Every k-cycle in S_n can be written as a product of transpositions.

Defn A permutation in S_n is called an even permutation if it is a product of even number of transpositions o.w it is called odd permutation.

Justification of the above defn:

Let $k[x_1, x_2, -, x_n]$ be a poly ring.

consider the poly

$$\boxed{P(x_1, x_2, \ldots, x_n) = \prod_{n \geq i > j \geq 1} (x_i - x_j)}$$

if $n=3$ $\quad P(x_1, x_2, x_3) = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$

let $\sigma \in S_n$ we define

$$\sigma P(x_1, \ldots, x_n) = \prod_{n \geq i > j \geq 1} \left(x_{\sigma(i)} - x_{\sigma(j)} \right)$$

Example $\sigma = (1\,2\,3) \in S_3$.

$$\sigma P(x_1, x_2, x_n) = \sigma\left((x_3 - x_2)(x_3 - x_1)(x_2 - x_1)\right)$$

$$= \left(x_{\sigma(3)} - x_{\sigma(2)}\right)\left(x_{\sigma(3)} - x_{\sigma(1)}\right)\left(x_{\sigma(2)} - x_{\sigma(1)}\right)$$

$$= (x_1 - x_3)(x_1 - x_2)(x_3 - x_2).$$

$$= P(x_1, x_2, x_3)$$

If σ is a transposition then $\sigma P = -P$

If σ is a product of even no of transpositions then $\sigma P = P$

Note that every k-cycle is a product of transposition

$$\boxed{(a_1 \cdots a_k) = (a_1\, a_k)(a_1\, a_{k-1}) \cdots (a_1\, a_3)(a_1\, a_2)}$$

$$(1\,2\,3) = (1\,3)(1\,2). \qquad (*)$$

$\sigma \in S_n$ is said to be an even permutation if $\sigma P = P$ and σ is said to be an odd permutation if $\sigma P = -P$.

From (*) if σ is a k-cycle and k is even then σ is an odd permutation and if k is odd then σ is an even permutation.

Consider S_n and let A_n denote the set of all even permutations of S_n.

Q Is A_n forms a group?

Yes A_n is a subgp of S_n.

Hint: X = Set of all even permutations

Y = Set of all odd permutations

$$f : X \longrightarrow Y$$

$$f(\sigma) = (12)\sigma.$$

(Ex) check that f is injective and surjective.

$$\therefore \quad |X| = |Y| = \frac{n!}{2}.$$

Group Homomorphism:

Let G and G' be two groups.

A map $f : G \to G'$ is called a homomorphism of groups if

$$f(gh) = f(g) \cdot f(h) \quad \forall g, h \in G.$$

gp op of G gp op of G'.

Examples of group homomorphism :

(1) $f : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^\times$

$$f(A) = \det A.$$

$f(AB) = \det AB$

$f(A) \cdot f(B) = \det A \cdot \det B.$

$$f(AB) = f(A) f(B)$$

$\therefore$ f is a gp homomorphism.

(2) $f : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^\times, \cdot)$

$$f(x) = e^x.$$

$$f(x+y) = e^{x+y} = e^x \cdot e^y = f(x) f(y).$$

(3) $\phi : \mathbb{Z} \rightarrow G$ where G is gp written multiplicatively

$\phi(n) = a^n$ for a fixed elt $a \in G$

$$\phi(n+m) = a^{n+m} = a^n \cdot a^m = \phi(n) \cdot \phi(m)$$

(4) Let H be any subgp of a gp G.

then $i : H \to G$ defined by

$i(x) = x$ is a gp homo.

(5) $f : S_n \longrightarrow \{1, -1\}$

$$f(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Properties of gp homomorphism:

(1) If $f : G \to G'$ is a gp homo then

$f(1_G) = 1_{G'}$.

and $f(a^{-1}) = f(a)^{-1}$

[use $1 \cdot 1 = 1$.

$f(1 \cdot 1) = f(1)$

$= f(1) \cdot f(1) = f(1)$

$\Rightarrow f(1) = 1_{G'}$]

(2) Let $G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3$

where f and g are group homo

then $g \circ f : G_1 \longrightarrow G_3$ is also a gp

homo.

<u>Remark</u>: Every group homo ϕ determine two important subgps its image and kernel.

The image of a gp homo is defined as $\operatorname{im}\phi = \{ x \in G' \mid x = \phi(a) \text{ for some } a \in G \}$

<u>Ex</u> Check $\operatorname{im}\phi$ is a subgp of G'.

The kernel of a gp homo is defined as $\ker\phi = \{ x \in G \mid \phi(x) = 1_{G'} \}$.

Note that $1_G \in \ker \phi$.

and if $f, g \in \ker\phi$ then

$$\phi(fg) = \phi(f)\cdot\phi(g) = 1_{G'}\, 1_{G'} = 1_{G'}$$

$$\Downarrow$$

$$fg \in \ker\phi$$

and if $a \in \ker\phi$ then

$$f(a^{-1}) = f(a)^{-1} = 1_{G'}^{-1} = 1_{G'}.$$

$$\therefore \quad a^{-1} \in \ker\phi$$

Thus $\ker\phi$ is a subgp of G.