Dat
07/11/2019

Lecture 14

Defⁿ :-

Length or Magnitude of a vector

The magnitude (or length) A of a contravariant vector A^i is defined by

$$(A)^2 = e_{(A)} \, g_{ij} A^i A^j, \quad \to (1)$$

where $e_{(A)}$ is the indicator $+1$ or -1 which makes A real.

The magnitude A is an invariant (ie, $\bar{A} = A$). (how? EX)

Slly, the magnitude B of a covariant vector B_i is defined by the eqⁿ

$$(B)^2 = e_{(B)} \, g^{ij} B_i B_j, \quad \to (2)$$

where $e_{(B)}$ is the indicator of the vector B_i & it is clear that B is an invariant (EX) (ie, $\overline{B} = B$)

Also, a vector whose magnitude is unity, is called a Unit vector & a vector whose magnitude is zero, is called a null vector.

Unit tangent vector:

We have,
$$ds^2 = g_{ij}\, dx^i\, dx^j,$$
where from $g_{ij} \dfrac{dx^i}{ds} \dfrac{dx^j}{ds} = e = \pm 1$

This shows that

$\frac{dx^i}{ds}$ is a contravariant vector of magnitude one,

(ie, unit contravariant vector)

& this unit vector is denoted as **unit tangent vector** to some curve C in a Riemannian space V_N.

---

§/ Associate Tensor

The inner product of the fundamental tensor g_{ij} & the contravariant vector A^i

is the covariant vector $g_{ij} A^j$ which is said to associate to A^j. We define

$$A_i = g_{ij} A^j \longrightarrow (1).$$

Slly, we define $A^i = g^{ij} A_j$ for covariant vector A_j

The vectors A^i & A_i are associate to each other

The relation betn a vector & its associate is reciprocal. For

$$\sum_{j=1}^{N} g^{ij} A_j = \underset{(=0)}{g^{i1}} A_1 + \underset{(=0)}{g^{i2}} A_2 + \cdots + \underset{(=1)}{g^{ii}} A_i + \cdots + g^{iN} A_N = A^i$$

the vector associate to A_i is

$$g^{ij} \underbrace{A_j} = \underbrace{g^{ij} g_{jk}} A^k \quad [\text{by } 1]$$

$$= \delta^i_k A^k = A^i$$

This process of association is often referred to as 'lowering the superscript' or 'raising the subscript' resp.

Also, $e_{(A)} g_{ij} A^i A^j = e_{(A)} g_{ij} g^{ik} A_k \, g^{jl} A_l$

$$= e_{(A)} g^{kl} A_k A_l \quad \left[\begin{array}{l} g_{ij} g^{ik} = g_{kj} \\ g_{kj} g^{jl} = g^{kl} \end{array} \right.$$

which shows that the magnitude of associate vectors are equal.

Angle betⁿ vectors:

Defⁿ:- The angle betⁿ two unit vectors A^i & B^i is defined by

$$\cos\theta = g_{ij} A^i B^j = A_j B^j$$
$$= g^{jk} A_j B_k = A^k B_k. \quad (1)$$

It follows from (1) that the angle betⁿ two vectors A^i & B^i which are not necessarily unit vectors, is given by

$$\therefore \cos\theta = \frac{g_{ij} A^i B^j}{\sqrt{e_{(A)} e_{(B)} g_{lm} A^l A^m g_{ns} B^n B^s}} \longrightarrow (2)$$

Orthogonality :-

Two vectors are said to be orthogonal to one another if the angle bet^n them is a right angle. [ie., $\theta = 90°$, so $\cos\theta = 0$.

The N/S condition for the orthogonality of the two vectors

(8)

A^i & B^i is

$g_{ij} A^i B^j = 0$.

$\longrightarrow$ (3)

---

§/ Christoffel symbols or, Brackets

We first investigate the two functions formed from the fundamental tensor g_{ij}. These are called the Christoffel symbols or brackets of the first & second kinds, defined respectively by

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \rightarrow (1)$$

& $$\left\{ {l \atop ij} \right\} = g^{lk}\,[ij,k]$$ (2)

★★★★★

Properties :-

2

I) Christoffel symbols of both kinds are symmetric w.r.t the indices i & j,

i.e., $$[ij,k] = [ji,k]$$

& $$\left\{ {l \atop ij} \right\} = \left\{ {l \atop ji} \right\}.$$

Soln :- By defn

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

As g_{ij} is symmetric $\therefore g_{ij} = g_{ji}$

$$= \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^k}\right)$$

$$= [ji,k] \rightarrow (3)$$

3

Again, $\begin{Bmatrix} l \\ i\ j \end{Bmatrix} = g^{lk} [ij, k]$

$= g^{lk} [ji, k]$

$\therefore \begin{Bmatrix} l \\ i\ j \end{Bmatrix} = \begin{Bmatrix} l \\ j\ i \end{Bmatrix} \longrightarrow (4)$

II) $[ij, m] = g_{lm} \begin{Bmatrix} l \\ i\ j \end{Bmatrix}$

Sol^n:- We have, $\begin{Bmatrix} l \\ i\ j \end{Bmatrix} = g^{lk} [ij, k] \longrightarrow (1)$

Inner multiplication of (1) by g_{lm} gives,

$g_{lm} \begin{Bmatrix} l \\ i\ j \end{Bmatrix} = g_{lm} g^{lk} [ij, k]$

$= \delta_m^k [ij, k]$ $[\because g_{lm} g^{lk} = \delta_m^k$

$\therefore g_{lm}\begin{Bmatrix} l \\ i\,j \end{Bmatrix} = g^{k}_{m}\,[ij,k]$ (4)

$= [ij,m]$ $\left[\sum_{k=1}^{N} g^{k}_{m}[ij,k] = [ij,m]\right]$

Thus, $[ij,m] = g_{lm}\begin{Bmatrix} l \\ i\,j \end{Bmatrix}$,

---

~~HW:~~ show that

EX/ (iii) $\dfrac{\partial g_{ik}}{\partial x^{j}} = [ij,k] + [kj,i]$

EX/ (IV) $\dfrac{\partial g^{mk}}{\partial x^{l}} = -g^{mi}\begin{Bmatrix} k \\ i\,l \end{Bmatrix} - g^{ki}\begin{Bmatrix} m \\ i\,l \end{Bmatrix}$

Hint :- We know, $g^{ik}g_{ij} = \delta^{k}_{j} = \begin{cases} 1, & j=k \\ 0, & j\neq k. \end{cases}$

Diff. w.r.t x^{l}.