VECTOR SPACES: A non-empty set V with two operations

(i) $\forall\, u, v \in V, \quad u+v \in V$

(ii) $\forall\, u \in V, \quad \lambda u \in V \quad \forall\, \lambda \in \mathbb{R}$.

and some axioms hold like a) $u+v = v+u$

b) ⋮

Ex: • Space $\mathbb{R}^n$, $\mathbb{R}_m$
- Polynomial space
- matrix space
- Function space

LINEAR COMBINATION: A vector $v \in V$ is a linear combination of $u_1, u_2, \dots, u_m$ from V if there exist scalars $\alpha_1, \alpha_2, \dots \alpha_m \in \mathbb{R}$ such that

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_m u_m.$$

Ex: Express $v = (3, 7, -4)^T$ in $\mathbb{R}^3$ as a linear combination of the vectors $u_1 = (1, 2, 3)^T$ $u_2 = (2, 3, 7)^T$ $u_3 = (3, 5, 6)^T$.

We seek scalars x, y, z such that

$$v = x u_1 + y u_2 + z u_3$$

$$\Rightarrow \begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} + z \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}.$$

OR

$$\left.\begin{aligned} x + 2y + 3z &= 3 \\ 2x + 3y + 5z &= 7 \\ 3x + 7y + 6z &= -4 \end{aligned}\right\}$$

Solve them using Gauss-Elimination to get $x = 2,\ y = -4,\ z = 3$.

Thus $v = 2u_1 - 4u_2 + 3u_3$.

Remark: expressing a vector as a linear combination of other given vectors is equivalent to solving system of linear equation. This system may not have solution or it may have unique or infinitely many solutions.

Q: In the above exemple if u_1, u_2 & u_3 are linearly independent then what can you say about uniqueness of the expression $v = xu_1 + yu_2 + zu_3$.

SPANNING SETS: If every $v \in V$ is a linear combination of vectors $u_1, u_2, \cdots u_m$. then vectors $u_1, u_2, \cdots u_m$ are said to span V.

Remark: 1. Suppose $u_1, u_2, \cdots, u_m$ span V. Then, for any vector w, then set $w, u_1, u_2, \cdots u_m$ also spans V.

2. Suppose $u_1, u_2, \cdots u_m$ span V and suppose u_k is a linear combination of some of the other u's. Then the u's without u_k also span V.

3. Suppose $u_1, u_2, \cdots u_m$ span V and suppose one of the u's is the zero vector. Then the u's without zero vector also span V.

SUBSPACES: $W \subset V$. Then W is a subspace of V if W is itself a vector space.

Ex: Solution space of homogeneous system:

Consider $Ax = \theta$ where $A \in \mathbb{R}^{n\times n}$ $x \in \mathbb{R}^{n\times 1}$ $\theta \in \mathbb{R}^{n\times 1}$

Every solution x can be viewed as a vector in $\mathbb{R}^n$.

This implie that the set of solutions is a subset of $\mathbb{R}^n$.

Check whether the solution set is a vector space (subspace of $\mathbb{R}^n$).

Let X_1 & X_2 are solutions of $Ax = \theta$ that is $AX_1 = 0$ & $AX_2 = 0$

Then $A(x_1+x_2) = Ax_1 + Ax_2 = 0$

$\Rightarrow (x_1+x_2)$ is also a solution of $Ax = 0$.

Further: λx_1 is also a solution of $Ax = 0$ for any $\lambda \in \mathbb{R}$.

$\Rightarrow$ The solution space of $Ax = 0$ is a subspace of $\mathbb{R}^n$.

THEOREM: Row equivalent matrices have the same row space.

IDEA OF PROOF: Let M is the matrix obtained by applying one of the following elementary row operations on A:

i) interchange of R_i and R_j

ii) Replace ~~of~~ R_i by λR_i

iii) Replace R_j by $\lambda R_i + R_j$.

Then each row of M is a row of A or a linear combination of rows of A. Hence the row space of M is contained in the row space of A.

On the other hand, we can apply the inverse elementary row operation on M to obtain A; hence the row space of A is contained in the row space of M.

$\Rightarrow$ A and M have the same row space.

Example: Consider the following two sets of vectors in $\mathbb{R}^4$:

$u_1 = (1, 2, -1, 3)$ $u_2 = (2, 4, 1, -2)$ $u_3 = (3, 6, 3, -7)$

$w_1 = (1, 2, -4, 11)$ $w_2 = (2, 4, -5, 14)$.

Let $U = \text{span}(u_i)$ and $W = \text{span}(w_i)$, show that

$U = W$.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{bmatrix} \qquad B \sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

We see that $\text{span}\overset{\text{rowspace of}}{(A)} = \text{span}\overset{\text{rowspace of}}{(B)}$

Basis and Dimension:

- A set $S = \{u_1, u_2, \ldots, u_n\}$ of vectors is a basis of V, if,
  i) S is linearly independent 2) S spans V.
- No of elements in S (basis) is called dimension of V.

Note: The vector space $\{0\}$ is defined to have dimension zero.

EXAMPLES of Basis:

a) Vector space $\mathbb{R}^n$: Consider the following n vectors.
$e_1 = (1, 0, 0 \ldots 0)^T$, $e_2 = (0, 1, 0, \ldots, 0)^T$, $\ldots$ $e_n = (0, 0 \ldots, 1)^T$
These vectors are linearly independent and for any vector
We have: $v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$. Here $v = (v_1, v_2, \ldots v_n)$

b) Vector space of all $r \times s$ matrices (M):

Ex: 2×3 matrices:

consider: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdots \cdots \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

These vectors are linearly independent and span M.

Hence the dimension of $M = rs$.

Note:

Bases in the examble a) & b) above are called usual or standard basis.

c) Vector space $P_n(t)$ of all polynomials of degree $\leq n$.

The set $S = \{1, t, t^2, \cdots, t^n\}$ of $(n+1)$ polynomials is a basis of $P_n(t)$.

Some useful theorems:

Th: Let V be a vector space of finite dimension n. Then:

(i) any $(n+1)$ or more vectors in V are linearly dependent

ii) Any linearly independent set $S = \{u_1, u_2, \cdots, u_n\}$ with n elements is a basis of V.

iii) any spaning set $\{v_1, v_2, \cdots v_n\}$ of V with n elements is a basis of V.

Th: Suppose S spans a vector space V. Then:

i) any maximum number of linearly independent vectors in S form a basis of V.

ii) Suppose one deletes from S every vector that is a linear combination of preceding vectors in S. Then the remaining vectors form a basis of V.

Th: Let V be a vector space of finite dimension and let $S = \{u_1, u_2, \dots, u_r\}$ be a set of linearly independent vectors in V. Then S is a part of a basis of V: that is, S may be extended to a basis of V.

Ex: 1. The following set in $\mathbb{R}^4$

$$(1,1,1,1)^T, (0,1,1,1)^T, (0,0,1,1)^T (0,0,0,1)^T$$

forms a basis of $\mathbb{R}^4$.

Because $\dim(\mathbb{R}^4) = 4$ & these vectors are linearly independent.

2. Consider any four vectors in $\mathbb{R}^3$, say

$$(1,2,1), (0,1,3), (3,4,0), (6,1,8).$$

These vectors must be lineary dependent since $\dim(\mathbb{R}^3) = 3$.

Dimension of the solution space of a homogeneous system $Ax=0$:

Ex: Solve the system of equations:

$$x_1 + 3x_2 + 3x_3 - x_4 + 2x_5 = 0$$
$$2x_1 + 6x_2 - 2x_3 + 14x_4 - 3x_5 = 0$$
$$4x_1 + 12x_2 + 2x_3 + 16x_4 + x_5 = 0$$

Sol: Augmented matrix

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 3 & 3 & -1 & 2 & 0 \\ 2 & 6 & -2 & 14 & -3 & 0 \\ 4 & 12 & 2 & 16 & 1 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{cccccc} x_1 & & x_3 & & x_5 & \end{array} \\ \left[\begin{array}{cccccc} \boxed{1} & 3 & 3 & -1 & 2 & 0 \\ 0 & 0 & \boxed{8} & -16 & 7 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right] \\ \begin{array}{cccccc} & x_2 & & x_4 & & \end{array} \end{array}$$

$x_5 = 0 \qquad x_4 = \alpha_1 \qquad x_2 = \alpha_2$

$x_3 = 2\alpha_1 \qquad x_1 = -5\alpha_1 - 3\alpha_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha_1 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \alpha_1, \alpha_2 \in \mathbb{R}.$$

The solution space is spanned by two linearly independent vectors. $[-5, 0, 2, 1, 0]^T$ & $[-3, 1, 0, 0, 0]^T$. Hence the dimension of the solution space is $2\,(=5-3) =$ [no. of unknowns $-$ rank(A)]

Theorem: The dimension of the solution space W of a homogeneous system $Ax=0$ is $n-r$, where n is the number of unknowns and r is the rank of the coeff. matrix A.

Note: Solution space of $Ax=0$ is also called null space of A.

Theorem: Solutions of nonhomogeneous linear system are of the form: $X = X_0 + X_h$

where X_0 is any fixed solution of $Ax=b$ and X_h runs through all the solutions of the corresponding homogeneous system $Ax=0$.

Ex:

$$[A|b] \sim \left[\begin{array}{ccccc|c} 1 & 3 & 3 & -1 & 2 & 17 \\ 2 & 6 & -2 & 14 & -3 & -19 \\ 4 & 12 & 2 & 16 & 1 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & 8 & -16 & 7 & 53 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Chosing $x_4 = \alpha_1$ & $x_2 = \alpha_2$

$$\underset{\Uparrow}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} = \underset{\Uparrow}{\begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \\ 3 \end{bmatrix}} + \underbrace{\alpha_1 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$X = X_0 + X_h$$

LINEAR MAPPING (LINEAR TRANSFORMATION)

Let X and Y be any two vector space. A mapping $F: X \to Y$ is called a linear mapping or linear transformation if it satisfies the following two conditions:

i) for any two vectors $x, v \in X$, $F(x+v) = F(x) + F(v)$

ii) for any scalar k and vector $x \in X$, $F(kx) = k\,F(x)$.

<u>Remark:</u>

1) Note that for $k=0$: $F(0) = 0$. Thus every linear mapping takes the zero vector into the zero vector.

2) The two conditions above can be combined into one:

$$F(k_1 x + k_2 v) = k_1 F(x) + k_2 F(v).$$

Example: Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ with

$$F(x, y, z) = (x, y, 0)$$

Let $u = (a, b, c)$ & $v = (a', b', c')$

then
$$\begin{aligned} F(u+v) &= F(a+a', b+b', c+c') \\ &= (a+a', b+b', 0) \\ &= (a, b, 0) + (a', b', 0) \\ &= F(u) + F(v) \end{aligned}$$

and for any constant k.

$$F(ku) = F(ka, kb, kc) = (ka, kb, 0) = k(a, b, 0) = kF(u).$$

Thus F is linear.

<u>Ex:</u> Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ with $F(x, y) = (x+1, y+2)$.
Check whether F is linear or not?

<u>Sol:</u> It is not linear since $F(0,0) = (1,2) \neq 0$.

Matrices as linear mapping (transformation)

Let $X = \mathbb{R}^n$ & $Y = \mathbb{R}^m$. Then any real $m \times n$ matrix A gives a transformation of $\mathbb{R}^n$ into $\mathbb{R}^m$,

$$y = Ax$$

Since.

$$A(u+x) = Au + Ax$$

$$\& \; A(\lambda x) = \lambda Ax$$

This is a linear transformation.

KERNEL and IMAGE of a linear mapping:

Let $F: X \to Y$ be a linear mapping.

$$\operatorname{Ker} F = \{x \in X : F(x) = 0\}$$

$$\operatorname{Im} F = \{y \in Y : \text{there exists } x \in X \text{ for which } F(x) = y\}.$$

Theorem: Let $F: X \to Y$ be a linear mapping. Then the kernel of F is a subspace of X and image of F is a subspace of Y.

Theorem: Suppose $x_1, x_2, \cdots x_m$ span a vector space X and Suppose $F: X \to Y$ is linear. Then $F(x_1), F(x_2) \cdots F(x_m)$ span $\operatorname{Im} F$.

IDEA: Let $y \in \operatorname{Im} F$. Then $\exists \; x \in X : F(x) = y$.

Also: $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m$

Therefore. $y = F(x) = \alpha_1 F(x_1) + \alpha_2 F(x_2) + \cdots + \alpha_m F(x_m)$

$\Rightarrow$ The vectors $F(x_1), F(x_2), \cdots F(x_m)$ span $\operatorname{Im} F$.

Ex: $F(x,y,z) = (x,y,0)$

$\operatorname{Im} F = \{(a,b,c) : c = 0\} = xy$ plane

$\operatorname{Ker} F = \{(a,b,c) : a = 0, b = 0\} = z$-axis.

KERNEL & IMAGE OF MATRIX MAPPINGS:

Let us consider $A: \mathbb{R}^4 \to \mathbb{R}^3$ with

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

Take usual basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

of $\mathbb{R}^4$.

Then: Ae_1, Ae_2, Ae_3, Ae_4 spans the image of A.

$$Ae_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}; \quad Ae_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \quad Ae_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}; \quad Ae_4 = \begin{bmatrix} a_4 \\ b_4 \\ c_4 \end{bmatrix}.$$

Thus the image of A is precisely the column space of A.

The kernel of A consists all vectors x for which $Ax = 0$. This implies that kernel of A is the solution space of the homogeneous system $AX = 0$, called the null space of A.

Rank and Nullity of a linear Mapping: Let $F: X \to Y$ be a linear mapping. Then.

$$\operatorname{rank}(F) = \dim(\operatorname{Im} F) \quad \& \quad \operatorname{nullity}(F) = \dim(\ker F).$$

Theorem: Let X be of finite dimension and let $F: X \to Y$ be a linear map. Then: $\operatorname{rank}(F) + \operatorname{nullity}(F) = \dim X$

Proof: Think about matrix mapping: $r + (n - r) = n$ (Page 28 & 12)

Ex: Let $F: \mathbb{R}^4 \to \mathbb{R}^3$ be the linear mapping defined by

$$F(x,y,z,t) = (x-y+z+t,\ 2x-2y+3z+4t,\ 3x-3y+4z+5t)$$

Find a basis and dimension of a) the image of F b) Kernel of F.

Sol: We know that the vectors

a)
$F(1,0,0,0) = (1,2,3)$ $F(0,1,0,0) = (-1,-2,-3)$

$F(0,0,1,0) = (1,3,4)$ $F(0,0,0,1) = (1,4,5)$.

span Im F.

Now consider $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Thus $(1,2,3)$ & $(0,1,1)$ form a basis of Im F and

$\dim(\mathrm{Im}\,F) = 2$ and $\mathrm{rank}(F) = 4-2 = 2$.

b) Set: $F(x,y,z,t) = 0 \Rightarrow$
$$x-y+z+t=0$$
$$2x-2y+3z+4t=0$$
$$3x-3y+4z+5t=0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} x & & z & \end{array} \\ \begin{bmatrix} \boxed{1} & -1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} & y & & t \end{array} \end{array}$$

Free variables are y & t

$\dim(\mathrm{Ker}\,F) = 2$. or $\mathrm{nullity}(F) = 2$.

Let $t = \alpha_1$ & $y = \alpha_2$

$z = -2\alpha_1$ $x = \alpha_2 + 2\alpha_1 - \alpha_1$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus $(1,0,-2,1)$ & $(1,1,0,0)$ form a basis for Ker F.

MATRIX REPRESENTATION OF A LINEAR OPERATOR:

Cordinates of a vector:

Let X be an n-dimensional vector space (over $\mathbb{R}$) with basis $\{x_1, x_2, \cdots, x_n\}$. Then any vector $x \in X$ can be expressed uniquely as a linear combination of the basis vectors i.e,

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

The n scalars $\alpha_1, \alpha_2 \cdots \alpha_n$ are called the coordinates of x with respect to $S = \{x_1, x_2, \cdots x_n\}$.

Notation: $[x]_S = [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$.

Suppose $F: X \to Y$ is a linear mapping. Let X ~~and~~ Y ~~be~~ vector spaces of dimensions m and n with bases $S = \{x_1, x_2, \cdots x_m\}$ and $S' = \{y_1, y_2, \cdots y_n\}$ respectively.

The vectors $F(x_1), F(x_2) \cdots F(x_m)$ belongs to Y so

$$F(x_1) = a_{11} y_1 + a_{12} y_2 + \cdots + a_{1n} y_n$$
$$F(x_2) = a_{21} y_1 + a_{22} y_2 + \cdots + a_{2n} y_n$$
$$\vdots$$
$$F(x_m) = a_{m1} y_1 + a_{m2} y_2 + \cdots + a_{mn} y_n$$

The transpose of the above matrix of coefficients, denoted by $[F]_S^{S'}$ is called the matrix representation of F relative to the bases S and S'.

<u>Theorem</u>: For any vector $x \in X$, $[F]_S^{S'} [x]_S = [F(x)]_{S'}$

Ex: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+z \\ y-z \end{bmatrix}$$

Determine the matrix of the linear transformation T, with respect to the ordered basis:

$$X = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ in } \mathbb{R}^3 \text{ and } Y = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \text{ in } \mathbb{R}^2$$

[JIJ, P3.41]

Sol: $$T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(1) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}(1)$$

$$T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(0) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}(1)$$

$$T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(1) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}(0)$$

Therefore the matrix of the linear transformation T w.r.t the given basis is
$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Ex: Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be linear operator defined by $F(x,y) = (2x+3y, 4x-5y)$

a) Find the matrix representation of F relative to $S = \{(1,2)^T, (2,5)^T\}$

b) " " $E = \{(1,0)^T, (0,1)^T\}$.

Ex: Consider $A = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}$ as linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Find the matrix representation of A relative to the basis
$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\} \ \& \ E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$