Recall, Let f be integrable on $\mathbb{R}^d$,
Then, given $\epsilon > 0$, $\exists \delta > 0$ s.t.
$$\int_E |f| < \varepsilon \quad \text{whenever } m(E) < \delta.$$
(Absolute continuity)

Dominated Convergence theorem

DCT

$\{f_n\}$ is a sequence of meas. functions s.t. $f_n \to f$ a.e. x,
If $|f_n(x)| \leq g(x)$ where g is integrable, then

$$\int f_n \to \int f \quad \text{as } n \to \infty$$

(Since, $\left|\int f_n - \int f\right| = \left|\int f_n - f\right| \leq \int |f_n - f|$

To show $\int |f_n - f| \to 0$ as $n \to \infty$,) Proof :- For $N \geq 0$, let

$$E_N = \left\{ x : |x| \leq N,\ g(x) \leq N \right\}$$

For $\varepsilon > 0$, $\exists$ N so. that

$$\int_{E_N^c} g < \varepsilon \quad , \quad \text{— (i)}$$

(Part (i) of previous result)

Consider the sequence $f_n \chi_{E_N}$ & note that this sequence is bounded by N.

$$\int_{E_N} |f_n - f| < \varepsilon \quad \text{for } n \text{ large.}$$

BCT

$|f_n \chi_{E_N}| \leq N$

Now,

$$\int |f_n - f|$$

$$= \int_{E_N} |f_n - f| + \int_{E_N^c} |f_n - f|$$

$$\leq \int_{E_N} |f_n - f| + 2\int_{E_N^c} |g|$$

$$\leq \varepsilon + 2\varepsilon = 3\varepsilon \quad \text{for } n \text{ large.}$$

$$\Longrightarrow \int f_n \longrightarrow \int f \quad \text{as } n \to \infty.$$

$\rightsquigarrow$ <u>Complex Valued Functions:-</u>

$f : \mathbb{R}^d \longrightarrow \mathbb{C}$

$f(x) = u(x) + i\,v(x), \quad x \in \mathbb{R}^d.$

$x = \underbrace{(x_1, x_2, \dots x_d)}_{d\text{-tuple}}$

$u : \mathbb{R}^d \longrightarrow \mathbb{R}$

$v : \mathbb{R}^d \longrightarrow \mathbb{R}$

f is Lebesgue integrable iff u & v are Lebesgue integrable.

$$\left[\ |u| \leq |f|, \quad |v| \leq |f| \qquad |f| \leq |u| + |v|\ \right]$$

$\ominus$ $\int f(x) := \int u(x) + i \int v(x)$

$\rightsquigarrow$ <u>Riesz- Fischer theorem</u>

<u>L^1 - metric :-</u>

$f, g \in L^1$ $\quad d : L^1 \times L^1 \to \mathbb{R}^+$

i) $d(f,g) \geq 0$, $d(f,g) = 0 \iff f = g$

ii) $d(f,g) = d(g,f)$

iii) $d(f,g) \leq d(f,h) + d(h,g)$

For any Lebesgue int. function f on $\mathbb{R}^d$, we define norm

$$\|f\| = \|f\|_{L^1} = \int_{\mathbb{R}^d} |f| \, dx$$

$\longrightarrow$ $\|f\| = 0 \iff f = 0$

$\int |f| = 0$

$\iff f = 0$ a.e.

$\boxed{\|cf\| = |c| \|f\|.}$

We write, $f = g$ in L^1 means $f = g$ a.e.

$[f]$

$= \{g : g = f \text{ a.e.}\}$

$\int |f| = 0$

$\Rightarrow f = 0$ a.e.

$\Rightarrow [f] \equiv 0$

$f = 0$

$f, g \in L^1$

i) $\|af\| = |a| \, \|f\|_{L^1}$

ii) $\|f+g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}$

iii) $\|f\|_{L^1} = 0 \Leftrightarrow f = 0$ a.e.

iv) $d(f, g) = \|f - g\|$

Verify that d is a metric on L^1.

$\{x_k\} \to$ Cauchy sequence in V

$x_n \to x \iff d(x_k, x) \to 0$

as $k \to \infty$

$\Longrightarrow \quad x \in V$

$\Rightarrow$ (V, d) is a complete metric space.

$\rightsquigarrow \quad \underline{C[a,b]} \subseteq R[a,b] \subseteq L^1[a,b]$

($C[a,b]$, with L^1 - metric) $\qquad L^1$-metric

Quest. Whether ($C[a,b]$, L^1-metric) is complete or not.

($R[a,b]$, L^1 metric) is complete or not.

$C[a,b]$ is not complete in L^1-metric

$$d(f,g) = \int_a^b |f(x) - g(x)|\, dx \checkmark$$

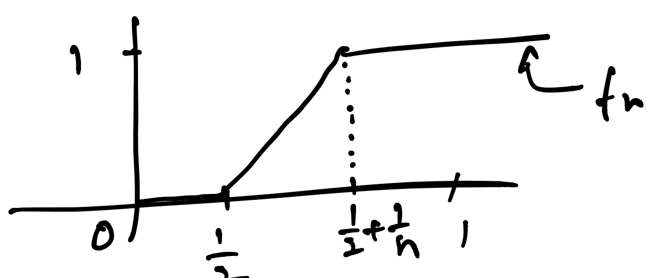
$\{f_n\}$ in $C[a,b]$

$d(f_n, f) \longrightarrow 0$, but, $f \notin C[a,b]$

Define

$f_n(x) : [0,1] \longrightarrow \mathbb{R}$,

$$f_n(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ n\left(x - \frac{1}{2}\right), & \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{n} \\ 1, & \frac{1}{2} + \frac{1}{n} \le x \le 1 \end{cases}$$



Suppose $\exists$ a cont. function f s.t.

$d(f_n, f) \longrightarrow 0$ as $n \to \infty$

$$\int_0^{\frac{1}{2}} |f(x)|\, dx = \int_0^{\frac{1}{2}} |f_n - f|\, dx$$

$$\leq \int_0^1 |f_n - f| \, dx$$

$$\Rightarrow \int_0^{\frac{1}{2}} |f(x)| \, dx = 0$$

$$\Rightarrow \quad f(x) = 0 \quad \forall \quad x \in [0, \tfrac{1}{2}].$$

——— $(*)$

If, $\frac{1}{2} < x \leq 1$. Suppose $\exists$ n_0 s.t. $\frac{1}{2} + \frac{1}{n_0} < x$. Then $\forall n \geq n_0$

$$\int_x^1 |1 - f(x)| \, dx$$

$$= \int_x^1 |f_n(x) - f(x)| \, dx$$

$$\leq \int_0^1 |f_n - f| \, dx \longrightarrow 0$$

$$\Rightarrow \int_x^1 |1 - f(x)| \, dx = 0 \ \forall \ \tfrac{1}{2} < x \leq 1$$

$$\Rightarrow \quad f(x) = 1 \quad \forall \quad \tfrac{1}{2} < x \leq 1$$

$$f = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ 1, & \frac{1}{2} < x \le 1. \end{cases}$$

$\Rightarrow f \notin C[0,1]$

$\Rightarrow$ $C[0,1]$ is not complete in L^1 metric.

Question :-

$C[a,b]$, sup norm.

Q: $C[a,b]$ is complete w.r.t. metric induced by sup norm?

Yes

Convergence in Sup-norm $\equiv$ Uniform convergence

$\{f_n\} \in C[a,b]$

$f_n \xrightarrow[\text{uniformly}]{} f$

f is continuous.

$f \in C[a,b]$

$(C[a,b], \|\cdot\|_\infty)$ is a complete metric space

$d(f,g) = \|f-g\|_\infty$

$= \sup_{x \in [a,b]} |f(x) - g(x)|$

Riesz-Fischer theorem

$L_1[a,b]$ is complete in L_1 metric.

Proof :- $\{f_n\}$ Cauchy in $L_1[a,b]$, i.e. $\|f_n - f_m\| \to 0$ as $n, m \to \infty$.

If $f_n \to f$, i.e $\|f_n - f\| \to 0$, then, $f \in L_1[a,b]$

Idea of the proof :-

To exarct a subsequence (f_{n_k}) of (f_n) s.t. $\|f_{n_k} - f\| \to 0$

as $k \to \infty$.

$$\|f_n - f\| = \|f_n - f_{n_k} + f_{n_k} - f\|$$
$$\leq \underbrace{\|f_n - f_{n_k}\|}\ + \|f_{n_k} - f\|$$
$$\leq \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon$$

$f_n \to f$ in norm.

Aim :-

a) $f_{n_k} \longrightarrow f$ pt. wise a.e. x.

b) $\|f_{n_k} - f\| \longrightarrow 0$ as $k \to \infty$

since $\{f_n\}$ is Cauchy, For each $k \geq 1$, we must have,

$$\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k}$$

(a_n)

$|a_n - a_m| < \varepsilon$

$n > m \geq n_0$

n_k

$\varepsilon = \frac{1}{2^k}$

~~Note~~

$$f_{n_k} = \cancel{f_{n_1}} + (\cancel{f_{n_2}} - \cancel{f_{n_1}}) + (f_{n_3} - \cancel{f_{n_2}}) + \cdots + (f_{n_k} - \cancel{f_{n_{k-1}}})$$

$$= f_{n_1} + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j})$$

$$f = f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j})$$

T.S.

$f = \lim_{k\to\infty} f_{n_k}$, (by construction)

T.S. $f \in L^1$

$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

$$\int |f_{n_1}(x)| + \sum_{k=1}^{\infty} \int |f_{n_{k+1}} - f_{n_k}|$$

$$\leq \int |f_{n_1}| + \sum_{k=1}^{\infty} \frac{1}{2^k}$$

$< \infty$

By the corollary of MCT, the the function $\underline{g \text{ is}}$ integrable.

We see that $|f| \leq g$

$\implies f \in L^1$

To prove $\underline{f_{n_k} \to f \text{ in } L^1}$

$|f - f_{n_k}| \leq g \ \forall k.$

$g \in L^1 \implies$ Apply DCT,

We set $\|f_{n_k} - f\|_{L^1} \to 0$ as $k \to \infty$

This proves, $\{f_n\} \to f$ in L^1 norm. $\square$

Thm: $C[a,b]$ is dense in $L_1[a,b]$.

Proof: $C[a,b] \subseteq L_1[a,b]$

$f \in L_1[a,b]$, Given $\varepsilon > 0$,

need to find $g \in C[a,b]$ s.t. $\|f-g\|_{L^1} < \varepsilon$.

Now, $f \in L_1[a,b]$

$f = f^+ - f^-$ where both

f^+ & f^- are positive,

$f^+ \in L^1[a,b]$, $f^- \in L^1[a,b]$

Claim :-

$\|f^+ - g\| < \varepsilon/2$

$\|f^- - h\| < \varepsilon/2$

$\|f - (g+h)\|$

$< \|f^+ - g\|$

$+ \|f^- - h\|$

$< \varepsilon$

$g+h \in C[a,b]$

It is enough to consider $f \geq 0$

$\Longleftarrow$

$$\boxed{\begin{array}{l} \int f < \infty \\ f \geq 0 \end{array}}$$

We can choose a sequence (s_n) of non-negative simple functions s.t. $s_n \nearrow f$ &

(*) $\lim_{n \to \infty} \int s_n = \int f$ (by MCT)

$s_n \in L_1[a,b]$, we can chose n_0 s.t. $\|s_{n_0} - f\|_{L^1} < \varepsilon$. by (*)

$\Rightarrow$

$\hookrightarrow$ This proves that simple meas. functions are dense in $L^1[a,b]$

$L^1 = L_1$

We can now prove the theorem with f replaced by

$f = \chi_A$, $A \subseteq [a,b]$, A is measurable.

$$\left(f = \sum_{i=1}^{n} a_i \chi_{A_i} , \quad a_i \in \mathbb{R} \right.$$

A_i's measurable, $\bigcup_{i=1}^{n} A_i = [a,b]$

$\left. A_i \cap A_j = \phi , \ i \neq j \right)$

By Littlewood's 1st principle, we get, $\exists$ $F \subseteq [a,b]$ where F is finite union of intervals, $m(A \Delta F) < \varepsilon$.

$$\| \chi_A - \chi_F \|_{L^1} \leq m(A \Delta F) < \varepsilon$$

<u>Write</u>, $F = \bigcup_{i=1}^{m} I_i$, I_i

intervals s.t. $I_i \cap I_j \neq \phi$.

$$\chi_F = \chi_{\bigcup_{j=1} I_i} = \sum_{i=1}^{m} \chi_{I_i}.$$

To complete the proof of our thm, we need to show that given $I \subseteq [a,b]$, $\exists$ a continuous function g on $[a,b]$ s.t.

$$\|\chi_I - g\| < \varepsilon. \quad (**)$$

$f \in L$

$\| f - g \| < \varepsilon$

$$
\begin{aligned}
&\underline{\| x_F - mg \|} \\
&= \left\| \sum_{i=1}^{m} x_{I_i} - mg \right\| \\
&\le \sum_{i=1}^{m} \| x_{I_i} - g \| \\
&\le \underline{m\varepsilon} \qquad \text{(i)}
\end{aligned}
$$

$$
\begin{aligned}
&\| x_A - ng \| \\
&\le \| x_A - x_F + x_F - ng \| \\
&\le \| x_A - x_F \| + \| x_F - ng \| \\
&\le \varepsilon + n\varepsilon
\end{aligned}
$$

(3) $\|f - m_n g\|$

$$\leq \sum_{i=1}^{n} \left(|a_i| \chi_{A_i} - m g \right)$$

$$< \underline{n \, (\varepsilon + m \varepsilon)}$$

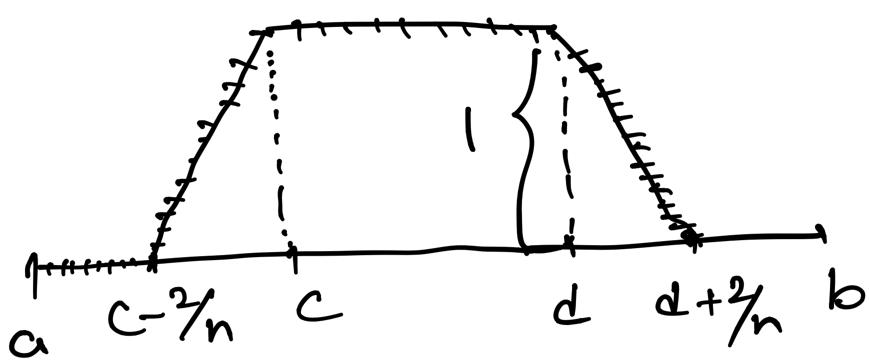
Our revised goal is :

We only need to show that given $I \subseteq [a,b]$, $\exists$ a continuous function g on $[a,b]$ such that $\|\chi_I - g\| < \varepsilon$

$f \in L^1$

$$I = (c, d) \subseteq [a, b]$$

$$a < c < d < b$$



$$g(x) = \begin{cases} 0, & a \le x \le c - \frac{2}{n} \\ \frac{n}{2}\left(x - c + 2/n\right), & c - 2/n \le x \le c \\ 1, & c \le x \le d \\ \frac{n}{2}\left(d + 2/n - x\right), & d \le x \le d + 2/n \\ 0, & d + 2/n \le x \le b \end{cases}$$

$$\| \chi_I - g \|_{L^1}, \qquad I = (c, d)$$

$$= \int_a^b |\chi_I - g| \, dx$$

$$= \frac{2}{n}$$

depending on $\varepsilon > 0$, we can choose n large enough say n_0 such that $2/n_0 < \varepsilon$.

$$C[a,b] \subseteq R[a,b] \subseteq L_1[a,b]$$

$$\overline{C[a,b]} = L_1[a,b]$$

$$\Rightarrow \overline{R[a,b]} = L_1[a,b]$$

$\Rightarrow L_1[a,b]$ is the completion of $R[a,b]$.

$\rightsquigarrow$

(Riemann-Lebesgue Lemma)

(Weaker version)

<u>Problem</u> :- Let f be a bounded meas. function defined on (a,b). Show that

$$\lim_{\beta \to \infty} \int_a^b f(x) \sin \beta x \, dx = 0$$

<u>Proof</u> :- f is bdd + meas

$$\Rightarrow f \in L^1[a,b]$$

$\forall \, \varepsilon > 0 \;\; \exists \; h = \sum_{i=1}^{n} \xi_i \chi_{(a_i, b_i)}$

s.t. $\int_a^b |f-h| \, dx < \varepsilon$. (by our previous work)

$$\left| \int_a^b f \sin\beta x \, dx \right|$$

$$= \left| \int_a^b (f - h + h) \sin\beta x \, dx \right|$$

$$\leq \int_a^b \left| (f - h) \sin\beta x \right| dx + \left| \int_a^b h \sin\beta x \, dx \right|$$

$$= \int_a^b |f - h| \, dx + \left| \int_a^b h \sin\beta x \, dx \right|$$

$$< \underline{\underline{\varepsilon}} + \left| \int_a^b h \sin\beta x \, dx \right|$$

$$\left| \int_a^b \chi_{(a_i, b_i)} \sin\beta x \, dx \right| = \left| \int_{a_i}^{b_i} \sin\beta x \, dx \right|$$

$$= \left| \frac{1}{\beta} \int_{\beta a_i}^{\beta b_i} \sin y \, dy \right| \qquad \begin{aligned} \beta x &= y \\ dx &= \frac{dy}{\beta} \end{aligned}$$

$$\leq \frac{1}{|\beta|} \times \left. -\cos y \right|_{\beta a_i}^{\beta b_i}$$

$$= \frac{1}{|\beta|} \left[-\cos(\beta b_i) + \cos \beta (a_i) \right]$$

$$\leq \frac{2}{|\beta|}$$

$$\leq \frac{\varepsilon}{nM} \quad , \text{ for } \beta > \beta_0$$

where $M = \max \{ \underline{s_1}, \underline{s_2}, \ldots, \underline{s_n} \}$

$$\left| \int_a^b f \sin \beta x \right|$$

$$< \frac{\varepsilon}{nM} \times \underline{\underline{nM}} + \varepsilon$$

$$= \varepsilon + \varepsilon \quad \text{for}$$

$$= 2\varepsilon \quad , \beta > \beta_0$$

$$\lim_{\beta \to \infty} \int_a^b f \sin \beta x \, dx = 0$$

$\square$

<u>Prove</u>,

$$\lim_{\beta \to \infty} \int_a^b f(x) e^{i\beta x} \, dx = 0$$

<u>Riemann-Lebesgue lemma</u>..

$$\mathbb{R}^d = \underline{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \text{ where}$$

$d_1 + d_2 = d$, $d_1, d_2 \geq 1$

$(x, y) \in \mathbb{R}^d$, $x \in \mathbb{R}^{d_1}$
$y \in \mathbb{R}^{d_2}$

Slice If f is a function in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, then the slice corresponding to $y \in \mathbb{R}^{d_2}$, is the function f^y of the $x \in \mathbb{R}^{d_1}$ variable given by,

$$f^y(x) = f(x, y)$$

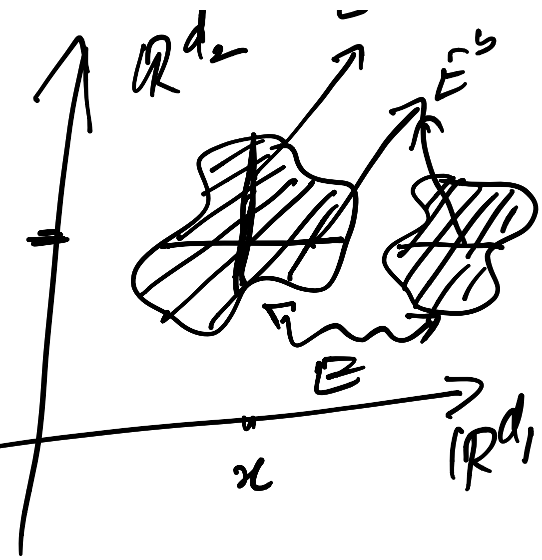
Ily, the slice of f for a fixed $x \in \mathbb{R}^{d_1}$ is

$$f_x(y) = f(x, y)$$

F^x

Slice of a set

$E \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ y



We define its slice by

$$E^y = \{x \in \mathbb{R}^{d_1} : (x,y) \in E\}$$

$$E^x = \{y \in \mathbb{R}^{d_2} : (x,y) \in E\}$$

Theorem :-

Suppose $f(x,y)$ is integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$

i) The slice f^y is integrable on $\mathbb{R}^{d_1}$

ii) The function defined by

$$y \longrightarrow \int_{\mathbb{R}^{d_1}} f^y(x)\,dx \quad \text{is}$$

integrable on $\mathbb{R}^{d_2}$

iii) $$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \right] dy$$

$$= \int_{\mathbb{R}^d} f$$

$$= \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,dy \right) dx$$

Assignment- 4

① MCT $\Longrightarrow$ Fatou's lemma

Proof :- $\{f_K\}$ - seq. of non-negative meas. functions.

To prove, $\int \liminf f_K \leq \liminf \int f_K$

$$g_K = \inf \{f_K, f_{K+1}, f_{K+2}, \dots\} \quad K \in \mathbb{N}$$

g_K non-negative.

g_K measurable

$g_K \leq g_{K+1}$

$g_K \uparrow$

MCT $\Rightarrow$

$$\int \lim_{k\to\infty} g_k = \underline{\lim_{k\to\infty} \int g_k}$$

$$\Rightarrow \int \liminf f_k \, dx = \underline{\lim_{k\to\infty} \int g_k}$$

<u>Now</u>, $g_k \leq f_n \;\; \forall \; n \geq k$

$$\Rightarrow \int g_k \leq \underline{\int f_n \;\; \forall \; n \geq k}$$

$$\Rightarrow \int g_k \leq \inf \int f_n \;\; \forall n \geq k$$

<u>So</u>, $\int \liminf f_k \, dx$

$$= \lim_{k\to\infty} \int g_k \, dx$$

$$\leq \lim_{k\to\infty} \inf_{n\geq k} \left(\int f_n\right),$$

$$= \liminf \int f_k$$

✓

1. (iii) $$f_n(x) = \begin{cases} -n, & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \text{otherwise} \end{cases}$$

$$\lim_{n\to\infty} f_n = f = 0 \text{ on } [0,1]$$

$$\int_0^1 f_n \, dx = -1 \quad \forall n.$$

$$\liminf \int_0^1 f_n = -1$$

$$\underline{0 < -1 \Rightarrow \Leftarrow}$$

1. (iv) Fatou's Lemma $\Rightarrow$ D.C.T $\Rightarrow$ B.C.T.

$|f_n| \leq g$, $g \in L^1$

$f_n \to f$

$\Rightarrow |f| \leq g$

$(g - f_n)$, $(g + f_n)$

$\downarrow$ $\quad\quad\quad\quad$ $\downarrow$

$g - f$ $\quad\quad\quad$ $g + f$

$\liminf \int (g - f_n) \geq \int (g - f)$

& $\liminf \int (g + f_n) \geq \int (g + f)$

But, $\liminf \int g + f_n$

$= \liminf \int g + \liminf \int f_n$ ✓

$$\liminf \int (g - f_n)$$

$$= \liminf \int g - \limsup \int f_n$$

$$\cancel{\int g} - \limsup \int f_n \geq \cancel{\int g} - \int f$$

$$\Rightarrow \int f \geq \limsup \int f_n \quad \text{--- } (*)$$

$$\cancel{\int g} + \liminf \int f_n \geq \cancel{\int g} + \int f$$

$$\Rightarrow \int f \leq \liminf \int f_n \quad \text{--- } (**)$$

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n$$

$$\Rightarrow \lim \int f_n = \int f.$$