Lecture 11

Measurable functions.

Def:- Let f be an extended real valued function defined on a measurable set $E \subseteq \mathbb{R}$,

i.e. $f : E \to \mathbb{R} \cup \{\pm\infty\}$.

Then f is said to be a measurable function or a Lebesgue measurable function, if for each $\alpha \in \mathbb{R}$, the set $\{x \in E \mid f(x) > \alpha\} = f^{-1}((\alpha, \infty))$ is a measurable set.

Theorem:- Let $E \subseteq \mathbb{R}$ be a measurable set & $f : E \to \mathbb{R} \cup \{\pm\infty\}$ be a function. Then the following are equivalent.

(i) f is a measurable function

(ii) for any $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) \geq \alpha\}$ is measurable.

(iii) for any $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) < \alpha\}$ is measurable

(iv) for any $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) \leq \alpha\}$ is measurable.

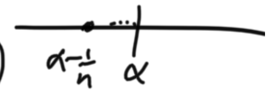
proof:-

(i) $\Rightarrow$ (ii): Assume f is measurable.

To show: For $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) \geq \alpha\} \in \mathcal{M}$.

$$\{x \in E \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > \alpha - \tfrac{1}{n}\}$$

is measurable. ($\because f$ is measurable).



as required.

(ii) $\Rightarrow$ (iii): Assume for $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) \geq \alpha\} \in \mathcal{M}$.

$$\{x \in E \mid f(x) < \alpha\} = \{x \in E \mid f(x) \geq \alpha\}^c \in \mathcal{M}.$$

(iii) $\Rightarrow$ (iv): Assume $\{x \in E \mid f(x) < \alpha\}$ is measurable.

To show: $\{x \in E \mid f(x) \leq \alpha\}$ is measurable.

$$\{x \in E \mid f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) < \alpha + \tfrac{1}{n}\} \in \mathcal{M},$$

(iv) $\Rightarrow$ (i): Assume (iv).

To show: f is measurable.

i.e, for $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) > \alpha\}$ is measurable.

But $\{x \in E \mid f(x) > \alpha\} = \{x \in E \mid f(x) \leq \alpha\}^c \in \mathcal{M}$.

Proposition :— Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function

Then $\{x \in E \mid f(x) = \alpha\}$ is measurable, for each $\alpha \in \mathbb{R}$.

proof:- For $\alpha \in \mathbb{R}$,

$$\{x \in E \mid f(x) = \alpha\} = \{x \in E \mid f(x) \le \alpha\} \cap \{x \in E \mid f(x) \ge \alpha\}$$

$\in \mathcal{M}$ (by using above Thm)

if $\alpha = +\infty$,

$$\{x \in E \mid f(x) = +\infty\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > n\}.$$

$\|^{ly}$ if $\alpha = -\infty$. $\in \mathcal{M}$.

Examples:-

① Every constant function is measurable.

Pf:- For $\alpha \in \mathbb{R}$, f is a constant function,

Then $\{x \in \mathbb{R} \mid f(x) > \alpha\} = \phi$ or $\mathbb{R}$

$\in \mathcal{M}$.

② $A \subseteq \mathbb{R}$, χ_A is measurable $\iff$ A is measurable.

proof:- For $\alpha \in \mathbb{R}$,

$$\{x \in \mathbb{R} \mid \chi_A(x) > \alpha\} = \phi \text{ or } A \text{ or } \mathbb{R} \in \mathcal{M}$$

where $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$ $\chi_A : \mathbb{R} \to \mathbb{R}$, Thus $\{x \in \mathbb{R} / \chi_A(x) > \alpha\}$ is measurable, $\forall \alpha \in \mathbb{R}$

iff

A is measurable.

(3) Every continuous function defined on a measurable set is measurable.

proof: Let $f : E \to \mathbb{R}$ be a continuous function.

To show for any $\alpha \in \mathbb{R}$,

$\{x \in E / f(x) > \alpha\}$ is measurable.

$$\{x \in E / f(x) > \alpha\} = f^{-1}(\underbrace{(\alpha, \infty)}_{\text{open}})$$ is an open set

$\Rightarrow$ This set is measurable, as required.

Theorem:- Let $f, g : E \to \mathbb{R} \cup \{\pm\infty\}$ be measurable functions. Let $c \in \mathbb{R}$. Then

$f + c$, cf, $f+g$, $f-g$, fg are measurable

functions.

proof: (i) To show: $f + c$ is measurable.

For $\alpha \in \mathbb{R}$,

$$\{x \in E \,/\, (f+c)(x) > \alpha\} = \{x \in E \,/\, f(x) + c > \alpha\}$$
$$= \{x \in E \,/\, f(x) > \alpha - c\} = f^{-1}((\alpha - c, \infty])$$

is measurable because f is measurable.

(ii) To show: cf is measurable.

if $c = 0$, then nothing to prove.

Assume $c \neq 0$.

First assume $c > 0$. For $\alpha \in \mathbb{R}$,

$$\{x \in E \,/\, (cf)(x) > \alpha\} = \{x \in E \,/\, c\, f(x) > \alpha\}$$
$$= \{x \in E \,/\, f(x) > \alpha/c\}.$$

is a measurable set

($\because f$ is measurable).

if $c < 0$, then still true. (try it!)

(iii) To show: $f+g$ is measurable.

Let $\alpha \in \mathbb{R}$ & $A = \{x \in E \,/\, (f+g)(x) > \alpha\}$

$$= \{x \in E \,/\, f(x) + g(x) > \alpha\}.$$

Note that $x \in A$ only if $f(x) > \alpha - g(x)$

i.e. only if there exists a rational number r_i such that $f(x) > r_i > \alpha - g(x)$

where $\{r_i \mid i = 1, 2, \dots\}$ is an enumeration of $\mathbb{Q}$.

Then $g(x) > \alpha - r_i$ & $f(x) > r_i$.

Thus if $x \in A$, then

$$x \in \{x \in E \mid f(x) > r_i\} \cap \{x \in E \mid g(x) > \alpha - r_i\}.$$

$$\Rightarrow A \subseteq \bigcup_{i=1}^{\infty} \left(\{x \in E \mid f(x) > r_i\} \cap \{x \in E \mid g(x) > \alpha - r_i\}\right)$$ ('i' depends on x)

& $= B$ (say)

$B \subseteq A$. (check it)

Thus $A = B$.

But B is a measurable set

Thus A is measurable, as required.

$f - g = f + (-g)$ $\therefore$ measurable.

~~$f \circ g$~~

$\boxed{\begin{array}{l} fg : E \to \mathbb{R} \cup \{\pm\infty\} \\ (fg)(x) = f(x)g(x) \end{array}}$

(iv) To show fg is measurable.

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$$

We already proved that $f+g$, $f-g$ are measurable.

Enough to show: f^2 is measurable.

For $\alpha \in \mathbb{R}$,

$$\{x \in E \mid f^2(x) > \alpha\} = \{x \in E \mid (f(x))^2 > \alpha\}$$

if $\alpha < 0$, then this set is equal E. which is measurable.

Suppose $\alpha \geq 0$.

$$\{x \in E \mid f(x)^2 > \alpha\} = \{x \in E \mid f(x) > \sqrt{\alpha}\} \cup \{x \in E \mid f(x) < -\sqrt{\alpha}\}$$

which is measurable.
