Recurrence relations / Difference equations :-

Ex:- $\{a_r\}$:- $\boxed{a_r = 3^r}$ → Numeric f^n.

$\boxed{a_r = 3a_{r-1}}$ → recurrence relation.

Ex:- 1, 1, 2, 3, 5, 8, . . .

$a_r = a_{r-1} + a_{r-2}$. $\boxed{a_0 = 1 = a_1}$

$$a_r = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{r+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{r+1}$$

★ Generating function:-

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$$

Ex:- Suppose we toss a coin r times. There are 2^r possible sequences of outcomes. We want to know the no. of sequences of outcomes in which heads never appear on successive tosses.

Sol:- Let a_r denotes this no., find a_r.

$$a_r = a_{r-1} + a_{r-2}; \quad a_0 = 1, \quad a_1 = 2.$$

Linear recurrence relations with const. coeff :-

$$\boxed{C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots\cdots + C_k a_{r-k} = f(r).} \quad (1)$$

This is k^{th} order linear recurrence relation

→ It needs (k) initial conditions to initialise the recurrence solution.

→ if 'k' consecutive values $a_{m-k}, a_{m-k+1}, \ldots a_{m-1}$ are known for some 'm', then the numeric f^n can be found unique.

Ex:- $\boxed{3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5}$ (2^{nd} order). → (i)

Let, $a_3 = 3$, $a_4 = 6$. } any 2 value will be given.

★ Homogenous Solution :- (Method of characteristic roots)

$$a_r = \underbrace{(\text{homo. par of } a_r)}_{a_r^{(h)}} + \underbrace{(\text{partial of } a_r)}_{a_r^{(p)}}.$$

put, $\boxed{a_r^{(h)} = A\alpha^r}$ in (i) with $f(r) = 0$.

we get:-

$$\boxed{C_0\alpha^k + C_1\alpha^{k-1} + C_2\alpha^{k-2} + \cdots + C_{k-1}\alpha + C_k = 0} \quad (ii)$$

→ characteristics eq^n of the diff. eq^n (i).

Ex: $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = 1$, $a_1 = 1$.

characteristic eqⁿ:- (putting $a_n = \alpha^n$)

$$\alpha^2 - 5\alpha + 6 = 0.$$
$$\alpha = +3, +2.$$

$a_n = a_n^{(h)} + 0$;

$a_n = A_1 3^n + A_2 2^n$; A_1, A_2 are const.

Now apply initial given conditions and find the constants.

→ Case of multiple roots:-

(i) has roots → $\alpha_1, \alpha_2, \dots, \alpha_q$.

multiplicities → $m_1, m_2, \dots, m_q$.

$$\left.\begin{matrix} \alpha_1^r, & r\alpha_1^r, & r^2\alpha_1^r, & \dots, & r^{m_1-1}\alpha_1^r \\ \alpha_2^r, & r\alpha_2^r, & \dots & \dots, & r^{m_2-1}\alpha_2^r \\ \vdots & & & & \\ \alpha_q^r, & r\alpha_q^r, & \dots, & & r^{m_q-1}\alpha_q^r \end{matrix}\right\}$$

basic solution ~~bas~~ say $b_1, b_2, \dots, b_p$.

$a_r^n = \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_p b_p$

Ex: $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$.

$a_0 = 1$, $a_1 = 2$, $a_2 = 0$.

Solution: characteristic equation (put $a_n = x^n$)

$$x^3 - 7x^2 + 16x - 12 = 0.$$
$$(x-2)^2(x-3) = 0.$$
$$x = 2, 2, 3. \longrightarrow 2^n, n2^n, 3^n.$$

General solution is:

$$a_n = \lambda_1 2^n + \lambda_2 n 2^n + \lambda_3 3^n.$$

by apply given conditions we get:

$$a_n = 5\cdot 2^n + 2n\cdot 2^n - 4\cdot 3^n.$$

⊙ Homogenous linear recurrence relation:-

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_p a_{n-p}}, \quad n \geq p$$

Solⁿ:- $a_n = A_1\alpha_1^n + A_2\alpha_2^n + \dots + A_n\alpha^n$

characteristic root method:-

put; $a_n = x^n$.

$x^n = c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_p x^{n-p}$ → find roots

$$\boxed{x^p - c_1 x^{p-1} - c_2 x^{p-2} \dots - c_p = 0},$$

$\underbrace{x = \alpha_0, \alpha_1, \dots, \alpha_{p-1}}_{\text{all distinct}}$

Case I

$\underbrace{a_1 = \lambda_1, a_2 = \lambda_2, \dots, a_{p-1} = \lambda_{p-1}}_{\text{given}}$

$$\begin{cases} A_0 + A_1 + \dots + A_{p-1} = \lambda_0 \\ A_1\alpha_0 + A_2\alpha_1 + \dots + A_{p-1}\alpha_{p-1} = \lambda_1 \\ \vdots \\ A_1\alpha_0^{p-1} + A_2\alpha^{p-1} + \dots + A_{p-1}\alpha_{p-1}^{p-1} = \lambda_{p-1} \end{cases}$$

$AX = b$.

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_{p-1} \\ \vdots & & & & \\ \alpha_0^{p-1} & \alpha_1^{p-1} & \dots & \dots & \alpha_{p-1}^{p-1} \end{bmatrix}$$

→ $|A|$ = Vandermonde determinant

$= \prod (\alpha_i - \alpha_j) \neq 0$

$0 \leq i, j \leq p-1$

$i \neq j$.

Case II:- α root of the characteristic eqⁿ of multiplicity (2) → m

$f(x) = x^p - c_1 x^{p-1} \dots - c_p$.

$a_n = \alpha^n$

Then $f(\alpha) = 0 = f'(\alpha)$.

$a_n = n\alpha^n$

$\vdots$

→ $x f'(x) = px^p - c_1(p-1)x^{p-1} \dots - c_{p-1}x$

$a_n = n^{m-1}\alpha$

$\boxed{\alpha f'(\alpha) = 0}$ characteristic eqⁿ.

* Finding Particular Solⁿ :-

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \dots + c_k a_{r-k} = f(r) \quad ; f(r) \neq 0$$

function of r

general solⁿ :-

$$\boxed{a_r = a_r^{(h)} + a_r^{(p)}}$$

where, $f(r) = F_1 r^t + F_2 r^{t-1} + \dots + F_t r + F_{t+1}$

$F_i \to$ const.

poly. in r

→ the corresponding particular solⁿ will be of the form.

$$P_1 r^t + P_2 r^{t-1} + \dots + P_t r + P_{t+1}$$

Ex:- Find parti. solⁿ :-

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2.$$

$$\boxed{a_r^{(p)} = P_1 r^2 + P_2 r + P_3}$$

$$P_1 r^2 + P_2 r + P_3 + 5[P_1(r-1)^2 + P_2(r-1) + P_3] + 6[P_1(r-2)^2 + P_2(r-2) + P_3] = 3r^2.$$

→ $12 P_1 = 3$

→ $12 P_2 - 34 P_1 = 0$.

→ $12 P_3 - 17 P_2 + 29 P_1 = 0$.

→ $P_1 = 1/4$; $P_2 = \frac{17}{24}$; $P_3 = \frac{115}{288}$.

◉ **General Rule :- (i)** when, $f(r) = \beta^r$ (β is not a root of the chara. eqⁿ).

the corresponding particular solⁿ is of the form.

$$a_r^{(p)} = P\beta^r.$$

(ii) when, $f(r) = (F_1 r^t + F_2 r^{t-1} + \dots + F_t r + F_{t+1})\beta^r$ {$\beta \to$ not a root of chara. eqⁿ.}

then,

$$a_r^{(p)} = (P_1 r^t + P_2 r^{t-1} + \dots + P_t r + P_{t+1})\beta^r.$$

Ex:- Find particular solⁿ of :- $a_r + a_{r-1} = 3r2^r$.

(ii) case:-

$$a_r^{(p)} = (P_1 r + P_2) 2^r.$$

$$a_r^{p} = \left(2r + \frac{2}{3}\right) 2^r.$$

(iii) when, $f(r) = (F_1 r^t + F_2 r^{t-1} + \dots + F_{t+1})\beta^r$ {β is a root of chara. eqⁿ of multiplicity 'm'}

then,

$$\boxed{a_r^{(p)} = r^m (P_1 r^t + P_2 r^{t-1} + \dots + P_t r + P_{t+1})\beta^r.}$$

Ex:- $a_r - 2a_{r-1} = 3 \cdot 2^r$; $x - 2 = 0$

$x = 2$ root

$$\boxed{a_r^{(p)} = Pr2^r}$$

$$Pr2^r - 2[P(r-1)2^{r-1}] = 3 \cdot 2^r.$$

$$\boxed{P = 3}.$$

Ex:- Find particular solⁿ :-

$$a_r = a_{r-1} + 7.$$

$$a_r - a_{r-1} = 7 \cdot 1^r.$$

∴ here, 1 is the root with multiplicity 1.

$$a_r^{(p)} = r^1 \cdot P \cdot 1^r = rP$$

→ $rP - (r-1)P = 7$

$P = 7$.

→ particular solⁿ is $\boxed{a_r^{(p)} = 7r}$.

*. Solution by the method of generating fⁿ :-

$G(x) = a_0 + a_1 x + a_2 x^2 + \dots$

$\{a_r\}_{r=0,1,\dots}$

Ex:- $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r$.

→ $\sum_{r=2}^{\infty} a_r x^r - \sum_{r=2}^{\infty} 5a_{r-1} x^r + 6\sum_{r=2}^{\infty} a_{r-2} x^r = \sum_{r=2}^{\infty} 2^r x^r + \sum_{r=2}^{\infty} r x^r$

→ $[G(x) - a_0 - a_1 x] - 5x[G(x) - a_0] + 6x^2 G(x) = \frac{4}{1-2z} + z\left[\frac{1}{(1-z)^2} - 1\right]$

Ex: {Legitimate codewords}

$\{0,1,2,3\} \longrightarrow$ alphabet

codewords from the alphabet $\{0,1,2,3\}$ are to be recognized legitimate they have an even no. of 0's.

→ How many legitimate codewords of length k are there?

Let a_k be the answer:-

a_{k+1}

# of illegitimate codewords of length $k \Rightarrow 4^k - a_k$.

sequence rel.^{n}:- $\boxed{a_{k+1} = (4^k - a_k) + 3a_k}$

Now solve this recurrence rel^n:

By solving we get:- $a_k = \frac{1}{2}\cdot 4^k + \frac{1}{2}\cdot 2^k$.

⇒ In the above que. we first make recurrence rel^n then solve it.

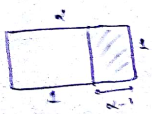
Ex: $\alpha = \frac{1+\sqrt{5}}{2}$; $F_{n+1} = F_n + F_{n-1}$; $F_0 = 0$, $F_1 = 1$.

Sol:- $\boxed{x^2 = x+1} \rightarrow \alpha, \beta$.

$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \rightarrow$ (Binet function)

$G(z) = F_0 + F_1 z + F_2 z^2 + \cdots$

$= \frac{z}{1-z-z^2}$.



$\rightarrow \frac{\alpha}{1} = \frac{1}{\alpha - 1}$

$\rightarrow \alpha^2 = \alpha + 1$

$\{F_n\}_{n=0,1,\ldots}$

$\rightarrow G(z) = F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots$

$z G(z) = F_0 z + F_1 z^2 + F_2 z^3 + \cdots$

$z^2 G(z) = F_0 z^2 + F_1 z^3 + \cdots$

$G(z) - zG(z) - z^2 G(z) = \underbrace{F_0}_{0} + \underbrace{(F_1 - F_0)}_{1} z + \underbrace{(F_2 - F_1 - F_0)}_{0} z^2 + \cdots$

$\rightarrow \boxed{G(z) = \frac{z}{1-z-z^2}}$

$\rightarrow z^2 + z - 1 = 0$

$z = \frac{-1 \pm \sqrt{5}}{2}$

$= \frac{1}{\alpha}, \frac{1}{\beta}$.

$\Rightarrow G(z) = \frac{z}{(1-\alpha z)(1-\beta z)} = \frac{z}{\sqrt{5}}\left[\frac{1}{1-\alpha z} - \frac{1}{1-\beta z}\right]$

$G(z) = \frac{1}{\sqrt{5}}\left[(1 + \alpha z + (\alpha z)^2 + \cdots) - (1 + \beta z + \beta^2 z^2 + \cdots)\right]$.

↳ coeff of z^n.

$\boxed{F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)}$

Exercise:- (i) $\sum_{k=0}^{n} F_k F_{n-k}$; $F_{n+1} = F_n + F_{n-1}$.

$\Rightarrow \frac{1}{5}\left[(n+1)(\alpha^n + \beta^n) - 2F_{n+1}\right]$

Hint:- compute $G(z) \times G(z)$.

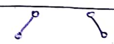
(ii) $\alpha^n = F_n \alpha + F_{n-1}$

$\beta^n = F_n \beta + F_{n-1}$.

(iii). $\sum_{k=0}^{n} F_k F_{n-k} = \frac{1}{5}(n-1)F_n + \frac{2n}{5} F_{n-1}$

Example: Number of distinct binary trees with 'n' nodes. → (catalan number)** $= b_n$

$n=2 \rightarrow$



$n=3 \rightarrow$

n	b_n
0	1
1	1
2	2
3	5
4	14
⋮	⋮
10	16796
⋮	

$$\boxed{n \Rightarrow \frac{1}{n+1}\binom{2n}{n}}$$

$$b_n = \sum_{i=0}^{n-1} b_i \cdot b_{n-i-1}$$

$B(x) = b_0 + b_1 x + b_2 x^2 + \ldots\ldots$; $b_0 = 1$.

$B(x) - 1 = b_1 x + b_2 x^2 + \ldots\ldots$

$x B(x) = b_0 x + b_1 x^2 + \ldots\ldots$

$= d_0 + d_1 x + d_2 x^2 + \ldots\ldots$; $d_0 = 0$, $\boxed{d_n = b_{n-1}}$

$[B(x)][x B(x)] = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots\ldots$

when, $\boxed{c_n = b_n * d_n} = \sum_{i=0}^{n-1} b_i d_{n-i} \neq \sum_{i=0}^{n-1} b_i b_{n-i-1}$

$\Rightarrow x B(x)^2 = B(x) - 1.$

$x B(x)^2 - B(x) + 1 = 0$

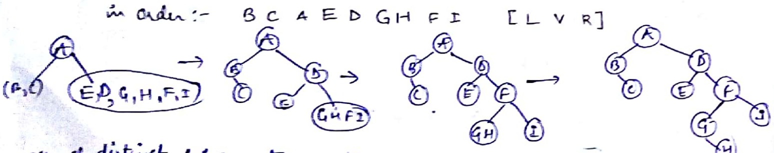
$$B(x) = \frac{1 - \sqrt{1-4x}}{2x} = \frac{1}{2x}\left[1 - \sum_{\substack{n\geq 0 \\ int}} \binom{\frac{1}{2}}{n} (-4x)^n\right]$$

$$= \sum_{n\geq 1} \binom{\frac{1}{2}}{n} (-1)^{n-1} 2^{2n-1} x^{n-1}$$

$$= \sum_{m\geq 0} \frac{(2m)!}{m!\,(m+1)!} x^m \quad \text{(check)}.$$

Ex: preorder :- A B C D E F G H I [V L R]

inorder :- B C A E D G H F I [L V R]



# of distinct binary trees with n nodes $\Rightarrow$

# of distinct inorder permutations given a fixed preorder permutation of n nodes.

e.g., 1, 2, 3 preorder permutation [V L R].

inorder → [L V R].

Example. Regions in the plane

$f(n)$ = # of regions in the plane separated by 'n' lines.

$f(1) = 2$; $f(2) = 4$; $f(3) = 7$; $f(n) = ?$

$$\boxed{f(n+1) = f(n) + n + 1}$$

$= f(n-1) + n + (n+1)$

$= \vdots$

$$f(n+1) = 1 + \frac{(n+1)(n+2)}{2}.$$

* Counting Permutations

$\{a_n\}$.

$G(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots$ {Generating fn}.

$H(z) = a_0 + \frac{a_1 z}{1!} + \frac{a_2 z^2}{2!} \ldots\ldots$ {exponential generating fn}

Example. a, b, c → length 5 codewords.

- A codeword can use three diff. letters a, b or c.
- A sequence of five or fewer letters gives a codeword
- The codeword can use at most one b, at most one c and upto three a's.
- How many possible codewords are there of length 'k'.

Ordinary generating fn.

→ $(1 + ax + a^2x^2 + a^3x^3)(1 + bx)(1 + cx).$

→ $1 + (a+b+c)x + (bc + a^2 + ab + ac)x^2 + \boxed{(a^3 + abc + a^2b + a^2c)x^3}$

$+ (a^2bc + a^3b + a^3c)x^4 + a^3bc\,x^5.$

$\Rightarrow$ coeff. of x^k → ways of obtaining 'k' letters.

3 letters can be obtained here as:

3 a's → 1 perm.

1a, 1b, 1c → abc, acb → 3! perm.

2 a's, 1b → $\frac{3!}{2!\,1!}$

2 a's, 1c → $\frac{3!}{2!\,1!}$

$\left[\frac{3!}{3!}a^3 + \frac{3!}{1!\,1!\,1!}abc + \frac{3!}{2!\,1!}a^2b + \frac{3!}{2!\,1!}a^2c\right]x^3$

$$H(x) = \left(1 + \frac{ax}{1!} + \frac{a^2x^2}{2!} + a^3\frac{x^3}{3!}\right)\left(1 + \frac{bx}{1!}\right)\left(1 + \frac{cx}{1!}\right).$$

coeff of $\frac{x^3}{3!}$.

$\underline{a=b=c=1}$

→ # codewords of length 3.

$$= 3!\left(\frac{1}{3!} + \frac{1}{1!\,2!\,3!} + \frac{1}{2!\,1!} + \frac{1}{2!\,1!}\right).$$

$$= 13.$$

⇒ Suppose that we have 'p' types of objects, with n_i indinguishable objects of type i, $i = 1, 2, \dots, p$.

Then # of distinguishable permutation of ~~type~~ length k with upto n_i objects of type i is the coeff. of $\frac{x^k}{k!}$ in the exponential generating f^n.

$$\rightarrow \left(1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n_1}}{n_1!}\right)\left(1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n_2}}{n_2!}\right)\cdots\left(1 + x + \frac{x^2}{2!} + \dots + \frac{x^{p}}{p!}\right)$$

** Probability generating f^{n}s (P_i)

$$G(x) = \sum_{k=0}^{\infty} P_k x^k \quad ; \quad P_0 + P_1 + \dots + P_k + \dots = 1.$$

Ex: coin toss:-

$$P_0 = \tfrac{1}{2} = P(H) \quad ; \quad P_1 = \tfrac{1}{2} = P(T).$$

$$G(x) = \frac{1}{2} + \frac{1}{2}x.$$

Ex: (Bernoulli trials)

'n' independent repeated trials of an experiment with each trial leading to a success with prob 'p' & 'q' failure with prob. $\boxed{q = 1-p}$

→ prob. that in 'n' trials there will be 'k' success.

$$= b(k, n; p) = \binom{n}{k} p^k q^{n-k}.$$

prob. generating funct:-

$$G(x) = \sum_{k=0}^{n} b(k, n, p)\, x^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} x^k = (px+q)^n.$$

$$G(1) = (p+q)^n = 1$$

$$G'(1) = ?$$

$$\rightarrow G(x) = \sum_k P_k x^k$$

$$G'(x) = \sum_k k P_k x^{k-1}.$$

$$G'(1) = \sum_k k P_k \rightarrow \text{expectation.} \Rightarrow \begin{pmatrix} 1 & 2 & \dots & n \\ P_1 & P_2 & \dots & P_n \end{pmatrix}.$$

$$G(1) = (p+q)^n = 1.$$

$$G'(1) = \text{expectation.}$$

→ variance = $G''(1) + G'(1) - [G'(1)]^2$.

→ $\boxed{\text{variance} = \sum k^2 P_k - (\sum k P_k)^2}$ → expectation.

⇒ (Bernoulli, trials) → $G(x) = (px+q)^n$.

Expectation = $G'(1)$. $= \left[np(px+q)^{n-1}\right]_{x=1}$

$$= np.$$

** Hatcheck Problems

Imagine that n gentlemen attend a party and check their hats. The checker returns the hats at random. What is the probab. that no gentlemen recieves his own hat.

Derangement :- A permutation or an arrange. in which object is not placed in the i^{th} place for any i.

→ D_n → # of derangement of n objects.

$$D_0 = 0 \; ; \quad D_1 = 0 \; ; \quad D_2 = 1$$

Final Results:

2^{nd} order rec. rel^n → $D_{n+1} = n(D_n + D_{n-1})$

1^{st} order rec. rel^n → $D_{n+1} = (n+1)D_{n} + (-1)^{n+1}$.

derange. of $n+1$ objects :- k^{th} $(n+1)k$

$\to$ k^{th} posi. having 1 $\to D_{n-1}$

$\to$ k^{th} posi. not having 1 $\to D_n$.

place 1 at the position & then derange elements $2, 3, \ldots, k-1, 1, k+1, \ldots, n+1$.

$\Rightarrow$ <u>Using ordinary generating f^n</u> :- (<u>solving prev. results</u>).

$$\sum_{n=0}^{\infty} D_{n+1} x^n = \sum_{n=0}^{\infty} (n+1) D_n x^n + \sum_{n=0}^{\infty} (-1)^{n+1} x^n$$

$$D_1 + D_2 x + D_3 x^2 + \ldots \qquad \to \frac{-1}{1+x}$$

$$= \frac{1}{x}\left[G(x) - D_0\right] \qquad \Rightarrow \sum_{n=0}^{\infty} n D_n x^n + \sum D_n x^n$$

$$\Rightarrow x G'(x) + G(x).$$

$$G(x) = \sum_{n=0}^{\infty} D_n x^n.$$

$$G'(x) = \sum_{n=0}^{\infty} n D_n x^{n-1}.$$

$$\Rightarrow \frac{1}{x}\left[G(x) - D_0\right] \Rightarrow x G'(x) + G(x) - \frac{1}{1+x}.$$

or.

$$\boxed{G'(x) + \left(\frac{1}{x} - \frac{1}{x^2}\right) G(x) = \frac{1}{x(1+x)} - \frac{1}{x^2}.} \to \text{not easy to solve}$$

$\to$ <u>Using exponential generating f^n</u>

$$\sum_{n=0}^{\infty} D_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} \cancel{(n+1)} D_n \frac{x^{n+1}}{\cancel{(n+1)!}\, n!} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{(n+1)!}$$

$$H(x) - D_0 \qquad\qquad x H(x) \qquad\qquad e^{-x} - 1.$$

$$\Rightarrow H(x) - 1 = x H(x) + e^{-x} - 1.$$

$$H(x) = \frac{e^{-x}}{1-x} = \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right]\left[1 + x + x^2 + x^3 + \ldots\right].$$

$$D_n = \text{coeff. of } \frac{x^n}{n!} \text{ in } H(x).$$

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right] \text{ ——— (1)}$$

Prob. in <u>Hatcheck prob</u> :- $\dfrac{\text{eq}^n \text{ (1)}}{n!}$.

★ Distribution of distinguishable balls into indistinguishable cells (occupancy problem), with no cell empty. "$S(n, k)$" $\to$ Stirling's no. of 2^{nd} kind.

$\to$ $\underbrace{n \text{ balls}}_{\text{disting.}}$, $\underbrace{k \text{ cells}}_{\text{indisting.}}$.

1 2 3 ... k

Final result :- $$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.$$

$\Rightarrow$ Let,

$T(n, k) \to$ no. of ways to put n distinguishable balls into 'n' distinguishable cells, labelled $1, 2, \ldots, k$. with no cells empty.

$$\boxed{T(n, k) = k!\, S(n, k)}$$

$c_i \to i^{th}$ balls goes to cell c_i

$$\boxed{C(1)\; C(2)\; C(3)\; C(4) \ldots C(n)}$$

$1 \leq c_i \leq k$.

$\to$ # of such permutation is given by the co-eff. of $\frac{x^n}{n!}$. in exponential generating f^n.

$$H(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)^k.$$

$\to$ '<u>k' types of object.</u> $1, 2, \ldots, k$.

$$\to H(x) = (e^x - 1)^k$$

$$= \sum_{i=0}^{k} \binom{k}{i} (-1)^i (e^x)^{k-i}$$

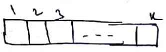
$$= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \sum_{n=0}^{\infty} \frac{1}{n!} (k-1)^n x^n.$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{i=0}^{k} \binom{k}{i} (-1)^i (k-1)^n.$$

★ Classification of occupancy problems :-

		Distinguishable balls?	Distinguishable cells?	Can cell be empty?	# of ways to place n balls into k cells.
Case I :-	1.a)	Yes	Yes	Yes	k^n
	1.b)	Yes	Yes	No	$T(n,k) = n!\, S(n,k)$ coeff. of $\frac{x^n}{n!}$ in $H(x) = \left(x + \frac{x^2}{2!} + \dots\right)^k$ $\Rightarrow \sum \binom{k}{i}(-1)^i (k-i)^n = (e^x - 1)^k$
Case II :-	2.a)	No	Yes	Yes	$C(k+n-1, n)$
	2.b)	No	Yes	No	$C(n-1, k-1)$
Case III :-	3.a)	Yes	No	Yes	$S(n,1) + S(n,2) + \dots + S(n,k)$
	3.b)	Yes	No	No	$S(n,k)$
Case IV)	4.a)	No	No	Yes	# of partition of n into k or fewer parts
	4.b)	No	No	No	# of partition of n into exactly k parts-

#
1 2 3 k



k cells, n balls.

$T(n,k) \Rightarrow n!\, S(n,k)$.

$S(n,k) \Rightarrow$ Stirling's no. of 2^{nd} kind

$\Rightarrow$ # of distri of n distinguishable balls into k indistinguishable cells.

⊙ k types of objects :-

of ways to choose 'n' cells with unlimited repeatations allowed $\Rightarrow$ coeff of x^n in $\Rightarrow (1 + x + x^2 + \dots)^k$

$\Rightarrow (1-x)^{-k}$.

Case 4)

$n = 5, k = 3$ → partition of $n = 5$:-

→ $\{1,1,1,1,1\}$ → $\{1,1,1,2\}$

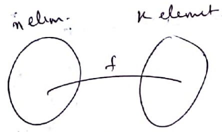
→ $\{1,2,2\}$

→ $\{1,1,3\}$

→ $\{2,3\}$

→ $\{1,4\}$

→ $\{5\}$



# of surjective f^n or onto f^n

$\Rightarrow T(n,k)$

Ex :- A elevator with 8 passangers stops at 6 diff floors. Find the no. of ways the passanger who get off together.

Sol :- 6^8.

Ex :- For non-negative integers m and n, let,

$s(m,n) =$ coeff of x^m in $x(x-1)(x-2)\dots(x-n+1)$

→ stirling's no. of 1st kind.

$s(0,0) = 1, \quad s(m,0) = 0\ (m>0) \quad ; \quad s(0,n) = 0\ (n>0)$.

$S(m,n)$ → Stirling's no. of 2^{nd} kind.

$S(0,0) = 1; \quad S(m,0) = 1\ (m>0); \quad S(0,n) = 1\ (n>0)$.

Exercise :-

Prove that :-

(i) $s(m,n) = s(m-1, n-1) - (m-1)\, s(m-1, n) \qquad m, n > 0$.

(ii) $S(m,n) = S(m-1, n-1) + n\, S(m-1, n) \qquad n, m > 0$.

(iii) $M = (s(m,n))_{4\times 4} \quad 1 \le m, n \le 4$.

$N = (S(m,n))_{4\times 4} \quad 1 \le m, n \le 4$

Then $\boxed{M^{-1} = N}$

★ Simultaneous Eqn for Generating f^n :-

Ex :- i) alphabet $\{0,1,2,3\}$

i) a_k = # of codewords of length k with an even no. of 0's & an even no. of 3's.

i) b_k = # of —— " —— even no. of 0's & odd no. o

i) c_k = # of —— " —— odd no. of 0's & an even no. o

→ $a_k + b_k + c_k + d_k \Rightarrow 4^k$.

→ $a_{k+1} \Rightarrow 2a_k + b_k + c_k$.

→ $b_{k+1} \Rightarrow 2b_k + a_k + d_k$.

→ $c_{k+1} \Rightarrow 2c_k + a_k + d_k$

$A(x) = \sum_{n=0}^{\infty} a_n x^n$.

$B(x) = \{b_n\}_{n=0}^{\infty}$

$C(x) = \{c_n\}_{n=0}^{\infty}$

$$a_{k+1} = 2a_k + b_k + c_k$$
$$b_{k+1} = b_k - c_k + 4^k$$
$$c_{k+1} = c_k - b_k + 4^k$$
— (1)

choose a_0, b_0, c_0 so that (1) holds.

$k=0$.

$a_1 = 2$
$b_1 = 1$
$c_1 = 4$

$$2 = 2a_0 + b_0 + c_0$$
$$1 = b_0 - c_0 + 1$$
$$1 = c_0 - b_0 + 1$$

$b_0 = c_0 = 0$
$\Rightarrow a_0 = 1$.

$\Rightarrow$ multiply x^k both sides in (1):-

$$\frac{1}{x}[A(x) - a_0] = 2A(x) + B(x) + C(x)$$
$$\frac{1}{x}[B(x) - b_0] = B(x) - C(x) + \frac{1}{1-4x}.$$
$$\frac{1}{x}[C(x) - c_0] = C(x) - B(x) + \frac{1}{1-4x}.$$

$$\Rightarrow B(x) = C(x) = \frac{x}{1-4x}.$$

$$A(x) = \frac{2x^2-4x+1}{(1-2x)(1-4x)} = 1 + \frac{x}{1-4x} + \frac{x}{1-2x}.$$

$$\Rightarrow b_k = c_k = 4^{k-1}, \quad k>0$$
$$= 0 \text{ for } k=0.$$

$$a_k = \begin{cases} 4^{k-1} + 2^{k-1} & \text{for } k>0 \\ 1 & \text{for } k=1. \end{cases}$$

Example:- The number of labelled graphs:-
i) n vertices
ii) e edges.

$$= {}^rC_0 + {}^rC_1 + {}^rC_2 + \cdots {}^rC_r.$$
$$= 2^r = 2^{{}^nC_2}.$$

$\to L(n) = \#$ of labelled graphs $\bar{e}$ n vertices.

$\to 0 \le e \le ({}^nC_2)$.
$\Downarrow$
r

$\to a_k = L(n,k)$.
$= \#$ of labelled graphs with number of k edges.

$$\to G(x) = \sum_{k=0}^{({}^nC_2)} a_k x^k = \sum_{k=0}^{r} ({}^rC_k)\, x^k = (1+x)^r; \quad r = ({}^nC_2).$$

Ex: # of labelled digraphs:-

n vertices $\to$ $n(n-1)$ possible ~~edges~~ arcs.

$M(n,a) \to \#$ of labelled digraphs with 'n' vertices & 'a' arcs.

$$M(n,a) = \binom{n(n-1)}{a}, \quad 0 \le a \le n(n-1).$$

$M(n) \to \#$ of labelled digraphs with 'n' vertices.

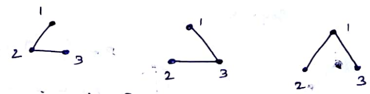
$$M(n) = \sum_{a=0}^{n(n-1)} \binom{n(n-1)}{a}.$$

*. Labelled graphs with 3 vertices:-

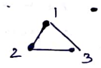
0 edges

1 edge

2 edge

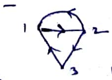


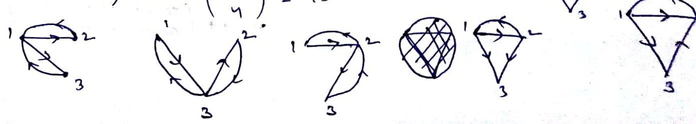
3 edges



labelled digraphs $\bar{e}$ 3 vertices, 4 edges:-

$$M(3,4) = \binom{3\times 2}{4} = 15$$





Ex: # of labelled trees with 'n' vertices.

Cayley Theorem:

If $n \geq 2$, then there are n^{n-2} distinct labelled trees of 'n' vertices.

proof: (using Prufer code).

★ trees $\bar{c}$ 'n' vertices.
↓
(i) connected graph $\bar{c}$ no circuit.
(ii) $n-1$ edges.
(iii) has at least pendant vertices.

Prufer sequence – a seq. of $n-2$ numbers, each being one of the no's has $1, 2, \ldots n$.
↓
collection of this seq^n. "P_n"

$|P_n| = n^{n-2}$.

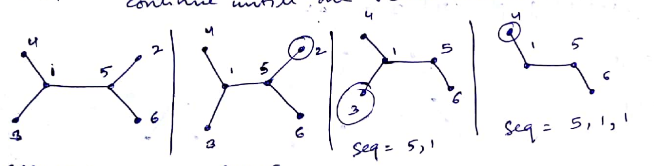
$T_n \to$ set of all labelled trees with 'n' vertices.

★. **Algo. to encode any tree into a prufer sequence.**

i) Take a tree $T \in T_n$, where vertices are labelled from 1 to n in any manner.
ii) Take the pendant vertex with smallest label. Delete it from the tree & write down the label of its only neighbour.
iii) Repeat this process with the new smaller tree continue untill one vertex remains.

a seq. of length $(n-1)$ – drop the $(n-1)^{th}$ element as it is always 'n' → $(n-2)$ length seq.

Ex:

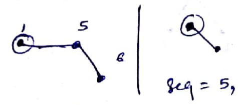


seq - - -

Seq = 5

Seq = 5, 1

Seq = 5, 1, 1



Seq = 5, 1, 1, 5

Seq = 5, 1, 1, 5, 6.

$P = \{5, 1, 1, 5\}$

→ each is a neighbour of a pendant vertex.

⟹

→ no vertex of deg. 1 in P.

→ deg. of a vertex in P, = 1 + (# of times it appears in P).

★. **Reconstruction of encoding tree from a Prufer seq. 'p'.**

1.) Find the smallest no. from 1 to n that is not in the seq. P and ~~attach~~ attach the vertex with that no. to the vertex with the first no. in P'.

2.) Remove the 1st no. of 'p' from the seq^n. Repeat the process considering only the nos whose vertices have not attained their correct deg.

3.) Do it untill there are no nos left in P. Finally attach in the last no. in 'P' to the vertex 'n'.

Ex: Draw the tree whose Prufer code is:– 1, 1, 1, 1, 6, 5

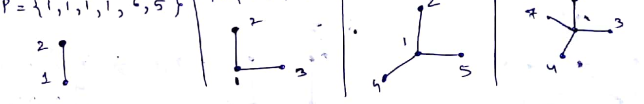
8 vertices.

deg 1 → 5

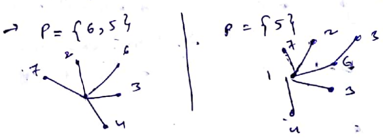
deg 2, 3, 4, 7, 8 → 1

deg 6, 5 → 2.

→ $P = \{1, 1, 1, 1, 6, 5\}$ | $P = \{1, 1, 1, 6, 5\}$ | $P = \{1, 1, 6, 5\}$ | $P = \{1, 6, 5\}$



→ $P = \{6, 5\}$ | $P = \{5\}$



→ $f : T_n \to P_n$ $\quad |T_n| = |P_n| = n^{n-2}$.

$f^{-1} : P_n \to T_n$

★ Chromatic Polynomial :

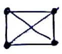
Suppose G is a graph & $P(G,x)$ counts the no. of ways to color G in at most x colors.

→ $\chi(G)$ = chromatic numbers.
= smallest 'k' such that G is k-colorable.

⊙ coloring 'G' means vertex coloring so that adjacent vertices get different colors.

⊙ 'G' is k-colorable if 'G' can be colored in this way using at most 'k' colors.

K_4



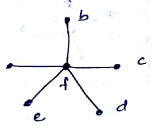
$P(K_4,x) = 0$; $x = 1,2,3$.

$P(K_4,x) = x(x-1)(x-2)(x-3)$

$\boxed{\chi(K_4) = 4}$

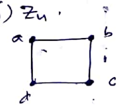
⇒ roots of $P(G,x) \xrightarrow{\text{given}}$ estimates of $\chi(G)$.

(ii)



$P(G,x) = x(x-1)^5$

(iii) Z_4.



$P(Z,x) \Rightarrow x(x-1)^2(x-1)$
$+$
$x(x-1)(x-2)(x-2)$

$\Rightarrow x^4 - 4x^3 + 6x^2 - 3x$.
here sum of coeffs is 0.

Reduction Theorem :

★ $G = G_1 \cup G_2$.

G_1 , G_2

$V(G) = W_1 \cup W_2$
$W_1 \cap W_2$
→ no edges from W_1 to W_2.

$P(G,x) = P(G_1,x)\,P(G_2,x)$

⊙ α be an edge in 'G' joining vertices a, b.

G_α' = the graph attained from 'G' by deleting edge α, but retaining the vertex a, b.

G_α'' = the graph obtained from 'G' by contracting the edge α.

Ex :

(G) G_α' G_α''.

$P(G_\alpha',x)$
↳ property poly for above.

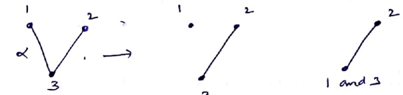
i) a, b get different colors in G_α'.

ii) a, b get same color in G_α'

⇒ Fundamental Reduction Theorem :-

$P(G,x) = P(G_\alpha',x) - P(G_\alpha'',x)$

Ex :



(G) (G_α') (G_α'')

$\Rightarrow P(G_\alpha',x) = P(K_1,x)\,P(K_2,x)$
$= x \cdot x(x-1)$
$\Rightarrow x^2(x-1)$

$P(G,x) = x^2(x-1) - x(x-1)$
$\Rightarrow x(x-1)^2$.

$P(G_\alpha'',x) = P(K_2,x)$
$\Rightarrow x(x-1)$.

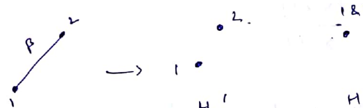
$P(K_n,x) = x(x-1)\cdots(x-n+1)$.

$P(I_n,x) = x^n$

K_n →

I_n →

★★ Again,



$G_\alpha'' = K_2$
↳ H

H_β' $= I_2$

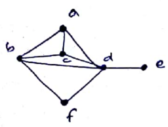
H_β'' $= I_1$

$P(G_\alpha'',x) = P(I_2,x) - P(I_1,x)$
$= x^2 - x$.

$P(G',x) = P(I_1,x)\,P(K_2,x)$

$P(G',x) \Rightarrow P(I_1,x)[P(I_2,x) - P(I_1,x)]$.

★. **Clique**:- a collection of vertices in a graph in which each vertex is joined to other vertices.



→ clique no.

$\{d,e\}$

$\{a,b,c,d\}$

$\{a,b,c\}$

→ independent set of a graph 'G' :- a collection of ~~graph G~~ vertices where no two vertices are adjacent. Independent no. of G. $\alpha(G)$ is the size of a largest independent set of 'G'.

e.g.) $\{a,f,e\}$, $\{a,f\}$, $\{e,c,f\}$.

$\alpha(G) = 3$.

∴ $\{m_1, m_2, \ldots, m_k\}$ av. value

⇒ A coloring of $n = |V(G)|$ vertices of 'G' in $\chi(G)$ colors partitions the vertices into $k = \chi(G)$ "color classes", each defining an independent set.

The av. size of such an independent set $= \frac{n}{k}$; $\chi(G) = k$.

∴ By an application of pigeonhole principle, there is at least one independent set with size at least $\frac{n}{k}$.

$$\alpha(G) \geq \frac{n}{k} = \frac{|V(G)|}{\chi(G)}.$$

★. **Pigeonhole principle & its generalisation**:-

"many" pigeons and "few" pigeons.

"Theorem" (Pigeonhole principle):-

If $k+1$ pigeons are placed in 'k' pigeonholes, then at least one pigeonhole will contain two or more pigeons.

Ex:- 13 people in a room at least two people have a birthday in a same month.

Ex:- 677 people from the telephone book.

$26 \times 26 = 676$

there will be atleast two people whose first and last names begin with the same letter.

Ex:- Manufacturing P.C's.

A manufacturer of PCs makes at least one PC every day over a period of 30 days, does not start a new PC on a day when it is impossible to finish it and averages no more than $1\frac{1}{2}$ PC's per day. Show that there must be a period of consecutive days during which exactly 14 PC's are made?

⇒ $a_1, a_2, \ldots, a_i, \ldots a_{30}$.

$1 \leq a_1 < a_2 < \cdots < a_{30} \leq 30 \times \frac{3}{2} = 45$.

$a_1, a_2, \ldots a_{30}$

$[\, 1 \leq a_1+14, a_2+14, \ldots \leq 45+14 = 59 \,]$.

→

pigeon	pigeonhole
$k+1$	k
$2k+1$	k
$3k+1$	k
$mk+1$	k

Theorem:- If 'm' pigeons are placed in 'k' pigeonholes then at least one pigeonhole will contain more than $\left[\frac{m-1}{k}\right]$ pigeons.

Theorem (Erdo"s & Szekeres [1935]).

Given a sequence of n^2+1 distinct integers, either there is an increasing subsequence of $n+1$ terms or a decreasing subsequence of $n+1$ terms.

$x_1, x_2, \dots, x_p \rightarrow$ a seq. of nos.

$x_{i_1}, x_{i_2}, \dots x_{i_q}$, $1 \le i_1 < i_2 < i_3 \dots < i_q \le p$.

↳ a subsequence of $x_1, x_2, \dots x_p$.

e.g., 9, 6, 14, 8, 17.

→ Longest increasing subsequent → 9, 14, 17.

→ longest decreasing subsequent → 9, 8, 6

Ex: 10, 3, 2, 1, 6, 5, 4, 9, 8, 7. $10 = 3^2+1$ $\boxed{n=3}$

t_i = # of largest inc. subseq. starting at x_i.

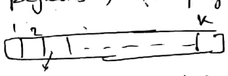
i	x_i	t_i	sample subsequ
1	10	1	10
2	3	3	3, 6, 9
3	2	3	2, 6, 7
4	1	3	1, 6, 7
5	6	2	6, 9
6	5	2	5, 9
7	4	2	4, 9
8	9	1	9
9	8	1	8
10	7	1	7

10, 9, 8, 7 → dec. subsequ. of length $n+1 = 4$.

★ **Generalization of pigeonhole principle**

'm' pigeons, 'K' pigeonholes.

Then,



at least one pigeonhole should contain more than $\lfloor \frac{m-1}{k} \rfloor$ pigeons.

→ proof by contradiction:-

if our assumption is not true, suppose each pigeonhole contain at most $\lfloor \frac{m-1}{k} \rfloor$ many pigeons.

$\Rightarrow$ total # of pigeons $\le k \lfloor \frac{m-1}{k} \rfloor \le m-1 < m$

★ **Theorem:**

$\{a_n\} \rightarrow$ subseq.

+ve integers ↳ inc./dec. subseq. of length $n+1$.

n^2+1.

→ Given a subsequence of n^2+1 distinct integers, either there is an increasing subsequ. of $n+1$ terms or a decreasing subsequence of $n+1$ terms.

proof

$x_1, x_2, \dots, x_{n+1}$

$\downarrow t_1 \quad \downarrow t_2$

$t_i \rightarrow$ length of the longest inc. subseq. starting at x_i.

eg, 9, 6, 14, 8, 17

$t_1 = 3$, $t_2 = 3$, $t_3 = 2$, $t_4 = 1$, $t_5 = 1$

→ If any t_i is $n+1$ → decreasing.

Assume that each t_i is in between 1 to n.

n^2+1 → many t_i → pigeons.

$1, 2, \dots, n$ → pigeonholes.

Then at least one pigeonhole containing at least $\lfloor \frac{(n^2+1)-1}{n} \rfloor + 1$ many pigeons.

$\Rightarrow$ '$n+1$' many t_i's are equal.

proof. Claim: corresponding "$n+1$" many x_i's associated with t_i's form a decreasing subsequence.

→ Suppose $t_i = t_j$ with $i<j$, then $x_i > x_j$.

$t_{i_1} = t_{i_2} = \dots t_{i_{n+1}}$.

$\{x_{i_1}, x_{i_2}, \dots, x_{i_{n+1}}\} \downarrow$

if $x_i \not> x_j$, then with $t_i = t_j$ & $i<j$

⇓

$x_i < x_j$

x_i followed by longest subseq. beginning at x_j.

$t_i = t_j + 1 = t_j + 1$.

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☆ The principle of inclusion and exclusion :-

Theorem :- If 'N' is the no. of objects in a set 'A', the no. of objects in 'A' having none of the properties $a_1, a_2, \dots, a_n$ is given by.

$$N(a_1' a_2' \dots a_r') = N - \sum N(a_i) + \sum N(a_i a_j) - \sum N(a_i a_j a_k \dots) \dots + (-1)^r N(a_1 a_2 \dots a_r)$$

$N(a_i)$ = # of objects having property a_i.

$N(a_i a_j)$ = # ----- having the properties a_i, a_j.

$N(a_i' a_j')$ = # --- having none of the properties a_i, a_j.

Ex :- How many integers between 1 & 1000 are

(a) not divisible by 2? $N(a_i') = N - N(a_i) = 1000 - \frac{1000}{2} = 500.$

(b) not divisible by 2 or 5? $N(a_1' a_2') = 400$

(c) not divisible by 2, 5 or 11? $N(a_1' a_2' a_3') = 363$

☆ Derangements of 'n' objects :- (using principle of inclusion & exclusion).

D_n = # of derangements of 'n' objects.

A = set of all permutations of 'n' objects.

a_i = the property that objects 'i' is placed at the i^{th} position.

Then, $D_n = N(a_1' a_2' \dots a_n')$. $\Rightarrow n! - ({}^nC_1)(n-1)! + \binom{n}{2}(n-2)! - \dots$

$N = n!$

$N(a_i) = (n-1)!$

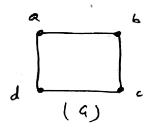
$N(a_i a_j) = (n-2)!$

$\sum N(a_{i_1} a_{i_2} a_{i_3} \dots a_{i_k}) = \binom{n}{t}(n-t)!$

$$D_n = n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{(n-2)!\,2!}(n-2)! - \dots \dots (-1)^n \frac{n!}{n!(n-n)!}(n-n)!$$

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right]$$

Ex :- consider all possible coloring of the vertices of the graph 'G' with 'x' or fewer colours.



→ we shall even allow coloring where two vertices joined by an edge get the same color, but we shall call such coloring improper and all other proper.

a_1 = property that a & b get the same color.

a_2 = - - - - b & c.

a_3 = ——"—— c & d

a_4 = - - — d & a - —

Then, $N = x^4$ such colorings.

∴ The $P(G, x) = N(a_1' a_2' a_3' a_4')$

$= N - \sum N(a_i) + \sum N(a_i a_j) - \sum N(a_i a_j a_k) + N(a_1 a_2 a_3 a_4)$

i) $N = x^4$

i) $N(a_1) = x^3 = N(a_2) = N(a_3) = N(a_4)$.

i) $N(a_1, a_2) = x^2 = N(a_1 a_3) = N(a_2 a_3)$

i) $N(a_1 a_2 a_3) = x$

i) $N(a_1 a_2 a_3 a_4) = x$.

∴ $P(G, x) = x^4 - 4x^3 + 6x^2 - 4x + x$.

$P(G, x) = x^4 - 4x^3 + 6x^2 - 3x$.

★. The principle of inclusion and exclusion :-

$A \rightarrow$ a set of n objects.

$a_1, a_2, \ldots, a_r \rightarrow$ 'r' ~~permutations~~ properties.

$$\boxed{N(a_1' a_2' \ldots a_r') = N - \sum N(a_i) + \sum_{i \neq j} N(a_i a_j) - \sum_{i \neq j \neq k} N(a_i a_j a_k) + \cdots + (-1)^r N(a_1 a_2 \ldots)}$$

$N = |A|$

$N(a_i) = \#$ of objects in 'A' having property a_i.

$N(a_i a_j) = \#$ of ------ both properties $a_i a_j$.

⋮

proof :- LHS = # of objects in 'A' having none of the properties $a_1 a_2 \ldots a_r$.

RHS = every object having none of the properties $a_1, a_2 \ldots a_r$ counted once.

+

every object having at least one of the properties $a_1 a_2 \ldots a_r$ is counted exactly zero times (in a net same).

- An object has none of the properties $a_1 a_2 a_3 \ldots a_r$
  - it is counted once in computing 'N';
  - not counted in computing $\sum N(a_i)$, $\sum N(a_i a_j)$ so on.

- An object has exactly 'p' of the properties $a_1 \ldots a_p$; $p > 0$
  - it is counted $\binom{p}{0}$ times in computing N; # of objects in A.
  - it is counted $\binom{p}{1}$ times in computing $\sum_i N(a_i)$
  - — " — $\binom{p}{2}$ times in — " — $\sum_{i \neq j} N(a_i a_j)$
  - — " — $\binom{p}{3}$ — " — $\sum N(a_i a_j a_k)$

⋮ so on.

Hence counted in the RHS exactly :-

$$\Rightarrow \binom{p}{0} - \binom{p}{1} + \binom{p}{2} - \binom{p}{3} + \cdots + (-1)^r \binom{p}{r}.$$

$$\Rightarrow \binom{p}{0} - \binom{p}{1} + \binom{p}{2} - \cdots + (-1)^p \binom{p}{p}. \qquad \because \binom{p}{k} = 0 \text{ if } k > p. :-$$

$$\Rightarrow \{1 + (-1)\}^p = 0.$$

★. Generalisation of chromatic polynomial :-

Suppose 'G' is any graph and we want to compute $P(G, x)$:-

- consider the set A of all colorings, proper or improper, of all the vertices of G in x or fewer colors.
- For each edge i, let a_i be the property that the end vertices of edge i get the same color.

$|V(G)| = n$, $\quad |E(G)| = r$

$N = |A| = x^n$, $\quad P(G, x) = N(a_1' a_2' \ldots a_n')$

$$\therefore P(G, x) \Rightarrow \underset{\substack{\| \\ x^n}}{N} - \sum N(a_j) + \sum N(a_i a_j) - \cdots + (-1)^r N(a_1 a_2 \ldots a_r)$$

Let us consider the term $N(a_{i_1} a_{i_2} \ldots a_{i_t})$.

Spanning subgraph H of G with edges $i_1, i_2, \ldots, i_t$.

Let $C(H) = \#$ of connected components of H.

Then, # of coloring of vertices in $H \Rightarrow \#$ of colorings of vertices of 'G'.

$$\Rightarrow x^{C(H)}.$$

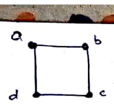
Theorem :- If 'G' is a graph and $h(t, c)$ is the no. of spanning subgraphs of 't' edges & 'c' components, then,

$$P(G, x) = \sum (-1)^t x^c h(t, c)$$

t → no. of edges

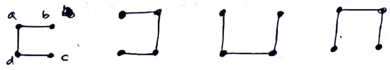
c → no. of connected components in a spanning subgraphs of 'G' with 't' edges and 'c' connected components.

$$\rightarrow h(0, c) = \begin{cases} 1, & \text{if } c = n \\ 0, & \text{o.w} \end{cases}$$

Ex:- 

$h(4,1) = 1$

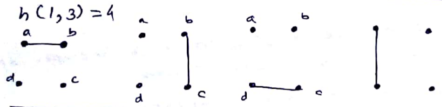
edge components



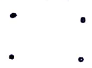
$h(3,1) = 4.$

$h(2,2) = 6$

$h(1,3) = 4$



$h(0,4) = 1$



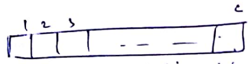
$\rightarrow P(G,x) = (-1)^4 \cdot x^1 \cdot 1 + (-1)^3 \cdot x^1 \cdot 4 + (-1)^2 x^2 \cdot 6 + (-1)^1 \cdot x^3 \cdot 4 + (-1)^0 \cdot x^4 \cdot 1.$

$= x - 4x + 6x^2 - 4x^3 + x^4.$

$$\boxed{P(G,x) = x^4 - 4x^3 + 6x^2 - 3x}$$

Ex:- $T(n,c)$ = # of distribution of n distinguishable balls into 'c' distinguishable cells with no cells empty.

using principle of inclusion and exclusion.



c distinguishable cells.

n distinguishable balls

A = the set of distribution of balls to cells.

a_i = the property that i^{th} cell empty.

$\therefore N = c^n$

$N(a_i) = (c-1)^n$

$N(a_i a_j) = (c-2)^n$

$\vdots$

$T(n,c) = N(a_1' a_2' \cdots a_c')$

$= N - \sum N(a_i) + \cdots \cdots (-1)^c N(a_1 a_2 a_3 \cdots a_c).$

$= c^n - \binom{c}{1}(c-1)^n + \binom{c}{2}(c-2)^n + \cdots + (-1)^c \binom{c}{c}(c-c)^n.$

$= \sum_{t=0}^{c} (-1)^t \binom{c}{t} (c-t)^n.$

$\therefore \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^c$ = coeff. of $\frac{x^n}{n!}$.

Ex:- Interviewing, 3 teachers
4 plumbers
6 autoworkers.

# of way to choose 11 workers to interview? Assume two workers with the same job to be indistinguishable.

$\rightarrow (1+x+x^2+x^3)(\cdots + x^4)(\cdots + x^5 + x^6). \rightarrow x^{11}$

$\rightarrow$ Ans 6.

★. **Alternative counting (using principle of inclusion and exclusion) :-**

Ex:- consider the case where there are infinitely many teachers, plumbers & autoworkers available. Let A = the set consisting of all the ways of choosing 11 workers to interview, A particular element of A is said to satisfy.

- property a_1 if it uses at least 4 teachers.
- a_2 —— " —— 5 plumbers.
- a_3 —— " —— 7 autoworkers.

★. **Theorem:-** If there are 'p' types of objects, the # of distinguishable ways to choose 'K' objects if we allow unlimited repititions of each type, is given by:-

$$\binom{p+K-1}{K}.$$

in previous ex:-

$N = 8$; $P = 3$, $k = 11$.

$\rightarrow$ $N = \binom{3+11-1}{11} = {}^{13}C_{11} = 78.$

$\rightarrow$ $P = 3$, $k = 11-4 = 7$ $\quad N(a_1) = \binom{3+7-1}{7} = {}^{9}C_{7} = 36.$

$\rightarrow$ $P = 3$, $k = 11-5 = 6.$ $\quad N(a_2) = \binom{3+6-1}{6} = {}^{8}C_6 = 28.$

$\rightarrow$ $P = 3$, $k = 11-7 = 4$ $\quad N(a_3) = \binom{3+4-1}{4} = {}^{6}C_4 = 15.$

$\rightarrow$ $P = 3$, $k = 11-9 = 2$ $\quad N(a_1a_2) = \binom{3+2-1}{2} = {}^{4}C_2 = 6$

$N(a_1a_3) = \binom{3+0-1}{0} = 1$

$\rightarrow$ $P = 3$, $k = 11-11 = 0.$ $\quad N(a_2a_3) = 0.$

$\therefore N(a_1'a_2'a_3') = 78 - (36+28+15) + (6+1+0) - 0$
$= 6.$

$\therefore D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \frac{(-1)^n}{n!}\right].$

$\Rightarrow$ The number of objects having exactly 'm' properties:-

Theorem:- The # of objects having exactly m properties, if there are r properties & $m \le r$ is given,

$$e_m = s_m - \binom{m+1}{1} s_{m+1} + \binom{m+2}{2} s_{m+2} - \binom{m+3}{3} s_{m+3} + \cdots + (-1)^p \binom{m+p}{p} s_{m+p} + \cdots + (-1)^{r-m} \binom{m+r-m}{r-m} s_r.$$

where, $s_t = \sum N(a_{i_1} a_{i_2} \cdots a_{i_t})$; $t \ge 1$.
$i_1\, i_2\, i_3 \ldots$ all are diff.

Special case:- Take $m = 0$. & $s_0 = N$.

Example:- (Hatcheck problem)
# of ways in which exactly one gentleman gets his hat back $= nD_{n-1}$.

Exercise:- Apply the above formula to count this
i.e, find $e_1 = s_1 - \binom{2}{1} s_2 + \binom{3}{2} s_3 - \binom{4}{3} s_4 + \cdots (-1)^{n-1} \binom{n}{n-1} s_n.$

$\rightarrow s_1 = \sum N(a_i)$; $r = n$, $m = 1$. define the problem.
a_i = i^{th} man gets his hat back.

$\rightarrow \binom{n}{1}(n-1)! = s_1.$

① Theorem:- $e_m \rightarrow$ # of objects of a set A of size N having exactly 'm' properties $a_1, a_2, \ldots, a_m$.
$s_i = \sum N(a_{i_1} a_{i_2} a_{i_3} \ldots a_{i_t})$
$i_1\, i_2 \ldots$ all are diff

Then,
$$e_m = s_m - \binom{m+1}{1} s_{m+1} + \binom{m+2}{2} s_{m+2} + \cdots + (-1)^{r-m} \binom{m+r-m}{r-m} s_r.$$
$(-1)^p \binom{m+p}{p} s_{m+p}.$

proof:-
(i) — an object has fewer than 'm' properties.
(ii) — an object has exactly 'm' properties.
(iii) — an object has more than 'm' properties, m+j properties say.
$\rightarrow$ it is not counted in calculating e_m.

On RHS:-
(i) it is counted $\binom{m+j}{m}$ times in calculating s_m.
(ii) — " — $\binom{m+j}{m+1}$ times in calculating s_{m+1}.
(iii) — " — $\binom{m+j}{m+p}$ times in calculating s_{m+p}.

claim :-

$$= \binom{m+j}{m} - \binom{m+1}{1}\binom{m+j}{m} + \binom{m+2}{2}\binom{m+j}{m+2} + \cdots + (-1)^p \binom{m+j}{p}\binom{m+j}{m+p} + \cdots$$

$$\cdots (-1)^j \binom{m+j}{j}\binom{m+j}{m+j} = 0.$$

$$= \binom{m+j}{m}\left[1 - \binom{j}{1} + \binom{j}{2} - \binom{j}{3} + \cdots + (-1)^p\binom{j}{p} + \cdots (-1)^j\binom{j}{j}\right]$$

$$= \binom{m+j}{m}\left[1 + (-1)\right]^j = 0.$$

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claim :-

$$\binom{m+j}{m+p}\binom{m+p}{p} = \binom{m+j}{m}\binom{j}{p}$$

$$\text{LHS} = \frac{(m+j)!}{\cancel{(m+p)!}\,(j-p)!} \times \frac{\cancel{(m+p)!}}{p!\,m!} = \frac{(m+j)!}{m!\,j!} \times \frac{j!}{(j-p)!\,p!} = \binom{m+j}{j}\binom{j}{p}$$

Theorem. If there are 'r' properties then the no. of objects having an even no. of properties is given by.

$$= e_0 + e_2 + e_4 + \cdots$$

$$= \frac{1}{2}\left[s_0 + \sum_{t=0}^{r} (-2)^t s_t\right]$$

$\therefore$ here, $s_0 = N =$ total no. of objects.

& the no. of objects having an odd no. of properties is given by.

$$e_1 + e_3 + e_5 + \cdots = \frac{1}{2}\left[s_0 - \sum_{t=0}^{r}(-2)^t s_t\right].$$

Ex: We define a codeword from the alphabet $\{0, 1, 2, 3\}$ is legitimate if it had an even no. of 0's and we let, "b_k" be the no. of legitimate codewords of length 'k'. Then using generating f^n, we have shown that, $\boxed{b_k = \frac{1}{2}(2^k + 4^k)}$

Alternative method :- (using principle of inclusion and exclusion)

$A =$ set of all sequences of length 'k' from $\{0, 1, 2, 3\}$

$a_i =$ the property that the i^{th} digit is 0, $i = 1, 2, \ldots k$.

$e_m =$ # of objects having exactly 'm' properties.

$b_k =$ # of objects of 'A' having an even no. of these properties.

$$= e_0 + e_2 + e_4 + \cdots$$

$$= s_0 + \sum_{t=0}^{r} (-2)^t s_t.$$

$$= \frac{1}{2}\left[4^k + \sum_{i=0}^{k} (-2)^t \binom{k}{t} 4^{k-t}\right]$$

$$= \frac{1}{2}\left[4^k + \{4 + (-2)\}^k\right]$$

$$= \frac{1}{2}\left[4^k + 2^k\right]$$

proof: $\{e_m\}_{m=0}^{r}$

ordinary generating f^n.

$$E(x) = e_0 + e_1x + e_2x^2 + e_3x^3 + \cdots + e_rx^r.$$

$$E(1) = e_0 + e_1 + e_2 + e_3 + \cdots + e_r.$$

$$E(-1) = e_0 - e_1 + e_2 - e_3 + \cdots + (-1)^r e_r.$$

$$\frac{1}{2}[E(1) + E(-1)] = e_0 + e_2 + e_4 + \cdots$$

$$\frac{1}{2}[E(1) - E(-1)] = e_1 + e_3 + e_5 + \cdots$$

$\rightarrow$ $E(x) = e_0 + e_1x + e_2x^2 + \cdots + e_rx^r.$

$\underline{m=0}$; $= [s_0 - s_1 + s_2 \cdots + (-1)^r s_r]$

$$\therefore e_m = \left[s_m - \binom{m+1}{1}s_{m+1} + \binom{m+2}{2}s_{m+2} + \cdots + (-1)^r\binom{r}{r-m}s_r\right]$$

$$+ \left[s_1 - \binom{2}{1}s_2 + \binom{3}{2}s_3 + \cdots + (-1)^r\binom{r}{r-1}s_r\right]x$$

$$+ \cdots$$

$$+ \left[s_m - \binom{m+1}{1}s_{m+1} + \binom{m+2}{2}s_{m+2} - \cdots + (-1)^{r-m}\binom{r}{r-m}s_r\right]x^m$$

$$+ \cdots + s_rx^r.$$

$$e_m = s_0 + s_1(x-1) + s_2\left[x^2 - \binom{2}{1}x + 1\right] + s_3\left[x^3 - \binom{3}{1}x^2 + \binom{3}{1}x - \binom{3}{3}\right]$$

$$+ \cdots + s_r[\ \ ]$$

Scanned by CamScanner

→ $\frac{1}{2}[E(1) + E(-1)]$

→ $\frac{1}{2}\left[s_0 + \sum_{m=0}^{r} s_m (-2)^m \right]$

Ex:- • Suppose every library in the US is preparing a database of all its collections.

• From around the country, all the databases are submitted to the library of congress.

• Now the national librarian starts to sort through the database and found :-

- Some of the databases include themselves in the listing
- others do not.

~~What~~ Where to pute database 2?

→ If she put it in the listing, she has to put it in Database 1 that do include themselves.

$\boxed{S = \{x \mid x \notin S\}}$.
↓
is not well defined.

Russel's paradox.
↓
shows inconsistency in set theory.
↓
Resolved by having levels of sets.
↓
sets exists in Hierchy.

• $S \to$ the set of sets.
Does S contains itself?

$S = \{x \mid x \notin x\}$,

- S is not a member of itself :-
  it is satisfied.

- S is a member of itself.

★. **Cardinality of infinite sets :-**

• If A, B are infinite sets and $A \subseteq B$:

$|A| \overset{?}{=} |B|$

Ex:- Let A = set of all natural nos.

B = the set of even integers.

$f(n) = 2n$.

$A = 1, 2, 3, 4, 5, \ldots$

$B = 2, 4, 6, 8, 10, \ldots$

1-1 correspondence
⇓
bijection. $A \to B$.

$\boxed{f(n) = 2n}$

Ex:- (Finite coffee beans) :-

• dark and light roasted coffee beans.

• without counting them explicitly, is there an equal nos of dark roasted & light roasted coffee beans?

• matching.

Ex:- (Hilbert's Hotel).

$1 \to 2 \to 3 \to 4 \to 5$

$|N| = \aleph_0$ (aleph null)
↳ transfinite no.

$\aleph_0 + 1 = \aleph_0$
$\aleph_0 + 2 = \aleph_0$
$\aleph_0 + 3 = \aleph_0$
⋮
$\aleph_0 + \aleph_0 = \aleph_0$

$|N| = \aleph_0$
$|P(N)| = \aleph_1$

→ consider an infinite bus arrives at Hilbert's Hotel.

1	2	3	4	----
2	4	6	8	----

$\therefore$ n^{th} guest $\rightarrow$ $2n-1^{th}$ room.

$\aleph_0 \times 1 = \aleph_0$

$\aleph_0 \times 2 = \aleph_0$

$\vdots$

$\aleph_0 \times \aleph_0 = \aleph_0$

★ **How to compare infinite sets:**

1) <u>countable set</u> :- set that is either finite or has the same no. of elements as the set of +ve integers.

$\{a_n\}_{n=0}^{\infty}$

2) <u>uncountable set</u> :- not countable.

Ex:

1) set of odd integers countable $\rightarrow f(n) = 2n-1$

2) set of integers countable.

$\hookrightarrow$ $f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ even.} \\ -\frac{(n-1)}{2} & \text{if } n \text{ odd.} \end{cases}$

3) set of +ve rational no's countable.

Ex: Show that the set of +ve rational no's is countable.

Sol^n: $\frac{p}{q}$, p, q are +ve integers.

$$\begin{array}{cccccccl} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots & & q=1 \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots & & q=2 \\ & & & & & & & q=3 \end{array}$$

these are already conted as $\frac{1}{1}$ & $\frac{2}{1}$

real p/q such that

$p+q = 2$

$p+q = 3$

$\vdots$

& so on

★ **Enumeration:**

Let 'S' be a set. An enumeration of 'S' is a surjection 'f' from initial line segment of N to S.

$\rightarrow f : N \rightarrow Q$. <u>surjective mapping.</u>

$\downarrow$

enumeration with repetition.

Fact: A set 'S' is countable iff there exists an enumeration of 'S'.

Ex:

$S \rightarrow$ the set of natural nos of the form $3n$, $n \in N$.

$f(n) = 3n$.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 3 & 6 & 9 & 12 & \cdots \end{array}$$

$\rightarrow$ <u>bijection.</u>

$\hookrightarrow$ enumeration without repetition.

$g(n) = \begin{cases} 3n+6 & \text{if } n = 3k \text{ for some } k. \\ 3n & \text{if } n = 3k+1, \; k \in N. \\ 3n-6 & \text{if } n = 3k+2, \; k \in N. \end{cases}$

$k=0$; $\rightarrow$ 6, 3, 0.

$k=1$; $\rightarrow$ 15, 12, 5

$k=2$; $\rightarrow$

Ex: $\Sigma = \{a, b\}$

$\Sigma^* =$ set of all integers over Σ.

$\{\tau(n)\}_{n=0}^{\infty} \leftarrow \Sigma^* =$ countable?

$\tau(0) = \varepsilon$

$\tau(1) = a$

$\tau(2) = b$

$\tau(3) = aa$

$\tau(4) = ab$

$\tau(5) = ba$

$\tau(6) = bb$

$\tau(7) = aaa$

$\vdots$

Proof techniques:-

- → proof by contradiction.
- → proof by well ordering principle.
- → —"— mathematical induction
- → —"— diagonalization argument.

Well ordering principle:-

Every non-empty subset of natural nos. has a least element.

Ex:- N is archimedes.

∴ Archimedean property of S.

For any $x, y \in S$ with $x > 0$, $\exists$ a +ve integer n. such that $nx > y$.

→ i.e, to prove that $\forall\, x, y \in \exists\ n \in N$. s.t $nx > y$.

Sol:- If possible, let N be not archimedian.

i.e, $\exists$ same x, y for which $nx \le y\ \forall\ n \in N$.

Let $S = \{ y - nx \mid n$ is a +ve integer $\}$.

$S \neq \phi$

S is a non-empty subset of N.

∴ By well-ordering principle, S must have a smallest element, say $y - mx \in S$, $m \in N$.

consider, $y - (m+1)x < y - mx$.

Then, $y - (m+1)x \in S$, but it is smaller than the smallest element of S. $(\rightarrow \leftarrow)$

Example:- Q Archimedean

i.e, to prove that if $a > 0$, $a \in Q$, $b \in Q$ any rational no.

then $\exists$ a +ve integer 'n' s.t, $na > b$.

Sol: Trivial cases. if $\left.\begin{array}{l} b < 0 \\ \text{or } 0 \le b \le a \end{array}\right\}$ Take $n = 1$.

case 2:- $(b > 0$ & $b > a)$

claim. $\exists$ a +ve int. n st $na > b$

↳ (by contradiction) If possible, let claim is false,

i.e, $na \le b\ \forall\ n \in N$.

$(na)\, b^{-1} \le 1 <$ any +ve int. m other than 1.

$(na)\, b^{-1} \le m\ \forall\ m \in N$.

$nm^{-1} \le ba^{-1}\ \forall\ n \in N$, $m \in N$

⇒ for any +ve $x \in Q$ we have $x \le a$ fixed rational no, (ba^{-1}) → cannot be trivial.

Ex:- Prove that $n < 2^n\ \forall\ n \ge 1$ using the well ordering principle.

Sol:- Let, A = set of natural nos. for which the inequality does not hold.

$\Rightarrow \{ n \in N \mid n \ge 2^n \}$.

claim $A = \phi$.

↳ (by contradiction)

if not, let $A \neq \phi$.

So $A \neq \phi$, $A \subseteq N$.

∴ By well ordering principle, A must have a smallest element, say 'm'.

∴ Then $m \neq 1$, $1 \ge 2^1$.

$m \ge 2$ → $m + m \ge 2 + m$, $2m - 2 \ge m$, $m - 1 \ge \frac{m}{2}$

Also, $m \ge 2^m \Rightarrow \boxed{\frac{m}{2} \ge 2^{m-1}}$

$2^{m-1} \le \frac{m}{2} \le m - 1 \Rightarrow \underbrace{m-1}_{} \in A$.

which is smaller than the smallest element m of A $(\rightarrow \leftarrow)$

Theorem ①:- (First principle of Finite induction).

Let 'S' be a set of +ve integers with the following properties.

a) The integer $1 \in S$.

b) whenever the integer $K \in S$, the next integer $K+1 \in S$ (Induction step).

$\Rightarrow$ Then the ~~set~~ 'S' is the set of all +ve integers.

Theorem ②:- (Second principle of Finite induction).

change condition (b) by (b')

(b') If K is a +ve integer s.t $1, 2, \ldots, K \in S$ then $K+1 \in S$.

Proof of th. ①:- (by contradiction)

Let $T \neq \phi$ be the set of all +ve integers not in S.

$T = N \setminus S$

T is a non-empty subset of N.

↳ by well-ordering principle, T has atleast element, say 'a'.

$\rightarrow a \neq 1$ as $1 \in S$. Also $a > 1$.

$a - 1 < a$ ..., $a - 1 > 0$

↓

least element of T. $a-1$ is the smallest than a.

$\rightarrow$ S. $(a-1) \notin T$.

$(a-1) \in S$.

Then, by ~~contradiction~~ condition b; $a \in S$ ($\rightarrow\leftarrow$).

Ex:- (well-ordering principle).

In a round-robin tournament, every player plays every other player exactly once and each match has a winner & a looser. We say that the player, $P_1, P_2, \ldots, P_m$ ~~from~~ form a cycle if P_1 beats P_2, P_2 beats P_3, ..., P_{m-1} beats P_1.

$\rightarrow$ Use well ordering principle to show that if there is a cycle of length m $(m \geq 3)$ among the players in a round-robin tournament, then there must be a cycle of 3 players.

Sol^n:

Let $S = \{ n \in N \mid \exists$ a cycle of length 'n' $\}$.

$S \neq \phi$ as $m \in S$ by given condition of the problem.

So the well-ordering principle, S must have a smallest element say K.

Let $P_1 P_2 \ldots P_K$ be the cycle. $(K \geq 3)$.

$\rightarrow P_1, P_2, P_3$ $\rightarrow$ P_1 beats P_3 or P_3 beats P_1 A cycle of length 3.

Theorem:- Every infinite set contains a countably infinite set.

Proof:- Let 'S' be an infinite set.

Select a sequence of elements $\{a_n\}_{n=0}^{\infty}$ from 'S' as follows:-

pick a_0 from 'S'.

" a_1 " $S - \{a_0\}$.

" a_2 " $S - \{a_0, a_1\}$.

⋮

so on.

→ a_{i+1} selected from $S - \{a_0, a_1, \dots, a_i\}$.

→ Each $S - \{a_0, a_1, \dots, a_i\}$ is infinite,

otherwise $(S - \{a_0, a_1, \dots, a_i\}) \cup \{a_0, a_1, \dots, a_i\}$

$\Rightarrow S$. finite $(\rightarrow \leftarrow)$.

→ The set $\{a_n\}_{n=0}^{\infty}$ is countable subset of 'S'.

Theorem:- The union of a countable collection of countable sets is countable.

proof:- $S_0, S_1, S_2, \dots$ be a countable collection of countable sets &

$S_i = \langle a_{i0}, a_{i1}, a_{i2}, \dots \rangle$

$$\begin{array}{ll}
S_0 : & (a_{00} \quad a_{01} \quad a_{02} \quad a_{03} \quad \dots\dots) \\
S_1 : & (a_{10} \quad a_{11} \quad a_{12} \quad a_{13} \quad \dots\dots) \\
S_2 : & (a_{20} \quad a_{21} \quad a_{22} \quad a_{23} \quad \dots\dots) \\
S_3 : & (a_{30} \quad a_{31} \quad a_{32} \quad a_{33} \quad \dots\dots)
\end{array}$$

→ countable set 'S'

$S \Leftrightarrow N$

$a_{ij} \rightarrow$ height $i+j$

Enumerate the elements as shown in the diagram.

Let,

$$\left.\begin{array}{l} b_0 = a_{00} \\ b_1 = a_{01} \\ b_2 = a_{10} \end{array}\right\}$$

$$\left.\begin{array}{l} b_3 = a_{02} \\ b_4 = a_{11} \\ b_5 = a_{20} \end{array}\right\}$$

& so, on.

→ $\{b_n\}_{n=0}^{\infty} \longrightarrow \bigcup_{i=0}^{\infty} S_i$

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⇒ $\mathbb{Q}^+$ → set of +ve rational nos.

$\frac{p}{q}$

$$\underbrace{\overset{a_0}{\frac{1}{1}}}_{p+q=2} \quad \underbrace{\overset{a_1}{\frac{1}{2}}, \overset{a_2}{\frac{2}{1}}}_{p+q=3} \quad \underbrace{\overset{a_3}{\frac{1}{3}}, \overset{*}{\frac{2}{2}}, \overset{a_4}{\frac{3}{1}}}_{p+q=4} \quad \underbrace{\overset{a_5}{\frac{1}{4}}, \overset{a_6}{\frac{2}{3}}, \overset{a_7}{\frac{3}{2}}, \overset{a_8}{\frac{4}{1}}}_{p+q=5} \quad \underbrace{\overset{a_9}{\frac{1}{5}}, \overset{*}{\frac{2}{4}}, \overset{*}{\frac{3}{3}}, \overset{*}{\frac{4}{2}}, \overset{a_{10}}{\frac{5}{1}}}_{p+q=6} \cdots$$

<u>Claim</u>. $\mathbb{Q}$ → countable.

⇒ $\mathbb{Q}^-$ → set of −ve rational nos.

$$\begin{matrix} a_0 & a_1 & \cdots \\ \downarrow & \downarrow & \\ -a_0 & -a_1 & \end{matrix}$$

$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$

countable by the Theorem.

<u>Ex</u>. $N \times N$ is countable

Let, $A_n = \{(1,1), (1,2), (1,3) \ldots (n,n)\}$ → countable

$\bigcup_{n=1}^{\infty} A_n = N \times N$ is countable.

∴ <u>Alternatively</u>.

→ <u>bijection from $N \times N \to N$</u>.

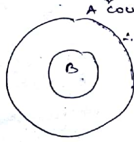
$(i,j) \to \frac{1}{2}(i+j-1)(i+j-2) + i$.

$(1,1) \to 0+1 = 1$

$(1,2) \to 1+1 = 2$

$(1,3) \to 3+1 = 4$

$(2,1) \to 1+2 = 3$



Given:- A countable.

∴ Then B is countable.

→ <u>Alternate proof</u>: ($\mathbb{Q}^+$ is countable)

⇒ $\frac{p}{q}$; p & q are co-prime.

⇒ $\mathbb{Q}^+$ → the set of all +ve rational nos.

⇒ Define $A' = \{(p,q) \mid p, q$ are coprime, p, q are integers$\} \subseteq N \times N$;

$A \to \mathbb{Q}^+$, $(p,q) \to \frac{p}{q}$.

<u>Claim</u>:- Any subset of a countable set is countable

'A' countable ⇒ elements of 'A' can be listed as $\{a_n\}_{n=0}^{\infty}$

Take the subseq. corresp. to elements of 'B'.

※ <u>Another proof</u>

('$\mathbb{Q}$' is countable)

$A_n = \left\{ \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \frac{-1}{n}, \frac{-2}{n}, \frac{3}{n}, \frac{-3}{n}, \ldots \right\}$ → countable set.

$\mathbb{Q} = \bigcup_{n=1}^{\infty} A_n$ → union of countable ~~set~~ collection of countable set → countable.

→ $N \times N$ is countable.

$|N \times N| = |N|$ → cardinality.

$|N^2| = |N|$.

→ 'R' → set of real nos.

→ <u>Theorem</u>: The set of real nos 'R' is uncountable.

(Cator's diagonalization arguement).

<u>proof</u>: By contradiction:-

Let the set of real nos 'R' be countable.

Then $[0,1] \subseteq R$ is also countable.

So, the real nos in $[0,1]$ can be listed in some order, say $r_1, r_2, r_3, \ldots$.

the decimal representation of these no's be.

$r_1 = d_{11} d_{12} d_{13} d_{14} \ldots$

$r_2 = d_{21} d_{22} d_{23} d_{24} \ldots$ $\qquad d_{ij} \in \{0, 1, \ldots, 9\}$.

$r_3 = d_{31} d_{32} d_{33} d_{34}$

$r_4 = d_{41} d_{42} d_{43} d_{44}$

so on.

→ construct a new decimal no. 'r' as follows.

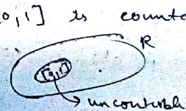
$r = \square \square \square \square \ldots$

diff from d_{11}, diff from d_{22}, " d_{33}, " d_{44}

⇒ The new decimal no. 'r' is different from any decimal no. in the list ①.

(→←) that $[0,1]$ is countable

⇒ $[0,1]$ uncountable.



⇒ <u>claim</u> → R is uncountable

Claim: Any superset of a uncountable set is uncountable.

Theorem: Power set of Σ^* is uncountable.

$\Sigma = \{a, b\}$ finite alphabet

Σ^* = collection of all strings over the alphabet Σ.

Σ^* countable $\rightarrow$ why?

$b_0 = \epsilon$
$b_1 = a$
$b_2 = b$
$b_3 = ab$
$\vdots$

$\Sigma^* = \{\epsilon, a, b, aa, ab, ba, \ldots\}$

proof: (By contradiction)
(This is wrong!)
(Assume that $P(\Sigma^*)$ is countable.) Let $\langle A_0, A_1, A_2, \ldots \rangle$ be an enumeration of the set $P(\Sigma^*)$.

Let $\langle x_0, x_1, x_2, \ldots \rangle$ be an enumeration of Σ^*.

$M = (a_{ij})$

construct a binary matrix 'M' as follows:-

$$\begin{array}{cc|cccc}
 & & x_0 & x_1 & x_2 & x_3 \; \cdots \\
\hline
 & A_0 & a_{00} & a_{01} & a_{02} & a_{03} \; \cdots \\
P(\Sigma^*) \rightarrow & A_1 & a_{10} & a_{11} & a_{12} & a_{13} \; \cdots \\
 & A_2 & a_{20} & a_{21} & a_{22} & a_{23} \; \cdots \\
 & A_3 & a_{30} & a_{31} & a_{32} & a_{33} \; \cdots \\
 & \vdots & & & &
\end{array}$$

Now construct a set 'A' as follows:

$x_i \in A$ iff $a_{ii} = 0$.

$A = \{x_i \mid x_i \notin A_i, \; i \in \mathbb{N}\}$.

Claim: 'A' cannot be any A_j in the list (1).

$A \in P(\Sigma^*)$.

$\rightarrow$ 'A' cannot appear in the enumeration $\in P(\Sigma^*)$.

$\langle A_0 \; A_1 \; A_2 \ldots \rangle$ of $P(\Sigma^*)$

$(\rightarrow\leftarrow)$

that $P(\Sigma^*)$ is countable.

$\rightarrow$ **Cardinality** $[0,1] \longleftrightarrow [a,b]$
uncountable.

$f(x) = \dfrac{x-a}{b-a}$

$0 \le f(x) \le 1$ when, $a \le x \le b$

$[0,1] \rightarrow$ continuum.

$|[0,1]| = c$ (cardinality)

$|\mathbb{N}| = \aleph_0$

S countable $\rightarrow |S| = |\mathbb{N}| = \aleph_0$.

S uncountable.

uncountable
$S \xrightarrow{\text{bijection}} [0,1]$

$|S| = |[0,1]| = c$.

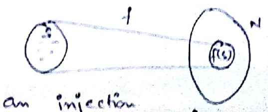
Defn:- Let, S & T be two sets.
Then, S, T are equipotent or have same cardinality denoted by $|S| = |T|$, if there is a bijection from S to T.

Theorem:-
Equipotence relation is an equivalence relation.

Defn:-
- $|S| \le |T|$ if $\exists$ an injection from S to T.
- $|S| < |T|$ if $\exists$ an injection from S to T, but no bijection from S to T.

$\rightarrow$ $\mathbb{R}$; $|[0,1]| = c$; $P(\Sigma^*)$.

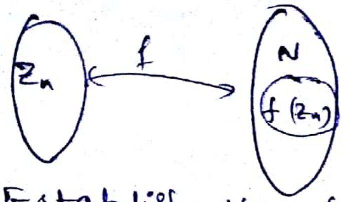
Theorem:- Let 'S' be a finite set.
then $|S| < \aleph_0 < c$.



Establish an injection, $f : S \rightarrow \mathbb{N}$.

$S = \{y_1, y_2, \ldots, y_n\}$; $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$

$$\boxed{f(x) = x+1 \;;\; x \in \mathbb{Z}_n}$$



- Establish the fact that no bijection from N to Z_n.
  - ↳ clear as 'N' is infinite set.
  - ↳ $|S| \le |N| = \aleph_0$.

$\Rightarrow$ $|N| < |[0,1]|$ $\longrightarrow$ define injective mapping.

$$\boxed{f(n) = \frac{1}{n+3}}$$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ 1/4 & 1/5 & 1/6 & 1/7 & \cdots \end{array}$$

$\longrightarrow$ **theorem:-** If 's' is infinite set, then $\aleph_0 \le |S|$

Theorem:- Let 'S' be any set.
Then $|S| < |P(S)|$.