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Partial Differential Equations

Q1 $u_t = u_{xx}$, $\quad u(x,0) = \sin(\pi x)$, $\quad 0 < x < 1$.

$u(0,t) = u(1,t) = 0$

Ans Discretizing using central difference approx (FTCS)

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \frac{\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}{(\delta x)^2} \quad \text{(Explicit scheme)}.$$

$\Rightarrow$ Here, $\dfrac{\delta t}{(\delta x)^2} = \nu$

$$\Rightarrow u_i^{n+1} = u_i^n + \nu\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$\boxed{u_i^{n+1} = \nu\left(u_{i+1}^n + u_{i-1}^n\right) + (1-2\nu)\,u_i^n}$$

we have $u(0,t) = u(1,t) = 0$.

we take, $\delta x = 1/4$, $\delta t = 1/32$ $\quad (\because \nu \le 1/2)$ $\quad \nu = 1/2$

x_0 (0), x_1 (0.25), x_2 (0.5), x_3 (0.75), x_4 (1)

$u_0^n = u_4^n = 0 \quad \forall n$

& $u_i^0 = \sin(\pi x_i)$

Solving this, we get,

$x_i \downarrow$ $n \rightarrow$	0	1	2	3	4	----
x_0	0	0	0	0	0	
x_1	$1/\sqrt{2}$	0.5	$1/2\sqrt{2}$	0.25	$1/4\sqrt{2}$	
x_2	1	$1/\sqrt{2}$	0.25	$1/2\sqrt{2}$	0.25	
x_3	$1/\sqrt{2}$	0.5	$1/2\sqrt{2}$	0.25	$1/4\sqrt{2}$	
x_4	0	0	0	0	0	

Using Implicit Scheme

(b) Discretizing using BTCS

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\delta x^2}$$

$\delta x = 1/4$, $\delta t = 1/32$

$$\gamma = \frac{\delta t}{(\delta x)^2}$$

$$\Rightarrow u_i^{n+1} - u_i^n = \gamma\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)$$

$$\Rightarrow \underbrace{(\gamma)}_{a_i}\left(u_{i-1}^{n+1}\right) + \underbrace{(-2\gamma-1)}_{b_i}u_i^{n+1} + \underbrace{\gamma}_{c_i} u_{i+1}^{n+1} = \underbrace{-u_i^n}_{d_i}$$

$$\Rightarrow a_i u_{i-1}^{n+1} + b_i u_i^{n+1} + c_i u_i^{n+1} = d_i$$

↳ $(N-1)\times(N-1)$ tri-diagonal matrix.

Solving this, we get.

$i \downarrow$ $n \rightarrow$	0	1	2	3	4
x_0	0	0	0	0	0
x_1	$1/\sqrt{2}$	0.5469	0.4230	0.3272	0.2531
x_2	1	0.7735	0.5982	0.4627	0.3579
x_3	$1/\sqrt{2}$	0.5469	0.4230	0.3272	0.2531
x_4	0	0	0	0	0

(c) Using the Crank Nicolson Scheme:

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \frac{1}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\delta x^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\delta x^2}\right]$$

taking $\frac{\delta t}{\delta x^2} = \gamma$

$$\left(\frac{\gamma}{2}\right)u_{i-1}^{n+1} + (-2\gamma - 1)u_i^{n+1} + \left(\frac{\gamma}{2}\right)u_{i+1}^{n+1} = -\frac{\gamma}{2}\left(u_{i+1}^n + u_{i-1}^n\right) + (2\gamma - 1)u_i^n$$

↳ we get a tri-diagonal system. $Ax = d$.

Solving this, we get

$i \downarrow$ $n \rightarrow$	0	1	2	3	4	5
x_0	0	0	0	0	0	0
x_1	$1/\sqrt{2}$	0.5265	0.3920	0.2978	0.2173	0.1618
x_2	1	0.7445	0.5543	0.4127	0.3073	0.2280
x_3	$1/\sqrt{2}$	0.5265	0.3920	0.2918	0.2173	0.1618
x_4	0	0	0	0	0	0

(★) For ~~Explicit~~ Implicit Scheme:

Let $u(x_i, t_{n+1}) \rightarrow u_i^{n+1}$ be the exact solution of the PDE, then, T.E $= \frac{(u_i^{n+1} - u_i^n)}{\delta t} - \frac{(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})}{(\delta x)^2} = 0$

$$= \frac{\left(u_i^{n+1} - u(x_i, t_{n+1} - \delta t)\right)}{\delta t} - \left(\frac{u(x_i + \delta t, t_{n+1}) - 2u_i^{n+1} + u(x_i - \delta t, t_{n+1})}{(\delta x)^2}\right)$$

$$= \left(\cancel{u_i^{n+1}} - \cancel{u_i^{n+1}} + \delta t \frac{\partial u}{\partial t}\Big|_i^{n+1} - (\delta t)^2 \frac{\partial^2 u}{\partial t^2}\Big|_i^{n+1} + \cdots\right) \times \frac{1}{\delta t}$$

$$-\frac{1}{\delta x^2}\Big[\cancel{u_i^{n+1}} + \cancel{\delta x \cdot \frac{\partial u}{\partial x}\Big|_i^{n+1}} + \frac{(\delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+1} + \cancel{\frac{(\delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_i^{n+1}} + \frac{(\delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4}\Big|_i^{n+1}$$

$$+ u_i^{n+1} - \cancel{\delta x \frac{\partial u}{\partial x}\Big|_i^{n+1}} + \frac{(\delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+1} - \cancel{\frac{(\delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_i^{n}} + \frac{(\delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4}\Big|_i^{n+1}$$

$$- 2\cancel{u_i^{n+1}}\Big)$$

$$= \left(\frac{\partial u}{\partial t}\Big|_i^{n+1} - \frac{\partial^2 u}{\partial x^2}\Big|_i^{n+1} \right) + O(\delta t, \delta x^2) \rightarrow 0$$

since $u(x_i, t_{n+1})$ is an exact sol

$\therefore$ $TE \equiv O(\delta t, \delta x^2)$

(*) Now, For the Crank-nicolson scheme:-

$$T.E = \frac{u_i^{n+1} - u_i^n}{\delta t} - \frac{\gamma}{2(\delta x)^2}\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n} \right]$$

Expanding by Taylor series about $u(x_i, t_{n+1/2})$.
i.e., we assume $u(x_i, t_{n+1/2})$ is an exact sol.

$$T.E = \left(\frac{u(x_i, t_{n+1/2} + \frac{\delta t}{2}) - u(x_i, t_{n+1/2} - \frac{\delta t}{2})}{\delta t} \right) - \frac{\gamma}{2}\Bigg[$$

$$\Bigg[\left(\frac{u(x_i + \delta x, t_{n+1/2} + \delta t/2) - 2u(x_i, t_{n+1/2} + \delta t/2) + u(x_i - \delta x, t_{n+1/2} + \delta t/2)}{(\delta x)^2} \right)$$

$$+ \left(\frac{u(x_i + \delta x, t_{n+1/2} - \delta t/2) - 2u(x_i, t_{n+1/2} - \delta t/2) + u(x_i - \delta x, t_{n+1/2} - \delta t/2)}{(\delta x)^2} \right) \Bigg]$$

$$= \frac{1}{\delta t}\Bigg[\cancel{u_i^{n+1/2}} + \left(\frac{\delta t}{2}\right)\frac{\partial u}{\partial t}\Big|_i^{n+1/2} + \frac{1}{2}\left(\frac{\delta t}{2}\right)^2 \cancel{\frac{\partial^2 u}{\partial t^2}}\Big|_i^{n+1/2} + \frac{1}{3!}\left(\frac{\delta t}{2}\right)^3 \frac{\partial^3 u}{\partial t^3}\Big|_i^{n+1/2}$$

$$- \cancel{u_i^{n+1/2}} + \left(\frac{\delta t}{2}\right)\frac{\partial u}{\partial t}\Big|_i^{n+1/2} - \frac{1}{2!}\left(\frac{\delta t}{2}\right)^2 \cancel{\frac{\partial^2 u}{\partial t^2}}\Big|_i + \frac{1}{3!}\left(\frac{\delta t}{2}\right)^3 \frac{\partial^3 u}{\partial t^3}\Big|_i^{n+1/2}$$

$$-\frac{\gamma}{2(\delta x)^2}\Bigg[\cancel{u_i^{n+1/2}} + \cancel{\delta x \cdot \frac{\partial u}{\partial x}\Big|_i^{n+1/2}} + \cancel{\left(\frac{\delta t}{2}\right)\frac{\partial u}{\partial t}\Big|_i^{n+1/2}} + \frac{(\delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+1/2} + \cancel{(\delta x)\left(\frac{\delta t}{2}\right)\frac{\partial^2 u}{\partial t \partial x}}\Big|_i$$

$$\cancel{\frac{1}{3!}\left(\frac{\delta t}{2}\right)^3 \frac{\partial^3 u}{\partial t^3}\Big|_i^{n+1/2}} + \cancel{\frac{(\delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_i^{n+1/2}} + \cancel{\left(\frac{\delta t}{2}\right)^2\frac{(\delta x)}{3!}\frac{\partial^3 u}{\partial t^2 \partial x}}\Big|_i^{n+1/2} + \left(\frac{\delta t}{2}\right)\frac{\delta x^2}{3!}\frac{\partial^3 u}{\partial t \partial x^2}\Big|_i$$

$$+ \cdots O \cdots$$

$$-2\Big\{ \cancel{u_i^{n+1/2}} + \cancel{\frac{\delta t}{2}\frac{\partial u}{\partial t}\Big|_i^{n+1/2}} + \left(\frac{\delta t}{2}\right)^2\frac{\partial^2 u}{\partial t^2}\Big|_i^{n+1/2} + \cdots \Big\}$$

$$+ \cancel{u_i^{n+1/2}} - \cancel{\delta x \frac{\partial u}{\partial x}\Big|_i^{n+1/2}} + \cancel{\left(\frac{\delta t}{2}\right)\frac{\partial u}{\partial t}\Big|_i^{n+1/2}} + \frac{(\delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+1/2} - \cancel{\frac{(\delta x)\,\delta t}{2 \cdot 2!}\frac{\partial^2 u}{\partial x \partial t}}\Big|_i$$

$$+ \cancel{\frac{1}{3!}\left(\frac{\delta t}{2}\right)^3\frac{\partial^3 u}{\partial t^3}\Big|_i^{n+1/2}} - \frac{(\delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_i^{n+1/2} - \cancel{\frac{(\delta t)^2}{2}\frac{\delta x}{3!}\frac{\partial^3 u}{\partial t^2 \partial x}}\Big|_i^{n+1/2} + \cancel{\left(\frac{\delta t}{2}\right)(\delta x)^2\frac{\partial^3 u}{\partial x}}$$

$$+ \cancel{u_i^{n+1/2}} + \cancel{\delta x \frac{\partial u}{\partial x}\Big|_i^{n+1/2}} - \cancel{\left(\frac{\delta t}{2}\right)\frac{\partial u}{\partial t}\Big|_i^{n+1/2}} + \frac{(\delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_i + \cancel{\cdots}$$

$$-2\Big\{ \cancel{u_i^{n+1/2}} - \cancel{\frac{\delta t}{2}\frac{\partial u}{\partial t}} + \left(\frac{\delta t}{2}\right)^2\frac{\partial u}{\partial t}\Big|_i + \cdots \Big\}$$

$$+ \cancel{u_i^{n+1/2}} - \cancel{\delta x \frac{\partial u}{\partial x}\Big|_i} - \cancel{\left(\frac{\delta t}{2}\right)\frac{\partial u}{\partial t}\Big|_i^{n+1/2}} + \cancel{\cdots}$$

$$= \left[\frac{\partial u}{\partial t}\Big|_i^{n+1/2} - \gamma\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+1/2} \right] + O(\delta t^2, \delta x^2) \rightarrow 0$$

Since, $u(x_i, t_{n+1/2})$ is an exact s.

$\therefore$ $T.E \equiv O(\delta t^2, \delta x^2)$

Von-Neumann Stability Analysis

(a) Explicit Scheme

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$$\epsilon_j^{n+1} - \epsilon_j^n = \gamma\left(\epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n\right)$$

$$\epsilon_j^n = A^n e^{i\theta j}$$

$$\left(A^{n+1} - A^n = \gamma\left(A^n e^{i\theta} - 2A^n + A^n e^{-i\theta}\right)\right.$$

~~let~~ $\xi = \dfrac{A^{n+1}}{A^n}$

$$\xi - 1 = \gamma\left(e^{i\theta} + e^{-i\theta} - 2\right)$$

$$\xi = 1 + 2\gamma(\cos\theta - 1)$$

Now, for stabilit,

$$|\xi| \le 1$$

$$\Rightarrow -1 \le \xi \le 1$$

$$1 + 2\gamma(\cos\theta - 1) \le 1$$

$$2\gamma(\cos\theta - 1) \le 0$$

$$\cos\theta \le 1$$

$$\Rightarrow (\cos\theta - 1) \le 0$$

$$\Rightarrow 2\gamma(\cos\theta - 1) \le 0 \quad \forall\ \gamma \ge 0$$

Now, $-1 \le 1 + 2\gamma(\cos\theta - 1)$

$$\Rightarrow -2 \le 2\gamma(\cos\theta - 1)$$

$$(1 - \cos\theta)\gamma \le 1$$

$$-1 \le -\cos\theta \le 1$$

$$0 \le 1 - \cos\theta \le 2 \Rightarrow \gamma(1 - \cos\theta) \le 2\gamma \le 1$$

$$\Rightarrow \boxed{\gamma \le 1/2}$$

$\therefore$ Explicit Scheme is conditionally stable for $\gamma \le 1/2$

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(b) Implicit Scheme:

$$\epsilon_j^{n+1} - \epsilon_j^n = \gamma\left(\epsilon_{j+1}^{n+1} - 2\epsilon_j^{n+1} + \epsilon_{j-1}^{n+1}\right)$$

$$\epsilon_j^n = A^n e^{i\theta j}$$

$$\Rightarrow (A^{n+1} - A^n) = \gamma\left(A^{n+1} e^{i\theta} - 2A^{n+1} + A^{n+1} e^{-i\theta}\right)$$

Let. $\xi = \dfrac{A^{n+1}}{A^n}$

$$\xi - 1 = \gamma\left(\xi(e^{i\theta} + e^{-i\theta}) - 2\xi\right)$$

$$\Rightarrow \xi - 1 = \gamma\,\xi\,(2\cos\theta - 2).$$

$$\xi + \xi(2 - 2\cos\theta)\gamma = 1$$

$$\xi = \frac{1}{1 + 2\gamma(1 - \cos\theta)}$$

$$0 \le (1 - \cos\theta) \le 2 \Rightarrow 0 \le 2\gamma(1 - \cos\theta) \le 4\gamma$$

$$1 \le 1 + 2\gamma(1 - \cos\theta) \le 4\gamma + 1$$

$$\frac{1}{1 + 4\gamma} \le \frac{1}{1 + 2\gamma(1 - \cos\theta)} \le 1$$

$$\frac{1}{1 + 4\gamma} \le \xi \le 1$$

$$|\xi| \le 1 \quad \forall\ \gamma$$

$\therefore$ it is unconditionally stable.

(c) Crank-Nicolson Scheme:

$$\epsilon_j^{n+1} - \epsilon_j^n = \frac{\gamma}{2}\left[\epsilon_{j+1}^{n+1} - 2\epsilon_j^{n+1} + \epsilon_{j-1}^{n+1} + \epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n\right]$$

$$\epsilon_j^n = A^n e^{i\theta j}$$

$$A^{n+1} - A^n = \frac{\gamma}{2}\left[A^{n+1} e^{i\theta} - 2A^{n+1} + A^{n+1} e^{-i\theta} + A^n e^{i\theta} - 2A^n + 2e^{-i\theta}\right]$$

Let $\xi = \dfrac{A^{n+1}}{A^n}$

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$$\xi - 1 = r\left[\xi(\cos\theta - 1) + (\cos\theta - 1)\right]$$

$$\xi - 1 = r(\cos\theta - 1)\xi + r(\cos\theta - 1)$$

$$\xi\left(1 + r(1-\cos\theta)\right) = 1 + r(\cos\theta - 1)$$

$$\xi = \frac{1 - r(1-\cos\theta)}{1 + r(1-\cos\theta)}$$

$$|\xi| \le 1 \quad \forall r$$

$\therefore$ it is unconditionally stable.

(*) Here we have seen that

for implicit scheme $\rightarrow O(\delta t, \delta x^2) \rightarrow 0$ (consistent) as $\delta t, \delta x \rightarrow 0$

& it is unconditionally stable

$\therefore$ it is convergent (By Lax Equiv. Theorem)

Similarly for Crank-Nicolson $\rightarrow O(\delta t^2, \delta x^2) \rightarrow 0$ (consistent) as $\delta t, \delta x \rightarrow 0$

& it is unconditionally stable

$\therefore$ it is convergent (by Lax. the)

(Q)
$$\frac{u_j^{n+1} - u_j^{n-1}}{2\delta t} = \frac{u_{j+1}^{n} - 2\{\theta u_j^{n+1} + (1-\theta)u_j^{n-1}\} + u_{j-1}^{n}}{\delta x^2}$$

We have to check for consistency, if

(i) $\delta t = r\,\delta x$ (ii) $\delta t = r\,\delta x^2$

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let's consider $u(x_i, t_{n+1}) \rightarrow$ exact solution

$$\Rightarrow T.E = \frac{u_j^{n+1} - u_j^{n-1}}{2\delta t} - \left(\frac{u_{j+1}^{n} - 2\{\theta u_j^{n+1} + (1-\theta)u_j^{n-1}\} + u_{j-1}^{n}}{\delta x^2}\right)$$

Expanding By Taylor Series about u_j^{n+1}

$$TE = \frac{u_j^{n+1} - u(x_i, t_{n+1} - 2\delta t)}{2\delta t} - \frac{1}{\delta x^2}\Big(u(x_j + \delta x, t_{n+1} - \delta t) - 2\{\theta u_j^{n+1} + (1-\theta)u(x_j, t_{n+1} - 2\delta t)\} + u(x_j - \delta x, t_{n+1} - \delta t)\Big)$$

$$= \left(\frac{\cancel{u_i^{n+1}} - \cancel{u_i^{n+1}} + (2\delta t)\frac{\partial u}{\partial t}\Big|_i^{n+1} - \frac{(2\delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}\Big|_i^{n+1} + \cdots}{2\delta t}\right)$$

$$-\frac{1}{\delta x^2}\Bigg[u_i^{n+1} + \cancel{\delta x\frac{\partial u}{\partial x}\Big|_i^{n+1}} - \delta t\frac{\partial u}{\partial t}\Big|_i^{n+1} + \frac{(\delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+1} \cancel{(\delta x)\frac{\delta t}{2}\frac{\partial^2 u}{\partial x \partial t}\Big|_i^{n+1}}$$

$$+ \frac{(\delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}\Big|_i^{n+1} + \cancel{\frac{(\delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_i^{n+1}} - \frac{(\delta x)^2(\delta t)}{3!}\frac{\partial^3 u}{\partial x^2 \partial t}\Big|_i^{n+1}$$

$$+ \cancel{\frac{(\delta x)(\delta t^2)}{3!}\frac{\partial^3 u}{\partial x \partial t^2}\Big|_i^{n+1}} - \frac{(\delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3}\Big|_i^{n+1} + \cdots$$

$$-2\left\{\theta u_i^{n+1} + (1-\theta)\left(u_i^{n+1} - 2\delta t\frac{\partial u}{\partial t}\Big|_i^{n+1} + \frac{(2\delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}\Big|_i^{n+1} - \cdots\right)\right\}$$

$$+ u_i^{n+1} - \cancel{\delta x\frac{\partial u}{\partial x}\Big|_i^{n+1}} - \delta t\frac{\partial u}{\partial x}\Big|_i^{n+1} + \frac{(\delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2} + \cancel{\frac{\delta x\,\partial t}{2!}\frac{\partial^2 u}{\partial x \partial t}}$$

$$+ \frac{(\delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}\Big|_i^{n+1} - \cancel{\frac{(\delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_i^{n+1}} - \frac{(\delta x)^2}{3!}\frac{\delta t\,\partial^3 u}{\partial^2 x\,\partial t}\Big|_i$$

$$- \cancel{\frac{\delta x(\delta t)^2}{3!}\frac{\partial^3 u}{\partial x \partial t^2}} - \frac{(\delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3} + \cdots\Bigg]$$

$$= \frac{\partial u}{\partial t}\Big|_i^{n+1} - 2\delta t\left[\cdots\right]$$

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$$-\frac{1}{\delta x^2}\left[\cancel{2u_i^{n+1}} - 2\delta t \frac{\partial u}{\partial t}\Big|_i^{n+1} + \frac{(\delta x)^2}{}\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+1} + (\delta t)^2 \frac{\partial^2 u}{\partial t^2}\Big|_i^{n+1}\right.$$

$$- \frac{(\delta x)^2}{3}(\delta t)\frac{\partial^3 u}{\partial^2 x \partial t}\Big|_i^{n+1} - \frac{(\delta t)^3}{3}\frac{\partial^3 u}{\partial t^3}\Big|_i$$

$$-2\left\{\theta\, \cancel{u_i^{n+1}} + \cancel{u_i^{n+1}} - 2\delta t \frac{\partial u}{\partial t}\Big|_i^{n+1} + \frac{(2\delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}\Big|_i^{n+1}\right)$$

$$- \theta\, \cancel{u_i^{n+1}} + 2\theta\, \delta t \frac{\partial u}{\partial t}\Big|_i^{n+1} + \theta(\cdots)$$

$$= \cancel{\left(\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2}\right)}^{0} - \frac{1}{\delta x^2}\left[\delta t \frac{\partial u}{\partial t}\left(-2 + 4\theta\right)\right.$$

$$+ (\delta t)^2 \frac{\partial^2 u}{\partial t^2}\left(1 - 4 + 4\theta\right)$$

$$\left. + \cdots \right].$$

if (i) $\delta t = r\, \delta x$

then $4\theta - 3 = 0$ $\theta = 3/4$

(ii) if $\delta t = r\, \delta x^2$

$4\theta - 2 = 0$ $\theta = 1/2$

Q **Leap frog scheme :**

$$\epsilon_j^{n+1} = \epsilon_j^{n-1} + 2r\left(\epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n\right)$$

$\epsilon_j^n \to A^n e^{i\theta j}$

$$\Rightarrow A^{n+1} = A^{n-1} + 2r\left(A^n e^{i\theta} - 2A^n + A^n e^{-i\theta}\right).$$

$$\Rightarrow A^2 = 1 + 2r\left(A(2\cos\theta) - 2A\right)$$

$$\xi^2 - \xi\, 4r(\cos\theta - 1) - 1 = 0$$

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$$\xi = \frac{4r(\cos\theta - 1) \pm \sqrt{(4r(\cos\theta - 1))^2 + 4}}{2}$$

$$= 2r(\cos\theta - 1) \pm \sqrt{1 + 4r^2(\cos\theta - 1)^2}$$

$$|\xi| \leq 1 + 8r^2(\cos^2 - 1)$$

$$1 - 8r^2(1 - \cos\theta) \leq 1 \quad \forall r$$

$\therefore$ it's unconditionaly stable.

Q $\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}$, $0 < x < 1$, $u = 0.1$, $\alpha = 0.01$

$T(0,t) = 0$, $T(1,t) = 100$

$T(x, 0) = 100x$

$$\Rightarrow \cancel{\frac{T_j^{n+1} - T_j^n}{\delta t} + u\left(\frac{T_{j+1}^n - T_{j-1}^n}{2\delta x}\right) =}$$

$$\frac{T_j^{n+1} - T_j^n}{\delta t} + \frac{u}{2}\left\{\frac{T_{j+1}^{n+1} - T_{j-1}^{n+1}}{2\delta x} + \frac{T_{j+1}^n - T_{j-1}^n}{2\delta x}\right]$$

$$= \frac{\alpha}{2}\left[\frac{T_{j-1}^{n+1} - 2T_j^{n+1} + T_{j+1}^{n+1}}{\delta x^2} + \frac{T_{j-1}^n - 2T_j^n + T_{j+1}^n}{\delta x^2}\right]$$

$$\Rightarrow \left[\frac{-u}{4\delta x} - \frac{\alpha}{2\delta x^2}\right]T_{j-1}^{n+1} + \left[\frac{1}{\delta t} + \frac{\alpha}{\delta x^2}\right]T_j^{n+1}$$

$$+ \left[\frac{u}{4\delta x} - \frac{\alpha}{2\delta x^2}\right]T_{j+1}^{n+1} = \frac{\alpha}{2}\left[\frac{T_{j-1}^n - 2T_j^n + T_{j+1}^n}{\delta x^2}\right] + \frac{T_j^n}{\delta t}$$

$$- \frac{u}{2}\left[\frac{T_{j+1}^n - T_{j-1}^n}{2\delta x}\right]$$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$0 \quad 0.25 \quad 0.5 \quad \dots$

(Here, we take $\delta t = 0.1$, $\delta x = 0.25$)

we have got the tri-diagonal system

$$a_i T_{i-1}^{n+1} + b_i T_i^{n+1} + c_i T_{i+1}^{n+1} = d_i$$

↳ Solving this, we get

$t \to$	0	0.1	0.2	0.3	0.4	0.5
$n \to$	0	1	2	3	4	5
x_0	0	0	0	0	0	0
x_1	25	24.0177	23.07	22.1567	21.2757	20.426
x_2	50	49.003	48.019	47.0059	46.0135	45.025
x_3	75	73.998	72.992	71.9828	70.9699	69.953
x_4	100	100	100	100	100	100

Alternating direction Implicit Scheme (ADI)

For equation, $\dfrac{\partial u}{\partial t} = \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$, $t>0$, $x,y \in R$

$R: \; 0<x<a$
$\quad\;\; a<y<b.$

I.C: $u(0,x,y) = f(x,y)$, $\quad x,y \in R$

B.C: $u(t,x,y)$ is prescribed at the boundary of R.

For Step I: $\quad t_n \to t_{n+1/2}$

$$\frac{u_{ij}^{n+1/2} - u_{ij}^{n}}{\delta t/2} = \nu\left[\frac{u_{i+1j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i-1j}^{n+1/2}}{\delta x^2} + \frac{u_{ij+1}^{n} - 2u_{ij}^{n} + u_{ij-1}^{n}}{\delta y^2}\right]$$

& then Step II:

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+1/2}}{\delta t/2} = \nu\left[\frac{u_{i-1j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i+1j}^{n+1/2}}{\delta x^2} + \frac{u_{ij-1}^{n+1} - 2u_{ij}^{n+1} + u_{ij+1}^{n+1}}{\delta y^2}\right]$$

for Step I, let us consider $u(t_{n+1/2}, x_i, y_j)$ to be exact sol.

$$T.E = \frac{u_{ij}^{n+1/2} - u(t_{n+1/2} - \delta t/2, x_i, y_j)}{(\delta t/2)} = \nu\left[\frac{u(t_{n+1/2}, x_i+\delta x, y_j) - 2u_{ij}^{n+1/2} + u(t_{n+1/2}, x_i - \delta x, y_j)}{\delta x^2} + \frac{u(t_{n+1/2} - \delta t, x_i, y_j - \delta y) - 2u(t_{n+1/2} - \delta t, x_i, y_j) + u(t_{n+1/2} - \delta t, x_i, y_j + \delta y)}{\delta y^2}\right]$$

$$\Rightarrow \frac{\cancel{u_{ij}^{n+1/2}} - \cancel{u_{ij}^{n+1/2}} + (\delta t/2)\dfrac{\partial u}{\partial t}\Big|_{ij}^{n+1/2} - \dfrac{1}{2}\left(\dfrac{\delta t}{2}\right)^2 \dfrac{\partial^2 u}{\partial t^2}\Big|_{ij}^{n+1/2} + \dfrac{1}{3!}\left(\dfrac{\delta t}{2}\right)^3 \dfrac{\partial^3 u}{\partial t^3}\Big| - \cdots}{(\delta t/2)}$$

$$= \nu\left[\frac{\cancel{u_{ij}^{n+1/2}} + \cancel{\delta x \dfrac{\partial u}{\partial x}\Big|_{ij}^{n+1/2}} + \dfrac{\delta x^2}{2}\dfrac{\partial^2 u}{\partial x^2} + \cancel{\dfrac{\delta x^3}{3!}\dfrac{\partial^3 u}{\partial x^3}} + \cdots - \cancel{2u_{ij}^{n+1/2}} + \cancel{u_{ij}^{n+1/2}} - \cancel{\delta x \dfrac{\partial u}{\partial x}} + \dfrac{\delta x^2}{2}\dfrac{\partial^2 u}{\partial x^2} - \cancel{\dfrac{1}{3!}\delta x^3 \dfrac{\partial^3 u}{\partial x^3}} + \cdots}{\delta x^2}\right.$$

$$+ \frac{\left(u_{ij}^{n+1/2} - \cancel{\delta y \dfrac{\partial u}{\partial y}} - \cancel{\delta t \dfrac{\partial u}{\partial t}} + \dfrac{(\delta y)^2}{2}\dfrac{\partial^2 u}{\partial y^2} + \cancel{\dfrac{(\delta y)(\delta t)}{2!}\dfrac{\partial^2 u}{\partial y \partial t}} + \cancel{\dfrac{(\delta t)^2}{2}\dfrac{\partial^2 u}{\partial t^2}} + \cdots\right) - 2\left(u_{ij}^{n+1/2} - \cancel{\delta t \dfrac{\partial u}{\partial t}} + \cancel{\dfrac{(\delta t)^2}{2!}\dfrac{\partial^2 u}{\partial t^2}} - (\delta t)^3 \dfrac{\partial^3 u}{\partial t^3} + \cdots\right) + u_{ij}^{n+1/2} + \cancel{\delta y \dfrac{\partial u}{\partial y}} - \cancel{\delta t \dfrac{\partial u}{\partial t}} + \dfrac{(\delta y)^2}{2!}\dfrac{\partial^2 u}{\partial y^2} - \cancel{\dfrac{\delta y\,\delta t}{2!}\dfrac{\partial^2 u}{\partial y \partial t}} + \cancel{\dfrac{(\delta t)^2}{2}\dfrac{\partial^2 u}{\partial t^2}} \cdots}{\delta y^2}$$

As, $\dfrac{\partial u}{\partial t} - \nu\left[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right] = 0$

$T.E = O(\delta t^2, \delta x^2, \delta y^2)$

Similarly we can show for Step II the T.E

$= O(\delta t^2, \delta x^2, \dots$

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Q

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad -1 < x, y < 1 \quad t > 0$$

$$u(x, y, 0) = \cos\frac{\pi x}{2} \cdot \cos\frac{\pi y}{2}$$

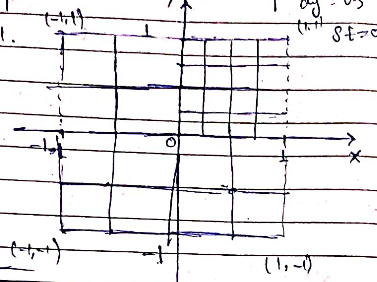
$dx = 0.5$

$dy = 0.5$

$\delta t = 0.1$

$u = 0, \ x = \pm 1$

$u = 0, \ y = \pm 1.$



$$\frac{u_{ij}^{n+1/2} - u_{ij}^{n}}{\delta t/2} = \left[\frac{u_{i+1j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i-1j}^{n+1/2}}{\delta x^2} + \frac{u_{ij+1}^{n} - 2u_{ij}^{n} + u_{ij-1}^{n}}{\delta y^2} \right]$$

$$\Rightarrow \left(\frac{1}{\delta x^2}\right) u_{i-1j}^{n+1/2} + \left(\frac{-2}{\delta x^2} - \frac{2}{\delta t}\right) u_{ij}^{n+1/2} + \left(\frac{1}{\delta x^2}\right) u_{i+1j}^{n+1/2}$$

$$= -\left(\frac{u_{ij+1}^{n} + u_{ij-1}^{n}}{\delta y^2}\right) + u_{ij}^{n}\left(\frac{2}{\delta y^2} - \frac{2}{\delta t}\right)$$

After solving this we get $u(t_{n+1/2}, x_i, y_j) \ \forall \ x_i, y_j$

$i = 0, 1, \ldots N$

$j = 0, 1. \ldots M.$

Now Step II

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+1/2}}{\delta t/2} = \left[\frac{u_{i+1j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i-1j}^{n+1/2}}{\delta x^2} + \frac{u_{ij+1}^{n+1} - 2u_{ij}^{n+1} + u_{ij-1}^{n+1}}{\delta y^2} \right]$$

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$$\Rightarrow \left(\frac{1}{\delta y^2}\right) u_{ij-1}^{n+1} + \left(\frac{-2}{\delta y^2} - \frac{2}{\delta t}\right) u_{ij}^{n+1} + \left(\frac{1}{\delta y^2}\right) u_{ij+1}^{n+1}$$

$$= -\left(\frac{u_{i+1j}^{n+1/2} + u_{i-1j}^{n+1/2}}{\delta x^2}\right) + \left(\frac{2}{\delta x^2} - \frac{2}{\delta t}\right) u^{n+1/2}$$

Solving these equations using ADI scheme we get

at $t = 0$,

$t=0$	$y=-1$	$y=-0.5$	$y=0$	$y=0.5$	$y=1$
$x=-1$	0	0	0	0	0
$x=-0.5$	0	0.5	0.70711	0.5	0
$x=0$	0	0.70711	1	0.70711	0
$x=0.5$	0	0.5	0.70711	0.5	0
$x=1$	0	0	0	0	0

$t=0.05$	$y=-1$	$y=-0.5$	$y=0$	$y=0.5$	$y=1$
$x=-1$	0	0	0	0	0
$x=-0.5$	0	0.5294	0.74868	0.5294	0
$x=0$	0	0.74868	1.0588	0.74868	0
$x=0.5$	0	0.5294	0.74868	0.5294	0
$x=1$	0	0	0	0	0

$t=0.1$					
$x=-1$	0	0	0	0	0
$x=-0.5$	0	0.56053	0.7927	0.56053	0
$x=0$	0	0.7927	1.12105	0.7927	0
$x=0.5$	0	0.56053	0.7927	0.56053	0
$x=1$	0	0	0	0	0

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(Q) $u_t + u_x = \nu u_{xx}$, $\nu = 1$.

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 0 & x > 0 \end{cases}$$

$u(-5,t) = 1$, $u(5,t) = 0$, $t \geq 0$

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{1}{2}\left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\delta x} + \frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}\right)$$

$$= \frac{\nu}{2}\left[\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\delta x^2} + \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\delta x^2}\right]$$

Solving this we get

-5   0   5

$x \downarrow$ \ $t \rightarrow$	0	0.1	0.2	0.3	0.4
$x_0 = -5$	1	1	1	1	1
$x_1 = -4$	1	1	1	1	1
$x_2 = -3$	1	1	0.9999	0.9991	0.9994
$x_3 = -2$	1	0.999	0.9966	0.9934	0.4901
$x_4 = -1$	1	0.9515	0.9280	0.9077	0.8941
$x_5 = 0$	0	0.1337	0.2304	0.3228	0.3908
$x_6 = 1$	0	0.091	0.0330	0.0656	0.1030
$x_7 = 2$	0	0.006	0.0034	0.0095	0.0194
$x_8 = 3$	0	0	0.003	0.0011	0.0029
$x_9 = 4$	0	0	0	0.001	0.0004
$x_{10} = 5$	0	0	0	0	0.

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(Q) $u_t + u u_x = \nu u_{xx}$

(i) $u(x,0) = \sin \pi x$, $0 < x < 1$

$u(0,t) = u(1,t) = 0$, $t > 0$.

↳ Discretizing using Crank Nicolson Scheme.

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{1}{2}\left[u_j^{n+1}\left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\delta x}\right)\right] - \frac{\nu}{2}\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\delta x^2}\right]$$

$$= \frac{\nu}{2}\left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\delta x^2}\right] - \frac{1}{2}\left[u_j^n\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}\right)\right]$$

$$\left(u_j^{n+1}\right)^{(k+1)} = \left(u_j^{n+1}\right)^{(k)} + \Delta u_j^{n+1}.$$

using this :

$$\frac{\left(u_j^{n+1}\right)^{(k)} + \Delta u_j^{n+1} - u_j^n}{\delta t} + \frac{1}{2}\left[\left(\left(u_j^{n+1}\right)^{(k)} + \Delta u_j^{n+1}\right)\left(\frac{\left(u_{j+1}^{n+1}\right)^{(k)} + \Delta u_{j+1}^{n+1} - \left(u_{j-1}^{n+1}\right)^{(k)} - \Delta u_{j-1}^{n+1}}{2\delta x}\right)\right]$$

$$-\frac{\nu}{2}\left[\left(u_{j+1}^{n+1}\right)^{(k)} + \Delta u_{j+1}^{n+1} - 2\left(u_j^{n+1}\right)^{(k)} - 2\Delta u_j^{n+1} + \left(u_{j-1}^{n+1}\right)^{(k)} + \Delta u_{j-1}^{n+1}\right]$$

$$= \frac{\nu}{2}\left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\delta x^2}\right] - \frac{1}{2}\left[u_j^n\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}\right)\right] \rightarrow d_i$$

↳

$$\left(-\frac{\left(u_j^{n+1}\right)^{(k)}}{4\delta x} - \frac{\nu}{2\delta x^2}\right)\Delta u_{j-1}^{n+1} + \left(\frac{1}{\delta t} + \frac{\left(u_{j+1}^{n+1}\right)^{(k)} - \left(u_{j-1}^{n+1}\right)^{(k)}}{4\delta x} + \frac{\nu}{\delta x^2}\right) + \left(\frac{\left(u_j^{n+1}\right)^{(k)}}{4\delta x} - \frac{\nu}{2\delta x^2}\right)\Delta u_{j+1}^{n+1} = d_i$$

this is the Tri-diagonal fn

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Von-Neumann Analysis on ADI scheme

$$\frac{\epsilon_{ij}^{n+1/2} - \epsilon_{ij}^{n}}{\delta t/2} = \nu\left[\frac{\epsilon_{i+1,j}^{n+1/2} - 2\epsilon_{ij}^{n+1/2} + \epsilon_{i-1,j}^{n+1/2}}{\delta x^2} + \frac{\epsilon_{i,j+1}^{n} - 2\epsilon_{ij}^{n} + \epsilon_{i,j-1}^{n}}{\delta y^2}\right]$$

$$\epsilon_{jk}^{n} = A^n e^{i\theta j} e^{i\phi k}$$

$$\frac{A^{n+1/2} - A^n}{\delta t/2} = \nu\left[\frac{A^{n+1/2}e^{i\theta} - 2A^{n+1/2} + A^{n+1/2}e^{-i\theta}}{\delta x^2} + \frac{A^n e^{i\phi} - 2A^n + A^n e^{-i\phi}}{\delta y^2}\right]$$

$$\Rightarrow \frac{\xi - 1}{\delta t} = \frac{\nu}{2}\left[\xi\left(\frac{2\cos\theta - 2}{\delta x^2}\right) + \left(\frac{2\cos\phi - 2}{\delta y^2}\right)\right]$$

$\delta x = \delta y$ & $\frac{\delta t}{\delta x^2} = \gamma$

$$\xi - 1 = \frac{\nu\gamma}{2}\left[\xi(2\cos\theta - 2) + 2\cos\phi - 2\right]$$

$$\xi - \xi(\cos\theta - 1)\nu\gamma = 1 + \nu\gamma(\cos\phi - 1)$$

$$\xi = \frac{1 + \nu\gamma(\cos\phi - 1)}{1 - \nu\gamma(\cos\theta - 1)}$$

$$= \frac{1 - \nu\gamma(1 - \cos\phi)}{1 + \nu\gamma(1 - \cos\theta)} < 1$$

$|\xi| \le 1$.

thus Step I is ~~condition~~ stable

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Now, for Step II

$$\frac{\epsilon_{ij}^{n+1} - \epsilon_{ij}^{n+1/2}}{\delta t} = \frac{\nu\gamma}{2}\left[\epsilon_{i+1,j}^{n+1/2} - 2\epsilon_{ij}^{n+1/2} + \epsilon_{i-1,j}^{n+1/2} + \epsilon_{i,j+1}^{n+1} - 2\epsilon_{ij}^{n+1} + \epsilon_{i,j-1}^{n+1}\right]$$

$$\Rightarrow A^{n+1} - A^{n+1/2} = \frac{\nu\gamma}{2}\left[A^{n+1/2}e^{i\theta} - 2A^{n+1/2} + A^{n+1/2}e^{-i\theta} + A^{n+1}e^{i\phi} - 2A^{n+1} + A^{n+1}e^{-i\phi}\right]$$

$$\Rightarrow \xi - 1 = \frac{\nu\gamma}{2}\left[\xi(2\cos\theta - 2) + 2\cos\phi - 2\right]$$

$$\Rightarrow \xi + \nu\gamma(1 - \cos\phi)\xi = 1 - \nu\gamma(1 - \cos\theta)$$

$$\Rightarrow \xi = \frac{1 - \nu\gamma(1 - \cos\theta)}{1 + \nu\gamma(1 + \cos\phi)} \le 1$$

$\Rightarrow |\xi| \le 1$

thus ADI scheme is unconditionally stable

Q) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, $0 \le x, y \le 1$. $t > 0$

$\delta x = \delta y = 0.25$

$\delta t = 0.1$

$u(x, y, 0) = \sin\pi x \cdot \sin\pi y$

$$\frac{u_{ij}^{n+1/2} - u_{ij}^{n}}{\delta t/2} = \left[\frac{u_{i-1,j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i+1,j}^{n+1/2}}{\delta x^2} + \frac{u_{i,j-1}^{n} - 2u_{ij}^{n} + u_{i,j+1}^{n}}{\delta y^2}\right]$$

↳ converting it into Tri diagonal form & solving it, we get at $t=0$

	$y=0$	$y=0.25$	$y=0.5$	$y=0.75$	$y=1$
$x=0$	0	0	0	0	0
$x=0.25$	0	0.5	0.70711	0.5	0
$x=0.5$	0	0.70711	1	0.70711	0
$x=0.75$	0	0.5	0.70711	0.5	0
$x=1$	0	0	0	0	0

at $t = \delta t/2 = 0.05$

	0	0	0	0	0
$x=0$	0	0.69159	0	0	0
$x=0.25$	0	0.69159	0.97805	0.69159	0
$x=0.5$	0	0.97805	1.38177	0.97805	0
$x=0.75$	0	0.69159	0.97805	0.69159	0
$x=1$	0	0	0	0	0

Now,

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+1/2}}{\delta t/2} = \left[\frac{u_{i-1j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i+1j}^{n+1/2}}{\delta x^2} + \frac{u_{ij-1}^{n+1} - 2u_{ij}^{n+1} + u_{ij+1}^{n+1}}{\delta y^2} \right]$$

$$\Rightarrow \left(\frac{1}{\delta y^2}\right) u_{ij-1}^{n+1} + \left(\frac{-2}{\delta y^2} - \frac{2}{\delta t}\right) u_{ij}^{n+1} + \frac{1}{\delta y^2} u_{ij+1}^{n+1}$$

$$= -\left(\frac{u_{i-1j}^{n+1/2} + u_{i+1j}^{n+1/2}}{\delta x^2}\right) + \left(\frac{2}{\delta x^2} - \frac{2}{\delta t}\right) u_{ij}^{n+1/2}$$

$i = 1, 2, \ldots, N-1$

$j = 1, 2, \ldots, M-1$

~~Solving this~~ w

This is in Tri-diagonal form

Solving this eq.

$\delta t = 0.1$

at $t = 0.1$

	$y=0$	$y=0.25$	$y=0.5$	$y=0.75$	$y=1$
$x=0$	0	0	0	0	0
$x=0.25$	0	0.95659	1.35262	0.95659	0
$x=0.5$	0	1.35282	1.91317	1.35262	0
$x=0.75$	0	0.95659	1.35282	0.95659	0
$x=1$	0	0	0	0	0

$\Rightarrow$

Q $\dfrac{\partial T}{\partial t} + T\dfrac{\partial T}{\partial x} = \nu \dfrac{\partial^2 T}{\partial x^2}, \quad 0 < x < 1, \quad t > 0$

$T(x,0) = f(x), \quad T(0,t) = T(1,t) = 0.$

Discretizing using Crank-Nicolson Scheme:

$$\frac{T_j^{n+1} - T_j^n}{\delta t} + \frac{1}{2}\left[T_j^{n+1} \frac{(T_{j+1}^{n+1} - T_{j-1}^{n+1})}{2\delta x}\right] - \frac{\nu}{2}\left[\frac{T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}}{\delta x^2}\right]$$

$$= \frac{\nu}{2}\left[\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\delta x^2}\right] - \frac{1}{2}\left[T_j^n \frac{(T_{j+1}^n - T_{j-1}^n)}{2\delta x}\right]$$

Now, we have to use the newton's linearization technique

$$\left(T_j^{n+1}\right)^{(k+1)} = \left(T_j^{n+1}\right)^{(k)} + \Delta T_j^{n+1}$$

$$\frac{(T_j^{n+1})^{(k)} + \Delta T_j^{n+1} - T_j^n}{\delta t} + \frac{1}{2}\left[\left((T_j^{n+1})^{(k)} + \Delta T_j^{n+1}\right)\left(\frac{(T_{j+1}^{n+1})^{(k)} + \Delta T_{j+1} - (T_{j-1})^{(k)} - \Delta T_{j-1}}{2\delta x}\right)\right]$$

$$- \frac{\nu}{2}\left[\frac{(T_{j+1}^{n+1})^{(k)} + \Delta T_{j+1} - 2(T_j^{n+1})^{(k)} - 2\Delta T_j^{n+1} + (T_{j-1}^{n+1})^{(k)} + \Delta T_{j-1}}{\delta x^2}\right]$$

$$= \frac{\nu}{2}\left[\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\delta x^2}\right] - \frac{1}{2}\left[T_j^n \frac{(T_{j+1}^n - T_{j-1}^n)}{2\delta x}\right]$$

$$= \underbrace{\left(-\frac{(T_j^{n+1})^{(k)}}{4\delta x} - \frac{\nu}{2\delta x^2}\right)}_{a_j} \Delta T_{j-1}^{n+1} + \underbrace{\left(\frac{1}{\delta t} + \frac{(T_{j+1}^{n+1})^{(k)} - (T_{j-1}^{n+1})^{(k)}}{4\delta x} + \frac{\nu}{\delta x^2}\right)}_{b_j} * \Delta T_j^{n+1}$$

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$$+\left(\frac{(T_j^{n+1})^{(k)}}{4\delta x} - \frac{\nu}{2\delta x^2}\right) T_{j+1}^{n+1} = \frac{\nu}{2}\left[\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\delta x^2}\right]$$

d_i

$$\left(\frac{(T_j^{n+1})^{(k)} - T_j^n}{\delta t}\right) - T_j^n\left(\frac{T_{j+1}^n - T_{j-1}^n}{2\delta x}\right)$$

$$-\frac{1}{2}\left((T_j^{n+1})^{(k)}\left(\frac{(T_{j+1}^{n+1})^{(k)} - (T_{j-1}^{n+1})^{(k)}}{2\delta x}\right)\right) \Rightarrow d_i$$

$$+\frac{\nu}{2}\left[\frac{(T_{j+1}^{n+1})^{(k)} - (2T_j^{n+1})^{(k)} + (T_{j-1}^{n+1})^{(k)}}{\delta x^2}\right]$$

This is the ensuing Tri-diagonal system that needs to be solved

Q. $\nabla^2 u - 2\frac{\partial u}{\partial x} = -2$ in R, $0<x<1$, $0<y<1$

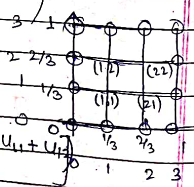
Discretising we get

$$\left(\frac{u_{i-1j} - 2u_{ij} + u_{i+1j}}{\delta x^2} + \frac{u_{ij-1} - 2u_{ij} + u_{ij+1}}{\delta y^2}\right)$$

$$-\cancel{2}\times\left(\frac{u_{i+1j} - u_{i-1j}}{\cancel{2}\delta x}\right) = -2$$

$i = 1, 2,\ j = 1, 2$

$\delta x = 1/3 = \delta y$



$i=1,\ j=1$

$$9\left[(\cancel{u_{01}}^{0} - 2u_{11} + u_{21}) + \cancel{u_{10}} - 2u_{11} + u_{12}\right]$$

$$-3(u_{21} - \cancel{u_{01}}^{0}) = -2$$

$\Rightarrow 9\left[-4u_{11} + u_{12} + u_{21}\right] - 3u_{21} = -2$

$-36u_{11} + 9u_{12} + 9u_{21} - 3u_{21} = -2$

$-36u_{11} + 9u_{12} + 6u_{21} = -2$ — (I)

$i=1,\ j=2$

$$9(\cancel{u_{02}}^{0} - 2u_{12} + u_{22} + u_{11} - 2u_{12} + \cancel{u_{13}}^{0})$$

$$-3(u_{22} - \cancel{u_{02}}^{0}) = -2$$

$-36u_{12} + 9u_{22} + 9u_{11} - 3u_{22} = -2$

$\boxed{-36u_{12} + 6u_{22} + 9u_{11} = -2}$ — (II)

$i=2,\ j=1$

$$9(u_{11} - 2u_{21} + \cancel{u_{31}}^{0} + \cancel{u_{20}}^{0} - 2u_{21} + u_{22})$$

$$-3(\cancel{u_{31}}^{0} - u_{11}) = -2$$

$\Rightarrow -36u_{21} + 9u_{11} + 9u_{22} + 3u_{11} = -2$

$12u_{11} - 36u_{21} + 9u_{22} = -2$

$i=2,\ j=2$

$$9(u_{12} - 2u_{22} + \cancel{u_{32}}^{0} + u_{21} - 2u_{22} + \cancel{u_{23}}^{0})$$

$$-3(\cancel{u_{32}}^{0} - u_{12}) = -2$$

$\Rightarrow 12u_{12} - 36u_{22} + 9u_{21} = -2$

$u_{11} = \frac{22}{219}$ $\quad u_{21} = \frac{26}{219}$

$u_{12} = \frac{22}{219}$ $\quad u_{22} = \frac{26}{219}$

$$u_j^{n+1} = (1-\nu)\, u_j^n + \nu\, u_{j-1}^n$$

$$\xi = \frac{A^{n+1}}{A^n} = 1-\nu + \nu(\cos\theta - i\sin\theta)$$

for stability

$$|\xi| \le 1$$

$$\Rightarrow \quad 1 - 2\nu(1-\nu)(1-\cos\theta) \le 1$$

if $\quad (1-\nu) \geq 0$

$\Rightarrow \quad 1 \geq \nu \quad \Rightarrow \quad \nu(1-\nu)(1-\cos\theta)$ will be +ve

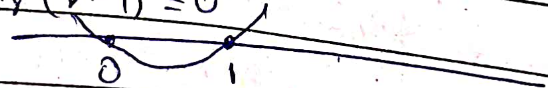
$\Rightarrow 1 - 2\nu(1-\nu)(1-\cos\theta) \le 1$

then $\underline{\underline{\nu \le 1}}$

(b) $\underline{\underline{c<0}}$

$\Rightarrow \quad \nu(1-\nu) \geq 0$

$\nu(\nu-1) \le 0$



$\underline{\underline{\nu = 0}}$