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Lecture 5

Sol^n:- I) Using the def^n of $J_n(x)$, we have

$$\frac{d}{dx}\left[x^n J_n(x)\right] =$$

$$= \frac{d}{dx}\left\{x^n \sum_{m=0}^{\infty} (-1)^m \cdot \frac{1}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n}\right\}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \frac{1}{2^{2m+n}} \cdot \frac{d}{dx}\left(x^{2m+2n}\right)$$

$$= \sum_{m=0}^{\infty} \left[\frac{(-1)^m (2m+2n) \cdot x^{2m+2n-1}}{m!\,\Gamma(n+m+1)\, 2^{2m+n}}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \cdot 2\cancel{(m+n)} \cdot x^n \cdot x^{2m+n-1}}{m!\,\cancel{(n+m)}\,\Gamma(n+m)\, 2^{2m+n}}$$

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$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n-1+m+1)} \cdot x^{2m+(n-1)}$$

$2m+n-1$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n-1+m+1)} \left(\frac{x}{2}\right)^{2m+(n-1)}$$

$$= x^n J_{n-1}(x)$$

$$\left[\Gamma(n+1) = n\Gamma(n)\right]$$

[by the defn of $J_{n-1}(x)$]

(II) Using the defn of $J_n(x)$, we have

$$\frac{d}{dx}\left\{x^{-n} J_n(x)\right\} = \frac{d}{dx}\left\{x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n}\right\}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \cdot \frac{1}{2^{2m+n}} \cdot \frac{d}{dx}\left(x^{2m}\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \cdot \frac{1 \cdot (2m)\, x^{2m-1}}{2^{2m+n}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \cancel{2} \cdot x^{2m-1}}{\cancel{2} \cdot (m-1)!\,\Gamma(n+m+1)} \cdot \frac{1}{2^{2m+n-1}}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m \cdot x^{2m-1}}{(m-1)!\,\Gamma(n+m+1)} \cdot \frac{1}{2^{2m+n-1}}$$

[since $(m-1)! = \infty$ when $m=0$, so the term corresponding to $m=0$ vanishes]

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} \cdot x^{2m+2-1}}{m!\,\Gamma(n+m+2)} \frac{x^{m} \cdot x^{-n}}{2^{2m+2+n-1}}$$

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!\,\Gamma(n+m+2)} \times \left(\frac{x}{2}\right)^{2m+n+1}$$

[Let $m = M-1$ so that $M = m+1$ then $m=0$, when $M=1$ when $m=\infty$, $M=\infty$]

$$= -x^{-n} \sum_{M=0}^{\infty} \frac{(-1)^{M}}{M!\,\Gamma(n+1+M+1)} \left(\frac{x}{2}\right)^{n+1+2M}$$

$$= -x^{-n} J_{n+1}(x),$$

by the def^n of $J_{n+1}(x)$.

[On changing the variable of summation from m to M.]

(III) Recurrence Relation :-

$$\frac{d}{dx}\left\{x^n J_n(x)\right\} = x^n J_{n-1}(x)$$

or, $n x^{n-1} J_n(x) + x^n J_n'(x) = x^n J_{n-1}(x)$

Dividing both sides by x^{n-1},

$$n J_n(x) + x J_n'(x) = x J_{n-1}(x)$$

or, $\left(\frac{n}{x}\right) J_n(x) + J_n'(x) = J_{n-1}(x)$ $(\div x)$

or, $\boxed{J_n'(x) = J_{n-1}(x) - \left(\frac{n}{x}\right) J_n(x)}$

IV) Recurrence relation II is

$$\frac{d}{dx}\left\{x^{-n} J_n(x)\right\} = -x^{-n} J_{n+1}(x)$$

or, $-n x^{-n-1} J_n(x) + x^{-n} J_n'(x)$

$$= -x^{-n} J_{n+1}(x),$$

Dividing both sides by x^{-n},

$$-n x^{-1} J_n(x) + J_n'(x)$$

$$= -J_{n+1}(x)$$

or, $-\frac{n}{x} J_n(x) + J_n'(x) = -J_{n+1}(x)$

or, $\boxed{J_n'(x) = \left(\frac{n}{x}\right) J_n(x) - J_{n+1}(x)}$

V) From recurrence relations III & IV, we get

$$J_n'(x) = J_{n-1}(x) - \left(\frac{n}{x}\right) J_n(x)$$

$$J_n'(x) = \left(\frac{n}{x}\right) J_n(x) - J_{n+1}(x)$$

(+) $$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$\Rightarrow \boxed{J_n'(x) = \frac{1}{2}\left\{J_{n-1}(x) - J_{n+1}(x)\right\}}$$

(VI) (−)

$$0 = J_{n-1}(x) + J_{n+1}(x) - \frac{2n}{x} J_n(x)$$

or, $$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

yyxy

Q1) a) Show that $x^n J_n(x)$ is a solution of

$$x\left(\frac{d^2y}{dx^2}\right) + (1-2n)\frac{dy}{dx} + xy = 0.$$

H.W Ex b) Show that $x^{-n} J_n(x)$ is a solution of

$$x\left(\frac{d^2y}{dx^2}\right) + (1+2n)\frac{dy}{dx} + xy = 0.$$

Sol^n:- 1) a) Given,

$$x\left(\frac{d^2y}{dx^2}\right) + (1-2n)\frac{dy}{dx} + xy = 0. \rightarrow (1)$$

Let $y = x^n J_n(x) \rightarrow (2)$

From recurrence relation I, we have

$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$$

$$\Rightarrow \frac{dy}{dx} = x^n J_{n-1}(x) \quad [\text{by } (2)] \rightarrow (3)$$

From (3)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{x^n J_{n-1}(x)\right\}$$

$$= x^n J'_{n-1} + n x^{n-1} J_{n-1}$$

Substituting the above values of y, $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in (1), we get

$$x\left(x^n J'_{n-1} + n x^{n-1} J_{n-1}\right) + (1-2n)\, x^n J_{n-1}$$

$$+ x^{n+1} \underset{(=J_n)}{J_n} = 0$$

or, $x^{n+1} J'_{n-1} - (n-1)\, x^n J_{n-1} + x^{n+1} J_n = 0$

or, $x^{n+1}\left[J'_{n-1} - \left(\frac{(n-1)}{x}\right) J_{n-1}\right] + x^{n+1} J_n = 0$

$\longrightarrow$ (4)

From recurrence relation IV, we have

i.e., $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

so that $J_{n-1}'(x) = \frac{(n-1)}{x} J_{n-1}(x) - J_n(x)$

$$\Rightarrow J_{n-1}'(x) - \frac{(n-1)}{x} J_{n-1} = -J_n(x) \quad \rightarrow (5)$$

Using (5), eqn (4) reduces to

$$-x^{n+1} J_n + x^{n+1} J_n = 0$$

$\Rightarrow 0 = 0$, which is true

Hence, $x^n J_n$ is a solution of eqn (1).

Q2) (Lommel theorems)

Prove that

(i) $$\left(\frac{1}{x}\frac{d}{dx}\right)^m (x^n J_n) = x^{n-m} J_{n-m}$$

where m is a positive integer & $m < n$

HW (ii) $$\left(\frac{1}{x}\frac{d}{dx}\right)^m \left(x^{-n} J_n\right) = (-1)^m x^{-n-m} J_{n+m}$$

HW (iii) $$J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$$

n being positive integer.

Sol^n (i) From recurrence relation I we have

$$\frac{d}{dx}(x^n J_n) = x^n J_{n-1}$$

so that $\left(\frac{1}{x}\frac{d}{dx}\right)(x^n J_n)$

$$= x^{n-1} J_{n-1} \longrightarrow (1)$$

Now, $\left(\frac{1}{x}\frac{d}{dx}\right)^m (x^n J_n)$

$$= \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1} \underbrace{\left(\frac{1}{x}\frac{d}{dx}\right)(x^n J_n)}$$

$$= \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1} \left(x^{n-1} J_{n-1}\right) \quad [\text{by } (1)]$$

$$= x^{n-m} J_{n-m}$$

on proceeding as before
m times more $(m<n)$

Q2) Prove that

(i) $\boxed{J_0' = -J_1}$

(ii) $J_2 - J_0 = 2J_0''$

HW (iii) $J_2 = J_0'' - \left(\frac{1}{x}\right)J_0'$

HW (iv) $J_2 + 3J_0' + 4J_0''' = 0$.

Soln:- (i) Using recurrence relation IV,

$$xJ_n' = nJ_n - xJ_{n+1} \rightarrow (1)$$

Replacing n by 0 in eqn (1), we get

$$xJ_0' = -xJ_1$$

or, $\boxed{J_0' = -J_1}$

(ii) Recurrence relation V is

$$J_n' = \frac{1}{2}(J_{n-1} - J_{n+1})$$

$$\Rightarrow 2J_n' = J_{n-1} - J_{n+1} \rightarrow (2)$$

Diff. (2) w.r.t 'x', we get

$$2J_n'' = J_{n-1}' - J_{n+1}' \rightarrow (3)$$

Replacing n by (n-1) & (n+1) successively in eqn (2), we get

$$2J_{n-1}' = J_{n-2} - J_n \rightarrow (4)$$

$$\& \; 2J_{n+1}' = J_n - J_{n+2} \rightarrow (5)$$

Putting the values of J_{n-1}' & J_{n+1}' from (4) & (5) in eqn (3), we get

$$2J_n'' = \frac{1}{2}(J_{n-2} - J_n) - \frac{1}{2}(J_n - J_{n+2})$$

or, $$\boxed{4J_n'' = J_{n-2} - 2J_n + J_{n+2}} \longrightarrow (6)$$

Replacing n by 0 in (6), we get

$$4J_0'' = J_{-2} - 2J_0 + J_2$$

$$= (-1)^2 J_2 - 2J_0 + J_2 \qquad \left[J_{-n} = (-1)^n J_n \right.$$

$$\Rightarrow 4J_0'' = 2(J_2 - J_0)$$

$$\Rightarrow \boxed{2J_0'' = J_2 - J_0}$$

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Q3) Prove that

(i) $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x} - \sin x\right)$

HW (ii) $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$

i.e., $\sqrt{\frac{\pi x}{2}}\, J_{3/2}(x) = \frac{\sin x}{x} - \cos x.$

Soln :- (i) we know that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \rightarrow (1)$$

$$\& \; J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\cdot \sin x \rightarrow (2)$$

Using Recurrence relation (VI)

i.e., $J_{n-1}(x) + J_{n+1}(x) = \left(\frac{2n}{x}\right) J_n(x)$

$\rightarrow (3)$

Replacing n by $(-1/2)$ in eqn (3)

$$J_{-3/2}(x) + J_{1/2}(x) = -\frac{1}{x} J_{-1/2}(x)$$

$$\Rightarrow J_{-3/2}(x) = -J_{1/2}(x) - \frac{1}{x} J_{-1/2}(x)$$

$$= -\sqrt{\frac{2}{\pi x}} \sin x - \frac{1}{x^2}\sqrt{\frac{2}{\pi x}} \cos x$$

$$\boxed{= \sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x} - \sin x\right)}$$

H.W

(ii)

Q 4) If $n > -1$, show that

HW

$$\int_0^x x^{n+1} J_n(x)\,dx = x^{n+1} J_{n+1}(x)$$

Q 5) Prove that

(i) $J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots) = 1$

HW (ii) $|J_0(x)| \leq 1$

HW (iii) $|J_n(x)| \leq 2^{-1/2}$, when $n \geq 1$

HW Q 6) Prove that

(i) $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \left[(3 - x^2)/x^2 \right] \sin x - (3/x) \cos x \right]$

2 (ii) $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \left[(3-x^2)/x^2 \right] \cos x + (3/x) \sin x \right]$

Sol^n (i) From recurrence relation III & IV, we have

$$J_n' = (-n/x) J_n + J_{n-1} \rightarrow (1)$$

$$J_n' = (n/x) J_n - J_{n+1} \rightarrow (2)$$

Replacing n by $(n+1)$ in (1), we get

$$J_{n+1}' = -\frac{(n+1)}{x} J_{n+1} + J_n \rightarrow (3)$$

$$\therefore \frac{d}{dx}\left\{ J_n^2 + J_{n+1}^2 \right\} = 2 J_n J_n' + 2 J_{n+1} J_{n+1}'$$

$$\therefore \frac{d}{dx}\left\{J_n^2 + J_{n+1}^2\right\}$$

$$= 2J_n\left(\frac{n}{x}J_n - \cancel{J_{n+1}}\right)$$

$$+ 2J_{n+1}\left(-\frac{(n+1)}{x}J_{n+1} + \cancel{J_n}\right)$$

[by ② & ③]

$$= 2\left(\frac{n}{x}J_n^2 - \frac{(n+1)}{x}J_{n+1}^2\right) \longrightarrow ④$$

Replacing n by $0, 1, 2, 3, \ldots$ successively in ④, we get

$$\frac{d}{dx}\left(J_0^2 + J_1^2\right) = 2\left(0 - \frac{1}{x}J_1^2\right)$$

$$\frac{d}{dx}\left(J_1^2 + J_2^2\right) = 2\left(\frac{1}{x}J_1^2 - \frac{2}{x}J_2^2\right)$$

$$\frac{d}{dx}\left(J_2^2 + J_3^2\right) = 2\left(\frac{2}{x}J_2^2 - \frac{3}{x}J_3^2\right)$$

Adding these columnwise & noting that $J_n \to 0$ as $n \to \infty$, we get

$$\frac{d}{dx}\left[J_0^2 + 2(J_1^2 + J_2^2 + \cdots)\right] = 0$$

(m)

$$J_0^2(x) + 2\left[J_1^2(x) + J_2^2(x) + \cdots\right] = c \quad \rightarrow (5)$$

Replacing 'x' by '0' in (5) & noting that $J_0(0) = 1$ & $J_n(0) = 0$, $\forall n \geq 1$, we get

$$1 + 2(0 + 0 + \cdots) = c$$

$\Rightarrow c = 1$ $\therefore$ Hence, $J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = 1$