Date
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Lecture 10

S'lly, by (1) with $f(t) = y''(t)$, we have

$$\mathcal{L}\{t y''\} = -\frac{d}{ds}\left[s^2 Y - s y(0) - y'(0)\right]$$

$$= -2sY - s^2 \frac{dY}{ds} + y(0)$$

$$= -2sY - s^2 \frac{dY}{ds} + y(0) \longrightarrow (3)$$

Hence, if a d.eq^n has co-efficien. such as $(at+b)$, we will get a first-order d.eq^n for Y, which is sometimes simpler than the given eq^n!

But if the latter has co-efficients say (at^2+bt+c) we get by two applications of (1), a second-order L.d. eq^n for Y; & this shews that L.T methods works well only for very special eq^{n}s with variable co-efficients.

ing out why

Ex/ Laguerre's d. eq^n, Laguerre polynomials

Laguerre's d. eq^n is

$$ty'' + (1-t)y' + ny = 0 \longrightarrow (4)$$

we determine a sol^n of (4) with $n = 0, 1, 2, \ldots$

From eq^ns (2) & (3), we get

$$\mathcal{L}\{ty''\} + \mathcal{L}[(1-t)y'] + \mathcal{L}[ny] = 0.$$

$$\Rightarrow \left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + \left[sY - y(0)\right] - \left[-Y - s\frac{dY}{ds}\right] + nY = 0$$

Simplification gives —

$$\Rightarrow (s - s^2)\frac{dY}{ds} + (n + 1 - s)Y = 0.$$

—

Separating variables, using partial fraction, integrating (with the constant of integration taken zero) & taking exponentials, we get

$$\frac{dy}{y} = -\frac{(n+1-s)}{(s-s^2)} ds$$

$$= \left[\frac{n}{s-1} - \frac{(n+1)}{s}\right] ds$$

[using Partial Fraction]
(how?)

(on $\int$)

$$\ln y = n \ln(s-1) - (n+1) \ln s$$

$$\ln y = \ln (s-1)^n - \ln s^{n+1}$$

$$\Rightarrow \boxed{y = \frac{(s-1)^n}{s^{n+1}}} \rightarrow (*)$$

$\Rightarrow \; y(t) = \mathcal{L}^{-1}\{Y(s)\}$

we write $\quad l_n = \mathcal{L}^{-1}\{Y(s)\}$

& show that

$$\boxed{l_0 = 1, \quad l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right), \quad n = 1, 2, \dots \to (5)}$$

this is Rodrigues's Formula.

$$\therefore \mathcal{L}\{l_n(t)\} = \mathcal{L}\left[\frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right)\right].$$

These are polynomials because the exponential terms cancel (how?) if we perform the indicated differentiation. They are called Laguerre polynomials & are usually ~~denoted by~~ L_n, but we will conform to our convention. (what is it?)

We have

$$\mathcal{L}(t^n e^{-t}) = \frac{n!}{(s+1)^{n+1}}$$

$$\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$$ (how?)

Here, $f(t) = t^n e^{-t}$.

Hence, using the formula

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}\underbrace{f(0)}_{(=0)} - s^{n-2}\underbrace{f'(0)}_{(=0)} - \cdots - \underbrace{f^{(n-1)}(0)}_{(=0)}.$$

$$\therefore \mathcal{L}\{(t^n e^{-t})^{(n)}\} = \frac{n!\, s^n}{(s+1)^{n+1}}$$ [since the derivatives are zero at 0].

...w, we make another shift & divide by $n!$ to get

$$\mathcal{L}(l_n) = \mathcal{L}\left\{ \frac{e^t}{n!} \frac{d^n}{dt^n}\left(t^n e^{-t}\right) \right\}$$

$$= \frac{(s-1)^n}{(s\ [redacted])^{n+1}} \quad (\text{how?})$$

$$= \boxed{\frac{(s-1)^n}{s^{n+1}}} \quad n = 1, 2, \ldots$$

$$= Y(s).$$

(? Find out more about them?)

EX2

Bessel Functions

We define a Bessel function of order n by

$$J_n(t) = \frac{t^n}{2^n \Gamma(n+1)}\left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4.(2n+2)(2n+4)} - \cdots\right]$$

Some important properties are:-

H.W

(1) $J_{-n}(t) = (-1)^n J_n(t)$, if n is a positive integer.

$$J_{n+1}(t) = \frac{2n}{t} J_n(t) - J_{n-1}(t)$$

(3) $$\frac{d}{dt}\left\{t^n J_n(t)\right\} = t^n J_{n-1}(t).$$

If $n=0$, we have

$$J_0'(t) = -J_1(t).$$

(4) $$e^{\frac{1}{2}t(u - 1/u)} = \sum_{n=-\infty}^{\infty} J_n(t)\, u^n$$

This is called the generating function for the Bessel functions.

$J_n(t)$ satisfies the Bessel's d. eqn given by

$$t^2 y''(t) + t y'(t) + (t^2 - n^2) y(t) = 0$$

$$n = 0, 1, 2, \ldots$$

It is convenient to define

$$J_n(it) = i^{-n} I_n(t),$$

where $I_n(t)$ is called the modified Bessel function of order n.

Q1) Find $\mathcal{L}\{J_0(t)\}$, where $J_0(t)$ is the

a) Bessel function of order zero.

b) Use the result of (a) to find

$\mathcal{L}\{J_0(at)\}$

Sol^n:- Method 1 / Using series

Letting $n=0$ in eq^n (i), we find

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

$$\mathcal{L}\{J_0(t)\} = 1/s - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2\cdot 4^2}\cdot\frac{4!}{s^5} - \frac{1}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{6!}{s^7} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\cdot\left(\frac{1}{s^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{s^4}\right) - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{s^6}\right) + \cdots\right]$$

$$= \frac{1}{s}\left\{\left(1+\frac{1}{s^2}\right)^{-1/2}\right\}$$

$$= \frac{1}{\sqrt{s^2+1}},$$ using the Binomial theorem (how!)

Method 2 / Using d. eqn's!

The function $J_0(t)$ satisfies the d. eqn

$$t\,J_0''(t) + J_0'(t) + t\,J_0(t) = 0. \quad \to (1)$$

(how?)

(with $n=0$).

Taking the Laplace transform of both sides of (1), & using $J_0(0)=1$, $J_0'(0)=0$

$$Y(s) = \mathcal{L}\{J_0(t)\}$$

$$\mathcal{L}\{t\,J_0''(t)\} + \mathcal{L}\{J_0'(t)\} + \mathcal{L}\{t\,J_0(t)\} = 0$$

$$\Rightarrow -\frac{d}{ds}\left[s^2 Y - s(1) - 0\right] + \{sY - 1\} - \frac{dY}{ds} = 0.$$

$$\Rightarrow -\left[2sY + s^2\frac{dY}{ds} - \cancel{1}\right] + \left[sY - \cancel{1}\right] - \frac{dY}{ds} = 0.$$

$$\Rightarrow -(s^2+1)\frac{dY}{ds} - sY = 0.$$

$$\Rightarrow \frac{dY}{ds} = -\frac{sY}{(s^2+1)}$$

$$\Rightarrow \frac{dY}{Y} = -\frac{s\,ds}{(s^2+1)}$$

(or)

$$\ln Y = -\frac{1}{2}\int \frac{d(s^2+1)}{(s^2+1)} + \ln c$$

$$\Rightarrow Y(s) = \frac{c}{\sqrt{s^2+1}}$$, to find $c = ?$

Now, $$\lim_{s\to\infty}\{sY(s)\} = \lim_{s\to\infty}\left\{\frac{cs}{\sqrt{s^2+1}}\right\}$$

$$= \lim_{s\to\infty}\frac{c}{\sqrt{1+1/s^2}}$$

$$\left[\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)\right]$$

$= c.$ (how?)

$$c = \lim_{t \to 0} J_0(t) = 1 \quad \text{(how?)}$$

Thus, by the initial-value theorem, we have

$$c = 1$$

& so $\mathcal{L}\{J_0(t)\} = \dfrac{1}{\sqrt{s^2+1}}$

(b) $\mathcal{L}\{J_0(at)\} = \dfrac{1}{a} \cdot \dfrac{1}{\sqrt{(s/a)^2+1}}$

$$= \frac{1}{\sqrt{s^2+a^2}}.$$

$\mathcal{L}\{f(t)\} = F(s)$
then $\mathcal{L}\{f(at)\} = \frac{1}{a} F(s/a)$

Q1) Find $\mathcal{L}\{J_1(t)\}$, where $J_1(t)$ is Bessel's function of order one.

Hint:- Use $J_0'(t) = -J_1(t)$.

Hence, $\mathcal{L}\{J_1(t)\} = -\mathcal{L}\{J_0'(t)\} = -[s\,\mathcal{L}\{J_0(t)\} - 1] = 1 - \frac{s}{\sqrt{s^2+1}} = \frac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}}$

Q2) Show that

$$\int_0^\infty J_0(t)\,dt = 1.$$

Hint:- $\mathcal{L}\{J_0(t)\} = \int_0^\infty e^{-st} J_0(t)\,dt = \frac{1}{\sqrt{s^2+1}}$

—x—

Let $s \to 0^+$, we get

$$\int_0^\infty J_0(t)\,dt = 1.$$

Generalized

Convolution product of f & g. Integral eq^ns

$F(s)$, $G(s)$

— inverses of $f(t)$ & $g(t)$

Q) To calculate the inverse of the product

$$H(s) = F(s) \cdot G(s).$$

$$h(t) = (f * g)(t)$$

Q) How can we find h from f & g?

Th-16

(Convolution Theorem)

Let $f(t)$ & $g(t)$ satisfy the hypothesis of the

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theorem. Then the product

of their transforms

$F(s) = \mathcal{L}\{f\}$ & $G(s) = \mathcal{L}\{g\}$

is the transform

$$H(s) = \mathcal{L}\{h\}$$

of the <u>convolution</u> $h(t)$

of $f(t)$ & $g(t)$, which

is denoted by $(f*g)(t)$ &

is defined by

$$h(t) = (f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

i.e., $\mathcal{L}(f*g) = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$
$= F(s) \cdot G(s)$.

Using convolution, find the inverse of

$$H(s) = \frac{1}{(s^2+1)^2} = \frac{1}{(s^2+1)} \cdot \frac{1}{(s^2+1)}$$

Sol^n:- We know that each factor on the R.H.S has the inverse transform $\sin t$. Hence, by the Convolution theorem, we get

$$h(t) = \mathcal{L}^{-1}\{H(s)\}$$

$$= \mathcal{L}^{-1}(F) * \mathcal{L}^{-1}(G)$$

$$= \sin t * \sin t.$$

$$= \sin t * \sin t$$

$$= \int_0^t \sin\tau \, \sin(t-\tau)\, d\tau$$

$$= \frac{1}{2}\int_0^t (-\cos t \, d\tau) \qquad \left[\begin{array}{l} 2\sin A \sin B \\ = \cos(A-B) \\ \quad - \cos(A+B) \end{array}\right.$$

$$+\frac{1}{2}\int_0^t \cos(2\tau - t)\, d\tau$$

$$= -\frac{1}{2} t\cos t + \frac{1}{4}\Big[\sin(2\tau - t)\Big]_{\tau=0}^{t}$$

$$= -\frac{1}{2} t\cos t + \frac{1}{2}\sin t$$

EX2/ $\frac{1}{s^3}$ (Using convolution)

$$\frac{1}{s^3} = \frac{1}{s^2} \cdot \frac{1}{s}$$

has the inverse

$$t * 1 = \int_0^t \tau \cdot 1 \, d\tau$$

(how?)

$$= \boxed{t^2/2}.$$

C.W
EX3/ Let $H(s) = \dfrac{1}{s^2(s-a)}$

Find $h(t)$. (Using convolution).

Sol:- We know $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$, $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$.

$$h(t) = t * e^{at}$$

$$= \int_0^t \underline{\tau} \cdot \underline{e^{a(t-\tau)}} \, d\tau$$

$$= \frac{1}{a^2}(e^{at} - at - 1).$$ //

H.W

Q) Find the value of

$\cos t * \sin t$.

Sol^n:- $\cos t * \sin t = \int_0^t \cos(\tau) \sin(t-\tau)\, d\tau$

$= \frac{1}{2} t \sin t$

Q) $\sin t * t^2$

Sol^n:- $\sin t * t^2 = \int_0^t \sin \tau \, (t-\tau)^2 \, d\tau$

$= t^2 + 2\cos t - 2$

(~~be~~ observe the problem)