Discussion

Tutorial 6

Tut-6.

Q2. Let R be ring. Let $a \in R$ be nilpotent i.e $a^n = 0$ for some $n \geq 0$.

WTS $(1+a)$ is an unit

$$(1+a)\left(1 - a + a^2 - a^3 + \cdots + (-1)^{n-1} a^{n-1}\right) = 1.$$

$\therefore$ $(1+a)$ is an unit.

If u is an unit and a is nilpotent then $u^{-1}a$ is nilpotent

$1 + u^{-1}a$ is an unit by previous argument.

$u(1+u^{-1}a) = u + a$ is an unit.

Q4. Consider the ring

$$B = C[y_1, y_2, \ldots]$$

where C is an int domain and y_i's are indeterminates.

$$I = (y_1, y_2^2, y_3^3, \ldots y_n^n, \ldots)$$

Consider the ring $A = B/I$.

let z_n denote the image of y_n in A.

$$z_n = y_n + I.$$

Then in A $z_n^n = 0$. and $z_n^{n-1} \neq 0$.

let $f = \sum_{n=0}^{\infty} z_n x^n \in A[|x|]$ then each coeff of f is nilpotent.

check $f^n \neq 0$ $\forall n$.

Q8. $a \in \mathbb{Z}/12\mathbb{Z}$ is nilpotent

if $a^n = \bar{0}$

$\Rightarrow a^n \in 12\mathbb{Z}$.

$12 = 2^2 \cdot 3$.

$n = p_1^{n_1} \cdots p_r^{n_r}$.

Then $nil(\mathbb{Z}/n\mathbb{Z}) = p_1 p_2 \cdots p_r$.

Ex. $Nil(R)$ = Intersection of all the prime ideals of R.

$nil(\mathbb{Z}/n\mathbb{Z}) = \bigcap_{\substack{P \text{ is prime} \\ \text{ideal of } \mathbb{Z}/n\mathbb{Z}}} P$.

$nil(\mathbb{Z}/12\mathbb{Z}) = (2\mathbb{Z}) \cap (3\mathbb{Z}) = 6\mathbb{Z}$.

$n = 12$. $m\mathbb{Z}$ where $n\mathbb{Z} \subseteq m\mathbb{Z}$ and m is prime.

Q9 Show that all ideals of $\mathbb{R}[|x|]$ are principal

$I \subseteq k[x]$

WTS $I = (f(x))$

lowest deg poly present in I.

In $\mathbb{R}[|x|] \ni f(x)$

$$f(x) = \sum_{n \geq 0} a_n x^n$$

$$\text{ord}(f) = \min\{i \mid a_i \neq 0\}.$$

Let $I \subseteq \mathbb{R}[|x|]$.

WTS. $I = (f(x))$

$f(x)$ is an elt with smallest order prsnt in I.

If you consider I a proper ideal of $\mathbb{R}[|x|]$.

then $\forall f(x) \in I$ $f(x)$ will not have const term.

Because if $f(x)$ has the constant term then it will be an unit elt by Ex 4 and hence

$I = \mathbb{R}[|x|]$.

<u>Q12</u>. $\underline{\varphi(x) = x \quad \& \quad \varphi(x) = -x.}$

$$\begin{cases} \varphi(x) = a_0 + x \\ \text{or } \varphi(x) = a_0 - x. \end{cases}$$

where $a_0 \in \mathbb{Z}$.

Q12. φ is an automorphism
let φ^{-1} be its inverse then
φ^{-1} is also an automorphism.

Let $p(x) \in \mathbb{R}[x]$ and

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

φ is a ring homo.

$$\varphi(p(x)) = \varphi(a_0 + a_1 x + \cdots + a_n x^n)$$

$$= \varphi(a_0) + \varphi(a_1)\varphi(x) + \cdots + \varphi(a_n)\varphi(x)^n,$$

Moreover $\varphi(1) = 1$.

$$\Rightarrow \varphi(n) = n \quad \forall n \in \mathbb{Z}$$

$$= a_0 + a_1 \varphi(x) + \cdots + a_n \varphi(x)^n$$

Thus we need to only find $\varphi(x) = ?$

Let $\varphi(x) = b_0 + b_1 x + \cdots + b_m x^m$.

Then $\varphi(p(x))$ is of deg nm.

Thus if $m > 1$ φ cannot be an automorphism because we can't find a preimage of x.

$\therefore \quad \varphi(x) = b_0 + b_1 x$.

Similar $\quad \varphi^{-1}(k) = k \quad \forall k \in \mathbb{Z}$.

$\quad \varphi^{-1}(x) = c_0 + c_1 x$.

$\underline{\varphi^{-1}(\varphi(x)) = x}$.

$$\varphi^{-1}(b_0 + b_1 x) = b_0 + b_1(c_0 + c_1 x)$$

$$= b_0 + b_1 c_0 + b_1 c_1 x.$$

$$= x$$

$\Rightarrow b_1 c_1 = 1 \quad \Rightarrow b_1 = \pm 1.$

Any automorphism is of the form

$$\begin{cases} \varphi(x) = b_0 \pm x, & \text{where } b_0 \in \mathbb{Z}. \\ \varphi(k) = k & \forall\, k \in \mathbb{Z} \end{cases}$$

check φ is inj & surjective.