Linear Algebra

Lecture 12

Linear transformation

Example:

Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation.

Let $\{e_1, e_2\}$ be the ordered basis under consideration.

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta\, x_1 - \sin\theta\, x_2 \\ \sin\theta\, x_1 + \cos\theta\, x_2 \end{pmatrix}$$

Matrix associated with this transformation.

$$T(e_1) = \cos\theta\, e_1 + \sin\theta\, e_2$$

$$T(e_2) = -\sin\theta\, e_1 + \cos\theta\, e_2$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ represents perpendicular reflection of every vector in $\mathbb{R}^2$ thro' the line $y = x$.

(Consider the standard basis $\{e_1, e_2\}$ as ordered basis).

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example :

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation defined as

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 - 2x_2 \\ x_1 + x_2 \end{pmatrix}.$$

Construct the matrix A corresponding to the transformation T where $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is the ordered basis for $\mathbb{R}^2$ and $\{e_1, e_2, e_3\}$ is the ordered basis for $\mathbb{R}^3$.

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1.e_1 + 2e_2 + 1e_3$$

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 1.e_1 + 0.e_2 + 2.e_3$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$$

Construct the matrix $\hat{A}$ corresponding to the transformation T when $\{e_1, e_2\}$ is the ordered basis for $\mathbb{R}^2$ & $\{e_1, e_2, e_3\}$ is the ordered basis for $\mathbb{R}^3$.

$$Te_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1e_1 + 2e_2 + 1e_3$$

$$Te_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = 0.e_1 - 2e_2 + 1.e_3$$

$$\hat{A} = \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ 1 & 1 \end{bmatrix}$$

<u>Question:</u> Let $T: \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ be a linear transformation with matrix representation:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

under the ordered basis $\{1, x, x^2, x^3\}$

Then compute $T(\overbrace{3x^3 - 7x^2 + 9x - 13}^{P_3(\mathbb{R})})$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -13 \\ 9 \\ -7 \\ 3 \end{pmatrix} = \begin{array}{l} -13T(1) + 9T(x) \\ \quad -7T(x^2) \\ \quad\quad +3T(x^3) \end{array}$$

$T(1)$ $\quad$ $T(x)$

$$\underbrace{\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{C} = c_1a_1 + c_2a_2 + \cdots + c_na_n$$

Exercise: $V = \mathbb{R}^2$

$B_1 = \{e_1, e_2\}$

$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

Suppose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is co-ordinate of vector in $\mathbb{R}^2$ with respect to B_1.

Compute the coordinate w.r.t. B_2 ??

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 1 \cdot e_1 + 0 \cdot e_2 \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= 1 \cdot e_1 + 1 \cdot e_2 \end{aligned}}$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{B_2} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Change of Basis.

$$A \boxed{B_2^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

$$\downarrow$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Let V be an n-dimensional vector space.
Let $B_1 = \{v_1, \dots, v_n\}$ and
$B_2 = \{u_1, \dots, u_n\}$ be two ordered
bases of V.

Let $x = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ be the coordinate vector of x w.r.t. B_1

and $x = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$ be the coordinate vector of x w.r.t. B_2.

$$x = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_n u_n$$

$$x = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$= U \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

<u>Theorem</u>; Let V & W be vector spaces over a field $\mathbb{F}$ and let $T_1, T_2 : V \to W$ be linear maps.

(a) For any $a \in \mathbb{F}$, $aT_1 + T_2$ is also linear.

for some $\alpha \in \mathbb{F}$, $x, y \in V$

<u>Pf:</u> $\underbrace{(aT_1 + T_2)}_{T}(\alpha x + y) = \alpha T(x) + T(y)$

$$(aT_1 + T_2)(\alpha x + y)$$

$$= aT_1(\alpha x + y) + T_2(\alpha x + y)$$

$$= a\alpha T_1(x) + aT_1(y) + \alpha T_2(x) + T_2(y)$$

$$= \alpha[aT_1(x) + T_2(x)] + aT_1(y) + T_2(y)$$

$$= \alpha(aT_1 + T_2)(x) + (aT_1 + T_2)(y)$$

∎

Let V & W be vector spaces over a field $\mathbb{F}$. Let $\mathcal{L}(V, W)$ denote the set of all linear transformations from V to W. Then prove that $\mathcal{L}(V, W)$ is also a vector space over $\mathbb{F}$.

(Note $0 \in \mathcal{L}(V, W)$ is infact the zero function / zero linear transformation from V to W.

<u>Theorem</u>: Let V and W be finite dimensional vector spaces. Then $\mathcal{L}(V, W)$ is also finite dimensional.

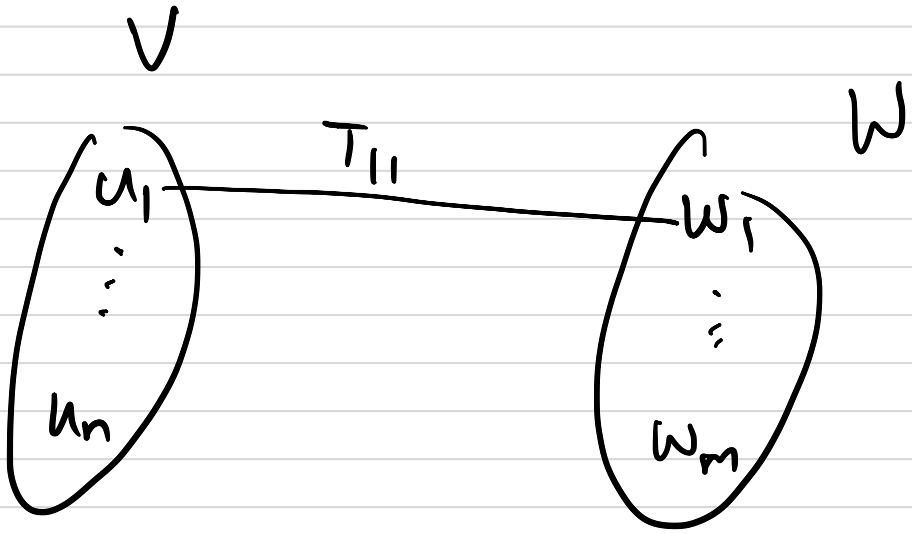
<u>Proof</u>: Let $\{v_1, \ldots, v_n\}$ be a basis for V and let $\{w_1, \ldots, w_m\}$ be a basis for W.

Define linear transformation

$T_{ij} : V \longrightarrow W$ as

$T_{ij}(v_i) = w_j$ for $i = 1, \dots, n$
$j = 1, \dots, m$

$T_{ij}(v_k) = 0$ for $k = 1, \dots, n$
$k \neq i$

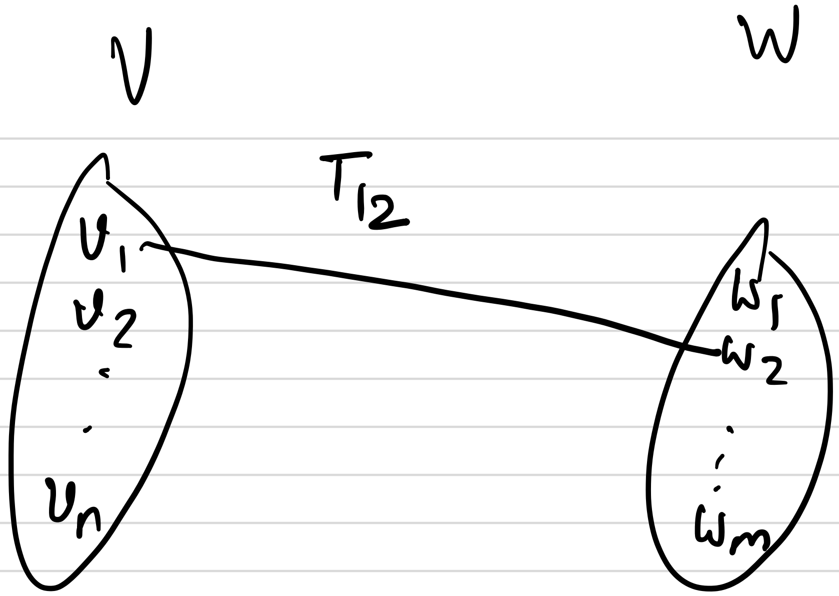


$T_{11}(v_1) = w_1$

$T_{11}(v_2) = 0$

$\vdots$

$T_{11}(v_n) = 0$



$$T_{12}(v_1) = w_2 \;, \; T_{12}(v_2) = 0, \ldots, T_{12}(v_n) = 0$$

$$T_{ij} \in \mathcal{L}(V, W) \quad \text{for} \quad \begin{array}{l} i = 1, \ldots, n \\ j = 1, \ldots, m \end{array}$$

$$\mathcal{L}(V, W) \overset{??}{=} \operatorname{span} \left\{ T_{ij} \;;\; \begin{array}{l} i = 1, \ldots, n \\ j = 1, \ldots, m \end{array} \right\}$$

Is $\left\{ T_{ij} \;;\; \begin{array}{l} i = 1, \ldots, n \\ j = 1, \ldots, m \end{array} \right\}$ linear independent set ??

Independence:

Consider a linear combination

$$\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} T_{ij} = 0$$

for $a_{ij} \in \mathbb{F}$.

To prove: $a_{ij} = 0$ for every i & j

Evaluate both sides at v_i, $i = 1, \dots, n$

$$\sum_{j=1}^{m} a_{ij} w_j = 0$$

$$\Rightarrow a_{ij} = 0 \qquad j = 1, \dots, m$$

$$a_{ij} = 0$$

$\Rightarrow \left\{ T_{ij} : \begin{matrix} i = 1, \dots, n \\ j = 1, \dots, m \end{matrix} \right\}$ is a linearly independent set.

To prove: $\mathcal{L}(V, W) = \text{span}\{T_{ij}\}$

Let $T \in \mathcal{L}(V, W)$

$$T(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$$

$$T(v_2) = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m$$

$$\vdots$$

$$T(v_n) = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m$$