Lecture 19

Date
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EX1/ Find the complex Fourier series of

$$f(x) = e^{x}, \text{ if } -\pi < x < \pi$$

& $f(x+2\pi) = f(x)$, & obtain from it the usual Fourier series.

Soln:- Since $\sin n\pi = 0$, for integer n

& $e^{\pm in\pi} = \cos n\pi \pm i \underset{(=0)}{\sin n\pi}$

$= \cos n\pi$

$= (-1)^{n}$.

With this we obtain from $eq^{n}(8)$ (how?) by integration

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x} \cdot e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x-inx}\, dx$$

$$= \frac{1}{2\pi}\cdot\frac{1}{(1-in)}\left[e^{\pi-in\pi} - e^{-\pi+in\pi}\right]$$

$$= \frac{1}{2\pi}\underbrace{\frac{1}{(1-in)}}\left(e^{\pi}-e^{-\pi}\right)(-1)^n$$

(how?)

Now, $\dfrac{1}{1-in} = \dfrac{1+in}{(1-in)(1+in)} = \dfrac{1+in}{1+n^2}$

& $e^{\pi} - e^{-\pi} = 2\sinh(\pi)$

Hence, the complex Fourier series is

$$e^{x} = \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx}$$

$$\therefore\ e^{x} = \frac{\sinh(\pi)}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(1+in)}{(1+n^2)} e^{inx}$$

$$(-\pi < x < \pi). \quad \longrightarrow (10)$$

From this we now derive the real Fourier Series.

Using (2) & $\boxed{i^2 = -1}$, we have in eq^n (10).

$$(1+in)\, e^{inx} = (1+in)(\cos nx + i \sin nx)$$

$$= (\cos nx - n \sin nx) + i(n \cos nx + \sin nx)$$

[Now, eq^n (10) also has a corresponding term with $(-n)$ instead of n.

Since $\cos(-n\pi) = \cos n\pi$

& $\sin(-n\pi) = -\sin n\pi$.

we obtain in this term

$$(1-in)e^{-in\pi} = (1-in)(\cos n\pi - i\sin n\pi)$$

$$= \left(\boxed{\cos n\pi - n\sin n\pi}\right) \pm i\left(n\cos n\pi + \sin n\pi\right).$$

If we add these two expressions,

$$(1+in)e^{in\pi} + (1-in)e^{-in\pi}$$

$$= 2(\cos n\pi - n\sin n\pi)$$

$n = 1, 2, \dots$

not

For $n=0$, we get 1 (not $\underline{2}$) because there is only one term.

Hence, the real Fourier series is given by

$$e^{x} = \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(\cos nx - n\sin nx)}{1+n^2} + \frac{\sinh \pi}{\pi}$$

$n = 1, 3 \ldots \infty$

$$= \frac{2\sinh(\pi)}{\pi}\left[\frac{1}{2} - \frac{1}{1+1^2}(\cos x - \sin x) + \frac{1}{1+2^2}(\cos 2x - 2\sin 2x) - \ldots\right]$$

where $-\pi < x < \pi$.

Properties Fourier Series.

Integration & Differentiation of Fourier Series

Intuitively, it is the differentiation of Fourier series that poses more problems than integration.

$$\frac{d}{dx}\left\{\sin(nx)\right\} = n\cos(nx)$$

& $$\frac{d}{dx}\left\{\cos(nx)\right\} = -n\sin(nx)$$

which for large n are both larger in magnitude than the original terms.

$$\int \sin(nx)\,dx = -\frac{\cos(nx)}{n}$$

& $$\int \cos(nx)\,dx = \frac{\sin(nx)}{n}$$, both smaller in magnitude.

Note:- The integration of a Fourier series poses less of a problem & can virtually always take place. A minor problem arises because the result is not necessarily another F. Series. A term linear in x is produced by integrating the constant term when it is not zero.

Th-1 / If f is continuous on $[-\pi, \pi]$ & piecewise differentiable in $(-\pi, \pi)$ which means that the ~~first~~ derivative f' is piecewise continuous on $[-\pi, \pi]$

& if $f(x)$ has the Fourier series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

then the Fourier series of the derivative of $f(x)$ is given by

$$f'(x) = \sum_{n=1}^{\infty} \left[-n\, a_n \sin(nx) + n\, b_n \cos nx \right].$$

PHD :- EX

Th-2 / If f is piecewise continuous on the interval $[-\pi, \pi]$ & has the Fourier series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]$$

then for each

$$x \in [-\pi, \pi]$$

$$\int_{-\pi}^{x} f(t)\,dt = \frac{1}{2} a_0 (\underline{\underline{x}} + \pi) + \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin nx - \frac{b_n}{n}(\cos nx - \cos n\pi)\right]$$

& the f^n on the R.H.S converges uniformly to the f^n on the left.

Let us discuss the details of differentiating & integrating Fourier series via the three series

x^3, x^2 & x, in $[-\pi, \pi]$

viz.,

$$f(x) = f(x+2\pi).$$

The three F-series can be derived as

$$x^3 = \sum_{n=1}^{\infty} (-1)^n \frac{2}{n^3}\left(6 - \pi^2 n^2\right)\sin(nx)$$

(EX)

$$\boxed{x^2 = \pi^2/3 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)}$$

(EX)

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$$

for $-\pi < x < \pi$.

The series for x^2 is uniformly convergent

Neither x nor x^3 is uniformly convergent.

(why?)

~~All~~ But all these 3 series are pointwise convergent.

It is therefore legal to differentiate the series for x^2 but not either of the other two. All the series can be integrated.

$$x^2 = \pi^2/3 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$

Diff^n w.r.t. x term by term

$$2x = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2} \cdot \cancel{n} \sin nx$$

$$\Rightarrow 2x = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n} \sin nx$$

which is the same as the F. series for x.

$$x^2 = \pi^2/3 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$

Integrating w.r.t. x the series for x^2 term by term.

$$\frac{x^3}{3} = \frac{\pi^2}{3}x + A$$

$$+ \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^3} \cdot \sin(nx).$$

where A is an arbitrary constant.

$\Rightarrow$ putting $x=0$ gives $A=0$.

$$\therefore \frac{x^3}{3} = \frac{\pi^2}{3}x + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^3} \sin(nx).$$

Substituting the F.S for x, we get

$$\therefore \frac{x^3}{3} = \frac{\pi^2}{3}\left[\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx)\right]$$

$$+ \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^3} \sin(nx)$$

Integrating the F.S for x is not useful as it is not easy to evaluate the arbitrary constant that is generated

$$\Rightarrow x^3 = \sum_{n=1}^{\infty} (-1)^n \frac{2}{n^3}(6 - \pi^2 n^2) \sin(nx).$$

which is the F.S for x^3 already given.

Thus integration of F.S is not always productive

Let us now derive a more general result involving the integration of Fourier series.

Now

$$\int_{-\pi}^{t} G(x)\,dx - \int_{-\pi}^{\xi} G(x)\,dx.$$

$$= \left(\int_{-\pi}^{t} + \int_{\xi}^{-\pi}\right) G(x)\,dx$$

$$= \int_{\xi}^{t} G(x)\,dx$$

[What is $G(x)$?!]

$F(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos(nt) + B_n\sin(nt)\right)$

we define $G(x)$ as $\quad -\pi < t < \pi$

$\int_{-\pi}^{t} G(x)\,dx = \frac{1}{2}a_0 t + F(t)$

$\therefore F'(t) = G(t) - \frac{1}{2}a_0$

$$= \frac{1}{2}a_0(t-\xi) + \sum_{n=1}^{\infty}\frac{1}{n}\Big[b_n\{\cos(n\xi) - \cos(nt)\} + a_n\{\sin(nt) - \sin(n\xi)\}\Big]$$

which gives the form of the integral of a Fourier series.

(how??)

11) Use the Fourier series

$$x^2 = \pi^2/3 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$$

to deduce the value of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3}.$$

Sol^n:- Using the result just derived on the integration of Fourier series,

we put $\xi = 0$ & $t = \pi/2$, $(\xi = 0)$

we can write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

& $x^2 = \pi^2/3 + 4\sum_{n=1}^{\infty} (-1)^n/n^2 \cos nx.$

Here, $\underline{a_0/2} = \pi^2/3$

$\sum a_n = 4(-1)^n/n^2$

$$\therefore \int_0^{\pi/2} x^2\,dx$$

$$= \frac{\pi^2}{3}\left(\frac{\pi}{2} - 0\right) + 4\sum_{n=1}^{\infty} \frac{1}{n}\left[\frac{(-1)^n}{n^2}\left\{\sin(n\pi/2) - 0\right\}\right]$$

$$\Rightarrow \frac{\pi^3}{24} = \frac{\pi^3}{6} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}\sin(n\pi/2)$$

since $\sin(n\pi/2)$ takes the values

$0, 1, 0, -1, \ldots$ for

$n = 0, 1, 3, \ldots$

we deduce that (EX)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(n\pi/2) = -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3}$$

which gives

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(n\pi/2) \qquad \left[+1, \frac{-1}{3^3}, \ldots \right.$$

$$= \frac{1}{4}\left(\pi^3/6 - \pi^3/24\right) = \pi^3/32.$$

Hence, putting $n = 2k-1$, will give the result

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} = \boxed{\pi^3/32}$$

Th-3/ If $f(t)$ & $g(t)$ are continuous in $(-\pi, \pi)$ & provided

$$\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$$

$$\& \int_{-\pi}^{\pi} |g(t)|^2 dt < \infty$$

& if a_n, b_n are the Fourier co-efficients of $f(t)$ & α_n, β_n those of $g(t)$, then

$$\int_{-\pi}^{\pi} f(t)\, g(t)\, dt$$

$$= \frac{1}{2}\pi a_0 \alpha_0 + \pi \sum_{n=1}^{\infty} (\alpha_n a_n + \beta_n b_n)$$

proof 1.- Since the F. series for $f(t)$ & $g(t)$ are given by

$$f(t) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$g(t) = \alpha_0/2 + \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt)$$

we can write

$$f(t)g(t) = \frac{a_0}{2} g(t) + \sum_{n=1}^{\infty} (a_n g(t) \cos nt + b_n g(t) \sin nt)$$

Integrating this series from $-\pi$ to π, gives

$$\int_{-\pi}^{\pi} f(t) g(t)\, dt = \frac{1}{2} a_0 \int_{-\pi}^{\pi} g(t)\, dt$$

$$+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi} f(t)\cos nt\,dt\right.$$

$$\left.+b_n\int_{-\pi}^{\pi} f(t)\sin nt\,dt\right]$$

provided the Fourier series for $f(t)\,g(t)$ is uniformly convergent, enabling the summation & integration operations to be interchanged. This follows from the Cauchy–Schwarz's inequality

$$\int_{-\pi}^{\pi}|f(t)\,g(t)|\,dt \le \left(\int_{-\pi}^{\pi}|f(t)|^2dt\right)^{1/2}.$$

$$\left(\int_{-\pi}^{\pi}|g(t)|^2dt\right)^{1/2} < \infty$$

However, we know that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \alpha_n$$

& $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \beta_n$$

& that $$\frac{\alpha_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

so this implies

$$\int_{-\pi}^{\pi} f(t)\, g(t)\,dt$$

$$= \frac{1}{2}\pi a_0 \alpha_0 + \pi\sum_{n=1}^{\infty}(\alpha_n a_n + \beta_n b_n)$$

$\rightarrow (*)$ ✓

If we substitute $f(t) = g(t)$ in the

above result (*), the following important theorem follows

Th-4 / Ex (Parseval's) Identity French mathematician [Marc Antoine Parseval]

If $f(t)$ is continuous in the range $(-\pi, \pi)$, is square integrable (ie, $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$) & has the Fourier coefficients a_n, b_n then

$$\int_{-\pi}^{\pi} [f(t)]^2 dt = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

Note: - This is a very useful result in mathematics but perhaps its most helpful attribute lies in its interpretation. The L·H·S represents the mean square value of $f(t)$ (once it is divided by 2π). It can therefore be thought of in terms of energy if $f(t)$ represents a signal.

Then Parseval's theorem states that the energy of a signal expressed as a waveform is proportional to the sum of the squares of its Fourier co-efficients.

Ex/

Conservation law

The sum of the squares of the modulli of the complex Fourier coefficients is equal to the average value of $|f(x)|^2$ within the period.

i.e., $$\frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 .$$

Ex/ Given the Fourier series

$$t^2 = \pi^2/3 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nt)$$

deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Sol^n:- Applying Parseval's theorem to this series

the L.H.S becomes

$$\int_{-\pi}^{\pi} [f(t)]^2 dt$$

$$= \int_{-\pi}^{\pi} (t^2)^2 dt$$

$$= \int_{-\pi}^{\pi} t^4 dt$$

$$= \left[\frac{t^5}{5}\right]_{-\pi}^{\pi} = 2/5\,\pi^5$$

Here

$f(t) = t^2$

$a_0/2 = \pi^2/3$

$\Rightarrow a_0 = 2\pi^2/3$

$a_n = 4(-1)^n/n^2$

e R·H·S becoms

$$= \pi \left[a_0^2/2 + \sum_{n=1}^{\infty} a_n^2 \right]$$

$$= \pi \left[\frac{4\pi^4}{2.9} + \sum_{n=1}^{\infty} \frac{16(-1)^{2n}}{n^4} \right]$$

$$= \pi \left[2\pi^4/9 + \sum_{n=1}^{\infty} \frac{16}{n^4} \right]$$

Equating ~~these~~ two expressions leads to

$$\frac{2}{5}\pi^5 = \frac{2}{9}\pi^5 + \pi \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\Rightarrow 16 \sum_{n=1}^{\infty} \frac{1}{n^4} = \pi^4 \left(\frac{2}{5} - \frac{2}{9} \right) = \frac{8}{45}.$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}.$$

(H.W)
Ex.

Parseval's identity for Fourier Series (of length $2l$)

It states that

$$\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2 dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

where a_n & b_n are given by

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\frac{n\pi}{l}x\,dx.$$

An important consequence is that

$$\begin{cases} \lim_{n\to\infty} \int_{-l}^{l} f(x) \sin \frac{n\pi}{l} x \, dx = 0 \\ \lim_{n\to\infty} \int_{-l}^{l} f(x) \cos \frac{n\pi}{l} x \, dx = 0 \end{cases}$$

This is called Riemann's theorem.

Finite Fourier Transforms.

The finite Fourier sine transforms of $f(x)$, $0 < x < l$, is defined as

$$f_s(n) = \int_0^l \underline{f(x)} \sin \frac{n\pi}{l} x \, dx$$

where n is an integer $\rightarrow (1)$

The function $f(x)$ is then called the inverse finite Fourier sine transform of $f_s(n)$ & is given by

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi}{l} x. \quad \rightarrow (2)$$

(ie, $f(x)$)

The finite Fourier cosine transform of $f(x)$, $0<x<l$, is defined as

$$f_c(n)=\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

where n is an integer.

The function $f(x)$ is then called the inverse finite Fourier cosine transform of $f_c(n)$ & is given by

$$f(x)=\frac{1}{l}f_c(0)+\frac{2}{l}\sum_{n=1}^{\infty}f_c(n)\cos\frac{n\pi}{l}x$$

They are useful in solving d.e's (how?)

Gibbs Phenomenon.

To discuss the Gibbs phenomenon, let us consider the F. series expansion of the fn

$f,\ f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 \le x < \pi \end{cases}$

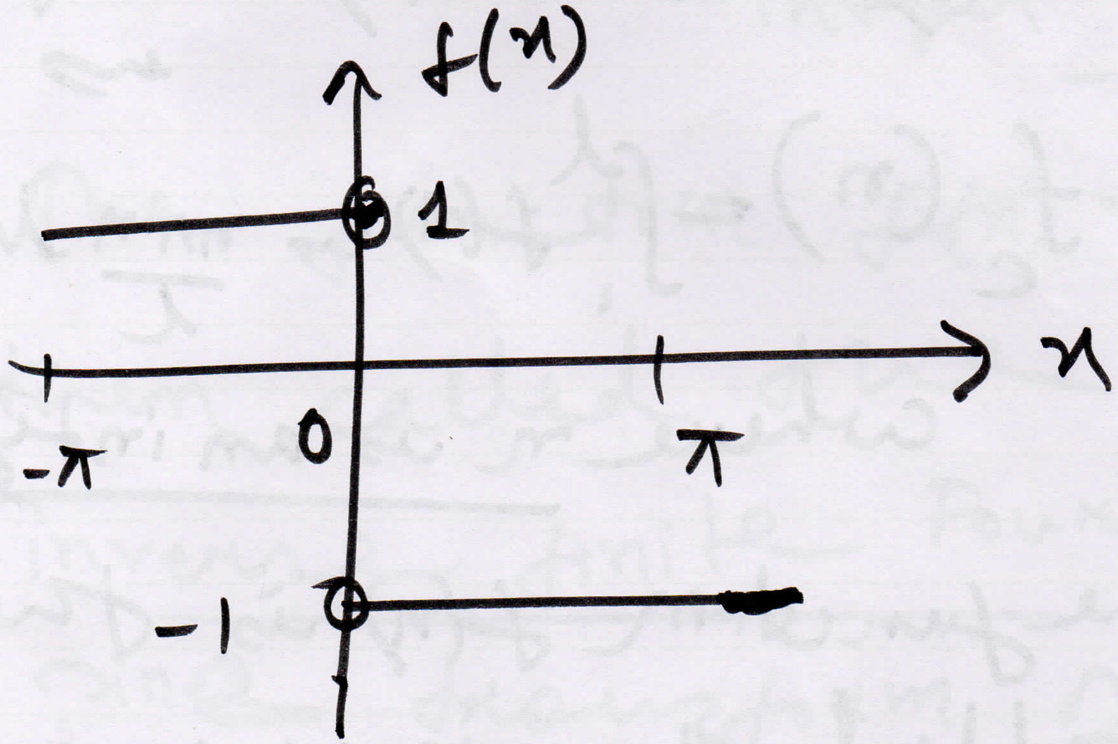

Fig 1:- Graph of f(x).

The function is odd.
(as it is symmetric w.r.t the origin O)

∴ we have a sine series. We have,

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\sin(nx)}_{\text{odd}}}_{(\text{even } f^n)} dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin(nx)\, dx$$

[By property of definite integral]

$$= -\frac{2}{\pi}\int_0^{\pi} \sin nx \, dx.$$

$$= \frac{2}{n\pi}\Big[\cos nx\Big]_0^{\pi} = \frac{2}{n\pi}\left[\cos n\pi - 1\right]$$

$$= \frac{2}{n\pi}\left[(-1)^n - 1\right]$$

$$\Rightarrow b_n = \begin{cases} 0, & n \text{ is even} \\ -4/n\pi, & n \text{ is odd.} \end{cases}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= -\frac{4}{\pi}\Big[\overbrace{\overbrace{\underbrace{\sin x}_{=S_1} + \frac{\sin 3x}{3}}^{S_2} + \frac{\sin 5x}{5}}^{S_3} + \dots\Big]$$

— x —