Legendre Polynomial $P_n(x)$:-

Q) If n^{th} degree polynomial $P_n(x)$ is a solution of legendre equation, show that :-

$$P_n(x) = \sum_{r=0}^{p} \frac{(-1)^r (2n-2r)!}{2^n \cdot r! \cdot (n-r)! \, (n-2r)!} \cdot x^{n-2r},$$

where the integer p is $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Sol^n:- Substituting, $y = \sum_{r=0}^{\infty} a_r \cdot x^r$ in legendre eq^n,

we have,

$$a_{r+2} = \frac{(r-n)(r+n+1)}{(r+1)(r+2)} a_r .$$

$$y(x) = a_n \left[x^n + \frac{a_{n-2}}{a_n} . x^{n-2} + \frac{a_{n-4}}{a_n} . x^{n-4} + \dots \right]$$

$$\frac{a_{n-2}}{a_n} = -\frac{(n-1).n}{2 \cdot (2n-1)} \; ; \quad \frac{a_{n-4}}{a_{n-2}} \Rightarrow \frac{(n-3)(n-2)}{4 \cdot (2n-3)} \; ; \dots$$

$$\therefore y(x) = a_n \left[x^n - \frac{n(n-1)}{2 \cdot (2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} \cdot x^{n-4} - \dots \right]$$

↳ choosing; $a_n = \dfrac{(2n)!}{2^n (n!)^2}$ by putting $r=0$.

$$\Rightarrow \frac{(2n)!}{2^n (n!)^2} \cdot \frac{n(n-1)}{2 \cdot (2n-1)} = \frac{(2n-2)!}{2^n \, (n-2)! \, (n-1)!},$$

$$\Rightarrow \frac{(2n)!}{2^n \cdot (n!)^2} \cdot \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} = \frac{(2n-4)!}{2^n \cdot 2! \, (n-4)! \, (n-2)!}$$

to determine 'p' :- for even 'n', Last term must be 1.

$$\therefore n - 2p = 0$$
$$p = n/2$$

Similarly for odd 'n' :- Last term must be x.

$$\therefore n - 2p = 1$$
$$p = \frac{n-1}{2}$$

$\therefore$ Hence, $P_n(x) = \sum \dots$

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• Rodrigue's formula for $P_n(x)$:-

$$P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

Solⁿ:- Proof:-

$$P_n(x) = \sum_{r=0}^{p} \frac{(-1)^r (2n-2r)!}{2^n \cdot r!\,(n-r)!\,(n-2r)!} x^{n-2r},$$

where p is $\frac{n}{2}$ or $\frac{n-1}{2}$ according to n is even or odd.

$$\frac{d^n}{dx^n}\left(x^{2n-2r}\right) = (2n-2r)(2n-2r-1)\dots\dots\{(2n-2r)-n+1\}x^{n-2r}$$

$$y = x^n$$
$$y_1 = n \cdot x^{n-1}$$
$$y_2 = n\cdot(n-1) \cdot x^{n-2}$$
$$\vdots \qquad \vdots$$
$$y_r = n(n-1)\dots\dots(n-r+1)\,x^{n-r}.$$

⊙ INTERESTING PROPERTIES :-

Orthogonal property of legendre polynomials

$$\int_{-1}^{1} P_m(x) \cdot P_n(x)\,dx = 0 \quad, m \neq n$$
$$= \frac{2}{2n+1}, \quad m = n.$$

Proof:- $$\frac{d}{dx}\left[(1-x^2)\frac{d P_m(x)}{dx}\right] + m(m+1)P_m = 0 \quad \text{—— (1)}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{d P_n(x)}{dx}\right] + n(n+1)P_n = 0 \quad \text{—— (2)}$$

→ $\cancel{n(n+1)P_n \frac{d}{dx}\left[(1-x^2)\frac{dP_m(x)}{dx}\right]} - \cancel{m(m+1)P_m \frac{d}{dx}\left[(1-x^2)\cdot\frac{dP_n(x)}{dx}\right]} = 0$

(2). P_m − (1).P_n :-

$$P_m \cdot \frac{d}{dx}\left[(1-x^2)\cdot\frac{dP_n}{dx}\right] - P_n \cdot \frac{d}{dx}\left[(1-x^2)\cdot\frac{dP_m}{dx}\right] + P_m \cdot P_n\left[n(n+1) - m(m+1)\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)\left\{P_m\cdot\frac{dP_n}{dx} - P_n\cdot\frac{dP_m}{dx}\right\}\right] + P_m\cdot P_n\left[n(n+1)-m(m+1)\right] = 0 \quad —(3)$$

Integrate (3) wrt x, in $[-1,1]$

$$[n(n+1)-m(m+1)]\int_{-1}^{1} P_n\cdot P_m\cdot dx = 0 \quad — (4)$$

[$\because$ first term disappears due to $(1-x^2)$ term]

for $m \neq n$:- $\int_{-1}^{1} P_m\cdot P_n\, dx = 0$

Proof:- To prove, $\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$

$u_n = (x^2-1)^n$, $u_n^{(m)} = \frac{d^m}{dx^m}(x^2-1)^n$

Rodrigue's formula for P_n : $P_n(x) = \frac{1}{2^n\cdot n!}u_n^{(n)}$

$$\therefore \int_{-1}^{1} P_n^2(x)\, dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} u_n^{(n)}\cdot u_n^{(n)}\, dx$$

$$= \frac{1}{2^{2n}\cdot(n!)^2}\left[\underbrace{u_n^{(n)}\cdot u_n^{(n-1)}\Big|_{-1}^{1}}_{0} - \int_{-1}^{1} u_n^{(n+1)}\cdot u_n^{(n-1)}\, dx\right]$$

$$\rightarrow \int_{-1}^{1} P_n(x)^2 dx = -\int_{-1}^{1} u_n^{(n+1)}\cdot u_n^{(n-1)}\, dx$$

$$= (-1)^2\int_{-1}^{1} u_n^{(n+2)}\cdot u_n^{(n-2)}\, dx$$

$$\vdots$$

$$= (-1)^n\int_{-1}^{1} u_n^{(2n)}\cdot u_n^{(0)}\, dx.$$

Recurrence relation:-

(1.) $P'_{n+1}(x) - x \cdot P'_n(x) = (n+1) \cdot P_n(x)$ {∴ $P_n \to$ Legendre Polynomial

→ can be proved by Rodrigue's formula.

→ Leibneitz formula for n^{th} derivative of product:

$$y = uv.$$

$$\therefore y_n \equiv (uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + u \cdot v_n.$$

Proof:
$$P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} \left[(x^2-1)^n \right]$$

LHS: ⇒ $P'_{n+1}(x) - x \cdot P'_n(x)$.

$$\Rightarrow \frac{1}{2^{n+1} \cdot (n+1)!} \frac{d^{n+2}}{d(x)^{n+2}} \left((x^2-1)^{n+1} \right) - \frac{x}{2^n \cdot n!} \cdot \frac{d^{n+1}}{dx^{n+1}} \left[(x^2-1)^n \right].$$

$$\Rightarrow \frac{1}{2^{n+1} \cdot (n+1)!} \left[\frac{d^{n+1}}{dx^{n+1}} \left\{ (n+1) \cdot 2x \cdot (x^2-1)^n \right\} + (n+1) \left(\frac{d^n}{dx^n} (x^2-1)^n \right) \right\}$$

$$\Rightarrow (n+1) \cdot \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n - 2(n+1) \cdot x \cdot \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n \Big]$$

$$\Rightarrow (n+1) P_n(x) = \text{RHS}.$$

(2.) $(n+1) P_{n+1}(x) - (2n+1) x \cdot P_n(x) + n \cdot P_{n-1}(x) = 0$.

Proof.
$$P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} \left[(x^2-1)^n \right]$$

LHS:- $(n+1) P_{n+1}(x) - (2n+1) x \cdot P_n(x) + n \cdot P_{n-1}(x)$

$$= \frac{(n+1)}{2^{n+1} \cdot (n+1)!} \frac{d^{n+1}}{dx^{n+1}} \left[(x^2-1)^{n+1} \right] - \frac{(2n+1) x}{2^n \cdot n!} \frac{d^n}{dx^n} \left[(x^2-1)^n \right] + n P_{n-1}(x).$$

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$$\Rightarrow \frac{1}{2^{n+1} n!}\left[2\cdot(n+1)\cdot\frac{d^n}{dx^n}\left[\underset{u}{(x^2-1)^n}\cdot \underset{v}{x}\right] - 2x\cdot(2n+1)\frac{d^n}{dx^n}\left[(x^2-1)^n\right]\right] + n\cdot P_{n-1}(x).$$

LHS $\frac{1}{2^{n+1}\cdot n!}\left[2(n+1)\left\{x\cdot\frac{d^n}{dx^n}(x^2-1)^n + n\cdot\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right\}\right.$

$$\left. - 2(2n+1)x\frac{d^n}{dx^n}(x^2-1)^n\right] + n\cdot P_{n-1}(x).$$

$$\Rightarrow \frac{1}{2^n\cdot n!}\left[n(n+1)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - nx\cdot\frac{d^n}{dx^n}(x^2-1)^n\right] + nP_{n-1}(x)$$ Now substitute its value.

$$\Rightarrow \frac{1}{2^n(n-1)!}\left[(n+1)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - x\cdot\frac{d^n}{dx^n}(x^2-1)^n + 2n\cdot\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right]$$

$$\Rightarrow \frac{1}{2^n(n-1)!}\cdot\frac{d^{n-1}}{dx^{n-1}}\left[(n+1)(x^2-1)^n - \frac{d}{dx}\left\{x(x^2-1)^n\right\} + n(x^2-1)^n + 2n(x^2-1)^{n-1}\right].$$

$$\Rightarrow \frac{1}{2^n(n-1)!}\cdot\frac{d^{n-1}}{dx^{n-1}}\left[(n+1)(x^2-1)^n - 2nx^2(x^2-1)^{n-1} - (x^2-1)^n + n(x^2-1)^n + 2n(x^2-1)^{n-1}\right].$$

$$\Rightarrow \frac{1}{2^n(n-1)!}\cdot\frac{d^{n-1}}{dx^{n-1}}\left[2n(x^2-1)^n + 2n(1-x^2)(x^2-1)^{n-1}\right]$$

$$\Rightarrow \frac{2n}{2^n(n-1)!}\cdot\frac{d^{n-1}}{dx^{n-1}}\left[(x^2-1)^n - (x^2-1)^n\right] = 0 \quad = \text{RHS}.$$

③ $x P_n'(x) - P_{n-1}'(x) = n\cdot P_n(x)$.

→ We have proved following two recurrence relations:-

a.) $(n+1)P_{n+1}(x) - (2n+1)xP_n + n\cdot P_{n-1} = 0$ ——— (1)

b.) $P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x)$. ——— (2)

Differentiate (1) wrt x,

$$(n+1)P_{n+1}'(x) - (2n+1)\left[xP_n' + P_n\right] + n\cdot P_{n-1}' = 0 \quad \text{——— (3)}$$

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→ Eliminate P'_{n+1} from (2) & (3) :-

③ − (n+1) ②,

$$\Rightarrow -(2n+1)P_n + \{(n+1) - (2n+1)\}xP_n' + nP_{n-1}' = -(n+1)^2 P_n.$$

$$\Rightarrow -n\cdot xP_n' + nP_{n-1}' = (2n+1-n^2-2n-1)P_n$$

$$\Rightarrow \underline{xP_n' - P_{n-1}' = nP_n} \quad \text{———} (4)$$

④ $$P'_{n+1} - P'_{n-1} = (2n+1)P_n \quad \text{———} (5)$$

add. eq^n (2) and eq^n (4).

⑤. $$(x^2-1)P_n' = nxP_n - n\cdot P_{n-1}$$

Replace n by $n-1$ in eq^n (2).

$$P_n' - xP_{n-1}' = nP_{n-1} \quad \text{———} (6)$$

Eliminate P'_{n-1} from (4) and (6) :-

$x\cdot(4) - (6)$:-

$$(x^2-1)P_n' = nxP_n - nP_{n-1}.$$

★. Sturm-Liouville Theory :-

Consider the operator

$$L = p_0(x)\cdot\frac{d^2}{dx^2} + p_1(x)\cdot\frac{d}{dx} + p_2(x).$$

Suppose L acts on $u(x) \in [a,b]$

we assume that the first $2-i$ derivatives of $p_i(x)$ are continous and $p_0(x) \neq 0$ in $[a,b]$.

For any two functions $u(x)$, $v(x)$, the inner product is defined as,

$$\langle v|u\rangle = \int_a^b v^*(x)\cdot w(x)\cdot u(x)\,dx.$$

where, $w(x)$ is some weight function related with eigenvalue equation.

$$\boxed{L u(x) = -\lambda \cdot w(x) \cdot u(x)}$$

λ is eigen value of L (ev) and $u(x)$ is eigen ~~value~~ function of L. corresponding to ev λ.

The operator L is called <u>self-adjoint</u>.

if $\langle v | L u \rangle = \langle L v | u \rangle$

holds $\forall$ u, v.

<u>Demo</u> :-

Legendre eq^n :-

$$(1-x^2)\frac{d^2y}{dx^2} - 2x \cdot \frac{dy}{dx} + n(n+1)y = 0 \quad , \quad x \in [-1,1]$$

$$\Rightarrow \quad L = (1-x^2)\frac{d^2}{dx^2} - 2x \cdot \frac{d}{dx} + n(n+1) .$$

$$\Rightarrow \quad \frac{d}{dx}\left[(1-x^2) \cdot \frac{d}{dx}\right] + n(n+1) .$$

$\Rightarrow$ <u>condition for self-adjointness</u> :-

$$\langle v | L u \rangle = \int_a^b v^* w \, L u \, dx$$

$$= \int_a^b v^* w \, (p_0 u'' + p_1 u' + p_2 u) \, dx$$

$$= \left[p_0 w v^* u' \Big|_a^b - \int_a^b (p_0 v^* w)' u' dx \right] + \int_a^b p_2 v^* w u \, dx .$$

$$+ \left[p_1 w v^* u \Big|_a^b - \int_a^b (p_1 w v^*)' u \, dx \right]$$

$$\Rightarrow \underbrace{\left[p_0 w v^* u' - (p_0 v^* w)' u + p_1 w v^* u \right]_a^b}_{\text{Residue terms.}} - \int_a^b (p_0 v^* w)'' u \, dx - \int_a^b (p_1 w v^*)' u \, dx$$

$$+ \int_a^b p_2 v^* w u \cdot dx .$$

finally we will get $(p_0 w)' = p_1 w$

and b.c, $v^* p_0 w u' \Big|_{x=a} = v^* p_0 w u' \Big|_{x=b}$.

★ Condition for self-adjointness of a class of operators,
$L = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$, $x \in [a,b]$ acting on the function $u(x)$ belonging to same function space.

We know, L is self-adjoint if :-

$$\langle v|Lu\rangle = \langle Lv|u\rangle, \forall\ u, v. \quad —(1)$$

where,

$$\langle v|u\rangle = \int_a^b v^*(x) \cdot w(x) \cdot u(x)\, dx.$$

→ We start from LHS $\langle v|Lu\rangle$ and to remove u', u'' and to get v', v'' we apply integration by parts (for 1st integral, twice and for 2nd integral, once).

$$\langle v|Lu\rangle = \left[v^*w(p_0u' + p_1u) - (p_0v^*w)'u\right]_a^b + \int_a^b \left[(p_0v^*w)'' - (p_1v^*w)' + p_2v^*w\right]u\,dx$$

$$\Rightarrow \left[(p_0v^*w)'' = p_0wv^{*''} + 2(p_0w)'v^{*'} + (p_0w)''v^*\right]$$

$$\Rightarrow \left[(p_1v^*w)' = (p_1w)'v^* + (p_1w)v^{*'}\right].$$

→ Now we start from RHS :-

$$\langle Lv|u\rangle = \int_a^b (p_0v^{*''} + p_1v^{*'} + p_2v^*)\,wu\,dx \qquad [p_i\text{'s are real}$$

If two integrals are equal, then integrals are equal.

$$2(p_0w)'v^{*'} + (p_0w)''v^* - (p_1w)'v^* - (p_1w)v^{*'} = p_1v^{*'}wu.$$

$$\Rightarrow 2\left[(p_0w)' - (p_1w)\right]v^{*'} + \left[(p_0w)'' - (p_1w)'\right]v^* = 0$$

v is arbitrary $\Rightarrow \boxed{(p_0w)' = p_1w}$ → this is the required condition.

The non-integral term in $\langle v|Lu\rangle$,

$\Rightarrow p_0 v^* w u' + p_1 u v^* w - (p_0 w)' v^* u - (p_0 w) v^{*\prime} u.$

$\Rightarrow p_0 v^* w u' + \cancel{p_1 u v^* w} - \cancel{p_1 w v^* u} - p_0 w v^{*\prime} u \quad [\because (p_0 w)' = p_1 w]$

$\Rightarrow p_0 w (v^* u' - v^{*\prime} u)$ which vanishes if we assume a boundary condition (b.c)

$$p_0 w v^* \frac{du}{dx}\Big|_{x=a} = 0, \quad p_0 w v^* \frac{du}{dx}\Big|_{x=b} = 0.$$

here it should be $u.\frac{dv^*}{dx}$ but u, v^* are arbitrary so, no changes if they are interchanged.

or more generally:

$$v^* p_0 w \frac{du}{dx}\Big|_{x=a} = v^* p_0 w \frac{du}{dx}\Big|_{x=b}.$$

★, So, a 2nd order ordinary homogenous linear ODE,

$$p_0 u'' + p_1 u' + p_2 u = 0.$$

is in self-adjoint form if,

$$(p_0)' = p_1$$

e.g., Legendre eqn: $(1-x^2)u'' - 2xu' + n(n+1)u = 0$. is in self-adj form.

Ques: If some ODE is not in self-adjoint form, can we ~~write~~ make it self-adjoint?

Ans:- Yes, by multiplying by suitable factors.

★. $L = p_0 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_2$ is not self-adjoint. Find 'f' so that $f.L$ is self-adjoint. find f.

Hint:- Given $(p_0)' \neq p_1$ Now, $(f p_0)' = f p_1$

Ans:- $f = e^{\int \frac{p_1}{p_0} dx}$.

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$L = p_0 \frac{d^2}{dx^2} + p_1 \cdot \frac{d}{dx} + p_2$, $x \in [a,b]$

here, a, b can be ∞.

→ is not self-adjoint i.e., $(p_0)' \neq p_1$

find $f(x)$, so that (fL) becomes self adjoint.

Solⁿ :- $fL = f p_0 \frac{d^2}{dx^2} + f p_1 \frac{d}{dx} + f p_2$.

$$(fp_0)' = fp_1$$

$$\Rightarrow p_0 f' + p_0' f = f p_1$$

$$\Rightarrow \frac{f'}{f} = \left(\frac{p_1}{p_0} - \frac{p_0'}{p_0}\right)$$

Integrate, both sides.

$$\rightarrow \ln(f) = \int \frac{p_1}{p_0} dx - \ln(p_0)$$

$$\rightarrow f = \cancel{\frac{1}{p_0} \exp\left[\int \frac{p_1}{p_0} dx\right]}$$

$$\Rightarrow \boxed{f = \frac{1}{p_0} \cdot \exp\left[\int \frac{p_1}{p_0} dx\right]} \rightarrow \text{remember}$$

$\Rightarrow L = p_0 \cdot \frac{d^2}{dx^2} + p_1 \cdot \frac{d}{dx} + p_2$, $x \in [a,b]$.

is self-adjoint if $\langle v|Lu\rangle = \langle Lv|u\rangle$

In general, → dagger.

$$\langle v|Lu\rangle = \langle L^{+} v|u\rangle$$

where, L^{+} is called the adjoint operator of L.

Actually, $L^{+} = (L^{*})^{T}$

{ Remember :- $\left(\frac{d}{dx}\right)^{+} = -\frac{d}{dx}$

$$\left(\frac{d^2}{dx^2}\right)^{+} = \frac{d^2}{dx^2}$$

in general, $\left(\frac{d^n}{dx^n}\right)^{+} = (-1)^n \cdot \frac{d^n}{dx^n}$. }

$(AB)^{+} = B^{+}A^{+}$

here, A, B are two

$(A+B)^{+} = A^{+} + B^{+}$

Q.| Find the adjoint for operator $L = p_0 \cdot \frac{d^2}{dx^2} + p_1 \cdot \frac{d}{dx} + p_2$.

$\rightarrow L^{+} = \left(p_0 \cdot \frac{d^2}{dx^2}\right)^{+} + \left(p_1 \cdot \frac{d}{dx}\right)^{+} + p_2^{+}$

$L^{+} = \left(\frac{d^2}{dx^2}\right)^{+} \cdot p_0^{+} + \left(\frac{d}{dx}\right)^{+} \cdot (p_1)^{+} + p_2^{+}$

$L^{+}u = \left[\left(p_0 \cdot \frac{d^2}{dx^2}\right)^{+} + \left(p_1 \cdot \frac{d}{dx}\right)^{+} + p_2\right] u$

$= \left[\left(\frac{d^2}{dx^2}\right)^{+} p_0^{+} + \left(\frac{d}{dx}\right)^{+} \cdot (p_1)^{+} + p_2\right] u.$

$= \frac{d^2}{dx^2}(p_0 u) - \frac{d}{dx}(p_1 u) + p_2 u$

$= \left(p_0' u + p_0 \cdot \frac{du}{dx}\right)'$

$L^{+}u = \left(p_0'' u + 2p_0' \frac{du}{dx} + p_0 \cdot \frac{d^2u}{dx^2}\right) - \left(p_1' u + p_1 \frac{du}{dx}\right) + p_2 u.$

$L^{+} = p_0 \frac{d^2}{dx^2} + (2p_0' - p_1) \cdot \frac{d}{dx} + (p_0'' - p_1' + p_2).$

Q.| Reduce and simplify your L^{+} when $L^{+} = L$ (self-adjoint).

$L^{+} = p_0 \cdot \frac{d^2}{dx^2} + \underbrace{(2p_0' - p_1)}_{\downarrow\, p_0'} \cdot \frac{d}{dx} + \underbrace{(p_0'' - p_1' + p_2)}_{\downarrow\, p_2}$

cause, $(p_0)' = p_1$

$L^{+} = \frac{d}{dx}\left(p_0 \cdot \frac{d}{dx}\right) + p_2.$

Q.) Check whether the foll. diff eq^n are in self-adjoint form or not? If not make it self adjoint. Write down the eigen value ~~for~~ eq^n for the corresponding diff. eq^n.

1.) **Legendre eq^n.**

$$(1-x^2)u'' - 2xu' + \ell(\ell+1)u = 0 . \text{ here, } n = 0,1,2,\ldots$$

$x \in [-1,1]$.

2.) **Associated legendre eq^n:-**

$$(1-x^2)u'' - 2xu' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]u = 0 .$$

$\ell = 0,1,2,\ldots$

$-\ell \le m \le \ell$ are integers.

3.) **Shifted legendre eq^n:-**

$$x(1-x)u'' - (2x-1)u' + \ell(\ell+1)u = 0 , \quad x \in [0,1]$$

$\ell = 0,1,2,\ldots$

4.) **Lagurre eq^n:-**

$$xu'' + (1-x)u' + \alpha y = 0 , \quad x \in [0,\infty)$$

$\alpha = 0,1,2,\ldots$

5.) **Associated Lagurre eq^n:-**

$$xu'' + (\beta+1-x)u' + \alpha y = 0 , \quad x \in [0,\infty)$$

$\alpha, \beta = 0,1,2,\ldots$

6.) **Hermite eq^n:-**

$$u'' - 2xu' + 2ky = 0 , \quad k = 0,1,2,\ldots$$

$x \in (-\infty,\infty)$

7.) **Simple Harmonic oscillator:-**

$$u'' + n^2 u = 0 , \quad x \in [0,\infty)$$

8.)

9.)

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8.) <u>Bessel eq^n</u> :-

$$x^2 u'' + x u' + (k^2 x^2 - \nu^2) u = 0 \quad , \nu, k \in R, \ x \in [0, a] \ a \in R.$$

9.) <u>chebyeev eq^n</u> :

$$(1 - x^2) u'' - x u' + n^2 u = 0 \quad , \ n = 0, 1, 2, \ldots \quad x \in [-1, 1].$$

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Boundary conditions :- and integration interval :

consider again the eigenvalue equation;

$$L\,u(x) = -\lambda\,\omega(x).\,u(x).$$

where, $L = p_0(x)\dfrac{d^2}{dx^2} + p_1(x).\dfrac{d}{d(x)} + p_2(x)$

L will be self-adjoint. if $(p_0\,\omega)' = p_1$

alongwith B.C,

$$v^* p_0 \omega \frac{du}{dx}\Big|_{x=a} = u v^* p_0 \omega \frac{du}{dx}\Big|_{x=b}, \quad \forall \; v, u$$

Sometimes vanishing B.C is also taken.

→ In this case, eigenfunctions of L are orthogonal

i.e., $\langle u|v\rangle \equiv \int_a^b v\, u^* \omega\, dx. = 0, \; \forall\; u, v.$

Moreover, if $\langle u|u\rangle = 1$ holds $\forall\, u$,

then ~~a set of~~ eigenfunctions are orthonormal.

☆. Hermitian operators :-

consider self-adjoint eigenvalue equation.

$$Lu \equiv (p_0 u')' + p_2 u = -\lambda \omega u \;; \quad v^* p_0 \omega u'\Big|_{x=a} = v^* p_0 \omega u'\Big|_*$$

Then, Hermiticity property of the operator L is integral

$$\int_a^b v^*(Lu)\,\omega\, dx = \int_a^b u\,(Lv)^*\,\omega\, dx.$$

→ i.e., $\boxed{\langle u|Lv\rangle = \langle Lv|u\rangle}$

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Properties of Hermitian operators :-

1) Eigenvalues of Hermitian operator is real.

Proof: Let, $\{\psi_n, \lambda_n\}_{n=0}^{N}$ be eigenfunctions and eigenvalues of following eigenvalues problem:

$$A\psi = -\lambda\omega\psi.$$

where, $\omega(x)$ is a non-negative real function.

then, for arbitrary n,

$$A\psi_n = -\lambda_n\omega\psi_n \quad \text{--- (1)}$$

$$\Rightarrow \omega\psi_n^* A\psi_n = -\lambda_n\omega^2|\psi_n|^2 \quad \text{--- (2)}$$

Taking complex conjugate of (1),

$$A^*\psi_n^* = -\lambda_n^*\omega\psi_n^*.$$

$$\Rightarrow \omega\psi_n A^*\psi_n^* = -\lambda_n^*\omega^2|\psi_n|^2 \quad \text{--- (3)}$$

$\therefore$ (2) − (3), and integrating over $[a,b]$.

$$\int_a^b \psi_n^*(A\psi_n)\,\omega\,dx - \int_a^b \psi_n(A\psi_n)^*\,\omega\,dx = (\lambda_n^* - \lambda_n)\int_a^b \omega^2|\psi_n|^2\,dx.$$

$$\Rightarrow 0 = (\lambda_n^* - \lambda_n)\int_a^b \omega^2|\psi_n|^2\,dx, \quad \text{since } A \text{ is Hermitian.}$$

$$\Rightarrow \lambda_n^* = \lambda_n, \quad \text{since integral is non-negative.}$$

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2) A Hermitian operator posses an orthogonal set of eigenfunction.

proof:- Consider the eigenvalue eqn for Hermitian operator A :

$$A\psi = -\lambda\omega\psi. \quad \text{on } [a,b], \ \omega \text{ is real.}$$

Let ψ_m, ψ_n be two distinct ~~eigenval~~ eigenfunctions corresponding to eigenvalues λ_m, λ_n.

Case I:- Non-Degenerate case $(\lambda_m \neq \lambda_n)$

$$\therefore A\psi_m = -\lambda_m\omega\psi_m, \qquad A\psi_n = -\lambda_n\omega\psi_n$$

$$\Rightarrow \omega\psi_n^* A\psi_m = -\lambda_m\omega^2\psi_n^*\psi_m, \qquad \omega\psi_m A^*\psi_n^* = -\lambda_n\omega^2\psi_m\psi_n^*$$

Multiply first by ψ_n^* and complex conjugation of second by ψ_m, and integrating the difference over $[a,b]$,

$$\Rightarrow \int_a^b \psi_n^*(A\psi_m)\,w\,dx - \int_a^b \psi_m (A^*\psi_n^*)\,w\,dx = (\lambda_n-\lambda_m)\int_a^b w^2 \psi_n^* \psi_m \,dx$$

$$\Rightarrow \langle \psi_n | A\psi_m\rangle - \langle A\psi_n | \psi_m\rangle = (\lambda_n-\lambda_m)\int_a^b w^2 \psi_n^* \psi_m \,dx.$$

$$\Rightarrow 0 = (\lambda_n-\lambda_m)\langle \psi_n|\psi_m\rangle \qquad [\because A \text{ is Hermitian so, LHS} = 0.]$$

$$\Rightarrow \langle \psi_n|\psi_m\rangle = 0, \quad \text{for } m\neq n \quad [\lambda_n \neq \lambda_m]$$

Thus $\{\psi_n\}$ is orthogonal set.

Note:- for above proof we replace $w=1$. then only sol^n comes.

Proof: Case 2:- Degenerate case $(\lambda_n = \lambda_m)$:-

In this case, two linearly independent eigenfunctions ψ_n, ψ_m need not be automatically orthogonal. But we can always construct an orthogonal set by Gram-Schmidt orthogonalization method as follows:-

Suppose, $\{\psi_n, \lambda_n\}_{n=0}^N$ be the eigen functions and eigenvalues of operator A. Eigen functions are linear indep. Some of which may be degenerate. We now construct orthonormal set $\{\phi_n,\lambda_n\}$ from given $\{\psi_n\}$

Set, $\tilde{\phi}_0 = \psi_0$, $\phi_0 = \dfrac{\tilde{\phi}_0}{\|\tilde{\phi}_0\|} = \dfrac{\psi_0}{\sqrt{\int_a^b |\psi_0|^2 dx}}$; $\left[\|\phi_0\| = \sqrt{\langle \phi_0|\phi_0\rangle}\right]$

$\tilde{\phi}_1 = \psi_1 + c\phi_0$ s.t., $\langle \tilde{\phi}_1, \phi_0\rangle = 0$.

$$\Rightarrow \langle \tilde{\phi}_1, \phi_0\rangle = \langle \psi_1|\phi_0\rangle + c\langle \phi_0|\phi_0\rangle$$

$$\Rightarrow 0 = \langle \psi_1|\phi_0\rangle + c\|\phi_0\|^2$$

$$\Rightarrow c = -\int_a^b \psi_1^* \phi_0 \,dx.$$

$$\therefore \phi_1 = \frac{1}{\|\tilde{\phi}_1\|}\tilde{\phi}_1.$$

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$\tilde{\Phi}_2 = \Psi_2 + c\phi_0 + d\phi_1$,

$\rightarrow \langle \tilde{\Phi}_2, \phi_0 \rangle = 0$, $\langle \tilde{\Phi}_2, \phi_1 \rangle = 0$.

31/10. Imp. Problem :

★ Legendre Polynomials by Gram-schmidth othogonalisation :

consider : $\{\Psi_n\} = \{x^n\}$, $n = 0, 1, 2, \ldots$ in $[-1, 1]$ with weight function $\omega(x) = 1$. Note that Ψ_n's are non-orthogonal.

⊙ $\tilde{\Phi}_0 = \Psi_0 = 1$, $\phi_0 = \dfrac{\tilde{\Phi}_0}{\|\tilde{\Phi}_0\|} = \dfrac{1}{\sqrt{\int_{-1}^{1} 1^2 dx}} = \dfrac{1}{\sqrt{2}} \equiv \dfrac{1}{\sqrt{2}} P_0$

⊙ $\tilde{\phi}_1 = \Psi_1 + c\phi_0 \Rightarrow \langle \tilde{\phi}_1, \phi_0 \rangle = 0 = \langle \Psi_1, \phi_0 \rangle + c\|\phi_0\|^2$

$\Rightarrow c = -\int_{-1}^{1} \dfrac{x}{\sqrt{2}} dx = 0$

$\Rightarrow \tilde{\Phi}_1 = x$, $\phi_1 = \dfrac{\tilde{\phi}_1}{\|\tilde{\phi}_1\|} = \dfrac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} = \sqrt{\dfrac{3}{2}}\, x \equiv \sqrt{\dfrac{3}{2}}\, P_1(x)$.

⊙ $\tilde{\phi}_2 = \Psi_2 + c\phi_0 + d\phi_1 \Rightarrow \langle \tilde{\phi}_2, \phi_0 \rangle = \langle \Psi_2, \phi_0 \rangle + c = 0$

$\Rightarrow c = -\int_{-1}^{1} \dfrac{x^2}{\sqrt{2}} dx = -\dfrac{2}{3} \cdot \dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{3}$.

$\langle \tilde{\phi}_2, \phi_1 \rangle = \langle \Psi_2, \phi_1 \rangle + d = 0$

$\Rightarrow d = -\int_{-1}^{1} \sqrt{\dfrac{3}{2}}\, x \cdot x^2 dx = 0$

$\therefore \tilde{\phi}_2 = x^2 - \cancel{4}\dfrac{\sqrt{2}}{3} \cdot \dfrac{1}{\sqrt{2}} = x^2 - \dfrac{1}{3}$.

$\therefore \phi_2 = \dfrac{\tilde{\phi}_2}{\|\tilde{\phi}_2\|} = \dfrac{(x^2 - 1/3)}{\sqrt{\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx}} = \dfrac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1) = \sqrt{\dfrac{5}{2}}\, P_2$

$\therefore$ we get $\left\{ \dfrac{P_0}{\sqrt{2}}, \sqrt{\dfrac{3}{2}}\, P_1, \sqrt{\dfrac{5}{2}}\, P_2, \ldots \right\}$

Assignment

→ construct foll. set of polynomials listed below from $\{\psi_n\} = \{x^n\}$ $n=0,1,2,\dots$ by Gram-Schmidth procedure:

Polynomial:	Interval	$w(x)$
→ Legendre $\{P_n\}$	$[-1,1]$	1
→ Shifted Legendre $\{\tilde{P}_n\}$	$[0,1]$	1
→ Chebyshev $\{T_n\}$	$[-1,1]$	$(1-x^2)^{-1/2}$
→ Laguerre $\{L_n\}$	$[0,\infty)$	e^{-x}
→ Associated Legendre $\{P_n^m\}$	$[0,\infty)$	$x^k \cdot e^{-x}$
→ Hermite $\{H_n\}$	$(-\infty,\infty)$	e^{-x^2}

★. TENSOR ANALYSIS

A point in N-dimensional space is a set of N numbers $(x^1, x^2, \dots, x^N)$.

Coordinate transformation :-

Let $(x^1, x^2, \dots, x^N)$ and transformed coordinate are $(\bar{x}^1, \bar{x}^2, \dots)$ and these ~~to~~ are the coordinate of a point in two different frames of reference. Suppose there exists N independent relations between the coordinates of the two systems having the form.

$$\bar{x}^k = \bar{x}^k(x^1, x^2, \dots, x^N) \quad \text{———} (1)$$

$$, k = 1, 2, \dots, N$$

Then conversely to each set of coordinates $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^N)$, there will correspond a unique set $(x^1, x^2, \dots, x^N)$ given by,

$$x^k = x^k(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^N), \quad k = 1, 2, \dots, N. \quad \text{—} (2)$$

The relations (1) ~~and~~ or (2) define a transformation of coordinates from one frame to another.

The summation convention :-

$$a_p x^p \equiv a_1 x^1 + a_2 x^2 + \dots + a_N x^N$$

→ here p is called dummy index.

* **Contravarient and covarient vectors** (tensor of rank 1)

→ tensor of rank 0 is scalar.

Contravariant :-
$$\boxed{\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} \cdot A^q}$$

A^q, $\bar{A}^p$ are contravariant vectors in two frames.

covariant :-
$$\boxed{\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} \cdot A_q}$$

$\bar{A}_p$, A_q are covariant vector in two frames.

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* **Tensors of rank two** :- (known as tensors)

contravariant :
$$\bar{A}^{pq} = \frac{\partial \bar{x}^p}{\partial x^r} \cdot \frac{\partial \bar{x}^q}{\partial x^s} A^{rs}.$$

covariant :
$$\bar{A}_{pq} = \frac{\partial x^r}{\partial \bar{x}^p} \cdot \frac{\partial x^s}{\partial \bar{x}^q} \cdot A_{rs}$$

Mixed :
$$\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^r} \cdot \frac{\partial x^s}{\partial \bar{x}^q} A^r_s.$$

{ contravarient of rank 1 and covarian[t] of rank 1 }

Mixed tensor of rank 5 — **Assignment**

(2) Write down a transformation law for a mixed tensor (contravariant of rank 3, covariant of rank 2).

$$\bar{A}^{prm}_{ij} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} \frac{\partial \bar{x}^m}{\partial x^t} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A^{qst}_{kl}.$$

The Kronecker Delta:-

$$\delta^j_k = \begin{cases} 0 & , \text{ if } j \neq k \\ 1 & , \text{ if } j = k. \end{cases}$$

Prob:- δ^j_k is a mixed tensor of rank 2.

Ans:- coordinates x^p are functions of coordinates $\bar{x}^q$, which are in turn functions of coordinates x^r.

Then By chain rule:-

$$\frac{\partial x^p}{\partial x^r} = \frac{\partial x^p}{\partial \bar{x}^q} \cdot \frac{\partial \bar{x}^q}{\partial x^r}$$

$$\Rightarrow \delta^p_r = \frac{\partial x^p}{\partial \bar{x}^q} \cdot \frac{\partial \bar{x}^q}{\partial x^r}.$$

$$\Rightarrow \delta^p_r = \frac{\partial x^p}{\partial \bar{x}^r} \cdot \frac{\partial \bar{x}^r}{\partial x^p}.$$

$$\Rightarrow \delta^p_r = \frac{\partial x^p}{\partial \bar{x}^q} \left(\frac{\partial \bar{x}^s}{\partial x^r} \cdot \delta^q_s \right)$$

*** Chain Rule:**

$x = x(r, \theta)$, where $r = r($

$\theta = \theta($

$$\frac{\partial x}{\partial u} = \frac{\partial x}{\partial r} \cdot \frac{\partial r}{\partial u}.$$

3/11/18 $\Rightarrow \delta_i^j$ is a mixed tensor of rank 2.

Proof:- To show $\bar{\delta}_i^j = \frac{\partial \bar{x}^j}{\partial x^q} \cdot \frac{\partial x^p}{\partial \bar{x}^i} \cdot \delta_p^q$

By defn:- $\bar{\delta}_i^j = \delta_i^j = 1$, if $i=j$ and, 0 if $i \neq j$.

Now, $\bar{x}^j = \bar{x}^j(x^1, x^2, \dots, x^N)$, $x^i = x^i(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^N)$

By chain rule, $\frac{\partial \bar{x}^j}{\partial \bar{x}^i} = \frac{\partial \bar{x}^j}{\partial x^k} \cdot \frac{\partial x^k}{\partial \bar{x}^i}$.

$$\Rightarrow \bar{\delta}_i^j = \frac{\partial \bar{x}^j}{\partial x^k} \cdot \frac{\partial x^q}{\partial \bar{x}^i} \cdot \delta_q^k$$

* Scalars (or invariants) $\phi = \bar{\phi}$. (tensor of rank 0).

Skew
^ Symmetric tensor :-

If $A_{qs}^{mpr} = A_{qs}^{pmr}$ or (rpm), then we can say that A_{ij}^{pqr} is symmetric wrt p, q.

* Fundamental Operations:-

1.) Addition :- Between two tensors of same rank (of same type) gives a new tensor of same rank (of same type), and.

$$C_q^{mp} = A_q^{mp} + B_q^{mp}$$

N=3.

$$A_i^i \Rightarrow A_1^1, A_1^2, A_1^3, A_2^1, A_2^2, A_2^3, A_3^1, A_3^2, A_3^3.$$

$$B_i^i \Rightarrow B_1^1, B_1^2, \dots, B_2^1, \dots$$

$$C_i^j = A_i^j + B_i^j = A_1^1 + B_1^1 (= C_1^1), A_1^2 + B_1^2 (= C_1^2), \dots$$

2) Subtraction :- b/w two tensors of same rank (of same type) gives a new tensor of same rank (of sand type) and,

$$C_q^{mp} = A_q^{mp} - B_q^{mp}.$$

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3.) Outer multiplication

$$A^{pr}_{q} \cdot B^{m}_{s} = C^{prm}_{qs}.$$

i.e., rank of product tensor is sum of ranks of two given tensors.

4) Contraction :-

Suppose A^{mpr}_{qs} is a tensor. Its rank is 5.

Set $r = s$; $A^{mpr}_{qr} \equiv B^{mp}_{q}$ is a new tensor of rank 3.

further, set $p = q$,

$B^{mq}_{q} \equiv A^{m}$ is a new tensor of rank 1.

N=2. ~~Set i=k~~, $A^{jk}_{1} \longrightarrow A^{11}_{1}, A^{12}_{1}, A^{21}_{1}, A^{22}_{1}, A^{11}_{2}, A^{12}_{2}, A^{21}_{2}, A^{22}_{2}$

Set $i = k$, $A^{jk}_{k} = B^{j} \longrightarrow B^{1} (= A^{11}_{1})$, $B^{2} (=$.

5.) Inner Multiplication:-

Outer multiplication followed by contraction.

e.g., given tensors :- A^{mp}_{q}, B^{r}_{st}.

outer product :- $A^{mp}_{q} \cdot B^{r}_{st}$.

Let, $q = r$, Inner product:- $A^{mp}_{r} \cdot B^{r}_{st}$.

6.) Quotient Law :-

Suppose it is not known that

If an inner product of X with an arbitrary tensor is itself a tensor,

*. Line element and metric tensor:-

A quantity is scalar if it does not

trog

→ In N-dimensional space with co-ordinates $(x^1, x^2, \ldots, x^N)$, the line element dS is a scalar, defined by quadratic form, called the metric form or metric as follows.

3. $$(ds)^2 = g_{pq}\, dx^p\, dx^q$$

→ The quantities g_{pq} are the components of a covariant tensor of rank 2 called the metric tensor of fundamental tensor

→ if $N=3$, then we have cartesian coordinate system (x^1, x^2, x^3).

orthogonal ⇒ Eucledean.

$g_{pq} \rightarrow$ diagonal.

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow (ds)^2 = d(x)^2 + (dx^2)^2 + (dx^3)^2.$$

→ <u>Spherical polar (r, θ, ϕ)</u>

$$g = \begin{pmatrix} \overset{dr\cdot dr}{1} & \overset{dr\cdot d\theta}{0} & \overset{dr\cdot d\phi}{0} \\ \overset{d\theta\cdot dr}{0} & \overset{d\theta\cdot d\theta}{r^2} & \overset{d\theta\cdot d\phi}{0} \\ \overset{d\phi\cdot dr}{0} & \overset{d\phi\cdot d\theta}{0} & \overset{d\phi\cdot d\phi}{r^2 \sin\theta} \end{pmatrix}$$

$$(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin\theta\, (d\phi)^2.$$

⇒ <u>Let us be given the relⁿ in some space:-</u>

$$(ds)^2 = 2(dx^1)^2 + 6(dx^1)(dx^2) - (dx^2)^2 + 7(dx^2 \cdot dx^3)$$

<u>find g.</u>

$$g = \begin{pmatrix} 2 & 3 & 0 \\ 3 & -1 & 7/2 \\ 7/2 & 0 & 0 \end{pmatrix}$$

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Prob:- i) Show that g_{pq} is symmetric.

ii) Show that g_{pq} is a covariant tensor of rank 2.

Sol:- i) $(ds)^2 = g_{pq}\, dx^p dx^q$ is a scalar.

$$(ds)^2 = g_{pq}\, dx^p \cdot dx^q = g_{qp}\, dx^q dx^p \quad (p \to q,\ q \to p)$$

$$= g_{qp}\, dx^p dx^q \quad \text{(ordinary product is commutative.)}$$

$$\Rightarrow 2(ds)^2 = g_{pq}(dx)^p dx^q + g_{qp}\, dx^p dx^q$$

$$(ds)^2 = \frac{1}{2}(g_{pq} + g_{qp})\, dx^p dx^q$$

$$= \tilde{g}_{pq}\, dx^p dx^q .$$

$$\tilde{g}_{pq} = \frac{1}{2}(g_{pq} + g_{qp})$$

↳ symmetric.

ii) $(ds)^2$ is invariant.

$$\Rightarrow \bar{g}_{pq} \cdot d\bar{x}^p \cdot d\bar{x}^q = g_{pq}\, dx^p dx^q .$$

$$g_{ij}\, \partial\bar{x}^i \partial\bar{x}^j = g_{pq}\left(\frac{\partial x^p}{\partial \bar{x}^j} \cdot \partial\bar{x}^j\right)\left(\frac{\partial x^q}{\partial \bar{x}^i} \cdot \partial\bar{x}^i\right)$$

$$\therefore\ dx^p = \frac{\partial x^p}{\partial \bar{x}^j} \cdot \partial\bar{x}^j$$

$\Rightarrow$ $\bar{g}_{pq}$ is a covariant tensor of rank 2.

☆ Conjugate and reciprocal tensor:-

Let $g = |g_{pq}|$ denote the determinant with elements g_{pq} and suppose $g \neq 0$. Define g^{pq} by

$$g^{pq} = \frac{G(p,q)}{g} \quad ; \quad G(p,q) \text{ is cofactor of } g_{pq} \text{ in } g$$

→ g^{pq} is a reciprocal tensor.

Prob: i) g^{pq} is symmetric
ii) g^{pq} is contravariant tensor of rank 2.

Ans i) g_{jk} is symmetric $\Rightarrow$ $G(j,k)$ is symmetric

$\Rightarrow g^{jk} = G(j,k)/g$ is symmetric

ii) Let B^p be an arbitrary contravariant tensor.

$\therefore$ $\cancel{B^p}$ $B_q = g_{pq} \cdot B^p$ is also an arbitrary covariant tensor.

→ Multiply by g^{jq},

$$g^{jq} \cdot B_q = g^{jq} \cdot g_{pq} B^p = \delta^j_p \cdot B^p = B^j$$

$$= B^j = g^{jq} \cdot B_q.$$

$\therefore$ B^j, B_q are arbitrary tensor

By quotient law g^{jq} is a tensor.

Imp

CRISTOFFEL SYMBOL: (These are not generally tensors)

These are of two types:-

1st kind :- $[pq, r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r}\right)$ (space)

2nd kind :- $\left\{ {s \atop pq} \right\} = g^{sr} [pq, r]$ (space)

Ex: Consider $N = 3$,

compute all components of Christoffel symbol of first kind.?

$[11,1], [11,2], [11,3], [12,1], [12,2], [12,3], [13,1], [13,2],$

$[13,3]$.

$$[pq, r] = [qp, r] \quad ; \quad \left\{ {s \atop pq} \right\} = \left\{ {s \atop qp} \right\} \quad ; \quad [pq, r] = g_{rs} \left\{ {s \atop pq} \right\}$$

I proof: $\left\{ {s \atop pq} \right\} = g^{sr}[pq, r]$

Multiplying by g_{ks} $\left\{ {s \atop pq} \right\} = (g_{ks}\, g^{sr})[pq, r]$

$= \delta_k^r \cdot [pq, r]$

$= [pq, k]$.

Problem: 1) Rectangular (cartesian) :-

$g_{pp} = 1$, otherwise $= 0$.

$\therefore \left\{ {s \atop pq} \right\}$

For orthogonal, $g_{pq} = 0$; $p \neq q$.

$\Rightarrow g^{jj} = \frac{1}{g_{jj}}$ (not summed).

$\Rightarrow g^{ij} = 0$; $i \neq j$.

$\Rightarrow \left\{ {s \atop pq} \right\} = g^{sr}[pq, r]$

$p = q = s$. $\therefore \left\{ {p \atop pp} \right\} = g^{sp}[pp, p] = \frac{[pp, p]}{g^{pp}} = \frac{1}{2 \cdot g^{pp}} \cdot \frac{\partial g_{pp}}{\partial x^p}$

$\Rightarrow \frac{1}{2} \cdot \frac{\partial}{\partial x^p}(\ln g_{pp})$

★ Christoffel's symbol:- (Not tensors)

Euclidean space $N=3$.

$(ds)^2 = g_{pq}\, dx^p dx^q$ ($\because$ these are orthogonal coordinates $g_{pq} = 0$, $p \neq q$. $g^{pq} = 0$, $p \neq q$; $g^{pp} = \frac{1}{g_{pp}}$)

cartesian (x, y, z) : $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$
$[x^1 = x, x^2 = y, x^3 = z; g_{pp} = 1; p = 1, 2, 3]$

spherical polar (r, θ, ϕ) : $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2$
$[x^1 = r, x^2 = \theta, x^3 = \phi, g_{11} = 1, g_{22} = r^2, g_{33} = r^2\sin^2\theta]$

cylindrical (ρ, ϕ, z) : $(ds)^2 = (d\rho)^2 + (\rho)^2(d\phi)^2 + (dz)^2$.

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Ex: compute i) $[1\ 3, 2]$ ii) $[2\ 2, 3]$ iii) $\left\{ {2 \atop 2\ 3} \right\}$ for $N=3$ in
a) cylindrical (ρ, ϕ, z)
b) spherical polar (r, θ, ϕ)
c) cartesian.

Ans: a) cylindrical (ρ, ϕ, z).

$$(ds)^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2$$

$x^1 = \rho$, $x^2 = \phi$, $x^3 = z$, $g_{pq} = 0$, $p \neq q$,
$g_{11} = 1$, $g_{22} = \rho^2$, $g_{33} = 1$.

i) $[1\ 3, 2]$: $$\frac{1}{2}\Bigg(\underset{0}{\underset{\parallel}{\frac{\partial g_{12}}{\partial x^3}}} + \underset{0}{\underset{\parallel}{\frac{\partial g_{32}}{\partial x^1}}} - \underset{0}{\underset{\parallel}{\frac{\partial g_{13}}{\partial x^2}}} \Bigg) = 0$$

ii) $[2\ 2, 3]$: $$\frac{1}{2}\Bigg[\underset{0}{\underset{\parallel}{\frac{\partial g_{23}}{\partial x^2}}} + \underset{0}{\underset{\parallel}{\frac{\partial g_{23}}{\partial x^2}}} - \frac{\partial g_{22}}{\partial x^3} \Bigg]$$

$$\Rightarrow -\frac{1}{2} \frac{\partial}{\partial z}(\rho^2) = 0$$

if we take $[2\ 2, 1]$

$$\Rightarrow \frac{1}{2}\left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial x_{22}}{\partial x^1} \right]$$

$$\Rightarrow \frac{1}{2} \frac{\partial(\rho^2)}{\partial(\rho)} = \rho.$$

iii) $$\left\{ {2 \atop 2\ 3} \right\} = g^{2\gamma}[2\ 3, \gamma]$$
$$= \underset{0}{\underset{\parallel}{g^{21}}}[2\ 3, 1] + g^{22}[2\ 3, 2] + \underset{0}{\underset{\parallel}{g^{23}}}[2\ 3, 3]$$

$g^{11} = 1$, $g^{22} = \frac{1}{\rho^2}$, $g^{33} = 1$.

$$\Rightarrow \frac{1}{\rho^2} \cdot \frac{1}{2} \cdot \frac{\partial g_{22}}{\partial x^3} = \frac{1}{2\rho^2} \cdot \frac{\partial}{\partial z}(\rho^2) = 0$$

- g_{pq} is a covariant tensor of rank 2.
- g_{pq} is symmetric.
- δ_i^j is a mixed tensor of rank 2.

$\mathcal{L} = P_0(x)\frac{d^2}{dx^2} + P_1(x)\cdot\frac{d}{dx} + P_2(x)$; $x \in [a,b]$

$\mathcal{L}$ Eigenvalue equation:-

$$\mathcal{L}u = -\lambda\, w(x)\, u \quad , \quad w(x) > 0$$

Defi for self-adjoint of $\mathcal{L}$:- in terms of inner product in in Dirac bra-ket notation $x=\infty$; and correspond integral refers.

Cond^n for self-adjointness; $(P_0 w)' = P_1$;

b.c: a) $P_0 w v^* \frac{du}{dx}\Big|_{x=a} = P_0 w v^* \frac{du}{dx}\Big|_{x=b}$

Adjoint of $\mathcal{L}$: $\mathcal{L}^\dagger$ def^n: $\langle\psi|\mathcal{L}\phi\rangle = \langle\mathcal{L}^\dagger\psi|\phi\rangle$

$$\Rightarrow \int_a^b w\psi(\mathcal{L}\phi)^* dx = \int_a^b \phi^* w(\mathcal{L}^\dagger\psi)$$

computation of $\mathcal{L}^\dagger$ in the form:-

$$\mathcal{L}^\dagger = q_0\frac{d^2}{dx^2} + q_1\frac{d}{dx} + q_2$$

Sol^n:- $\mathcal{L}^\dagger = \left(P_0\frac{d^2}{dx^2}\right)^\dagger + \left(P_1\frac{d}{dx}\right)^\dagger + P_2^\dagger$

$$\left[\because \left(\frac{d^n}{dx^n}\right)^\dagger = (-1)^n\frac{d^n}{dx^n} \quad (AB)^\dagger = B^\dagger A^\dagger\right]$$

$$= \frac{d^2}{dx^2}P_0 - \frac{d}{dx}P_1 + P_2$$

$$\Rightarrow \frac{d}{dx}\left(P_0\frac{d}{dx} + P_0'\right) \cdots$$

★ Hermitian operator $\hat{A}$ Def^n :-

$\langle \Psi_m | A \Psi_n \rangle = \langle A \Psi_m | \Psi_n \rangle$

Self- adjoint form :-

If $\mathcal{L}u = 0$ is not in self- adjoint form,

$f\mathcal{L}u = 0$ is self - adjoint.

where, $f = \frac{1}{P_0} e^{\int \frac{P_1}{P_0} dx}$.

★ Hermite eq^n

$$\mathcal{L} = \frac{d}{dx}\left(P_0 \frac{dy}{dx}\right) + P_2$$

$$\mathcal{L} H_n = -\lambda_n \omega H_n.$$

Q) Show H_0, H_1, H_2 are eigen f^n of $\mathcal{L}$, with eigenvalue $\lambda_1, \lambda_2, \lambda_0$.

★★. Legendre polynomials from $\{x^n\}$, $n = 0, 1, \cdots$, $\omega(x) = 1$.

$[a, b] = [-1, 1]$ using Gram- Schmidth.

$\rightarrow \phi_0, \phi_1, \phi_2$

$\phi_0 = (\quad) P_0$, $\phi_1 = (\quad) P_1$, $\phi_2 = (\quad) P_2$.

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10 Marks Tensor