MA 51002: Measure Theory and Integration
Assignment - 2 (Spring 2021)
Lebesgue integration
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Note: Unless otherwise stated, $\int f$ will denote the Lebesgue integral of a measurable function f.

1. Show that if f is a non-negative measurable function, then $f = 0$ a.e. if and only if $\int f\,dx = 0$. If f is any measurable function, then show that only one side implication is true.

2. Show that for a bounded measurable function $f : S \to \mathbb{R}$ with $m(S) < \infty$, one has
   (i) $\int_S f \leq \int_S |f|$
   (ii) if $f \geq 0$, then $\int_S f \geq 0$
   (iii) if $f \geq 0$ a.e., then $\int_S f \geq 0$
   (iv) if $f \geq 0$ a.e., then $\int_A f \geq 0$ for any $A \subset S$
   (v) show that if $f(x) > 0$ a.e. on A and $\int_A f = 0$, then $m(A) = 0$.

3. If f and g be non-negative measurable functions then prove that
$$\int (f+g)\,dx = \int f\,dx + \int g\,dx$$
   Prove the result when f and g are any two integrable functions.

4. Let $\{f_n\}$ be a sequence of non-negative measurable functions. Prove that
$$\int \sum_{n=1}^{\infty} f_n\,dx = \sum_{n=1}^{\infty} \int f_n\,dx.$$
   When the corresponding result is true for a sequence of arbitrary integrable functions?

5. If E and F are two disjoint measurable sets and f be a non-negative measurable function, then prove that
$$\int_{E\cup F} f\,dx = \int_E f\,dx + \int_F f\,dx.$$
   Prove the result when f is any integrable function.

   Using the problem 4, prove that if $\{E_n\}$ be a sequence of pairwise disjoint measurable sets and f be a non-negative measurable function, then
$$\int_{\cup_{n=1}^{\infty} E_n} f\,dx = \sum_{n=1}^{\infty} \int_{E_n} f\,dx.$$

6. Explain why the Bounded Convergence Theorem applies to the sequences on $[0,1]$
   (i) $f_n(x) = e^{-nx^2}$
   (ii) $f_n(x) = \arctan(nx)$.
   Conclude from the Bounded Convergence Theorem about the $\lim_{n\to\infty} \int_0^1 f_n\,dx$ in each cases.

7. If $\{f_n\}$ be a sequence of bounded measurable functions on S converging pointwise to a function f, then is it always true that f must be bounded? If $\{f_n\}$ converge to f uniformly, then show that f is bounded.
   Give an example of a sequence of bounded measurable functions $\{f_n\}$ on S converging pointwise to a bounded function f, but $\lim_{n\to\infty} \int_S f_n\,dx \neq \int_S f\,dx$. Explain why the Bounded Convergence Theorem is not applicable here.

8. (a) Let $f(x) = \frac{1}{x}\sin(\frac{1}{x})$, on $[0,1]$.
   (i) Is f improperly Riemann integrable?
   (ii) Is $|f|$ improperly Riemann integrable?
   (iii) Is f Lebesgue integrable?

(b) Answer the above question when $f(x) = \frac{\sin x}{x}$ on $[0, \infty)$.

(c) Observing (b) prove that, if a function $f : [a, \infty) \to \mathbb{R}$ is such that f is Riemann integrable on $[a, b]$, for all $b > a$ and the improper Riemann integral $R \int_a^\infty f(x)\, dx$ is conditionally convergent, then f is not Lebesgue integrable over $[a, \infty)$.

9. Let the improper Riemann integral $R \int_a^\infty f(x)\, dx$ is absolutely convergent, i.e. $R \int_a^\infty |f(x)|\, dx < \infty$. Then prove that $f : [a, \infty) \to \mathbb{R}$ is Lebesgue integrable and the two integrals are same, i.e.
$$R \int_a^\infty f(x)\, dx = \int_a^\infty f(x)\, dx.$$

10. Prove that if f is an (Lebesgue) integrable function and $A = \{x : f(x) = \pm\infty\}$, then $m(A) = 0$.