Lecture 9

Proposition:-

(i) Every non-empty open set has +ve measure.

(ii) Let $\mathbb{Q} = \{q_1, q_2, \cdots\}$ &

$$G = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2},\ q_n + \frac{1}{n^2}\right).$$

Then for any closed set $F \subseteq \mathbb{R}$ we have

$$m(G \Delta F) > 0.$$

proof:- (i) Let $U \subseteq \mathbb{R}$ be a non-empty open set.

We know that U is the union of disjoint open intervals, at most countable in number.

Say $U = \dot{\bigcup}_{j=1}^{\infty} U_j$, U_j = open interval

$$\therefore m(U) = m\left(\dot{\bigcup}_{j=1}^{\infty} U_j\right) = \sum_{j=1}^{\infty} m(U_j)$$

$$= \sum_{j=1}^{\infty} \ell(U_j) > 0$$

(ii) Let $F \subseteq \mathbb{R}$ be a closed set.

To prove: $m(G \Delta F) > 0$.

where $G = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2},\ q_n + \frac{1}{n^2}\right)$

$G \Delta F = (G \setminus F) \cup (F \setminus G)$.

$G \in \mathcal{M}$

$F \in \mathcal{M}$

$G \setminus F = G \cap F^c \in \mathcal{M}$

If $m(G \setminus F) > 0$, then $m(G \Delta F) > 0$ as required.

$\left(\because G \setminus F \subseteq G \Delta F, \; \underset{0 <}{m(G \setminus F)} \leq m(G \Delta F) \right)$

Assume $m(G \setminus F) = 0$.

$G \setminus F = G \cap F^c$ an open set.

if $G \setminus F \neq \phi$, then by (i), $m(G \setminus F) > 0$ but this not the case.

$\therefore \; G \setminus F = \phi$.

$\Rightarrow \; G \subseteq F$

Also $\mathbb{Q} \subseteq G \subseteq F$

$\Rightarrow \underset{\underset{\mathbb{R}}{\shortparallel}}{\overline{\mathbb{Q}}} \subseteq \overline{G} \subseteq \overline{F} = F$ (taking closure)

$\Rightarrow \; \mathbb{R} = F$.

$\Rightarrow \; m(F) = \infty$.

And,
$$m(G) = m\left(\bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2} \right) \right)$$
$$\leq \sum_{n=1}^{\infty} \left(q_n + \frac{1}{n^2} - \left(q_n - \frac{1}{n^2} \right) \right)$$
$$= \sum_{n=1}^{\infty} \frac{2}{n^2}, < \infty$$

Thus $\underline{m(G) < \infty} \quad m(F) = \infty$.

$\Rightarrow$ $\boxed{m(F\setminus G)=\infty}$

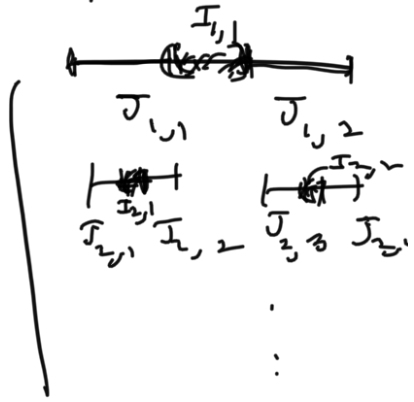
$\Rightarrow$ $m(F\setminus G)>0$

$\Rightarrow$ $m(G\triangle F)>0$, as required.

Proposition:— There exists an uncountable set of measure zero.

We show that the Cantor set has measure zero.

proof: Cantor set $= P = \bigcap_{n=1}^{\infty} P_n$, $\quad P_n = \bigcup_{r=1}^{2^n} J_{n,r}$.



$$P^c = [0,1]\setminus P$$
$$= \bigcup_{n=1}^{\infty} P_n^c.$$
$$= \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{2^{n-1}} I_{n,r}. \quad \text{disjoint unions.}$$

$$m(P^c) = m\left(\dot{\bigcup_{n=1}^{\infty}} \dot{\bigcup_{r=1}^{2^{n-1}}} I_{n,r}\right)$$
$$= \sum_{n=1}^{\infty} \sum_{r=1}^{2^{n-1}} m(I_{n,r})$$
$$= \sum_{n=1}^{\infty} \sum_{r=1}^{2^{n-1}} \frac{1}{3^n}.$$

$$= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$$

$$\therefore \quad m\left([0,1] \setminus P\right) = 1$$

$$\Rightarrow \quad m(P) = 0 \text{.} \left(\because [0,1] = P \cup \left([0,1]\setminus P\right) \right.$$

$$m([0,1]) = m(P) + m(P^c)$$

$$\left. \Rightarrow 1 = m(P) + 1 \right)$$

We already proved that P is uncountable

The following result states that the measurable sets in $\mathbb{R}$ are those which can be approximated closely, in terms of m^*, by open or closed sets.

$$[0,1] = (\mathbb{Q} \cap [0,1]) \cup (\mathbb{Q}^c \cap [0,1])$$

$$m([0,1]) = m(\mathbb{Q} \cap [0,1]) + m(\mathbb{Q}^c \cap [0,1])$$

$$\Rightarrow 1 = 0 + \text{''}$$

$$\Rightarrow m(\mathbb{Q}^c \cap [0,1]) = 1$$

Theorem:— Let $E \subseteq \mathbb{R}$. Then the following are equivalent.

(i) E is measurable.

(ii) Given $\varepsilon > 0$, there exists an open $U \underset{\mathbb{R}}{\subseteq}$ such that

$$E \subseteq U \quad \& \quad m^*(U \setminus E) < \varepsilon.$$

(iii) There exists a G_δ-set G such that $G \supseteq E$ & $m^*(G \setminus E) = 0$.

$(ii)^*$ Given $\varepsilon > 0$, there exists a closed set F such that $F \subseteq E$ & $m^*(E \setminus F) \leq \varepsilon$.

$(iii)^*$ There exists an F_σ-set, $F \subseteq E$ such that $m^*(E \setminus F) = 0$.

Def:- A non-ve countably additive set function satisfying the above equivalent conditions (ii) to $(iii)^*$ is said to be a <u>regular measure</u>.

The above theorem says that m^* is a regular measure.

$\boxed{m^*(E) = 0 \ \forall E \subseteq \mathbb{R}, \text{ trivial.}}$

<u>proof</u>:-

<u>(i) $\Rightarrow$ (ii)</u>: Assume E is measurable.

To show: Given $\varepsilon > 0$, there exists an open set U such that $U \supseteq E$ & $m^*(U \setminus E) \leq \varepsilon$.

Let $\varepsilon > 0$.

Suppose $m(E) < \infty$, a finite number.

There exists an open set $U \subseteq \mathbb{R}$ such that $U \supseteq E$ & $m^*(U) \leq m^*(E) + \varepsilon$.

$$\Rightarrow m^*(U \setminus E) = m^*(U) - m^*(E) \quad (\because m^*(E) < \infty)$$
$$\leq \varepsilon.$$

as required.

Suppose $m(E) = \infty$,

Let $\mathbb{R} = \dot{\bigcup_{n=1}^{\infty}} I_n$, a disjoint union of finite intervals.

Let $E_n = E \cap I_n$. $\forall n$.

Then $m(E_n) < \infty$ $\forall n$.

$\therefore$ There exists an open set U_n such that $E_n \subseteq U_n$ & $m(U_n \setminus E_n) \leq \frac{\varepsilon}{2^n}$. $\forall n$.

Let $U = \bigcup_{n=1}^{\infty} U_n$ open set.

Now $U \setminus E = \bigcup_{n=1}^{\infty} U_n \setminus \left(\bigcup_{n=1}^{\infty} E_n \right)$
$$\subseteq \bigcup_{n=1}^{\infty} (U_n \setminus E_n)$$

$$\therefore m(U \setminus E) \leq m\left(\bigcup_{n=1}^{\infty} (U_n \setminus E_n) \right)$$

$$\leq \sum_{n=1}^{\infty} m(U_n \setminus E_n)$$

$$\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

$$\therefore \quad m(U \setminus E) \leq \varepsilon.$$