$x-1 \le k < x$

Corollary (iii). For $x \in \mathbb{R}$ $\exists$ an integer k s.t.

Proof: for $x \in \mathbb{R}$ $\exists$ $n \in \mathbb{N}$ s.t. $x < n$ .... (i)

$\exists$ $m \in \mathbb{Z}$ s.t. $x > m$ .... (ii)

from (i) and (ii) $m < x < n$

choose largest k from $m, m+1, m+2, \dots, n$ for which $k < x$, $k+1 \ge x$

$k < x$ is possible as the set $\{m, m+1, \dots, n\}$ is finite

$\Rightarrow$ $k < x$, $k \ge x-1$ $\Rightarrow$ $x-1 \le k < x$

Rational density theorem: Between any 2 real no.s a, b with $a < b$ $\exists$ a rational no. $r \in \mathbb{Q}$ s.t. $a < r < b$

$\star$ — for arbitrary +ve real no ϵ a natural no. n s.t. $\frac{1}{n} < \epsilon$

$\lim_{n \to \infty} \frac{1}{n} = 0$ proof: $\epsilon > 0$. $\exists$ $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \epsilon$

if $\epsilon > 0$, $a = \frac{1}{\epsilon} \in \mathbb{R}$ using archimedian principle

$\exists$ $n \in \mathbb{N}$ s.t. $n > \frac{1}{\epsilon}$ i.e. $\frac{1}{n} < \epsilon$

Proof: $b - a > 0$ by above result i.e. $\frac{1}{n} < \epsilon$

$\exists$ $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$ .... (i)

$\Rightarrow$ $a < b - \frac{1}{n}$ $\Rightarrow$ $na < nb - 1$ ....

$nb \in \mathbb{R}$ put $x = nb$ in corollary (iii)

$\Rightarrow$ $\exists$ a $k \in \mathbb{Z}$ s.t. $nb - 1 \le k < nb$ ....

$na < nb - 1 \le k < nb$

$\Rightarrow$ $na < k < nb$

$a < \frac{k}{n} < b$

$a < r < b$

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ set A, $P(A)$ – powerset of A

Cantor's theorem: does there exist a subjective map b/w A and $P(A)$

set value map/fn

$\phi : A \to P(A)$ Is it onto? NO

Proof: assume there's an onto fn

if $a \in A$, $\phi(a)$ is a set

Scanned by CamScanner

Example 1: Let b denote any real number and let
(i) $S = \{ x \mid x$ is a rational no. and $x < b \}$. Find lub of S.

$S = \{ x \mid x \in \mathbb{Q} : x < b \}$, b is an upper bound.
S is bounded above so it has lub in $\mathbb{R}$ [Any bounded above subset of $\mathbb{R}$ has an lub in $\mathbb{R}$ here its $\mathbb{Q}$]
so it has a supremum.
let $c = \text{lub } S$ $\quad c \leq b \to$ upper bound
if $c = b$ it is proved and lub $S = b$
if $c < b$ $\exists$ a rational no. r $\quad c < r < b \quad r \in \mathbb{Q}$

27/7/18 **Th 1:** If a and b are real no.s & $a > 1$ then $\exists$ a natural no. n s.t. $a^n > b$

Proof: suppose $a^n \leq b \ \forall n \in \mathbb{N}$. consider set $S = \{ x \mid x = a^n, n \in \mathbb{N} \}$
then the set S is bounded above [$\because$ there is a 'b' s.t. $a^n \leq b \ \forall n \in \mathbb{N}$]
by completeness principle lub S exists. let $c = \text{lub } S$

as $a > 1$, $ac > c$ or $c < ac \Rightarrow \frac{c}{a} < c$

$\exists a^n \in S$ s.t. $a^n > \frac{c}{a}$ (if $a^n \leq \frac{c}{a}$, $\frac{c}{a}$ becomes upper bound and $a > 1 \Rightarrow \frac{c}{a} < c$ ie. c is not lub of S $\therefore \rightarrow\leftarrow$)

$a^{n+1} > c$ so $a^{n+1} \in S$

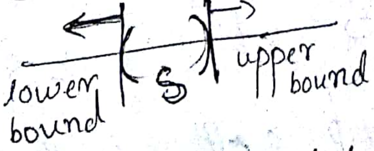
hence $\rightarrow\leftarrow$ because c is an upper bound but $\exists$ an element in S which is greater than c

$\therefore$ set $\{ a^n > b, n \in \mathbb{N}, a, b \in \mathbb{R} \}$ is an unbounded above set.

Corollary $S = \{ 10^n \mid n \in \mathbb{N} \}$ this set S is unbounded above [ie. we cant find α in $\mathbb{R}$ for which $10^n < \alpha \ \forall n \in \mathbb{N}$]

Proof: take $a = 10$ in th 1

— let S be a nonempty set in $\mathbb{R}$. A real no. b where b is not necessarily in S is said to be a lower bound if
$x \geq b \ \forall x \in S$



Greatest lower bound (glb / inf / infimum) of S.
$\alpha = \text{glb } S$ if (i) $x \geq \alpha \ \forall x \in S$ (ii) if β is a lower bound then $\alpha > \beta$ for all such β of S.

Scanned by CamScanner

• if $\alpha \in S$, then glb S = minimum of S

ex: sup of $\{\frac{n}{n+1} \mid n \in \mathbb{N}\} = 1$, $1 \notin \{\frac{n}{n+1} \mid n \in \mathbb{N}\}$

Completeness axiom for bounded below set:
Any bounded below subset of $\mathbb{R}$ has a greatest lower bound.

Bounded set: A set is said to be bounded if it is both bounded from below & bounded above.

in general $\mathbb{R}$ is not a bounded set but extended set $\mathbb{R} \cup \{\pm\infty\}$ [extended real no. system] is a bounded set.

$a + \infty = \infty$
$a - \infty = -\infty$ where $a \in \mathbb{R}$
$a \cdot \infty = \infty$, $a > 0$
$a \cdot (\infty) = -\infty$, $a < 0$

• if S is unbounded above, sup $S = +\infty$
similarly if S is unbounded below, glb $S = -\infty$

Prob 1:
Show that lub $\{\frac{n}{n+1} : n \in \mathbb{N}\} = 1$

Proof: $\frac{n}{n+1} < \frac{n+1}{n+1}$ $\forall n \in \mathbb{N}$ $\Rightarrow \frac{n}{n+1} < 1$ $\forall n \in \mathbb{N}$ i.e. 1 is an upper bound of S

suppose lub $S = c < 1$ then $1 - c > 0$ so $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < 1 - c$

$\Rightarrow c < 1 - \frac{1}{n} = \frac{n-1}{n}$

for $n > 1$, $n \in \mathbb{N}$
we see that $\frac{n-1}{n} \in S$ so ($\to\leftarrow$) since c is smaller than an element

for $n = 1$ $\frac{1}{1} < 1 - c \Rightarrow 0 < -c \Rightarrow c < 0 \Rightarrow c$ is smaller than all elements of S $\therefore \to\leftarrow$ hence lub $S \geq 1$ //

Prob 2: An upper bound c of $S \neq \phi$ in $\mathbb{R}$ is the lub of S iff $\forall \varepsilon > 0$ $\exists$ an element $s_\varepsilon \in S$ s.t. $c - \varepsilon < s_\varepsilon \leq c$

Prob 3. A lower bound b of $S \neq \phi$, $S \subset \mathbb{R}$ is the glb of S iff $\forall \varepsilon > 0$ $\exists s_\varepsilon \in S$ s.t. $b \leq s_\varepsilon < b + \varepsilon$

Scanned by CamScanner

Prob 1 ... if $0 \le a-b < \epsilon$ for every $\epsilon > 0$, then $a = b$

Proof: Assume $a \neq b$ $\quad 0 \le a-b \Rightarrow a \ge b$

let $c = a-b$ $\quad 0 \le c < \epsilon \;\; \forall \epsilon > 0 \quad \frac{c}{\epsilon} < 1$

Prob 2 proof: suppose c is an upper bound of S satisfying the following conditions i.e. $\forall \epsilon > 0 \; \exists s_\epsilon \in S$ s.t. $c - \epsilon < s_\epsilon \le c$

if $b < c$, suppose lub $S = b$ let $\epsilon = c-b$, $\exists s_\epsilon$ s.t.

$b = c - \epsilon < s_\epsilon$ i.e. b is not an upper bound $(\rightarrow\leftarrow)$

$\Rightarrow$ lub of $S = c$ $\leftarrow$ i.e. $b < c$ is wrong.

$b \ge c$

$\Rightarrow b = c$

converse: given c is the lub of S

to show that $\forall \epsilon > 0 \; \exists s_\epsilon \in S$ s.t. $c - \epsilon < s_\epsilon \le c$

c is lub, $\epsilon > 0$, $c - \epsilon < c \Rightarrow c - \epsilon$ is not an upper bound because if it is an upper bound then $c - \epsilon$ is lub which is $\rightarrow\leftarrow$ as c is lub.

$\therefore$ $c - \epsilon$ is not upper bound, $\exists s_\epsilon \in S$ s.t. $s_\epsilon > c - \epsilon$

$\Rightarrow c - \epsilon < s_\epsilon \le c$

Prob 3 proof: suppose b is an lower bound of S satisfying the following condition: i.e. $\forall \epsilon > 0 \; \exists s_\epsilon \in S$ s.t. $b \le s_\epsilon < b + \epsilon$

suppose $c > b$, glb $S = c$ let $\epsilon = c - b$ $\exists s_\epsilon$ s.t.

$s_\epsilon < b + (c-b) \Rightarrow s_\epsilon < c$

$\Rightarrow c$ is not an lower bound

$\Rightarrow c > b$ is wrong

$\Rightarrow c \le b$ $\quad \because$ if $c < b$ then c cant be glb

$\Rightarrow c = b$

$\therefore$ ~~lub~~ glb of $S = b$ //

Scanned by CamScanner

Let A, B be two non-empty ... one-to-one correspondence between A and B if there exists a fn $f: A \to B$ $\left.\begin{array}{l}(i)\ f \text{ is one-to-one} \\ (ii)\ f \text{ is onto}\end{array}\right\}$ bijective

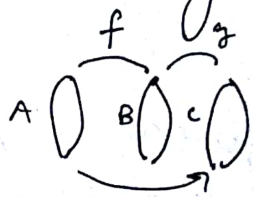
Two sets A and B are said to be equivalent if $\exists$ a one-to-one correspondence between A and B & we denote it by $A \approx B$
↳ equivalence relation

$A \approx A$ $\quad f(x) = x \ \forall x \in A$; is a reflexive relation

$A \approx B \Rightarrow B \approx A$

$f: A \to B$ is bijective then $f^{-1}: B \to A$ is also bijective
hence $B \approx A$

$A \approx B, B \approx C \Rightarrow A \approx C$



$g \circ f(x) = g[f(x)]$

$f: A \to B$ (bijective) $\quad g: B \to C$ (bijective)
$g \circ f: A \to C$ is also bijective $\because$ compositⁿ of two bijective fn is also bijective.

$J = \mathbb{N} = \{1, 2, 3, \ldots\}$ $\quad$ whole numbers $= \mathbb{N} \cup \{0\}$

$I = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

J_n = set of n elements $= \{1, 2, \ldots n\}$

Finite set: A set A is said to be finite if $\exists$ a set J_n s.t. $A \approx J_n$

Infinite set: A set which is not finite is an infinite set

Countable set / Denumerable / Enumerable: A set A is countable set if $A \approx J$

. A set 'A' is said to be atmost countable if it is either finite or countable.

Scanned by CamScanner

A set which is not countable is uncountable set.

A sequence in set A is a map $f: \mathbb{N} \to A$ $\quad f(n) = x_n \quad x_n \in A \ \forall n$

$\{1, 2, \dots, n\}$

$$f(1) = x_1$$
$$f(2) = x_2$$
$$\vdots$$
$$f(n) = x_n$$

if all these $x_n \in A$ then its a sequence in A

J: 1 2 3 4 5 6 7 8 9 ....

I: 0, ±1, ±2, ±3, ±4, ...

$f: J \to I$ given $f(n) = \frac{n}{2}$ if n is even

$= -\left(\frac{n-1}{2}\right)$ if n is odd

hence set of natural no.s, integers are equivalent sets.

J^e: set of even numbers $f(n) = 2n$ $\quad J \to 2J$ so $J \simeq 2J$

[here $2J \subsetneq J$ (proper subset) still $J, 2J$ are equivalent sets]

A set A is said to be infinite iff it has one-to-one correspondence with one of its proper subsets.

$\mathbb{Q}^+$: set of pos. rational no.s $= \left\{ \frac{p}{q} : p, q > 0 \ \ p, q \in J \right\}$

$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$	...
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$	...
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$	...
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	...
$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$	...
$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{7}{7}$	$\frac{8}{7}$	....
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

(cantor approach)

by following indicated path & omitting those fractions which have already been encountered in the path, we define a function f

$f(1) = 1/1$ $\quad f(2) = 2/1$ $\quad f(3) = 1/2$

$f(4) = 1/3$ $\quad f(5) = 3/1$ $\quad f(6) = 4/1$

$f(7) = 3/2$ $\quad f(8) = 2/3$ $\quad f(9) = 1/4$

$f(10) = 1/5$ $\quad f(11) = \frac{5}{1}$ ....

An arbitrary rational no. a/b is located in a^{th} column and b^{th} row of this arrangement $\therefore$ is reached after finitely many steps of the

Scanned by CamScanner

Thm: The set of all positive ...

$\mathbb{Q}^+ = \{\frac{p}{q} : p, q \in J\}$ $\quad f : J \to \mathbb{Q}^+$ bijective

$\mathbb{Q} = \mathbb{Q}^+ \cup (-\mathbb{Q}^+) \cup \{0\}$

• A is countable, B is countable, $A \cup B$ is countable

$A_\alpha \quad \alpha \in I \quad \alpha$ is an index

$\cup A_\alpha$: arbitrary union of A_α

$\cap A_\alpha$: " intersection of A_α

$\underline{\underline{\cup A_\alpha}} = \{x \mid x \in A_\alpha$ for some $\alpha\}$ $\quad \underline{\underline{\cap A_\alpha}} = \{x \mid x \in A_\alpha \ \underline{\underline{\forall}}\ \alpha\}$

when $I = J$

$\bigcup_{i=1}^{\infty} A_i \to$ countable union $\quad \bigcap_{i=1}^{\infty} A_i \to$ countable intersection

$f : \mathbb{Q}^+ \xrightarrow[\text{bijective}]{} \mathbb{Q}^-$ $\quad f(x) = -x$ $\quad \Rightarrow \mathbb{Q}^-$ is countable ($\because \mathbb{Q}^+$ is countable)

countable union of countable sets is countable hence $\mathbb{Q}$ is countable

Proof: By defn. of $\mathbb{Q}^+$, every element of $\mathbb{Q}^+$ can be written in the form p/q, $p, q \in J$. Now consider the following arrangement in which the n^{th} row consists of all fractions with denominator n and numerators with 1, 2, 3, ... successively and the n^{th} column consists of all fractions with numerator m and denominator 1, 2, 3, 4 ... etc.

type listed above. That means some +ve no. n is associated with a/b.

hence proved.

Theorem: set of all real numbers between 0 and 1 is uncountable.

$E = \{x : 0 < x < 1\}$ is uncountable.

Scanned by CamScanner

(ii) $T \subseteq S$, T is uncountable $\Rightarrow$ S is uncountable

3/8/18 [skipped] $I = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is uncountable

Theorem: The set of all real no.s between 0 and 1 is uncountable

Proof: we know every real no. can be represented as non-terminating decimal (some are repeating, others are non-repeating

Assume that the set of all real numbers b/w 0 and 1 is countable

so let $\exists$ a one-to-one correspondence

$f: \mathbb{N} \to I$ $\quad f(1) = 0 \cdot a_{11} a_{12} \cdots a_{1n} \cdots$

$f(2) = 0 \cdot a_{21} a_{22} \cdots a_{2n} \cdots$

$f(n) = 0 \cdot a_{n1} a_{n2} \cdots a_{nn} \cdots$

each a_{ij} satisfies $0 \le a_{ij} \le 9$

Note: In case where two representations are possible such that $1/4 = 0.2499\ldots = 0.2500\ldots$ & $4/5 = 0.7999\ldots = 0.8000\ldots$, choose the repetition of 9 than 0 in each case.

This establishes the uniqueness of representation

trailing '9' > trailing '0'

let us consider a no. $0 < c < 1$ s.t. $c \in f(\mathbb{N})$ $f: \mathbb{N} \to I$

let $c = 0. c_1 c_2 c_3 c_4 c_5 c_6 c_7 \ldots$, construct 'a' as follows

let 'a' be a no. b/w 1 & 9 not equal to a_{ii}

by construction 'c' differs from all the numbers already on the list.

so we reject the assumption that the real no. b/w 0 & 1 are countable.

Theorem: show that $\mathbb{R}$ is uncountable

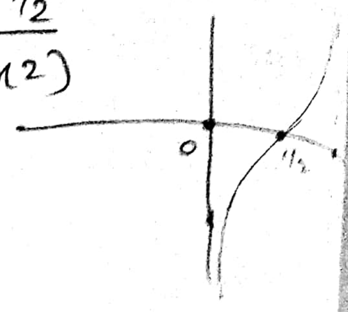
Proof: we know that the set $I = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is uncountable and also $I \subset \mathbb{R}$ $\therefore$ I is uncountable $\Rightarrow \mathbb{R}$ " "

Scanned by CamScanner

$(0,1) \approx [0,1] \approx [0,1)$

Thm: There is a one-to-one correspondence from the open interval $(0,1)$ and $\mathbb{R}$ (entire real set)

Proof: define a map $f:(0,1) \to \mathbb{R}$ $\quad f(x) = \dfrac{x - 1/2}{(x - x^2)}$



$0 < x_1 < x_2 < 1 \Rightarrow f(x_1) < f(x_2)$

$\therefore f$ is strictly increasing

Thm: the interval $(0,1)$ & $(0,1]$ have the same cardinality

Proof: let $A = \{ \frac{1}{n} \mid n > 1,\ n \text{ is an } \mathbb{Z} \}$

if $x \in (0,1) - A$ $\quad f(x) = x$

$f(\frac{1}{n}) = \frac{1}{n-1} \quad n > 1$

Well ordering principle: every non empty set of +ve integers contains a least element.

Thm: If $A \to \infty$ subset of $\mathbb{N}$, then $A \approx |\mathbb{N}|$ ie. A is countable

Proof: since $A \neq \phi$ & $A \subseteq \mathbb{N}$ By well ordering principle $\exists$ a_1 (smallest integer) s.t. $a_1 \in A$ since A is ∞

$A - \{a_1\} \neq \phi$

let a_2 be the smallest element of $A - \{a_1\}$ since A is not finite we continue this process to get $a_{n+1} \in A - \{a_1 \ldots$

so the sequence $a_1, a_2, \ldots, a_n$ contains every element of A and the mapping $f: A \to |\mathbb{N}|$

$f(a_n) = n$

f is a bijection ie. A is countable

Thm: Infinite subset of a countable set is countable.

Proof: suppose $E \subset A$ and E is infinite. As A is countable we arrange the elements 'x' of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows, let n_1 be the smallest +ve Z s.t. $n_1 \in E$

having chosen $n_1, n_2, \ldots n_{k-1}$ $(k = 2, 3, \ldots)$

let n_k be the smallest integer greater than n_{k-1}

set x_{n_k} s.t. $x_{n_k} \in E$

$f(k) = x_{n_k}$ $(k = 1, 2, \ldots)$ $\quad \mathbb{N} \to E$ $\quad f(k) = x_{n_k}$

Thm: A_α is countable then $\bigcup_{n=1}^{\infty} A_n$ is countable

Thm: Show that set of rational no. is countable

Proof: we know Q^+ is countable and Q^- is equivalent to Q^+

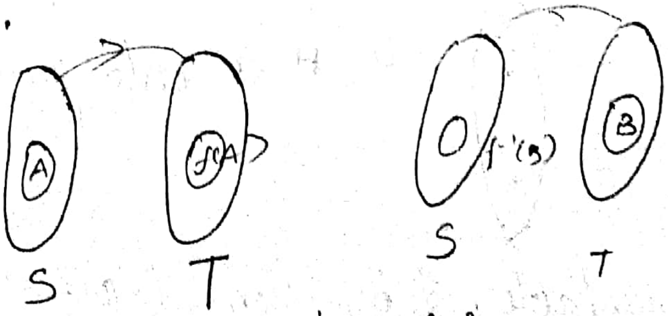
$f(Q^+) \to Q^-$ $\quad f(x) = -x$

$f \circ g \quad \mathbb{N} \to Q^-$ is a bijection

$Q = Q^+ \cup Q^- \cup \{0\}$ $\quad K: \mathbb{N} \to Q^+ \xrightarrow{+} Q^-$

hence Q^- countable.

$f: S \to T$



$f(A)$ = Image of A : $f(A) = \{ f(x) \mid x \in A \}$

$f^{-1}(B)$ = [illegible] image of B

$0 \qquad x \in A \cap B$

$x \in A \Rightarrow f(x) \in f(A) \quad y \in f(A) \Rightarrow$

$x \in B \Rightarrow f(x) \in f(B) \quad y \in f(B)$

Thm: $f(A \cap B) \neq f(A) \cap f(B)$

Proof: $S = \{s_1, s_2, s_3\} \quad T = \{t_1, t_2\}$

$A = \{s_1, s_2\} \quad B = \{s_2, s_3\}$

$f: S \to T \quad f(s_1) = t_1 \quad f(s_2) = t_2 \quad f(s_3) = t_1$

$A \cap B = \{s_2\}$

$f(A \cap B) = t_2 \quad f(A) = \{t_1, t_2\} \quad f(B) = \{t_2, t_1\}$

$t_2 \neq \{t_1, t_2\}$

<u>8/8/18</u> The following statements are equivalent

(a) S is atmost countable

(b) there exists a surjection of $\mathbb{N}$ onto S

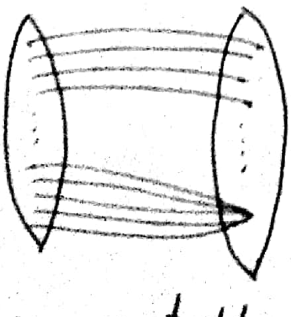
(c) there exists an injection of S into $\mathbb{N}$

Proof: (a) $\Rightarrow$ (b): Given: S is atmost countable.

- If S is finite, $\exists$ a bijection h from set $\mathbb{N}$ onto S and we define a map H on $\mathbb{N}$ by $H: \mathbb{N} \to S$ by

$$H(k) = \begin{cases} h(k) & \text{for } k = 1, 2, 3 \ldots n \\ h(n) & \text{for } k > n \end{cases}$$

$S = \{x_1, x_2, \ldots x_n\}$
$N = \{1, 2, \ldots n\}$
$h(1) = x_1 \quad h(2) = x_2 \ldots$
$h(n) =$



ie. H is onto from $\mathbb{N}$ to S

- If S is countable, $\exists$ bijection ...

Scanned by CamScanner

Proof: (b) ⇒ (c): If $H: \mathbb{N} \to S$ is onto
define $H_1 : S \to \mathbb{N}$ by taking $H_1(s)$ as the least element in the set. $H^{-1}(s) = \{n \in \mathbb{N} : H(n) = s\}$ (well ordering principle)

To show that H_1 is one-to-one, $H_1(s) = H_1(t) \Rightarrow s = t$

now if $s, t \in S$
$n_{st} = H_1(s) = H_1(t)$ $(n_{st} \in \mathbb{N})$

i.e. $H(n_{st}) = s$, $H(n_{st}) = t$ $\Rightarrow s = t \Rightarrow$ Hence, H_1 is one-one.

Proof: (c) ⇒ (a): If $H_1 : S \to \mathbb{N}$ is one-to-one then
$H_1 : S \to \mathbb{N}$ is one-to-one then $H_1 : S \to H_1(S)$ is bijective
then $H_1(S) \subset \mathbb{N}$

$\mathbb{N}$ is countable $\Rightarrow H_1(S)$ is countable

$f : \mathbb{N} \to H_1(S)$ $\quad f^{-1} : H_1(S) \to \mathbb{N}$

$H_1 : S \to H_1(S)$ $\quad H_1 : S \to H_1(S) \xrightarrow{f^{-1}} \mathbb{N}$

$f^{-1} \circ H_1 : S \to \mathbb{N} \Rightarrow$ i.e. S is countable.

Thm: Let A be the set of all sequences whose elements are digits 0 and 1. This set A is uncountable?
[the elements of A are sequences like 1, 0, 0, 1, 1, 0, 1, 1, ...

Proof: Let E be a countable subset of A.
As E is countable, E consists of the sequences s_1, s_2, s_3

$E = \{s_1, s_2, s_3, \dots\}$

We construct a sequence s as follows. If the n^{th} digit in s_n is 1, we let the n^{th} digits of s be 0 or vice versa. Then the sequence 's' differs from every element of E in atleast

one place. i.e. $s \notin E$ but $s \in A$ hence $E \subsetneq A$

$\Rightarrow$ Any countable subset of A is a proper subset of A
hence A is not countable.
(for otherwise A would be a proper subset of A, which is absurd).

9/8/18 A_m is a sequence of atmost countable sets so
$\bigcup_{m=1}^{\infty} A_m$ is atmost countable $\qquad$ $\bigcup_{\alpha \in I} A_\alpha$ - is arbitrary union
countable union

Proof: Given A_m is atmost countable so for each $m \in \mathbb{N}$, let
$\phi_m : \mathbb{N} \to A_m$ be a surjection
Define a map $\psi : \mathbb{N} \times \mathbb{N} \to A$ by $\psi(m,n) = \phi_m(n)$, $A = \bigcup_{m=1}^{\infty} A_m$
To show that ψ is onto, if $a \in A$ ie $a \in A_m$ for some $m \in \mathbb{N}$
Hence $\exists$ a least $n \in \mathbb{N}$ s.t. $a = \phi_m(n)$
$\therefore$ $a = \psi(m,n)$, ψ is onto.

$\rightarrow$ $P(\mathbb{N})$: Power set of natural no. is uncountable
show that there is no onto/surjective map from A to $P(A)$
suppose $\exists$ $\psi : A \to P(A)$ which is surjective
if $a \in A$ $<$ $a \in \psi(a)$, $a \notin \psi(a)$ $\qquad$ $\psi(a)$ is a set $\in P(A)$

$D = \{a \in A \mid a \notin \psi(a)\}$ then $D \subset A \Rightarrow D \in P(A)$
as ψ is onto for $D \in P(A)$ $\exists$ $a_0 \in A$ s.t. $\psi(a_0) = D$
if $a_0 \in \psi(a_0)$, $a_0 \notin D$ $\quad$ if $a_0 \notin \psi(a_0)$ $a_0 \in D$
but $\psi(a_0) = D$
$\therefore$ There is no onto map from a set to its power set.

Prove that if $0 \le a-b < \epsilon \ \forall \epsilon > 0$ then $a = b$

If $0 \le a < \epsilon \ \forall \epsilon$, $a = 0$

suppose $a \ne 0$ ie $a > 0$ define $\epsilon_a = \frac{a}{2} > 0$ (WLOG)

$\Rightarrow 0 \le a < \frac{a}{2} \Rightarrow a < a/2$ (not possible) $(\to\leftarrow)$ $\therefore a \ne 0$

$\Rightarrow a = 0$

$\hookrightarrow$ let $c = a-b$, using $\uparrow$ $c = 0$ $\therefore a-b=0$ ie $a = b$

Scanned by CamScanner

METRIC

Let $X \neq \phi$, a mapping $d: X \times X \to \mathbb{R}$ is called ... if the following conditions hold:

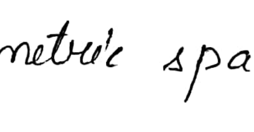
1. $d(x,y) \geq 0 \;\forall\, x,y \in X$
   $d(x,y) = 0$ iff $x = y$
2. $d(x,y) = d(y,x) \;\forall\, x,y \in X$
3. $d(x,y) + d(y,z) \geq d(x,z)$
   Triangle inequality

Metric space: (X, d) is called a metric space where X is a non-empty set and d is a metric.

- $(\mathbb{R}, d = |\,,|)$ is a metric space
  (1) $d(x,y) = |x-y| \geq 0 \;\forall\, x,y \in \mathbb{R}$
  $d(x,x) = |x-x| = 0 \;\forall\, x \in \mathbb{R}$
  (2) $d(x,y) = |x-y| = |y-x| = d(y,x)$
  (3) $d(x,y) + d(y,z) \geq d(x,z)$
  $d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z|$
  $\leq d(x,y) + d(y,z)$
- $(\mathbb{R}, d)$ with $d = |x^2 - y^2|$ is not a metric space
  $\because d(1,-1) = |1^2 - (-1)^2| = 0,\; 1 \neq -1$
- $(\mathbb{R}, d)$ $\quad d(x,y) = \dfrac{|x-y|}{1+|x-y|} \quad d(y,z) = \dfrac{|y-z|}{1+|y-z|} \quad d(z,x) = \dfrac{|x-z|}{1+|x-z|}$

10/8/18

- $(\mathbb{R}^n, d)$
  $x \in \mathbb{R}^n,\; x = (x_1, x_2, \dots x_n);\; x_j \in \mathbb{R} \quad j = 1,2,\dots,n$
  $y \in \mathbb{R}^n \quad y = (y_1, y_2, \dots y_n);\; y_j \in \mathbb{R} \quad j = 1,2,\dots,n$
  Dot product or scalar product of x and y
  $$x \cdot y = \sum_{j=1}^{n} x_j y_j$$



$|x| = \sqrt{(x_1-0)^2 + (x_2)^2} = \sqrt{x_1^2 + x_2^2}$

$$\|x\| = \sqrt{\sum_{j=1}^{n} x_j^2} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$x = (x_1, x_2) \quad y = (y_1, y_2)$
$x \cdot y = x_1 y_1 + x_2 y_2$ $\quad |x|$ = length of vector from origin to x

Scanned by CamScanner

$\ldots + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2$

$x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, to prove $d(x,y) + d(y,z) \geq d(x,z)$

usual metric

$d(x,y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$

Prob: for $x, y \in \mathbb{R}$ define $d^*(x,y) = \dfrac{|x-y|}{1+|x-y|}$

test whether d^* is a metric space or not

Soln: $|x-y| \geq 0 \Rightarrow d^*(x,y) \geq 0$ ✓ $\quad 1+|x-y| > 0$

$d^*(x,y) = 0 \Leftrightarrow \dfrac{|x-y|}{1+|x-y|} = 0 \Rightarrow |x-y| = 0 \Rightarrow x = y$

$\therefore d^*(x,y) = 0$ iff $x = y$. ✓

$d^*(y,x) = \dfrac{|y-x|}{1+|y-x|} = \dfrac{|x-y|}{1+|x-y|} = d^*(x,y)$ ✓

$d^*(x,y) + d^*(y,z) \geq d^*(x,z)$!!

$\dfrac{|x-y|}{1+|x-y|} + \dfrac{|y-z|}{1+|y-z|} \qquad \dfrac{|x-z|}{1+|x-z|}$

$\dfrac{|x-y| \; 2|x-y||y-z| + 1}{(1+|x-y|)(1+|y-z|)} \qquad \dfrac{|x-z|}{1+|x-z|}$

$1 + |x-y| + |y-z| + |x-y||y-z|$

$1 + |x-y| + |y-z| \geq 1 + |x-y|$

$\dfrac{|x-y|}{1+|x-y|+1}$

$$\boxed{\begin{array}{c} m \geq k \geq 0 \\ \dfrac{m}{1+m} \geq \dfrac{k}{1+k} \end{array}}$$

$\dfrac{|x-y|}{1+|x-y|} + \dfrac{|y-z|}{1+|y-z|} \geq \dfrac{|x-y|}{1+|x-y|+|y-z|} + \dfrac{|y-z|}{1+|y-z|+|x-y|}$

$= \dfrac{|x-y|+|y-z|}{1+|x-y|+|y-z|}$ — (1)

$x \geq'$

$\overbrace{|x-y|+|y-z|}^{m} \geq \overbrace{|x-z|}^{k}$

$\dfrac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \geq \dfrac{|x-z|}{1+|x-z|}$ — (2)

$\therefore$ from (1) and (2).

$\dfrac{|x-y|}{1+|x-y|} + \dfrac{|y-z|}{1+|y-z|} \geq \dfrac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \geq \dfrac{|x-z|}{1+|x-z|}$ //

Scanned by CamScanner

✓ In $\mathbb{R}$, $d_1(x,y) = |x^2 - y^2|$ — not a metric
$d_2(x,y) = 2|x-y|$ — is a metric
$d_3(x,y) = glb\{2, |x-y|\}$

Pg 44 | Ex 11 | W. rudin: ✓ for $x,y \in \mathbb{R}$, define $d_1(x,y) = (x-y)^2$, $d_2(x,y) = \sqrt{|x-y|}$
$d_3(x,y) = |x-2y|$ $d_4(x,y) = \dfrac{|x-y|}{1+|x-y|}$ is a metric

d_1: $d_1(x,y) \geq 0$ ✓ $d_1(x,y) = 0 \Rightarrow (x-y)^2 = 0 \Rightarrow x = y$ ✓ $d_1(x,y) = d_1(y,x)$
$d_1(x,z)$: $(x-y)^2 + (y-z)^2 \geq (x-z)^2$?
not true in all cases.

$x - y = a$
$y - z = b$
$z - x = c$
$a + b + c = 0$
$a^2 + b^2 \geq c^2$
$a + b = -c$
$a^2 + b^2 + 2ab = c^2$
$a^2 + b^2 \geq c^2$

Ex: $x = 0,\ y = 1,\ z = 2$
$1 + 1 \geq 4$ ✗ ∴ d_1 is not a metric

d_2: $\sqrt{|x-y|} \geq 0$ $\sqrt{|x-y|} = 0 \Rightarrow x = y$
d_2 is a metric

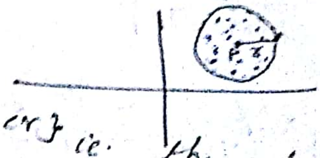
10 in midsem: some $d(x,y)$ is metric or not
solve rudin!!

d_3: not symmetric $|x-2y| = d_3(x,y) \neq d_3(y,x) = |y-2x|$ d_3 - not metric

→ Let X be a metric space (X,d), $Y \subset X$ then (Y,d) is also a metric space. i.e. a subset of metric space is also a metric space.

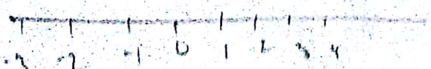
1.
Neighbourhood: of a point p with radius r $r > 0$ $p \in X$ is a set $N_r(p) = \{q \in X \mid d(p,q) < r\}$

deleted neighbourhood: $N_r'(p) = N_r(p) \setminus \{p\} = \{q : 0 < d(p,q) < r\}$ i.e. the circle without centre



2. **Limit point**
A point p is said to be an limit point of E if every neighbourhood of p has a point, $q \neq p$
if every nbhd $N_r(p) = N(p,r)$ intersects the set at pt q other than p
$q \neq p$

$\mathbb{Z}$ $1 \in \mathbb{R}$



In $\mathbb{R}$ $d_1(x,y) = |x-y|$ - is a metric
$d_2(x,y) = 2|x-y|$
$d_3(x,y) = glb\{2, |x-y|\}$

Pg 44 | Ex 11 | W. rudin :- for $x,y \in \mathbb{R}$, define $d_1(x,y) = (x-y)^2$ $d_2(x,y) = \sqrt{|x-y|}$
$d_3(x,y) = |x-2y|$ $d_4(x,y) = \frac{|x-y|}{1+|x-y|}$ is a metric

d_1: $d_1(x,y) \geq 0$ ✓ $d_1(x,y) = 0 \Rightarrow (x-y)^2 = 0 \Rightarrow x = y$ ✓ $d_1(x,y) = d_1(y,x)$

~~$d_1 + d$~~ $(x-y)^2 + (y-z)^2 \geq \cancel{y}\cdot(x-z)^2$?

not true in all cases.

Ex: $x = 0$, $y = 1$, $z = 2$
$1 + 1 \geq 4$ ✗ $\therefore d_1$ is not a metric

$x - y = a$
$y - z = b$
$z - x = c$
$a + b + c = 0$
$a^2 + b^2 \geq c^2$
$a + b = -c$
$a^2 + b^2 + 2ab = c^2$
$a^2 + b^2 \geq c^2$

d_2: $\sqrt{|x-y|} \geq 0$ $\sqrt{|x-y|} = 0 \Rightarrow x = y$

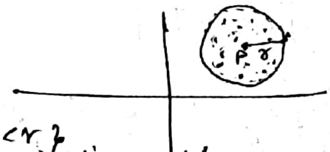
d_2 is a metric

1Q in midsem: some $d(x,y)$ is metric or not
solve rudin!!

d_3: not symmetric $|x-2y| = d_3(x,y) \neq d_3(y,x) = |y-2x|$ d_3: not metric

→ Let X be a metric space (X,d), $Y \subset X$ then (Y,d) is also a metric space. i.e. a subset of metric space is also a metric space

1. Neighbourhood: of a point p with radius r $r > 0$ $p \in X$ is a set $N_r(p) = \{q \in X \mid d(p,q) < r\}$



deleted neighbourhood: $N_r'(p) = N_r(p) \setminus \{p\} = \{q : 0 < d(p,q) < r\}$ i.e. the circle without centre

2. Limit point
A point p is said to be an limit point of E if every neighbourhood of p has a point, $q \neq p$

if every nbhd $N_r(p) = N(p,r)$ intersects the set at pt q other than p
$q \neq p$

$\mathbb{Z}$ $1 \in \mathbb{R}$

-3 -2 -1 0 1 2 3 4

Scanned by CamScanner

3. Isolated pt: if $p \in E$ and p is not a limit point, it is called an isolated point.

4. E is closed if every limit of E is a pt. of E

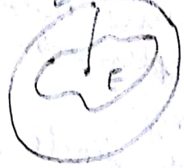
5. A point p is said to be an interior pt of $\exists$ a nbhd $N_r(p)$ s.t. $N_r(p) \subset E$

6. E is open if every point of E is an interior pt. OPEN SET

7. $E^c = \{x \mid x \notin E\}$

8. Perfect set: E is perfect if it is closed and every pt of E is a limit pt of E.

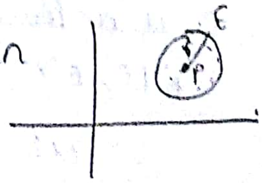
9. Bounded set: A set E is said to be bounded if $\exists\, q \in X$, $M \in \mathbb{R}$ s.t. $d(p,q) < M \;\forall\, p \in E$



10. Dense: A set E is said to be dense in X if every point of X is a limit pt of E or pt. of E or both.

Thm: Every neighbourhood is an open set.

let $E = N_r(p)$ be a neighbourhood of p with radius r



to show that E is open i.e every pt. of E is an interior pt. of E. Let $q \in E$ i.e $d(p,q) < r$

$$r - d(p,q) > 0$$

let $h = r - d(p,q)$. so $h > 0$

$$d(p,q) = r - h$$

for all points s in metric space, construct nbhd taking q as centre & s as radius. such that $d(q,s) < h$

$$d(p,s) \le d(p,q) + d(q,s)$$
$$\le r - h + d(q,s) \quad [d(q,s) < h]$$

$$d(p,s) < r \Rightarrow s \in E$$

Scanned by CamScanner

3. **Isolated pt**: if $p \in E$ and p is not a limit point, it is called an isolated point.

4. E is closed if every limit of E is a pt of E

5. A point p is said to be an interior pt if $\exists$ a nbhd $N_r(p)$ s.t. $N_r(p) \subset E$

6. E is open if every point of E is an interior pt. OPEN SET

7. $E^c = \{x \mid x \notin E\}$

8. Perfect set: E is perfect if it is closed and every pt of E is a limit pt of E.

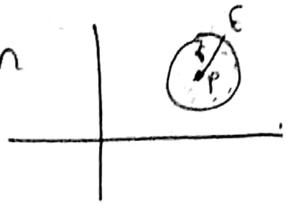
9. Bounded set: A set E is said to be bounded if $\exists\, q \in X$, $M \in \mathbb{R}$ s.t. $d(p,q) < M \;\forall\, p \in E$

10. Dense: A set E is said to be dense in X if every point of X is a limit pt of E or pt of E or both.

Thm: Every neighbourhood is an open set.

let $E = N_r(p)$ be a neighbourhood of p with radius r

to show that E is open i.e every pt of E is an interior pt of E. Let $q \in E$ i.e $d(p,q) < r$



$$r - d(p,q) > 0$$

let $h = r - d(p,q)$ so $h > 0$

$$d(p,q) = r - h$$

for all points s in metric space, construct nbhd taking q as centre & s as radius.

such that $d(q,s) < h$

$$d(p,s) \le d(p,q) + \underbrace{d(q,s)}_{<h}$$
$$\le r - h + d(q,s)$$

$$d(p,s) < r \Rightarrow s \in E$$

Scanned by CamScanner

and radius
$N(p, \epsilon)$ is the neighbourhood with centre 'p' and radius

16/8/18 $N_p(\epsilon) = \{q \in X \mid d(p,q) < \epsilon\}$

$N(p, \epsilon)$ is an open set i.e. every pt. of $N(p, \epsilon)$ is an interior ... then

Thm.
Let E be a subset of X. If p is a limit pt. of E if every neighbourhood of p contains infinitely many pts. of E. ($)

Corollary: If the set E is finite, it doesn't have a limit pt.

Proof: Because if it has a limit pt., then clearly nbhd of the pt. intersects the set into infinitely many pts. but it's not possible as the set is finite.

i.e. $N \cap E$ - contains infinite elements

$\Rightarrow$ ~~Hence~~ Every finite set is closed

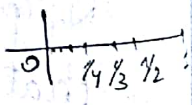
Proof: It doesn't have a limit point so it's vacuously true.

· consider · $E = \{\frac{1}{n} \mid n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \rightarrow$ infinite set

Is it closed?

0 is a limit point of E as in any neighbourhood $N(0, \epsilon)$ but $0 \notin E$

$\therefore$ set is not closed



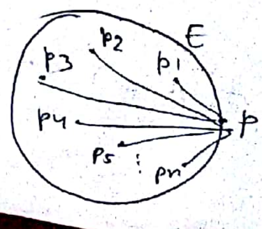
Proof ($): suppose $\exists$ a deleted neighbourhood N of p s.t. N $(N(p, \epsilon) - \{p\})$ contains only finite no. of pts $p_1, p_2, \dots p_n$ of E

$d(p, p_1) > 0, \ d(p, p_2) > 0, \dots d(p, p_n) > 0$

let $r = \min\{d(p_1, p_2), d(p_1, p_2) \dots d(p, p_n)\}$

min. of finite numbers is finite i.e. $r > 0$

Now $N_r(p)$ will not intersect any pt. other than p. Hence p is not a limit pt.



Examples: let us consider the following ~~examples~~ subsets of $\mathbb{R}^2$:

(a) the set of all complex number z s.t. $|z| < 1$

(b) " " " " $|z| \leq 1$

(c) A finite set

(d) The set of integers

(e) $E = \{\frac{1}{n} : n = 1, 2, \ldots\}$

(f) The set of all complex no.s (ie. $\mathbb{R}^2$)

(g) Open interval $(a,b) = \{x : a < x < b\}$

d, e, g are subsets of $\mathbb{R}$

	closed	open	perfect	bounded
(a)	NO	YES	NO	Yes
(b)	Yes	No	Yes	Yes
(c)	Yes	No	No	Yes
(d)	Yes	No	No	No
(e)	NO	~~Yes~~ No	No	Yes
(f)	Yes	Yes	Yes	No
(g)	No	~~No~~	No	Yes

Yes - subset of $\mathbb{R}$ - open
No - " " $\mathbb{R}^2$ - closed

17/8/18 $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

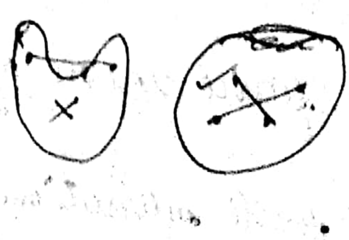
$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$a_j < b_j \;\forall\; j = 1, 2, 3, \ldots, k$

$x = (x_1, x_2, \ldots x_k) \in \mathbb{R}^k$

k-cell $= \{x : a_j \leq x_j \leq b_j \; ; \; j = 1, 2, \ldots k\}$

- cells are closed bounded interval
cells are rectangular



→ A subset $E \subset \mathbb{R}^k$ is said to be convex if $\forall\; x, y \in E$

$\lambda x + (1-\lambda) y \in E \qquad 0 \leq \lambda \leq 1$

Scanned by CamScanner

open ball in $\mathbb{R}^k$: $B(x,r) = \{ y \in \mathbb{R}^k \mid |y-x| < r \}$

closed " " $\bar{B}(x,r) = \{ y \in \mathbb{R}^k \mid |y-x| \leq r\}$

B $\bar{B}$ is convex sets

- $x_j \in E$, $\boxed{\sum \lambda_j x_j}$ linear combination
$\lambda_j \geq 0$ $\sum \lambda_j = 1$ $\Rightarrow$ convex combination

- let $y, z \in B(x,r)$ i.e. $|y-x| < r$ $|z-x| < r$

prove $\lambda y + (1-\lambda) z \in B(x,r)$ (or) $\lambda z + (1-\lambda) y \in B(x,r)$

$$|\lambda y + (1-\lambda) z - x|$$
$$= |\lambda (y-x) + \lambda x - x + (1-\lambda) z|$$
$$= |\lambda (y-x) + (1-\lambda)(z-x)| \leq \lambda |y-x| + (1-\lambda)|z-x|$$
$$< \lambda r + (1-\lambda) r$$
$$= r$$

By Δle inequality

applies for B & $\bar{B}$ ($<r$, $\leq r$)

- let $\{G_\alpha\}$ be a collection of subsets of $\mathbb{R}$
$\{F_\alpha\}$ " " " "

suppose $\{G_\alpha\}$ is open $\forall \alpha$, $\{F_\alpha\}$ is closed $\forall \alpha$

consider the arbitrary union $\bigcup_{\alpha \in I} G_\alpha$ if G is open then this union " loss

arbitrary union of open set is open

- " intersection " closed " " closed $\bigcap_{\alpha \in I} F_\alpha \to$ closed

finite union of closed sets is closed $\bigcap_{j=1}^{n} G_j \to$ open

finite intersection of open set is open $\bigcup_{j=1}^{n} F_j \to$ closed

$G_n = (-1/n, 1/n)$

$\bigcap G_n = \{0\} \to$ not closed

$\Rightarrow$ infinite intersection of open set is not open.

. $\mathbb{R}$ is both open and closed

. compliment of open set is closed

" " closed " open

→ A collection $\{G_\alpha\}$ of open sets τ (subsets of $\mathbb{R}$) i.e. $\tau = \{G_\alpha\}$ is said to be a topology if

(a) $\phi, \mathbb{R} \in \tau$

(b) $\bigcup_{\alpha \in I} G_\alpha \in \tau$

(c) $\bigcap_{j=1}^{n} G_j \in \tau$

(b), (c) are prop. of open sets. hence τ is an open collection.

$(\mathbb{R}, \tau)$ is called a topological space.

consider $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$

τ is a topology ∵ (a), (b), (c) are satisfied.

$\tau = \{\phi, X, \{a\}, \{b\}\}$ → not a topology ∵ ?

Thm: let $\{E_\alpha\}$ be a (finite or infinite) collection of sets.

then $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$

arbitrary union

Proof: let $A = \left(\bigcup_\alpha E_\alpha\right)^c$, $B = \bigcap_\alpha E_\alpha^c$: To show that $A = B$ we prove $A \subseteq B$ and $B \subseteq A$

let $x \in A$ i.e. $x \notin \bigcup_\alpha E_\alpha$ i.e. $x \notin E_\alpha$ for any α

i.e. $x \notin \bigcap_\alpha E_\alpha$ ⇒ $x \in E_\alpha^c$ for any α

i.e. $x \in \bigcap_\alpha E_\alpha^c$ ⇒ $x \in \bigcap_\alpha E_\alpha^c$ i.e. $x \in B$

i.e. $A \subseteq B$.

similarly let $x \in B$ i.e. $x \in \bigcap_\alpha E_\alpha^c$ i.e. $x \in E_\alpha^c$ for all α

i.e. $x \notin E_\alpha$ for all α

i.e. $x \notin \bigcup E_\alpha$

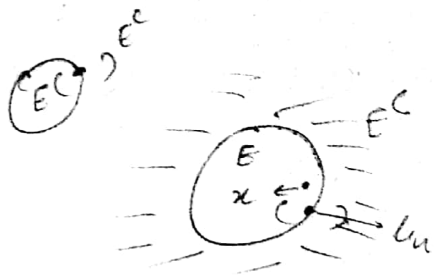
⇒ $x \in \left(\bigcup E_\alpha\right)^c$ ⇒ $x \in A$

∴ $B \subseteq A$

Scanned by CamScanner

$\left(\bigcup_\alpha A_\alpha\right)^c = \bigcap A_\alpha^c$ to prove countable union of sets consider the sets $A_1, A_2, \dots A_n, A$

$A_{n+1} = \phi = A_{n+2} = A_{n+3} \dots$



Thm:

- A set E is open iff E^c is closed.

Proof: suppose E^c is closed. To show that E is open, we have to prove every pt. of E is an interior pt.

let $x \in E \Rightarrow x \notin E^c$. Given E^c is closed, x is not a limit point of E^c i.e. $\exists$ a nbhd N of x s.t. $N \cap E^c = \phi$.

$\Rightarrow x \in N \subset E$. i.e. E is open.

$\hookrightarrow$ contained inside E so every pt. is interior.

Thm: ~~A set E is closed if E^c is open.~~ suppose E is open, to show that E^c is closed

i.e. every pt is a limit pt. let x be a limit pt. of E^c i.e. every nbhd of x contains pts of E^c. hence x is not an interior pt. of E, since E is open, $x \notin E$ i.e. $x \in E^c$

hence E^c is closed.

corollary: A set F is closed iff F^c is open.

replaced E^c by F in above thm.

Thm: (a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open

(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed

(c) " " finite collection $G_1, G_2, \dots, G_n$ of open sets $\bigcap_{j=1}^{n} G_j$ is open

(d) For any finite collection $F_1, F_2, \dots F_n$ of closed sets, $\bigcup_{j=1}^{n} F_j$ is closed.

Scanned by CamScanner

(a) $A = \bigcup_{\alpha \in I} G_\alpha$ prove every pt. of A is an interior pt of A

i.e. ~~G~~ A is open.

let $x \in A \Rightarrow x \in G_\alpha$ for some α

as G_α is open $\forall \alpha$ $\exists$ neighbourhood N_r

s.t. $x \in N_r \subset G_\alpha$ i.e. x is an interior pt. of G_α

union - some α
intersectn. $\forall \alpha$

(a) & (c) ; (b) & (d) $\exists$ Topology $\phi, \mathbb{R}$ are both open and closed

23/8/18 ~~Theorem:~~ A Topology on $\mathbb{R}$: A collection $\mathcal{J}$ is said to be a topology on $\mathbb{R}$ if (a) $\phi, \mathbb{R} \in \mathcal{J}$ (b) $\{A_\alpha\} \in \mathcal{J} \Rightarrow \bigcup A_\alpha \in \mathcal{J}$

(c) $\bigcap_{j=1}^{n} A_j \in \mathcal{J}$

$(\mathbb{R}, \mathcal{J})$ is called a topological space

Proof: (a) let $G = \bigcup_{\alpha \in I} G_\alpha$ to show that G is open i.e. every pt. of G is an interior pt. of G.

let $x \in G \Rightarrow x \in G_\alpha$ for some α i.e. x is an interior pt. of G_α

i.e. $\exists N(x, t) \subseteq G_\alpha$ $\{A \subset B \Rightarrow \bigcup A \subset \bigcup B\}$

$\bigcup N(x, \alpha) \subset \bigcup_\alpha G_\alpha = G$ $\therefore N(x, \alpha) \subset G$

(b) $\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha F_\alpha^c$

F_α is closed $\Rightarrow F_\alpha^c$ is open $\Rightarrow \bigcup_\alpha F_\alpha^c$ is open

$\Rightarrow \left(\bigcup_\alpha F_\alpha^c\right)^c$ is closed

$\Rightarrow \bigcap_\alpha F_\alpha$ is closed

(c) let $G = \bigcap_{i=1}^{n} G_i$ $x \in G \Rightarrow x \in G_i$ $\forall i$ but each G_i is open

$\exists r_i > 0$ s.t. $N(x, r_i) \subset G_i$ $i = 1, 2, \ldots n$

define $r = \min(r_1, r_2, \ldots, r_n)$ $r > 0$

now $N(x, r) \subset N(x, r_j)$ $\forall j$

$N(x, r) \subset N(x, r_1)$
$N(x, r) \subset N(x, r_2)$
$\vdots$
$N(x, r) \subset N(x, r_n)$

$\Rightarrow N(x, r) \subset G$

Scanned by CamScanner

(d) $\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c$

F_i^c is open set for each i

so $\bigcap_{i=1}^{n} F_i^c$ is open [proved in (c)]

$\Rightarrow \left(\bigcap_{i=1}^{n} F_i^c\right)^c = \bigcup_{j=1}^{n} F_j$ is closed

→ we cannot drop finiteness in (c) and (d) respectively.

→ let $G_n = (-1/n, 1/n)$, G_n is open subset of $\mathbb{R}$ for each n

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$$

let $x \in \bigcap_{n=1}^{\infty} (-1/n, 1/n)$ $\quad -1/n < x < 1/n$ $\quad$ similarly $x > -1/n \ \forall n$

$x < 1/n \ \forall n$ $\quad \Rightarrow x \geq 0 \ \forall n$

$\Rightarrow x \leq 0$

$\therefore x = 0$

→ set $F_n = \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$

$F_1 = [0, 0] = \{0\}$ $\qquad F_3 = [-2/3, 2/3]$

$F_2 = [-1/2, 1/2]$ $\qquad F_4 = [-3/4, 3/4]$

$\bigcup_{i=1}^{\infty} F_i = (-1, 1)$ proof: let $x \in \bigcup_{n=1}^{\infty} F_n$

i.e. $x \in F_n$ for some n

$-1 + 1/n \leq x \leq 1 - 1/n$

$-1 < -1 + 1/n \leq x \leq 1 - 1/n < 1$

$\Rightarrow -1 < x < 1$

$\Rightarrow x \in (-1, 1)$

Scanned by CamScanner

24/8/18

$E \subset X$

let E' denotes the set of all limit pts. of E

$\bar{E} = E \cup E'$ — closure of a set

$\bar{E} = E \cup E'$, $E' \subset \bar{E}$, $E \subset \bar{E}$

$E = (0,1)$, $E' = [0,1]$

$E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, $E' = \{0\}$

Theorem: If X is a metric space and $E \subset X$,

(a) $\bar{E}$ is closed

(b) $E = \bar{E}$ iff E is closed

(c) $\bar{E} \subset F$ for every closed set $F \subset X$ s.t. $E \subset F$

Proof: a) If $p \in X \Rightarrow p \in \bar{E}$, the result follows

ie $\bar{E}$ is closed.

$\bar{E}$ is E along with boundary

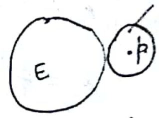
suppose $p \in X$ and $p \notin \bar{E}$

Hence p is neither a point of E nor a limit pt. of E.

Hence p has a neighbourhood which doesnt intersect E.

ie p is an interior pt. of $\bar{E}^c$ ie $\bar{E}^c$ is ~~clo~~ open

$\Rightarrow \bar{E}$ is closed.



$\exists$ an open nbhd doesn't intersect E, it lies in $\bar{E}^c$

closure of a set is always closed in a metric space//

b) If $E = \bar{E}$, then E is closed [from (a)] $\because \bar{E}$ is closed

If E is closed, to show that $E = \bar{E}$ ie $\bar{E} \subset E$

as E is closed, $E' \subset E$

$E \cup E' \subset E \cup E$

$\Rightarrow \bar{E} \subset E$

we know $E \subset \bar{E}$ $\therefore$ $E = \bar{E}$

(c) $A \subset B$ then $A' \subset B'$

proof: let $x \in A'$ ie every nbhd of x intersects A.

as $A \subset B$ it intersects B also.

for every closed set ~~$E \subset$~~ $F \supset E$ (F contains E)

$\Rightarrow F \supset E'$

$\bar{F} = F \cup F'$

hence $F \supset E'$

$F \cup E \supset E' \cup E$

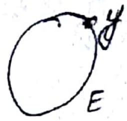
as F is closed, F contains all its limit pts ie. $F \supset F'$ as $F \supset E$

Scanned by CamScanner

Theorem: let 'E' be a nonempty set of real no.s which is bounded above. let $y = \sup E$. then $y \in \bar{E}$ hence $y \in E$ if E is closed.

Proof: if $y \in E$ then $y \in \bar{E} = E \cup E'$

assume $y \notin E$. for any $h > 0$ $\exists$ a pt. $x \in E$ s.t. $y - h < x < y$



$x \le y$ if ... $x < y$ ...

as $x \in E$, y is a least upper bound, $x \le y$

but $x \ne y$ as $x \in E$ and $y \notin E$

hence $x < y$

suppose $y - h \ge x$ i.e. $x \le y - h$ $\Rightarrow$ $y - h$ is an upper bound of E which is smaller than y ($\rightarrow\leftarrow$)

$\therefore y = \sup E$

$\therefore y - h < x$ is true i.e. y is a limit pt. of E $\Rightarrow y \in \bar{E}$

→ $E \subset Y \subset X$ X - metric space Y - subset of X

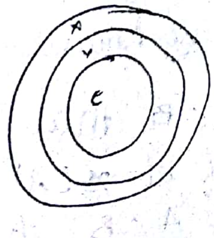
suppose E is open relative to Y then is E open relative to X? NO

let $Y = \mathbb{R}$, $X = \mathbb{R}^2$, $E = (a,b)$ $\mathbb{R} \subset \mathbb{R}^2$, $a \to (a,0)$

(a,b) is open subset of $\mathbb{R}$ but (a,b) is not an open subset of $\mathbb{R}^2$

Hence no nbhd of any pt. $x \in (a,b)$ contains inside (a,b)

Thm: suppose $Y \subset X$. A subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset G of X.



$E \subset Y \subset X$

E is open in X

$\forall p \in E$, $\exists r_p$ s.t. $d(p,q) < r_p$, $q \in X$

$\Rightarrow q \in E$

Scanned by CamScanner

bounded

$\subseteq Y$ $\forall y \in$

$\in Y$ $\because$ $\forall \in E$

E

$\Rightarrow \Leftarrow$)

NO

set of $\mathbb{R}^2$

Y iff

E is open in Y. $\forall p \in E$ $\exists$ $r_p > 0$ s.t. $d(p,q) < r_p$, $q \in Y$ $\Rightarrow q \in E$

Proof: suppose G is open in X and $E = Y \cap G$. To show that E is open relative to Y, i.e. every pt. of E is an interior pt. relative to Y.

$\forall p \in E$, $p \in G$ as G is open, every nbhd V_p of p is contained in G. i.e. $V_p \subset G$

$V_p \subset G$

$p \in V_p$ $(V_p \cap Y) \subset (G \cap Y)$

$(V_p \cap Y) \subset E$ $(\because E = Y \cap G)$

• suppose E is open relative to Y. To each $p \in E$ there is a +ve no. r_p s.t. $d(p,q) < r_p$; $q \in Y \Rightarrow q \in E$

let V_p be the set of all $q \in X$ s.t. $d(p,q) < r_p$ and define

$G = \bigcup_{p \in E} V_p$, V_p is open and arbitrary union of open sets is open

as G is open subset of X

Now, since $p \in V_p$, $\forall p \in E$ $E \subset (G \cap Y)$ — (i)

$E \subset Y$ now by the choice of V_p, we have $V_p \cap Y \subset E$ for every $p \in E$

$E \subset G$

~~$\cup (V_p \cap Y) \subset E$~~

$\Rightarrow$ ~~$G \cap Y$~~

$[\cup V_p \cap Y] \subset E$

$[G \cap Y] \subset E$ — (ii)

(i) & (ii) $\Rightarrow$ $G \cap Y = E$

OPEN COVERING

→ $\{G_\alpha\}$ is a collection of open sets. It is called an open cover of E if $\boxed{E \subset \bigcup_{\alpha \in I} G_\alpha}$

29/8/18 (absent)

→ A subset E of a metric space X is said to be compact if every open covering of E has a finite subcovering.

Scanned by CamScanner

$E \subset \bigcup_{\alpha \in I} V_\alpha$ $\{V_\alpha\}$ is an open covering

then $E \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n}$

eg: set $(0,1)$ is not compact as the open covering $I_n = (0, \frac{1}{n+1})$ does not have a finite subcovering.

$(0,1)$ can't be covered by finite no. of $I_1, I_2, \dots I_k$

Theorem: Suppose $K \subset Y \subset X$ then K is compact relative to X iff K is compact relative to Y.

Proof: Let K be compact relative to X. To show that K is compact relative to Y.

Let $\{V_\alpha\}$ be an open covering relative to Y. As V_α is open in Y, $\exists$ G_α open in X s.t. $V_\alpha = Y \cap G_\alpha$

$\because K$ is compact $\exists$ finite indices $\alpha_1, \alpha_2, \dots \alpha_n$ s.t.

$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$

($K \subset Y$ given)

$(K \cap Y) \subset (G_{\alpha_1} \cup G_{\alpha_2} \dots) \cap Y$

$\Rightarrow K \subset (G_{\alpha_1} \cap Y) \cup (G_{\alpha_2} \cap Y) \cup \dots \cup (G_{\alpha_n} \cap Y)$

$K \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n}$

Converse: Let $\{G_\alpha\}$ be collection of open subset of X s.t. covers K and put $V_\alpha = Y \cap G_\alpha$

V_α is open in Y, $V_\alpha \subset G_\alpha$ $\forall \alpha$ (K is compact relative to Y)

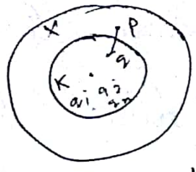
$K \subset (V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n}) \subset (G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n})$

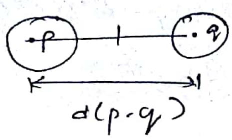
Theorem: Compact subsets of metric spaces are closed.

Proof: Let K be a compact subset of a metric space X to show that K^c is open.

let $p \in X$ and $p \notin K$

if $q \in K$, let V_q and W_q be the neighbourhood of p & q of radius $< \frac{1}{2} d(p,q)$





as K is compact $\exists\ q_1, q_2, \ldots, q_n$ s.t. K is ~~cove~~ cover $\bar{c}$

$W_{q_1} \cup W_{q_2} \cup \cdots \cup W_{q_n} = W$

$p \in V_{q_1} \cap V_{q_2} \cap \cdots \cap V_{q_n} = V \subset K^c$

i.e. K^c is open $\Rightarrow$ K is closed

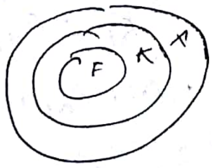
30/8/18

Theorem: closed subsets of compact sets are compact.

Proof: let $F \subset K \subset X$. F is closed subset of X. K is compact

let $\{V_\alpha\}$ be an open covering of F.

Now. adjoin F^c to the collection $\{V_\alpha\}$

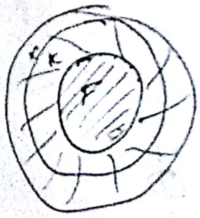


$\Omega = \{V_\alpha \cup F^c\}$

now Ω is a covering of K

K is compact $\exists$ a finite subcovering Φ which covers K

and also covers F as $F \subset K$



$\{F^c \cup V_\alpha\}$

In Φ if F^c is there we can remove it. still the subcollection is finite. i.e. F is compact

Corollary: F is closed and K is compact subsets of X. Then $F \cap K$ is compact.

Proof: $F \cap K \subset K$ [i.e. show that intersectn of closed set and compact set is closed]

K is compact $\Rightarrow$ K is closed.

$F \cap K$ (intersection of closed sets is closed)

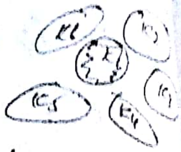
$\hookrightarrow$ closed

Scanned by CamScanner

closed subset of a compact set is compact $\Rightarrow F \cap K$ is compact

Theorem: Let $\{K_\alpha\}$ be a collection of compact subsets of X with every finitely many intersections of K_α is non-empty then $\cap K_\alpha \neq \phi$. [if $\bigcap_{j=1}^{m} K_j \neq \phi$ then $\bigcap_{j=1}^{p} K_j \neq \phi$ or $\bigcap_{j=k}^{p} K_j \neq \phi$]

for each α, $K_{\alpha_1} \cap K_{\alpha_2} \cap \dots \cap K_{\alpha_n} \neq \phi \Rightarrow \cap K_\alpha \neq \phi$ [so compact cannot be disjoint]



Proof: fix K_1, from the collection $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^c$

so G_α is an open covering (i.e. collectⁿ of open sets)

assume that no point of K_1 belongs to K_α

hence $\{G_\alpha\}$ is an open covering of K_1

as K_1 is compact $\exists$ indices $\alpha_1, \alpha_2, \dots, \alpha_n$ s.t.

$$K_1 \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$$

$K_1 \cap K_{\alpha_1} \cap K_{\alpha_2} \dots \cap K_{\alpha_n} = \phi$ $(\Rightarrow\Leftarrow)$ $\therefore \cap K_\alpha \neq \phi$

Corollary: If $\{K_\alpha\}$ is a sequence of non-empty compact sets s.t. $K_n \supset K_{n+1}$ $(n = 1, 2, \dots)$ then $\bigcap_{n=1}^{\infty} K_n \neq \phi$

i.e. $K_1 \supset K_2$ (K_1 contains K_2)
$K_2 \supset K_3$ (K_2 " K_3)
$\vdots$

Proof: $K_n \supset K_{n+1} \Rightarrow \bigcap_{n=1}^{m} K_n = K_m \neq \phi \Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \phi$

Theorem: If E is an infinite subset of a compact set K then E has a limit point in K.

$E \subset K$

$E = \{\frac{1}{n}\}$ $K = [0, 1]$

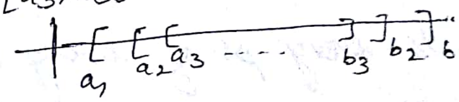
21/8/18

Proof: suppose no point of K were a limit pt. of E, then each $q \in K$ would have a nhbd V_q containing atmost one pt of E (namely $q \in E$). Hence E can't be covered by finitely many V_q ~~is open $\forall q \in X$~~ and hence K is not covered by finitely many V_q (as $E \subset K$). So K is not compact which contradicts as K is compact.

Theorem: If $\{I_n\}$ is a sequence of closed & bounded intervals i.e. $[a,b]$ in $\mathbb{R}$ such that $I_n \supset I_{n+1}$, $(n = 1, 2, \dots)$ then $\bigcap_{n=1}^{\infty} I_n \neq \phi$

Proof: let $I_n = [a_n, b_n]$, a_n, b_n are finite real no.s

$I_1 = [a_1, b_1]$ $I_2 = [a_2, b_2]$ $I_3 = [a_3, b_3] \dots$



let $E = \{a_n \mid a_n \in I_n\}$ then $E \neq \phi$ and bounded above by b_1. $\therefore$ supremum exists. let $x = \sup E$ (due to completeness principle)

if m and n are +ve integers then $a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$

i.e. $x \leq b_n$; $a_n \leq x$ $\Rightarrow$ $a_n \leq x \leq b_n$ i.e. $x \in I_n$ $\therefore \bigcap_{n=1}^{\infty} I_n \neq \phi$

Theorem: let $\{I_n\}$ be a sequence of k-cells with $I_n \supset I_{n+1}$ $(n = 1, 2, \dots)$ then $\bigcap_{n=1}^{\infty} I_n \neq \phi$

Proof: let $I_n = \{(x_1, x_2, \dots x_k) \mid a_{n,j} \leq x_j \leq b_{n,j} \; ; 1 \leq j \leq k\}$ $n = 1, 2, \dots$

$I_{nj} = [a_{n,j}, b_{n,j}]$ $n = 1, 2, \dots$ $j = 1, 2, \dots, k$

for fixed j the sequence satisfies the hypothesis of the prev. thm

hence $\exists\, x_j^*$ $(1 \leq j \leq k)$ s.t. $a_{nj} \leq x_j^* \leq b_{n,j}$ $(n = 1, 2, \dots$ $1 \leq j \leq k)$

$x^* = (x_1^*, x_2^*, \dots x_k^*)$ $\therefore \bigcap_{n=1}^{\infty} I_n \neq \phi$.

$x^* \in I_n$ for $n = 1, 2 \dots$

Scanned by CamScanner

A complex no. z is said to be algebraic if there are integers $a_0, a_1, \ldots a_n$ not all zero such that

$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n = 0$

ie a complex no. is algebraic if its a root of a polynomial eqn. with integer coefficients.

$\therefore$ for any +ve natural no. N there are only finitely many eqns. with $|a_0|+|a_1|+|a_2|+\ldots+|a_n| = N-n$

Construct a bounded set of real no.s with exactly 3 limit points.

$A = \{\frac{1}{n}\} \cup \{1+\frac{1}{n}\} \cup \{2+\frac{1}{n}\}$ $n \in \mathbb{N}$

0, 1, 2 are limit pts of A

8. Is every pt. of every open set $E \subset \mathbb{R}^2$ a limit pt. of E? Yes

" " " " closed " " " ? No

$A = \{(0,0)\} \subset \mathbb{R}^2$

$A' = \{(0,0)\}' = \phi$

6. let E' be the set of all limit pts of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ ~~are~~ have the same limit pt.

Do E and E' always have the same limit pt.? No

$E = \{1/n\}$ $E' = \{0\}$ $(E')' = \phi$

★ 10. let X be an infinite set. define $d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$

i) prove that d is a metric

ii) which subsets of the resulting metric space are open?

--- closed?

--- compact?

$d(p,q) = 0$ or $d(p,q) = 1$ $\therefore d(p,q) \geq 0$ ✓

$d(p,q) = 0 \Leftrightarrow p = q$ ✓

$d(p,q) = 0$ if $p = q$ $\qquad$ $d(p,q) = 1$ if $p \neq q$

$d(q,p) = 0 \because p = q$ $\qquad$ $d(q,p) = 1 \because p \neq q \Rightarrow q \neq p$.

$\Rightarrow q = p$

$\therefore d(p,q) = d(q,p)$

Scanned by CamScanner

$d(p,q) \leq d(p,w) + d(w,q) \quad \forall\, p,q,w \in X$

- $p = q = w$: $0 \leq 0+0$ ✓ $\qquad p \neq w \neq q$: $1 \leq 1+1$ ✓
- $p = w \neq q$: $1 \leq 0+1$ ✓ $\qquad p \neq w = q$: $1 \leq 1+0$ ✓
- $p = q \neq w$: $0 \leq 1+1$ ✓

every subset of X is open as well as closed w.r.t discrete metric

$p \in X$, p is not a limit pt.

since $d(p,q) = 1 > 1/2$ if $p \neq q$

there exists a nhbd $N_{\frac{1}{2}}(p)$ of p containing no pt. of $q \neq p$ s.t. $q \in X$

$$N_{1/2}(p) = \{q \in X ;\ d(p,q) < 1/2\} = \{p\}$$

$A \subset X$, A is open. every subset is open

$X - \underbrace{(X-A)}_{\text{subset is open}} = A$ is closed ; so its complement is closed.

$\therefore$ It is not compact.