s)|$

$= |c|\,|s_n - s|$

$< |c| \frac{\epsilon}{|c|}$

$< \epsilon$

given $\epsilon > 0, \exists N$ s.t.

$n \geq N$

$|s_n - s| < \frac{\epsilon}{|c|}$, $c \neq 0$

$|(c + s_n) - (c+s)|$

$= |s_n - s| < \epsilon$

(c) $|s_n t_n - st|$

$$= |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$$

$$\leq |(s_n - s)||(t_n - t)| + |s||t_n - t| + |t||s_n - s|$$

Given $\varepsilon > 0$, $\exists$ integers N_1, N_2 s.t.

$n \geq N_1 \Rightarrow |s_n - s| < \sqrt{\epsilon}$

$n \geq N_2 \Rightarrow |t_n - t| < \sqrt{\epsilon}$

Then take $N = \max(N_1, N_2)$

i.e. $n \geq N$, $\underline{|(s_n - s)(t_n - t)| < \epsilon}$

i.e. $\lim_{n\to\infty} [s_n - s)(t_n - t) = 0$

$$\left.\begin{array}{l} |t_n - t| < \epsilon/|s| \\ \underline{|s||t_n - t| < \cdot\epsilon} \end{array}\right\} \Rightarrow \lim_{n\to\infty} s(t_n - t) \to 0$$

$$\left.\begin{array}{l} |s_n - s| < \epsilon/|t| \\ \underline{|t||s_n - s| < \epsilon} \end{array}\right\} \Rightarrow \lim_{n\to\infty} t(s_n - s) \to 0$$

(d) As $s_n \to s$, for $\epsilon = \frac{|s|}{2}$, $\exists$ k_1

s.t. $|s_n - s| < \frac{|s|}{2}$ for $n \geq k_1$

$$||s_n| - |s|| \leq |s_n - s| < \frac{|s|}{2}$$

$|s| - |s_n| < \frac{|s|}{2}$ and $|s_n| - |s| < \frac{|s|}{2}$

$\Rightarrow \underline{|s_n| > \frac{|s|}{2}}$ and $|s_n| < \frac{3}{2}|s|$

$\hookrightarrow$ we need this $\hookrightarrow$ this we don't need as it doesn't matter

$\hookrightarrow (n \geq k_1)$

Now $\epsilon > 0$, $\exists\ k_2 \in \mathbb{N}$ s.t. for $n \geq k_2$

$$|s_n - s| < \frac{\epsilon}{2} |s|^2$$

let $K = \max(k_1, k_2)$

$$\cancel{\frac{1}{|s_n|}}\ \cancel{- \frac{1}{s}}$$

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s_n - s}{s_n s} \right| < \frac{\epsilon/2 |s|^2}{|s|^2/2} = \epsilon$$

$x \in \mathbb{R}^k$

$$x_n = (\alpha_{1,n}, \alpha_{2,n} \cdots \alpha_{k,n}) \in \mathbb{R}^k$$

k-cell

Th: (a) Suppose $x_n \in \mathbb{R}^k$ $(n = 1, 2, \ldots)$ and $x_n = (\alpha_{1,n}, \ldots, \alpha_{k,n})$

Then $\{x_n\}$ converges to $x = (\alpha_1, \alpha_2, \ldots \alpha_k)$ iff

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \quad (1 \leq j \leq k)$$

(b) Suppose $\{x_n\}, \{y_n\}$ are sequences in $\mathbb{R}^k$, and $\{\beta_n\}$ a sequence of real numbers $x_n \to x$, $y_n \to y$, $\beta_n \to \beta$, then $\lim_{n\to\infty} (x_n + y_n) = x + y$,

$$\lim_{n\to\infty} \beta_n x_n = \beta x.$$

$x_n = (\alpha_{1,n}, \alpha_{2,n} \ldots \alpha_{k,n})$, $y_n = (\gamma_{1,n}, \gamma_{2,n}, \ldots \gamma_{k,n})$.

norm $\leftarrow$ $|x - y| = \sqrt{|\alpha_1 - \gamma_1|^2 + |\alpha_2 - \gamma_2|^2 + \ldots + |\alpha_k - \gamma_k|^2}$

(a) If $x_n \to x$ given $\epsilon > 0$ $\exists$ an integer N

s.t. $n \geq N$, $|x_n - x| < \epsilon$.

$$|x_n - x| = |(\alpha_{1,n} - \alpha_1, \alpha_{2,n} - \alpha_2, \ldots, \alpha_{k,n} - \alpha_k)|$$

$$= \left(\sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j|^2 \right)^{1/2}$$

$$|\alpha_{j,n} - \alpha_j| \leq |x_n - x| < \epsilon$$

for $n \geq N$

$\Rightarrow |\alpha_{j,n} - \alpha_j| < \epsilon$ for $n \geq N$.

$\Rightarrow \lim_{n \to \infty} \alpha_{j,n} = \alpha_j \quad (1 \leq j \leq k)$

(Converse $\Leftarrow$)

Suppose $\lim_{n \to \infty} \alpha_{j,n} = \alpha_j \quad (1 \leq j \leq k)$ holds.

$\epsilon > 0$, $\exists$ a N s.t. $n \geq N$ implies

$$|\alpha_{j,n} - \alpha_j| < \epsilon/\sqrt{k} \quad (1 \leq j \leq k)$$

Hence $n \geq N$

$$|x_n - x| = \left\{ \sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j|^2 \right\}^{1/2}$$

$$< (\epsilon^2)^{1/2}$$

$$= \epsilon$$

Subsequences

Let $\{p_n\}$ be a sequence. A sequence of the integers $\{n_k\}$ with the properties $n_1 < n_2 < \ldots < n_k$.
Then $\{p_{n_j}\}$ is called a subseq. of $\{p_n\}$.

eg> $p_1, p_2, \ldots$ seq

$p_2, p_4, p_6 \ldots$ ✓ subseq.

$p_4, p_6, p_5 \ldots$ X subseq

The limit of the subsequence is said to be subsequential limits.

eg> $p_n = (-1)^n : -1, 1, -1, 1 \ldots$

$p_{2n} = (p_2, p_4, \ldots) = (1, 1, 1 \ldots)$ subseq limit $= 1$

$p_{2n+1} = (p_1, p_3, \ldots) = (-1, -1, -1 \ldots)$ subseq limit $= -1$

A ~~sub~~sequence $\{p_n\}$ converges to point 'p' iff every subsequence converges to same limit.

Thm (a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a pt of X.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subseq. (Balzano-Weierstrass Th.)

<u>Proof</u> Let E be the range of the seq $\{p_n\}$. If E is finite then there is a pt $p \in E$ and a sequence $\{n_i\}$ with $n_1 < n_2 \ldots$ such that

$p_{n_1} = p_{n_2} = p_{n_3} \ldots p_{n_i} = p \qquad p_{n_i} \to p$

If E is Infinite, E has a limit pt in X.

Let p be a limit pt in X.

Choose n_1 s.t. $d(p_{n_1}, p) < 1$. Having choosen $n_1, n_2, \dots n_{i-1}$ (This is possible as p is a limit pt)

As every nhbd of p contains infinitely many pts of E, $\exists\, n_i > n_{i-1}$ s.t. $d(p, p_{n_i}) < \frac{1}{i}$.

$\lim_{i \to \infty} p_{n_i} = p$

(b) Since Every bdd sequence in $\mathbb{R}^k$ is contained in a k-cell, which is compact subset of $\mathbb{R}^k$, by (a) every bounded sequence in $\mathbb{R}^k$ Contains a convergent subseq.

Th: The subsequential limits of a sequence $\{P_n\}$ in a metric space X form a closed subset of X.

<u>proof</u>: Let E^* denote the set of all sequential limits of the sequence $\{P_n\}$. Let q be a limit pt of E^*, To ~~shoo~~ show that $q \in E^*$.

- Choose n_1 such that $P_{n_1} \neq q$ (if no such n_1 exists, then E^* contains single pt only, it is closed)

$$d(P_{n_1}, q) \neq 0$$

- Choose $\overrightarrow{n_1, n_2, \dots n_{i-1}}$, Since q is a limit pt of E^*, $\exists\, x \in E^*$, s.t. $d(x, q) < \frac{\delta}{2^i}$, $\delta = d(q, P_{n_1}) \neq 0$

- Since $x \in E^*$, $\exists\, n_i > n_{i-1}$ s.t. $d(x, P_{n_i}) < \frac{\delta}{2^i}$

- $d(q, P_{n_i}) \leq d(q, x) + d(x, P_{n_i})$

$$\leq 2 \cdot \frac{\delta}{2^i} = 2^{1-i}\delta \quad \text{for } i = 1, 2, \dots$$

i.e. $P_{n_i} \to q$

eg) $S_n = 1 + (-1)^n$

$E^* = \{0, 1\} \to$ Closed as finite set.

Let X be a Metric space.

→ ① A sequence $\{P_n\}$ in X is said to be a <u>Cauchy sequence</u> if there exist an integer N s.t. for $n \geq N$, $m \geq N$,

$$d(P_n, P_m) < \epsilon$$

→ ② Let E be a subset of X. $S = \{d(P, q), P, q \in E\}$

Sup S exists. diameter $E = \sup S$.

$E_N = \{ P_N, P_{N+1}, P_{N+2}, \dots \}$

$n, m \geq N$

$d(P_{N+i}, P_{N+j}) < \epsilon$

$\forall\ i, j \to 1, 2, \dots$

$\{P_n\}$ is cauchy sequence iff $\lim_{N \to \infty} \dim E_N = 0$

→ A metric space X is said to be complete if every cauchy sequence converges in X.

eg> $X = (0, 1)$, $P_n = (\frac{1}{n}) \Rightarrow X$ is not complete

eg> $\mathbb{R}^k$ is a complete metric space.

Th: (a) If $\bar{E}$ is the closure of a set E in a metric space X, then $\dim \bar{E} = \dim E$.

(b) If K_n is a sequence of compact sets in X s.t. $K_n \supset K_{n+1}$ $(n = 1, 2, \dots)$ and if $\lim_{n \to \infty} \text{diam}\, K_n = 0$, then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

* (a) To show that $\text{diam}\,\overline{E} = \text{diam}\,E$

→ We know, $E \subset \overline{E}$

$\Rightarrow \text{diam}\,E \leq \text{diam}\,\overline{E}$ (i)

→ Fix $\varepsilon > 0$, and choose $p \in \overline{E}$, $q \in \overline{E}$.

By definition of $\overline{E}$, there are pts p', q' in E such that $d(p, p') < \epsilon$, $d(q, q') < \epsilon$

→ $d(p,q) \leq d(p,p') + d(p',q') + d(q',q)$

$\leq 2\epsilon + d(p', q')$

$\leq 2\epsilon + \text{diam}(E)$

$\text{diam}(\overline{E}) \leq \text{diam}(E) + 2\epsilon$ (ii) $\left[\begin{array}{l}\text{if } \forall a,\ a \leq b \\ \Rightarrow \sup(a) \leq b\end{array}\right]$

combining (i) and (ii) $\Rightarrow$ $\boxed{\text{diam}(E) = \text{diam}(\overline{E})}$.

* (b) $\{K_n\}$ is a sequence of compact sets in X.

$K_n \supset K_{n+1}$ $(n = 1, 2, \ldots)$

$K = \bigcap_{n=1}^{\infty} K_n \neq \phi$. To show that K contains exactly one point.

→ Suppose K contains more than one pt. Then $\text{diam}\,K$ is positive.

→ But for each n, $K_n \supset K$. $\Rightarrow \text{diam}\,K_n \geq \text{diam}\,K$ which contradicts to $\lim_{n\to\infty} \text{diam}\,K_n = 0$

[or Assume $\text{diam}\,K_n < \epsilon$ $\forall \epsilon$, $n \geq N$ and since $\text{diam}\,K > 0$]
let $\epsilon_1 = 2\epsilon \Rightarrow \epsilon \geq 2\epsilon$
Contradicti

$\Rightarrow \bigcap_{n=1}^{\infty} K_n$ consists exactly 1 pt. (hence $\text{diam}\,K = 0$)

-ace and if $\{P_n\}$ then $\{P_n\}$ converges

If

$\left[\begin{array}{l} \text{eg. } X = (0,1) \\ P_n = \{\frac{1}{n}\} \end{array}\right.$ So if X is not compact, we cannot claim anything

convergent seq. is a cauchy seq. (any metric space

$\mathbb{R}^k$, every cauchy seq. converges.

Proof: (a) Suppose $P_n \to P$ in X. i.e. given $\varepsilon \geq 0$.

$\exists N$ s.t. $n \geq N$ $d(P_n, P) < \epsilon$

$m \geq N$ $d(P_m, P) < \epsilon$

$$d(P_m, P_n) \leq d(P_n, P) + d(P, P_m)$$

$$< 2\epsilon$$

as soon as $n \geq N$, $m \geq N$. i.e. $\{P_n\}$ is a cauchy sequence.

Proof: (b) Let $\{P_n\}$ be a cauchy sequence in the compact set X.

For $N = 1, 2, 3, \ldots$, let E_N be the set consisting of $P_N, P_{N+1}, \ldots$ i.e. $E_N = \{P_N, P_{N+1}, \ldots\}$

$\left[\lim_{n\to\infty} \text{diam } E_N = \lim_{n\to\infty} \text{diam } \overline{E_N} = 0 \right.$

$\rightarrow$ being $\overline{E_N}$ a closed subset of the compact set X, it is compact.

$E_N \supset E_{N+1}$

$\Rightarrow \overline{E_N} \supset \overline{E_{N+1}}$

$\Rightarrow \bigcap_{N=1}^{\infty} \overline{E_N}$ consisting one pt.

i.e. $\exists$ a unique $p \in X$ s.t. $p \in \overline{E}_N$, $\forall$ N

$\lim\limits_{N \to \infty} \overline{E}_N = 0$, means, $\epsilon > 0$ $\exists$ N_0 s.t.

→ diam $\overline{E}_N < \epsilon$ for $N \geq N_0$

→ Since $p \in \overline{E}_N \Rightarrow d(p, q) < \epsilon$, $\forall$ $q \in \overline{E}$,

hence $\exists$ $P_n \to q$

i.e. $d(p, p_n) < \epsilon$, $n \geq N_0$.

i.e. $P_n \to P$

Proof: (c) Let P_n be a cauchy seq in $\mathbb{R}^K$. for $N = 1, 2, 3, \ldots$

Let $E_N = \{P_N, P_{N+1}, P_{N+2}, \ldots\}$

→ As $\lim\limits_{N \to \infty}$ diam $E_N = 0$, for $\epsilon = 1$, $\exists$ some N,

s.t. diam $E_N < 1$.

→ Range of $\{P_n\}$ is the union of E_N and the finite set $\{P_1, P_2, \ldots P_{N-1}\}$.

Hence $\{P_n\}$ is bounded.

→ Since Every bounded Subset of $\mathbb{R}^K$, its closure is compact in $\mathbb{R}^K$, $\{P_n\}$ converges in $\mathbb{R}^K$.

→ Hence $\mathbb{R}^K$ is complete Metric space.

* A real sequence $\{s_n\}$ is said to be monotonically increasing if

$$s_n \leq s_{n+1} \quad (n = 1, 2, \ldots)$$

monotonically decreasing if

$$s_n \geq s_{n+1} \quad (n = 1, 2, \ldots)$$

* A sequence is said to be monotone if either it is monotonicaly increasing or monotonically d.

<u>Th:</u> Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ convergent iff it is bounded.

<u>Proof:</u> Suppose $\{s_n\}$ is monotonically increasing

$$s_n \leq s_{n+1}$$

→ let $E = \{s_1, s_2, s_3 \ldots\}$, E is the range of $\{s_n\}$.

→ If $\{s_n\}$ is bounded, then E has a <u>lub</u> or Supremum.

→ Let us denote it s. i.e. $s_n \leq s \; (n = 1, 2, \ldots)$

for $\varepsilon > 0$, $\exists N$ s.t.

$$s - \varepsilon < s_N \leq s \longrightarrow (i)$$

→ Since $\{s_n\}$ is increasing $n \geq N$

$$\Rightarrow s_n \geq s_N$$

from (i), $s - \varepsilon < s_n \leq s$, $n \geq N$

i.e. $\underline{s_n \to s}$.

Application

① Let $\{y_n\}$ be a sequence defined to $y_1 := 1$,

$$y_{n+1} := \frac{1}{4}(2y_n + 3), \quad n \geq 1$$

Show that $\lim_{n\to\infty} y_n = 3/2$

$y_1 = 1$, $y_2 = 5/4$, $y_1 < y_2 < 2$

→ To show $y_n < 2$, $\forall n$.

$y_1 < 2$, $y_2 < 2$, $y_k < 2$ (suppose)

$$y_{k+1} = \frac{1}{4}(2y_k + 3) < \frac{1}{4}(2\times 2 + 3) = 7/4 < 2$$

$y_n < 2$, $\forall n$ (Induction)

→ Show that $y_n < y_{n+1}$ $\forall n$

$$\lim_{n\to\infty} y_{n+1} = \frac{1}{4}(2\lim_{n\to\infty} y_n + 3)$$

$$y = \frac{1}{4}(2y + 3)$$

$$\boxed{y = 3/2}$$

② If $s_1 = \sqrt{2}$, $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ $(n = 1, 2, \dots)$

Show that $\{s_n\}$ converges, and then $s_n < 2$ for $n = 1, 2,$

$s_2 = \sqrt{2 +}$ - - -

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$\rightarrow \{S_n\}$ be a bounded seq.

$$M_n = \sup\{S_n, S_{n+1}, S_{n+2}, \ldots\}$$

Since $\{S_{n+1}, S_{n+2}, \ldots\} \subset \{S_n, S_{n+1}, \ldots\}$

$$\sup\{S_{n+1}, \ldots\} \leq \sup\{S_n, S_{n+1}, \ldots\}$$

$M_{n+1} \leq M_n$ [$\{M_n\}$ is decreasing seq.]

$$\lim_{n\to\infty} \sup S_n = \lim_{n\to\infty} M_n$$

$\{P_n\}$ infinite series

Suppose there is a real number S satisfying : (1) for every $\varepsilon > 0$ there is N for $n > N$, $P_n < S + \epsilon$

(2) for every $\varepsilon > 0$ and $M > 0$ there is $n \geq M$ s.t. $P_n > S - \epsilon$. Then, $S = \lim_{n \to \infty} \sup P_n$
$= \overline{\lim_{n \to \infty}} P_n$

10-oct Some Special Sequences.

1. If $0 \leq x_n \leq S_n$ for $n \geq N$, where N is some fixed number and if $S_n \to 0$, then $x_n \to 0$.

2. If $p > 0$, then $\lim_{n \to \infty} \frac{1}{n^p} = 0$

3. If $p > 0$, then $\lim_{n \to \infty} \sqrt[n]{p} = 1$

4. $\lim_{n \to \infty} \sqrt[n]{n} = 1$

5. If $p > 0$ and α is a real number, then
$\lim_{n \to \infty} \frac{n^\alpha}{(1+p)^n} = 0$

6. If $|x| < 1$, then $\lim_{n \to \infty} |x|^n = 0$.

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* $\sum a_n$ converges iff for every $\epsilon > 0$ $\exists$ an integer N s.t. $\sum_{k=n}^{m} |a_k| \leq \epsilon$ if $m \geq n \geq N$

→ Cauchy criterion of convergence.

Th: If $\sum_{n=1}^{\infty} a_n < \infty$, then $\lim_{n \to \infty} a_n = 0$

[for complex we use $|a_n|$, not a_n]

$S_n = a_1 + \dots + a_n$

$S_{n-1} = a_1 + \dots + a_{n-1}$

$\lim_{n \to \infty} (S_n - S_{n-1}) = \lim_{n \to \infty} a_n$

$S - S = \boxed{\lim_{n \to \infty} a_n = 0}$ ✓

!! Converse is not true, as $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, yet $\lim_{n \to \infty} \frac{1}{n} = 0$.

Comparison-Test

① If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is fixed +ve integer and if $\sum_{n=1}^{\infty} c_n < \infty \Rightarrow \sum a_n < \infty$

② $0 \geq a_n \geq b_n$ & $\sum b_n = \infty \Rightarrow \sum a_n = \infty$

Ratio-test

The series $\sum a_n$

(a) Converges, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

(b) diverges, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

* $1 \to$ test fails.

Root-test

Given $\sum_{n=1}^{\infty} a_n$, put $\alpha = \overline{\lim_{n\to\infty}} \sqrt{|a_n|}$

Then,

(a) If $\alpha < 1$, $\sum a_n < \infty$

(b) If $\alpha > 1$, $\sum a_n = \infty$

(c) $\alpha = 1$, test fails.

Alternating Series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

Suppose the seq. $\{a_n\}$ of real numbers satisfy the following conditions.

(a) Alternative +ve & −ve sign.

(b) $|a_n| \leq |a_{n-1}|$ for every n.

(c) $\lim_{n \to \infty} a_n = 0$. Then $\sum_{n=0}^{\infty} a_n$ is convergent.

★ for series $\{a_n\}$, $\sum_{n=1}^{\infty} |a_n| < \infty$

then $\sum a_n$ is called convergent Absolutely.

★ If $a_n < \infty$, but $\sum |a_n| = \infty$, then $\sum a_n$ is conditionally convergent.

⇒ eg) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ → is conditionally converge

11-oct :

$f : E \to Y$, X, Y are metric spaces and $E \subset X$, let p be a limit pt of E. We write $f(x) \to q$ as $x \to p$, or

$\lim_{x \to p} f(x) = q$ if there is pt q in Y with the following property. For every $\epsilon > 0$ $\exists\, \delta > 0$ s.t. $d_Y(f(x), q) < \epsilon$, whenever $0 < d_X(x, p) < \delta$

$$\left.\begin{cases} q = f(p) \\ \lim_{x \to p} f(x) = q \end{cases}\right\} f \text{ is continuous at } x = p.$$

$$\begin{cases} f \text{ is continuous at } x = p \\ \epsilon > 0, \ \exists\, \delta > 0 \ \text{ s.t.} \\ d_Y(f(x), f(p)) < \epsilon, \text{ whenever } d_X(x, p) < \delta \end{cases}$$

$$\left.\begin{aligned} \lim_{x \to p} f(x) &= q \\ \lim_{x \to p} g(x) &= q_1 \end{aligned}\right\} \lim_{x \to p} f(x) + g(x) = q + q_1$$

$$\& \lim_{x \to p} (\alpha f(x)) = \alpha q$$

$$\& \lim_{x \to p} \frac{f(x)}{g(x)} = q/q_1 \quad (\text{provided } q_1 \neq 0)$$

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$f : E \to \mathbb{R}^k$, $E \subset X$

f is a vector valued function.

$f(x)$ is an element of $\mathbb{R}^k$.

$f(x) = \left(f_1(x), \ldots, f_k(x)\right)$

f is continuous iff $f_i(x)$ is continuous, $i = 1, \ldots k$.

<u>Thm</u>: Let $X, Y, \epsilon, f, \delta$ be as in defn.

Then, $\lim_{x \to p} f(x) = q$, iff $\lim_{n \to \infty} f(P_n) = q$,

for every sequence $\{P_n\}$ in E

s.t. $P_n \neq P$ & $\lim_{n \to \infty} P_n = P$

<u>Pf</u> $\longrightarrow$ Suppose $\lim_{x \to p} f(x) = q$. Choose

a sequence $\{P_n\}$ satisfying $P_n \neq P$

& $\lim_{n \to \infty} P_n = P$. let $\epsilon > 0$ be given,

then $\exists \delta > 0$, s.t. $d_Y(f(x), q) < \epsilon$

if $x \in E$ & $0 < d_X(x, p) < \delta$

As $P_n \to P$, $\exists N$ s.t. $n \geq N$

$d_Y(P_n, P) < \delta$. Thus for $n \geq N$, we have $d_Y(f(P_n), q) < \epsilon$

i.e. $\lim_{n \to \infty} f(P_n) = q$, for $\{P_n\}$ in E

with $\lim_{n \to \infty} P_n = P$.

$(\Leftarrow)$ $\lim_{n\to\infty} f(p_n) = q$ for $\{p_n\}$ in E with $p_n \neq p$

$\lim_{n\to\infty} p_n = p$

To show that $\lim_{x\to p} f(x) = q$

Suppose $\lim_{x\to p} f(x) \neq q$. Then $\exists\ \epsilon > 0$, s.t.

for every $\delta > 0$, $\exists$ a pt. $x \in E$ depending on δ, for which $d_Y(f(x), q) \geq \epsilon$,

but $0 < d_X(x, p) < \delta$.

Taking, $\delta_n = \frac{1}{n}$ $(n = 1, 2, \ldots)$

$0 < d_X(p_n, p) < \frac{1}{n}$ &

$d_Y(f(p_n), q) \geq \epsilon$;

which Contradicts the assumption.

12-10-18 Continuity

$f: E \subset X \to Y$

A pt. $p \in E$.

f is said to be continuous at the pt. p is for all $\epsilon > 0$, there exists a $\delta > 0$ s.t. $d_Y(f(x), f(p)) < \epsilon$ whenever $d_X(x, p) < \delta$.

→ f is not con. at a point if $\exists\ \epsilon > 0\ \forall\ \delta > 0$ & $\exists$ pt. $x \in E$ with $d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) \geq \epsilon$.

→ f is continuous on E if f is continuous at every pt. of E.

Ex:- $f(x) = \begin{cases} 1/x ; & x \neq 0 \\ 0 , & x = 0 \end{cases}$

→ $|x| < \delta \Rightarrow \frac{1}{|x|} > \frac{1}{\delta} > \epsilon$.

⇒ $f: E \to Y$

E is isolated set.

To show: f is continuous on E.

Let $p \in E$, Now, for any $\epsilon > 0$, we choose a $\delta > 0$ s.t. $d_X(x, p) < \delta \Rightarrow x = p$.

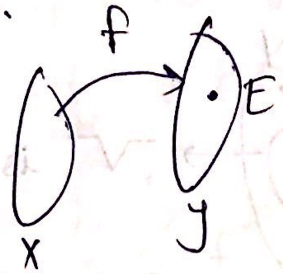
$d_Y(f(x), f(p)) = 0 < \epsilon$

$I, N \rightarrow$ isolated set $\rightarrow N \rightarrow \mathbb{R}$ & $Z \rightarrow \mathbb{R}$

automatically with $f: Z \rightarrow R$

$\rightarrow f: X \rightarrow Y$

f is continuous on X iff for all open subset V of Y, $f^{-1}(V)$ is open in X.



$\rightarrow E \subset X$

$f(E) = \{ f(x) \mid x \in E \}$

$f^{-1}(V) = \{ x \in X \mid f(x) \in V \}$

i.e. $\forall y \in Y$

$f^{-1}(y) = \{ x \mid f(x) = y \}$

$f^{-1}(y)$ contains atmost one pt.

24 – Oct

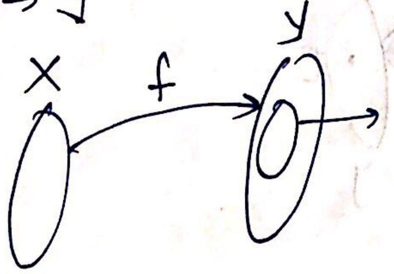
★ $f: \mathbb{R} \to \mathbb{R}$

f is said to be cont. at $x = a$

if given $\epsilon > 0$, $\exists\ \delta(\epsilon)$ s.t.

$|f(x) - f(a)| < \epsilon$, $|x - a| < \delta$

★ $f: X \to Y$



→ V is open subset of Y.

→ $f^{-1}(V)$ is open in X

, $\forall$ V open in Y.

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

↳ inverse image of a set.

Very important Result ⇓

★★ A mapping $f: X \to Y$ is continuous on X,

iff $f^{-1}(V)$ is open in X for

every open set V in Y.

$(\longrightarrow)$

Pf: Suppose f is continuous in X and V is open in Y. We have to show $f^{-1}(V)$ is open in X, i.e. every pt of $f^{-1}(V)$ is an interior pt. So, suppose $p \in X$ and $f(p) \in V$. Since V is open, $\exists$ $\epsilon > 0$ s.t. $y \in V$ if $d_Y(f(p), y) < \epsilon$.

There $d(,)$ is used instead $|\ |$.

and since f is continuous at p, $\exists$ $\delta > 0$ s.t. $d_X(x, p) < \delta$, then $d_Y(f(x), f(p)) < \epsilon$.

($\Leftarrow$) Converse part.

Suppose $f^{-1}(V)$ is open in X for all open subset V in Y. To show that f is continuous on X.

- fix $p \in X$ and $\epsilon > 0$. Let V be set of all $y \in Y$ s.t. $d_Y(y, f(p)) < \epsilon$ i.e. V is open subset of Y. $f^{-1}(V)$ is open in X, and hence $\exists\ \delta > 0$ s.t. $x \in f^{-1}(V)$ as soon as $d_X(x, p) <$

But $f(x) \in V$, $d_Y(f(x), f(p)) < \epsilon$.

★ Corollary.

$f : X \to Y$ is conti iff $f^{-1}(E)$ is closed in X $\forall$ E is closed in Y.

$$f^{-1}(E^c) = [f^{-1}(E)]^c$$

E^c is open

$f^{-1}(E^c)$ is open $\Rightarrow (f^{-1}(E^c))^c$ is closed

$\Rightarrow ((f^{-1}(E))^c)^c$ is closed.

$\Rightarrow f^{-1}(E)$ is closed.

Theorem: Let f be a continuous map from a compact matric space X into a matric space Y. Then $f(X)$ is compact.

proof: Let $\{V_\alpha\}$ be an open covering of $f(X)$

To show that it has a finite subcover

$\{V_\alpha\}$ is open in Y. As f is continuous, $f^{-1}(V_\alpha)$ is open in X.

As X is compact, $X \subset f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \ldots f^{-1}(V_{\alpha_n})$

$$f(X) \subset f\left(f^{-1}(V_{\alpha_1}) \cup \ldots f^{-1}(V_{\alpha_n})\right)$$

$$= f\left[f^{-1}(V_{\alpha_1} \cup \ldots V_{\alpha_n})\right]$$

$$\subset V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}.$$

$f(f^{-1}(A)) \subset A$

$f^{-1}(f(A)) \supset A$

set theory?

i.e. $f(X) \subset V_{\alpha_1} \cup V_{\alpha_2} \ldots \cup V_{\alpha_n}$.

Hence $f(X)$ is compact.

★ If $f: X \to \mathbb{R}^K$

Continuous, compact $\to$ $f(X)$ is compact subset of $\mathbb{R}^k$.

$\to$ $f(X)$ is closed + bdd.

$\Rightarrow$ f is bounded.

act

Th: Suppose f is continuous 1-1 mapping of a compact set X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x \ \forall x \in X$ is continuous mapping of Y onto X.

i.e. $\left[\begin{array}{l} f : X \xrightarrow{\text{bijective}} Y \\ \quad \downarrow \\ \text{Compact} \end{array} \Rightarrow f^{-1} \text{ is cont.} \right]$

Proof: $f : X \to Y$

As f is bijective

$f^{-1} : Y \to X$

rom Th — ition-Open before.

Let V be open in X. To show that $(f^{-1})^{-1}(V)$ is open. f(V) is open.

[open mapping ⇒ any subset of ope set is mapped to open set]

V^c is closed in X.

$\Rightarrow V^c$ is compact. (As closed subset of compact set is compact)

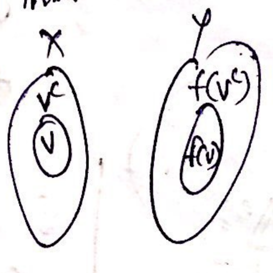
$\Rightarrow f(V^c)$ is compact. (from previous Thm.)

$\Rightarrow f(V^c)$ is closed.

$\Rightarrow f(V^c)^c$ is open.

$\Rightarrow f(V)$ is open

[Since f is one-one and onto, $f(V)$ is complement of $f(V^c)$].



* $f : [0, 2\pi) \longrightarrow S^1$

$$S^1 = \{ (x_1, x_2) \mid x_1^2 + x_2^2 = 1 \}$$

$$f(t) = (\cos t, \sin t)$$

$$0 \le t < 2\pi$$

f^{-1} is not continuous.

$$f^{-1} : S^1 \longrightarrow [0, 2\pi)$$

Any bijective continuous function's inverse may not be continuous. (Counter example Above)

Continuous takes → open set to open set

Uniformly continuous function:

$f : X \to Y$

f is said to be uniformly continuous on X if given $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.

$$d_Y(f(p), f(q)) < \epsilon, \text{ whenever } \forall p, q \in X$$

$$d_X(p, q) < \delta$$

Check difference

Def^n Cont. & Uni. Cont.

$\boxed{\delta \text{ depends on point } p \text{ \& } \epsilon \text{ in case of continuity def}^n.}$

$\boxed{\delta \text{ depends only on } \epsilon \text{ in uniformly cont. def}^n.}$

Continuous def^n is for a point $x = p$, where as eg) $f(x) = \frac{1}{x}$ on $(0,1)$

we ... it is not uniformly conti.

$\exists\ \varepsilon$ s.t. $\forall\ \delta > 0$ $\exists\ x, y \in (0,1)$ with

$$|x - y| < \delta$$

and $|f(x) - f(y)| \geq \epsilon$

Choose a +ve integer n s.t.

$\frac{1}{n} < \delta$. let $x_0 = 1/n$, $y_0 = 1/n+1$

$x_0, y_0 \in (0,1)$

$$|x_0 - y_0| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n} < \delta$$

$$|f(x_0) - f(y_0)|$$

$$= |n - (n+1)| = 1 > \epsilon_0$$ (where we choose $\epsilon_0 < 1$)

So $f(x) = \frac{1}{x}$ is not uniformly continuous

Thm: Let f be a continuous map from a compact matric space to a matric space Y. Then X is uniformly continuous.

Pf: Let $\epsilon > 0$ be given. Since f is Continuous we can associate to each pt. $p \in X$ a positive number $\phi(p)$ s.t. $q \in X$, $d_X(p, q) < \phi(p)$

$$\Rightarrow d_Y(f(p), f(q)) < \epsilon/2 \quad \text{——} (*)$$

Let $J(p) = \{q \in X \mid d_X(p, q) < \frac{1}{2}\phi(p)\}$

$J(p) \neq \phi$ as $p \in J(p)$

$$\left(d_X(p, p) = 0 < \frac{1}{2}\underset{\text{non -ve}}{\underset{\downarrow}{\phi(p)}} \right)$$

$J(p)$ is a open set by defn.

X is compact,

and the collection of all sets $J(p)$ is an open covering of X;

and since X is compact, there is a finite set of points $p_1, \dots p_n$ in X Such that X is contained in $\cup(J(p_i))$

$$X \subset J(p_1) \cup J(p_2) \dots \cup J(p_n) \quad \text{——} (**)$$

Put $\delta = \frac{1}{2}\min(\phi(P_1), \dots, \phi(P_n))$, $\delta > 0$

Now take p, q in X s.t. $d_X(p,q) \le \delta$

By (**), there is an integer m,

$1 \le m \le n$ s.t.

$p \in J(P_m)$, hence

$$d_X(p, P_m) < \frac{1}{2}\phi(P_m)$$

$$\begin{aligned} d_X(q, P_m) &\le d_X(q,p) + d_X(p, P_m) \\ &< \delta + \frac{1}{2}\phi(P_m) \\ &\le \phi(P_m) \end{aligned}$$

$$\begin{aligned} d_Y(f(p), f(q)) &\le d_Y(f(p), f(P_m)) \\ &\quad + d_Y(f(P_m), f(q)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

31-oct f is continuous Mapping

* If E is a connected subset of X then

f(E) is connected.

Proof: Suppose f(E) is not connected.

$$\Rightarrow \quad f(E) = A \cup B, \; A, B \neq \phi, \; \bar{A} \cap B = \phi,$$
$$A \cap \bar{B} = \phi$$

$$G = f^{-1}(A) \cap E$$
$$H = f^{-1}(B) \cap E$$

$$G \cup H = (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B))$$
$$= E \cap (f^{-1}(A) \cup f^{-1}(B))$$
$$= E \cap f^{-1}(A \cup B)$$
$$= E$$

$\rightarrow A \subset \bar{A}$

$f(A) \subset f(\bar{A})$

We know $G \subset f^{-1}(A)$ & $G \subset E$

now, $\Rightarrow \underline{G \subset f^{-1}(\bar{A})}$ ↳ closed

$\underline{\bar{G} \subset f^{-1}(\bar{A})}$ (bcoz f is continuous fun. ↑ pulls closed to closed.

↳ already closed.

$H \subset f^{-1}(B)$

~~$\bar{G}$~~ $\bar{G} \cap H = \phi \longrightarrow$ says E is

$\bar{G} \cap H \subset f^{-1}(\bar{A} \cap B)$

Th: let f be a continuous function on $[a,b]$. If $f(a) < f(b)$ and λ is a number s.t. $f(a) < \lambda < f(b)$; then $\exists\ x \in (a,b)$ s.t. $f(x) = \lambda$.

Pf: As $[a,b]$ is connected (as it is an interval in $\mathbb{R}$)

$E = f([a,b])$ is connected subset of $\mathbb{R}$ (as f is continuous). Now $f(a) \in E$, $f(b) \in E$.

Since $f(a) < \lambda < f(b)$, and $\left[f(A) = \{f(a) \mid a \in A\} \right]$

E is connected. $\lambda \in E$, so $\exists\ p \in [a,b]$

$f(p) = \lambda$.

→ $p \neq a$, if $p = a$ then $f(a) = f(p) = \lambda$ but it is given $\lambda > f(a)$, so $p \neq a$.

→ Same $p \neq b$. $\lambda < f(b)$

Type of discontinuity.

$f(x^+) \Rightarrow$ right limit.

$f(x^-) \Rightarrow$ left limit.

(i) $f(x^+) \neq f(x^-)$

(ii) $f(x^+) = f(x^-) \neq f(x)$

1. Type one discontinuity
2. Type two disco

Th: Let f be monotonically increasing on (a,b). Then $f(x+)$ and $f(x-)$ exists at every pt of x of (a,b).

Same is true for monotonically decreasing function.

Corollary: Monotonic functions have no discontinuities of second type. (As $f(x^+)$, $f(x^-)$ exists)

Th: Let f be monotonic on (a,b). Then the set of points of (a,b) at which f is discontinuous is at most countable.

$f: (a,b) \to \mathbb{R}$

f is diff at x:

$\phi(t) = \dfrac{f(x) - f(t)}{x-t}$

$\lim_{t \to x} \phi(t) = f'(x)$

But if $f: \mathbb{R}^2 \to \mathbb{R}$, this def^n of $f'(x)$ won't work.

1-Nov

Th: Suppose f is differentiable in (a,b)

(a) If $f'(x) \geq 0$ $\forall x \in (a,b)$, then f is monotonically increasing.

(b) If $f'(x) = 0$, $\forall x \in (a,b)$, then f is consta

(c) If $f'(x) \leq 0$, $\forall$ —, f is monoti- decreasing

Let $f: X \to \mathbb{R}$ be a given function we say f has a local maximum at pt $p \in X$ if $\exists \delta > 0$ s.t. $f(q) \leq f(p)$ $\forall q$ with $d(p,q) < \delta$

(Same if $f(q) \geq f(p)$ f has a local minimum)

Thm: Let f be defined on $[a,b]$, if f has a local maximum at a pt $x \in (a,b)$ and $f'(x)$ exists, then $f'(x) = 0$.

Thm: Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a pt $x \in (a,b)$ s.t. $f'(x) = \lambda$.

Pf:
$$g(t) = f(t) - \lambda t$$
$$g'(t) = f'(t) - \lambda$$

$$g'(a) = f'(a) - \lambda < 0$$
So $g(t_1) < g(a)$ for $t_1 \in (a,b)$

and $g'(b) > 0$, So that $g(t_2) < g(b)$ for some $t_2 \in (a,b)$

(Ex) Suppose ~~f(x) >~~ $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be the inverse of f. Prove that g is differentiable and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a<x<b)$$

$\rightarrow$ Let $a < x_1 < x_2 < b$, on $[x_1, x_2]$, f is differentiable

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(x^*), \quad x^* \in (x_1, x_2)$$

$f'(x^*) > 0$

$\Rightarrow f(x_2) - f(x_1) > 0$

$\Rightarrow \underline{f(x_1) < f(x_2)}$

As g is the inverse of f

$g(f(x)) = x, \; x \in X$

$f(g(y)) = y, \; y \in Y$

$g'(f(x)) \cdot f'(x) = 1$

Mean value Theorem and L'Hospital rule are not valid for complex valued function.

Counter Ex.) $f(x) = e^{ix}$

$f : [0, 2\pi] \to \mathbb{C}$

$f(2\pi) - f(0) = e^{i(2\pi)} - e^{i.0}$

$= 1-1 = 0$

$f'(x) = i e^{ix}$.

$|f'(x)| = 1 \to f'(x) \neq 0, \forall x$

$\underbrace{f(2\pi) - f(0) = f'(c)(2\pi - 0)}_{\text{will not hold true.}}$

$f : (0,1) \to \mathbb{R}$

$g : (0,1) \to \mathbb{C}$

$f(x) = x$

$g(x) = x + x^2 e^{\frac{i}{x^2}}$

$\lim_{x \to 0} \frac{x}{x + x^2 e^{(i/x^2)}}$

$= \frac{x}{x(1 + x e^{i/x^2})}$

$= 1$

But if we apply
L'Hon—

$$\frac{|f'(x)|}{|g'(x)|} = \frac{1}{|g'(x)|} \le \frac{x}{2-x}$$

$$g'(x) = 1 + 2xe^{i/x^2} + ie^{(i/x^2)}\left(\frac{-2}{x}\right)$$

$$|g'(x)| \geq \left|2x - \frac{2i}{x}\right| - 1$$

$$= \frac{2}{x}\sqrt{x^4+1} - 1$$

$$|g'(x)| \geq \frac{2}{x} - 1$$

2-Nov

Riemann Integration

Let $f: [a,b] \to \mathbb{R}$ be a bounded function.

let $P = \{x_0, x_1, x_2, \dots x_n\}$ with $\Delta x_i = x_i - x_{i-1}$

$$a = x_0 \le x_1 \le \dots \le x_n = b$$

be a partition in $[a,b]$.

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

$$m_i = \inf (f(x)) \quad x \in [x_{i-1}, x_i]$$

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i \quad \text{(Upper Riemann sum)}$$

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i \quad \text{(Lower ")}$$

$$\overline{\int_a^b} f\,dx = \inf_{\forall \text{ partition } P} U(f, P) \quad \text{(Upper Riemann integral)}$$

$$\underline{\int_a^b} f\,dx = \sup_{\forall P} L(f, P) \quad \text{(Lower Riemann integral)}$$

** $f \in R$ (Riemann integrable) if $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$

$\int_a^b f$ Common.

Since f is bounded on $[a, b]$

$\exists$ bounds M, and m s.t.

$m \le f(x) \le M \quad (a \le x \le b)$

$\Rightarrow \underline{\underline{m(b-a)}} \le L(P, f) \le U(P, f) \le M(b-a)$

$\hookrightarrow \left\{ \sum_{i=1}^{n} m \Delta x_i \le \sum_{i=1}^{n} m_i \Delta x_i \right\}$

$\left\{ \sum_{i=1}^{n} M \Delta x_i = M \sum_{i=1}^{n} \Delta x_i = M(b-a) \right\}$

$\underline{Def^n}$: Let α be a monotonically inc. function on $[a,b]$. Since $\alpha(a)$, $\alpha(b)$ are finite, α is bounded. Corresponding to each partition P of $[a,b]$, we write

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$, $\Delta\alpha_i \geq 0$, as α is monotonically increasing.

→ Now for any real function f which is bounded on $[a,b]$.

We put
$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i,$$
$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i.$$

(3) $$\overline{\int_a^b} f\,d\alpha = \inf_{\forall P} U(P,f,\alpha)$$

(4) $$\underline{\int_a^b} f\,d\alpha = \sup_{\forall P} L(P,f,\alpha)$$

if LHS. of (3) & (4) are equal, we denote their common value by $\int_a^b f\,d\alpha$ or $\int_a^b f(x)\,d\alpha(x)$

→ Riemann – Stieltjes Integral, when $\alpha(x) = x$, it reduces to Riemann Integral.

if, $f \in \mathcal{R}(\alpha) \to f$ is Riemann–Stieltjes integral.

{ $\alpha(a)$, $\alpha(b)$ are finite, becoz monotonic Inc. fun. on closed bounded subset }

★ P_1 and P_2 are two partition in $[a, b]$

$P_1 \supset P_2$

$\Rightarrow$ we say P_1 is refinement of P_2.

Given two partition P_1 and P_2, we say $P^* = P_1 \cup P_2$ is their common refinement.

<u>Th:</u> If P^* is a refinement of P, then

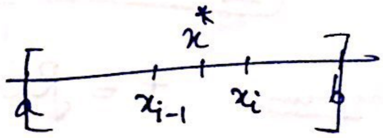
ⓐ $L(P, f, \alpha) \le L(P^*, f, \alpha)$ and

ⓑ $U(P^*, f, \alpha) \le U(P, f, \alpha)$

$$\left\{ \begin{array}{l} S_0 \subset S, \ S_0 \neq \phi, \ S \text{ is bounded} \\ \inf S \le \inf S_0 \le \sup S_0 \le \sup S \end{array} \right\}$$

To prove ⓐ

Suppose P^* contains just one pt more than P. Let the extrapoint be x^* and Suppose $x_{i-1} < x_* < x_i$, x_{i-1}, x_i are two consecutive pts of P.



put $\omega_1 = \inf\limits_{x \in [x_{i-1}, x_*]} f(x)$

$\omega_2 = \inf\limits_{x \in [x^*, x_i]} f(x)$

$m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$

$\omega_1 \ge m_i$

$\omega_2 \ge m_i$

Scanned with CamScanner

Hence $L(P^*, f, \alpha) - L(P, f, \alpha)$

$$= \omega_1[\alpha(x^*) - \alpha(x_{i-1})] + \omega_2[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x_{i-1})]$$

$$= \omega_1[\alpha(x^*) - \alpha(x_{i-1})] + \omega_2[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1})]$$

$$= \underbrace{(\omega_1 - m_i)}_{\geq 0}[\underbrace{\alpha(x^*) - \alpha(x_{i-1})}_{\geq 0}] + \underbrace{(\omega_2 - m_i)}_{\geq 0}[\underbrace{\alpha(x_i) - \alpha(x^*)}_{\geq 0}]$$

$$\geq 0$$

$$\Rightarrow L(P^*, f, \alpha) \geq L(P, f, \alpha)$$

Same can be proved for (b) part.

★ $\boxed{\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha}$ Always.

Theorem: $f \in \mathcal{R}(\alpha)$ on $[a, b]$ iff $\forall\, \epsilon > 0$, $\exists$ a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \quad (*)$$

Proof: Suppose (*) holds.

We know

$$L(P, f, \alpha) \leq \underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha \leq U(P, f$$

$$L(P, f, \alpha) - L(P, f, \alpha) \leq \underline{\int_a^b} f\, d\alpha - L$$

$$\leq \overline{\int_a^b} f\, d\alpha - L \leq U(P, f, \alpha) - L(P, f, \alpha)$$

$\Rightarrow 0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f < \epsilon$

$\left\{ b\cos z \quad 0 \leq \underline{\int_a^b} f - L \leq \epsilon, \quad 0 \leq \overline{\int_a^b} f - L < \epsilon \right\}$

difference of these two $0 \leftrightarrow \epsilon$

$$\Rightarrow \overline{\int_a^b} f - \underline{\int_a^b} f = 0$$

$\Rightarrow f \in \mathcal{R}(\alpha)$

---

now, let us say $f \in \mathcal{R}(\alpha)$

to show that $\forall \epsilon > 0$, $\exists$ a partition P st. $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

given $\epsilon > 0$, $\exists$ a partition P_1 s.t.

$$\frac{\epsilon}{2} + \int_a^b f\,d\alpha > U(P_1, f, \alpha)$$

$$U(P_1, f, \alpha) - \int_a^b f\,d\alpha < \epsilon/2 \longrightarrow (1)$$

Same, $(\exists P_2)$ $\int_a^b f\,d\alpha - L(P_2, f, \alpha) < \epsilon/2 \longrightarrow (2)$

Let $P = P_1 \cup P_2$

Adding (1) & (2)

$$\begin{aligned} U(P, f, \alpha) &\leq U(P_1, f, \alpha) \\ &< \int f\,d\alpha + \epsilon/2 \\ &< L(P_1, f, \alpha) + \epsilon/2 + \epsilon/2 \\ &\leq L(P_1, f, \alpha) + \epsilon \end{aligned}$$

$\Rightarrow U(P, f, \alpha) < L(P, f, \alpha) + \epsilon$ Proved

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$$f(x) = \begin{cases} 1 & x \text{ is rational in } [a,b] \\ 0 & \text{Otherwise.} \end{cases}$$

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = 1$$

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = 0$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = b-a.$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = 0$$

$$\overline{\int_a^b} f\,dx = b-a$$

$$\underline{\int_a^b} f\,dx = 0$$

two are different So not a Reimann integral.

<u>Thm:</u> If f is continuous on $[a,b]$, then $f \in R(\alpha)$ on $[a,b]$.

<u>Proof:</u> As α is monotonically increasing

$a < b$

$\Rightarrow \alpha(a) \leq \alpha(b)$

i.e. $\alpha(b) - \alpha(a) \geq 0$

Let $\epsilon > 0$ be given. Choose $\eta > 0$ s.t.

$$[\alpha(b) - \alpha(a)]\eta < \epsilon$$

$$\left(\eta < \frac{\epsilon}{\alpha(b) - \alpha(a)} \right)$$

for $\alpha(b) - \alpha(a) = 0$

$0\eta < \epsilon$

$0 < \epsilon$ ✓

Since f is uniformly continuous on $[a,b]$,

$\exists\ \delta > 0$ s.t.

$$|f(x) - f(t)| < \eta \quad \text{if } x \in [a,b],\ t \in [a,b] \text{ and } |x-t| < \delta$$

If P is any partition of $[a,b]$ such that

$\Delta\alpha_i < \delta \quad \forall i,$

then $|f(x) - f(t)| < \eta$

$\Rightarrow |M_i - m_i| \leq \eta \quad (i = 1, 2, \ldots n)$

and therefore

$U(P, f, \alpha) - L(P, f, \alpha)$

$$= \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i \leq \eta \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \eta[\alpha(b) - \alpha(a)] < \epsilon$$

$\Rightarrow f \in \mathcal{R}(\alpha)$ from the theorem before.

Th: Suppose f is bounded on $[a,b]$, f has only finitely many points of disconti- on $[a,b]$ and α is continuous at every pt at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.

→ f is integrable on $[a,b]$ iff f has countable number of discontinuity pts

Properties of Integral.

If $f_1 \in \mathcal{R}(\alpha)$, $f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in \mathcal{R}(\alpha)$, $cf \in \mathcal{R}(\alpha)$ for every constant and $\int_a^b (f_1+f_2)\, d\alpha$

$$= \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha, \quad \int_a^b c f\, d\alpha = c\int_a^b f\, d\alpha$$

(b) $f_1(x) \le f_2(x)$ on $[a,b]$

$$\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$$

(c) $$\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha \quad (a<c<b)$$

(d) $$\left|\int_a^b f\, d\alpha\right| \le M\{\alpha(b) - \alpha(a)\}$$

$|f(x)| \le M$.

(e) If $f \in \mathcal{R}(\alpha_1)$, $f \in \mathcal{R}(\alpha_2)$

$\Rightarrow f \in \mathcal{R}(\alpha_1 + \alpha_2)$

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$\rightarrow \int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$

$\rightarrow \int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha$, c is +ve constant

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8-oct

Thm: If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a,b]$, then

(a) $fg \in R(\alpha)$ and (b) $|f| \in R(\alpha)$ and

$$\left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha.$$

Unit Step

$$I(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

Thm: If $a<s<b$, f is bounded on $[a,b]$, f is continuous at s and $\alpha(x) = I(x-s)$, then

$$\boxed{\int_a^b f\, d\alpha = f(s)}$$

Thm: Suppose $c_n \ge 0$ for $n = 1, 2, \dots$, $\sum c_n < \infty$. $\{s_n\}$ is a sequence of distinct pts in (a,b) and $\alpha(x) = \sum_{n=0}^{\infty} c_n I(x-s_n)$.

Let f be continuous on $[a,b]$, then

$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

Thm : Assume α inc. monotonically and $\alpha' \in \mathbb{R}$ on $[a, b]$. Let f be a bounded real fun. on $[a, b]$. Then $f \in \mathbb{R}(\alpha)$ iff $f\alpha' \in \mathbb{R}$.

In that case $\int_a^b f\, d\alpha = \int_a^b f(x)\, \alpha'(x)\, dx$

Integration & diff

Th: $f \in R$ on $[a, b]$. For $a \leq x \leq b$, put

$$F(x) = \int_a^x f(t)\, dt.$$

Then F is continuous on $[a, b]$, furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differ at x_0 and $F'(x_0) = f(x_0)$.

Fundam thm of calculus

If $f \in R$ on $[a, b]$ and f there is a differentiable fun F on $[a, b]$ s.t. $F' = f$, then $\int_a^b f(x)\, dx = F(b) - F(a)$

Vector integration

$\vec{f} = (f_1, f_2, \dots f_n)$

$\int_a^b \vec{f}\, dx = \int_a^b (f_1 dx, f_2 dx, \dots, f_n dx)$

$\vec{f}\, dx$ is a pt in $\mathbb{R}^n$.

Seq^n^ce and Series of functions

$\langle f_n \rangle$ is a sequence of continuous fun.

$f_n : E \to \mathbb{R}$

(f_n) converges ptwise

$\epsilon > 0$, $\exists N$ s.t. $\{ N = N(\epsilon, x) \}$

iff $|f_n(x) - f(x)| < \epsilon$, $n \geq N$

$\{ f_n(x) \}$ converges for all $x \in E$.

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eg $f_n(x) = \frac{2n}{1+x^2}$

$f_n(0) = 2n$, $f_1(0) = 2$, $f_2(0) = 4$, $f_3(0) = 6 \dots$

Continu

$f_n \to f$

does not mean f is contin.

Double Sequence.

$m = 1, 2, \ldots, n = 1, 2, \ldots$

$$S_{m,n} = \frac{m}{m+n}$$

Fixed n

$$\lim_{m\to\infty} S_{m,n} = \lim_{m\to\infty} \frac{m}{m+n} = 1$$

$$\lim_{n\to\infty} \lim_{m\to\infty} S_{m,n} = 1$$

Fixed m

$$\lim_{n\to\infty} S_{m,n} = \lim_{n\to\infty} \frac{m}{m+n} = 0$$

$$\lim_{m\to\infty} \lim_{n\to\infty} S_{m,n} = 0$$

$$1 \neq 0$$

We cannot interchange m & n.