$$\Rightarrow d\left(\frac{z}{1+xy}\right) = d\left(\frac{x}{1+xy}\right) = 0$$

$$\Rightarrow \boxed{z - x = c(1+xy)}$$ is the common soln.

Charpit's Method

[General method of solving equation of order one but any degree]

$f(x, y, z, p, q) = 0$ —(1)

The fundamental idea in charpits method is the introduction of second PDE of 1st order

$g(x, y, z, p, q, a) = 0$ —(2)

which contains an arbitary const. 'a' and which is such that

(a) Eqn (1) and (2) can be solved to give

$p = p(x, y, z, a)$ and $q = q(x, y, z, a)$

(b) The eqn,

$dz = p(x, y, z, a)\, dx + q(x, y, z, a)\, dy$ is integrable, (3)

When such a function 'g' has been found, the soln of (3),

$F(x, y, z, a, b) = 0$ —(4) containing two arbitary consts a, b with a soln (complete integral) of eqn (1).

The main problem then is the determination of second eq^n (2), so we need to seek an eq^n $g=0$ compatible with $f=0$.

Expanding the compatibility condition we see that it is equivalent to the linear PDE

$$f_p \frac{\partial g}{\partial x} + f_q \frac{\partial g}{\partial y} + (p f_p + q f_q) \frac{\partial g}{\partial z} - (f_x + p f_z) \frac{\partial g}{\partial p}$$

$$- (f_y + q f_z) \frac{\partial g}{\partial q} = 0 \quad \text{—} (5)$$

for the determination of g.

Our problem then is to find the sol^n of this equation, as simple as possible, involving an arb. constant 'a' and this we do by finding an integral of the subsidiary eq^n,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p f_p + q f_q} = \frac{dp}{-(f_x + p f_z)} = \frac{dq}{-(f_y + q f_z)} \quad \text{—} (6)$$

These eq^ns (6) are known as Charpit's ~~meth~~ eq^ns.

Once an integral $g(x, y, z, p, q, a)$ of this kind has been found, the problem reduces to solve for p and q and then integrating eq^n (3). It should be noted that not all of the Charpit eq^n (6) need to be used, but that p and q must occur in the solution obtained.

$p^2 y + q^2 y = 2$

Working Rule to use Charpit's method

Step 1: $f = 0$.

Transfer all terms of the given eqⁿ to LHS, and denote the entire expression by f

Step 2: Write down Charpit's auxiliary eqⁿ.

Step 3: Using the values of f in step 1, write down the values of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, etc occuring in step 2 and put them in Charpit's eqⁿ;

Step 4: ~~Sto~~ After simplifying the step 3, select two proper fractions so that the resulting integral may come out to be the simplest relation involving atleast one of p and q.

Step 5: The simplest relation of step 4 is solved along with the given eqⁿ to determine p and q. Put these values of p and q in $dz = p\,dx + q\,dy$ which on integration gives the complete integral of the given eqⁿ.

Ex: Find a complete integral of $q = 3p^2$.

Solⁿ → Here the given eqⁿ is

$$f(x, y, z, p, q) = 3p^2 - q = 0 \quad \text{--- (1)}$$

Charpit's auxiliary equation

$$\frac{dx}{6p} = \frac{dy}{-1} = \frac{dz}{6p^2 - q} = \frac{dp}{0} = \frac{dq}{0}$$

$dp = 0$

$\Rightarrow p = a$ —— (ii)

$\therefore q = 3a^2$ —— (iii)

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Now,

$dz = p\,dx + q\,dy$

$= a\,dx + 3a^2 dy$

$\Rightarrow z = ax + 3a^2 y + b$, which is a complete integral, a and b being arb. constant.

Ex! Find ~~the~~ a complete integral of

$px + qy = pq$.

Sol^n →

$f(x, y, z, p, q) = px + qy - pq = 0$

Charpit's auxiliary eq^n :

$$\frac{dx}{x-q} = \frac{dy}{y-p} = \frac{dz}{(px-pq)+(qy-pq)} = \frac{dp}{-(p+p\cdot 0)} = \frac{dq}{-(q+q\cdot 0)}$$

$\Rightarrow$ Taking last two,

$$\frac{dp}{-p} = \frac{dq}{-q}$$

$\Rightarrow p = aq$

$aqx + qy = aq^2$

$\Rightarrow q = \frac{ax+y}{a}$

$p = ax + y$

$dz = (ax+y)\,dx + \left(\frac{ax+y}{a}\right) dy$

$a\,dz = a^2 x\,dx + a(y\,dx + x\,dy) + y\,dy$

$az = \frac{a^2x^2}{2} + a\,xy + \frac{y^2}{2}$ $\Rightarrow$ $x^2a^2 + y^2 + 2axy = 2az + b$

is the complete integral,

Lecture-20

4.9.17

Ex: Find a complete integral of
$$p^2 - y^2 q = y^2 - x^2.$$

Soln →
$$f(x,y,z,p,q) = p^2 - y^2 q - y^2 + x^2 = 0 \quad \text{—(1)}$$

Charpit's eq^n are:

$$\frac{dx}{2p} = \frac{dy}{-y^2} = \frac{dz}{2p^2 - y^2 q} = \frac{dp}{-(2x)} = \frac{dq}{-(2yq - 2y)}$$

Taking first and fourth fraction,
$$\Rightarrow \quad p\,dp + x\,dx = 0$$
$$\text{or} \quad p^2 + x^2 = a^2 \quad \text{or } p = \sqrt{a^2 - x^2}.$$

Solving with (1),
$$q = a^2 y^{-2} - 1.$$

$$dz = p\,dx + q\,dy$$
$$= \sqrt{a^2 - x^2}\,dx + (a^2 y^{-2} - 1)\,dy$$
$$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{a^2}{y} - y + b.$$

Ex: Find a complete and singular integral of
$$(p^2 + q^2)y = qz$$

Soln →
$$f(x,y,z,p,q) = (p^2 + q^2)y - qz = 0$$

Charpit's eq^n:

$$\frac{dx}{2p} = \frac{dy}{2qy - z} = \frac{dz}{2p^2 + 2q^2 y - qz} = \frac{dp}{-(pq)} = \frac{dq}{(p^2 + q^2 - q^2)}$$

$$\frac{dp}{pq} = \frac{dq}{-p^2}$$

$$\Rightarrow \quad p\,dp = q\,dq$$

$$\Rightarrow \quad p^2 + q^2 = a^2 \text{ (say)}$$

$$\Rightarrow \quad q = \frac{a^2 y}{z}$$

$$p = \sqrt{a^2 - \frac{a^4y^2}{z}} = \frac{a}{z}\sqrt{z^2 - a^2y^2}$$

Now,

$$dz = p\,dx + q\,dy$$

$$dz = \left(a - \frac{a^2y^2}{z}\right)dx + \frac{ay}{z}\,dy$$

$$\Rightarrow \quad z\,dz = az\,dx - a^2y^2\,dx + ay\,dy$$

$$\Rightarrow \quad xz\,dz - axz\,dx = axy\,dy - a^2y^2\,dx$$

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$z\,dz = a\sqrt{z^2 - a^2y^2}\,dx + a^2y\,dy$$

$$\Rightarrow \quad \frac{2z\,dz - a^2 2y\,dy}{2\sqrt{z^2 - a^2y^2}} = a\,dx$$

$$\Rightarrow \quad \sqrt{z^2 - a^2y^2} = ax + b$$

$$\Rightarrow \quad z^2 - a^2y^2 = (ax + b)^2$$

$z^2 = a^2y^2 + a^2x^2 + 2abx + b^2$ — (A)

Diff wrt a and b partially, we get

~~$2zp0 = 2a^2x + 2ab$~~

~~$2zq0 = 2a^2y$~~

$0 = 2ay^2 + 2(ax+b)x$ — (B)

$0 = 2(ax+b)$ — (C)

$\Rightarrow$ $ax + b = 0$

$\Rightarrow$ $a = 0,\ b = 0.$

Eliminating a & b from (A), (B) & (C) we get $z=0$ which satisfies the given eqn and hence it is singular integral.

Special Types of 1st order Equations :

[Mid-before this]

(a) Eqns involving only p and q.

$f(p, q) = 0$ — (1)

Charpits eqns reduce to

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

$\Rightarrow$ $p = a$ — (2) as the second PDE.

$f(a, q) = 0$ — (3)

$q = Q(a)$

$dz = p\,dx + q\,dy$ will give

$\Rightarrow \boxed{z = ax + Q(a)y + b}$ — (4)

Lecture 21 & 22.

6.8.2017.

Ex. Find a complete integral of $f(p,q) = p + q - pq = 0$.

Soln → Put $q = a$, $p = \frac{a}{a-1}$.

$\therefore\ z = \frac{a}{a-1}x + ay + b$.

Ex. Solve $p^2 + q^2 = 1$ for its complete integral.

Soln → $f(p,q) = p^2 + q^2 - 1$

put $p = a$, $q = \sqrt{1-a^2}$.

$\therefore\ z = ax + \sqrt{1-a^2}\,y + b$.

Aliter,

$q = a$,

$z = \sqrt{1-a^2}\,x + ay + b$.

Singular integral:

$\frac{\partial z}{\partial a} = 0 \Rightarrow \frac{-\cancel{2}a}{\cancel{2}\sqrt{1-a^2}}x + y = 0 \Rightarrow \frac{a}{\sqrt{1-a^2}} = \frac{y}{x}$.

$\frac{\partial z}{\partial b} = 0 \Rightarrow \boxed{0 = 1}$??? 404, which is inadmissible.

↓

We cannot find singular integral in these cases.

So, this eqn has no singular integral.

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(b) Eq^{n}s not involving the independent variable

$$f(z, p, q) = 0 \quad \text{—(i)}$$

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

From last two fractions,

$$p = aq \quad \text{—(ii)}$$

Solve (i) and (ii) and integrate to find z.

Ex. Find a complete Integral of $p^2 = zq$.

Sol^n → This is of type $f(z, p, q) = 0$

$$\Rightarrow p = aq$$

$$\therefore a^2q^2 = zq$$

$$\Rightarrow q = \frac{z}{a^2}$$

$$\Rightarrow p = \frac{z}{a}$$

$$\therefore dz = \frac{z}{a}dx + \frac{z}{a^2}dy$$

$$\Rightarrow \ln z = \frac{x}{a} + \frac{y}{a^2} + \ln b$$

$$\Rightarrow \boxed{z = b\, e^{(x/a + y/a^2)}}$$

Ex. $z = p^2 - q^2$.

Soln. It is of the form $f(z,p,q) = 0$.

$\Rightarrow p = aq$.

Now, $z = (a^2-1) q^2$.

$\Rightarrow q = \sqrt{\dfrac{z}{a^2-1}}$

$\Rightarrow p = \sqrt{\dfrac{a^2 z}{a^2-1}}$

$$\Rightarrow dz = \sqrt{z}\left(\frac{a}{\sqrt{a^2-1}}dx + \frac{1}{\sqrt{a^2-1}}dy\right)$$

$$\Rightarrow 2\sqrt{z} = \frac{ax}{\sqrt{a^2-1}} + \frac{y}{\sqrt{a^2-1}} + \frac{b}{\sqrt{a^2-1}}$$

$$\Rightarrow \boxed{4(a^2-1)z = (ax+y+b)^2}$$

(c) Separable Equations

$f(x,p) = g(y,q)$

Soln. Charpit's eqn are:

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{dp}{-f_x} = \frac{dq}{+g_y}$$

Considering 1st and 4th fraction,

$$\frac{dx}{f_p} + \frac{dp}{f_x} = 0$$

$\Rightarrow f_p\,dp + f_x\,dx = 0$

$\Rightarrow f(x,p) = a$ (say).

$$\boxed{f(x,p) = g(y,q) = a}$$

Ex. Solve $p^2+q^2 = x+y$.

Soln.

$$p^2 - x = y - q^2.$$

It is of seperable form $f(x,p) = g(y,q)$.

$\therefore$ $p^2 - x = a$ $\quad y - q^2 = a$.

$\Rightarrow$ $p = \sqrt{x+a}$ $\quad q = \sqrt{y-a}$.

$\Rightarrow$ $dz = \sqrt{x+a}\,dx + \sqrt{y-a}\,dy$

$\Rightarrow$ $z = \frac{2}{3}\left((x+a)^{3/2} + (y-a)^{3/2}\right) + b$.

Ex: Solve $pe^y - qe^x = 0$

Soln.

$$\frac{p}{e^x} = \frac{q}{e^y} = a \quad \text{(Variable separable form)}$$

$\Rightarrow$ $p = ae^x$ $\quad q = ae^y$

$\Rightarrow$ $dz = ae^x\,dx + ae^y\,dy$

$\Rightarrow$ $\boxed{z = ae^x + ae^y + b}$

Ex:

(d) Clairaut's Equation

$$z = px + qy + f(p,q) \quad \text{—①}$$

Charpit's eqⁿ are:

$$\frac{dx}{} = \frac{dy}{} = \frac{dz}{} = \frac{dp}{0} = \frac{dq}{0}$$

So that we can put,

$p = a, \quad q = b$

Substituting in ①

$$\boxed{z = ax + by + f(a,b)}$$

Ex: Find a complete integral of the pde

$$pqz = p^2(xq + p^2) + q^2(yp + q^2)$$

Solⁿ→

$$z = \cancel{p\left(\frac{x}{p}\right.} + px + qy + \left(\frac{p^4}{q} + \frac{q^4}{p}\right)$$

Putting

$p = a$ and $q = b$, we get,

$$\therefore \quad z = ax + by + \frac{a^4}{b} + \frac{b^4}{a}$$ is the required complete integral.

Ex: Find a complete integral of

$$z = px + qy - 2\sqrt{pq}$$

Solⁿ→ Putting $p = a, \; q = b$,

$$z = ax + by - 2\sqrt{ab}$$ is a complete integral.

Singular integral:

$$\frac{\partial z}{\partial a} = x - \sqrt{\frac{b}{a}} \qquad \frac{\partial z}{\partial b} = y - \sqrt{\frac{a}{b}}$$

$$\Rightarrow \quad x = \frac{1}{y} \Rightarrow \boxed{xy = 1} \Rightarrow \boxed{xy - 1 = 0}$$

No z is here

Not a integral

Homogeneous Linear pde with constant coefficients

$$\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + A_n \frac{\partial^n z}{\partial y^n} = f(x,y)$$

where $A_1, A_2, \ldots A_n$ are constants.

Denoting, $\frac{\partial}{\partial x}$ by D, $\frac{\partial}{\partial y}$ by D', it can be written as

$$\left(D^n + A_1 D^{n-1}D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n\right) \cdot z = f(x,y)$$

$$\Rightarrow \quad \phi(D, D')z = f(x,y) \quad \text{---} (1)$$

Complete solution of (1) will consist of two parts

(a) Complementary function (C.F.) and

(b) Particular Integral (P.I.)

The C.F. is the solution of

$$\phi(D, D')z = 0$$

By putting $D = m$ and $D' = 1$ in $\phi(D, D')z = 0$, the auxiliary eqn is obtained i.e.

$$m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0$$

If $m_1, m_2, \ldots m_n$ are distinct roots of the auxiliary eqn, the C.F of (1) is given by

$$z = \psi_1(y + m_1 x) + \psi_2(y + m_2 x) + \cdots + \psi_n(y + m_n x)$$

If the auxiliary eq^n has 'r' equal roots then,

$$z = f_1(y+mx) + x f_2(y+mx) + \dots + x^{r-1} f_r(y+mx)$$

Ex: Solve $2r + 5s + 2t = 0$. $\quad r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$

$\Rightarrow 2r + 5s + 2t = 0$.

Putting,

$(2D^2 + 5DD' + 2D'^2)z = 0$

Putting $D = m$, $D' = 1$, auxiliary eq^n is

$2m^2 + 5m + 2 = 0$

$\Rightarrow (2m+1)(m+2) = 0$

or $m = -\frac{1}{2}, -2$

$$\boxed{z = f\left(y - \frac{x}{2}\right) + \psi(y - 2x)}$$

Ex. Solve $r = a^2 t$.

$\Rightarrow r - a^2 t = 0$.

$\Rightarrow (D^2 - a^2 D'^2)z = 0$

Auxiliary eq^n:

$\Rightarrow m^2 - a^2 = 0$

or $m = \pm a$.

$\therefore$ $$\boxed{z = f(y+ax) + \psi(y-ax)}$$

Particular Integral (P.I.)

$$f(D, D')z = \phi(x,y)$$

$$\Rightarrow \quad P.I. = \frac{1}{f(D,D')}\phi(x,y)$$

Ex: Solve $(D^2 - 2DD' + D'^2)z = 12xy$.

Sol^n →

C.F.

A.E. is

$m^2 - 2m + 1 = 0$

$\Rightarrow \quad (m-1)^2 = 0$

$\therefore$ C.F. is $f(y+x) + x\psi(y+x)$

P.I.

$$= \frac{1}{D^2 - 2DD' + D'^2} 12xy$$

$$= \frac{1}{(D-D')^2} 12xy.$$

$$= \frac{1}{D^2}\left(1 - \frac{D'}{D}\right)^{-2} 12xy$$

$$= \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \cdots\right] 12xy$$

$$= \left(\frac{1}{D^2} + \frac{2D'}{D^3} + \cdots\right) 12xy$$

$$= 2x^3y + x^4$$

$\therefore$ G.S. is $f(y+x) + xf_1(y+x) + 2x^3y + x^4$.

Lecture - 23

11.09.2018

Ex: Solve $(D^2+3DD'+2D'^2)z = x+y$

Sol^n: A.E:

$m^2+3m+2=0$

$\Rightarrow (m+1)(m+2)=0$

$\therefore m=-1,-2.$

$\therefore$ C.F is $f(y-x)+\psi(y-2x)$.

$$P.I = \frac{1}{(D^2+3DD'+2D'^2)}(x+y)$$

$$= \frac{1}{D^2\left(1+\frac{3D'}{D}+2\frac{D'^2}{D^2}\right)}(x+y)$$

$$= \frac{1}{D^2}\left[1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right]^{-2}(x+y)$$

$$= \frac{1}{D^2}\left[1-\frac{3D'}{D}\right](x+y)$$

$$= \frac{1}{D^2}(x+y) - 3\frac{D'}{D^3}(x+y)$$

$$= \frac{x^3}{6}+\frac{yx^2}{2} - \frac{x^3}{2}$$

$$= \frac{yx^2}{2} - \frac{x^3}{3}.$$

G.S. is $f(y-x)+\psi(y-2x)+\frac{yx^2}{2}-\frac{x^3}{3}$

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Solve: $(2D^2 - 5DD' + 2D'^2)z = 24(y-x)$

Sol^n →

A.E :

$2m^2 - 5m + 2 = 0$

$\Rightarrow m = \frac{1}{2}, 2.$

$\therefore$ C.F. $= f_1\left(y - \frac{x}{2}\right) + f_2(y - 2x)$

$$P.I = \frac{24}{2D^2}\left[1 - \frac{5D'}{2D} + \frac{D'^2}{2D^2}\right]^{-1}(y-x)$$

$$= 12 \times \frac{1}{D^2}\left[1 + \frac{5D'}{2D}\right](y-x)$$

$$= 12\left[\frac{1}{D^2}(y-x) + \frac{5D'}{2D^3}(y-x)\right]$$

$$= 12\left[\frac{yx^2}{2} - \frac{x^3}{6} + \frac{5 \cdot x^3}{2 \times 2 \times 3}\right]$$

$$= 6yx^2 + 3x^3 \quad \left(-2x^3 + 5x^3\right)$$

G.S. is $f_1(y - x/2) + f_2(y - 2x) + 6yx^2 + 3x^3$.

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Other method for finding P.I.

When $f(x,y)$ is of the form $f(ax+by)$,

Theorem: If $F(D, D')$ be homogeneous function of D and D' of degree n, then

$$\frac{1}{F(D,D')}\phi^n(ax+by) = \frac{1}{F(a,b)}\phi(ax+by)$$

Provides. $F(a,b) \neq 0$, ϕ^n being nth derivative of ϕ wrt $ax+by$ as a whole.

When $F(a,b) = 0$

$$\frac{1}{(bD-aD')^n}\phi(ax+by) = \frac{x^n}{b^n n!}\phi(ax+by)$$

Q Solve: $(D^2+3DD'+2D'^2)z = x+y$.

→ P.I. $= \dfrac{1}{D^2+3DD'+2D'^2}(x+y)$

Put $D=1, D'=1$

$$= \frac{1}{6}\,\frac{(x+y)^3}{6}$$

$$= \frac{(x+y)^3}{36}$$

$$z = \phi_1(y-x) + \phi_2(y-2x) + \frac{1}{36}(x+y)^3$$

Verification of P.I.

$$z = \frac{1}{36}(x+y)^3$$

$$\frac{\partial z}{\partial x} = \frac{1}{36}(3x^2+6xy+3y^2)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{36}(6x+6y) = \left(\frac{x+y}{6}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x+y}{6}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{(x+y)}{6}$$

putting in $D^2+3DD'+2D'^2$.

$$= 6\times\frac{(x+y)}{6}$$

$$= x+y$$

Ex: $(D^2+2DD'+D'^2)z = e^{2x+3y}$

C.F: $\phi_1(y-x)+x\phi_2(y-x)$

$$P.I. = \frac{1}{(D^2+2DD'+D'^2)} e^{2x+3y}.$$

$D=2,\ D'=3$

$$= \frac{1}{25} e^{2x+3y}.$$

Verificatin

$$\frac{\partial^2 z}{\partial x^2} = \frac{4}{25} e^{2x+3y} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{9}{25} e^{2x+3y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{6}{25} e^{2x+3y}$$

$$\therefore (D^2+2DD'+D'^2)z = e^{2x+3y}$$

Lecture 24 & 25 — 13.9.2017

Ex: $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = e^{5x+6y}.$

Solⁿ → A.E:

$m^3 - 6m^2 + 11m - 6 = 0$

$\Rightarrow (m-1)(m-2)(m-3) = 0$

$\therefore$ C.F. is $f_1(y+x) + f_2(y+2x) + f_3(y+3x)$.

$$P.I. = \frac{1}{(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)} e^{5x+6y}.$$

$$F(a,b) = \frac{1}{(5^3 - 6\times5^2\times6 + 11\times5\times6^2 - 6\times6^3)} e^{5x+6y}$$

$$= -91$$

$$\therefore P.I. = \frac{-1}{91} e^{5x+6y}$$

Ex: $r - 2s + t = \sin(2x+3y)$

A.E:

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow (m-1)^2 = 0$$

$$\Rightarrow m = 1, 1.$$

$\therefore$ C.F. is $f_1(y+x) + x f_2(y+x)$.

$$F(a,b) = 2^2 - 2\times2\times3 + 3^2.$$

$$= 1.$$

$$\therefore \frac{1}{F(D,D')} \phi(2x+3y) = \frac{1}{1}\iint \sin(2x+3y)$$

$$= -\sin(2x+3y).$$

$\therefore$ G.S. is

$f_1(y+x) + x f_2(y+x) - \sin(2x+3y)$.

where f_1, f_2 are arbitrary functions.

Verification

$\frac{\partial z}{\partial x} = -2\cos(2x+3y)$

$r = 4\sin(2x+3y)$

$\frac{\partial z}{\partial y} = -3\cos(2x+3y)$

$t = 9\sin(2x+3y)$

$s = 6\sin(2x+3y)$

$\therefore r - 2s + t$

$= (4 - 2\times6 + 9)\sin(2x+3y)$

$= \sin(2x+3y)$

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Ex: $r - t = x - y$.

Solⁿ → A·E :

$m^2 - 1 = 0$.

$\Rightarrow \quad m = 1, -1$.

$\therefore$ C.F is $\phi_1(y+x) + \phi_2(y-x)$

$$P.I. = \frac{1}{D^2 - D'^2}(x-y)$$

$$= \frac{1}{(D+D')(D-D')}(x-y)$$

$$= \frac{1}{(D+D')} \underbrace{\frac{1}{(D-D')}(x-y)}$$

$\longrightarrow \frac{1}{F(a,b)} \int^n \phi(u, v)$

$$= \frac{1}{(D+D')} \frac{1}{2} \frac{(x-y)^2}{2}$$

$$= \frac{1}{4} \frac{1}{(D+D')}(x-y)^2.$$

$$= \frac{1}{(-1)4} \frac{1}{((-1)D - 1 \cdot D')^1}(x-y)^2$$

$\frac{1}{(D+D')}(y-x)^2$.

$D - D'$

$= \frac{x^1}{1^1 \times 1!} \cdot (y-x)$

$$= \frac{-1}{4} \times \frac{x^1}{(-1)^1 \times 1!}(x-y)^2$$

$$= \frac{x(x-y)^2}{4}.$$

$$P.I. = \frac{1}{D-D'} \frac{1}{D+D'}(x-y)$$

$$= \frac{1}{(D-D')}(-1) \times \frac{1}{((-1)D - 1 \cdot D')^1}(x-y)$$

$$= \frac{1}{(D-D')} \cdot (-1) \cdot \frac{x(x-y)}{(-1)\, 1!}$$

$$P.I. = \frac{1}{(D-D')} \, x(x-y).$$

$$= \frac{1}{D\left(1-\frac{D'}{D}\right)} x(x-y)$$

$$= \frac{1}{D}\left(1+\frac{D'}{D}\right) x(x-y).$$

$$= \left(\frac{1}{D} + \frac{D'}{D^2}\right) x(x-y)$$

$$= \left(\frac{x^3}{3} - \frac{x^2y}{2}\right) + \left(-\frac{x^3}{6}\right)$$

$$= \frac{x^3}{6} - \frac{x^2y}{2}$$

Verification

$$\frac{\partial z}{\partial x} = \frac{x^2}{2} - xy. \qquad \frac{\partial z}{\partial y} = -\frac{x^2}{2} \qquad \frac{\partial^2 z}{\partial x \partial y} = -x.$$

$$\frac{\partial^2 z}{\partial x^2} = x-y \qquad \frac{\partial^2 z}{\partial y^2} = 0$$

$$\therefore \boxed{r - t = x-y.}$$

Solve: $(D^2 - 6DD' + 9D'^2) z = 6x + 2y.$

A.E. : $m^2 - 6m + 9 = 0$

$= (m-3)^2 = 0$

$\Rightarrow m = 3, 3$

$\therefore$ C.F. is $\phi_1(y+3x) + x\,\phi_2(y+3x).$

$$P.I. = \frac{1}{(D-3D')^2}(6x+2y)$$

$$= 2 \times \frac{1}{(D-3D')^2}(3x+y)$$

$bD - aD'$

$1 \cdot D - 3 \cdot D'$.

$$= 2 \times \frac{1}{(1 \cdot D - 3D')^2}(3x+y)$$

$$= 2 \times \frac{x^2}{1^2 \times 2!}(3x+y)$$

$$= x^2(3x+y)$$

Ex: Solve : $(D^2 - 5DD' + 4D'^2)Z = \sin(4x+y)$

Soln

A.E:

$m^2 - 5m + 4 = 0$

$\Rightarrow (m-1)(m-4) = 0$

$\Rightarrow m = 1, m = 4,$

C.F. is $f_1(y+x) + f_2(y+4x)$

$$P.I = \frac{1}{(D-D')(D-4D')}\sin(4x+y)$$

$$= \frac{1}{(D-4D')}\frac{1}{(D-D')}\sin(4x+y)$$

$$= \frac{1}{(D-4D')} \times \frac{1}{3} - \cos(4x+y)$$

$$= \left(\frac{-1}{3}\right) \times \frac{1}{(D-4D')}\cos(4x+y).$$

$$P.I = \frac{-1}{3} \times \frac{1}{(D-4D')} \cos(4x+y)$$

$$= \left(\frac{-1}{3}\right) \times \frac{x^1}{(1)\times 1} \cos(4x+y)$$

$$= -\frac{x}{3} \cos(4x+y)$$

$\therefore$ G.S. is

$$f_1(y+x) + f_2(y+4x) - \frac{x}{3}\cos(4x+y).$$

Ex: Solve $(4r - 4s + t) = 16 \ln(x+2y)$

Soln: A.E:

$$4m^2 - 4m + 1 = 0$$

$$\Rightarrow (2m-1)^2 = 0.$$

$$\Rightarrow m = 1/2, 1/2.$$

$\therefore$ C.F. is $f_1(y+\frac{x}{2}) + x f_2(y+x/2)$.

$$P.I = \frac{1}{(2D-D')^2} 16 \ln(x+2y)$$

$$= 16 \frac{1}{(2.D - 1.D')^2} \ln(x+2y)$$

$\int \ln x = x(\ln x - 1)$

$$= 16 \times \frac{x^2}{2^2 \times 2!} \ln(x+2y)$$

$$= 2x^2 \ln(x+2y).$$

$\therefore$ G.S. is

$$f_1(y+\frac{x}{2}) + x f_2(y+x/2) + 2x^2 \ln(x+2y).$$

Other Method: [General Method]

· Take the P.I. corresponding to $\frac{1}{(D-mD')}\phi$ as $\int \phi(x, a-mx)\,dx$ and replace y by $y+mx$ after integration.

Ex: Solve
$$(D^2-2DD'-15D'^2) = 12xy.$$

Soln: **1st method:**

A.E.

$$m^2-2m-15=0$$
$$\Rightarrow m = 5, -3$$

$\therefore$ C.F. $= \phi_1(y+5x) + \phi_2(y-3x)$.

$$P.I. = \frac{1}{D^2-2DD'-15D'^2}\,12xy$$

$$= \frac{1}{D^2}\left[1+\frac{2D'}{D}+\cdots\right]12xy$$

$$= \frac{1}{D^2}12xy + \frac{2D'}{D^3}12xy$$

$$= 2x^3y + x^4$$

G.S. $\Rightarrow$ $z = f_1(y+5x)+f_2(y-3x) + 2x^3y + x^4$.

2nd method:

$$P.I. = \frac{1}{(D-5D')(D+3D')}\,12xy$$

$$= \frac{1}{(D-5D')}\int 12x(a+3x)\,dx.$$

$$= \frac{1}{(D-5D')}\,6x^2a + 12x^3$$

$$= \frac{1}{(D-5D')}\,6x^2(y-3x) + 12x^3.$$

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$$= \int 6x^2(a_1 - 5x) - 6x^3 \, dx.$$

$$= 2x^3 a_1 - 9x^4.$$

$$= 2x^3(y+5x) - 9x^4$$

$$= 2x^3 y + 10x^4 - 9x^4.$$

$$= 2x^3 y + x^4..$$

$$\frac{\partial z}{\partial x} = 4xy + 30x^2 - 36x^3.$$

$$\frac{\partial^2 z}{\partial x^2} = 4y + 60x - 108x^2$$

$$\frac{\partial^2 z}{\partial y^2} = 0, \quad 4y -$$

$$\frac{\partial^2 z}{\partial y \partial x} = 4x.$$

<u>other way round:</u>

$$P.I. = \frac{12 \quad 1}{(D+3D')(D-5D')} xy.$$

$$= 12 \frac{1}{D+3D'} \int x(a-5x)\,dx.$$

$$= 12 \frac{1}{(D+3D')} \left(\frac{x^2 a}{2} - \frac{5}{3}x^3\right) dx.$$

$$= \frac{12}{D+3D'} \left(\frac{x^2}{2}(y+5x) - 5/3\, x^3\right)$$

$$= \frac{2}{(D+3D')} (3x^2 y + 5x^3).$$

$$= 2 \int 3x^2(b+3x) + 5x^3 \, dx.$$

$$= 2\left[x^3 b + \frac{9}{4}x^4 + \frac{5x^4}{4}\right].$$

$$= 2\left[x^3(y-3x) + \frac{14x^4}{4}\right]$$

$$= 7x^4 + 2x^3 y - 6x^4$$

$$= x^4 + 2x^3 y.$$

Ex: $(r+s-6t) = y\cos x$.

Solⁿ → A·E:

$m^2+m-6=0$

$\Rightarrow (m+3)(m-2)=0$

$\Rightarrow m=2,-3$.

$\therefore$ C·F is $\phi_1(y+2x)+\phi_2(y-3x)$.

$$P.I. = \frac{1}{(D-2D')}\frac{1}{(D+3D')} y\cos x$$

$$= \frac{1}{(D-2D')}\int (a+3x)\cos x\,dx$$

$$= \frac{1}{(D-2D')}\left[a\sin x + 3\int x\cos x\,dx\right]$$

$$= \frac{1}{(D-2D')}\left[(y-3x)\sin x + 3\left[x\sin x + \cos x\right]\right]$$

$$= \frac{1}{D-2D'}\left[\sin x y + 3\cos x\right]$$

$$= \int (b-2x)\sin x + 3\cos x\,dx$$

$$= bx - x^2 + 3\sin x$$

$$= (y+2x)x - x^2 + 3\sin x$$

$\frac{\partial z}{\partial x} = 2x+y+3\cos x$

$r = 2-3\sin x$.

$s = 1$ $\quad \frac{\partial z}{\partial y} = x$.

$t = 0$.

$P.I. = x^2 + xy + 3\sin x$.

$\therefore$ G.S. is $\phi_1(y+2x)+\phi_2(y-3x) + x^2+xy+3\sin x$.

$$= \int (b \sin x - 2x \sin x + 3\cos x)\, dx.$$

$$= -b\cos x - 2\left[-x\cos x + \sin x\right] + 3\sin x \, dx.$$

$$= -(y+2x)\cos x + 2x\cos x - 2\sin x + 3\sin x \, dx.$$

$$P.I. = -y\cos x + \sin x.$$

Lecture - 26

PDE of order two (continued...)

Reduction To canonical form

$$R\frac{\partial^2 z}{\partial x^2} + S\frac{\partial^2 z}{\partial x \partial y} + T\frac{\partial^2 z}{\partial y^2} + F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0 \quad \text{—(1)}$$

$R, S, T \rightarrow$ function of x and y or const.

By suitable change of independent variables, we will show that eq^n can be transformed into 3 kinds of canonical form.

Let the independent variables

$$\xi = \xi(x,y), \quad \eta = \eta(x,y) \quad \Big\} \text{—(2)}$$

$$J = \frac{\partial(\xi, \eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0.$$

and we write $z(x,y)$ as $u(\xi, \eta)$.

then
$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= u_\xi \xi_x + u_\eta \eta_x$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= u_\xi \xi_y + u_\eta \eta_y$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(u_\xi \xi_x + u_\eta \eta_x\right)$$

$$= \xi_x \frac{\partial u_\xi}{\partial x} + u_\xi \frac{\partial \xi_x}{\partial x} + \eta_x \frac{\partial u_\eta}{\partial x} + u_\eta \frac{\partial}{\partial x}\eta_x$$

$$= \xi_x \left[\frac{\partial u_\xi}{\partial \xi} \times \frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta} \times \frac{\partial \eta}{\partial x}\right]$$

$$+ u_\xi \xi_{xx} + \eta_x \left[\frac{\partial u_\eta}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u_\eta}{\partial \xi}\frac{\partial \xi}{\partial x}\right]$$

$$+ u_\eta \eta_{xx}$$

$$= \xi_x \left[u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x\right] + u_\xi \xi_{xx}$$

$$+ \eta_x \left[u_{\eta\eta}\eta_x + u_{\xi\eta}\xi_x\right] + u_\eta \eta_{xx}$$

$$= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\eta_x\xi_x + u_\xi \xi_{xx}$$

$$+ u_{\eta\eta}\eta_x^2 + u_\eta \eta_{xx}.$$

$$\frac{\partial^2 z}{\partial y^2} = u_{\xi\xi}\,\xi_y^2 + 2u_{\xi\eta}\,\xi_y\eta_y + u_{\eta\eta}\,\eta_y^2$$
$$+ \xi_{yy}u_\xi + \eta_{yy}u_\eta$$

$$\frac{\partial^2 z}{\partial x\partial y} = u_{\xi\xi}\,\xi_x\xi_y + (\xi_x\eta_y + \xi_y\eta_x)$$
$$+ u_{\eta\eta}\,\eta_x\eta_y + \xi_{xy}u_\xi + \eta_{xy}u_\eta .$$

After substituting in eq^n 1

$$A(\xi_x, \xi_y)\,u_{\xi\xi} + 2B(\xi_x, \xi_y, \eta_x, \eta_y)\,u_{\xi\eta}$$
$$+ A(\eta_x, \eta_y)\,u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{— } (3)$$

where
$$A(u, v) = Ru^2 + Suv + Tv^2 \quad \text{— } (IV)$$

$$B(u_1, v_1, u_2, v_2) = Ru_1u_2 + \frac{1}{2}S(u_1v_2 + u_2v_1) + Tv_1v_2 \quad \text{— } (5)$$

$\phi(\xi, \eta, u, u_\xi, u_\eta)$ is the transform form of

$F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$.

(*) The problem is now to determine ξ and η so that the eq^n (3) takes the simplest possible form. The procedure is simple when the discriminate $S^2 - 4RT$ of the quadratic form (4) is +ve, −ve or zero and we will discuss these three cases separately.

Case a) $S^2 - 4RT > 0$:

When this condition is satisfied, the roots λ_1 and λ_2 of $R\lambda^2 + S\lambda + T = 0$ are real and distinct.

when $\lambda = \xi_x/\xi_y$ or η_x/η_y.

The coefficients of $\frac{\partial^2 u}{\partial \xi^2}$ and $\frac{\partial^2 u}{\partial \eta^2}$ in eqn (5) will vanish, if we choose ξ and η such that,

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y} \quad \text{— (7)}$$

$$\frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y} \quad \text{— (8)}$$

Rewriting (7),

$$\frac{\partial \xi}{\partial x} - \lambda_1 \frac{\partial \xi}{\partial y} = 0 \quad \text{— (9)}$$

Lagrange's auxiliary eqn of eqn 9

$$\frac{dx}{1} = \frac{dy}{-\lambda} = \frac{d\xi}{0}$$

Taking 3rd fraction, $\xi = c_1$

From 1st and 2nd fraction,

$dy + \lambda dx = 0$

$\therefore f(x, y) = c_2$

Soln of 7 is $\xi = f(x, y)$

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Similarly, soln of ⑧ can be written as

$$\eta = f_2(x,y).$$

It can be verified that

$$A(\xi_x, \xi_y)\,A(\eta_x \eta_y) - B^2(\xi_x, \xi_y, \eta_x, \eta_y)$$

$$= (4RT - S^2)\,(\xi_x \eta_y - \xi_y \eta_x)^2$$

when A's are zero. —— ⑫

$$B^2 = (S^2 - 4RT)\,(\xi_x \eta_y - \xi_y \eta_x)^2.$$

$S^2 - 4RT > 0$, so $B^2 \neq 0$..

We can divide both sides of ⑤ by B and ③ transforms to

$$\boxed{u_{\xi\eta} = \phi_1(\xi, \eta, u, u_\xi, u_\eta)} \quad — ⑬$$

which is the canonical form in this case.

(ii) Case (b) : $S^2 - 4RT = 0$

When this condition is satisfied, the roots λ_1, λ_2 of ⑥ are real and equal. We define ξ exactly as in Case (a) and take η to be any function of (x, y) which is independent of ξ. We have $A(\xi_x, \xi_y) = 0$. Also since $S^2 - 4RT = 0$, so from (12), $B^2 = 0$ i.e. $B = 0$.

Here $A(\eta_x, \eta_y) \neq 0$.

Dividing by $A(\eta_x, \eta_y)$ (3) transforms to

$$\boxed{\frac{\partial^2 u}{\partial \eta^2} = \phi_2(\xi, \eta, u, u_\xi, u_\eta)} \quad — ⑭$$

which is the canonical form of ① in this case.

Case(c): $S^2 - 4RT < 0$.

This is formally the same as case (a) except that now the roots of (6) are complex. To get a real canonical form we make further transform

$$\xi = \alpha + i\beta \qquad \Rightarrow \alpha = \frac{1}{2}(\xi + \eta) \quad \text{— (15)}$$
$$\eta = \alpha - i\beta. \qquad \Rightarrow \beta = \frac{1}{2}i(\eta - \xi)$$

$$\boxed{\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = \psi(\alpha, \beta, u, u_\alpha, u_\beta).}$$

Classification:

We classify the second order PDE of type (1) by their canonical forms. We say that an eq^n of this type is

(a) Hyperbolic if $S^2 - 4RT > 0$

(b) Parabolic if $S^2 - 4RT = 0$

(c) Elliptic if $S^2 - 4RT < 0$.

Hyperbolic equation:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = F_1\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)$$

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} = F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)$$

Parabolic equation:

$$\frac{\partial^2 u}{\partial \xi^2} = F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)$$

$$\partial^2 u/\partial \eta^2 = F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)$$

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Elliptic equation

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right).$$

The simplest example of the three cases are.

(i) Wave equation,

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} \Rightarrow \text{hyperbolic}.$$

(ii) Heat Equation:

$$\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2} \Rightarrow \text{parabolic}.$$

(iii) Laplace eqn:

$$\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow \text{elliptical}.$$

Ex: Classify the pde

$$\frac{\partial^2 u}{\partial x^2} + 2x \frac{\partial^2 u}{\partial x \partial y} + (1-y^2) \frac{\partial^2 u}{\partial y^2} = 0.$$

$$S^2 - 4RT = 4x^2 - 4(1-y^2)$$
$$= 4(x^2+y^2-1).$$

If $x^2+y^2-1 > 0$, the equation is hyperbolic

If $x^2+y^2-1 = 0$, the equation is parabolic

If $x^2+y^2-1 < 0$, the equation is elliptic.

Working rule for reducing a hyperbolic eqn to its canonical form

Step 1: Let the given eqn. $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ — (1) be hyperbolic such that $S^2 - 4RT > 0$.

Step 2: Write λ quadratic eqn. $R\lambda^2 + S\lambda + T = 0$ — (2) λ_1, λ_2 are distinct real roots.

Step 3: Corresponding characteristic eqn are:

$$\frac{dy}{dx} + \lambda_1 = 0 \quad , \quad \frac{dy}{dx} + \lambda_2 = 0$$

Solving $f_1(x, y) = c_1$ & $f_2(x, y) = c_2$ — (3)

Step 4: We select ξ, η such that

$$\left.\begin{aligned} \xi &= f_1(x, y) \\ \eta &= f_2(x, y) \end{aligned}\right\} \text{— (4)}$$

Step 5: Using (4) we find p, q, r, s, t in terms of ξ, η as shown in the theory.

Step 6: Substituting the values of p, q, r, s, t obtained in step 5 in (1) and simplifying we get the canonical form of (1) as

$$\frac{\partial^2 u}{\partial \xi \, \partial \eta} = \phi\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right).$$

Ex Reduce the eqn :

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$ to its canonical form.

Sol^n: $R=1,\ S=0,\ T=-x^2$.

$S^2-4RT = 4x^2 > 0$ ∴ Hyperbolic.

$$\frac{\partial^2 z}{\partial x^2} - x^2\frac{\partial^2 z}{\partial y^2} = 0$$

Quadratic eqn : $R\lambda^2+S\lambda+T=0$ gives

$\Rightarrow \lambda^2 - x^2 = 0$

$\Rightarrow \lambda_1 = x,\ \lambda_2 = -x$

$\Rightarrow \frac{dy}{dx}+\lambda_1 = 0$ | $\frac{dy}{dx}+\lambda_2 = 0$

$\Rightarrow \frac{dy}{dx}+x = 0$ | $\frac{dy}{dx}-x=0$

$\Rightarrow y+\frac{x^2}{2} = c_1$ | $y-\frac{1}{2}x^2 = c_2$.

$\xi = y+\frac{x^2}{2}$, $\eta = y-\frac{1}{2}x^2$.

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= x\frac{\partial u}{\partial \xi} - x\frac{\partial u}{\partial \eta}$$

$$\frac{\partial z}{\partial y} = \partial u/\partial\xi + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\partial z/\partial x) = \partial u/\partial\xi - \frac{\partial u}{\partial\eta} + x\left(x\frac{\partial^2 u}{\partial\xi^2} + x\frac{\partial^2 u}{\partial\eta^2} - x\left(2\frac{\partial^2 u}{\partial\xi\partial\eta}\right)\right)$$

$$= \left(\partial u/\partial\xi - \frac{\partial u}{\partial\eta}\right) + x^2\left(\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} - 2\frac{\partial^2 u}{\partial\xi\partial\eta}\right).$$

(*) &

$$= x^2\left(\frac{\partial^2 u}{\partial\xi^2} - 2\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{\partial^2 u}{\partial\eta^2}\right) + \left(\frac{\partial u}{\partial\xi} - \frac{\partial u}{\partial\eta}\right)$$

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$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial y}$$

$$= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

<u>Original eq^n:</u>

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$

<u>Substituting we get,</u>

$$x^2\left(\cancel{\frac{\partial^2 u}{\partial \xi^2}} - \frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta^2}}\right) + \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

$$= x^2\left(\cancel{\frac{\partial^2 u}{\partial \xi^2}} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta^2}}\right)$$

$$\Rightarrow 4x^2 \frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{1}{4(\xi - \eta)}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)}$$

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Ex: Classify and Reduce the eqⁿ:

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$$ to its canonical form.

$R = 1,\ S = 0,\ T = x^2.$

$S^2 - 4RT = -x^2 < 0.$

Quadratic eqⁿ: $R\lambda^2 + S\lambda + T = 0$

$\lambda^2 + x^2 = 0.$

$\Rightarrow \lambda = \pm ix$

$\frac{dy}{dx} + ix = 0 \qquad \frac{dy}{dx} - ix = 0$

$\Rightarrow y + \frac{i}{2}x^2 = c_1 \qquad y - \frac{i}{2}x^2 = c_2.$

$\xi = y + \frac{i}{2}x^2 = \alpha + i\beta$ (say)

$\eta = y - \frac{i}{2}x^2 = \alpha - i\beta$ (say)

$\alpha = y,\quad \beta = \frac{x^2}{2}.$

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = x\frac{\partial u}{\partial \beta}$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial \alpha}.$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial \beta}\right) = \frac{\partial u}{\partial \beta} + x\cdot\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \beta}\right)$$

$$= \frac{\partial u}{\partial \beta} + x\frac{\partial}{\partial \beta}\frac{\partial u}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} + 0$$

$$= x\frac{\partial^2 u}{\partial \beta^2} * x$$

$$= x^2\, \partial^2 u/\partial \beta^2 + \partial u/\partial \beta.$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 u}{\partial \alpha^2}$$

After substitution we get

$$x^2 \frac{\partial^2 u}{\partial \beta^2} + \frac{\partial u}{\partial \beta} + x^2 \frac{\partial^2 u}{\partial \alpha^2} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = \frac{-1}{x^2} \frac{\partial u}{\partial \beta}$$

$$= \frac{-1}{2\beta} \frac{\partial u}{\partial \beta}$$

Q. Reduce to canonical form:

$$\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

$\Rightarrow$ $R = 1, \quad S = 2, \quad T = 1.$

$S^2 - 4RT = 0.$

$Z(x,y) \rightarrow u(\xi, \eta)$

Quadratic equation:

$$R\lambda^2 + S\lambda + T = 0$$

$$\Rightarrow \lambda^2 + 2\lambda + 1 = 0$$

$$\Rightarrow \lambda = -1. \Rightarrow \lambda_1 = \lambda_2 = -1.$$

$$\Rightarrow \frac{dy}{dx} - 1 = 0 \Rightarrow x - y = c_1$$

$\therefore \xi = x - y.$

$\eta = x + y$ (say).

$$Z = (x+y) \phi_1(x-y) + \phi_2(x-y)$$

$$\frac{\partial^2 u}{\partial \eta^2} = 0$$

$$\Rightarrow \frac{\partial u}{\partial \eta} = \phi_1(\xi)$$

$$\Rightarrow u = \eta \phi_1(\xi) + \phi_2(\xi)$$

$\phi_1(\xi)$ and $(\phi_2)\xi$ are arb. f^n of ξ

Soln:

$(y-1)r - (y^2-1)s + y(y-1)t + p - q = 2ye^{2x}(1-y)^3$.

Now,

$R = y-1$, $S = -(y^2-1)$, $T = y(y-1)$

$$S^2 - 4RT = (y^2-1)^2 - 4y(y-1)^2$$
$$= (y-1)^2\{y^2+2y+1-4y\}$$
$$= (y-1)^4.$$

$\therefore$ $S^2 - 4RT > 0$, $\Rightarrow$ Hyperbolic function.

(*) Quadratic equation:

$R\lambda^2 + S\lambda + T = 0$

$\Rightarrow (y-1)\lambda^2 - (y^2-1)\lambda + y(y-1) = 0$.

$\Rightarrow \lambda^2 - (y+1)\lambda + y = 0$

$\Rightarrow (\lambda - y)(\lambda - 1) = 0$.

$\Rightarrow \lambda = y$, $\lambda = 1$.

$\Rightarrow \frac{dy}{dx} + 1 = 0$, $\frac{dy}{dx} + y = 0$.

$\Rightarrow x + y = c_1$

$\cancel{\frac{dy}{y} = -dx}$

$\cancel{\ln y = -x}$

$\Rightarrow$ $\cancel{y e^{-x} = c_2}$.

$\frac{dy}{-y} =$

$y = ke^{-x}$

$\Rightarrow$

$\xi = x + y$

$\eta = ye^{x}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} \times \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$\xi = x+y$

$\eta = ye^{x}$.

$$= \frac{\partial z}{\partial \xi} + ye^{x}\frac{\partial z}{\partial \eta}$$

$z(x,y) \longrightarrow z(\xi,\eta)$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi} \times \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$= \partial z/\partial \xi + e^{x}\,\partial z/\partial \eta$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial \xi}\left(\frac{\partial z}{\partial \xi} + ye^{x}\frac{\partial z}{\partial \eta}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \xi} + ye^{x}\frac{\partial z}{\partial \eta}\right)\frac{\partial \eta}{\partial y}$$

$$= \left(\frac{\partial^2 z}{\partial \xi^2} + ye^{x}\frac{\partial^2 z}{\partial \xi \partial \eta}\right)1 + e^{x}\left\{\frac{\partial^2 z}{\partial \eta \partial \xi} + ye^{x}\frac{\partial^2 z}{\partial \eta^2}\right.$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial^2 z}{\partial \xi^2} + ye^{x}\frac{\partial^2 z}{\partial \xi \partial \eta}\right) + e^{x}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\eta\frac{\partial^2 u}{\partial \xi \partial \eta} + \eta^2\frac{\partial^2 u}{\partial \eta^2} + \eta\frac{\partial u}{\partial \eta} = r$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2} + (e^{x}+\eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + \eta e^{x}\frac{\partial^2 u}{\partial \eta^2} + e^{x}\frac{\partial u}{\partial \eta} = s$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + 2e^{x}\frac{\partial^2 u}{\partial \xi \partial \eta} + e^{2x}\,\partial^2 u/\partial \eta^2 = t$$

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$\frac{\partial^2 u}{\partial\xi\partial\eta} = 2\eta$

$(y-1)r - (y^2-1)s + y(y-1)t + p - q = 2ye^{2x}(1-y)^3$

$\Rightarrow (y-1)\{u_{\xi\xi} + 2\eta u_{\xi\eta} + \eta^2 u_{\eta\eta} + \eta u_\eta\}$

$- (y^2-1)\{u_{\xi\xi} + (e^x+\eta) u_{\xi\eta} + \eta e^x u_{\eta\eta} + e^x u_\eta\}$

$+ y(y-1)\{u_{\xi\xi} + 2e^x u_{\xi\eta} + e^{2x} u_{\eta\eta}\}$

$+ (\cancel{u_\xi} + \eta u_\eta) - (\cancel{u_\xi} + e^x u_\eta) = 2ye^{2x}(1-y)^3.$

$\Rightarrow u_{\xi\xi} + 2\eta u_{\xi\eta} + \eta^2 u_{\eta\eta} + \eta u_\eta$

$- y\{u_{\xi\xi} + (e^x+\eta) u_{\xi\eta}$

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Solⁿ of one-dimensional wave equation from Canonical form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{— (A)}$$

Initial Conditions: $u(x,0) = f(x)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

Solⁿ →

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 . \qquad u(x,y) \to u(\xi,\eta)$$

$\therefore \ \xi = x - ct \ , \ \eta = x + ct .$

Canonical form:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 .$$

$$\Rightarrow \quad u(\xi,\eta) = F(\xi) + G(\eta)$$

$$\therefore \ u(x,t) = F(x-ct) + G(x+ct) .$$

$$u(x,0) = f(x)$$

$$\Rightarrow \ F(x) + G(x) = f(x) \quad \text{— (1)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

$$\Rightarrow \ -cF'(x) + cG'(x) = g(x) \quad \text{— (2)}$$

$$\Rightarrow \quad G'(x) - F'(x) = \frac{g(x)}{c} .$$

$$\Rightarrow \quad \int_{x_0}^{x} \left[G'(x) - F'(x)\right] dx = \int_{x_0}^{x} \frac{g(s)}{c}\, ds .$$

$$\Rightarrow \ G(x) - F(x) - \left[G(x_0) - F(x_0)\right] = \int^{x} \frac{g(s)}{c}\, ds$$

$$\Rightarrow G(x) - F(x) = \int_{x_0}^{x} \frac{g(s)}{c}\,ds + G(x_0) - F(x_0)$$

$$\Rightarrow F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds - \frac{1}{2}\left[G(x_0) - F(x_0)\right]$$

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + \frac{1}{2}\left[G(x_0) - F(x_0)\right]$$

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds - \frac{1}{2}\left[G(x_0) - F(x_0)\right] + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds + \frac{1}{2}\left[G(x_0) - F(x_0)\right]$$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.}$$

This is called D'Alembert's sol^n of the wave equation.

Fourier Series

The F.S. for a periodic function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

where
$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos(nx)\,dx.$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin(nx)\,dx.$$

Even function: $f(-x) = f(x)\ \forall x$ in $(-\pi,\pi)$.

$f(x)\cos(nx) \rightarrow$ even and $f(x)\sin(nx) =$ odd.

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

$\therefore$ F.S. of an even function consists of terms of cosines only.

Odd funcⁿ:

$$f(-x) = -f(x)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx.$$

$\therefore$ F.S. of an odd f^n consist of sine terms only.

Functions having arbitary period, $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{+l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Half Range Fourier Series

given $(0, l)$

can be extended either as even or as odd f^n in $(-l, l)$.

Half Range Sine Series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \text{ where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Half Range Cosine Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right),$$

$$a_0 = \frac{2}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{2}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

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Initial and boundary Value Problems

A PDE subject to certain conditions in the form of initial [Cauchy condition] or boundary conditions is known as Initial Value Problem (IVP) or Boundary Value Problem (BVP).

BVP → 3 types of B.C.

(i) Dirichlet conditions or boundary conditions of 1st kind
are the values of u prescribed at each point of the boundary.

(ii) Neumann Condition or boundary conditions of 2nd kind
are the values of u prescribed at each point of the boundary.

(iii) Robin conditions or mixed boundary condition
or boundary condition of 3rd kind are the values of a linear combination of u and its derivative prescribed at each point of the boundary.

Solution by the method of separation of variables
(for IVP and BVP)

We consider

$$Rr + Ss + Tt + Pp + Qq + Zz = F \quad \text{——— (1)}$$

Let the Sol^n of (1) be in the form

$$z = X(x)\,Y(y) \quad \text{——— (2) where,}$$

$X(x)$ and $Y(y)$ are functions of x and y alone respectively.

Substituting in (1) and simplifying:

$$\frac{1}{X}\{f(D)X\} = \frac{1}{Y}\{g(D')Y\} \quad \text{——— (3)}$$

Where $f(D)$ and $g(D')$ are functions of $D = \frac{d}{dx}$ and $D' = \frac{d}{dy}$

So, LHS of (3) is a function of x alone and RHS of (3) is a function of y alone.

So these two cannot be equal unless each is equal to a constant value λ.
i.e.

$$f(D)X = \lambda X \quad \text{and} \quad g(D')Y = \lambda Y \quad \text{——— (4)}$$

Thus the solⁿ of (1) reduces to, the solⁿ of a pair of ODE given by (4).

Solⁿ of one-dimensional wave eqⁿ: [EXAM TYPE]:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} (A) \qquad 0 < x < l, \ t > 0.$$

B.C.

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(l,t) &= 0 \end{aligned}\right\} \text{—}(B) \qquad \left.\begin{aligned} u(x,0) &= f(x) \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= g(x) \end{aligned}\right\} \text{—}(C).$$

Suppose, A has a solution of the form

$$u(x,t) = F(x)\,T(t) = FT \text{ (say)} \quad \text{——} (1)$$

where F is a function of x only and T is a fn of t only.

Then substituting in (A),

$$F \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 F}{dx^2}$$

$$\Rightarrow \frac{1}{F}\frac{d^2 F}{dx^2} = \frac{1}{c^2 T}\frac{d^2 T}{dt^2}$$

Now, the LHS is a fⁿ of the independent variable x and R.H.S. is a fⁿ of the independent variable t.

These two cannot be equal to each other, unless both reduce to a constant value.

Hence,
$$\frac{1}{F}\frac{d^2 F}{dx^2} = \frac{1}{c^2 T}\frac{d^2 T}{dt^2} = 0 \text{ or } k^2 \text{ or } -k^2.$$

1st Case:

$$\frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} = 0$$

2nd Case:

$$\frac{d^2 F}{dx^2} - k^2 F = 0 \qquad \frac{d^2 T}{dt^2} - k^2 c^2 T = 0$$

3rd Case:

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \qquad \frac{d^2 T}{dt^2} + k^2 c^2 T = 0$$

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The G.S. in these three cases are.

(i) $F = Ax + B \qquad T = Ct + D$

(ii) $F = Ae^{kx} + Be^{-kx} \qquad T = Ce^{kct} + De^{-kct}$.

(iii) $F = A\cos kx + B\sin kx \qquad T = C\cos kct + D\sin kct$.

Case (i) Using Boundary Conditions. and (1)

$u(0,t) = F(0)\,T(t) = 0$

$u(l,t) = F(l)\,T(t) = 0$

which gives either

$T(t) = 0$ or $F(0) = 0$ and $F(l) = 0$

But $T(t) \neq 0$, otherwise from (1) we have $u(x,t) = 0$.

$\therefore F(0) = 0$ and $F(l) = 0$

$B = 0$ and $Al = 0 \Rightarrow A = 0$.

$\Rightarrow$ which gives $A = B = 0$ leading to.

$\Rightarrow F = 0$

$\Rightarrow u(x,t) = 0$.

Case II:

$F(0) = 0 = A + B = 0$

$F(l) = Ae^{kl} + Be^{-kl} = 0$ $F(x) = 0$ and hence

Giving $A = B = 0$ so that

$u(x,t) = 0$

$\therefore$ Solutions (i) and (ii) do not constitute the solⁿ of the wave equation (A).

Case III:

$u(x,t) = (A\cos kx + B\sin kx)(C\cos kct + D\sin kct)$

$u(0,t) = A\,(C\cos kct + D\sin kct) = 0$

$\therefore A = 0$.

$u(l,t) = (0 + B\sin kl)\,(C\cos kct + D\sin kct) = 0$

$\Rightarrow B\sin kl = 0 \qquad [B \neq 0]$

$\Rightarrow \sin(kl) = 0$

$kl = n\pi$

$\Rightarrow \boxed{k = \frac{n\pi}{l}}, \qquad n = 1, 2, 3, \ldots$

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Sol^n of (A) satisfying (B) is

$$u_n(x,t) = \left(C_n \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l} \quad — (3)$$

Since the equation (A) is linear and homogenous, therefore the sum of any number of different solutions of (A) will be a sol^n of (A). Thus instead of (3) the sol^n may be taken in the form, [Principle of Superposition]

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l} \quad — (4)$$

Now, from (4) and the I. C. (C)

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = f(x) \quad — (5)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \frac{n\pi c}{l} D_n \sin\frac{n\pi x}{l} = g(x) \quad — (6)$$

The LHSides can be considered as the Fourier sine expansions of the RHS.

Hence,

$$C_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx \quad — (7)$$

$$\frac{n\pi c}{l} D_n = \frac{2}{l}\int_0^l g(x) \sin\frac{n\pi x}{l}\,dx \quad — (8)$$

Thus $u(x,t)$ given by (4) with the coefficients (7) and (8) is the sol^n of the wave eq^n (A) that satisfies the conditions (B) and (C).

Lecture - 32

23.10.2017

Particular case: [D'Alembert's sol^n]

If $g(x) = 0$, then $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = 0$.

$\Rightarrow D_n = 0$

Now, (4) reduces to the form,

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} C_n \left[\sin\left(\frac{n\pi}{l}(x-ct)\right) + \sin\left(\frac{n\pi}{l}(x+ct)\right)\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{l}(x-ct) + \frac{1}{2}\sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{l}(x+ct).$$

$$= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] \quad \text{— from (5)}.$$

which can be taken as sol^n of wave eq^n (A) where f is the odd periodic extension of f with period $2l$.

Ex: Obtain the sol^n of the wave-equation

$$u_{tt} = c^2 u_{xx}$$

under the following conditions.

(i) $u(0,t) = u(2,t) = 0$.

(ii) $u(x,0) = \sin^3\frac{\pi x}{2}$

(iii) $u_t(x,0) = 0$.

$\infty \rightarrow$ In exam, you have to show the procedure to reach to this sol^n.

Sol^n: Sol^n of the wave equation is of the form.

$$u(x,t) = (A\cos kx + B\sin kx)\left[C\cos(kct) + D\sin(kct)\right]$$

Now,

(i) $u(0,t)$ ~~$\cos(kct)$~~ $= 0 \Rightarrow A = 0$.

Now, $u(2,t) = 0$

$\Rightarrow (A\cos 2k + B\sin 2k)(C\cos(kct) + D\sin(kct)) = 0$

$\Rightarrow$ ~~$A\cos 2k +$~~ $B\sin 2k = 0$

$\Rightarrow 2k = n\pi \Rightarrow k = \frac{n\pi}{2}$ $\quad n = 1, 2, 3, \ldots$

Also,

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

$\Rightarrow$ $D = 0$.

$\therefore$ $u(x,t) = B\sin kx \; C\cos(kct)$.

$= A_n \sin kx \cos(kct)$ $\{A_n = BC\}$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{2} \cos\left(\frac{n\pi ct}{2}\right)$$

Now,

$$u(x,0) = \sin^3\frac{\pi x}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) = \sin^3\frac{\pi x}{2}.$$

$$= \frac{3}{4}\sin\frac{\pi x}{2} - \frac{1}{4}\sin\frac{3\pi x}{2}$$

$\therefore$ $A_1 = \frac{3}{4}$, $A_3 = -\frac{1}{4}$ and all other A_n's are zero,

Hence, the required soln is

$$u(x,t) = \frac{3}{4}\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi ct}{2}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{2}\right)\cos\left(\frac{3\pi ct}{2}\right)$$