Lecture 16

Def :- A subset $E \subseteq \mathbb{R}^d$ is said to be Lebesgue measurable or simply measurable if given $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}^d$ such that $E \subseteq U$ & $m^*(U \setminus E) \leq \varepsilon$.

If E is a measurable set, then we define the Lebesgue measure or measure $m(E)$ by $m(E) = m^*(E)$.

Theorem :-

① Every open set in $\mathbb{R}^d$ is measurable.

② If $m^*(E) = 0$, for a set $E \subseteq \mathbb{R}^d$, then E is measurable.

③ A countable union of measurable sets is measurable.

proof :- ① Directly follows from the def.

② Assume $m^*(E) = 0$.

Let $\varepsilon > 0$.

Recall that $m^*(E) = \inf\limits_{\substack{U \supseteq E \\ \text{open}}} \{ m^*(U) \} = 0$.

Then exists an open set U such that $U \supseteq E$ & $m^*(U) \leq \varepsilon$. (by inf. property)

But $U \setminus E \subseteq U$, This gives that

$$m^*(U \setminus E) \leq m^*(U) \leq \varepsilon$$

$\therefore$ There exists an open set U such that $U \supseteq E$ & $m^*(U \setminus E) \leq \varepsilon$.

$\therefore$ E is measurable.

(3) Let $\{E_j\}_{j=1,2,\ldots}$ be a collection of measurable sets in $\mathbb{R}^d$.

Let $E = \bigcup\limits_{j=1}^{\infty} E_j$, To show: E is measurable.

Let $\varepsilon > 0$.

Since E_j is measurable, there exists an open set U_j in $\mathbb{R}^d$ such that $U_j \supseteq E_j$ &

$$m^*(U_j \setminus E_j) \leq \frac{\varepsilon}{2^j}$$

Let $U = \bigcup_{j=1}^{\infty} U_j$. U is an open set.

& $E \subseteq U$. $\left(\because E_j \subseteq U_j \ \forall j \right)$

Now $U \setminus E \subseteq \bigcup_{j=1}^{\infty} (U_j \setminus E_j)$

$$\Rightarrow m^*(U \setminus E) \leq m^*\left(\bigcup_{j=1}^{\infty} (U_j \setminus E_j) \right)$$

$$\leq \sum_{j=1}^{\infty} m^*(U_j \setminus E_j)$$

$$\leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$$

$\therefore E = \bigcup_{j=1}^{\infty} E_j$ is measurable.

<u>Proposition</u>:- closed sets in $\mathbb{R}^d$ are measurable.

<u>proof</u>:- Let $F \subseteq \mathbb{R}^d$ be a closed set.

<u>Recall</u>:- $\mathbb{R}^d$, $\underline{a}, \underline{b} \in \mathbb{R}^d$

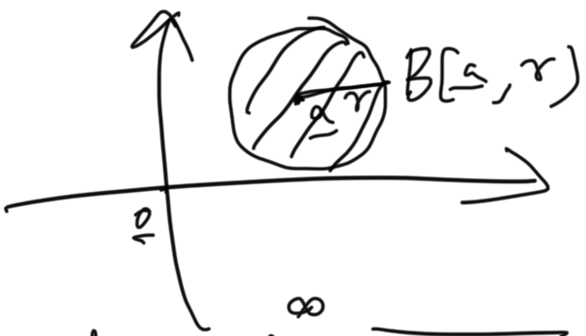
$$\|\underline{a} - \underline{b}\| = \sqrt{|a_1 - b_1|^2 + \cdots + |a_d - b_d|^2}$$

metric space Euclidean norm.

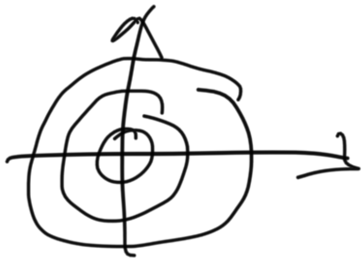
Open Ball centre at $\underline{a} \in \mathbb{R}^d$ & radius $r > 0$

$$B(\underline{a}, r) := \left\{ \underline{x} \in \mathbb{R}^d \,\middle|\, \|\underline{a} - \underline{x}\| < r \right\}$$

Closed Ball $\overline{B(\underline{a}, r)} = \left\{ \underline{x} \in \mathbb{R}^d \,\middle|\, \|\underline{a} - \underline{x}\| \leq r \right\}$



$$\mathbb{R}^d = \bigcup_{k=1}^{\infty} \overline{B(0,k)}$$



$$\Rightarrow F = F \cap \mathbb{R}^d = \bigcup_{k=1}^{\infty} F \cap \left(\overline{B(0,k)}\right).$$

where $\overline{B(0,k)}$ = the closed Ball with Centre at $\underline{0}$ & radius k.

Also note that $F \cap \overline{B(0,k)}$ closed & bounded is if is Compact.

To show F is measurable, it is suffices to show the compact sets are measurable.

Assume F is Compact (i.e, closed & bounded).

$\therefore m^*(F) < \infty.$

Let $\varepsilon > 0$. We have $m^*(F) = \inf\limits_{\substack{U \supseteq F \\ \text{open}}} \left(m^*(U) \right)$

There exists an open set $U \subseteq \mathbb{R}^d$ such that $U \supseteq F$ & $m^*(U) \leq m^*(F) + \varepsilon$

Since F is closed, we get $U\setminus F$ is open

$$= U\cap F^c.$$

Let $U\setminus F = \bigcup_{j=1}^{\infty} Q_j$ almost disjoint union of cubes Q_j.

For a fixed integer N, let $K = \bigcup_{j=1}^{N} Q_j$

K is compact

& $K \cup F \subseteq U$, this union is a disjoint union $\boxed{K \subseteq U\setminus F}$

$$\therefore\ m^*(U) \geq m^*(K\cup F) = m^*(K) + m^*(F) \quad (??)$$

$$= m^*(F) + \sum_{j=1}^{N} m^*(Q_j).$$

⊛ below.

$$\Rightarrow \sum_{j=1}^{N} m^*(Q_j) \leq m^*(U) \setminus m^*(F)$$

$$\leq \varepsilon.$$

True for any $N \in \mathbb{N}$.

$$m^*(U\setminus F) = m^*\Big(\bigcup_{j=1}^{\infty} Q_j\Big) \leq \sum_{j=1}^{\infty} m^*(Q_j) \leq \varepsilon.$$

Thus there exists an open set $U \supseteq F$ such that $m^*(U\setminus F) \leq \varepsilon$?

$\therefore$ F is measurable.

(*): Let $E = E_1 \cup E_2$ & $\text{dist}(E_1, E_2) > 0$,

where $\text{dist}(E_1, E_2) = \inf \{ \|\underline{x} - \underline{y}\| \mid \underline{x} \in E_1, \underline{y} \in E_2 \}$.

Then $m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$.

Apply (*) to K & F $\quad F \cap K = \emptyset$.

Compact closed.

Toshow: $d(K, F) > 0$. (Ex).