$$y'' + p(x)\, y' + q(x)\, y = 0$$

if $p(x)$ and $q(x)$ are singular, but $x\,p(x)$ and $x^2 q(x)$ are regular then $x=0$ is a regular singular point

$(x = x_0)$

For $(x-x_0)\,p(x)$ and $(x-x_0)^2 q(x)$ to be regular,

leading term of $\begin{cases} p(x) \sim \dfrac{1}{x-x_0} \\ q(x) \sim \dfrac{1}{(x-x_0)^2} \end{cases}$

We seek soln of the form

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^{m+n}$$

accordingly, the general soln depends on the roots of the "indicial eqn"

For a general m, above eqn is of the form

$$y(x) = a_0 x^m (1 + \alpha_1(m) x + \alpha_2(m) x^2 + \cdots)$$

$$y'' + py' + qy \sim a_0 x^{m-2} (\text{indicial equation}) + \sum_{n=0}^{\infty} \underbrace{F(\alpha_i(m))}_{\text{some relation of } \alpha_i(m)} x^{m+n}$$

$$\simeq 0$$

$$\cancel{x^2}\, xy'' + y' - xy$$

$$= a_0 x^{m-2} m^2 + \sum_{n=0}^{\infty} [a_{n+1} (m+n+1)^2 - a_{n-1}] x^{m+n}$$

indicial eqn $= 0$

recurrence relation

$$\underbrace{y'' + p(x) y' + q(x) \sim a_0 x^{m-2} \text{ indicial eqn} \sim a_0 x^{m-2} (m - m_1)(m - m_2)}_{*}$$

if $m_1 = m_2$, only one Frobenius series via this method

diff wrt m, $*$

$$y_m'' + p(x)\, y_m' + q(x)\, y_m$$

$$\simeq a_0 \frac{\partial}{\partial m}\left(x^{m-2} (m-m_1)(m-m_2) \right)$$

$$= a_0 \frac{\partial}{\partial m}\left(x^{m-2} (m-m_1)^2 \right)$$

still function of $(m-m_1)$ and hence vanish for $m = m_1$

$\Rightarrow$ $\frac{\partial y}{\partial m}$ is also a soln. of $y'' + py' + qy = 0$.

$$\Rightarrow \frac{\partial}{\partial m}\left[x^m \left(1 + \alpha_1(m) x + \alpha_2(m) x^2 + \cdots \right)\right]$$

is another soln.

$$= \frac{\partial}{\partial m}\left[e^{m \ln|x|} \left\{ 1 + \alpha_1 x + \alpha_2 x^2 + \cdots \right\}\right]$$

$$= x^{m_1} \ln|x| \, (1 + \alpha_1 x + \cdots) + x^{m_1} \left\{ \alpha_1' x + \alpha_2' x^2 + \cdots \right\}$$

repeated roots:

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$$

$$y_2 = x^{m_1} \ln|x| \, (1 + \alpha_1 x + \cdots) + x^{m_1} \sum_{n=1}^{\infty} b_n x^n$$

$$\simeq x^{m_1}\ln|x|\sum_{n=0}^{\infty}a_n x^n$$

$$+ x^{m_1}\sum_{n=0}^{\infty}b_n x^n$$

$$= y_1(x)\ln|x| + x^{m_1}\sum_{n=1}^{\infty}b_n x^n$$

roots differed by an integer

Recall $xy'' + 4y' - xy = 0$

roots $\underline{m = 0, -3}$, $a_1 = 0$

$\underline{m=0}$ $y_1(x) = x^0\sum_{n=0}^{\infty}a_n x^n$

recurrence relation $a_{n+2} = \dfrac{a_n}{(m+n+2)(m+n+5)}$

$\underline{m=0}$ a_n can be determined in terms of $\underline{a_0}$ $\boxed{a_{n+2} = \dfrac{a_n}{(n+2)(n+5)}}$

$\underline{n=0}$ $a_2 = \dfrac{a_0}{2\cdot 5}$.

$a_3 = \dfrac{a_1}{3\cdot 6} = 0$ $(\because a_1 = 0)$

$\underline{m=-3}$. $$a_{n+2} = \frac{a_n}{(n-1)(n+2)} \quad , n=0,1..$$

$\underline{n=0}$ $$a_2 = \frac{a_0}{-1\cdot 2}$$

$\underline{n=1}$ $$a_3 = \frac{a_1}{0} \quad \text{but } a_1 = 0$$

$\Rightarrow$ a_3 is arbitrary

$\underline{n=2}$ $$a_4 = \frac{a_2}{1\cdot 4} = -\frac{a_0}{1\cdot 2\cdot 1\cdot 4}$$

$\underline{n=3}$ $$a_5 = \frac{a_3}{2\cdot 5}$$

$\underline{n=4}$ $$a_6 = \frac{a_4}{3\cdot 6} = -\frac{a_0}{1\cdot 2\cdot 1\cdot 4\cdot 3\cdot 6}$$

$\underline{n=5}$ $$a_7 = \frac{a_5}{4\cdot 7} = \frac{a_3}{2\cdot 5\cdot 4\cdot 7}$$

$$y_2(x) \simeq \left(\begin{array}{c}\text{even}\\ \text{power}\end{array}\right)a_0 + \left(\begin{array}{c}\text{odd}\\ \text{powers.}\end{array}\right)a_3$$

C6

roots differed by an integer

$m_1 > m_2$, ~~m_2~~ $m_1 = m_2 + k$

for m_1 $\quad y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$

for m_2 $\quad y_2 = y_1 \ln|x| + x^{m_2} \sum_{0}^{\infty} b_n x^n$

Legendre's equation

Consider $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

α- constants

$x=0$ is an ordinary point

let $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$-\sum_{n=1}^{\infty} 2n a_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

equate coef.ts of x^k, $k \in \mathbb{N}_0$

$2a_2 + \alpha(\alpha+1)a_0 = 0 \quad \ldots \quad x^0$ — (e0)

$6a_3 - 2a_1 + \alpha(\alpha+1)a_1 = 0 \quad \ldots \quad x^1$ — $\cancel{(e_1)}$ (e1)

$(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + \alpha(\alpha+1)a_n = 0$

$n = 2, 3, \ldots$ — (en)

(e0) $\Rightarrow$ $a_2 = -\dfrac{\alpha(\alpha+1)a_0}{2}$

(e1) $\Rightarrow$ $a_3 = -\dfrac{(\alpha-1)(\alpha+2)}{2\cdot 3}a_1$

$$a_4 = \frac{(\alpha+3)(\alpha-2)}{3\cdot 4} a_2 = -\frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{1\cdot 2\cdot 3\cdot 4} a_0$$

$\cdot \qquad \cdot$

$$a_5 = -\frac{(\alpha-3)(\alpha+4)}{4\cdot 5} a_3$$

$$= -\frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{1\cdot 2\cdot 3\cdot 4\cdot 5} a_1$$

$$\therefore y(x) = a_0\left[1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 \cdots\right]$$

$$+ a_1\left[x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)\cdots}{5!}x^5 \cdots\right]$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

<u>α not an integer</u>

both y_1 & y_2 are power series

<u>α is an integer :</u>

<u>$\alpha = K$ even integer</u> y_1 is a polynomial of degree <u>K</u> and y_2 is a power series

<u>$\alpha = k$ even</u>

y_1 – poly of deg k, even powers

y_2 – power series odd powers

<u>$\alpha = k$ odd integer</u>

y_1 power series even powers

y_2 poly ~~of~~ odd powers

<u>For $\alpha = k$ +ve integer</u>

$$a_{n+2} \cancel{a_{\alpha+2}} = -\frac{(\alpha-n)(\alpha+n+1)}{(n+1)(n+2)} \cancel{a} \, a_n$$

$$a_{k+2} = \frac{-(k-n)(k+n+1)}{(n+1)(n+2)} a_n$$

$$a_{k-2} = -\frac{k(k-1)}{2(2k-1)} a_k$$

$$a_{k-4} = -\frac{(k-2)(k-3)}{4(2k-3)} a_{k-2}$$

$$= (-1)^2 \frac{k(k-1)(k-2)(k-3)}{2\cdot 4 \cdot (2k-1)(2k-3)} a_k$$

$a_k \neq 0$, $a_{k+2} = 0$ and hence

$a_{k+2j} = 0$, $j \in \mathbb{N}$

$$a_{k-4} = (-1)^2 \frac{k(k-1)(k-2)(k-3)}{2\cdot 4\cdot (2k-1)(2k-3)} a_k$$

for $2l < k$

$$a_{k-2l} = (-1)^l \frac{k(k-1)(k-2)\cdots(k-2l+1)}{[2\cdot 4\cdots(2l)][(2k-1)(2k-3)\cdots(2k-2l+1)]} a_k$$

$$= (-1)^l \frac{k!}{(k-2l)!} \frac{(2k-2)(2k-4)\cdots(2k-2l)}{2^l l! [(2k-1)(2k-2)(2k-3)\cdots(2k-2l)]} a_k$$

$$= (-1)^l \frac{k!}{(k-2l)!\, 2^l l!} \frac{2^l (k-1)(k-2)\cdots(k-l)}{(2k-1)(2k-2)\cdots(2k-2l)} a_k$$

$$= (-1)^l \frac{k!\,(k-1)!}{(k-2l)!\,(k-l-1)!} \frac{(2k-2l-1)!}{(2k-1)!} a_k$$

$$a_{k-2l} = \frac{(-1)^{\lambda}(2k-2\lambda)!}{2^k \lambda!(k-\lambda)!(k-2\lambda)!}$$

with $a_k = \dfrac{(2k)!}{2^k (k!)^2}$

$$a_{k-2l} = (-1)^{\lambda}\frac{(2k-2\lambda)!}{2^k \lambda!(k-\lambda)!(k-2\lambda)!}$$

<u>Def</u>: The polynomial

$$P_n(x) = \sum_{\lambda=0}^{M_n} \frac{(-1)^{\lambda}(2n-2\lambda)!}{2^n \lambda!(n-\lambda)!(n-2\lambda)!}\, x^{n-2\lambda}$$

is called Legendre polynomial of degree n, M_n: $n/2$ – n even

$(n-1)/2$ – n odd

$$\therefore P_n(x) = \sum_{\lambda=0}^{\lceil n/2 \rceil} (-1)^{\lambda} \frac{\overset{1}{\cancel{(2n-2\lambda)!}}}{2^n \lambda!(n-\lambda)!} \cdot \frac{d^n x^{2n-2\lambda}}{dx^n}$$

$$= \cancel{\frac{1}{2^n n!} d \sum_{\lambda=0}^{\lceil n/2 \rceil}} \frac{n!}{\lambda!(n-\lambda)!}$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{l=0}^{[n/2]} \frac{n!}{l!\,(n-l)!} (x^2)^{n-l} (-1)^l$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{l=0}^{n} \binom{n}{l} (x^2)^{n-l} (-1)^l \quad \left(\text{add terms to make } n\right)$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$ Rodrigue's formula

$P_0(x) = 1$, $\quad P_1(x) = x$,

$P_2(x) = \frac{1}{2}(3x^2-1) \sim \frac{3}{2}x^2 - \frac{1}{2}$

Q1: is there a function that generates P_n

Q2: can a given function be expanded in terms of P_n

Generating function of Legendre polynomials

Consider

$$f(x,h) = (1-2xh+h^2)^{-1/2}$$

$$= (1-y)^{-1/2}, \quad y = 2xh - h^2$$

(for $|y|<1$)

$$= 1 + \frac{1}{2}y + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}y^2 + \cdots$$

$$= 1 + \frac{1}{2}(2xh-h^2) + \frac{3}{8}(2xh-h^2)^2 + \cdots$$

$$= 1 + xh - \frac{1}{2}h^2 + \frac{3}{8}(4x^2h^2 + h^4 - 4xh^3) + \cdots$$

$$= 1 + xh + h^2\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \cdots$$

$$= P_0 + hP_1 + h^2P_2 + \cdots$$

$$= \sum_{n=0}^{\infty} h^n P_n(x)$$

$$\therefore (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Further, if $x=1$,

$$\underline{x=1} \Rightarrow (1-2h+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(1)$$

$$\Rightarrow \frac{1}{1-h} = \sum_{n=0}^{\infty} h^n P_n(1)$$

$$\Rightarrow 1+h+h^2+\cdots = P_0 + hP_1 + h^2 P_2 + \cdots$$

$$\Rightarrow \boxed{P_n(1) = 1} \quad \forall n.$$

Recurrence Relations for P_n

① we have $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

diff w.r.t h.

$$(x-h)(1-2xh+h^2)^{-3/2} = \sum_{n=1}^{\infty} h^{n-1} n P_n(x)$$

$$\Leftrightarrow (x-h)(1-2xh+h^2)^{-1/2} = (1-2hx+h^2)\sum_{n=1}^{\infty} h^{n-1} n P_n(x)$$

$$\Leftrightarrow (x-h)\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)\sum_{n=1}^{\infty} h^{n-1} n P_n(x)$$

$$\sum_{n=0}^{\infty} h^n x P_n(x) - \sum_{n=0}^{\infty} h^{n+1} P_n(x)$$

$$\sum_{n=0}^{\infty} h^n x P_n(x) - \sum_{n=0}^{\infty} h^{n+1} P_n(x)$$

$$= \sum_{n=1}^{\infty} h^{n-1} n P_n(x) - \sum_{n=1}^{\infty} h^n 2nx P_n(x)$$

$$+ \sum_{n=1}^{\infty} h^{n+1} n P_n(x)$$

$$\sum_{n=0}^{\infty} h^n x P_n(x) - \sum_{n=1}^{\infty} h^n P_{n-1}(x)$$

$$= \sum_{n=0}^{\infty} h^n (n+1) P_{n+1}(x) - \sum_{n=1}^{\infty} h^n 2nx P_n(x)$$

$$+ \sum_{n=2}^{\infty} h^n (n-1) P_{n-1}(x)$$

$$\cancel{(xp_0 - p_1) + (xp_1 - p_0 - 2p_2 + 2xp_1)h + \sum_{n=2}^{\infty}}$$

$$3xP_1 - P_0 - 2P_2 = 3x^2 - 1 - 2\,\frac{1}{2}(3x^2-1)$$

$$h^0(\underbrace{xP_0 - P_1}_{0}) + h^1(\underbrace{xP_1 - P_0 - 2P_2 + 2xP_1}_{0})$$

$$+ \sum_{n=2}^{\infty}\left[xP_n - P_{n-1} - (n+1)P_{n+1} + 2nxP_n - (n-1)P_{n-1}\right]h^n = 0$$

$$\Rightarrow \boxed{(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}}$$

$$n \geq 2$$

R.R 2

we have $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$ — (*)

diff w r t h

$$(x-h)(1-2xh+h^2)^{-3/2} = \sum_{n=1}^{\infty} n h^{n-1} P_n(x) \quad \text{— (A)}$$

diff (*) w r t x

$$h(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P_n'(x) \quad \text{— (B)}$$

from (A),

$$h(x-h)(1-2xh+h^2)^{-3/2} = \sum_{n=1}^{\infty} n h^n P_n(x) \quad \text{— (C)}$$

use (B) in (C)

$$(x-h)\sum_{n=0}^{\infty} h^n P_n'(x) = \sum_{n=1}^{\infty} h^n n P_n(x)$$

$$\Rightarrow \sum_{n=0}^{\infty} h^n x P_n'(x) - \sum_{n=0}^{\infty} h^{n+1} P_n'(x) - \sum_{n=0}^{\infty} h^n n P_n(x) = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} h^n x P_n' - \sum_{n=1}^{\infty} h^n P_{n-1}'(x) - \sum_{n=0}^{\infty} h^n n P_n = 0$$

$$\Rightarrow \boxed{n P_n = x P_n' - P_{n-1}'}$$

R.R 3:

diff R.R 1 wrt x and then use $x P_n'$ from R.R 2, to show

$$(2n+1) P_{n+1} = P_{n+1}' - n P_{n-1}' \quad \text{--- } (3)$$

R.R 4

diff R.R 1 wrt x

use R.R 2

Replace n by $(n-1)$

$$P_{n+1}' = x P_{n-1}' - n P_{n-1} \quad \text{--- } (4)$$

R.R 5 R.R 2 in 4

$$(1-x^2) P_n' = n [P_{n-1} - x P_n]$$

Orthogonality of Legendre polynomials

If P_n and P_m satisfy Legendre eqn.

$$[(1-x^2)P_n']' + n(n+1)P_n = 0 \quad \text{— (a)}$$

$$[(1-x^2)P_m']' + m(m+1)P_m = 0 \quad \text{— (b)} \qquad n \neq m$$

P_m (a) $- P_n$ (b) & integrate between $[-1, 1]$

$$\int_{-1}^{1}\left\{P_m[(1-x^2)P_n']' - P_n[(1-x^2)P_m']'\right\}dx$$
$$+ [n(n+1) - m(m+1)]\int_{-1}^{1} P_n P_m \, dx = 0$$

$\Rightarrow$ integration by parts of 1st term

$$P_m(1-x^2)P_n'\Big|_{-1}^{1} - P_n(1-x^2)P_m'\Big|_{-1}^{1}$$
$$\underbrace{-\int_{-1}^{1} P_m'(1-x^2)P_n'\,dx + \int_{-1}^{1} P_n'(1-x^2)P_m'\,dx}_{\text{cancel}}$$

P_n, P_n' are bounded, hence 1st, 2nd terms above $\to 0$

$$\therefore \int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0, \quad n \neq m$$

to calculate $\int_{-1}^{1} P_n^2(x)\, dx$

---

let $I_n = \int_{-1}^{1} P_n^2(x)\, dx = \dfrac{1}{2^{2n}(n!)^2} \int_{-1}^{1} \cancel{\dfrac{d^n}{dx^n}(1-x^2)^n \dfrac{d^n}{dx^n}(1-x^2)^n}\, dx$

$$= \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \cdot \frac{d^n}{dx^n}(x^2-1)^n\, dx$$

$$\Rightarrow 2^{2n}(n!)^2 I_n = \left. \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \cdot \frac{d^n}{dx^n}(x^2-1)^n \right|_{-1}^{1}$$

$$- \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \cdot \frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\, dx$$

diff $(x^2-1)^n$ less than n times leaves $(1-x^2)$ hence $\to 0$ between -1 to 1

repeating n times

$$I_n = \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n \frac{d^{2n}}{dx^{2n}}(x^2-1)^n \, dx$$

$(2n)$th derivative of $(x^2-1)^n$ is $(2n)!$

$$\therefore I_n = \frac{(-1)^n}{2^{2n}(n!)^2}(2n)!\boxed{\int_{-1}^{1}(x^2-1)^n \, dx \quad t = \frac{x+1}{2} \quad \downarrow \quad \int_0^1 \quad \text{Ex}}$$

$$= \frac{(n!)^2}{(2n+1)}(-1)^n$$

$$\therefore I_n = \int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$$

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0, & n \neq m \\ \dfrac{2}{2n+1}, & n = m \end{cases}$$

application of orthogonality

Let f be defined on $[-1, 1]$

Can we expand f as P_n?

Look for $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$ — (*)

$$x + 2 = 2P_0 + P_1$$

$$x^2 + 3x - 10 =$$

$\int_{-1}^{1} P_m$ (*) $\Rightarrow$

$$\int_{-1}^{1} f(x) P_m(x)\,dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n P_m\,dx$$

$$= \sum_{n=0}^{\infty} a_n \begin{cases} 0, & n \neq m \\ \frac{2}{2n+1}, & n = m \end{cases}$$

$$\Rightarrow a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\,dx$$

$$a_n = \frac{(2n+1)}{2} \int_{-1}^{1} \left(-4 + 2x + 9x^2\right) \cdot P_n(x)\, dx$$

$n = 0$

$$a_0 = \frac{1}{2} \int_{-1}^{1} \left(-4 + 2x + 9x^2\right)(1)\, dx$$

$$= \frac{1}{2} \left[-4x + x^2 + 3x^3 \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[-4 + 1 + 3 - (4 + 1 - 3) \right]$$

$$= -1$$

$n = 1$

$$a_1 = \frac{3}{2} \int_{-1}^{1} \left(-4 + 2x + 9x^2\right)(x)\, dx$$

$$= \frac{3}{2} \int_{-1}^{1} -4x + 2x^2 + 9x^3 \, dx$$

$$= \frac{3}{2} \left[-2x^2 + \frac{2x^3}{3} + \frac{9x^4}{4} \right]_{-1}^{1}$$

$$= \frac{3}{2} \left[-2 + \frac{2}{3} + \frac{9}{4} - \left(-2 - \frac{2}{3} + \frac{9}{4} \right) \right]$$

$$= \frac{3}{2} \left[\frac{4}{3} \right] = 2$$

$(x^2-2x)y'' + 2y = 0$, $x_0 = 1$

$x_0=1$ ordinary point, can try

$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$

instead $t = x-1 \Rightarrow t=0$ ordinary point

$x^2-2x = (t^2-1)$

$\Rightarrow (t^2-1)y'' + 2y = 0$, $t=0$ ordinary point

try $y(t) = \sum_{n=0}^{\infty} a_n t^n$

$y'' - \sin x \cdot y = 0$

$y = \sum_{n=0}^{\infty} a_n x^n$; $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Bessel Equation

$$x^2 y'' + xy' + (x^2 - k^2) y = 0 \quad \text{—} \; (*)$$

$\Rightarrow$ $x = 0$ is a regular singular point

Let $y = \sum_{n=0}^{\infty} a_n x^{m+n}$

$(*)$ $\Rightarrow$ $$x^2 \sum_{n=0}^{\infty} (m+n)(m+n) a_n x^{m+n-2}$$

$$+ x \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$+ x^2 \sum_{n=0}^{\infty} a_n x^{m+n} - k^2 \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n} + \sum_{n=0}^{\infty}(m+n)a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2} x^{m+n} - \sum_{n=0}^{\infty} k^2 a_n x^{m+n} = 0$$

<u>Coeff. of a_0:</u> $(n=0)$

$$m(m-1) + m - k^2 = 0 \implies m = \pm k$$

<u>for $m=k$</u>

$$\sum_{n=0}^{\infty}\left[(k+n)^2 - k^2\right] a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2} x^{m+n} = 0$$

$$(k+n)(k+n-1) + (k+n) - k^2 = (k+n)^2 - k^2$$

$$\implies \sum_{n=0}^{\infty} n(2k+n) a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2} x^{m+n} = 0$$

$$\implies a_1 (2k+1) x^{k+1} + \sum_{n=2}^{\infty}\left[n(2k+n)a_n + a_{n-2}\right] x^{k+n} = 0 \qquad (*)$$

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if $k = -1/2$, $a_n = -\dfrac{1}{n(n-1)} a_{n-2}$, $n \geq 2$

let $n = 2l$, $l = 1, 2 \ldots$

$$a_{2l} = \frac{-1}{2l(2l-1)} a_{2l-2}$$

$$= \frac{(-1)^l}{[2l(2l-1)][2(l-1)(2l-3)]\cdots[2\cdot 1\cdot 1]} a_0$$

$$\cancel{= \frac{(-1)^l}{2^l\, l!\, (2l-1)!} a_0}$$

$$\cancel{= \frac{(-1)^l}{(2l)!} \tilde{a}_0}$$

(~~choosing~~ $\cancel{\tilde{a}_0 = \frac{a_0}{2^l} \frac{2l}{l!}}$

$$= \frac{(-1)^l a_0}{\cancel{2}\,(2l)!}$$

lly $a_{2l+1} = \dfrac{(-1)^l}{(2l+1)!} a_1$

C28

$\therefore$ for $k = -1/2$,

$$y(x) = x^{-1/2} \sum_{n=0}^{\infty} a_n x^n$$

$$= x^{-1/2} \left[\sum_{n=1,3,5\ldots}^{\infty} a_n x^n + \sum_{n=2,4,\ldots}^{\infty} a_n x^n \right]$$

$$= x^{-1/2} \left[\sum_{l=0}^{\infty} a_{2l} x^{2l} + \sum_{l=0}^{\infty} a_{2l+1} x^{2l+1} \right]$$

$$= x^{-1/2} \left[a_0 \cos x + a_1 \sin x \right]$$

Bessel function of order $\pm 1/2$, by choosing $a_0 = a_1 = \sqrt{\frac{2}{\pi}}$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

general soln. for $m = \pm 1/2$ is

$$y = c_1 J_{-1/2}(x) + c_2 J_{1/2}(x)$$

$k \neq -1/2$, then $\circledast$ $\Rightarrow$ (29)

$a_1(2k+1) = 0 \quad \Rightarrow \quad a_1 = 0$

$a_n \, n(2k+n) + a_{n-2} = 0$

$$\Rightarrow \quad a_n = -\frac{1}{n(2k+n)} a_{n-2}, \quad n = 2, 3 \dots$$

$$n=3, \quad a_3 = \frac{-1}{3(2k+3)} a_1 = 0$$

$\Rightarrow \quad a_5 = a_7 = \cdot \cdot = 0$ (odd)

Let $n = 2\lambda$, $\lambda = 1, 2 \dots$

$$a_{2\lambda} = -\frac{1}{2\lambda(2k+2\lambda)} a_{2\lambda-2}$$

$$= -\frac{1}{4\lambda(k+\lambda)} a_{2\lambda-2}$$

$\underline{\lambda=1}$ $\quad a_2 = -\frac{1}{4(k+1)} a_0$

$\lambda=2$, $\quad a_4 = -\frac{1}{4 \cdot 2 \cdot (k+2)} a_2$

$$a_{2l} = \frac{(-1)^l}{[4l(k+l)][4(l-1)(k+l-1)]\cdots[4\cdot 1(k+1)]} a_0$$

$$= \frac{(-1)^l \, k!}{2^{2l} \, l! \, (k+l)!}$$

$$\therefore y = a_0 \sum_{l=0}^{\infty} \frac{(-1)^l \, k!}{2^{2l} \, l! \, (k+l)!} x^{2l+k}$$

define $$J_m(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+m}} \frac{x^{2l+m}}{l! \, (l+m)!}$$

$$y = a_0 2^m m! \, J_m(x) = \bar{a}_0 J_m(x)$$

$$y = a_0 \frac{m!}{\cancel{2^m}} 2^m \sum_{l=0}^{\infty} \frac{(-1)^l \, x^{2l+m}}{2^{2l+m} \, l! \, (m+l)!}$$

$k \neq -1/2$,

$$y = a_0 J_m(x)$$

<u>m integer</u>

$$\underline{J}_m(x) = \sum_{l'=0}^{\infty} \frac{(-1)^{l'} x^{2l'-m}}{2^{2l'-m} l'! (l'-m)!} \qquad \begin{pmatrix} l \sim l' \\ m \sim -m \end{pmatrix}$$

$l' = 0, 1, 2, \cdots (m-1)$ invites $-$ve factorial in the denominator, which are unbounded, hence $l' = 0, 1, \cdots (m-1)$ terms vanish

$$= \sum_{l'=m}^{\infty} \frac{(-1)^{l'} x^{2l'-m}}{2^{2l'-m} l'! (l'-m)!}$$

let $l = l' - m$

$$\Rightarrow \underline{J}_m(x) = \sum_{l=0}^{\infty} \frac{(-1)^{l+m} x^{2l+m}}{2^{2l+m} l! (l+m)!}$$

$$= (-1)^m J_m(x)$$

J_m, $\underline{J}_m$ are not linearly independent

C32

$$J_m(x) = \begin{cases} \displaystyle\sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda}\, x^{2\lambda+|m|}}{2^{2\lambda+|m|}\,\lambda!\,(\lambda+|m|)!}, & |m| \neq 1/2 \\ \sqrt{\dfrac{2}{\pi x}}\cos x, & m = -1/2 \\ \sqrt{\dfrac{2}{\pi x}}\sin x, & m = 1/2 \end{cases}$$

2nd soln other method

<u>properties</u>

① $$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(x/2\right)^{2n}$$

$$J_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$$

$$\frac{d}{dx} J_0(x) = \underline{-J_1(x)}$$

$$\frac{d}{dx} (x J_1(x)) = \underline{x J_0(x)}$$

(34)

$$\frac{d}{dx}\left(x J_1(x)\right) = x J_1'(x) + J_1(x)$$

$$= \sum_{l=0}^{\infty} \frac{(-1)^l (2l+1) x^{2l+1}}{2^{2l+1}\, l!\,(l+1)!} + J_1(x)$$

$$= \sum_{l=0}^{\infty} \frac{(-1)^l (2l+2) x^{2l+1}}{2^{2l+1}\, l!\,(l+1)!}$$

$$= \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l+1}}{2^{2l}\,(l!)^2} = x J_0(x)$$

(2) Consider $J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, (x/2)^{2n+m}}{n! \, (m+n)!}$ C35

$$\frac{d}{dx}\left(x^m J_m(x)\right) = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+2m} \cdot 2^m}{n! \, (m+n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+2m) \, x^{2n+2m-1} \cdot 2^m}{n! \, (m+n)! \, 2^{2n+2m}}$$

$$= 2 \sum_{n=0}^{\infty} (-1)^n \frac{(m+n)}{n! \, (m+n)(m+n-1)!} \, \frac{x^{2n+m-1} \cdot x^m}{2^{2n+m}}$$

$$= x^m \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+m-1}}{n! \, (m+n-1)!} \, \frac{1}{2^{2n+m-1}}$$

$$= x^m J_{m-1}(x)$$

Ex. $\frac{d}{dx}\left(x^{-m} J_m(x)\right) = -x^{-m} J_{m+1}(x)$

C76

We have

$$\frac{d}{dx}\left(x^{m} J_m\right) = x^{m} J_{m-1}$$

$$\frac{d}{dx}\left(x^{-m} J_m\right) = -x^{-m} J_{m+1}$$

from the above

$$J_m'(x) + \frac{m}{x} J_m(x) = J_{m-1}(x)$$

$$J_m'(x) - \frac{m}{x} J_m(x) = -J_{m+1}(x)$$