MA 20104 (Probability and Statistics)

Summary of lectures in the second week.

(Instructor: Swanand Khare)

Conditional Probability.

Example 1: Consider a box containing 'w' numbe[r] of white balls labelled $1, 2, \ldots, w$ and 'b' numb[er] of black balls labelled $1, 2, \ldots, b$. Then the ~~question~~ event of interest here is to be able to pick a ball with label 1. Then clearly we can see that this probability is $\frac{2}{b+w}$. However, if some added information is provided to us that if the ball picked is ~~red~~ white, then can the probability of the picked ball is labelled 1 b[e] "modified" in the light of this more informat[ion] which is made available?? More formally, define A be an event when a ball labelled 1 is picke[d] and B be an event when a white ball is pick[ed]. Then we are interested in computing probability ... that the event B has

Since we are in the setting of uniform probability space, clearly $P(A) = \frac{2}{b+w}$ and $P(B) = \frac{w}{w+b}$. With the information that event B has ocurred, then picking a ball labelled 1 has probability equal to $\frac{1}{w}$ which is computed using the formula

$$\frac{\#(A \cap B)}{\# B} = \frac{1}{w} = \frac{\#(A \cap B)/b+w}{\# B/b+w}$$

$$= \frac{P(A \cap B)}{P(B)} \quad \ldots \text{ since we are in the uniform probability space}$$

$$= \frac{1/b+w}{w/b+w}$$

$$P(A \mid B) = \frac{1}{w}$$

Note that the conditional probability $P(A \mid B)$ is "modified" from the probability $P(A)$ when additional information, is that the event B has occured, is made available.

Taking motivation from the computations perfor

Definition: Let A and B be two events with $P(B) > 0$. Then the conditional probability of A given B, written as $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} .$$

Example*: Consider the random experiment of throwing a fair die. Then $\Omega = \{1,2,3,4,5,6\}$. Let A be an event when the top face of the die shows a perfect square. Let B be the event when the top face of the die shows an even number. Then

$A = \{1,4\}$, $B = \{2,4,6\}$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{3/6} = \frac{1}{3} . \qquad (*)$$

Note that $P(A) = \frac{1}{3}$.

Further,
$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/6}{2/6} = \frac{1}{2} \qquad (**)$$

$P(B) = \frac{1}{2}$.

Note a very interesting thing here. In case $(*)$, the knowledge that even number has appeared

Similarly in case (**), occurrance of a perfect square does not "change" the probability of obtaing an even number on the top face. This particular situation is very important where occurrence of an event B does not effect occurrance or non-occurrance of A. In such a situation we call A and B independent events. ~~From the multiplication rule of probabilities~~ ~~P(A∩B)=~~ We will formally define the independence of events a little later.

<u>Multiplication Rule of probabilities.</u>

From the definition of conditional probability,

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad \text{if} \quad P(B) \neq 0,$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} \quad \text{if} \quad P(A) \neq 0.$$

Thus $\boxed{\begin{aligned} P(A\cap B) &= P(A|B)\,P(B) \\ P(A\cap B) &= P(B|A)\,P(A) \end{aligned}}$ multiplication rules

<u>Revisit to the total probability law.</u>

Let $B_1, B_2, \ldots, B_n$ be mutually disjoint events such tha

Thus,

$$A = \bigcup_{i=1}^{k} (A \cap B_i) \quad \cdots \text{ disjoint union}$$

$$\Rightarrow P(A) = \sum_{i=1}^{k} P(A \cap B_i)$$

$$\Rightarrow \boxed{P(A) = \sum_{i=1}^{k} P(B_i)\, P(A \mid B_i)} \quad \cdots$$

Another form of total probability in terms of conditional prob.

Baye's Theorem:

From the above disussions, we observe,

$$P(A \cap B_i) = P(A)\, P(B_i \mid A) = P(B_i)\, P(A \mid B_i)$$

$$\Rightarrow P(B_i \mid A) = \frac{P(B_i) P(A \mid B_i)}{P(A)}$$

$$\Rightarrow \boxed{P(B_i \mid A) = \frac{P(B_i)\, P(A \mid B_i)}{\sum_{i=1}^{k} P(B_i)\, P(A \mid B_i)}} \quad \cdots \text{ Baye's rule}$$

It is very striking result in statistics. It is obvious to note the reversal of conditional probabilities. Another interesting interpretation of Baye's rule can be thought as follows. Let events

Baye's rule basically is the probability that the event A is triggered by the cause B_i. It should be noted that typically probabilities ~~P~~ on the right hand side of Baye's rule are available or could be approximated from the measurement data. A typical case study where we can apply Baye's theorem is the following example.

Exercise: Suppose a factory has 2 machines A & B which make 60 % and 40 % of the total production respectively. Out of total production, machine A produces 3 % defective items while machine B produces 5 % defective items. Find the probability that a given faulty item is produced by machine B.

Soln: Let F denote the event that a faulty item is produced, A_F denote the machine A produces faulty item and B_F denote the event when the machine B_F produces faulty item.

We want to compute $P(B_F | F)$.

Independence of events

By ~~tak~~ observing the calculations from Example" in the previous discussion, we define independence of events in the following definition.

Definition: Two events A and B are independent if & only if $P(A \cap B) = P(A)P(B)$. More generally, for $n \geq 3$, events $A_1, A_2, \ldots, A_n$ are mutually independent if $P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i)$

and if any subcollection of A_i's containing at least two and at most $n-1$ events ~~is~~ are mutually independent.

Exercise: If A & B are independent events, then so are A^c and B, A^c and B^c, A and B^c.

$P(A \cap B) = P(A) \cdot P(B)$ ... given. (Also $P(A|B) = P(A)$)

To prove A^c and B are independent, see

$P(A^c \cap B) = P(B)\, P(A^c | B) = P(B)[1 - P(A | B)]$

$= P(B)[1 - P(A)]$ ... since A & B are