Homomorphisms

A homomorphism is a map $h : \Sigma^* \to \Gamma^*$ s.t. $\forall\, x, y \in \Sigma^*$

$$h(xy) = h(x)h(y)$$

$\Rightarrow$ $h(\varepsilon) = \varepsilon$

$$|h(\varepsilon)| = |h(\varepsilon\varepsilon)|$$
$$= |h(\varepsilon)h(\varepsilon)|$$
$$= |h(\varepsilon)| + |h(\varepsilon)|$$

$$\Rightarrow \quad h(\varepsilon) = \varepsilon$$

h acting on Σ gives all values of h acting on Σ^*

If $A \subseteq \Sigma^*$
we define $h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$
and if $B \subseteq \Gamma^*$
$$h^{-1}(B) = \{x \mid h(x) \in B\} \subseteq \Sigma^*$$

The set $h(A)$ is called the image of A under h,
and " " $h(B)$ " " " pre-image of B under h.

Lemma 1: Let $h : \Sigma^* \to \Gamma^*$ be a homomorphism. If $B \subseteq \Gamma^*$ is regular, then the preimage $h^{-1}(B)$ under h is also regular.

Lemma 2: Let $h : \Sigma^* \to \Gamma^*$ be a homomorphism. If $A \subseteq \Sigma^*$ is regular, then $h(A)$, the image of A under h, is also regular.

$M = (Q, \Gamma, \delta, s, F)$, s.t. $L(M) = B$.
$M' = (Q, \Sigma, \delta', s, F) \rightsquigarrow h^{-1}(B)$

$$\delta'(q, a) = \hat{\delta}(q, h(a))$$

$\hat{\delta}'(q, x) = \hat{\delta}(q, h(x))$ by induction on $|x|$

Base case: $x = \varepsilon$

$$\hat{\delta}'(q, \varepsilon) = q = \hat{\delta}(q, \varepsilon) = \hat{\delta}(q, h(\varepsilon))$$

Induction step: $\hat{\delta}'(q, x) = \hat{\delta}(q, h(x))$

$$\begin{aligned}
\hat{\delta}'(q, xa) &= \delta'(\hat{\delta}'(q, x), a) \quad \text{def}^{\underline{n}} \text{ of } \hat{\delta}' \\
&= \delta'(\hat{\delta}(q, h(x)), a) \\
&= \hat{\delta}(\hat{\delta}(q, h(x)), a) \\
&= \hat{\delta}(q, h(x)h(a)) \rightarrow \underline{\text{why?}} \\
&= \hat{\delta}(q, h(xa))
\end{aligned}$$

$$\begin{aligned}
x \in L(M') &\iff \hat{\delta}'(s, x) \in F \\
&\iff \hat{\delta}(s, h(x)) \in F \\
&\iff h(x) \in L(M) \\
&\iff x \in h^{-1}(L(M))
\end{aligned}$$

$\underline{a^n b^n}$ is not regular

$h(a) = a$
$h(b) = a$

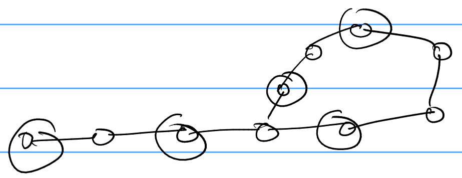
$\rightsquigarrow a^{2n}$

$b^{2n} \rightsquigarrow a^{2n}$

Ultimate periodicity

The set $U \subseteq \mathbb{N} \cup \{0\}$ is said to be <u>ultimately periodic</u> if $\exists$ numbers $n \geq 0$ and $p \geq 0$ s.t. $\forall\, m \geq n$, $m \in U$ iff $m+p \in U$. The number p is called the <u>period</u> of U.

Lemma: Let $A \subseteq \{a\}^*$. Then A is regular iff the set $\underbrace{\{m \mid a^m \in A\}}_{\text{set of lengths of strings in } A}$ is ultimately periodic.



Corollary: Let A be any regular set over any finite alphabet Σ (not necessarily a singleton alphabet). Then the set of lengths of strings in A is ultimately periodic.

Proof:
$$h : \Sigma \to \{a\}$$
$$h(b) = a \quad \forall\, b \in \Sigma$$
Then $h(x) = a^{|x|}$.

Lengths of strings in $h(A)$

Since A is regular, $h(A)$ (image of A under h) is also regular

Q.E.D.