Date
05/11/2019

# Lecture 13



## Inner multiplication :-

Using the process of outer multiplication of two tensors that is followed by a contraction we obtain a new tensor, called an "inner multiplication" or "inner product" of the given tensors. This process is called "inner multiplication"

eg., For the given tensors $A^{ijk}_{lm}$ & $B^{np}_{q}$, the outer multiplication / product is

$$A^{ijk}_{lm} \cdot B^{np}_{q} .$$ Let $m = p$

the inner product

$$A^{ijk}_{lm} B^{nm}_{q}.$$

Further setting $m=p$, $l=n$ & $q=k$

then another inner product is obtained as

$$A^{ijk}_{lm} B^{lm}_{k}.$$

Note :- It may be noted that inner & outer multiplication of tensors is commutative & associative

(E)

f. Quotient law :-

Suppose it is not known whether a quantity X is a tensor or not. If an inner product of X with an arbitrary tensor is itself a tensor, then the quantity X is also a tensor. This is called the 'Quotient law'

Ex/ Show that Kronecker delta is a mixed tensor of rank two. Also, show that it is invariant (i.e., it has the same components in every co-ordinate system).

Sol$^n$:- 1st part:- To show that $\delta^i_j$ (Kronecker delta) is a mixed tensor of rank two.

Let $u^k$ be an arbitrary vector.

Clearly, $\delta^i_j \cdot u^j = u^i$,

which is a contravariant tensor (or, vector) of rank one.

Thus, the inner product of $\delta^i_j$ with an arbitrary tensor/vector is a tensor.

Hence, by <u>Quotient law</u>, $\delta^i_j$ is tensor. As it has two distinct indices, one in upper & the other in lower position, hence $\delta^i_j$ is a mixed tensor of rank 2.

2nd part:-

To show that $\delta^i_j$ has the same components in every co-ordinate system, it is sufficient to show that $\bar{\delta}^i_j = \delta^i_j$.

By transformation law $x^i$ to $\bar{x}^i$ system,

$$\bar{\delta}^i_j = \frac{\partial \bar{x}^i}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^j} \delta^l_m$$

$$= \left(\delta^l_m \frac{\partial \bar{x}^i}{\partial x^l}\right) \frac{\partial x^m}{\partial \bar{x}^j}$$

$$= \frac{\partial \bar{x}^i}{\partial x^m} \cdot \frac{\partial x^m}{\partial \bar{x}^j} = \frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \delta^i_j$$

$$\therefore \boxed{\bar{\delta}^i_j = \delta^i_j}$$

HW

EX1/ If $A_M^{PQ}$ & $B_M^{PQ}$ are tensors then show that their sum & differences are tensors.

EX2/ The tensor product of the tensors of the type $(M, S)$ & $(M', S')$ is a tensor of the type $(M+M', S+S')$. Prove it.

** EX3/ A tensor $A_{ijk}$ is anti-symmetric in suffixes i & j, show that it has $\frac{n^2(n-1)}{2}$ independent components, n-2 indices.

Ex4/ Show that the contraction enables us to obtain a tensor of order $(M-2)$ from a mixed tensor of order $M$.

Q) State & prove the Quotient law of tensors.

Soln:- Quotient law :-

Statement :- A set of $N^P$ functions of co-ordinates $x^i$ $(i=1-N)$ form the components of a tensor of order $P$, provided that an inner multiplication (or, product)

of these $f^{ns}$ with an arbitrary tensor is itself a tensor.

proof :- It will suffice to consider the proof for the following particular case.

The set of $N^3$ - functions $A^{ijk}$ form the components of a tensor of the type by its indices. if

$$A^{ijk} \cdot B^{p}_{ij} = C^{pk} \longrightarrow (1)$$

provided that $B^{p}_{ij}$ is an arbitrary tensor & $C^{pk}$ is a tensor.

The transformed quantities referred to a system of co-ordinates $x^i$ to $\bar{x}^i$ $(i = 1 - N)$, satisfy the eq$^n$

$$\bar{A}^{ijk} \bar{B}^{p}_{ij} = \bar{C}^{pk} \quad \rightarrow (2)$$

This becomes by transformation law

$$\bar{A}^{ijk} \frac{\partial \bar{x}^p}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} B^{l}_{mn}$$

$$= \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^M} C^{qM}.$$

$$= \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^M} A^{ijM} B^{q}_{ij} \quad [\text{by } (1)]$$

(~~by (1)~~

With a change of dummy suffices [$q$ by $l$, $(i,j)$ by $(m,n)$]

$$\frac{\partial \bar{x}^p}{\partial x^l}\left[\bar{A}^{ijk}\frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j} - A^{mnq}\frac{\partial \bar{x}^k}{\partial x^q}\right]B^l_{mn} = 0$$

On multiplying this $\rightarrow$ (3) eqn (3) by $\frac{\partial x^s}{\partial \bar{x}^p}$ & summing over $p$ from 1 to $N$, we get

$$\left[\bar{A}^{ijk}\frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j} - A^{mnq}\frac{\partial \bar{x}^k}{\partial x^q}\right]B^s_{mn} = 0$$

$$= \delta^s_l B^l_{mn}\left[\because \frac{\partial \bar{x}^p}{\partial x^l}\frac{\partial x^s}{\partial \bar{x}^p}B^l_{mn} = \frac{\partial x^s}{\partial x^l}B^l_{mn} \rightarrow (4)\right]$$

$= B^s_{mn}$

Since $B^{s}_{mn}$ is an arbitrary tensor, we can arrange that only one of its components differs from zero. Now, each component of $B^{s}_{mn}$ may be chosen in turn as that one which does not vanish, ∴ this shows that the expression in brackets in eqn (4) is identically zero. ie,

$$\bar{A}^{ijk}\frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j} - A^{mnq}\frac{\partial \bar{x}^k}{\partial x^q} = 0$$

$$\Rightarrow \bar{A}^{ijk} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} = A^{mnn} \frac{\partial \bar{x}^k}{\partial x^n} \quad \rightarrow (5)$$

An inner multiplication of eq$^n$ (5) by $\frac{\partial \bar{x}^s}{\partial x^m} \frac{\partial \bar{x}^t}{\partial x^n}$ yields the result

$$\boxed{\bar{A}^{stk} = \frac{\partial \bar{x}^s}{\partial x^m} \frac{\partial \bar{x}^t}{\partial x^n} \frac{\partial \bar{x}^k}{\partial x^n} A^{mnn}} \quad \rightarrow (6)$$

$$\left[ \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} \cdot \frac{\partial \bar{x}^s}{\partial x^m} \frac{\partial \bar{x}^t}{\partial x^n} \right]$$

$$= \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial \bar{x}^s}{\partial x^m} \cdot \frac{\partial x^n}{\partial \bar{x}^j} \frac{\partial \bar{x}^t}{\partial x^n}$$

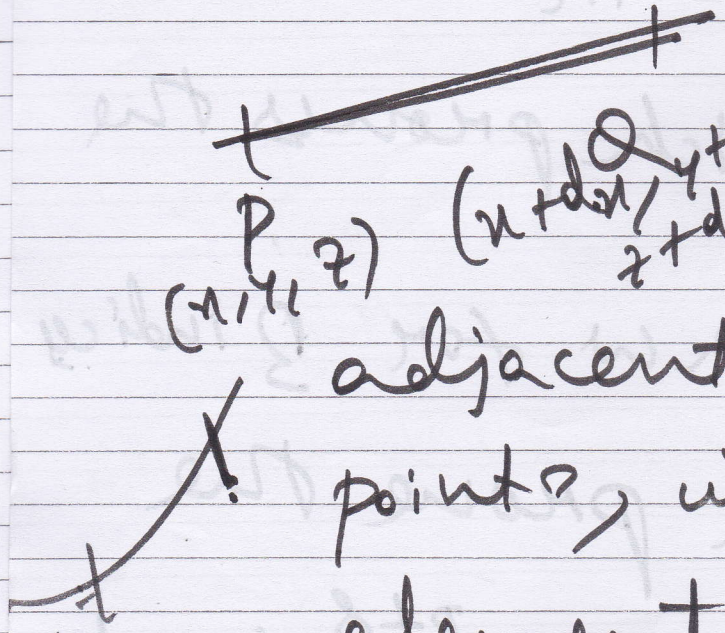
$$= \delta^s_i \, \delta^t_j =$$

Thus, $A^{mnr}$ is a tensor of the third order & contravariant in all its indices, which proves the quotient law for 3 indices

Slly, we can prove the theorem for $N^{p+s}$ functions

$$A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_s}$$

# The Line element & Metric Tensor




A formula which expresses the distance betn two adjacent or neighbouring points, is called a 'line element' or 'metric'

eg., The differential of arc-length ds in rectangular co-ordinates, betn two adjacent points (x, y, z) & (x+dx, y+dy, z+dz) is obtained from

$$ds^2 = dx^2 + dy^2 + dz^2$$

By transforming to general curvilinear co-ordinates, this becomes

$$ds^2 = \sum_{p=1}^{3} \sum_{q=1}^{3} g_{pq}\, du_p\, du_q,$$

which spaces are called 'three dimensional Euclidean spaces'

Its generalization to N dimensional space with co-ordinates $(x^1, x^2, \ldots, x^N)$ immediately follows.

We define the <u>line element</u> $ds$ in this space to be given by the quadratic form, called the <u>metric form</u> or <u>metric</u>

$$ds^2 = \sum_{p=1}^{N} \sum_{q=1}^{N} g_{pq}\, dx^p\, dx^q$$

or, using the summation convention

$$ds^2 = g_{pq}\, dx^p\, dx^q .$$

Thus, if the distance $ds$ betn two neighbouring points with

co-ordinates $x^i$ & $x^i + dx^i$ in space $V_N$, is expressed by the quadratic differential form

$$\boxed{ds^2 = g_{ij}\, dx^i\, dx^j} \quad (1)$$

Riemannian metric

where $g_{ij}$ are functions of $x^i$, subject only to the restriction

$g = |g_{ij}| \neq 0$, the space
$= \det(g_{ij})$ (i.e., $g^{-1}$ exist)

is said to be a <u>Riemannian space</u>

The quantity $g_{pq}$ or $g_{ij}$ are the components of a <u>covariant</u> tensor of rank two & these ($g_{ij}$) are called the '<u>Metric Tensor</u>' or, '<u>Fundamental Tensor</u>'

Also, $d\bar{s}^2 = ds^2$

or, $ds^2 = d\bar{s}^2$

from $x^i$ to $\bar{x}^i$, co-ordinate system.

[ie, $ds$ is an <u>invariant</u>

Also, from the Quotient law, it follows that

$(g_{ij} + g_{ji})$ is a covariant tensor of rank 2.

Also, $g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) + \frac{1}{2}(g_{ij} - g_{ji})$

so that the contribution of $\frac{1}{2}(g_{ij} - g_{ji})\, dx^i dx^j$ to $ds^2$ is zero.

Hence, w.o.l.o.g, we can assume that $g_{ij}$ is symmetric. Thus,

Thus, $g_{ij}$ is covariant symmetric tensor of the second order, called the Fundamental tensor of the Riemannian space.

Now, if we compare eqn (1) with

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

then all the components of the fundamental tensor $g_{ij}$ are zero, except $g_{11} = g_{22} = g_{33} = 1$.

It is clear that the metric of a Euclidean space is positive definite

ie, $ds^2 = 0$ when $dx^1 = 0 = dx^2 = dx^3$

& can only take positive values for all other real values of $dx^1, dx^2, dx^3$.

---

eg, In the case of special theory of Relativity (STR), there consider the 4 dim space with the line element given by

(24)

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2(dx^4)^2.$$

$\longrightarrow$ (4)

This metric is not positive definite, as it is positive for all curves along with $x^1, x^2, x^3$ are all constants.

(ie, $dx^1 = 0$, $dx^2 = 0$, $dx^3 = 0$)

but negative for all curves along which $x^4$ is constant (ie, $dx^4 = 0$)

Thus, along the latter curve the distance betn two neighbouring points cannot be real.

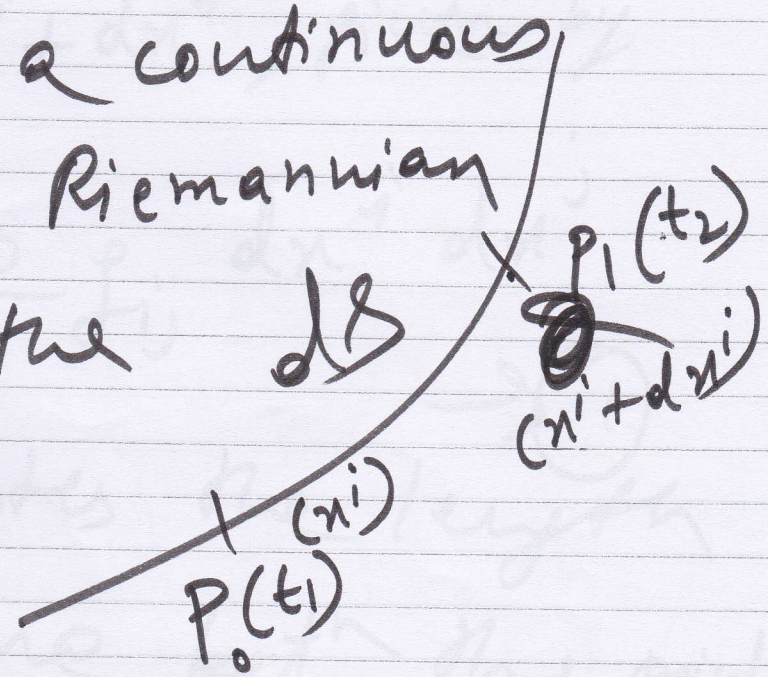
In order that the distance betn two neighbouring points be real, eqn (1) can be changed to the form

$$ds^2 = e\, g_{ij}\, dx^i dx^j \quad \rightarrow (5)$$

where the factor $e$, called the <u>indicator</u> takes the value +1 or −1 so that $ds^2$ is always positive.

# Length of a curve

We consider a continuous curve in a Riemannian space $V_N$ & the curve




$$x^i = x^i(t)$$

is a $f^n$ of the parameter $t$. Let '$s$' denotes the arc-length of the curve measured from a fixed point $P_0$ on the curve. The length $ds$ of the arc betn two neighbouring points whose

co-ordinates are $x^i$ & $x^i + dx^i$, given by

$$ds^2 = e\, g_{ij}\, dx^i\, dx^j \quad \rightarrow (1)$$

If 's' denotes the length of the curve betn the point, $P_0$ & $P_1$ corresponding to $t = t_1$ & $t = t_2$ then from eqn (1) we have (after integrating)

$$s = \int_{t=t_1}^{t_2} \sqrt{e\, g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}\, dt \quad \rightarrow (2)$$

In eqn (2), if $g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}=0$

along a curve then the two points $P_0$ & $P_1$ corresponding to $t_1$ & $t_2$ are at

zero distance from one another, despite the fact that they are not coincident. Such a curve is called 'minimal' or 'null' curve

8) Find out more about them

# Conjugate or Reciprocal Tensor

Let $g = |g_{pq}|$ denote the determinant with elements $g_{pq}$ & suppose that $g \neq 0$. We define $g^{pq}$ by

$$g^{pq} = \frac{\text{cofactor of } g_{pq}}{g} \rightarrow (1)$$

Then $g^{pq}$ is a symmetric contravariant tensor of rank 2

[Cofactor of $g_{pq}$ $= (-1)^{p+q} \times$ (Minor of $g_{pq}$ in $g = |g_{pq}|$)]

which is called the <u>conjugate or reciprocal tensor</u> of $g_{pq}$.

It can be shown that

$$\boxed{g^{pq} \cdot g_{pq} = \delta^p_q} \quad (2)$$

"dm · $D_q$ . Let m=p