Date
30/10/2017

Lecture 25

Topic
Partial Differential eqn
(P.D eqn)

Tentative

'13' the November (~~Monday~~) 'Class Test' 8:30 – 9:30 am
'14' Equivalent Mid-Sem

Theory of linear spaces
- Fourier Series
  ↓
- Fourier Integral
  ↓
- Fourier Transform

Fourier Transform as an eg. of an integral transform (F.T)

Deriving the F.T

$$\hat{f} : \mathbb{R} \to \mathbb{C},$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega.$$

time

frequency

→ Inverse Fourier Transform

$\hat{f}(\omega)$

Integral Transform :

The integral transform of a function $f(x)$ is defined as $\int_a^b f(x)\, \underbrace{K(s,x)}\, dx$, where $K(s,x)$ is termed as the kernel of the transform & it is a function of s & x.

We can observe that the given integral is a f^n of s.

So, by integral transform, the f^n $f(x)$ is transformed to $I(s)$, where

$$I(s) = \int_a^b f(x)\, \underset{\sim}{K(s,x)}\, dx .$$

<u>Kernel</u> :- In an integral transform, the kernel $\underline{K(s,x)}$ is a f^n of $\underline{s \,\&\, x}$, where the input f^n is a f^n of these. There are <u>various kernels</u> that define different kinds of transform as

(a) $K(s,x) = e^{-sx}$ for Laplace transform of $f(x)$

$$\int_a^b f(x)\, e^{-sx}\, dx.$$

ii) $K(s,x) = e^{-isx}$ for

Fourier transform of $f(x)$

iii) $K(s,x) = \sin(sx)$ for

Fourier sine transform of $f(x)$.

iv) $K(s,x) = \cos(sx)$ for

Fourier cosine transform

i.e., $\int_a^b f(x)\cos(sx)\,dx$.

$\hat{f} : \mathbb{R} \to \mathbb{C}$

spectrum f^n ff $\hat{f}(\omega) = \underbrace{A(\omega)}_{\text{amplitude}} \cdot e^{i\overbrace{\phi(\omega)}^{\text{phase}}}$

is a complex-valued f^n

where $A(\omega)$ & $\phi(\omega)$ are real f^ns of the real variable ω. $\hat{f}(\omega)$ is a complex-valued f^n of a real variable ω.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega)\, e^{i(\omega x + \phi(\omega))}\, d\omega.$$

F.T & F.S

(<u>EX</u> Investigate Find out more about them)

$\mathcal{F}(\cos x)$ δ → generalized f^n.

cont. f^n.

(Find out more about them)

Relationship bet F.T & L.T

Let us consider the f^n to be transformed

in the form $e^{-kx} f(x)$.

Fourier transform is

$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} e^{-kx} f(x)\, dx$.

then we can't the In this form, the F.T can be related to L.T.

$$\hat{f}_k(\omega) = \int_{0}^{\infty} e^{-i\omega x} \cdot e^{-kx} f(x)\, dx.$$

This $\hat{f}_k(\omega)$ will exist provided the f^n $f(x)$ is of exponential order (why?)

i.e.,
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \hat{f}_k(\omega)\, d\omega = \begin{cases} 0, & x < 0 \\ e^{-kx} f(x), & x \geq 0 \end{cases}$$

i.e.,
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(k+i\omega)x} \hat{f}_k(\omega)\, d\omega = \begin{cases} 0, & x < 0 \\ f(x), & x \geq 0 \end{cases}$$

Let $\underline{k} + i\omega = s$,

As k is not varying,

$$ds = i\,d\omega \Rightarrow d\omega = \frac{ds}{i}$$

∴ <u>limits of ∫</u>

$$\omega \to -\infty, \quad s \to k - i\infty$$

$$\omega \to \infty, \quad s \to k + i\infty.$$

∴ L.H.S becomes → Bromwich's integral formula

$$\boxed{f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} e^{sx} F(s)\,ds = \mathcal{L}^{-1}\{F(s)\}}$$

where, $F(s) = \hat{f}_k(\omega)$ is now a complex valued f^n of the complex variable s.

$$\Rightarrow \mathcal{L}\{f(x)\} = F(s).$$

which is indeed the general form of the inverse laplace transform.

(EX Investigate more about them).

Fourier Transform (F.T)

Note !

(in a broader sense)

F.T is a systematic way to decompose 'generic' f^{ns} into a superposition of "symmetric" f^{ns}

these symmetric f'ns are usually quite explicit

e.g., sin (nx) or, cos (nx).

$\downarrow$ associated with

physical concepts such as frequency or energy.

** What "symmetric" means here will be vague, but it will usually be associated with some sort of groups

(abelian)

lest

Algebraic Geometry + analysis

$\langle F, +, \cdot \rangle$

$V(F)$

abelian

$\langle A, + \rangle$

(1), (2) A, (3), (4) a^{-1}

Indeed F.T is a tool

in the study of groups

(i.e., representation theory of groups)

→ which says how a group can describe a notion of symmetry.

Q) How a group can define a notion of symmetry.

linked with linear Algebra

such as

eg: $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$.

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eg, 1) $f : \mathbb{R} \to \mathbb{R}$

even

$f(-x) = f(x)$

odd

$f(-x) = -f(x)$.

eg, $f(x) = x^3 + \underline{3x^2} + 3x + 1$

It is neither even nor odd.

$$f = f_e + f_o$$

$$f_e(x) = \frac{f(x) + f(-x)}{2}$$

$$f_o(x) = \frac{f(x) - f(-x)}{2}.$$

eg, $f_e(x) = 3x^2 + 1$

$f_o(x) = x^3 + 3x$.

A(no)

$f = f_e + f_o \xrightarrow{\text{is}}$ unique.

This basic F.T $\longrightarrow$ associated with the two-element multiplicative group $G = \{+1, -1\}, \times$.

$+1 \xrightarrow{\text{Identity map}} x \mapsto x$

associated

$(-1) \xrightarrow{\text{associated to}} x \mapsto -x$

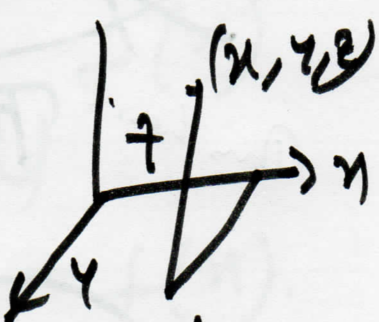
$\Rightarrow$ the reflection map

Partial Differential eq

(P. D. eqn)

Note :-

If a function depends on more than one variable then it is in general possible to differentiate it w.r.t one of them provided all the others are held constant while doing so.



eg:- $f(x, y, z)$ will have 3 derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \left\{ \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x} \right\}$$

y & z are held constant,

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \left\{ \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y} \right\}$$

x & z are held constant.

$$\frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \left\{ \frac{f(x, y, z+\Delta z) - f(x,y,z)}{\Delta z} \right\}$$

where x & y are held constant.

Ex/ Find all first order partial derivatives of the fns (a) x^2yz

Sol(n) 1 — The partial derivatives of first order are

$$\frac{\partial}{\partial x}(x^2yz) = 2xyz,$$

$$\frac{\partial}{\partial y}(x^2yz) = x^2z,$$

$$\& \frac{\partial}{\partial z}(x^2yz) = x^2y$$

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(3) $f(x,y,z) = x\sin(x+yz)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\{x\sin(x+yz)\}$$

$$= \sin(x+yz) + x\cos(x+yz)$$

(EX)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\{x\sin(x+yz)\}$$

$$= xz\cos(x+yz)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\{x \sin(x+yz)\}$$

$$= xy \cos(x+yz).$$

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There are chain rules for determining partial derivative when

$f = f(u, v, w)$,

where $u = u(x, y, z)$,

$v = v(x, y, z)$,

$w = w(x, y, z)$

new Jacobian

P.D. eq^n $\xrightarrow{\text{arise}}$ connection with various physical & geometrical problems when f^n depend on 2 or more independent variables usually the variables are time & space variables.

- simplest physical situation can be modelled by O.D. eqn.,

otherwise most problems in fluid mechanics, elasticity, heat transfer, electromagnetic theory, quantum mechanics, & other areas of physics.

- lead to p.d. eqns.

most important p.d. eq^n

I.V.P , B.V.P

P.D. eq^n

Def^n:— An eq^n involving one ore more partial derivative of an (unknown) f^n of two or more independent variables is a partial differential eq^n (P.D. eq^n)

The order of the highest derivative is called the order of the eq^n.

Just as in the case of o.d. eqn, we say that a p.d. eqn is linear if it is of the first degree in the dependent variable (i.e. the unknown fn) & its partial derivatives.

If each term of such an eqn contains either the dependent variable or one of its derivatives the eqn is said to be homogeneous, otherwise it is said to be non-homogeneous p.d. eqn xamples

Important linear p.d. eqn of the second order

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

— one — dimensional wave eqn.

(2) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

— one — dimensional heat eqn.

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

— Two — dimensional Laplace eqn.

(4) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Two — dimensional Poisson eqn.

(5) $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

– Two – dimensional wave eqn.

(6) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

– Three – dimensional Laplace eqn.

Here, c is a constant,

t → time,

x, y, z → cartesian co-ordinates

& dimension — the no. of these co-ordinates in the eqn.

Eqn (4) (with $f(x,y) \neq 0$) is non-homogeneous while the other eqns are homogeneous.

~~A~~ A <u>solution</u> of a p.d.eqn in some region R of the space of the independent variables is a <u>function</u> that has all the partial derivatives appearing in the eqn in <u>some</u> domain contain R & <u>satisfies</u> the eqn everywhere in R.

In general, the totality of solutions of a pd.eqn is very large.

eg) the f^{ns}

$$u = x^2 - y^2, \quad u = e^x \cos y$$

$$u = \ln(x^2+y^2)$$

which are entirely different from each other, are solutions of $u_{xx} + u_{yy} = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{check}$$

Unique soln of a p.d.eq.

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will be obtained by the use of additional cond^{n}s arising from the problem.

(boundary cond^{n}s)

t u, u_t when $t=0$ (initial cond^{n}s)

07

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n-1) (Fundamental Theorem) of P.D.E.
(superposition or linearity principle)

If u_1 & u_2 are any solutions of a linear homogeneous p.d.eqn in some region R, then

$$u = c_1 u_1 + c_2 u_2$$

with any constants c_1 & c_2 is also a solution of that eqn in R.

Ex/ Find a solution $u(x,y)$ of the p.d.eq^n

$$u_{xx} - u = 0.$$

Sol^n:- Since no y-derivative occur, we can solve this like $u'' - u = 0$

A.E $\underset{=0}{m^2 - 1} \Rightarrow m = \pm 1$

$\therefore u = Ae^{x} + Be^{-x}$, with constant A & B may be functions of y so that

$$u(x,y) = A(y)\,e^{x} + B(y)\,e^{-x}.$$

Where A & B are arbitrary f^{n}s.

Ex/ Solve the p.d.eqn

$$u_{xy} = -u_x$$

Soln:- Let $u_x = p$, we have

$$p_y = -p \qquad \left[u_x = \frac{\partial u}{\partial x} \text{ etc.} \right.$$

$$\Rightarrow \frac{p_y}{p} = -1$$

(on $\int$ w.r.t y, keeping x const)

$$\ln p = -y + \tilde{c}(x).$$

$$\Rightarrow p = c(x)\, e^{-y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = c(x)\, e^{-y}$$

($\int$ w.r.t x, keeping y constant)

$$\Rightarrow u(x,y) = \int c(x)\, e^{-y}\, dx + f(y).$$

$$\Rightarrow u(x,y) = f(x)e^{-y} + g(y)$$

where

$$f(x) = \int c(x)\,dx,$$

Here $f(x)$ & $g(y)$ are arbitrary f^{ns}.

Classification of p.d.eq^n

- Elliptic p.d.eq^n
- Parabolic p.d.eq^n
- Hyperbolic p.d.eq^n

H P E

0

The given second order P.D eqⁿ

$$a_1 \frac{\partial^2\phi}{\partial x^2} + b_1 \frac{\partial^2\phi}{\partial x \partial y} + c_1 \frac{\partial^2\phi}{\partial y^2} + d_1 \frac{\partial\phi}{\partial x} + e_1 \frac{\partial\phi}{\partial y} + f_1\phi = g_1 \quad \rightarrow (1)$$

where $a_1, b_1, c_1, d_1, e_1, f_1$ & g_1 are suitably well-behaved f^ns of x & y.

However, this is not a convenient form of the p.D eqⁿ for ϕ:

how? (Using Taylor's theorem)

EX

3 basic types of linear second order p.d.eqs

Three standard types of P.D. eqn are term hyperbolic, parabolic & elliptic.

Cannonical forms :-

We may $\phi_x = \frac{\partial \phi}{\partial x}$, $\phi_y = \frac{\partial \phi}{\partial y}$ etc.

$\phi_{xx} = \frac{\partial^2 \phi}{\partial x^2}$, $\phi_{yy} = \frac{\partial^2 \phi}{\partial y^2}$,

$\phi_{xy} = \frac{\partial^2 \phi}{\partial x \partial y}$.

Hyperbolic :- $a_2 \phi_{xy} + b_2 \phi_x + c_2 \phi_y + f_2 \phi = g_2$

Elliptic :- $a_3(\phi_{xx} + \phi_{yy}) + b_3 \phi_x + c_3 \phi_y + f_3 \phi = g_3$

Parabolic :- $a_4 \phi_{xx} + b_4 \phi_x + c_4 \phi_y + f_4 \phi = g_4$

(Normal forms)

In these eqns the a's, b's c's, f's & g's are fns of the variables x, y.

Note :- Laplace Transforms are useful in solving parabolic

& some hyperbolic p.d.eq^n.

They are not in general useful for solving Elliptic p.d.eq^n (or, P.D.eq^n)

Types & Normal forms of L.P.d.eq^ns.

A eq^n of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y) \rightarrow (1)$$

is said to be

Elliptic if $AC - B^2 > 0$

& hyperbolic if $AC - B^2 < 0$

& parabolic if $AC - B^2 = 0$.

—x—