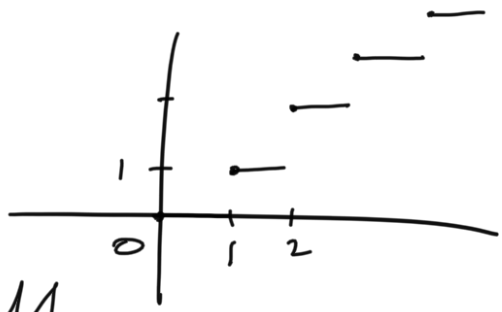
Lecture 13

Example:- Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = [x]$

$[x]$ = the integer part of x

Then f is measurable.



proof:-

For $\alpha \in \mathbb{R}$

to show: $\{x \in \mathbb{R} \mid f(x) > \alpha\} \in \mathcal{M}$.

$$\{x \in \mathbb{R} \mid f(x) > \alpha\} = \{x \in \mathbb{R} \mid [x] > \alpha\}$$

$$= [\alpha+1, \infty) \quad \in \mathcal{M}.$$

$\therefore f$ is measurable.

proposition:- Let $x \in [0,1]$, have the expansion to the base l, $x = 0.x_1 x_2 \cdots x_n \cdots$, for some +ve integer l. Then $f_n : [0,1] \to \mathbb{R}$,

$f_n(x) = x_n$ is measurable. $\forall n$

proof:-

$$x = 0.x_1 x_2 \cdots = \sum_{n=1}^{\infty} \frac{x_n}{l^n}.$$

$$0 \le x_n < l.$$

We want to write x_n as a function of x.

$lx = x_1 \cdot x_2 x_3 \cdots$

$[lx] = x_1 = f_1(x)$

Then $f_1(x) = [lx]$ (Check it!)

$\Rightarrow f_1$ is a measurable function.

$lx - [lx] = 0 \cdot x_2 x_3 \cdots$

$l(lx - [lx]) = x_2 \cdot x_3 x_4 \cdots$

$\boxed{l^2 x - l[lx]}$

$[l^2 x - l[lx]] = x_2 = f_2(x)$

$\Rightarrow f_2$ is a measurable function.

$\vdots$

Theorem: There exists a measurable set which is not a Borel set.

i.e., $\mathcal{B} \subsetneq \mathcal{M}$.

$\boxed{\mathcal{B} \subseteq \mathcal{M}}$

proof: Let $x \in [0,1]$

Then the binary expansion of x is

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}, \quad \text{with } \varepsilon_n = 0 \text{ or } 1$$

$\forall n$.

Define a function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n} \in P, \text{ the Cantor set}$$

$\varepsilon_n = 0$ or 1.

In fact $\operatorname{Im}(f) \subseteq P$.

We know $f_n(x) = \varepsilon_n$ are measurable function by above proposition.

$$\Rightarrow \sum_{i=1}^{n} \frac{2\varepsilon_i}{3^i} \text{ is also a measurable function } \forall n.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n} = f(x) \text{ is a measurable function}$$

$$\| \qquad \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{2\varepsilon_i}{3^i}\right)$$

f_n is measble
$\Rightarrow \lim f_n$ is measble

Thus f is a measurable function.

Also f is injective because the value $f(x)$ defines the sequence $\{\varepsilon_n\}$ in the expansion $\sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n}$ uniquely; so x is determined uniquely.

$$\boxed{\begin{matrix} f(x) = f(y) \\ \Downarrow \\ x = y \end{matrix}}$$

<u>To show:</u> $\mathcal{B} \subsetneq \mathcal{M}$.

Suppose not, i.e. $\mathcal{B} = \mathcal{M}$.

Since f is a measurable function, this implies that $f^{-1}(B)$ is measurable for any measurable set B in $\mathbb{R}$.

Let V be a non-measurable set in $[0,1]$

$B = f(V) \subseteq \operatorname{Im}(f) \subseteq P$, the Cantor set

But $m^*(P) = 0 \Rightarrow m^*(B) = 0$.

$\Rightarrow B$ is measurable.

But since f is injective, $f^{-1}(B) = f^{-1}(f(V)) = V$

$\Rightarrow V$ is measurable.

$\Rightarrow\Leftarrow$

$$\boxed{\begin{aligned} f^{-1}(B) &= \{x \in [0,1] \mid f(x) \in B\} \\ &= \{x \in [0,1] \mid f(x) \in f(V)\} \\ &= \{x \in [0,1] \mid f(x) = f(y),\ \exists\, y \in V\} \\ &= \{x \in [0,1] \mid x = y \in V\} = V \end{aligned}}$$

Thus $\mathcal{B} \neq \mathcal{M}$

$\Rightarrow \mathcal{B} \subsetneq \mathcal{M}$.

i.e. There exists a measurable set which is not a Borel set

Definition:- We say that a property P holds <u>almost everywhere</u> (a.e) if P holds except on a set of measure zero.

Theorem:- Let f be a measurable function & $f = g$ a.e for some function g. Then g is also measurable.

proof:- Given that f is measurable

$\Rightarrow$ for any $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) > \alpha\} \in \mathcal{M}$.

given $f = g$ a.e.

$\Rightarrow$ $\{x \in E \mid f(x) \neq g(x)\}$ has measure zero.

$\Rightarrow$ $m^*(\{x \in E \mid f(x) \neq g(x)\}) = 0$.

We have $\{x \in E \mid f(x) > \alpha\} \,\Delta\, \{x \in E \mid g(x) > \alpha\}$

$\subseteq \{x \in E \mid f(x) \neq g(x)\}$

Pf:-

$$\text{LHS} = \left(\{x \in E \mid f(x) > \alpha\} \setminus \{x \in E \mid g(x) > \alpha\}\right) \cup \left(\{x \in E \mid g(x) > \alpha\} \setminus \{x \in E \mid f(x) > \alpha\}\right)$$

$$= \left(\{x \in E \mid f(x) > \alpha \text{ \& } g(x) \le \alpha\}\right) \cup \left(\{x \in E \mid g(x) > \alpha \text{ \& } f(x) \le \alpha\}\right)$$

$$\subseteq \{x \in E \mid f(x) \neq g(x)\}$$

But ✓ has measure 0.

$$\Rightarrow m^*\left(\underbrace{\{x \in E \mid f(x) > \alpha\}}_{\in \mathcal{M}} \,\Delta\, \{x \in E \mid g(x) > \alpha\}\right) = 0.$$

$$\Rightarrow \{x \in E \mid g(x) > \alpha\} \in \mathcal{M}$$

(by using a prop).

$\therefore$ g is measurable.

Proposition:- Let $\{f_n\}$ be a sequence of measurable functions converging a.e to f. Then f is measurable.

proof:- Given $f_n \to f$ a.e.

$\Rightarrow f = \limsup (f_n)$ a.e.

But $\limsup (f_n)$ is measurable.

$\therefore$ By above Theorem, f is measurable.

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proposition:- Let f be a measurable function.

Then $f^+ = \max\{f, 0\}$ & $f^- = -\min\{f, 0\}$ are measurable.

proof:- $f, 0$ are measurable funtion.

$\Rightarrow \max\{f, 0\}, \min\{f, 0\}$ are measurable (Check it!)

$f^+(x) = \max\{f(x), 0\}$

$f^-(x) = -\min\{f(x), 0\}$.

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prop:- Let $\{f_n\}$ be a sequence of measurable functions. Then $\{x \in E \mid f_n(x) \text{ Converges}\}$ is a measurable set.

<u>proof:</u> $\{x \in E \mid f_n(x) \text{ converges}\} = \{x \in E \mid \limsup(f_n)(x) = \liminf(f_n)(x)\}$

$= \{x \in E \mid (\limsup(f_n) - \liminf(f_n))(x) = 0\}$

is a measurable set.

$\boxed{\begin{array}{l} g \text{ is measurable} \\ \Rightarrow \{x \in E \mid g(x) = \alpha\} \\ \in \mathcal{M} \end{array}}$