Ring Theory

Lecture 26

Propn An ideal M of R is a maximal ideal iff R/M is a field.

Pf: Let M be a maximal ideal.

WTS R/M is a field

Note that R/M has only two ideals the $(\bar{0})$ and $(\bar{1})$. If $x \notin M$ then $(\bar{x}) = (\bar{1})$.

$\Rightarrow \exists$ some $\bar{y} \in R/M$ s.t. $\bar{x}\bar{y} = \bar{1}$.

$\therefore$ R/M is a field.

You can prove the converse similarly.

Cor The zero ideal is a maximal ideal of R iff R is a field.

Remark: Note that every maximal ideal is a prime ideal.
But every prime ideal need not be a maximal ideal.

Example In $\mathbb{Z}$. a prime ideal is of the form $n\mathbb{Z}$ where n is a prime number. Note that $\mathbb{Z}/n\mathbb{Z}$ is a field if n is a prime number thus $n\mathbb{Z}$ is also a maximal ideal. In $\mathbb{Z}$, (0) is a prime ideal but it is not a maximal ideal.

Example In $\mathbb{C}[x,y]$ then (x)
$\mathbb{C}[x,y]/(x)$. $\cong$

$\mathbb{C}[x,y]/(x) \simeq \mathbb{C}[y]$ [Prove it EX]

$f(x,y) + (x)$.

$5x + 6xy + 2y + 5 + (x)$

$= 2y + 5 + (x)$.

Since $\mathbb{C}[y]$ is an integral domain thus (x) is a prime ideal. But it is not a maximal ideal as. $(x) \subsetneq (x,y)$

Q Is (x,y) a maximal ideal in $\mathbb{C}[x,y]$?

$$\frac{\mathbb{C}[x,y]}{(x,y)} \simeq \mathbb{C}$$ which is a field.

Hence (x,y) is a maximal ideal.

Example Consider $k[x]$ where k is a field.

Obs 1. Every ideal of $k[x]$ is gen by a single elt.

Pf: Let $(0) \subsetneq I \subsetneq k[x]$.

Since $I \neq k[x]$, $\exists$ a poly of (+)ve deg in I. Let $f(x) \in I$ having smallest (+)ve degree.

WTS $I = (f(x))$

Let $g(x) \in I$. By division algo we have $g(x) = q(x) f(x) + r(x)$ where $q(x), r(x) \in k[x]$ and $\deg r(x) < \deg f(x)$ or $r(x) = 0$.

Now $r(x) = g(x) - q(x) f(x) \in I$.

$\Rightarrow$ $r(x) = 0$ by minimality of deg of $f(x)$.

$\therefore$ $g(x) = f(x) q(x) \in (f(x))$.

Hence $I = (f(x))$.

In $k[x]$ the prime ideals are of the form $(f(x))$ where $f(x)$ is an irreducible poly.

Propn. Every maximal ideal of $k[x]$ is of the form $(f(x))$ where $f(x)$ is irreducible over $k[x]$.

Ex Show that if $f(x)$ is irreducible poly then $(f(x))$ is a maximal ideal.

Consider $\mathbb{C}[x]$.

Maximal ideals of $\mathbb{C}[x]$ are of the form $m_a = (x-a)$ where $a \in \mathbb{C}$. As any poly $f(x) \in \mathbb{C}[x]$ of (+)ve deg is a product of linear poly (by FTA) thus irreducible polys over $\mathbb{C}[x]$ are linear polys only.

<u>Remark</u> There is a 1-1 correspondence between the pts of $\mathbb{C}$ and the set of maximal ideals in $\mathbb{C}[x]$ given by

$$\begin{array}{ccc} \mathbb{C} & & \text{Maximal ideals of } \mathbb{C}[x] \\ a & \longrightarrow & m_a = (x-a) \\ a & \longleftarrow & \end{array}$$

Note that the ideal m_a is the kernel of the ring homo $\varphi: \mathbb{C}[x] \longrightarrow \mathbb{C}$ defined by

$$\varphi(f(x)) = f(a)$$

The extension of the above Thm to several variable is one of the most important Thm about alg geometry & commutative alg which is known as Hilbert's Nullstellensatz

maximal ideals of $\mathbb{C}[x_1, \ldots, x_n]$ $\qquad\qquad$ $\mathbb{C}^n$

$$(x_1 - a_1, \ldots, x_n - a_n) \longleftrightarrow (a_1, a_2, \ldots, a_n)$$

$$G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}.$$

$$G \cong \mathbb{Z}_{n_1} \qquad r = 1.$$

$$\underline{\underline{\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}.}}$$