Lecture 15

We have studied the measure in $\mathbb{R}$.

The Lebesgue measure in $\mathbb{R}^d$.

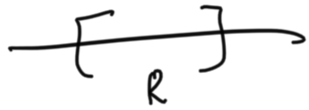
Defn A rectangle (closed) R in $\mathbb{R}^d$ is given by the product of d one dimensional closed & bounded intervals.

i.e. $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$

$$= \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; \begin{array}{c} a_1 \le x_1 \le b_1 \\ a_2 \le x_2 \le b_2 \\ \vdots \\ a_d \le x_d \le b_d \end{array} \right\}$$

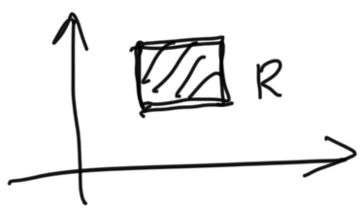
where $a_j \le b_j \quad \forall j = 1, 2, \ldots, d$.

$d=1$:

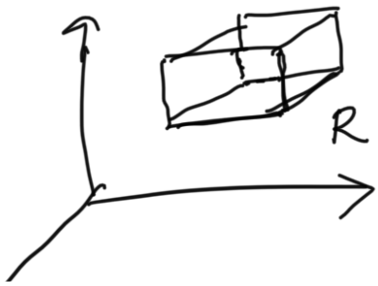


closed interval

$d=2$:



$d=3$:



Remark:- If R is a rectangle, in $\mathbb{R}^d$, then R is closed & has sides parallel to the co-ordinate axis.

Def:- A <u>cube</u> is a rectangle R for which all sides of R have the equal length.

i.e, $b_1 - a_1 = b_2 - a_2 = \dots = b_d - a_d$.

Lemma:- Let $R_1, R_2, \dots, R_n$, R be rectangles in $\mathbb{R}^d$ & $R \subseteq \bigcup_{j=1}^{n} R_j$. Then $|R| \leq \sum_{j=1}^{n} |R_j|$

where $|R| =$ the volume of the rectangle R.

Theorem:- Every open set in $\mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes. i.e, $U = \dot{\bigcup}_{j=1}^{\infty} C_j$, $\{C_j\}$ are almost disjoint.

Def:- A union of rectangles is said to be almost disjoint if the interiors of the rectangles are disjoint.

Eg:- $R_1 = [0,1] \times [0,1]$, $\underbrace{R_1^o}_{\text{interior of } R_1} = (0,1)\times(0,1)$

$R_2 = [0,1] \times [1,2]$, $\underbrace{R_2^o}_{\text{interior of } R_2} = (0,1)\times(1,2)$

$R_1^o \cap R_2^o = \phi$.

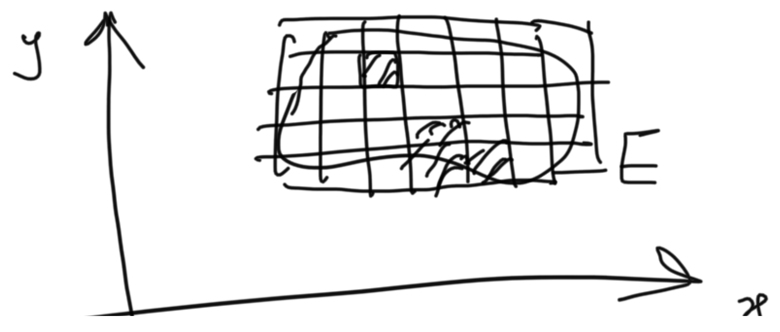
$\therefore$ R_1, R_2 are almost disjoint.

(Note that $\underbrace{R_1 \cap R_2}_{(1,1)} \neq \phi$)

Def:- (outer measure or exterior measure).

Let $E \subseteq \mathbb{R}^d$. Then the exterior measure of E (or outer measure) is defined as

$$m^*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j| \;\middle|\; E \subseteq \bigcup_{j=1}^{\infty} Q_j, \text{ \& } Q_j\text{'s are closed cubes} \right\}.$$



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<u>Remark:-</u> In definition, replace the coverings by cubes with coverings by rectangles or with coverings by balls then it yields the same outer measure.

<u>Examples:-</u>

① $\{\underline{a}\} \subseteq \mathbb{R}^d$. $m^*(\{\underline{a}\}) = 0$.

$\because$ $\underline{a}$ a point which is a cube with volume 0.

② $Q \subseteq \mathbb{R}^d$ closed cube. Then $m^*(Q) = |Q|$.

<u>proof:-</u> Q is closed by $\{Q\}$

$\Rightarrow m^*(Q) \leq |Q|$.

To prove the reverse inequality, consider an arbitrary covering $Q \subseteq \bigcup_{j=1}^{\infty} Q_j$ by closed cubes.

It suffices to prove: $|Q| \leq \sum_{j=1}^{\infty} |Q_j|$

Let $\varepsilon > 0$.
Choose for each j, an open cube S_j such that

$S_j \supseteq Q_j$ & $|S_j| \leq (1+\varepsilon)|Q_j|$.

$\therefore$ $Q \subseteq \bigcup_{j=1}^{\infty} S_j$ & Q is closed & bounded

(i.e. Q is compact)

$\Rightarrow$ there exists $N \in \mathbb{N}$ such that

$$Q \subseteq \bigcup_{j=1}^{N} S_j \subseteq \bigcup_{j=1}^{N} \overline{S_j}$$

closure of S_j.

$$\Rightarrow |Q| \leq \sum_{j=1}^{N} |S_j|$$

$$\leq (1+\varepsilon) \sum_{j=1}^{N} |Q_j|$$

$$\leq (1+\varepsilon) \sum_{j=1}^{\infty} |Q_j|$$

True for any $\varepsilon > 0$.

$\Rightarrow |Q| \leq \sum_{j=1}^{\infty} |Q_j|$. true for any $\{Q_j\}$ covering of Q.

$\Rightarrow |Q| \leq m^*(Q)$.

$\therefore$ $m^*(Q) = |Q|$.

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(3) Q is an open Cube. Then $m^*(Q) = |Q|$

proof: $\because Q \subseteq \overline{Q}$ & $|Q| = |\overline{Q}|$.

(4) For any rectangle R in $\mathbb{R}^d$, $m^*(R) = |R|$.

⑤ $m^*(\mathbb{R}^d) = \infty$

Properties of outermeasure in $\mathbb{R}^d$.

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① (Monotonicity): Suppose $E_1 \subseteq E_2 \subseteq \mathbb{R}^d$. Then

$$m^*(E_1) \leq m^*(E_2).$$

② (Countably subadditive) Let $E_j \subseteq \mathbb{R}^d$ $\forall j \in \mathbb{N}$.

$$m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m^*(E_j)$$

③ Let $E \subseteq \mathbb{R}^d$. Then $m^*(E) = \inf\limits_{\substack{U \supseteq E \\ \text{open}}} \left(m^*(U)\right)$

where infimum is taken over all open sets U containing E.

④ Let $E = \dot{\bigcup}_{j=1}^{\infty} Q_j$ almost disjoint cubes Q_j. Then

$$m^*(E) = \sum_{j=1}^{\infty} |Q_j|.$$
