Group Theory

Lecture 10

Example. $G = \mathbb{Z}/6\mathbb{Z} = \langle \bar{1} \rangle = \langle \bar{5} \rangle$.

$\phi : G \to G$

$\phi(x)$

If $|x| = n$, then $|\phi(x)| = n$.

if ϕ is an automorphism:

$G \longrightarrow G$

$\phi(\bar{2}) = \phi(\bar{1}) + \phi(\bar{1})$

$= \bar{1} + \bar{1}$

either $\bar{1} \longmapsto \bar{1}$

$\bar{2} \longmapsto \bar{2}$

or $\bar{1} \longmapsto \bar{5}$

$\phi_1(\bar{1}) = \bar{1}$

and $\phi_2(\bar{1}) = \bar{5}$

In $\mathbb{Z}/n\mathbb{Z}$. there will be $\phi(n)$ many automorphism.

$\phi : G \longrightarrow G$

$G =$

$D_n = \langle x, y \mid x^2 = 1,\ y^n = 1,$
$xy \cdots \rangle.$

$\phi : D_n \longrightarrow D_n.$

$x \longmapsto x$

$x \longmapsto x^{-1}$

$y \longmapsto y^{i}$ $\quad \gcd(i, n) = 1$

$\phi : G \longrightarrow G$

$\quad\ \ \rotatebox{90}{$\leq$}$

$G = \langle \cdots \cdots$

S_3

Ex. Write all the automorphism of D_4.

Product of Groups

Let G_1 & G_2 be two groups. Define their product as

$$G_1 \times G_2 = \{(a,b) \mid a \in G_1, b \in G_2\}$$

For $(a,b), (c,d) \in G_1 \times G_2$ define a binary operation

gp operation of G_1

$$(a,b) \cdot (c,d) = (ac, bd)$$

group of G_2.

Identity elt of $G_1 \times G_2$ $\quad (1_{G_1}, 1_{G_2})$.

Inverse of $(g_1, g_2) = (g_1^{-1}, g_2^{-1})$

$i_1(g) = (g, 1_{G_2})$ $\quad\quad$ $\pi_1(a,b) = a$

$$\begin{tikzcd}
G_1 \arrow[dr, "i_1"] & & G_1 \\
& G_1 \times G_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
G_2 \arrow[ur, "i_2"'] & & G_2
\end{tikzcd}$$

Then all the maps i_1, i_2, π_1 & π_2 are group homomorphisms.

If G_1 & G_2 are finite gp then

$$|G_1 \times G_2| = |G_1|\,|G_2|.$$

Let G be a gp and H and K are subgps of the group G.

(i) If $H \cap K = \{1\}$ then the product map $p : H \times K \longrightarrow G$ defined by $p(h,k) = hk$ is injective and its image is the subset HK.

__Pf:__ let (h_1, k_1) & (h_2, k_2) s.t

$$p(h_1, k_1) = p(h_2, k_2)$$

$$h_1 k_1 = h_2 k_2 .$$

WTS $\quad h_1 = h_2 \;\&\; k_1 = k_2 .$

$$\to h_2^{-1} h_1 k_1 = k_2$$

$$\Rightarrow h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K = \{1\}.$$

$\therefore \quad h_2^{-1} h_1 = 1 \quad$ and $\quad k_2 k_1^{-1} = 1.$

$$\Rightarrow h_1 = h_2 \quad \& \quad k_1 = k_2 .$$

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(2) If either H or K is normal subgp of G then the product sets HK and KH are equal and HK is a subgp of G.

Pf: Let H is a normal subgp of G.

WTS $\quad HK = KH.$

$\subseteq$

$\supseteq$

Let $h \in H$ and $k \in K$.

Then $hk = k(k^{-1}hk)$

$= kh'$ $\quad \because H \trianglelefteq G$, $k^{-1}hk \in H$

$\Rightarrow HK \subseteq KH$.

Similarly $KH \subseteq HK$.

$\therefore \quad HK = KH$.

<u>WTS</u> HK is a subgp of G.

Let $h_1 k_1, h_2 k_2 \in HK$. $\quad KH = HK$

Then $(h_1 k_1)\cdot(h_2 k_2) = h_1(k_1 h_2)k_2$

$= h_1 h' k' k_2$,

$\in HK$.

$\left[\begin{array}{l} \because KH = HK \\ k_1 h_2 = h'k' \end{array}\right]$

and $1 \cdot 1 = 1 \in HK$,

$(hk)^{-1} = k^{-1}h^{-1} = h'k' \in HK$,

Thus HK is a subgp of G.

(3) If H and K both are normal and $H \cap K = \{1\}$ and $G = HK$, then G is isomorphic to $H \times K$,

Pf: If both H & K are normal and $H \cap K = \{1\}$ then $hk = kh$ $\forall\, h \in H$, $k \in K$.

WTS $hk = kh$.

i.e $h^{-1}k^{-1}hk = 1$.

Note that

$$h^{-1}k^{-1}hk = h^{-1}\overbrace{(k^{-1}hk)}^{\in H} \in H,$$

and

$$h^{-1}k^{-1}hk = \underbrace{(h^{-1}k^{-1}h)}_{\in K} k \in K,$$

$$h^{-1}k^{-1}hk \in H \cap K = \{1\}.$$

$\therefore\ h^{-1}k^{-1}hk = 1 \Rightarrow hk = kh.$

WTS ϕ is a grp homomorphism.

WTS $\phi((h_1,k_1)\cdot(h_2,k_2)) = \phi(h_1,k_1)\,\phi(h_2,k_2)$

$$\begin{aligned}
&\phi((h_1,k_1)\cdot(h_2,k_2)) \\
&= \phi(h_1h_2,\ k_1k_2) \\
&= h_1h_2k_1k_2 = h_1k_1h_2k_2 \\
&= \phi(h_1,k_1)\cdot\phi(h_2,k_2).
\end{aligned}$$

Thus ϕ is a grp homo.

$\phi : H \times K \longrightarrow G = HK$ (assumption)

i.e ϕ is surjective.

Hence $H \times K \simeq G$.

Ex 1. Find $|H \times K|$.

Ex 2. Show that $\mathbb{Z}_r \times \mathbb{Z}_s \simeq \mathbb{Z}_{rs}$ if $\gcd(r,s) = 1$.

$$\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}.$$