Lecture 23

Recall, the Lebesgue integral of a simple function $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$, which is in the canonical representation, is defined as $\int_{\mathbb{R}^d} \varphi = \sum_{k=1}^{N} a_k m(E_k)$.

If $E \subseteq \mathbb{R}^d$ is a measurable set & $m(E) < \infty$, then $\int_E \varphi = \int_{\mathbb{R}^d} \varphi \chi_E$.

Example:- ① $\varphi = \chi_{[-1,0]} - 2\chi_{[1,2]} + 3\chi_{[3,4]}$.

$$\therefore \int_{\mathbb{R}} \varphi = m([-1,0]) - 2m([1,2]) + 3m([3,4])$$
$$= 1 - 2(1) + 3(1)$$
$$= 2$$

Properties of the Lebesgue integral of simple functions.

Proposition:-

① (Independence of the representation)

If $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of φ, then $\int \varphi = \sum_{k=1}^{N} a_k m(E_k)$

$k=1$

② (Linearity) If φ, ψ are simple funtions & $a, b \in \mathbb{R}$ then $\int (a\varphi + b\psi) = a\int \varphi + b\int \psi$.

③ (Additive) If E, F are disjoint measurable sets of $\mathbb{R}^d$ with finite measure, then

$$\int_{E \cup F} \varphi = \int_E \varphi + \int_F \varphi .$$

④ (Monotonicity) If $\varphi \leq \psi$, are simple funtions, then $\int \varphi \leq \int \psi$.

⑤ (Triangular inequality) If φ is a simple funtion, then $|\varphi|$ is also a simple funtion &

$$\left|\int \varphi\right| \leq \int |\varphi| .$$

proof: ① Suppose $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$

case (i): Suppose E_k are disjoint & a_k's are not distinct. & non-zero.

For each distinct non-zero value a,

among the $\{a_k\}_{k=1,\dots,N}$ we define

$E'_a = \dot{\bigcup} E_k$, where union is taken over those indices k such that $a_k = a$.

Note that E'_a are disjoint

& $m(E'_a) = m\left(\dot{\bigcup} E_k\right) = \sum m(E_k)$

where the sum is over those k, such that $a_k = a$. , $\therefore \varphi = \sum a\, \chi_{E'_a}$ Canonical repr.

Thus $\int \varphi = \sum a\, m(E'_a)$

$$= \sum a \left(\sum m(E_k)\right)$$

$$= \sum_{k=1}^{N} a_k \, m(E_k).$$

<u>Case (ii)</u>: Suppose E_k are not disjoint.

Then we can refine the decomposition $\bigcup_{k=1}^{N} E_k$ by finding the sets $E_1^*, \dots, E_n^*$ with the property that $\bigcup_{k=1}^{N} E_k = \dot{\bigcup}_{j=1}^{n} E_j^*$ & the sets E_j^* are disjoint.

And for each k, $E_k = \bigcup E_j^*$,

where the union is taken over those E_j^* that are contained in E_k.

$E_1 \quad E_2$

$E_1^* = E_1$

$E_2^* = E_2 \setminus E_1$

$E_1 \cup E_2 = E_1^* \cup E_2^*$

For each j, let $a_j^* = \sum a_k$, where the sum is taken over all k such that $E_k \supseteq E_j^*$. Then

$$\varphi = \sum_{k=1}^{N} a_k \chi_{E_k} = \sum_{j=1}^{n} a_j^* \chi_{E_j^*}$$

in this representation a_j^* need not be distincts but E_j^* are disjoint.

$\therefore$ By the case (i),

$$\int \varphi = \sum_{j=1}^{n} a_j^* \, m(E_j^*).$$

$$= \sum_{j=1}^{n} \sum_{E_k \supseteq E_j^*} a_k \, m(E_j^*)$$

$$= \sum_{k=1}^{N} a_k \, m(E_k).$$

Thus $\int \varphi$ is independent of the representation of φ.

(2) Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$

$\psi = \sum_{j=1}^{M} b_j \chi_{F_j}$

$$a\varphi + b\psi = \sum_{k=1}^{N} a a_k \chi_{E_k} + \sum_{j=1}^{M} b b_j \chi_{F_j}.$$

simple function.

$\therefore$ By (1), $\int (a\varphi + b\psi) = \sum_{k=1}^{N} a a_k m(E_k) + \sum_{j=1}^{M} b b_j m(F_j)$.

$$= a \int \varphi + b \int \psi.$$

(3) Let E, F be disjoint measurable sets of finite measure.

Then we have $\chi_{E \cup F} = \chi_E + \chi_F - \chi_{E \cap F = \phi}$

$= \chi_E + \chi_F$.

Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$

$\int \varphi \chi_{E \cup F}$

$= \int \varphi (\chi_E + \chi_F)$

$= \int \varphi \chi_E + \int \varphi \chi_F$

Now $\int_{E \cup F} \varphi = \int_{\mathbb{R}^d} \varphi \chi_{E \cup F}$

$$= \int_{\mathbb{R}^d} \left(\sum_{k=1}^{N} a_k \chi_{E_k} \chi_{E \cup F} \right)$$

$$= \int_{\mathbb{R}^d} \left(\sum_{k=1}^{N} a_k \chi_{E_k \cap (E \cup F)} \right)$$

$$= \sum_{k=1}^{N} a_k \, m\left((E_k \cap E) \,\dot{\cup}\, (E_k \cap F) \right)$$

$$= \sum_{k=1}^{N} a_k \left(m(E_k \cap E) + m(E_k \cap F) \right)$$

$$= \sum_{k=1}^{N} a_k \, m(E_k \cap E) + \sum_{k=1}^{N} a_k \, m(E_k \cap F).$$

$$= \int \varphi \chi_E + \int \varphi \chi_F.$$

$$= \int_E \varphi + \int_F \varphi.$$

(4) Suppose $\varphi \leq \psi$. $\Rightarrow \psi - \varphi \geq 0$.

& $\psi - \varphi$ is a simple function.

$\therefore$ $\psi - \varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$, where $a_k \geq 0$

canonical repr.

$\therefore$ $\int (\psi - \varphi) = \sum_{k=1}^{N} a_k \, m(E_k) \geq 0$

$$\Rightarrow \quad \int \psi - \int \varphi \geq 0$$
$$\Rightarrow \quad \int \psi \geq \int \varphi.$$

(5) Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$. Canonical form.

Then $|\varphi| = \sum_{k=1}^{N} |a_k| \chi_{E_k}$.

Now
$$\left| \int \varphi \right| = \left| \sum_{k=1}^{N} a_k \, m(E_k) \right|$$
$$\leq \sum_{k=1}^{N} |a_k \, m(E_k)| = \sum_{k=1}^{N} |a_k| \, m(E_k).$$
$$= \int |\varphi|.$$

Thus $\left| \int \varphi \right| \leq \int |\varphi|$.

Bounded functions supported on a set of finite measure.

Def:- The <u>support</u> of a function $f: E \to \mathbb{R}$ is defined as
$$\text{supp}(f) := \{ x \in E \mid f(x) \neq 0 \}.$$

• We say that f is supported on a set $A \subseteq \mathbb{R}^d$,

if $f(x) = 0$ whenever $x \notin A$.

(i.e, $A \supseteq \mathrm{supp}(f)$.)

$x \notin A \Rightarrow f(x) = 0$

if $f(x) \neq 0$, then $x \in A$.

<u>Remark</u>:- If f is measurable, then $\mathrm{supp}(f)$ is a measurable set.

$f : E \to \mathbb{R}$