Linear Algebra

Lecture 21

Diagonalization

$T : V \longrightarrow V$

$\downarrow$

$A \in F^{n \times n}$

$$A = QDQ^{-1}$$

Ex:

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$$

Compute: A^{100}

$$A = QDQ^{-1}$$

$$A^2 = (QDQ^{-1})^2 = QDQ^{-1}QDQ^{-1} = QD^2Q^{-1}$$

Example:

$$\dot{x} = Ax$$

e^{At}: matrix exponential.

Geometric interpretation of diagonalizability

Let $W_1, W_2, \dots, W_k$ be subspaces of a vector space V. Then

$$W_1 + W_2 + \cdots + W_k = \left\{ u_1 + \cdots + u_k \;\middle|\; \begin{array}{l} u_i \in W_i \\ i = 1, 2, \dots, k \end{array} \right\}$$

Definition: Let $W_1, \dots, W_k$ be subspaces of a vector space V. We call V is a direct sum of $W_1, \dots, W_k$, and denote it as

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k = \bigoplus_{i=1}^{k} W_i$$

if

$$V = \sum_{i=1}^{K} W_i$$

and

$$W_j \cap \sum_{\substack{i=1 \\ i \neq j}}^{k} W_i = \phi \quad \text{for } j = 1, \dots, k$$

<u>Thm</u>: Let $W_1, W_2, \ldots, W_k$ be subspaces of a finite dimensional vector space V. Then the following conditions are equivalent.

(a) $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$

(b) $V = \sum_{i=1}^{k} W_i$ and for any $v_1, \ldots, v_k \in V$ s.t. $v_i \in W_i$ for $i = 1, \ldots, k$ if $v_1 + v_2 + \cdots + v_k = 0$, then $v_i = 0 \ \forall i$.

(c) Each vector $v \in V$ can be uniquely written as

$$v = v_1 + v_2 + \cdots + v_k \quad \text{where } v_i \in W_i$$

(d) If B_i is an ordered basis of W_i, then $B = B_1 \cup B_2 \cup \cdots \cup B_k$ is an ordered basis of W.

Theorem: A linear operator T on a finite dimensional vector space V is diagonalizable if and only if V is direct sum of eigenspaces of T.

λ_1 $\quad E_{\lambda_1} = \text{span}\{e_1, e_2\}$ $\quad \mathbb{R}^n$

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\varepsilon > 0$

$$\begin{pmatrix} 1 \\ \cancel{0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1+\varepsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 100 \\ 101 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 99 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$x_{k+1} = A_k x_k + b_k$$

Matrix limits

$$\lim_{m\to\infty} A^m$$

Complex numbers.

$$\{z_m\}_{m=1}^{\infty}$$

$$z_m \to z \quad \text{as} \quad m \to \infty$$

$$\lim_{m\to\infty} z_m = z$$

$$z_m = r_m + i s_m$$

$$z_m \to z \qquad \text{where} \quad z = r + is$$

$$\underbrace{r_m \to r}_{\text{exists}} \quad \text{and} \quad \underbrace{s_m \to s}_{\text{exists}}$$

Definition:

Let $L, A_1, A_2, A_3, \ldots$ be $n \times p$ matrices having complex entries.

Then the sequence $A_1, A_2, \ldots$ is said to converge to the $n \times p$ matrix L, called as the limit of the sequence $A_1, A_2, \ldots$, if

$$\lim_{m \to \infty} (A_m)_{ij} = (L)_{ij} \qquad \begin{array}{l} 1 \le i \le n \\ 1 \le j \le p \end{array}$$

$(i,j)^{th}$ entry of A_m

$$\boxed{\lim_{m \to \infty} A_m = L}$$

Theorem: Let $A_1, A_2, \dots$ be a sequence of matrices $(n \times p)$ with complex entries. Let $P \in M_{r\times n}(\mathbb{C})$ and $Q \in M_{p \times s}(\mathbb{C})$ be any two matrices of some $r, s \in \mathbb{N}$. Let $\lim_{m\to\infty} A_m = L$

Then $\lim_{m\to\infty} P A_m = PL$

and $\lim_{m\to\infty} A_m Q = LQ$

Corollary: Let $A \in M_{n\times n}(\mathbb{C})$ be such that $\lim_{m\to\infty} A^m = L$. Then for any invertible matrix $Q \in M_{n\times n}(\mathbb{C})$,

$$\lim_{m\to\infty} (QAQ^{-1})^m = QLQ^{-1}$$

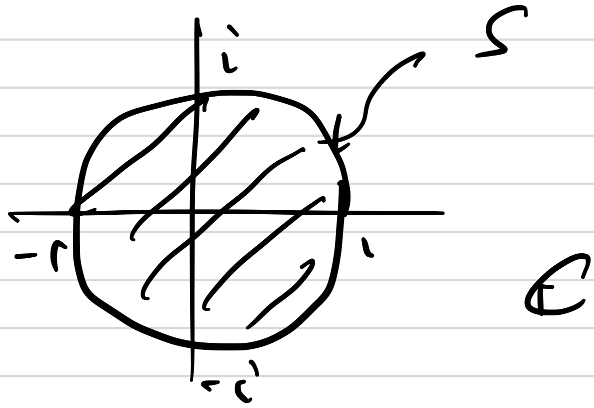
Pf: $(QAQ^{-1})^m = QAQ^{-1}QAQ^{-1} \cdots QAQ^{-1}$

$= QA^m Q^{-1}$

Definition :

$S \subseteq \mathbb{C}$ is defined as

$$S = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq 1 \}$$



Theorem: Let A be a square matrix with complex entries. Then

$\lim_{m \to \infty} A^m$ exists if and only if the following conditions hold.

(a) Every eigenvalue of A is contained in S.

(b) If 1 is the eigenvalue of A, the $\dim(E_1)$ equals multiplicity of 1.

$a \in \mathbb{R}$ / $a \in \mathbb{C}$

$\lim_{m \to \infty} a^m$ / $\lim_{m \to \infty} |a|^m$

1×1 complex matrix

$$\lim_{m \to \infty} A^m v = \left(\lim_{m \to \infty} A^m\right) v$$

$$= Lv$$

$$\lim_{m \to \infty} A^m v = \lim_{m \to \infty} (A^m v)$$

$$= \lim_{m \to \infty} \lambda^m v$$

does not exist for $|\lambda| > 1$

Theorem: Let $A \in M_{n \times n}(\mathbb{C})$ satisfy the following conditions

(a) Every eigenvalue of A is in the interior of $S^{\circ} \cup \{1\}$.

(b) A is diagonalizable.

Then $\lim_{m \to \infty} A^m$ exists.

Pf: Since A is diagonalizable, $\exists$ an invertible matrix Q such that

$$Q^{-1} A Q = D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$|\lambda_i| < 1$ or $\lambda_i = 1$

$$\lim_{m \to \infty} \lambda^i = \begin{cases} 1 & \text{if } \lambda_i = 1 \\ 0 & \text{if } |\lambda_i| < 1. \end{cases}$$

$$D^m = \begin{bmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix}$$

$$\lim_{m \to \infty} D^m = L$$

$$\lim_{m \to \infty} A^m = \lim_{m \to \infty} \left(Q D Q^{-1}\right)^m$$

$$= Q L Q^{-1}$$

$\blacksquare$

$x_{k+1} = A x_k$ $\quad A \in \mathbb{R}^{n \times n}$

$x_k \in \mathbb{R}^{n \times 1}$

x_0 : initial condition

$\in \mathbb{R}^{n \times 1}$

$x_1 = A x_0$

$x_2 = A x_1 = A^2 x_0$

$\vdots$

$x_m = A^m x_0$

$$\lim_{m \to \infty} A^m x_0$$

$$\parallel$$

$$\lim_{m \to \infty} x_m$$

$$\boxed{x_{k+1} = A x_n + b_k}$$

dynamical system
&
residuals.

$$Ax = b$$

$$A_{11} x_1 + \cdots + A_{1n} x_n = b_1$$

$$A_{11} x_1 = - A_{12} x_2 - A_{13} x_3 \cdots - A_{1n} x_n + b_1$$

$$A_{22} x_2 = - A_{11} x_1 - A_{13} x_3 \cdots A_{2n} x_n + b_2$$

$$\vdots$$

$$\underbrace{\begin{bmatrix} A_{11} & & & \\ & A_{22} & & \\ & & \ddots & \\ & & & A_{nn} \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{bmatrix} 0 & -A_{12} & \cdots & -A_{1n} \\ -A_{21} & 0 & & -A_{2n} \\ & & & \\ -A_{n1} & -A_{n2} & & 0 \end{bmatrix}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{k+1} = K^{-1} \tilde{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_k + K^{-1} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$