Lecture 12

<u>Proposition:-</u> Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function & $B \subseteq \mathbb{R}$, a Borel set. Then $f^{-1}(B)$ is a measurable set.

<u>proof:-</u> Let $\mathcal{F} = \{ A \subseteq \mathbb{R} \mid f^{-1}(A) \text{ is a measurable set} \}$

$f^{-1}(\mathbb{R}) = E \in \mathcal{M}$

$\therefore \mathbb{R} \in \mathcal{F}$.

Also $f^{-1}(A^c) = \{ x \in E \mid f(x) \in A^c \}$

$= \{ x \in E \mid f(x) \notin A \}$.

$\therefore A^c \in \mathcal{F}$ if $A \in \mathcal{F}$ $\quad = (f^{-1}(A))^c$

& $f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$ (Check it!)

$\therefore$ if $A_i \in \mathcal{F}$ $\forall i$, then

$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

$\therefore \mathcal{F}$ is a σ-algebra.

Since f is a measurable fun., we know that

$f^{-1}([\alpha, \infty)) \in \mathcal{M}$ $\quad$ ||ly $[a,b], (a,b], [a,b)$ $\in \mathcal{F}$.

$\Rightarrow$ all intervals belongs to $\mathcal{M}$

$\Rightarrow$ The Borel σ-algebra $\mathcal{B} \subseteq \mathcal{M}$.

$\Rightarrow$ all Borel sets $\in \mathcal{M}$

$\Rightarrow$ $f^{-1}(B)$ is measurable for any Borel set B.

<u>Remark</u>:- we have seen that if f is measurable, then

- $\{x \in E \mid f(x) \le \alpha\} = f^{-1}([-\infty, \alpha])$ is meas.
- $\{x \in E \mid f(x) < \alpha\} = f^{-1}([-\infty, \alpha))$ "
- $\{x \in E \mid f(x) \geq \alpha\} = f^{-1}([\alpha, \infty))$ "

ie $[-\infty, \alpha]$, (α, ∞), $[\alpha, \infty)$, $(-\infty, \alpha)$ are belongs to $\mathcal{M}$.

$$[a, b] = [-\infty, b] \cap [a, \infty) \in \mathcal{M}$$

$$(a, b) = (-\infty, b) \cap (a, \infty) \in \mathcal{M}$$

$\vdots$

Existance of a non-measurable set in $\mathbb{R}$.

<u>Theorem</u>:- Let $E \subseteq \mathbb{R}$ be a measurable. Then for each $y \in \mathbb{R}$, the set $E+y = \{x+y \mid x \in E\}$ is measurable. & $m(E) = m(E+y)$.

<u>proof</u>:- <u>To show</u>: $E+y$ is measurable.

<u>Enough to show</u>: given $\varepsilon > 0$. There exists an open set V such that $V \supseteq E+y$ & $m^*(V \setminus (E+y)) \leq \varepsilon$.

Let $\varepsilon > 0$.

Since E is measurable, there exists an open set $U \supseteq E$ such that $m^*(U \setminus E) \leq \varepsilon$.

$\Rightarrow$ $U+y \supseteq E+y$ & $U+y$ is open set.

& $(U+y) \setminus (E+y) = \{x+y \in U+y \,/\, x+y \notin E+y\}$

$$= \{x+y \in U+y \,/\, x \notin E\}$$

$$= (U \setminus E) + y.$$

$$\therefore \quad m^*((U+y)\setminus(E+y)) = m^*((U\setminus E)+y)$$

$$= m^*(U \setminus E) \quad \left(\because m^*(A) = m^*(A+y) \text{ for } A \subseteq \mathbb{R}\right)$$

$$= m(U \setminus E) \quad (\because U \setminus E \in \mathcal{M})$$

$$\leq \varepsilon.$$

Thus we find an open set $V = U+y$ such that $U+y \supseteq E+y$ & $m^*((U+y)\setminus(E+y)) \leq \varepsilon$

$\Rightarrow$ $E+y$ is measurable.

& $m(E+y) = m^*(E+y) = m^*(E) = m(E)$.

Theorem:- There exists a non-measurable set in $\mathbb{R}$.

proof:- For $x, y \in [0,1]$, define

$x \sim y$ if $y - x \in \mathbb{Q} \cap [-1,1] = \mathbb{Q}_1$. (Say).

Then $\sim$ is an equivalence relation.

$x \sim x$ $\because$ $x - x = 0 \in \mathbb{Q}_1$

$x \sim y \Rightarrow y - x \in \mathbb{Q}_1 \Rightarrow x - y \in \mathbb{Q}_1 \Rightarrow y \sim x$.

Suppose $x \sim y$ & $y \sim z$. (To show: $z \sim x$)

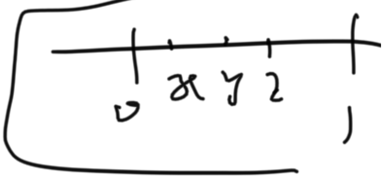
$a, b \in [-1,1]$, $a + b \in$

$\Rightarrow$ $y - x,\ z - y \in \mathbb{Q}_1$

$\Rightarrow$ $y - x = r_1$, $z - y = r_2$ for some $r_1, r_2 \in \mathbb{Q}_1$

$\underline{\underline{z - x}} = y - x + z - y = r_1 + r_2 \in \mathbb{Q} \cap [-1,1] = \mathbb{Q}_1$.

$\therefore$ $\sim$ is an equivalence relation on $[0,1]$



& say E_α are equivalence class

$\therefore$ $[0,1] = \dot{\bigcup}_\alpha E_\alpha$

We know that $\mathbb{Q}_1$ is countable this implies that

[equivalence class of x $= \{ y \in [0,1] \mid y \sim x \} = \{ y \in [0,1] \mid y - x \in \mathbb{Q}_1 \}$]

each $\underline{E_\alpha \text{ is countable}}$.

But we know $[0,1]$ is uncountable.

$\Rightarrow$ the union is an uncountable union.

Now by the axiom of choice, there exists a set V in $[0,1]$ containing just one element x_α from each E_α.

Let $Q_1 = \{r_1, r_2, \ldots\}$

& for each $n \in \mathbb{N}$, write $V_n = V + r_n$.

<u>Claim:-</u> $V_n \cap V_m = \phi$, for $n \neq m$.

<u>Pf:-</u> Suppose $y \in V_n \cap V_m$.

Then there exists $x_\alpha, x_\beta \in V$ such that $y = x_\alpha + r_n = x_\beta + r_m$.

$\Rightarrow x_\alpha - x_\beta = r_m - r_n \in Q_1$

$\Rightarrow x_\alpha \sim x_\beta$

$\Rightarrow x_\alpha, x_\beta$ lies in the same equivalence class.

$\Rightarrow x_\alpha = x_\beta$. (by def. of V).

$\Rightarrow m = n$.

Thm $V_n \cap V_m = \phi$ if $m \neq n$.

Now we have $[0,1] \subseteq \bigcup_{n=1}^{\infty} V_n$

for $x \in [0,1]$, $x \in E_\alpha$ for some α

$\Rightarrow$ $x = x_\alpha + r_n$ for some $r_n \in Q$,

$\Rightarrow$ $x \in V_n$ for some n

$$\therefore \boxed{[0,1] \subseteq \bigcup_{n=1}^{\infty} V_n \subseteq [-1,2]} \text{ — } (*)$$

$V_n = V + r_n$

$r_n \in [-1,1] \cap Q$.

Claim:- V is not measurable.

Pf:- Suppose V is measurable.

Then by above thm each $V_n = V + r_n$ is also measurable, $\forall n \in \mathbb{N}$.

& $m(V) = m(V_n)$ $\forall n$.

(by above Thm)

$\therefore$ By $(*)$, we have

$$m([0,1]) \leq m\left(\bigcup_{n=1}^{\infty} V_n\right) \leq m([-1,2])$$

$$\Rightarrow \quad 1 \leq \sum_{n=1}^{\infty} m(V_n) \leq 3$$

$$\Rightarrow \quad 1 \le \sum_{n=1}^{\infty} m(V) \le 3. \longrightarrow (**)$$

But $\displaystyle\sum_{n=1}^{\infty} m(V) = m(V) \sum_{n=1}^{\infty} 1.$

$$= 0 \text{ or } \infty$$

This contradicts $(**)$

$\therefore$ V is not measurable.