Duality Theory of LPP

Prof. M.P. Biswal

Department of Mathematics

IIT- Kharagpur
E-Mail: mpbiswal@maths.iitkgp.ac.in

February 8, 2021

Single Objective Linear Programming Models

General form of a linear programming problem is given by:

$$\text{(I)} \qquad \max : Z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, 2, ..., m$$

$$x_j \geq 0, \qquad j = 1, 2, ..., n$$

(II) $$\min : Z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \qquad i = 1, 2, ..., m$$

$$x_j \geq 0, \qquad j = 1, 2, ..., n.$$

(III) $$\max : Z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, 2, \ldots, m_1$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = m_1 + 1, m_1 + 2, \ldots, m_2$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \qquad i = m_2 + 1, m_2 + 2, \ldots, m$$

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n$$

(IV) $$\min : Z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, 2, ..., m_1$$
$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = m_1 + 1, m_1 + 2, ..., m_2$$
$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \qquad i = m_2 + 1, m_2 + 2, ..., m$$
$$x_j \geq 0, \qquad j = 1, 2, ..., n$$

For these models (I)-(IV), it is assumed that a_{ij}, b_i, c_j and d all are deterministic real number for all i and j. Since the objective function and the constraints of the models are linear, we may apply the following Linear programming methods to find the optimal solution:

- **Simplex Method**
- **Revised Simplex Method**
- **Dual Simplex Method**
- **Charne's Penalty Method (Big-M Method)**
- **Two-Phase Simplex Method**
- **Interior Point Method of Karmarkar (1984).**

Primal and Dual LPP:

$$(\boldsymbol{P}) \qquad \max : \boldsymbol{f} = \boldsymbol{c}^{\boldsymbol{T}} \boldsymbol{X}$$

Subject to $\boldsymbol{AX} \leq \boldsymbol{b}, \boldsymbol{X} \geq 0$

where $\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \ \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n}$$

In expanded form, it can be written as:

$$(\boldsymbol{P}) \qquad \max : \boldsymbol{f} = \sum_{\boldsymbol{j}=1}^{\boldsymbol{n}} \boldsymbol{c_j x_j}$$

Subject to

$$\begin{aligned} \sum_{\boldsymbol{j}=1}^{\boldsymbol{n}} \boldsymbol{a_{ij} x_j} &\leq \boldsymbol{b_i}, \quad \boldsymbol{i} = 1, 2, ..., \boldsymbol{m} \\ \boldsymbol{x_j} &\geq 0. \quad \boldsymbol{j} = 1, 2, ..., \boldsymbol{n} \end{aligned}$$

Dual of the Primal LPP:

$$(\boldsymbol{D}) \qquad \min : \boldsymbol{f}' = \boldsymbol{b}^T \boldsymbol{Y}$$

$$\textbf{s. t.} \quad \boldsymbol{A}^T \boldsymbol{Y} \geq \boldsymbol{c}, \boldsymbol{Y} \geq 0$$

$$\textbf{where } \boldsymbol{Y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_{\boldsymbol{n}} \end{pmatrix}.$$

In expanded form, it can be written as:

$$(D) \qquad \min : f' = \sum_{i=1}^{m} b_i y_i$$

s. t.

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &\geq c_j, \quad j = 1, 2, ..., n \\ y_i &\geq 0. \quad i = 1, 2, ..., m \end{aligned}$$

How to find Dual of an LPP ?
We will use Lagrange Multipliers Method to establish the Dual. Details are given below:

$$
\begin{aligned}
(P) \qquad \max : f &= \sum_{j=1}^{n} c_j x_j \\
\Rightarrow \quad \min : -f &= -\sum_{j=1}^{n} c_j x_j
\end{aligned}
$$

s. t.

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, ..., m \tag{1}$$

$$x_j \geq 0. \quad j = 1, 2, ..., n$$

Add a slack variable to (1)

$$\begin{aligned}
s_i^2 &\geq 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i &= 0, \quad i = 1, 2, ..., m \qquad (2) \\
\text{where} \quad x_j &\geq 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad - x_j &\leq 0, \quad j = 1, 2, ..., n \\
-x_j + t_j^2 &= 0, \quad j = 1, 2, ..., n \\
\text{Slack variable} \quad t_j^2 &\geq 0, \quad j = 1, 2, ..., n
\end{aligned}$$

Let $L(\ldots)$ be the Lagrange Function.

$$L(X, S, T, \lambda, \mu) = -\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i \right) + \sum_{j=1}^{n} \mu_j \left(-x_j + t_i^2 \right)$$

$$\begin{aligned} \textbf{where} \quad \lambda_1, \lambda_2, ..., \lambda_m &\geq 0 \\ \mu_1, \mu_2, ..., \mu_n &\geq 0 \\ x_1, x_2, ..., x_n &\geq 0 \\ s_1^2, s_2^2, ..., s_m^2 &\geq 0 \\ t_1^2, t_2^2, ..., t_n^2 &\geq 0 \end{aligned}$$

All the Lagrange multipliers $\lambda_i, \forall\ i$ and $\mu_j, \forall\ j$ are non-negative. Total number of variables are $2m + 3n$. There are $m + n$ number of Lagrange multipliers. Using the Necessary Conditions for the minimum we have:

$$
\begin{aligned}
\frac{\partial L}{\partial x_j} &= 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad -c_j + \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j &= 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} - c_j &= \mu_j, \ \text{but} \quad \mu_j \geq 0 \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} - c_j &\geq 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} &\geq c_j, \quad j = 1, 2, ..., n \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \lambda_i} &= 0, \quad i = 1, 2, ..., m \\
\sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i &= 0, \ \textbf{but} \ \ s_i^2 \geq 0 \\
\Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad i = 1, 2, ..., m \qquad (4) \\
\frac{\partial L}{\partial s_i} &= 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad 2 s_i \lambda_i &= 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad s_i \lambda_i &= 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad s_i^2 \lambda_i &= 0, \quad i = 1, 2, ..., m \qquad (5)
\end{aligned}
$$

$$\Rightarrow \quad \lambda_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) = 0, \quad i = 1, 2, \ldots, m$$

$$\Rightarrow \quad \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) = 0, \quad i = 1, 2, \ldots, m$$

$$\frac{\partial L}{\partial \mu_j} = 0, \quad j = 1, 2, \ldots, n$$

$$\Rightarrow \quad - x_j + t_j^2 = 0, \quad j = 1, 2, \ldots, n$$

$$\Rightarrow \quad x_j = t_j^2, \quad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial t_j} = 0, \quad j = 1, 2, ..., n$$

$$2\mu_j t_j = 0, \quad j = 1, 2, ..., n$$

$$\Rightarrow \quad \mu_j t_j = 0, \quad j = 1, 2, ..., n$$

$$\textbf{So} \quad \mu_j t_j^2 = 0, \quad j = 1, 2, ..., n$$

From the last equation $t_j^2 = x_j$

$$\mu_j x_j = 0, \quad j = 1, 2, ..., n \tag{6}$$

where

$$\sum_{i=1}^{m} \lambda_i a_{ij} - c_j = \mu_j$$

From (3), we know that

$$c_j \leq \sum_{i=1}^{m} \lambda_i a_{ij}$$

Multiply both side by $x_j (\geq 0)$

$$
\begin{aligned}
c_j x_j &\leq \left(\sum_{i=1}^{m} a_{ij} \lambda_i \right) x_j, \quad \forall\, j \\
\Rightarrow \quad \sum_{j=1}^{n} c_j x_j &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \lambda_i \right) x_j,
\end{aligned}
$$

We can interchange the summation notation

$$
\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j \right),
$$

$$
\begin{aligned}
\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \quad &\leq \quad \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j \right), \text{ but } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \forall\, i, \\
\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \quad &\leq \quad \sum_{i=1}^{m} b_i \lambda_i \qquad (7)
\end{aligned}
$$

where $\lambda_1, \lambda_2, ..., \lambda_m$ are multipliers called the dual variable ($\lambda_1, \lambda_2, ..., \lambda_m \geq 0$)

$$
\begin{aligned}
\textbf{Let} \quad y_i &= \lambda_i, \quad i = 1, 2, \ldots, m \\
y_i &\geq 0, \quad i = 1, 2, \ldots, m \\
\sum_{j=1}^{n} c_j x_j &\leq \sum_{i=1}^{m} b_i y_i, \quad \lambda_i = y_i \\
\textbf{Now} \quad \Rightarrow \quad c^T X &\leq b^T Y \\
\Rightarrow \quad f &\leq f' && (8) \\
\textbf{Also} \quad \max: \quad c^T X &\leq \min: \quad b^T Y && (9) \\
\max: \quad f &\leq \min: f' && (10) \\
\max: \quad f &= \min: f' && (11)
\end{aligned}
$$

(12)

This is called Strong Duality.

Also we have

$$\min: \; \boldsymbol{b}^T \boldsymbol{Y} = \sum_{i=1}^{m} \boldsymbol{b_i y_i}$$

s. to

$$\begin{aligned}
\sum_{i=1}^{m} a_{ij}\lambda_i &\geq c_j \quad j = 1, 2, ..., n, \quad \textbf{from } (3) \\
\textbf{Since } \lambda_i &= y_i \geq 0 \\
\sum_{i=1}^{m} a_{ij} y_i &\geq c_j \quad j = 1, 2, ..., n
\end{aligned}$$

Finally we have Dual LPP

$$(D) \qquad \min : f' = \sum_{i=1}^{m} b_i y_i$$

s. to

$$\begin{aligned} \sum_{i=1}^{m} a_{ji} y_j &\geq c_j, \quad j = 1, 2, ..., n \\ y_i &\geq 0. \quad i = 1, 2, ..., m \end{aligned}$$

m+n Pairs of Complementary Conditions:

$$(\sum_{j=1}^{n} a_{ij}x_j - b_i)y_i = 0, \quad i = 1, 2, ..., m$$

where

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \qquad y_i \geq 0, \qquad i = 1, 2, ..., m$$

$$(\sum_{i=1}^{m} a_{ij}y_i - c_j)x_j = 0, \quad j = 1, 2, ..., n$$

where

$$\sum_{j=1}^{m} a_{ij}y_i \geq c_j, \qquad x_j \geq 0, \qquad j = 1, 2, ..., n$$

Numerical Example:1
Primal LPP:

$$\max : \boldsymbol{Z} = \boldsymbol{X}_1 + 3\boldsymbol{X}_2$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 \leq 10$$

$$\boldsymbol{X}_1 + 2\boldsymbol{X}_2 \leq 11$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 \leq 16$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \geq 0$$

DUAL LPP:

$$\min : \boldsymbol{Z'} = 10\boldsymbol{Y}_1 + 11\boldsymbol{Y}_2 + 16\boldsymbol{Y}_3$$

Subject to

$$\boldsymbol{Y}_1 + \boldsymbol{Y}_2 + \boldsymbol{Y}_3 \geq 1$$

$$\boldsymbol{Y}_1 + 2\boldsymbol{Y}_2 + 4\boldsymbol{Y}_3 \geq 3$$

$$\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Y}_3 \geq 0$$

1. Please Check $X_1 = 6, X_2 = 2.5$ is a feasible solution of the LPP.
2. Please Check $X_1 = 6, X_2 = 2.5$ is an Optimal solution of the LPP using Duality Theory.
3. Can you find an alternate optimal solution of the given LPP ?

We have five pairs of complimentary conditions:

$$(\boldsymbol{X}_1 + \boldsymbol{X}_2 - 10)\boldsymbol{Y}_1 = 0$$

$$(\boldsymbol{X}_1 + 2\boldsymbol{X}_2 - 11)\boldsymbol{Y}_2 = 0$$

$$(\boldsymbol{X}_1 + 4\boldsymbol{X}_2 - 16)\boldsymbol{Y}_3 = 0$$

$$(\boldsymbol{Y}_1 + \boldsymbol{Y}_2 + \boldsymbol{Y}_3 - 1)\boldsymbol{X}_1 = 0$$

$$(\boldsymbol{Y}_1 + 2\boldsymbol{Y}_2 + 4\boldsymbol{Y}_3 - 3)\boldsymbol{X}_2 = 0$$

$$\boldsymbol{X}_1 = 6, \boldsymbol{X}_2 = 2.5, \boldsymbol{X}_1 + \boldsymbol{X}_2 < 10, \boldsymbol{Y}_1 = 0$$

$$(\boldsymbol{Y}_1 + \boldsymbol{Y}_2 + \boldsymbol{Y}_3 - 1) = 0, (\boldsymbol{Y}_1 + 2\boldsymbol{Y}_2 + 4\boldsymbol{Y}_3 - 3) = 0, \boldsymbol{Y}_2 = \boldsymbol{Y}_3 = 1/2$$

$$\min : \boldsymbol{Z'} = 13.5, \max : \boldsymbol{Z} = 13.5$$

Given solution is an optimal solution.

Numerical Example:2
Primal LPP:

$$\min : \boldsymbol{Z} = 2\boldsymbol{X}_1 + 6\boldsymbol{X}_2$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 \geq 10$$

$$\boldsymbol{X}_1 + 2\boldsymbol{X}_2 \geq 11$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 \geq 16$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory. Check X=(8,2)
2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example:3
Primal LPP:

$$\max : \boldsymbol{Z} = \boldsymbol{X}_1 + 4\boldsymbol{X}_2$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 \leq 10$$

$$\boldsymbol{X}_1 + 2\boldsymbol{X}_2 \leq 11$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 \leq 16$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory. Check X=(0,4)
2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example:4
Primal LPP:

$$\min : \boldsymbol{Z} = \boldsymbol{X}_1 + 4\boldsymbol{X}_2$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 \geq 10$$

$$\boldsymbol{X}_1 + 2\boldsymbol{X}_2 \geq 11$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 \geq 16$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory. Check X=(8,2),(0,4).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 5
Primal LPP:

$$\max : \boldsymbol{Z} = \boldsymbol{X}_1 + 6\boldsymbol{X}_2$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 \leq 10$$

$$\boldsymbol{X}_1 + 2\boldsymbol{X}_2 \leq 12$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 \leq 16$$

$$\boldsymbol{X}_1 + 6\boldsymbol{X}_2 \leq 20$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory. Check X=(8,2),(0,10/3).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 6
Primal LPP:

$$\min : \boldsymbol{Z} = \boldsymbol{X}_1 + 8\boldsymbol{X}_2$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 \geq 10$$

$$\boldsymbol{X}_1 + 2\boldsymbol{X}_2 \geq 12$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 \geq 16$$

$$\boldsymbol{X}_1 + 6\boldsymbol{X}_2 \geq 20$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(20,0)

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 7
Primal LPP:

$$\max : \boldsymbol{Z} = \boldsymbol{X}_1 + 10\boldsymbol{X}_2$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 \leq 10$$

$$\boldsymbol{X}_1 + 2\boldsymbol{X}_2 \leq 12$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 \leq 16$$

$$\boldsymbol{X}_1 + 6\boldsymbol{X}_2 \leq 20$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(8,2),(0,10/3).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 8
Primal LPP:

$$\min : \boldsymbol{Z} = \boldsymbol{X}_1 + 12\boldsymbol{X}_2$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 \geq 10$$

$$\boldsymbol{X}_1 + 2\boldsymbol{X}_2 \geq 12$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 \geq 16$$

$$\boldsymbol{X}_1 + 6\boldsymbol{X}_2 \geq 20$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(20,0)

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 9
Primal LPP:

$$\max : \boldsymbol{Z} = 6\boldsymbol{X}_1 + 6\boldsymbol{X}_2$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 \leq 10$$

$$\boldsymbol{X}_1 + 2\boldsymbol{X}_2 \leq 12$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 \leq 16$$

$$\boldsymbol{X}_1 + 8\boldsymbol{X}_2 \leq 24$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(8,2),(10,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 10
Primal LPP:

$$\max : \boldsymbol{Z} = 20\boldsymbol{X}_1 + 50\boldsymbol{X}_2$$

Subject to

$$3\boldsymbol{X}_1 + 2\boldsymbol{X}_2 \leq 25$$

$$2\boldsymbol{X}_1 + 5\boldsymbol{X}_2 \leq 30$$

$$2\boldsymbol{X}_1 + 3\boldsymbol{X}_2 \leq 20$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(5/2, 5).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 11
Primal LPP:

$$\max : \boldsymbol{Z} = 6\boldsymbol{X}_1 + 6\boldsymbol{X}_2 + 8\boldsymbol{X}_3$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 \leq 12$$

$$3\boldsymbol{X}_1 + 3\boldsymbol{X}_2 + 4\boldsymbol{X}_3 \leq 36$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(6,6,0),(0,0,9).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 12
Primal LPP:

$$\min : \boldsymbol{Z} = 6\boldsymbol{X}_1 + 6\boldsymbol{X}_2 + 8\boldsymbol{X}_3$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 \geq 12$$

$$3\boldsymbol{X}_1 + 2\boldsymbol{X}_2 + 3\boldsymbol{X}_3 \geq 30$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(6,6,0),(12,0,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 13
Primal LPP:

$$\min : \boldsymbol{Z} = 8\boldsymbol{X}_1 + 8\boldsymbol{X}_2 + 6\boldsymbol{X}_3$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 \geq 18$$

$$4\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + 3\boldsymbol{X}_3 \geq 60$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(6,0,12),(0,6,12).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 14
Primal LPP:

$$\max : \boldsymbol{Z} = 6\boldsymbol{X}_1 + 6\boldsymbol{X}_2 + 8\boldsymbol{X}_3$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 \leq 12$$

$$3\boldsymbol{X}_1 + 2\boldsymbol{X}_2 + 4\boldsymbol{X}_3 \leq 40$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,10)

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 15
Primal LPP:

$$\max : \boldsymbol{Z} = 6\boldsymbol{X}_1 + 9\boldsymbol{X}_2 + 6\boldsymbol{X}_3$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 \leq 20$$

$$3\boldsymbol{X}_1 + 3\boldsymbol{X}_2 + 4\boldsymbol{X}_3 \leq 48$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,10,9/2).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 16
Primal LPP:

$$\max : \boldsymbol{Z} = \boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 + 3\boldsymbol{X}_4$$

Subject to

$$\boldsymbol{X}_1 - \boldsymbol{X}_2 + \boldsymbol{X}_3 + 5\boldsymbol{X}_4 \leq 5$$

$$2\boldsymbol{X}_1 + 3\boldsymbol{X}_2 - 2\boldsymbol{X}_3 + 4\boldsymbol{X}_4 \leq 6$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,16,21,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 17
Primal LPP:

$$\min : \boldsymbol{Z} = \boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 + 3\boldsymbol{X}_4$$

Subject to

$$\boldsymbol{X}_1 - \boldsymbol{X}_2 + \boldsymbol{X}_3 + 5\boldsymbol{X}_4 \geq 10$$

$$2\boldsymbol{X}_1 + 3\boldsymbol{X}_2 - 2\boldsymbol{X}_3 + 4\boldsymbol{X}_4 \geq 12$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,0,3).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 18
Primal LPP:

$$\max : \boldsymbol{Z} = \boldsymbol{X}_1 + 2\boldsymbol{X}_2 + \boldsymbol{X}_3$$

Subject to

$$4\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 \leq 6$$

$$2\boldsymbol{X}_1 + \boldsymbol{X}_2 - \boldsymbol{X}_3 \leq 2$$

$$2\boldsymbol{X}_1 - \boldsymbol{X}_2 + 5\boldsymbol{X}_3 \leq 6$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,4,2).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 19
Primal LPP:

$$\min : \boldsymbol{Z} = \boldsymbol{X}_1 + 2\boldsymbol{X}_2 + \boldsymbol{X}_3$$

Subject to

$$4\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 \geq 18$$

$$2\boldsymbol{X}_1 + \boldsymbol{X}_2 - \boldsymbol{X}_3 \geq 6$$

$$2\boldsymbol{X}_1 - \boldsymbol{X}_2 + 5\boldsymbol{X}_3 \geq 18$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(4,0,2).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 20
Primal LPP:

$$\max : \boldsymbol{Z} = \boldsymbol{X}_1 + 3\boldsymbol{X}_2 + 4\boldsymbol{X}_3$$

Subject to

$$2\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 \leq 9$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + 3\boldsymbol{X}_3 \leq 12$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,4).
2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 21
Primal LPP:

$$\min : \boldsymbol{Z} = \boldsymbol{X}_1 + 3\boldsymbol{X}_2 + 4\boldsymbol{X}_3$$

Subject to

$$2\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 \geq 63$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + 3\boldsymbol{X}_3 \geq 84$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(24,15,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 22
Primal LPP:

$$\max : \boldsymbol{Z} = 2\boldsymbol{X}_1 + 3\boldsymbol{X}_2 + 2\boldsymbol{X}_3 + \boldsymbol{X}_4 + \boldsymbol{X}_5$$

Subject to

$$3\boldsymbol{X}_1 - 3\boldsymbol{X}_2 + 4\boldsymbol{X}_3 + 2\boldsymbol{X}_4 - \boldsymbol{X}_5 = 10$$

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4 + \boldsymbol{X}_5 = 20$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4, \boldsymbol{X}_5 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,10,10,0,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 23
Primal LPP:

$$\max : \boldsymbol{Z} = 2\boldsymbol{X}_1 + 3\boldsymbol{X}_2 + 2\boldsymbol{X}_3 + \boldsymbol{X}_4 + \boldsymbol{X}_5$$

Subject to

$$3\boldsymbol{X}_1 - 3\boldsymbol{X}_2 + 4\boldsymbol{X}_3 + 2\boldsymbol{X}_4 - \boldsymbol{X}_5 \leq 10$$

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4 + \boldsymbol{X}_5 \leq 20$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4, \boldsymbol{X}_5 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,20,0,0,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 24
Primal LPP:

$$\min : \boldsymbol{Z} = 2\boldsymbol{X}_1 + 3\boldsymbol{X}_2 + 2\boldsymbol{X}_3 + \boldsymbol{X}_4 + \boldsymbol{X}_5$$

Subject to

$$3\boldsymbol{X}_1 - 3\boldsymbol{X}_2 + 4\boldsymbol{X}_3 + 2\boldsymbol{X}_4 - \boldsymbol{X}_5 \geq 10$$

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4 + \boldsymbol{X}_5 \geq 20$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4, \boldsymbol{X}_5 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,0,20,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 25
Primal LPP:

$$\max : \boldsymbol{Z} = \boldsymbol{X}_1 + 2\boldsymbol{X}_2 + 3\boldsymbol{X}_3 + 4\boldsymbol{X}_4$$

Subject to

$$20\boldsymbol{X}_1 + 9\boldsymbol{X}_2 + 6\boldsymbol{X}_3 + \boldsymbol{X}_4 \leq 20$$
$$10\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + 2\boldsymbol{X}_3 + \boldsymbol{X}_4 \leq 10$$
$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,0,10).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 26
Primal LPP:

$$\min : \boldsymbol{Z} = \boldsymbol{X}_1 + 2\boldsymbol{X}_2 + 3\boldsymbol{X}_3 + 4\boldsymbol{X}_4$$

Subject to

$$20\boldsymbol{X}_1 + 9\boldsymbol{X}_2 + 6\boldsymbol{X}_3 + \boldsymbol{X}_4 \geq 20$$

$$10\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + 2\boldsymbol{X}_3 + \boldsymbol{X}_4 \geq 10$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(1,0,0,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 27
Primal LPP:

$$\max : \boldsymbol{Z} = 9\boldsymbol{X}_1 + 8\boldsymbol{X}_2 + 6\boldsymbol{X}_3 + 5\boldsymbol{X}_4$$

Subject to

$$7\boldsymbol{X}_1 + 10\boldsymbol{X}_2 + 4\boldsymbol{X}_3 + 9\boldsymbol{X}_4 \leq 120$$

$$3\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4 \leq 30$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,30,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 28
Primal LPP:

$$\min : \boldsymbol{Z} = 9\boldsymbol{X}_1 + 8\boldsymbol{X}_2 + 6\boldsymbol{X}_3 + 5\boldsymbol{X}_4$$

Subject to

$$7\boldsymbol{X}_1 + 10\boldsymbol{X}_2 + 4\boldsymbol{X}_3 + 9\boldsymbol{X}_4 \geq 120$$

$$3\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4 \geq 35$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,15/2,0,5)

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 29
Primal LPP:

$$\max : \boldsymbol{Z} = 5\boldsymbol{X}_1 + 6\boldsymbol{X}_2 + 4\boldsymbol{X}_3 + 2\boldsymbol{X}_4$$

Subject to

$$10\boldsymbol{X}_1 + 5\boldsymbol{X}_2 + 5\boldsymbol{X}_3 + 3\boldsymbol{X}_4 \leq 50$$

$$12\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + 6\boldsymbol{X}_3 + \boldsymbol{X}_4 \leq 48$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,10,0,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 30
Primal LPP:

$$\min : \boldsymbol{Z} = 5\boldsymbol{X}_1 + 6\boldsymbol{X}_2 + 4\boldsymbol{X}_3 + 2\boldsymbol{X}_4$$

Subject to

$$10\boldsymbol{X}_1 + 5\boldsymbol{X}_2 + 5\boldsymbol{X}_3 + 3\boldsymbol{X}_4 \geq 50$$

$$12\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + 6\boldsymbol{X}_3 + \boldsymbol{X}_4 \geq 48$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(5,0,0,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 31
Primal LPP:

$$\min : \boldsymbol{Z} = 3\boldsymbol{X}_1 + \boldsymbol{X}_2 + 2\boldsymbol{X}_3 + \boldsymbol{X}_4$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 - \boldsymbol{X}_3 + \boldsymbol{X}_4 \geq 6$$
$$\boldsymbol{X}_1 - \boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4 \geq 4$$
$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,1,0,5).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 32
Primal LPP:

$$\max : \boldsymbol{Z} = 3\boldsymbol{X}_1 + \boldsymbol{X}_2 + 2\boldsymbol{X}_3 + \boldsymbol{X}_4$$

Subject to

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 - \boldsymbol{X}_3 + \boldsymbol{X}_4 \leq 6$$

$$\boldsymbol{X}_1 - \boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4 \leq 4$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,16,10,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 33
Primal LPP:

$$\min : \boldsymbol{Z} = 8\boldsymbol{X}_1 + 3\boldsymbol{X}_2 + 8\boldsymbol{X}_3 + 6\boldsymbol{X}_4$$

Subject to

$$4\boldsymbol{X}_1 + 3\boldsymbol{X}_2 - \boldsymbol{X}_3 + 3\boldsymbol{X}_4 \geq 10$$

$$\boldsymbol{X}_1 - \boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4 \geq 15$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(15/4,0,0,0).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 34
Primal LPP:

$$\max : \boldsymbol{Z} = 8\boldsymbol{X}_1 + 3\boldsymbol{X}_2 + 8\boldsymbol{X}_3 + 6\boldsymbol{X}_4$$

Subject to

$$4\boldsymbol{X}_1 + 3\boldsymbol{X}_2 - \boldsymbol{X}_3 + 3\boldsymbol{X}_4 \leq 15$$

$$\boldsymbol{X}_1 - \boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4 \leq 10$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,9,8).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 35
Primal LPP:

$$\min : \boldsymbol{Z} = \boldsymbol{X}_1 + 2\boldsymbol{X}_2 + \boldsymbol{X}_3$$

Subject to

$$2\boldsymbol{X}_1 + \boldsymbol{X}_2 - \boldsymbol{X}_3 \geq 6$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + 5\boldsymbol{X}_3 \geq 18$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(4,0,2).

2. Can you find an alternate optimal solution of the given LPP ?

Numerical Example: 36
Primal LPP:

$$\max : \boldsymbol{Z} = \boldsymbol{X}_1 + 2\boldsymbol{X}_2 + \boldsymbol{X}_3$$

Subject to

$$2\boldsymbol{X}_1 + \boldsymbol{X}_2 - \boldsymbol{X}_3 \leq 2$$

$$\boldsymbol{X}_1 + 4\boldsymbol{X}_2 + 5\boldsymbol{X}_3 \leq 6$$

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,4,2).

2. Can you find an alternate optimal solution of the given LPP ?