Deterministic Turing Machines – δ is a function

Non " " " – δ is a relation

In terms of computability, DTM is equivalent to NDTM. Also a TM on one tape is equivalent to a TM on many tapes. So we will switch from one model to another whenever convenient

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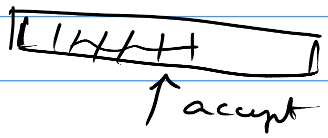
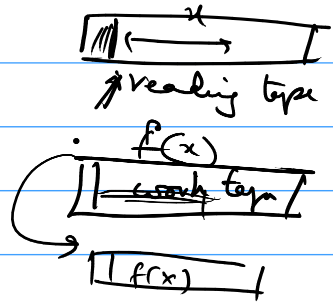
Accepting a language, and computing a given function from an input will also be considered as equivalent tasks.





$f(x) = O(g(x))$ if $\exists\, x_0, c$ s.t. $\forall x > x_0$, $f(x) \le c\, g(x))$

$f(x) = O(g(x)) \iff g(x) = \Omega(f(x))$

Thm: Let $\Sigma = \{0, 1, \#\}$

The set of palindromes

$$PAL = \{z \in \Sigma^* \mid z = \text{rev } z\}$$

requires $\Omega(n^2)$ time on a one-tape TM.

Let some TM M accept PAL.

Proof: Assume that M always moves to the rightmost non blank position before it enters accept state.

Consider n s.t. $4 \mid n$. Define:

$$PAL_n = \{x\#^{n/2}\text{rev } x \mid x \in \{0,1\}^{n/4}\}$$

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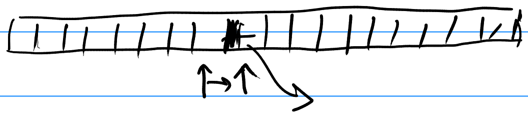


Let $c_i(x)$ be the sequence $q_1, q_2, \ldots, q_k$ of states $\in Q$ of M as it crosses the line between i & $i+1$th index of the tape/input

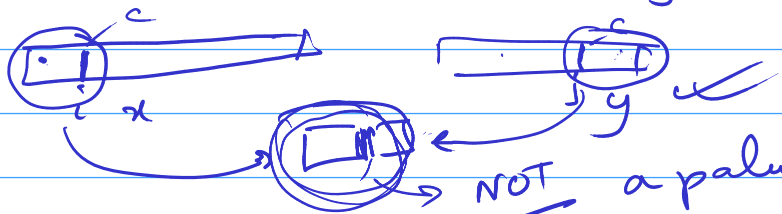
$c_i(x)$ is called the crossing sequence (of x) at i.

Let $\mathcal{C}(x) = \{ c_i(x) \mid n/4 \le i \le 3n/4 \}$

$$\overset{n/4}{0100}\,\boxed{\overbrace{1\#\#\#\#\#1}^{2n/4}}\,\overset{n/4}{0010}$$

Lemma: If $x, y \in PAL_n$, and $x \neq y$, then $\mathcal{C}(x) \cap \mathcal{C}(y) = \phi$

Suppose on the contrary, $c \in \mathcal{C}(x) \cap \mathcal{C}(y)$.
Say, $c = c_i(x) = c_j(y)$. Let x' be the prefix of x consisting of the first i symbols. Let y' be the suffix
" y " " " last $n-j$ " .



NOT a palindrome!

Contradiction

$\mathcal{C}(x) \cap \mathcal{C}(y) = \phi$

Let m_x be the length of the shortest crossing sequence in $\mathcal{C}(x)$. Let $m = \max\{ m_x \mid x \in PAL_n \}$

$$|PAL_n| = 2^{n/4}$$

How many possible crossing sequences are there of length $\le m$?

$$\sum_{i=1}^{m} |Q|^i = \frac{|Q|^{m+1} - 1}{|Q| - 1} \geq 2^{n/4}$$

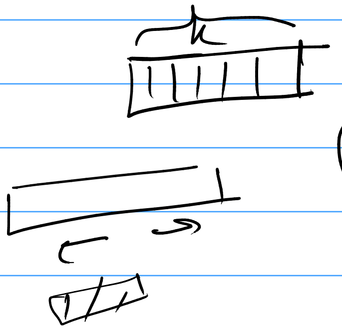
$$\sim |Q|^m \geq 2^{n/4}$$

$$m \log |Q| \geq \frac{n}{4} \log 2$$

$$m \geq \Omega(n)$$

time complexity $= \Omega(n \cdot \frac{n}{2}) = \Omega(n^2)$

Q. Given a TM M, with an input tape and a worktape. M is not allowed to delete or write anything in the input tape. However the worktape has only finitely many cells. If M accepts a language L, what can you comment on L?



$|\Sigma|$

$|\Sigma|^k \times |Q|$

L is regular.

$Q' = (s, q)$

$s \in \Sigma^k, q \in Q$

DTIME (def) : Let $T : \mathbb{N} \to \mathbb{N}$ be some function. A language L is in DTIME($T(n)$) iff $\exists$ a TM that runs in time $c \cdot T(n)$ for some constant $c > 0$ and decides L.

NTIME (def) " " " $\exists$ an NDTM "

Class P (def.) : $P = \bigcup_{c \geq 1} \mathrm{DTIME}(n^c)$

" NP (def) : $NP = \bigcup_{c \geq 1} \mathrm{NTIME}(n^c)$

Shortest path, Min span. tree, sorting $\in P$

$\Updownarrow$

class NP (def) : A Language $L \subseteq \{0,1\}^*$ is in NP if $\exists$ polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time TM M (called the verifier for L) s.t. for every $x \in \{0,1\}^*$
$x \in L \Leftrightarrow u \in \{0,1\}^{p(|x|)}$ s.t. $M(x, u) = 1$

A problem is in NP iff given a solution of it, the solution can be <u>verified</u> by a DTM in polynomial time.

Note : If a problem is in P, then it is also in NP.

A language $L \subseteq \{0,1\}^*$ is a polynomial-time Karp reducible to a language $L' \subseteq \{0,1\}^*$, denoted by $L \leq_p L'$, if there is a polynomial-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ s.t. for every $x \in \{0,1\}^*$, $x \in L$ if and only if $f(x) \in L'$.

We say that L' is NP_hard if $L \leq_p L'$ for every $L \in NP$.

We say that L' is NP complete if L' is NP hard and $L' \in NP$.

$x_1, x_2, x_3 \ldots x_n \in \{0,1\}$

OR , AND , NOT

$\cap$: $\cup$ + $\sim$ '

$\wedge$ $\vee$

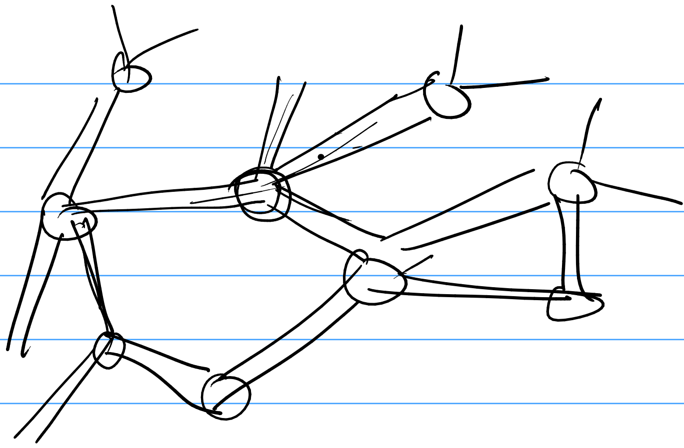
$x_1 \cup (x_2 \cap \bar{x}_3) \to$ boolean formula

A boolean formula is said to be satisfiable iff for some assignment of values (from $\{0,1\}$) to its constituent variables results in the boolean formula evaluating to 1 / True.

CNF form (Conjunctive Normal Form)
/ SAT
AND of ORs

$\underbrace{(x_1 \vee x_3 \vee x_4)}_{\text{clause}} \wedge \underbrace{(x_6 \vee \bar{x}_1 \vee x_5)}_{\text{clause}} \wedge (\ldots) \wedge (\ldots$

SAT / CNF and 3-SAT are both NP complete

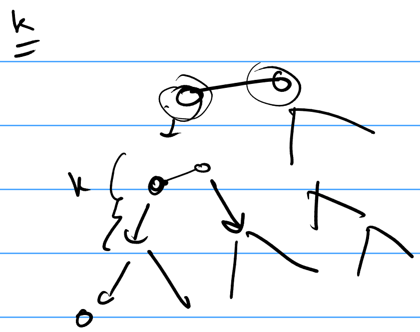


Given a graph $G(V, E)$, a vertex cover is a set $S \subseteq V$ s.t. $\exists (u,v) \in E$ $u \in S$ or $v \in S$

min VC : Find a vertex cover of minimum cardinality in a given graph.

Approximate answer: Take any (u,v) from E. Include both u & v in S. Delete (u,v) from G. Delete both u and v from G. Delete all edges incident on u or v from G.

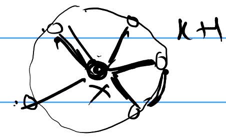
The above algorithm gives a v.c. which is atmost twice the size of a min. VC.



$2^k \cdot p(n)$

$p(n)\left(2^1 + 2^2 + 2^3 + \dots\right)$

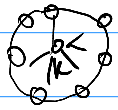
$\sim \boxed{2^{k+1}}\, p(n)$

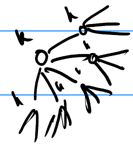


Is there a VC of size k?

Suppose there is.

Then put all vertices of degree $\geq k+1$ in S.





In the remaining graph, all vertices are of degree $\leq k$

If there are $> k^2$ edges then there is no VC of size k.

$|G'| \leq k^2$

$f(k)\, p(n)$

$k : 1 \to n$