Linear Algebra

Lecture 18

Orthogonal Complement.

Definition: Let S be a nonempty subset of an inner product space V. We define $S^\perp$ ("S perp") to be set of all vectors in V that are orthogonal to every vector in S.

$$S^\perp = \{ u \in V ; \langle x, v \rangle = 0 \ \forall x \in S \}.$$

This set $S^\perp$ is called as orthogonal complement of S in V.

Lemma: For any $S \subseteq V$, $S^\perp$ is a subspace of V.

- $\{0\}^\perp = V$, $V^\perp = \{0\}$ for any inner product space V.
- Ex: $\mathbb{R}^3$ Let $S = \{e_3\}$, then $S^\perp = \text{span}\{e_1, e_2\}$

Ex: $V = C([-1,1])$: the inner product space of all continuous fns on $[-1,1]$.

W_e : subspace of all even functions

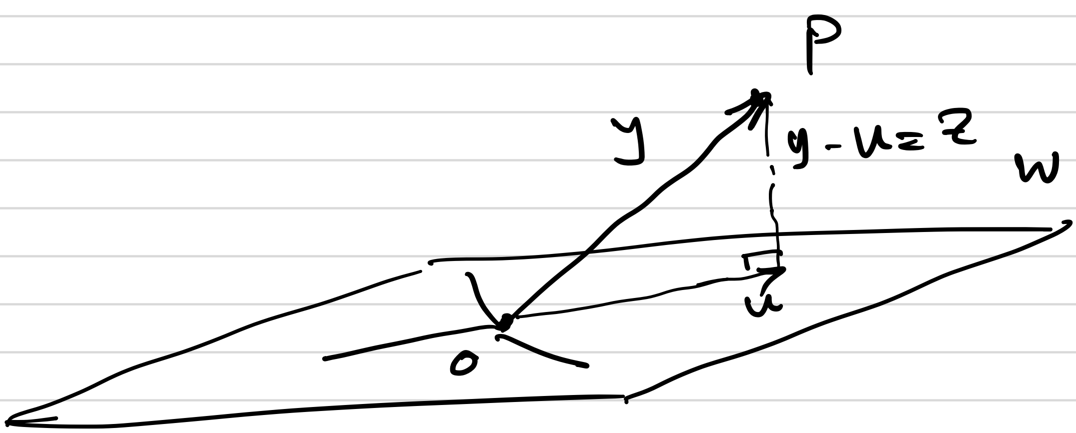
W_o : odd functions

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

Then prove: $W_e^{\perp} = W_o$ ★

Example:

Problem in $\mathbb{R}^3$ of computing distance of a point P from a plane W (subspace).



* u is an orthogonal projection of y onto W.

<u>Theorem</u>: Let W be a finite dimensional subspace of an inner product space V. Let $y \in V$. Then there exist <u>unique</u> vectors $u \in W$ and $z \in W^{\perp}$ such that $y = u + z$. Furthermore, if $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis of W, then

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

<u>Proof</u>: Let $\{v_1, \ldots, v_k\}$ be an orthornomal basis for W and

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i \in W$$

Then, let $z = y - u$

To show that $z \in W^{\perp}$.

For any $j = 1, 2, \ldots, k$

$$\langle z, v_j \rangle = \left\langle y - \sum_{i=1}^{k} \langle y, v_i \rangle v_i \, , v_j \right\rangle$$

$$= \langle y, v_j \rangle - \sum_{i=1}^{k} \langle y, v_i \rangle \langle v_i, v_j \rangle$$

$$= \langle y, v_j \rangle - \langle y, v_j \rangle = 0$$

To show uniqueness, let us assume

$y = u + z = u' + z'$ where $u' \in W$, $z' \in W^\perp$

$$\Rightarrow \underset{\substack{\uparrow \\ W}}{u - u'} = \underset{\substack{\uparrow \\ W^\perp}}{z - z'}$$

$$\Rightarrow u - u' = z - z' \in W \cap W^\perp = \{0\}$$

$$\Rightarrow u = u' \text{ and } z = z'.$$

$\blacksquare$

<u>Corollary</u>: In the notation of the above theorem, the vector u is the unique vector in W that is "closet" to y. In other words, for any $x \in W$, $\|y - x\| \geq \|y - u\|$ and the equality holds if and only if $x = u$.

<u>Proof</u>: From the above theorem, $y = \underset{\substack{\uparrow \\ W}}{u} + \underset{\substack{\uparrow \\ W^\perp}}{z}$

Let $x \in W$.

$u - x \perp z$ (*)

$$\|y - x\|^2 = \|u + z - x\|^2 = \|(u - x) + z\|^2$$

$$\overset{(*)}{=} \|u - x\|^2 + \|z\|^2$$

$$\Rightarrow \|y - x\|^2 \geq \|z\|^2 = \|y - u\|^2$$

$\blacksquare$

Example:

Let $V = P_3(\mathbb{R})$ with inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t) g(t)\, dt \quad \forall f, g \in V.$$

Compute orthogonal projection of $f(x) = x^3$ on $P_2(\mathbb{R})$.

Remember,

$$\left\{ \overset{= q_1}{\frac{1}{\sqrt{2}}},\ \overset{= q_2}{\sqrt{\frac{3}{2}}\, x},\ \overset{= q_3}{\sqrt{\frac{5}{8}}\,(3x^2 - 1)} \right\}$$

is an orthonormal basis for $P_2(\mathbb{R})$.

orthogonal projection of $f(x)$

$$= \sum_{i=1}^{3} \langle f(x), q_i \rangle q_i$$

$$\langle f(x), q_1 \rangle = \int_{-1}^{1} t^3 \cdot \frac{1}{\sqrt{2}}\, dt = 0$$

$$\langle f(x), q_2 \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}\, t^4\, dt = \frac{\sqrt{6}}{5}$$

$$\langle f(x), q_3 \rangle = 0$$

Orthogonal projection of $f(x)$ onto $P_2(\mathbb{R})$

$= \frac{3}{5} x$

Computer simulation :

generate a uniform random sample between -1 to 1.

$$\begin{array}{cccc} x_1, & x_2, & \cdots, & x_n \\ \downarrow & \downarrow & & \downarrow \\ x_1^3 & x_2^3 & & x_n^3 \\ \| & \| & & \| \\ y_1 & y_2 & & y_n \end{array}$$

$(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Fit a linear model to this date ..

Theorem: Let $S = \{v_1, v_2, \cdots, v_k\}$ be an orthonormal set in an n-dimensional inner product space V. Then

1) S can be extended to an orthonormal basis $\{v_1, \ldots, v_k, v_{k+1}, \cdots, v_n\}$ for V.
2) If $W = \text{span}(S)$. Then $S_1 = \{v_{k+1}, \cdots, v_n\}$ is an orthonormal basis for $W^\perp$.
3) If W is any subspace of V, then $\dim(V) = \dim(W) + \dim(W^\perp)$.

Adjoint of a linear operator.

Conjugate transpose of a matrix A is denoted as A^*.

For an inner product space V we want to define a similar transformation T^* called as adjoint of T.

Let V be an inner product space.

Let $y \in V$.

Define $g: V \longrightarrow \mathbb{F}$ as

$$g(x) = \langle x, y \rangle \quad \forall x \in V.$$

Then $g(x)$ is linear.

More interesting fact:

If V is finite dimensional, then every linear functional is of this form.

$\Downarrow$

linear transformation from V to $\mathbb{F}$.

<u>Theorem</u>: Let V be a finite dimensional inner product over $\mathbb{F}$, and $g : V \longrightarrow \mathbb{F}$ be a linear transformation. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle \quad \forall \; x \in V$.

<u>Proof:</u>

Let $B = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V.

let $\quad y = \sum_{i=1}^{n} \overline{g(v_i)} \, v_i$

Define $h : V \longrightarrow \mathbb{F}$ by $h(x) = \langle x, y \rangle$.

Clearly h is linear.

For $1 \leq j \leq n$,

$$h(v_j) = \langle v_j, y \rangle = \langle v_j, \sum_{i=1}^{n} \overline{g(v_i)} v_i \rangle$$

$$= \sum_{i=1}^{n} g(v_i) \langle v_j, v_i \rangle$$

$$= g(v_j)$$

$\Rightarrow h = g$.

To show: y is unique.

$g(x) = \langle x, y' \rangle = \forall x$

$\langle x, y \rangle = \langle x, y' \rangle \quad \forall x$

$\Rightarrow y = y'$ $\boxtimes$

<u>Definition</u>: Let V be a vector space over $\mathbb{F}$. An inner product on V is a function that assigns to every ordered pair of vectors x & y in V a scalar in $\mathbb{F}$, denoted as $\langle x, y \rangle$ s.t.

a) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

b) $\langle cx, y \rangle = c \langle x, y \rangle$

c) $\overline{\langle x, y \rangle} = \langle y, x \rangle$

d) $\langle x, x \rangle > 0$ if $x \neq 0$.

Ex: $x = (a_1, \ldots, a_n)$, $y = (b_1, \ldots, b_n) \in \mathbb{F}^n$

define

$$\langle x, y \rangle = \sum_{i=1}^{n} a_i \overline{b_i}.$$

Definition: Let $A \in M_{m \times n}(\mathbb{F})$. We define conjugate transpose or adjoint of A as $n \times m$ matrix A^* such that

$$(A^*)_{ij} = \overline{A_{ji}} \quad \text{for } i, j.$$

Ex:

$$A = \begin{pmatrix} i & 1+2i \\ 2 & 3-4i \end{pmatrix}$$

$$A^* = \begin{pmatrix} -i & 2 \\ 1-2i & 3+4i \end{pmatrix}$$