Lecture 21

Theorem (Egorov) (Littlewood 3rd principle)

Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set $E \subseteq \mathbb{R}^d$ with $m(E) < \infty$. Assume that $f_k(x) \to f(x)$ a.e on E. Given $\varepsilon > 0$ there exists a closed set $A_\varepsilon \subseteq E$ such that $m(E \setminus A_\varepsilon) \leq \varepsilon$ & $f_k \to f$ uniformly on A_ε

proof:- Given that $f_k(x) \to f(x)$ a.e on E, we may assume without loss of generality that $f_k(x) \to f(x)$ $\forall x \in E$.

For each pair $n, k \geq 1$, let

$$E_k^{(n)} = \left\{ x \in E \,/\, |f_j(x) - f(x)| < \frac{1}{n},\ \forall j > k \right\}$$

For a fixed n, $E_k^{(n)} \subseteq E_{k+1}^{(n)}$

For let $x \in E_k^{(n)} \Rightarrow |f_j(x) - f(x)| < \frac{1}{n}$ $\forall j > k$

$$\Rightarrow \quad |f_{k+1}(x) - f(x)| < \frac{1}{n}$$

$$|f_{k+2}(x) - f(x)| < \frac{1}{n}$$

$$\vdots$$

$$\Rightarrow \quad |f_j(x) - f(x)| < \frac{1}{n} \quad \forall\, j > k+1$$

$$\Rightarrow \quad x \in E_{k+1}^{(n)}$$

Thus $E_k^{(n)} \subseteq E_{k+1}^{(n)} \quad \forall k \geq 1.$

$$\Rightarrow E_1^{(n)} \subseteq E_2^{(n)} \subseteq \cdots\cdots \subseteq E$$

and all these sets are measurable.

$|f_k(x) - f(x)| < \varepsilon$ for $k \gg 0$

$$\therefore \lim_{k\to\infty} E_k^{(n)} = \bigcup_{k=1}^{\infty} E_k^{(n)} = E$$

(because $f_k(x) \to f(x)$ $\forall x \in E$)

$$\Rightarrow \lim_{k\to\infty} m\left(E_k^{(n)}\right) = m(E)$$

(proved earlier)

$\Rightarrow$ There exists $k_n \in \mathbb{N}$ such that

$$\left| m\left(E_k^{(n)}\right) - m(E) \right| < \frac{1}{2^n}, \quad \forall k \geq k_n.$$

$$\begin{aligned} &\quad \| \\ &m(E) - m\left(E_k^{(n)}\right) \\ &\quad \| \\ &m\left(E \setminus E_k^{(n)}\right) \quad (\because\ m(E) < \infty) \end{aligned}$$

Thus $\boxed{m\left(E \setminus E_k^{(n)}\right) < \frac{1}{2^n} \quad \forall\ k \geq k_n.}$

(Let $x_0 \in E$. To show $x_0 \in E_k^{(n)}$ $\exists\ k$

(Then $E \subseteq \bigcup_{k=1}^{\infty} E_k^{(n)}$)

We have $f_k(x_0) \to f(x_0)$ as $k \to \infty$

For $\varepsilon = \frac{1}{n} > 0$ there exists $k_0 \in \mathbb{N}$ such that $|f_k(x_0) - f(x_0)| < \varepsilon = \frac{1}{n}$ $\forall\ k \geq k_0$.

$$\Rightarrow \quad x_0 \in E_{k_0 - 1}^{(n)} \Big)$$

We have $|f_j(x) - f(x)| < \frac{1}{n}$, whenever $j > k_n$ & $x \in E_{k_n}^{(n)}$

Choose an integer $N > 0$ so that

$$\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}. \quad \& \text{ let}$$

$$\tilde{A}_\varepsilon = \bigcap_{n \geq N} E^{(n)}_{k_n}.$$

Now $m(E \setminus \tilde{A}_\varepsilon) = m\left(\bigcup_{n \geq N} (E \setminus E^{(n)}_{k_n}) \right)$

$$\leq \sum_{n \geq N} m(E \setminus E^{(n)}_{k_n})$$

$$\leq \sum_{n \geq N} \frac{1}{2^n}$$

$$< \frac{\varepsilon}{2}.$$

Thus $m(E \setminus \tilde{A}_\varepsilon) < \varepsilon/2$.

Claim:- $f_k \to f$ uniformly on $\tilde{A}_\varepsilon$.

proof of the Claim:- Let $\varepsilon' > 0$.

To show: there exists M such that
$\uparrow$
N

$$|f_k(x) - f(x)| < \varepsilon' \quad \forall k \geq M, \quad \forall x \in A'_\varepsilon.$$

choose $n \geq N$ such that $\frac{1}{n} < \varepsilon'$.

& note that $x \in \tilde{A}_\varepsilon$ implies that

$$x \in E_{k_n}^{(n)}$$

$\therefore$ $|f_j(x) - f(x)| < \frac{1}{n} < \varepsilon'$, whenever $j > k_n$.

i.e., $|f_j(x) - f(x)| < \varepsilon'$ $\forall j > k_n$ & $\forall x \in \tilde{A}_\varepsilon$.

$\Rightarrow$ $f_j \to f$ uniformly on $\tilde{A}_\varepsilon$ as $j \to \infty$.

this completes the proof the claim.

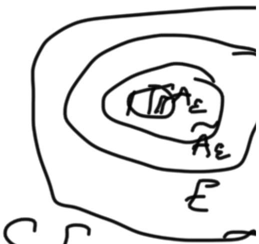
Now $\tilde{A}_\varepsilon$ is measurable.

then there exists a closed set $A_\varepsilon \subseteq \tilde{A}_\varepsilon$ such that $m(\tilde{A}_\varepsilon \setminus A_\varepsilon) < \varepsilon/2$.

Also $f_k \to f$ uniformly on $A_\varepsilon \subseteq \tilde{A}_\varepsilon$.

& $E \setminus A_\varepsilon \subseteq (E \setminus \tilde{A}_\varepsilon) \cup (\tilde{A}_\varepsilon \setminus A_\varepsilon)$

$$\Rightarrow m(E \setminus A_\varepsilon) \leq m(E \setminus \tilde{A}_\varepsilon) + m(\tilde{A}_\varepsilon \setminus A_\varepsilon)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$



Thus there exists a closed set $A_\varepsilon \subseteq E$ such that $m(E \setminus A_\varepsilon) < \varepsilon$ &

$f_k \to f$ uniformly on A_ε, as $k \to \infty$.

This completes the proof.

<u>Proposition</u>:- (<u>patching lemma</u> for continuous functions)

Let $B_i \subseteq \mathbb{R}^d$ be closed sets, $1 \leq i \leq s$.
(open)

Suppose $f : \bigcup_{i=1}^{s} B_i \to \mathbb{R}$ is a function

such that $f|_{B_i} : B_i \to \mathbb{R}$, (i.e., $f|_{B_i}(x) = f(x)$)

are continuous, $\forall\, i = 1, 2, \ldots, s$.

Then f is a continuous function.

proof:-

(Recall:- A function $f: X \to Y$, X, Y are metric spaces is said to be Continuous, if $f^{-1}(U)$ is open in X, for $U \subseteq Y$ are open)

Let $U \subseteq \mathbb{R}$ be any open set.

$$f^{-1}(U) = \left\{ x \in \bigcup_{i=1}^{s} B_i \;\middle|\; f(x) \in U \right\}.$$

$$= \bigcup_{i=1}^{s} \left\{ x \in B_i \;\middle|\; f(x) \in U \right\}$$

$$= \bigcup_{i=1}^{s} \left\{ x \in B_i \;\middle|\; f|_{B_i}(x) \in U \right\}$$

$$= \bigcup_{i=1}^{s} \left(f|_{B_i}\right)^{-1}(U)$$

But $f|_{B_i}$ are Continuous, & hence $\left(f|_{B_i}\right)^{-1}(U)$ are open

$\therefore$ $f^{-1}(U) = \bigcup_{i=1}^{s} \left(f|_{B_i}\right)^{-1}(U)$ is open.

$\therefore$ f is Continuous.