Lecture - 1

LINEAR ALGEBRA

☆

Linear Algebra (MA 30003, MA41003)

MA 30003 $\downarrow$ 3rd year of 5 year M.Sc.

MA41003 $\downarrow$ 1st year of 2 year M.Sc

Monday : 12:00 $\rightarrow$ 1:00

Tuesday : 10:00 $\rightarrow$ 12:00 (D4)

Thursday : 8:00 $\rightarrow$ 9:00

Class-tests (2)	10 marks each	
Mid-Semester	30 marks	
End-Semester	50 marks	

Text-books

1. Linear Algebra (Fourth Edition)
   Friedberg, Insel, Spence
2. Linear Algebra: a Geometric Approach
   S. Kumaresan

Reference

1. Linear Algebra : Hoffman & kunze

Linear equations.

Ex.1) $\boxed{2x + 3y = 0}$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = 1, 2, \dots, m$$

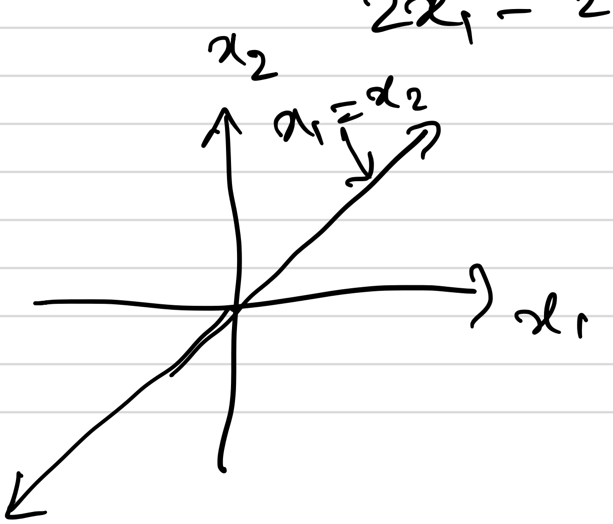
$$(*) \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Find all $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ which satisfy $(*)$.

Ex.2)

case (1)

$$x_1 - x_2 = 0$$
$$2x_1 - 2x_2 = 0$$



case (2)

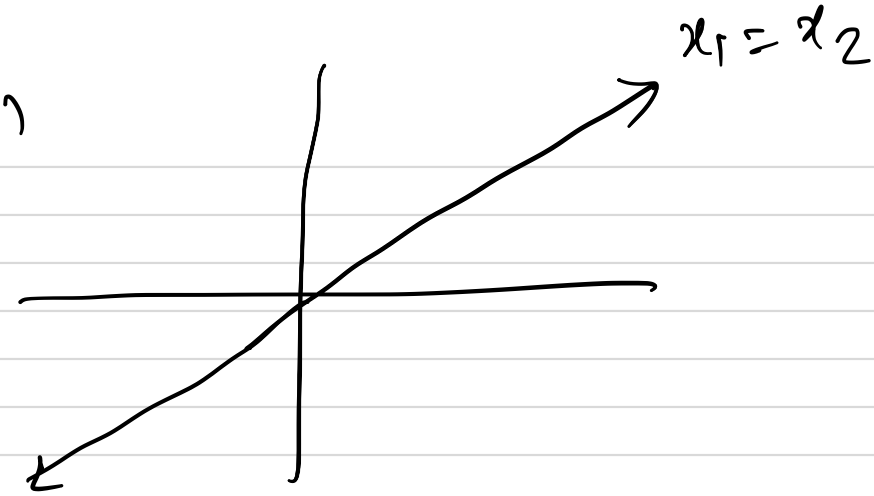
$$x_1 - x_2 = 0$$
$$2x_1 - 2x_2 = 2$$

no-solution

case (3)

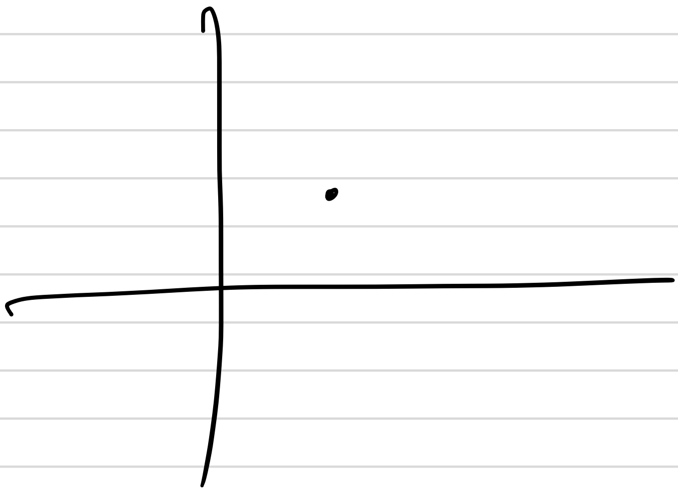
$$x_1 - x_2 = 0$$
$$2x_1 + 2x_2 = 2$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

Case (1)



Case (2) ϕ : empty set

Case (3) :



Question : How does solution set of the equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned}\right\} \quad (**)$$

"look" ??

origin or a line passing through origin.

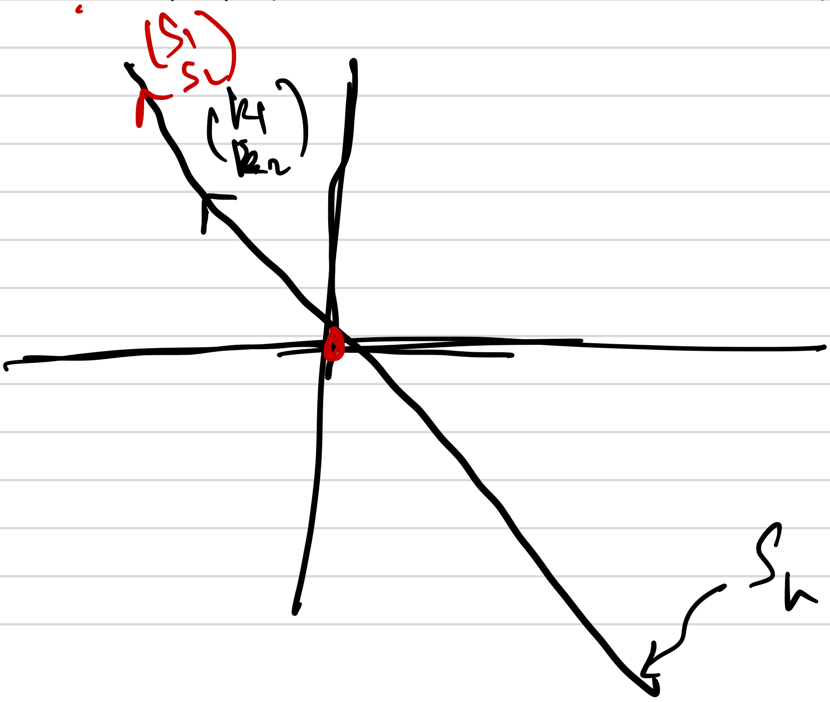
Let S_h be the set of all solutions of $(**)$

Let $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in S_h$ and $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in S_h$

$$\begin{pmatrix} k_1 + s_1 \\ k_2 + s_2 \end{pmatrix} \in S_h$$

For $\alpha \in \mathbb{R}$, $\alpha \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in S_h$

Observation : S_h is "closed" under vector addition and scalar multiplication.



"Assume" S_h is passing thru origin.

Example 2 :

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$

Let S_h be the set of all solutions to the above equation.

(Geometrically, if one of $a_i' \neq 0$, then this equation represents a plane in $\mathbb{R}^3$ which passes through origin).

Let $x, y \in S_h \subseteq \mathbb{R}^3$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Clearly, $x + y \in S_h$ and $\alpha x \in S_h$ for every $\alpha \in \mathbb{R}$.

S_h is "closed" under vector addition and scalar multiplication.

Example 3:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$

S_h = may be a plane passing thru' origin.

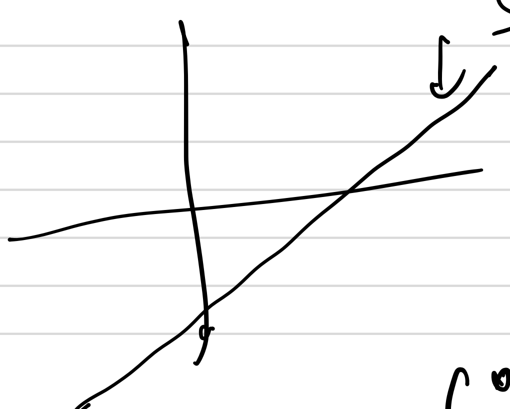
S_h = may be a line passing thru' origin.

$S_h = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$?? H.W.

Example 4:

$$x_1 - x_2 = 1$$

Let $S \subseteq \mathbb{R}^2$ be the set of all solutions to this equation.



Let $x, y \in S$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$x + y \notin S$

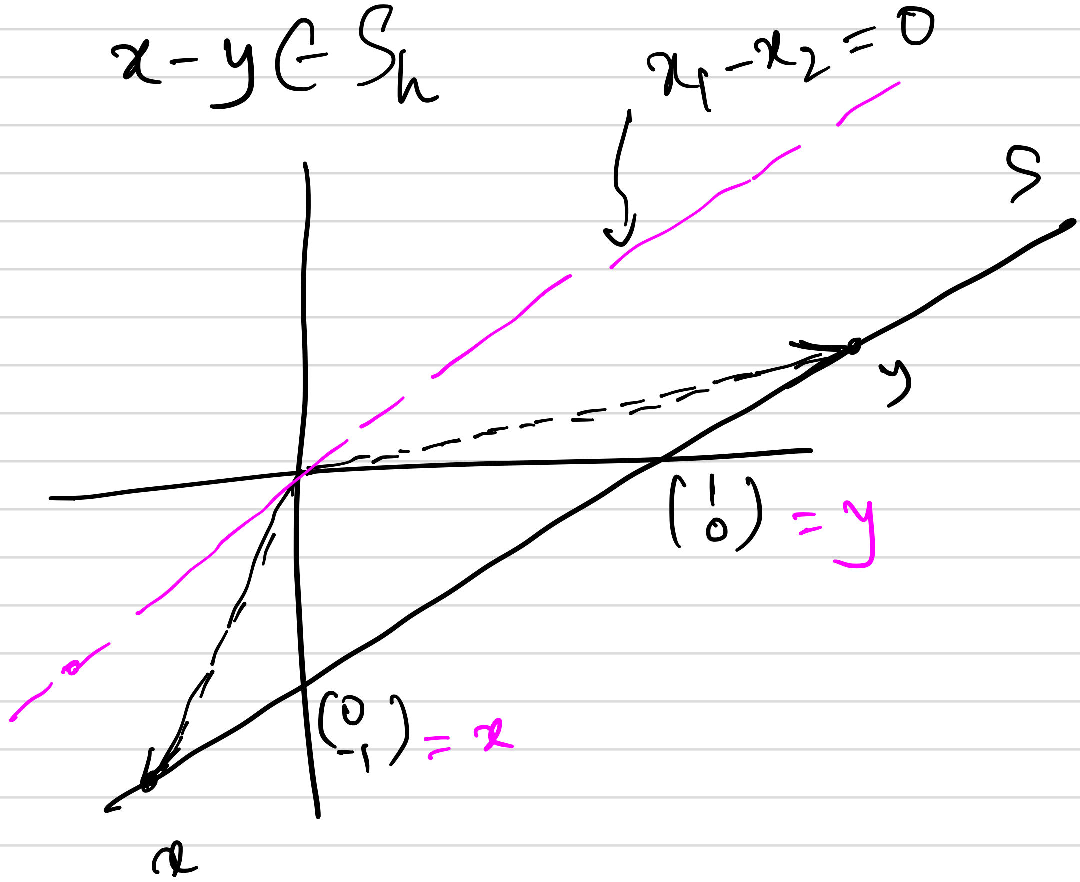
$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S$

S is <u>NOT</u> closed under vector addition and scalar multiplication.

In fact, $\begin{pmatrix}0\\0\end{pmatrix} \notin S$

<u>Observation</u>: For any x and $y \in S$

$x - y \in S_h$



S_h = set of all solutions of

$x_1 - x_2 = 0$

Homogeneous system $\sum_{j=1}^{n} a_{ij} x_j = 0$

$i = 1, \ldots, m$

Non-Homogeneous system

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

$i = 1, 2, \ldots, m$

Let S be the set of all solutions of non-homogeneous system & S_h be the solution set of the associated homogeneous system of eqn.

For a particular solution α of the non-homogeneous system,

$$S = \alpha + S_h$$

Example 5:

Let $[a,b] \subseteq \mathbb{R}$

$\mathcal{F}$ denotes the set of all real valued functions on $[a,b]$.

Define $f_1 + f_2$ for $f_1, f_2 \in \mathcal{F}$ as

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \forall x \in [a,b]$$

For a scalar $\alpha \in \mathbb{R}$,

$$(\alpha f_1) x = \alpha f_1(x) \quad \forall f_1 \in \mathcal{F}.$$
$$\forall x \in [a,b]$$

Clearly, $\mathcal{F}$ is closed under "function addition" and scalar multiplication.

Note, there exists "zero function",

$$0 : [a,b] \to \mathbb{R}$$
$$0(x) = \underline{\underline{0}} \quad \forall x \in [a,b]$$

$0 \in \mathcal{F}$

Example 6:

Consider $C[a,b] \subseteq \mathcal{F}$.

where $C[a,b]$ denotes the set of all continuous real valued functions from $[a,b]$ to $\mathbb{R}$.

Clearly, using properties from calculus, we can see that $C[a,b]$ contains "zero function" and is closed under "function addition" and scalar multiplication.

Example 7: $C^1[a,b] \subseteq C[a,b] \subseteq \mathcal{F}$

$C^n[a,b]$

$C^\infty[a,b]$

Example 7: Matrices

Consider $M_{m \times n}(\mathbb{R})$ to be the set of all matrices of size $m \times n$ with real entries.

Example 8: Sequences

Let S be a set of all real sequences.

In particular $s \in S$

$$s = \{s_n\}_{n=1}^{\infty}$$

$$s+t = \{s_n + t_n\}_{n=1}^{\infty} \quad \text{for } s,t \in S$$

$$\alpha s = \{\alpha s_n\}_{n=1}^{\infty} \quad \alpha \in \mathbb{R}.$$

Let S_0 be the subset of S such that $x \in S_0 \Rightarrow x = (x_n)_{n=1}^{\infty} \to 0$

<u>Example:</u> $\mathbb{F}_2 = \{0,1\}$

$\mathbb{P}^n_{\mathbb{F}_2}$ = set of all polynomials of degree $\leq n$ with coefficients from $\mathbb{F}_2$.

(less than or equal)

$p(x) \in \mathbb{P}^n_{\mathbb{F}_2}$

$p(x) = x^n + x^2 + x + 1$

$q(x) = x^n + 1$

$$(p+q)x = p(x) + q(x)$$

$$= \underbrace{\cancel{2x^n}}_{0} + x^2 + x + \underbrace{\cancel{2}}_{0}$$

$$= x^2 + x$$