Problems 1

① For any $\varepsilon > 0$, Construct an open set $U \subseteq \mathbb{R}$ such that $U \supseteq \mathbb{Q}$ & $m^*(U) \le \varepsilon$.

solution:-

$\mathbb{Q}$ is measurable & $m^*(\mathbb{Q}) = 0$

Let $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$

Let $\varepsilon > 0$.

Let $I_n = \left(r_n - \dfrac{\varepsilon}{2^{n+1}},\ r_n + \dfrac{\varepsilon}{2^{n+1}}\right)$ $\forall n \in \mathbb{N}$.

open set

& let $U = \bigcup_{n=1}^{\infty} I_n$ open set.

$r_n \in I_n\ \forall n \Rightarrow \mathbb{Q} \subseteq U$.

$$m^*(U) = m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \le \sum_{n=1}^{\infty} m^*(I_n)$$

$$= \sum_{n=1}^{\infty} l(I_n)$$

$$= \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$$

$$= \varepsilon.$$

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② (a) Show that $\mathbb{Q}$ is an F_σ-set.

(b) Show that there exists a G_δ-set G such that $G \supseteq \mathbb{Q}$ & $m^*(G) = 0$.

③ ... of all irrational

(c) show that the set of ... numbers is a G_δ-set.

Solution: (a) $\mathbb{Q} = \{r_1, r_2, \ldots\}$.

$= \bigcup_{j=1}^{\infty} \underbrace{\{r_j\}}_{\text{closed set}}$ is an F_σ-set

(b) $m^*(\mathbb{Q}) = 0$. & $\mathbb{Q}$ is measurable

$\Rightarrow \exists$ a G_δ-set G s.t. $G \supseteq \mathbb{Q}$ &

$m^*(G) = m^*(\mathbb{Q}) = 0$.

(c) $\mathbb{Q}^c = \bigcap_{j=1}^{\infty} \underbrace{\{r_j\}^c}_{\text{open set.}}$

is a G_δ-set.

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(3) Let $E \subseteq \mathbb{R}$ be a measurable set & $m^*(E) > 0$.

Prove that for every $\alpha \in (0,1)$, there exists a finite open interval I such that

$$\alpha\, m^*(I) \leq m^*(E \cap I) \leq m^*(I).$$

Solution:- The 2nd inequality follows from monotonicity property of m^*.

- Suppose $m^*(E) < \infty$.

For any $\alpha \in (0,1)$, set $\frac{1}{\alpha} = 1 + a$

$a > 0$, Let $\varepsilon = a\, m^*(E) > 0$

Since E is measurable, there exists an open set $U \supseteq E$ such that $m^*(U) \le m^*(E) + \varepsilon$

$$\Rightarrow m^*(U) \le m^*(E) + a\, m^*(E).$$
$$= m^*(E)(1+a).$$
$$= \left(\frac{1}{\alpha}\right) m^*(E).$$

Since U is an open set, we have,

$$U = \dot{\bigcup_{n=1}^{\infty}} I_n$$ disjoint union of finite open intervals in $\mathbb{R}$.

$$\therefore \quad m^*(U) = m^*\left(\dot{\bigcup_{n=1}^{\infty}} I_n\right)$$
$$= \sum_{n=1}^{\infty} m^*(I_n)$$

Since $E \subseteq U$, $m^*(E) = m^*(E \cap U)$

$$= m^*\left(E \cap \left(\dot{\bigcup_{n=1}^{\infty}} I_n\right)\right)$$
$$= m^*\left(\dot{\bigcup_{n=1}^{\infty}} (I_n \cap E)\right)$$
$$= \sum_{n=1}^{\infty} m^*(E \cap I_n)$$

$$\therefore \sum_{n=1}^{\infty} m^*(I_n) \le \left(\frac{1}{\alpha}\right) \sum_{n=1}^{\infty} m^*(E\cap I_n).$$

$\Rightarrow$ there exists at least one $n_0 \in \mathbb{N}$ such that $m^*(I_{n_0}) \le \frac{1}{\alpha} m^*(E\cap I_{n_0})$.

Let $I = I_{n_0}$.

Then $$\alpha\, m^*(I) \le m^*(E\cap I) \le m^*(I)$$ as required.

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(4) Let E be a measurable set in $\mathbb{R}$. & $m^*(E) = 1$. Show that there exists a measurable set $A \subseteq E$ such that $m^*(A) = \frac{1}{2}$.

<u>Solution:-</u> Define a function $f : \mathbb{R} \to [0,1]$

$$f(x) = m^*\left(E\cap(-\infty, x]\right)$$

f is an increasing function $\forall x \in \mathbb{R}$.

<u>claim:-</u> f satisfies the Lipschitz property.

i.e. $|f(x) - f(y)| \le |x-y| \quad \forall x, y \in \mathbb{R}$

Say $x<y$,

$$-f(x) + f(y) = -m^*\left(E \cap (-\infty, x]\right) + m^*\left(E \cap (-\infty, y]\right)$$

$$= m^*\left(E \cap (x, y]\right)$$

$$\leq m^*\left([x, y]\right)$$

$$= y - x$$

Thus $|f(y) - f(x)| \leq |y - x| \quad \forall x, y \in \mathbb{R}$

$\therefore$ f is a continuous function. on $\mathbb{R}$

Then by the intermediate value property,

$\exists x_0 \in \mathbb{R}$, such that $f(x_0) = \frac{1}{2}$

Set $A = E \cap (-\infty, x_0]$

then $m^*(A) = \frac{1}{2}$ & $A \subseteq E$.

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(5) Consider $\mathcal{M} = \{E \subseteq \mathbb{R} \,/\,$ either E is countable or E^c is countable $\}$

(a) show that $\mathcal{M}$ is a σ-algebra & $\mathcal{M} \subsetneq \mathcal{B}$

(b) Show that $\mathcal{M}$ is generated by $\{\{x\} \,/\, x \in \mathbb{R}\}$

Solution: (a) $\mathbb{R} = \phi^c$ & ϕ is countable

$\therefore \mathbb{R} \in \mathcal{M}$.

Suppose $E \in \mathcal{M} \Rightarrow$ either E or E^c is countable.

$\Rightarrow E^c \in \mathcal{M}$.

Suppose $E_1, E_2, \dots \in \mathcal{M}$.

<u>To show:</u> $\bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$.

Suffices to show: $\bigcup_{j=1}^{\infty} E_j$ is countable or

$\left(\bigcup_{j=1}^{\infty} E_j\right)^c = \bigcap_{j=1}^{\infty} E_j^c$ is countable

If all E_j's are countable, then $\bigcup_{j=1}^{\infty} E_j$ is also countable

$\therefore$ it belongs to $\mathcal{M}$.

Suppose $E_{n_0} \in \mathcal{M}$ which is not countable

$\Rightarrow E_{n_0}^c$ is countable.

Now $\left(\bigcup_{n=1}^{\infty} E_n\right)^c = \bigcap_{n=1}^{\infty} E_n^c \subseteq E_{n_0}^c$

which is also a countable set.

$\therefore \bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$.

$\therefore \mathcal{M}$ is a σ-algebra.

<u>To show:</u> $\mathcal{M} \subsetneq \mathcal{B}$.

$\mathcal{B}$ = the σ-algebra generated by open sets.

Suppose $E \in \mathcal{M}$.

$\Rightarrow$ E or E^c is countable.

If E is countable, then $E = \{x_1, x_2, \dots\}$

$$= \bigcup_{j=1}^{\infty} \underbrace{\{x_j\}}_{\text{closed sets}} \in \mathcal{B}.$$

||ly if E^c is countable.

$$\Rightarrow E^c \in \mathcal{B}$$

$$\Rightarrow E \in \mathcal{B}.$$

$\therefore \mathcal{M} \subseteq \mathcal{B}$.

$[0,1] \in \mathcal{B}$ but $[0,1] \notin \mathcal{M}$.

$\therefore \mathcal{M} \subsetneq \mathcal{B}$.

(b) Let $S = \{\{x\} \mid x \in \mathbb{R}\}$.

To show: $\mathcal{M}$ is generated by S.

Clearly $S \subseteq \mathcal{M}$.

$\Rightarrow$ $\sigma(S) =$ the smallest σ-alg gen by $S \subseteq \mathcal{M}$.

Now $E \in \mathcal{M}$ & $E \neq \phi$.

if E is countable then we can write

$$E = \bigcup_{n=1}^{\infty} \underbrace{\{x_n\}}_{S} \in \sigma(S)$$

||ly if E^c is countable $\therefore E \in \sigma(S)$

$\therefore \mathcal{M} \subseteq \sigma(S)$

$\therefore \mathfrak{F} \subseteq \sigma(S)$

$\therefore \mathfrak{F} = \sigma(S)$.