Group Action

Lecture 12

Group Action:

A gp action of a gp G on a set A is a map from $G \times A \longrightarrow A$

$$(g, s) \longmapsto g \cdot s$$

satisfying the following properties

(1) $g_1(g_2 s) = (g_1 g_2) s \quad \forall g_1, g_2 \in G$
$s \in A$.

(2) $1 \cdot s = s \quad \forall s \in A$

Example (1) Let $G = S_4$ and $A = \{1, 2, 3, 4\}$.

Then the gp action is defined as

$$G \times A \longrightarrow A$$
$$(\sigma, i) \longmapsto \sigma(i)$$

is a gp action.

$$\sigma_1(\sigma_2(i)) = (\sigma_1 \circ \sigma_2)(i).$$

(2) Let $G = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$

be the gp under matrix multiplication and set $A = \mathbb{R}^2$. Then the gp action is defined as

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix}$$

The above gp action rotate a vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ by angle θ.

(3) $G \times G \longrightarrow G$.

$(g, x) \longmapsto g \cdot x$ is a gp action by left multiplicat.

(4) $$G \times G \longrightarrow G$$

$$(g, x) \longmapsto g x g^{-1}$$

This is also define a gp action. and is know as conjugation operation.

(5) Let $G = \{1, r\}$ where r is the reflection wrt x-axis. G acts on $\mathbb{C}$ by

$$(g, z) \longmapsto g z$$

where $$gz = \begin{cases} z & \text{if } g = 1. \\ \overline{z} & \text{if } g = r \end{cases}$$

This defines a gp action.

Defn The kernel of a gp action is

$$= \{ g \in G \mid gs = s \ \forall s \in A \}.$$

Defn For each $s \in A$ the stabilizer of s $= \{ g \in G \mid gs = s \}$ which is denoted by G_s.

Question Is G_s a subgp of G?

Let $g_1, g_2 \in G_s$.

$$(g_1 g_2) s = g_1 (g_2 s) = g_1 s = s.$$

$\Rightarrow$ $g_1 g_2 \in G_s$.

$1 \in G_s$.

Let $g \in G_s$.

$$(g^{-1} g) s = s.$$

$$\Rightarrow \quad g^{-1}(gs) = g^{-1} s = s.$$

$\therefore$ G_s is a subgp.

<u>Remark</u> G_s is a subgp of G.

Let $s \in A$ be an elt. Then the orbit of s is defined as

$$O(s) = \{ gs \mid g \in G \} \subset A.$$

If G acts on A we define an equivalence relation on A as follows:

$s \sim t$ if $t = gs$ for some $g \in G$,

The equivalence class of $s = O(s)$.

The equivalence class is nothing but the orbit.

$\therefore$ A = disjoint union of orbits.

If A is finite then $|A| = |O(s_1)| + \cdots + |O(s_k)|$,

Example $G = S_4$ and $A = \{1, 2, 3, 4\}$.

$$\mathrm{stb}(\{2\}) = \{e, (1\,3), (1\,4), (3\,4), (1\,3\,4), (1\,4\,3)\}$$

$$O(\{2\}) = \{1, 2, 3, 4\} = \{\sigma(2) \mid \sigma \in S_4\}.$$

Defn A gp action of G on a set A is called transitive if there is only one orbit i.e given any two elts $x, y \in A$, $\exists\, g \in G$ s.t $x = gy$.

Example $G = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\}$.

and $A = \mathbb{R}^2$. Let $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

$$O(v) = \{ w \in \mathbb{R}^2 \mid |w| = |v| \}.$$

for non-zero vector v, $O(v)$ is a circle of radius $|v|$ and if v is the zero vector then the v is the only elt in the orbit.

Question Is there any relation between $O(s)$ and G_s. ?

Propn. let G be a gp acting on a set A. For each $s \in A$, the number of elts in $O(s)$ is index of the stabilizer of s. i-e $|O(s)| = [G : G_s]$