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$y'' + A(x)y' + B(x)y = C(x)$, $y(0) = y_0$, $y(a) = y_a$.

$A(x) \neq 0$, at $x = x_k$; $y_k'' + A_k y_k' + B_k y_k = C_k$.

$A_k y_k' = C_k - y_k'' - B_k y_k$

$p_k(x_k) = y_k'$, $p_{k-1}'(x_k) = y_k' \rightarrow$ (II)

$\hookrightarrow k = 0, 1, 2, \dots, n$ $\hookrightarrow$ (I) $\quad k = 1, 2, \dots, n$

$M_k, y_k \rightarrow$ unknowns

$2n$ relations, $M_0, M_1, \dots, M_n$, & $y_1, \dots, y_{n-1} \Rightarrow 2n$ unknowns.

System of equations (I) & (II) are coupled set of $2n$ linear equations involving $2n$ unknowns $M_0, M_1, \dots, M_n$ & $y_1, y_2, \dots, y_{n-1}$.

$p_k'(x_k) = y_k'$

$$\left(1 - \frac{h}{3}A_k\right)M_k - \frac{h}{6}A_k M_{k+1} = C_k - B_k y_k - \frac{A_k}{h}(y_{k+1} - y_k)$$

$k = 0, 1, \dots, n-1.$ (I)

$p_{k-1}'(x_k) = y_k'$; $\frac{h}{6}A_k M_{k-1} + \left(1 + \frac{h}{3}A_k\right)M_k = C_k - B_k y_k$

$$- \frac{A_k}{h}(y_k - y_{k-1}) \quad \text{(II)}$$

$k = 1, 2, \dots, n$

(Q) Solve the BVP : $y'' + 2y' + y = 30x$, $y(0) = 0 = y(1)$, $h = 0.5$.

$y_1 = ??$ $\quad$ ~~M_0~~ $\quad y_0 = 0$, $y_2 = 0$

$\rightarrow$ $\left(1 - \frac{h}{3}.2\right)M_0 - \frac{h}{6}.2.M_1 = 30(0) - y_0 - \frac{2}{h}(y_1 - y_0)$.

$\cancel{\frac{2M_0}{6} + \left(1 + \frac{2h}{3}\right)M_1} = 30(0.5) - y_1 - \frac{2}{h}(y_1 - y_0)$

~~$\left(\frac{1+2h}{3}\right)M_1 = \dots$~~

Spline →

(1) Higher order of accuracy.

(2) y_k' is also determined.

<Higher order compact difference scheme>

$y \sim p_k(x)$ in $[x_k, x_{k+1}]$

↳ Hermit interpolation polynomial.

Non-Linear BVP

$$\frac{d^2y}{dx^2} = f(x,y,y'),\quad 0<x<a$$

$$F(x, y, y', y'') = 0 \qquad y_0 = y(0),\quad y(a) = y_a$$

where F is any arbitrary function

$$y'' + 2yy' = 4 + 4x^2, \qquad y(1) = 2,\ y(2) = 4.$$

$$\frac{y_{i+1} - y_i + y_{i-1}}{h^2} + 2y_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) = 4 + 4x_i^2 \quad (*)$$

$i = 1, 2, \ldots, n-1$

which are $(n-1)$ algebraic equations involving $(n-1)$ unknowns, $y_1, \ldots, y_{n-1}$.

System of eq^n $(*)$ are non-linear, cannot be expressed in matrix form.

Nonlinear algebraic eq^n

Newton-Raphson method for root finding of a nonlinear eq^n $f(x)=0$, f is any function of x and if α is the root, then $f(\alpha)=0$, $\alpha = ??$

We determine the root iteratively. At any iteration level k, let $x^{(k)}$ is the approx. value of α. The root and let Δx is the error, i.e, $\alpha = x^{(k)} + \Delta x$.

$f(\alpha)=0$, $f(x^{(k)} + \Delta x) = 0$, Δx is unknown

Expand by Taylor series

$$0 = f(x^{(k)}) + \Delta x f'(x^{(k)}) + \left(\frac{(\Delta x)^2}{2!}\right) f''(x^{(k)}) + \cdots$$

which is an infinite degree polynomial in Δx.

Retaining only upto linear order terms in Δx, we get

$$f(x^{(k)}) + \Delta x f'(x^{(k)}) \simeq 0;\quad \Delta x = \frac{-f(x^{(k)})}{f'(x^{(k)})}$$

and the successive approx. for

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}, \quad k \geq 0$$

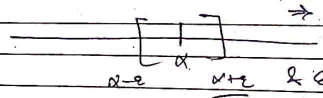
we repeat the process till $|x^{(k+1)} - x^{(k)}| < \varepsilon \Rightarrow |\Delta x| < \varepsilon$

Iteration (I) starts with an initial given $x^{(0)}$ for the root; at each iteration, the approx. for the root is modified.

$\{x_k \mid k = 0, 1, \ldots\}$. $x_k \to \alpha$, as $k \to \infty$

$x_k \to \alpha$ as $k \to \infty$

$\Rightarrow |x_k - \alpha| < \varepsilon \quad \forall k > K$



$K > 0$ a finite integer

& $\varepsilon > 0$ an arbitrary small quantity

$|x_k - x_{k+K}| < \varepsilon,\ \forall k \geq K,\ l > 0$

Non-linear BVP :- Newton's linearization technique.

$F(x, y, y', y'') = 0$, $0 < x < a$, y_0 & y_a are given

$x_i = i\,\delta x_i$, are grid points. $i = 0, 1, \ldots, n$.

Discretize the ODE to get $\to (*)$

$$f_i(y_{i-1}, y_i, y_{i+1}) = 0, \quad i = 1, 2, \ldots, n-1$$

where f_i are known nonlinear algebraic function of y_{i-1}, y_i & y_{i+1}

We solve the system of non linear algebraic eq^n $(*)$ iteratively by using the Newton's

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linearization technique i.e. at the $(k+1)^{th}$ iteration $y_i^{(k+1)} = y_i^{(k)} + \Delta y_i, \; k \geq 0 \rightarrow$ (a)

Δy_i is the error at the k^{th} iteration

$\Delta y_0 = \Delta y_n = 0$ at the two boundary pts.

Substitute (a) in the discretized eq^n (*) to get

$$f_i\left(y_{i-1}^{(k)} + \Delta y_{i-1},\; y_i^{(k)} + \Delta y_i,\; y_{i+1}^{(k)} + \Delta y_{i+1}\right) = 0 \quad i = 0, 1, \ldots, n-1 \rightarrow \text{(b)}$$

Δy_i for $i = 1, 2, \ldots, n-1$ are unknowns.

Expand (b) by Taylor series

$$f_i\left(y_{i-1}^{(k)}, y_i^{(k)}, y_{i+1}^{(k)}\right) + \left.\frac{\partial f_i}{\partial y_{i-1}}\right|^{(k)} \Delta y_{i-1} + \left.\frac{\partial f_i}{\partial y_i}\right|^{(k)} \Delta y_i + \left.\frac{\partial f_i}{\partial y_{i+1}}\right|^{(k)} \Delta y_{i+1} + O\left((\Delta y_{i-1})^2, (\Delta y_i)^2, (\Delta y_{i+1})^2\right) = 0$$

$$i = 1, 2, \ldots, n-1$$

which are $(n-1)$ eq^n involving $\Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}$ i.e. $(n-1)$ unknowns.

Retaining only upto linear orders in Δy_i's, we get a reduced set of linear algebraic eq^{n}s

$$\left.\frac{\partial f_i}{\partial y_{i-1}}\right|^{(k)} \Delta y_{i-1} + \left.\frac{\partial f_i}{\partial y_i}\right|^{(k)} \Delta y_i + \left.\frac{\partial f_i}{\partial y_{i+1}}\right|^{(k)} \Delta y_{i+1} = f_i\left(y_{i-1}^{(k)}, y_i^{(k)}, y_{i+1}^{(k)}\right).$$

$$i = 1, 2, \ldots, n-1.$$

which forms $(n-1)$ linear tri-diag. system for $\Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}$, which can be cast as $Ax = d$, $x^T = [\Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}]$

~~Substitute~~

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$\Delta y_0 = \Delta y_n = 0$

The solutions ~~of the~~ Δy_i's determined by solving $Ax = d$, we get the next approximation for y_i as

$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$

$\hookrightarrow$ modified sol^n , $\epsilon > 0 \rightarrow$ pre-assigned +ve q^{ty}.

Repeat the process, $\max\limits_{1 \leq i \leq n-1} |\Delta y_i| < \epsilon$

Steps $\rightarrow$ (1) Discretize the non linear ODE

(2) Solve the ensuing non linear algebraic eq^n iteratively.

(3) at each iteration, $y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$

(4) Solve the tridiag. system $Ax = d$ to get Δy_i's.

(5) repeat till $\max\limits_{1 \leq i \leq n-1} |\Delta y_i| < \epsilon$

(Q) $y'' + 2yy' = 4 + 4x^2$, $y(1) = 2$, $y(2) = 4$, $h = 0.25$.

Ans $\dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2y_i\left(\dfrac{y_{i+1} - y_{i-1}}{2h}\right) = 4 + 4x_i^2$

$x_0, x_1, x_2, \ldots$; 1, 0.5, 0.5, 2

To start the method, we need to guess $y_i^{(0)}$ $1 \leq i \leq n$

$y_0 = 2$, $y_4 = 4$. Let $y_2^{(0)} = 3$, $y_1^{(0)} = 2.5$, & $y_3^{(0)} = 3.5$

$$\frac{1}{h^2}\left(y_{i-1}^{(k)} - 2y_i^{(k)} + y_{i+1}^{(k)} + \Delta y_{i-1} - 2\Delta y_i + \Delta y_{i+1}\right) + \frac{1}{h}\left(y_i^{(k)} + \Delta y_i\right)\left(y_{i+1}^{(k)} + \Delta y_{i+1} - y_{i-1}^{(k)} - \Delta y_{i-1}\right) = 4 - 4x_i^2 = 0$$

$a_i \Delta y_{i-1} + b_i \Delta y_i + c_i \Delta y_{i+1} = e_i$

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Non-linear BVP

Newton's linearization technique: Iterative Method →

$$\max_{1\le i\le n-1} |\Delta y_i| = K < 10^{-5} = \epsilon \to \text{converged.}$$

$y_i^{(0)}\ \forall\, i$ needs to be provided.

$$\{y_i^{(k)} \mid k\ge 0\} \quad \lim_{k\to\infty} y_i^{(k)} \to y_i \Rightarrow |\Delta y_i| < \epsilon$$

$\epsilon = 0.5\times 10^{-5}$

k	Δy_i
1	--
2	--
3	--

decrease

$L[y]=0 \Rightarrow F_i(y_{i-1}, y_i, y_{i+1}) = 0,\ i=1,2,\dots,n-1.$

$$\Delta y_{i-1} \left.\frac{\partial F_i}{\partial y_{i-1}}\right|^{(k)} + \Delta y_i \left.\frac{\partial F_i}{\partial y_i}\right|^{(k)} + \Delta y_{i+1}\left.\frac{\partial F_i}{\partial y_{i+1}}\right|^{(k)} = d_i$$

$\Delta y_0 = \Delta y_n = 0$, $k\ge 0$, $i=1,2,\dots,n-1$

$\to y_i^{(k)}$

$$\boxed{a_i \Delta y_{i-1} + b_i \Delta y_i + c_i \Delta y_{i+1} = d_i} \Rightarrow \boxed{AX=d}$$

$A \to$ tri-diagonal.

$$X = \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \\ \vdots \\ \Delta y_n \end{bmatrix}; \quad y_i^{(k+1)} = y_i^{(k)} + \Delta y_i,\ i=1,2,\dots,n-1$$

Repeat till $\max_{1\le i\le n-1} |\Delta y_i| < \epsilon$.

Ex: $y'' - (y')^2 - y^2 + y + 1 = 0$, $y(0)=0.5$, $y(\pi) = -0.5$

Discretize by central diff. sch.

$$\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) - \frac{1}{4h^2}(y_{i+1} - y_{i-1})^2 - y_i^2 + y_i + 1 = 0 \quad (*)$$

$i=1,2,\dots,n-1.$

At $(k+1)^{th}$ iteration, $y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$, $k\ge 0$, $i=1,\dots$

Substitute in $(*)$

$$\frac{1}{h^2}\left[y_{i+1}^{(k)} + \Delta y_{i+1} - 2y_i^{(k)} - 2\Delta y_i + y_{i-1}^{(k)} + \Delta y_{i-1}\right]$$

$$-\frac{1}{4h^2}\left[y_{i+1}^{(k)} - y_{i-1}^{(k)} + \Delta y_{i+1} - \Delta y_{i-1}\right)^2 - \left(y_i^{(k)} + \Delta y_i\right)^2 + y_i^{(k)} + \Delta y_i = -1)$$

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$$+\left(y_{i+1}^{(k)} - y_{i-1}^{(k)}\right)^2 + 2(\Delta y_{i+1} - \Delta y_{i-1})\left(y_{i+1}^{(k)} - y_{i-1}^{(k)}\right)$$

$$+ (y_i^{(k)})^2 + 2\Delta y_i\, y_i^{(k)}$$

$=$

$$a_i \Delta y_{i-1} + b_i \Delta y_i + c_i \Delta y_{i+1} = d_i, \quad i=1,2,3,\dots,n-1.$$

$$a_i = \frac{1}{h^2} + \frac{1}{2h^2}\left(y_{i+1}^{(k)} - y_{i-1}^{(k)}\right), \quad b_i = \frac{-2}{h^2} - 2y_i^{(k)} + 1.$$

Lab Task

$$c_i = \frac{1}{h^2} - \frac{1}{2h^2}\left(y_{i+1}^{(k)} - y_{i-1}^{(k)}\right). \quad d_i = -1 - \frac{1}{h^2}\left[y_{i+1}^{(k)} - 2y_i^{(k)} + y_{i-1}^{(k)}\right]$$

$$+ \frac{1}{4h^2}\left[y_{i+1}^{(k)} - y_{i-1}^{(k)}\right]^2 + \left(y_i^{(k)}\right)^2 - y_i^{(k)}$$

$\Delta y_0 = \Delta y_n = 0$, $i=1,2,\dots,n-1$

$AX = d$, → solve to get Δy_is.

$y_i^{(0)}$ needs to be specified $\forall\, i$

$y^{0}(x) = f(x)$, should satisfy the boundary conditions

$$y^{(0)}(x) = \frac{x-\pi}{-\pi}(0.5) + \frac{x}{\pi}(0.5)(-1)$$

$$\boxed{y^{0}(x) = \frac{-x}{\pi} + 0.5}$$

Lab Task

Q) $3yy'' + (y')^2 = 0$, $y(0)=0$, $y(1)=1$. → unique

Discretize

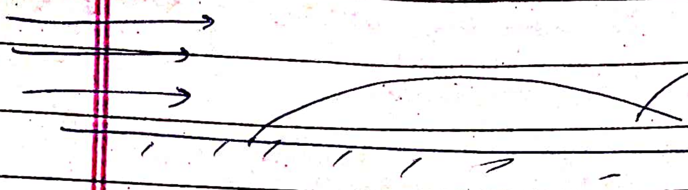
$h = 1/3$

$$3 y_i\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \left(\frac{y_{i+1} - y_{i-1}}{h}\right)^2 = 0 \quad (*)$$

At the $(k+1)^{th}$ iteration, $y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$.

Substituting in $(*)$,

wind or fluid movement 11-1 Lab [illegible]



Q $f''' + ff'' + 1 - (f')^2 = 0$

$f(0) = f'(0) = 0$

$f'(10) = 1$

get the reduced block-tridiagonal system which needs to be solved at every iteration.

$F = f'$

$F'' + fF' + 1 - F^2 = 0$

$X_i = \begin{pmatrix} f_i \\ F_i \end{pmatrix}$