(Last class Inner product spaces)

Lemma: Let $\langle , \rangle$ be an IPS on a vector space X

(a) Polarization Identity: For all $x,y \in X$

(i) $4\langle x,y\rangle = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x+iy\rangle$
$= \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

(ii) Let $x \in X$. Then $\langle x,y\rangle = 0 \ \forall y$ iff $x=0$

(iii) (Schwartz Inequality) For $x,y \in X$

$$|\langle x,y\rangle|^2 \le \langle x,x\rangle\langle y,y\rangle \le \|x\|^2\|y\|^2$$

(iv) RHS $\langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle$

$= \langle x, \cancel{x}\rangle + \langle x,y\rangle + \langle y,x\rangle + \cancel{\langle y,y\rangle}$
$- \langle \cancel{x,x}\rangle + \langle x,y\rangle + \langle y,x\rangle - \cancel{\langle y,y\rangle}$
$+ i\langle x,x\rangle - \langle y,x\rangle + \langle x,y\rangle + i\langle y,y\rangle$

$\cdot -$

$= 4\langle x,y\rangle$

(b) $(\Rightarrow)$ Let $x=0$

$\Rightarrow \langle 0,y\rangle = \langle 0+0, y\rangle$
$= \langle 0,y\rangle + \langle 0,y\rangle$

$\Rightarrow \langle 0,y\rangle = 0 \ \forall y$

$\Leftarrow$ Let $\langle x,y\rangle = 0 \ \forall y$
WLOG $y=x$
$\langle x,x\rangle = \|x\|^2 = 0$
As $\|x\|=0 \Leftrightarrow x=0$
$\Rightarrow x=0$.

(iii) Let $z = \langle y,y\rangle x - \langle x,y\rangle y$ for $x,y \in X$

$0 \le \langle z,z\rangle = \langle \langle y,y\rangle x - \langle x,y\rangle y, \langle y,y\rangle x - \langle x,y\rangle y\rangle$

$= \langle\langle y,y\rangle x, \langle y,y\rangle x\rangle - \langle\langle y,y\rangle x, \langle x,y\rangle y\rangle$
$- \langle\langle x,y\rangle y, \langle y,y\rangle x\rangle + \langle\langle x,y\rangle y, \langle x,y\rangle y\rangle$

$= \|y\|^4\langle x,x\rangle - \langle y,y\rangle\cancel{\overline{\langle x,y\rangle}\langle x,y\rangle}$
$-\langle x,y\rangle\langle y,y\rangle\overline{\langle x,y\rangle} + \cancel{\langle x,y\rangle\overline{\langle x,y\rangle}}\langle y,y\rangle$

$= \|y\|^2\left(\|y\|^2\|x\|^2 - \langle x,y\rangle\overline{\langle x,y\rangle}\right) \ge 0$

$\Rightarrow \boxed{\|y\|^2\|x\|^2 \ge |\langle x,y\rangle|^2}$

$|\langle x, y\rangle| = \|x\|\|y\|$ equality holds iff $\{x, y\}$ is linearly dependent

Equality holds when $\|z\|^2 = 0$

$\Rightarrow z = 0$

$\Rightarrow \langle y, y\rangle x - \langle x, y\rangle y = 0$

$\{x, y\}$ is L.D.

($\Leftarrow$) Suppose we set $\{x, y\}$ is l.D.

$x = \alpha y$

$y = \beta x$

$$
\begin{aligned}
|\langle x, y\rangle|^2 &= |\langle x, \beta x\rangle|^2 \\
&= |\bar{\beta}\langle x, x\rangle|^2 \\
&= |\beta|^2 \|x\|^2 \|x\|^2. \\
&= \|y\|^2 \|x\|^2.
\end{aligned}
$$

Every ips is a normed space.

Theorem: A norm on an IPS X satisfies the parallelogram law.

i.e. $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$

12th September

ℓ^p space $\quad \ell^p$ is not an IPS for $p \neq 2$

ℓ^2 is an Hilbert space, ℓ^p is Banach for $1 \le p < \infty$

$x = (1, 1, 0, 0, 0 \ldots) \in \ell^p$

$y = (1, -1, 0, 0, 0 \ldots) \in \ell^p$

$\|x\| = 2^{1/p}$ $\quad$ $x + y = (2, 0, 0 \ldots 0) \in \ell^p$

$\|y\| = 2^{1/p}$ $\quad$ $\|x+y\| = 2 \quad \|x-y\| = 2$

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

$$4 + 4 = \cancel{2}(2^{2/p} + 2^{2/p})$$

$$4 = 2 \cdot 2^{2/p}$$

$$\Rightarrow 2 = 2^{2/p}$$

$$\Rightarrow \boxed{p = 2} \rightarrow \text{IPS}.$$

$\Rightarrow$ Parallelogram law holds for only ℓ^2. In general Parallelogram law does not hold for ℓ^p $p \neq 2$

$C[a,b] = \{x(t) : x : [a,b] \longrightarrow \mathbb{R} / \mathbb{C}\}$

any one of the Scalar

$x \in C[a,b]$

$\|x\| = \sup_{t \in [a,b]} |x(t)|$

$$\langle x, y \rangle = \int_a^b x(t)\, \overline{y}(t)\, dt$$

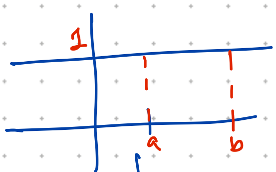
$$\|x\| = \left(\int_a^b |x(t)|^2 dt \right)^{1/2} \longrightarrow \text{Norm obtained by}$$

(Counterexample: Idea Parallelogram law does not hold)

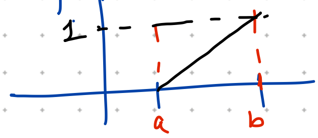
Let $x(t) = 1$

$y(t) = \dfrac{t - a}{b - a}$

$\|x\| = \sup_{t \in [a,b]} 1 = 1$ (at all pts)



$\|y\| = \sup_{t \in [a,b]} \left| \dfrac{t-a}{b-a} \right| = 1$ (at b)



$x + y = \dfrac{t - a + b - a}{b - a} = \dfrac{t + b - 2a}{b - a}$ $\quad$ $\|x + y\| = 2$

$x - y = \dfrac{t - a - b + a}{b - a} = \dfrac{t - b}{b - a}$ $\quad$ $\|x - y\| = 1$

$\|x\| = 1 \quad \|y\| = 1$

$\|x + y\| = 2 \quad \|x - y\|^{\prime}$

Checking the parallelogram law.

$$\|x-y\|^2 + \|x+y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

$$\Rightarrow \quad 4+1 = 2(1+1)$$

$$\Rightarrow \quad 5 = 4 \text{ (Not true)}$$

Hence parallelogram law does not hold

(Inner Product induces a norm)

Theorem: Let $\|\cdot\|$ be a norm on a vector space X. Then $\exists$ an inner product $\langle , \rangle$ on X s.t. $\|x\|^2 = \langle x, x\rangle \ \forall x \in X$.

iff the norm satisfies the $\|$gram law.

i.e. $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \ \forall x, y$

In this case, the unique inner product is given by

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

Proof: Suppose, $\exists$ an $\langle \cdot \rangle$ on X such that $\langle x, x\rangle = \|x\|^2$

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \langle x, x\rangle + \cancel{\langle x, y\rangle} + \cancel{\langle y, x\rangle} + \langle y, y\rangle \\ &\quad + \langle x, x\rangle - \cancel{\langle x, y\rangle} - \cancel{\langle y, x\rangle} + \langle y, y\rangle \\ &= 2(\|x\|^2 + \|y\|^2)\end{aligned}$$

And again by polarization identity

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

($\Leftarrow$) Suppose the $\|$gram law holds.

and $\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$

To show that $\langle , \rangle$ is an inner product.

1. $\langle x, x\rangle \geq 0$ $\quad x = 0$ iff $\langle x, x\rangle = 0$
2. $\langle x+y, z\rangle = \langle x, z\rangle + \langle y+z\rangle$
3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$
4. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

$$\langle x, x\rangle = \frac{1}{4}\{\|x+x\|^2 - \overset{\to 0}{\|x-x\|^2} + i\|x(\cancel{1+i})\|^2 - i\|x(\cancel{1-i})\|^2\}$$

$$= \|x\|^2 \geq 0$$

$\|x\|^2 = 0$ iff $x = 0$

$$\langle y, x\rangle = \frac{1}{4}\{\|x+y\|^2 - \|y-x\|^2 + i\|y+ix\|^2 - i\|y-ix\|^2\}$$

$$= \overline{\langle x, y\rangle}$$

Similarly, it follows that

$\langle -x, y\rangle = -\langle x, y\rangle$

$\langle ix, y\rangle = i\langle x, y\rangle$

<u>Show that</u> (To show)

$\langle x+z, y\rangle = \langle x, y\rangle + \langle z, x\rangle$

$$4\langle x+z, y\rangle = \|x+y+z\|^2 + i\|x+z+iy\| - \|x+z-y\|^2 - i\|x+z-iy\|$$

$$\|x+y+z\|^2 - \|x+z-y\|^2$$

$$= \|(x+y/_2) + (y/_2+z)\|^2 - \|(x-y/_2) + (z-y/_2)\|^2$$

<u>In IPS</u> $\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2$

$$\Rightarrow \quad = 2\|x+y/_2\|^2 + 2\|y/_2+z\|^2 - \cancel{\|x-z\|^2}$$

$$-2\|x-y/_2\|^2 - 2\|y/_2-z\|^2 + \cancel{\|x-z\|^2}$$

$$= 2(\|x+y/_2\|^2 - \|x-y/_2\|^2 + \|y/_2+z\|^2 - \|y/_2-z\|^2)$$

Now replacing y by iy in this result

$$i(\|x+z+iy\|^2 - \|x+z-iy\|^2)$$

$$= i(2\|x+iy/_2\|^2 - \|x-iy/_2\|^2 + \|iy/_2+z\|^2 - \|iy/_2-z\|^2)$$

$$\langle x+z, y\rangle = \left(2\langle x, y/_2\rangle + 2\langle z, y/_2\rangle\right) = \langle x, y\rangle + \langle z, y\rangle$$

set $z = 0$

$$\Rightarrow \langle x, y\rangle = 2\langle x, y/_2\rangle$$

$$\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$$

$z = x$

$\Rightarrow \langle 2x, y\rangle = 2\langle x, y\rangle$

$n \in \mathbb{N}\ \langle nx, y\rangle = n\langle x, y\rangle$

$\langle -x, y\rangle = -\langle x, y\rangle$

$n < 0$

$\langle -nx, y\rangle = -n\langle x, y\rangle$

$\langle nx, y\rangle = n\langle x, y\rangle$

$\forall n \in \mathbb{Z}.$

$$\langle nx, y\rangle = n\langle x, y\rangle \quad n \in \mathbb{Z}.$$

$$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle \quad \alpha \in \mathbb{K}$$

If $p, q \in \mathbb{Z},\ q \neq 0$

$$p\langle x, y\rangle = \langle px, y\rangle$$

$$= q\langle \tfrac{p}{q}x, y\rangle$$

$$\frac{p}{q}\langle x, y\rangle = \langle \tfrac{p}{q}x, y\rangle$$

<u>Claim:</u> $r_n \longrightarrow s$

$$r_n\langle x, y\rangle \longrightarrow s\langle x, y\rangle$$

$$r_n\langle x, y\rangle - s\langle x, y\rangle$$

$$|\langle x,y\rangle(r_n - s)| = |\langle x,y\rangle|\,|r_n - s| \leq |r_n - s|\,\|x\|\,\|y\|$$

if $r \in \mathbb{Q}$

$$\langle rx, y\rangle = r\langle x,y\rangle \quad \text{(Rational Density Thm)}$$

Now if $s \in \mathbb{R}$, $\exists (r_n) \in \mathbb{Q} \implies r_n \longrightarrow s \quad |r_n - s| \longrightarrow 0$

$$\|r_n x + y\| \longrightarrow \|sx + y\|$$

$$\big|\|x\| - \|y\|\big| \leq \|x - y\|$$

$$\big|\|r_n x + y\| - \|sx + y\|\big| \leq \|r_n x + y - sx + y\|$$

$$\Rightarrow \big|\|r_n x + y\| - \|sx + y\|\big| \leq \|(r_n - s)x\| = \underbrace{|r_n - s|\,\|x\|}_{\longrightarrow 0}$$

$$\Rightarrow \big|\|r_n x + y\| - \|sx + y\|\big| \longrightarrow 0$$

$$\Rightarrow \|r_n x + y\| \longrightarrow \|sx + y\|$$

$$\Rightarrow r_n x \longrightarrow sx.$$

$$\Rightarrow \langle r_n x, y\rangle \longrightarrow \langle sx, y\rangle$$

Since $\langle ix, y\rangle = i\langle x,y\rangle$, we have $\langle kx, y\rangle = k\langle x,y\rangle$

(9) Let (x_n) be a sequence in a metric space (X,d) and let $x \in X$. Show that if every subsequence of (x_n) converges to x, then $x_n \longrightarrow x$.

__Proof__: Suppose that $(x_n) \not\longrightarrow x$. Then $\exists\, \epsilon > 0$ s.t. for each $n \in \mathbb{N}$ there is some $m > n$ s.t. $d(x_m, x) \geq \epsilon$.

Choose $n_1 > 1$ s.t. $d(x_{n_1}, x) \geq \epsilon_0$. Then choose $n_2 > n_1$ with $d(x_n, x) \geq \epsilon_0$.

and so on proceeding in this way we get $n_1 < n_2 < \dots < n_k$ s.t. $d(x_{n_k}, x) \geq \epsilon_0 \;\forall\, k \in \mathbb{N}$.

Now (x_{n_k}) is a subsequence is a subsequence of (x_n) and no subsequence of (x_{n_k}) converges to x.

__Set 2__ (18) Let $f \neq 0$ be a linear functional on a vector space X. Show that $R(f)$ is a scalar of X.

Proof: $f: X \longrightarrow \mathbb{K}$ is linear

$\Rightarrow R(f)$ is a subspace of $\mathbb{K}$ (Property of linear transformation)

$\mathbb{K}$ is a vector space over $\mathbb{K}$

$\Rightarrow \dim(\mathbb{K}:\mathbb{K}) = 1$, since $f \neq 0 \Rightarrow \exists x \in X$ s.t. $f(x) \neq 0$

$\Rightarrow R(f) \neq 0 \Rightarrow R(f) = \mathbb{K}$

8(a) Let $\{x_n\}$ be a cauchy sequence in X (normed space)
show that $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ i.e. the series $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| < \infty$

Sol^n: Let $\{x_n\}$ be a cauchy sequence in X (normed space)

For $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ S.T. $n_{k+1} > n_k$ and $\|x_m - x_n\| < \frac{1}{2^k}$ for $m, n \geq n_k$

$$\sum_{i=1}^{\infty} \frac{1}{2^k} < \infty \qquad \|x_{n_k} - x_{n_{k+1}}\| < \frac{1}{2^k}$$

$$\sum_{k=1}^{\infty} \|x_{n_k} - x_{n_{k+1}}\| \leq 1.$$

(8b) Show that a normed space X is complete. If every absolutely convergent series is convergent.

Solution: Suppose that X is complete normed space.

Let $\{x_n\}$ be an absolutely convergent series, i.e.

$$\sum_{n=1}^{\infty} \|x_n\| < \infty$$

Let $S_n = x_1 + x_2 + \cdots + x_n$

at $t_n = \|x_1\| + \|x_2\| + \cdots + \|x_n\|$

Since $\{t_n\}$ is convergent, $\{t_n\}$ is a cauchy sequence.

For any $m, n \in \mathbb{N}$ with $n > m$

$$\|S_n - S_m\| = \|x_{m+1} + \cdots + x_n\|$$
$$\leq \|x_{m+1}\| + \cdots + \cdots \|x_n\|$$
$$= |t_n - t_m|$$

For $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ s.t.

$$|t_n - t_m| < \epsilon \quad \forall\, n, m > n_0$$

Then $\|S_n - S_m\| < \epsilon$, $m, n \geq n_1$.

Then $\{S_n\}$ is a cauchy - sequence in Complete space X.

Then $\{S_n\}$ is convergent.

($\Leftarrow$) X is not complete.

Then $\exists$ a cauchy sequence $\{x_n\}$ in X that is not convergent. Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. such that

$$\sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| < \infty$$

$$y_1 = x_{n_1}$$

$$y_2 = x_{n_2} - x_{n_1}$$

$$y_3 = x_{n_3} - x_{n_2}$$

$$\vdots$$

$$y_{k+1} = x_{n_{k+1}} - x_{n_k}$$

$$x_{n_{k+1}} = y_1 + y_2 + \cdots + y_k$$

and $\|y_1\| + \cdots + \|y_{k+1}\| = \sum$

$$= \sum_{i=1}^{k} \|x_{n_i} - x_{n_{i+1}}\|$$

since $\{x_n\}$ is a cauchy sequence that is not convergent,

$\exists$ no subsequence of $\{x_n\}$ is convergent

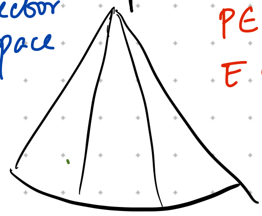
i.e. $\sum_{i=1}^{\infty} y_k$ does not converges.

But $\sum_{k=1}^{\infty} \|y_k\| < \infty$

$\Longrightarrow \Longleftarrow$.

$M \subset X \to$ Vector space

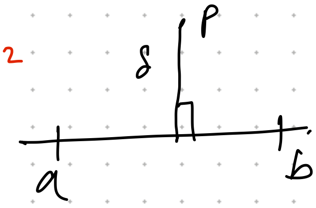
P



E

$P \in \mathbb{R}^2$

$E \subset \mathbb{R}^2$



$\delta = \inf d(x, \hat{y})$

$\hat{y} \in M$

Under what condition $\exists\, y \in \tilde{M}$

S.T. $\delta = d(x, y)$

X is a normed space

$\delta = \|x - y\|$

X is Hilbert.

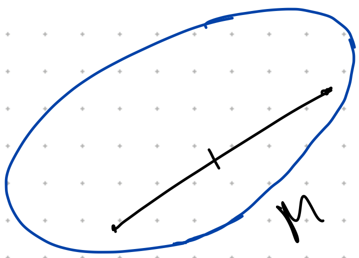
M is convex complete subset of X.

$M \subset X$ (vector space)

Let $x, y \in X$, segment b/w X and Y is the point $z = \alpha x + (1-\alpha) y$,

$0 \le \alpha \le 1$

A subset $M \subset X$ (Vector space) is said to be convex if the segment joining any two points of the set lies in the set.



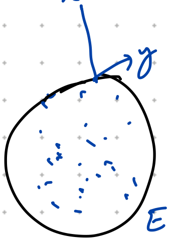
$y \in E$ is said to be the best approximation $\delta = \inf (x, \hat{y})$ $\hat{y} \in M$ of x into E.

$\|x - y\| \le \|x - z\|$ $x \in X$

$\forall z \in E$.

$\min_{z \in E} \|x - z\|$

$x \in X$



Let H be a Hilbert space

$E \neq \Phi$ be a convex complete (complete induced from the inner product) subset of X.

Then for every $x \in X$ $\exists$ a unique $y \in E$

S.T.

$\delta = \inf_{\hat{y} \in E} \|x - \hat{y}\| = \|x - y\|$

Proof: $\|x - y_1\|$ $y_1, y_2, \ldots \in E$

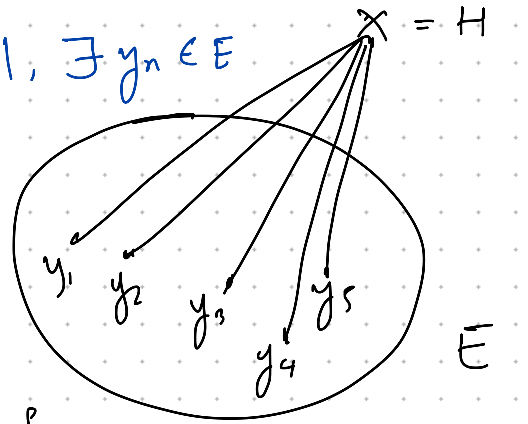
$\|x - y_2\|$

$\vdots$

$\|x - y_n\|$

Proof: As $\delta = \inf \|x - \tilde{y}\|$, $\exists\, y_n \in E$ s.t. $\tilde{y} \in E$

$\delta_n = \|x - y_n\| \longrightarrow \delta$ as $n \to \infty$



Next, to show that $\{y_n\}$ is a cauchy sequence in E.

Let $y_n - x = v_n$ $\quad \|v_n\| = \delta_n$

$y_m - x = v_m$

$\|v_n + v_m\| = \|y_n - x + y_m - x\| = 2\|\tfrac{1}{2}(y_n + y_m) - x\| \geq 2\delta$

(Use convexity)

$y_n, y_m \in E$, and E is convex $\Rightarrow \tfrac{1}{2}(y_n + y_m) \in E$

$\boxed{\inf_{\tilde{y} \in Y} \|x - \tilde{y}\| = \delta}$ $\quad \Rightarrow \|v_n + v_m\|^2 \geq 4\delta^2$

$y_n - y_m = v_n - v_m$

$\Rightarrow \|y_n - y_m\|^2 = \|v_n - v_m\|^2 = 2(\|v_n\|^2 + \|v_m\|^2) - \|v_n + v_m\|^2$ ⭐

$-\|v_n + v_m\|^2 \leq 4\delta^2$

$\Rightarrow 2(\|v_n\|^2 + \|v_m\|^2) - \|v_n + v_m\|^2 \leq -4\delta^2 + 2(\delta_n^2 + \delta_m^2) \longrightarrow 0$ As $n \to \infty$.

$\Rightarrow \|v_n - v_m\|^2 \longrightarrow 0$

$\Rightarrow \|y_n - y_m\| \longrightarrow 0$

Hence (y_n) is a cauchy sequence.

As E is complete,

$\lim_{n \to \infty} y_n = y \in E$

$\delta = \inf_{\tilde{y} \in E} \|x - \tilde{y}\| = \|x - y\|$

$\boxed{\text{Uniqueness}}$ Suppose $\exists\, y_1 \in E$ s.t. $\delta = \|x - y_1\|$.

To show that $y = y_1$

$\|y - y_1\| = 0$

$\|y - y_1\|^2 = \|(y - x) - (y_1 - x)\|^2$

$$= 2\|y-x\|^2 + 2\|y_1-x\|^2 - \|(y-x)+(y_1-x)\|^2$$

$$= 2\delta^2 + 2\delta^2 - 4\left\|\frac{(y+y_1)}{2} - x\right\|^2$$

As $\frac{1}{2}(y+y_1) \in E$

$$\Rightarrow \left\|\frac{y+y_1}{2} - x\right\| \geq \delta$$

$$\leq 4\delta^2 - 4\delta^2 = 0$$

$$\boxed{\|y-y_1\|^2 \leq 0} \Rightarrow \|y-y_1\| = 0$$

$\Rightarrow$ i.e. $\boxed{y_1 = y}$

26th Sept **Thm**: Let X be an IPS and $M \neq \phi$, a convex set which is complete $x \in X$ if $y \in M$ s.t.

$$\delta = \inf_{\hat{y} \in M} \|x-\hat{y}\| = \|x-y\|$$

$$\|x-y\| \leq \|x-z\| \ \forall z \in M$$

Thm (Lemma: Orthogonality) In theorem 1, let M be a complete subspace Y and $x \in X$. Then $z = x-y$ is orthogonal to Y.

Proof: As Y is complex and complete by Thm 1 $\forall x \in X, \exists y \in M = Y$ s.t.

$\delta = \|x-y\|$

To show: $\langle z, y^* \rangle = 0 \ \forall y^* \in Y$

Suppose $\exists$ some $y_1 \in Y$ s.t. $\langle z, y \rangle \neq 0$. Then

$\Rightarrow y_1 \neq 0 \ \because$ if $y_1 = 0 \Rightarrow \langle z, \overset{(y_1)}{0} \rangle = 0$

$$\Rightarrow \|z-\alpha y_1\|^2 = \langle z-\alpha y_1, z-\alpha y_1 \rangle \ \forall \alpha$$

$$= \langle z,z \rangle - \bar{\alpha}\langle z, y_1 \rangle - \alpha\left[\langle y_1, z \rangle - \bar{\alpha}\langle y_1, y_1 \rangle\right]$$

Let $\langle z, y_1 \rangle = \beta \neq 0$

$$= \|z\|^2 - \bar{\alpha}\beta - \alpha\left[\bar{\beta} - \bar{\alpha}\langle y_1, y_1 \rangle\right]$$

Then let $\bar{\alpha} = \left(\bar{\beta} / \langle y_1, y_1 \rangle\right)$ This term vanishes

$$= \|z\|^2 - |\beta|^2 / \|y_1\|^2 \leq \|z\|^2 < \delta^2 \quad \text{— (A)}$$

$$z - \alpha y_1 = x - y - \alpha y_1 = x - (y + \alpha y_1) = x - y_2$$

$(\because y_1 + \alpha y_2 = y_3 \in Y)$

$$\Rightarrow \|z - \alpha y_1\| = \|x - y_2\| \geq \delta$$ which contradicts (A)

Hence our assumption that $\langle z, y_1 \rangle \neq 0$ is not correct.

$$\Rightarrow \boxed{\langle z, y \rangle = 0}$$

26th Sept

$$H = Y \oplus Y^{\perp}$$

Y is a closed subspaces of H.

Direct sum: Let Y, Z be two vector subspaces of X

$X = Y \oplus Z$

iff every $x \in X$. $x = y + z$. $y \in Y, z \in Z$

A+B

$\boxed{\begin{array}{l} X = Y + Z \\ Y \cap Z = \{0\} \end{array}}$

$X = \mathbb{R}^3$, Y: plane passing through 'z'
Z: line passing through origin which does not lie in the plane.

Thm: If Y is closed subspace of H (Hilbert)
then $H = Y \oplus Y^{\perp}$
$Y^{\perp} = \{z \in H \mid z \perp Y\}$

Proof: Since Y is closed subspace of the Hilbert space H it is complete.
$H = \mathbb{R}$, $Y = (0, 1)$. By previous theorem and lemma (orthogonal property).
$\forall\, x \in H, \exists\, y \in Y$, s.t. $x = y \perp Y$
$y \in Y, z \in Y^{\perp}$.
Let $x - y \in Z$ i.e. $x = y + z$, $y \in Y$, $z \in Y^{\perp}$
Uniqueness, subspace $\exists\, y_1 \in Y, z_1 \in Z = Y^{\perp}$ S.T. $x = y_1 + z_1$

As $z, z_1 \in Z \Rightarrow z - z_1 \in Y$

$x = z + y = z_1 + y_1$

$\Rightarrow y - y_1 = z_1 - z \in Y$

$\Rightarrow y - y_1 \in Y$, $z - z_1 \in Z = Y^{\perp}$

$\Rightarrow z - z_1 \in Y \cap Y^{\perp}$ and $y - y \in Y \cap Y^{\perp}$

and $Y \cap Y^{\perp} = \{0\}$

$\Rightarrow z - z_1 = 0$, $y - y_1 = 0$

$\Rightarrow \underline{y = y_1, z = z_1}$

i.e.
$x = y + z$ is uniquely expressed.

$L^2[a, b] = \{f : \int_a^b |f|^2 < \infty\}$
Y = space of even function
$Y^{\perp}$ = space of odd f^n.

$L^2[a, b] = Y \oplus Y^{\perp}$

$$x = y + z$$

$$P : X \longrightarrow X$$
$$x \longmapsto Px = y$$

$P : X \longrightarrow Y$ is onto.
$$x \longmapsto y = Px$$

$$(x = y + z,\ y \in Y,\ z \in Y^{\perp})$$

$$P(x+y) = Px + Py$$
$$P(\alpha x) = \alpha Px.$$

$$x = y + z$$
$$x_1 = y_1 + z_1$$

$$x + x_1 = y_1 + y_2 + z_1 + z_2$$
$$p(x + x_1) = y_1 + y_2$$
$$= Px_1 + Px_2$$

$$P : X \longrightarrow X$$
$$x \longmapsto Px = y$$
$P : X \longrightarrow Y$ is onto.
$$P_Z : X \longrightarrow Z$$
$$P^2 = P(P)$$
$$P^2_x = P(Px)$$
$$= Px$$
$\forall x \in X$
$\Rightarrow P^2 = P \longrightarrow$ idempotent

Relationship between Range (P) and Null (P).

$$\boxed{H = R(P) \oplus N(P)}$$

Thm: Let X be a vector space.

(a) If $P : X \longrightarrow X$ is a projection, then $X = \text{Range}(P) \oplus \text{Null}(P)$.

(b) If $X = M \oplus N$, where M and N are vector subspaces of X, there is a projection $P : X \longrightarrow X$ with range$(P) = M$, $N(P) = N$

Proof: (a) We first show that $x \in \text{range}(P)$ iff $x = Px$

If $x = Px$, then $x \in \text{Range}(P)$. If $x \in \text{Range}(P)$, then $X = Py$, for some $y \in X$ and since $P^2 = P$

$$Px = P^2_x = P(Px) = Py = x$$

if $x \in \text{Range}(P) \cap N(P)$

$\Longrightarrow x \in \text{Range}(P)$ and $x \in N(P)$

$\Downarrow$

$x = Px \qquad Px = 0$

$\Rightarrow x = 0 \qquad$ i.e. $\text{Range}(P) \cap N(P) = \{0\}$

$\forall x \in X \qquad x = Px + (x - Px)$

In case of Hilbert space

find M s.t. $\boxed{H = M \oplus M^{\perp}}$

The projection of H into M along $M^{\perp}$ the orthogonal projection of H on M

If $x = y + z$ $y, y_1 \in M$

$x_1 = y_1 + z_1$ $z, z_1 \in M^{\perp}$

$$\langle Px, x_1 \rangle = \langle y, y_1 + z_1 \rangle$$
$$= \langle y, y_1 \rangle + \underbrace{\langle y, z_1 \rangle}_{0} = \langle y, y_1 \rangle$$
$$= \langle y + z, y_1 \rangle$$
$$= \langle x, y_1 \rangle = \langle x, Px_1 \rangle$$

$$\boxed{\langle Px_1, x_1 \rangle = \langle x, Px_1 \rangle}$$

i.e. Orthogonal projection is Self-Adjoint

Def: An orthogonal projection on a Hilbert space H is a linear map $P: H \longrightarrow H$ satisfies $P^2 = P$, $\langle Px, y \rangle = \langle x, Py \rangle$ for $x, y \in H$.

Proposition: If P is nonzero orthogonal projection, then $\|P\| = 1$.

Proof: $\|Px\|^2 = \langle Px, Px \rangle$

if $x \in H$ and $Px \neq 0$, $\|Px\| = \dfrac{\langle Px, Px \rangle}{\|Px\|} = \dfrac{\langle x, P^2x \rangle}{\|Px\|} = \dfrac{\langle x, Px \rangle}{\|Px\|}$

$$\leq \frac{\|x\| \, \|Px\|}{\|Px\|}$$

$\Longrightarrow \|Px\| \leq \|x\|$

$\Rightarrow \boxed{\|P\| = 1}$ — ①

if $P \neq 0$ $\exists x \in H$, s.t. $Px \neq 0$

$\|P(P(x))\| = \|Px\|$

$Px = y$

$\Rightarrow \|Py\| = \|y\|$

$\boxed{\|P\| \geq 1}$ — ②

$\boxed{\|P\| = 1}$

Thm: Let H be a Hilbert Space.

ⓐ If P is an orthogonal projection on H, then range(P) is closed and Range$(P) \oplus N(P)$

ⓑ If, M is closed subspace of N, then $\exists$ an orthonormal projection P on H with range$(P) = M$ and $N(P) = M^{\perp}$

11 Oct (FRIDAY)

(Pythagoras thm) Let $\{x_1, \dots x_n\}$ be an orthogonal set in an IPS X. Then

(a) $\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$

Proof: $\|\sum_{1}^{n} x_i\|^2 = \langle \sum_{i=1}^{n} x_i , \sum_{j=1}^{n} x_j \rangle$

$$= \langle x_1 + x_2 \dots x_n , x_1 + x_2 \dots x_n \rangle$$

$$= \langle x_1, x_1 \rangle + \langle x_1, x_2 \rangle \dots$$
$$\langle x_2, x_1 \rangle + \langle x_2, x_2 \rangle$$
$$\vdots$$
$$\langle x_n, x_1 \rangle + \langle x_n, x_2 \rangle + \dots \langle x_n, x_n \rangle$$

$$= \sum_{i=1}^{n} \|x_i\|^2$$

(b) E is an orthogonal subset of an IPS, then E is linearly independent.
which is orthonormal set in X.

Proof: Let $F = \{x_1, \dots x_n\}$ be a finite subset of E which is orthonormal set in X.

TS that the set $\{e_1, \dots e_n\}$ is linearly independent.

TS $\sum_{j=1}^{n} \alpha_j e_j = 0$

$$\Rightarrow \alpha_j = 0$$

$$\langle \sum_{j=1}^{n} \alpha_j e_j , e_j \rangle = \langle \alpha_1 e_1 + \alpha_2 e_2 + \dots \alpha_n e_n , e_j \rangle$$

$$= \langle \alpha_1 e_1 , e_j \rangle + \langle \alpha_2 e_2 , e_j \rangle + \dots \langle \alpha_n e_n , e_j \rangle$$

$$= \alpha_1 \cancelto{0}{\langle e_1, e_j \rangle} + \alpha_2 \cancelto{0}{\langle e_2, e_j \rangle} + \dots + \alpha_j \langle e_j, e_j \rangle + \dots \alpha_n \cancelto{0}{\langle e_n, e_n \rangle}$$

Since the set F is orthonormal

$$\Rightarrow \alpha_j \langle e_j, e_j \rangle = 0 \Rightarrow \alpha_j = 0$$

Since there is no restriction on α_j

$$\Rightarrow \alpha_j = 0 \quad \forall j \in \{1, \dots n\}$$

$\Rightarrow$ The set F is linearly independent.

(c) (Gram Schmidt Orthonormalization process)

Let $\{x_1, \dots x_n\}$ be a linearly independent set in an IPS X.

Define $y_1 = x_1$, $u_1 = \dfrac{y_1}{\|y_1\|}$

and for $n = 2, 3 \dots$

$$y_n = x_n - \sum_{i=1}^{n-1} c_i u_i \qquad c_i = \langle x_n, u_i \rangle$$

$$u_n = \frac{y_n}{\|y_n\|}$$

Then $\{u_n : n = 1, 2 \dots\}$ is an orthonormal set in X and span $\{x_1, \dots x_n\}$ = span $\{u_1 \dots u_n\}$

PROOF

Use induction

$$y_2 = x_2 - \langle x_2, u_1 \rangle u_1$$

$$u_2 = \frac{y_2}{\|y_2\|}$$

Example Let $X = \ell^2$ and for $n = 1, 2, \dots$

$$x_n = (1, 1, 1 \dots \underset{n^{th} \text{ position}}{1}, 0, 0, 0 \dots)$$

i.e. 1 occurs in the 1^{st} n entries 0's in remaining.

$x_n \in \ell^2$ $\qquad \|x_n\| = \sqrt{1^2 + 1^2 + 1^2 \dots} = \sqrt{n}$

$x_n \in \ell^2$

$$\sum \alpha_i x_i = 0$$

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0$$

$$\Rightarrow \alpha_1 (1, 0, 0 \dots) + \alpha_2 (1, 1, 0, 0 \dots) + \dots \alpha_n (1, 1, 1 \dots 1, 0, 0)$$

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \dots \alpha_n = 0$$

find the orthonormal set $\{v_1, v_2 \dots v_n\}$

$$y_1 = x_1 \quad \Rightarrow \quad v_1 = \frac{y_1}{\|y_1\|}$$

$$y_2 = x_2 - \langle x_2, u_1 \rangle u_1$$

$$= (1, 1, 0, 0 \dots) - \langle (1, 1, 0, 0, 0 \dots), (1, 0, 0, 0 \dots) \rangle (1, 0, 0, 0 \dots)$$

$$= (1, 1, 0, 0 \dots) - (1, 0, 0 \dots)$$

$$y_2 = (0, 1, 0, 0 \dots) \Rightarrow v_2 = \frac{y_2}{\|y_2\|} = (0, 1, 0, 0 \dots)$$

Bessel's Inequality $\sum_{i=1}^{\infty} |\langle x_i, e_i\rangle|^2 \leq \|x\|^2 \quad \forall x \in X.$

(Exam mein 90% chance)

Let $\{u_1, u_2 \dots u_n\}$ be an orthonormal set in X and X is an IPS. Then for every $x \in X$,

$$\sum_{i=1}^{\infty} |\langle u_n, x\rangle|^2 \leq \|x\|^2$$

where equality holds iff $x \in \text{span}\{u_1, u_2 \dots u_n\}$

Proof: Let $x \in X$ and define $x_m = \sum_{n=1}^{m} \langle x, u_n\rangle u_n$. Now by orthonormality of $\{u_1, \dots u_n\}$

We have

$$\langle x, x_m\rangle = \langle x_m, x\rangle = \langle x_m, x_m\rangle = \sum_{n=1}^{m} |\langle x, u_n\rangle|^2$$

$$\langle x, x_m\rangle = \langle x, \langle x,u_1\rangle u_1 + \langle x,u_2\rangle u_2 \dots + \langle x,u_m\rangle u_m\rangle$$

$$\begin{aligned}
\langle x, \sum_{n=1}^{m} \langle x,u_n\rangle u_n\rangle &= \langle x, \langle x,u_1\rangle u_1\rangle + \langle x, \langle x,u_2\rangle u_2\rangle + \dots + \langle x, \langle x,u_m\rangle u_m\rangle \\
&= \langle x, \langle x,u_1\rangle u_1\rangle + \langle x, \langle x,u_2\rangle u_2\rangle + \dots + \overline{\langle x, u_m\rangle} \langle x, u_m\rangle \\
&= \overline{\langle x, u_1\rangle} \langle x, u_1\rangle + \dots \\
&= \sum_{i=1}^{m} |\langle x, u_i\rangle|^2
\end{aligned}$$

$$\langle x_m, x_m\rangle = \langle \sum_{j=1}^{m} \langle x,u_j\rangle u_j, \sum_{i=1}^{m} \langle x,u_i\rangle u_i\rangle$$

As $\langle u_i, u_j\rangle = 0$ for $i \neq j$

$$= \sum_{n=1}^{m} |\langle x, u_n\rangle|^2$$

$$\|x - x_m\|^2 \geq 0$$

$$\begin{aligned}
= \langle x - x_m, x - x_m\rangle &= \langle x, x\rangle - \langle x, x_m\rangle - \cancel{\langle x_m, x\rangle} + \cancel{\langle x_m, x_m\rangle} \\
&= \langle x, x\rangle - \langle x, x_m\rangle \\
&= \|x\|^2 - \langle x, x_m\rangle \geq 0
\end{aligned}$$

$$\langle x, x_m\rangle \leq \|x\|^2$$

$$\Rightarrow \sum_{j=1}^{m} |\langle x, u_m\rangle|^2 \leq \|x\|^2$$

As RHS is independent of m

letting $m \to \infty$

$$\boxed{\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2}$$

Class Test on 25.10.19 (IPS)

Equality holds if $x_m \in \text{span}\{u_1, \dots u_n\}$

(Reisz's Representation Theorem)

(Functional on Hilbert Spaces)

Every bounded linear functional f on H (Hilbert space) can be represented in terms of inner product namely $f(x) = \langle x, z \rangle$, where z depends on f, is uniquely determined by f and has norm $\|z\| = \|f\|$

Proof (Uniqueness) $\langle x, z \rangle = \langle x, z_1 \rangle$, to show that $z = z_1$

$$\langle x, z \rangle - \langle x, z_1 \rangle = 0$$

$$\langle x, z - z_1 \rangle = 0$$

$\forall\, x \in H$

Take $x = z - z_1$ $\quad \langle z - z_1, z - z_1 \rangle = 0$

i.e. $\quad \|z - z_1\|^2 = 0$

$$\|z - z_1\| = 0$$

$$\Rightarrow \underline{\underline{z = z_1}}$$

16th Oct

To show that $\|f\| = \|z\|$

If $f = 0$, take $\langle x, z \rangle = 0 \ \forall \ x \Rightarrow x = 0$

$\|f\| = \|z\|$

$f \neq 0, \ z \neq 0$

$f(z) = \langle z, z \rangle = \|z\|^2$

$\|z\|^2 = f(z) \leq \|f\| \|z\| = \|z\| \leq \|f\|$ — (i)

Next, to show that $\|f\| \leq \|z\|$

$|f(x)| = |\langle x, z \rangle| \leq \|x\| \|z\|$

$\|f\| = \sup\limits_{\|x\| = 1} |\langle x, z \rangle| \leq \|z\|$

i.e. $\|f\| \leq \|z\|$ —— (ii)

$\|f\| = \|z\|$

$f(x) = \langle x, z \rangle$

Existence result :

If $f = 0$, $f(x) = \langle x, 0 \rangle = 0$, the thm follows:

Assume $f \neq 0$, $\exists \ z$ of H s.t. $f(x_0) \neq 0$

$F = f^{-1}\{0\} = \{x \in H : f(x) = 0\}$

F is a subspace of H (Null space)

F is a closed subspace of H (as f is bdd linear)

$H = F \oplus F^{\perp}$

But $f \neq 0$, $F \neq H$. Hence $F^{\perp}$ contains at least one non-zero elements and hence an element $\|y\| = 1$

$\forall \ x \in H$, $f(x)y - f(y)x \in F$ (To Prove)

$f(f(x)y - f(y)x) = f(x)f(y) - f(y)f(x) = 0$

$\Rightarrow f(x)y - f(y)x \in F$

$y \in F^{\perp}$

$$\langle f(x)y - f(y)x, y\rangle = 0$$

$$\Rightarrow \langle f(x)y, y\rangle - \langle f(y)x, x\rangle = 0$$

$$f(x)\underbrace{\langle y, y\rangle}_{=1} - f(y)\langle x, x\rangle = 0$$

$$\Rightarrow f(x) = \langle x, \underbrace{\overline{f(y)}x}_{=z}\rangle$$

$f(x) = \langle x, z\rangle$, which is the representation of f.

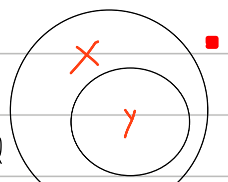
Reisz Representation

Bounded linear map

$f : H \longrightarrow \mathbb{K}$, $\exists$ a unique $x \in H$

$$f(x) = \langle x, z\rangle$$
$$x \in H$$
$$\|f\| = \|z\|$$

HAHN-BANACH THEOREM



X be a Hilbert space

Let $Y \subset X$, Y is a subspace of X

Let $f \in Y'$, then f can be extended to X, $\exists\, g : X'$ $\quad g : X \longrightarrow \mathbb{K}$

s.t. $g|_Y = f$, $\|g\| = \|f\|$

$g(y) = f(y)$

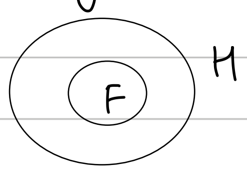
$\forall\, y \in Y$.

Arbitrary product of non-empty set is non empty.

Hahn-Banach extension thm in Hilbert Space.

Let H be a Hilbert space, F a subspace of H and g a bounded linear functional on F. Then there is a unique $f \in H'$ s.t. $f|_F = g$ and $\|f\| = \|g\|$

$$g : F \longrightarrow \mathbb{K}$$



<u>Proof:</u> Since g is a bdd linear on F, then g admits a norm preserving linear extension to the closure of F in H.

i.e.

$$\hat{g} : \overline{F} \longrightarrow \mathbb{K} \quad \text{and} \quad \|\hat{g}\| = \|g\|$$

WLOG, F is a closed subspace of H.

Since H is a hilbert space and F is a closed subspace of the Hilbert space, F is a Hilbert space.

Use Reisz Representation thm on F.

$$g(x) = (x, z) \,, \; x \in F.$$

Also $\|g\| = \|z\|$

Define $f : H \longrightarrow \mathbb{K}$

$$f(x) = \langle x, z \rangle \,, x \in H$$

Then $f \in H'$, $f|_F = g$, $\|f\| = \|g\| = \|z\|$

17th Oct

Hahn Banach Theorem

Let H be a hilbert space

F be a subspace of H. $g : F \longrightarrow \mathbb{K}$ is a bounded linear map then $\exists$ a unique $f \in H'$

S.T. $f|_F = g$

$\|f\| = \|g\|$

Uniqueness Let $h \in H'$ with $h|_F = g$ and $\|h\| = \|g\|$

Let $z \in H$ be the representer of h. so that $\|h\| = \|z\|$

Now $\frac{1}{2}(f+h)|_F = g$ and $\frac{1}{2}(y+z)$ is the representer of $\frac{1}{2}(f+h)$. Hence

$$\|y\| = \|g\| \leq \|\tfrac{1}{2}(f+h)\| = \|\tfrac{1}{2}(y+z)\| \leq \tfrac{1}{2}(\|y\| + \|z\|) = \|y\|$$

This implies that $\|\frac{1}{2}(y+z)\| = \|y\| = \|z\|$

$$\|y+z\|^2 + \|y-z\|^2 = 2(\|y\|^2 + \|z\|)^2$$

$$\|y-z\|^2 = 2(\|y\|^2 + \|z\|^2) - \|y+z\|^2$$

$$= 2(\|y\|^2 + \|z\|)^2 - 4\|y\|^2$$

$$= 4\|y\|^2 - 4\|y\|^2$$

$\Rightarrow \|y-z\|^2 = 0$

$\Rightarrow y = z$

$\Rightarrow f = h$

Any non zero normed space has a non zero function for which $\exists$ $x_0 \in X$ s.t. $f(x_0) \neq 0$

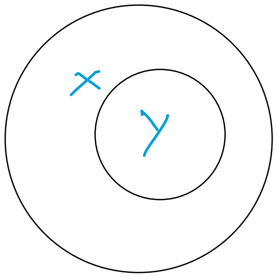
$$X' = \{ f : \underset{\text{bdd}}{f : X \longrightarrow \mathbb{K}} \}$$

$X \to$ Vector space

$Y \subset X$ is a subspace

$\|f\| \leq \alpha$

$\|g\| \leq \alpha$



$f \in Y^*$

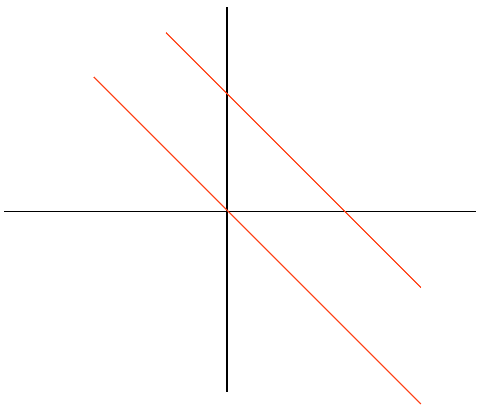
$g \in X^*$

S.T. $g|_Y = f$

$Y \subset X$, $X/Y = \{x + Y : x \in X\}$

$\dim(X/Y) = 1$

$\Rightarrow \operatorname{codim} Y = 1$



$Y \subset X$

$M = Y + x_0$

(Hyperspace)

Maximal linear variety / proper subspace

A hyperplane in a V.S. X is a maximal proper linear variety. i.e. there is a hyperspace $H \neq X$ and V is any hyperspace containing H, then $V = X$ or $V = H$.

Let H be a subspace of X, $x_0 \in X$, then a linear variety is written as $H + x_0$.

As maximal proper subspace of a V.S. is called a hyperspace

Shifting a hyperspace is called a hyperplane.

A proper subspace Z of X is said to be maximal

$\boxed{\text{iff}}$ $X = \text{span}\{Z \cup \{a\}\}$ $\forall$ $a \notin Z$.

18th Oct If Z is a hyperspace in X,
$Z + x$ where $x \in X$ is a hyperplane.

$$\text{Span}(Z \cup \{x\}) = X$$
$$\dim(X/Z) = 1$$

Prop 1 Let H be a hyperplane in a vector space X and a constant 'c'. Then there is a linear functional f on X and a constant 'c' S.T. $H = \{x \mid f(x) = c\}$. Conversely if f is a non zero functional on X, then the set $\{x : f(x) = c\}$ is a hyperplane in X.

Proof: Let H be a hyperplane in X. Then $H = x_0 + M$, where M is a hyperspace in X, $x_0 \notin M$.

$X = \text{span}(\{x_0\} \cup M)$, so every element $x \in X$ can be written as
$x = \alpha x_0 + m$, uniquely with $\alpha \in \mathbb{K}$ and $m \in M$.

Define $f(x) = \alpha$, then $H = \{x \mid f(x) = 1\}$

If $x_0 \in M$, take $x_1 \notin M$ and $X = \text{span}\{M \cup \{x_1\}\}$
$H = M$, and define $x = \alpha x_1 + m$, $f(x) = \alpha$, $H = \{x \mid f(x) = 0\}$

($\Leftarrow$) $f \neq 0$, i.e. $\exists\, x^* \in X$, such that $f(x^*) \neq 0$
Let $M = \{x \in X \mid f(x) = 0\}$. M is a subspace of X, called the null space. Let $x_0 \in X$, with $f(x_0) = 1$, then for $x \in X$,

Let $z = x - f(x)x_0$
$$\Rightarrow f(z) = f(x - f(x)x_0) = f(x) - f(x)\cancel{f(x_0)}^{1}$$
$$= 0$$

Hence $x - f(x)x_0 \in M$
$$\Rightarrow x - f(x)x_0 = m \text{ for some } m \in M$$
$$\Rightarrow x = m + f(x)x_0$$

$X = \text{span}\{M \cup \{x_0\}\}$

M is a maximal proper subspace. For any c, let x_1 be any element for which $f(x_1) = c$

$$\{x : f(x) = c\} = \{x : f(x - x_1) = 0\}$$
$$= M + x_1, \text{ which is a hyperplane}$$

A hyperplane H in a normed space X is either closed or dense in X, because as H is maximal proper subspace of X.

$$H = \overline{H} \text{ and } \overline{H} = X$$

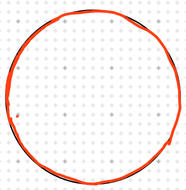
Prop 2 Let f be a non zero functional on X (normed space) then the hyperplane $H = \{x : f(x) = c\}$ is closed for every c iff f is bounded

If $f \neq 0$, linear functional on X, we associate with the hyperplane $H = \{x \mid f(x) = c\}$ for four sets $\{x : f(x) \leq c\}$, $\{x : f(x) < c\}$, $\{x : f(x) \geq c\}$, $\{x : f(x) > c\}$ called half spaces determined by H. The 1st two are called -ve half spaces and last two are +ve half spaces

$A \subseteq X$, metric space

$$(\overline{A})^{0} = \phi$$

Then A is said to be nowhere dense or rare if $(\overline{A})^{0} = \phi$.



Def: (Category) A subset M of X is said to be

ⓐ rare (nowhere dense) in X if its closure $\overline{M}$ has no interior pts (i.e.) $(\overline{M})^{0} = \phi$.

Ex: every finite set is nowhere dense
$\mathbb{Z}$ in nowhere dense in $\mathbb{R}$.

ⓑ meager (or the first category) in X if M is union of countable many sets each of which is rare in X

Ex: $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$.

$X = \bigcup_{i=1}^{\infty} X_i$, $\overline{X_i}^{\,0} = \phi \ \forall i$

Ⓒ Non meager (2^{nd} category) in X if M is not of 1^{st} category or meager.

Baire's Category Theorem (complete metric spaces)

If a metric space $X \neq \phi$ is complete, it is of 2^{nd} category or nonmeager

Note: If $X \neq \phi$ is complete and $X = \bigcup_{k=1}^{\infty} A_k$, ($A_k$ is closed), then at least one A_k contains a non empty open subsets.

A collection of functions $A = \{f_n : X \longrightarrow \mathbb{K}\}$ on X is said to be uniformly bounded if $\exists\, M \in \mathbb{R}$ s.t. $|f_i(x)| \leq M \;\forall\, f_i \in A, \forall\, x \in X$.

$$\Longrightarrow \|f_j\| \leq M$$

It is pointwise bdd. if $\exists\, M_x \in \mathbb{R}$

s.t. $|f_i(x)| \leq M_x$, $\forall\, f_i \in A$.

Ex: $A = \{f_1 = x,\ f_2 = 2x,\ f_3 = 3x \ldots,\ f_n = nx, \ldots\}$

$A \subset C[0,1]$ each function is ptwise bdd. but not uniformly bdd.

For M is any real number, however large, $\exists\, n_0 \in \mathbb{N}$, with $n_0 > M$ and $f_{n_0}(x_0) = n_0 > M$

$$x_0 = 1$$

Ex 2: $A = \{f_1 = \sin x,\ f_2 = \sin 2x, \ldots f_n = \sin nx \ldots\}$

$A \subset C(\mathbb{R})$ are uniformly bdd.

$$|f_j(x)| = |\sin j(x)| \leq 1$$
$$\forall j \;\&\; \forall x \in \mathbb{R}$$

Statement of uniform boundedness principle

Let T_n be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed space Y s.t. $(\|T_n x\|)$ is bdd $\forall\, x \in X$, 23rd Oct

Every $X \neq \phi$ complete metric space is of 2^{nd} category or nonmeager

$X \neq \bigcup_{n=1}^{\infty} A_i$: $\overline{A_i}^{\circ} = \phi$

$X = \bigcup_{i=1}^{\infty} A_i$, A_i is closed.

Then, $\exists$ $B(x,r) \subset A_K$

Proof of Uniform Boundedness principle

$T_n : X \longrightarrow Y$ $\|T_n(x)\| \leq C_x$ → Pointwise boundedness ⇓ Uniform boundedness

X ↓ banach

C_x ↑ independent of n

$\Rightarrow \|T_n\| \leq K$
$n = 1, 2, \ldots$

For any $k \in \mathbb{N}$
$$A_k = \{x \in X \mid \|T_n(x)\| \leq k\}$$
$n = 1, 2, \ldots$

To show that A_K is closed.

let $x \in \overline{A_k}$, $\exists$ $(x_j) \in A$ s.t. $x_j \longrightarrow x$.

i.e. for every fixed "n", we have $\|T_n(x_j)\| \leq k$

As T_n is continuous,
$$T_n(x_j) \longrightarrow T_n x$$

Again $\|\cdot\|$ is also continuous.

$\|T_n(x_j)\| \longrightarrow \|T_n x\|$ $\|T_n x\| = \|\lim_{j\to\infty} T_n(x_j)\|$

$\Rightarrow \|T_n x\| \leq k$

$\Rightarrow x \in A_K$, i.e. A_K is closed in X.

As X is complete and nonempty

$X = \bigcup_{k=1}^{\infty} A_K$, with atleast one $A_K^{\circ} \neq \phi$ (Baire Category thm)

i.e. some A_K contains an open ball say
$$B_0 = B(x_0; r) \subset A_{K_0}$$

Let x be an arbitrary non zero element,
$x \neq 0$,

$z = x_0 + \gamma x$, $\gamma = \frac{r}{2\|x\|}$

$\|z - x_0\| = \|\gamma x\| = \frac{r}{2\|x\|}\|x\| = \frac{r}{2} < r$ so $z \in B_0$

$\Rightarrow$ $z \in A_{K_0}$, $\|T_n z\| \leq k_0$ $\forall n$.

Also $\|T_n x_0\| \leq k_0$, $x = \frac{1}{\gamma}(z - x_0)$

$$\|T_n x\| = \frac{1}{\gamma}\|T_n(z - x_0)\| \leq \frac{1}{\gamma}(\|T_n z\| + \|T_n x_0\|) \leq \frac{2K_0}{\gamma} = \frac{2 \times 2\|x\| K_0}{r}$$

(Δ ineq.)

$$\leq \frac{4}{r}\|x\| k_0$$

$$\Rightarrow \|T_n x\| \leq \frac{4}{\gamma} k_0 \|x\| \quad \text{—(i)}$$

Hence for all n

$$\|T_n\| = \sup_{\|x\|=1} \|T_n(x)\| \leq \frac{4}{r} k_0 = C \text{ (Set)}$$

$$\Rightarrow \|T_n\| \leq C \quad \forall n.$$

Geometrical Representation (Uniform bddness principle)

It says that either T_n maps a given bdd. subset of a Banach space X into a fixed ball in the normed space Y, or else there is some $x \in X$ S.T. no ball in Y contains all $T_n(x)$

The normed space X of all polynomials with normed defined by $\|x\| = \max_j |a_j|$

$$x(t) = a_0 + a_1 t + \dots + a_n t^n$$

$$x(t) = \sum_{j=0}^{\infty} a_j t^j, \quad a_j = 0 \text{ for } j > N_x$$

x is a polynomial of degree N_x

Let $T_n = f_n \Rightarrow$ sequence of functionals on X.

$$T_n(0) = f_n(0) = 0$$

$$T_n(x) = f_n(x) = a_0 + \dots + a_{n-1}$$

24th Oct

Strong And Weak Convergence

$\{x_n\}$ is a sequence in NLS X

$x_n \longrightarrow x$

$\lim_{n\to\infty} \|x_n - x\| = 0$ Strong convergence.

(x_n) converges to x strongly

$x_n \longrightarrow x$ weakly

or $x_n \xrightarrow{w} x$ or $x_n \rightharpoonup x$ Weak convergence

iff $f(x_n) \longrightarrow f(x)$ $\forall f \in X'$.

X: Space of all polynomial

x, y, z

$x(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_n t^n$.

$\|x\| = \max_{s \in \{0,\dots,n\}} |\alpha_s|$, $(X, \|.\|)$ is a Normed linear space.

Let $T_n = f_n$ be a sequence of functionals defined by

$T_n(0) = f_n(0) = 0$

$T_n : X \longrightarrow \mathbb{K}$

$T_n x = f_n(x) = \alpha_0 + \dots + \alpha_n$

Check: $f_n = T_n$ is a linear functional.

(f_n) is a sequence of linear functionals

$$|f_n(x)| = |\alpha_0 + \alpha_1 + \dots + \alpha_n|$$
$$\leq |\alpha_0| + |\alpha_1| \dots + |\alpha_n|$$
$$\leq n\|x\| \quad \text{—} (1)$$

$\|x\| = \max_{s \in \{0,\dots n\}} |\alpha_s|$

$\Rightarrow |\alpha_i| \leq \|x\|$

Take supremum on both sides $\|x\| = 1$

$\|f\| \leq n$

Now for fixed $x \in X$ $(|f_n(x)|)$

$|f_n(x)| \leq C_x$, $n = 1, 2 \dots$

Any polynomial x of degree N_x has $N_x + 1$ coefficients.

So, $|f_n(x)| \leq (N_x + 1) \max |\alpha_j| = C_x$.

If X is Banach then, $\|f_n\|$ must be uniformly bounded.

Take $x(t) = 1 + t + t^2 + t^3 \dots + t^{n-1}$

$\|x\| = \max_{s \in \alpha_s} |\alpha_s| = 1$

$T x_n = f_n(x) = 1 + 1 + 1 + 1 \cdots = n$

$T x_n = n \cdot 1 = n \|x\|$

$\|f_n\| = \sup_{\|x\| = 1} |f_n(x)|$

$f_n(x) = n\|x\|$

$\|f_n\| = n$

$(\|f_n\|)$ is unbounded

$\Rightarrow$ X is a Banach space is wrong.

i.e. X is not a Banach space.

In sequences we have two types of convergence

1) Strong convergence
$\Uparrow \quad \Downarrow$
2) Weak convergence

Thm: Let X be a NLS.

(a) $x_n \longrightarrow x \Rightarrow x_n \longrightarrow x$

(b) $x_n \longrightarrow x \not\Rightarrow x_n \longrightarrow x$

(c) If $\dim X < \infty$, then $x_n \longrightarrow x \Rightarrow x_n \longrightarrow x$

Proof: (a) $x_n \longrightarrow x$ i.e. $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$

To show that $x_n \longrightarrow x$

$\forall f \in X'$, $f(x_n) \longrightarrow f(x)$ (To show)

$|f(x_n) - f(x)| = |f(x_n - x)| \leq \|f\| \, \underbrace{\|x_n - x\|}_{0 \text{ as } n \longrightarrow \infty}$

$\Rightarrow \lim_{n \to \infty} f(x_n) = f(x)$ or $f(x_n) \longrightarrow f(x)$ as $n \longrightarrow \infty$

ⓑ Let $X = H$ be a Hilbert space and (e_n) be sequence of orthonormal set

$\|e_n\| = 1$

$\langle e_n, e_m \rangle = 0, \; n \neq m$

Let $f \in H'$

$f(x) = \langle x, z \rangle$

$f(e_n) = \langle e_n, z \rangle$

By Bessel's inequality

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2$$

$\langle e_n, z \rangle \longrightarrow 0$

$f(e_n) \longrightarrow 0$

$\|e_n - e_m\|^2 = \langle e_n - e_m, e_n - e_m \rangle \quad n \neq m$

$= 2$

The cauchy sequence DNE.

$e_n \not\longrightarrow 0.$

See from BOOK.

30th Oct

$\dim X < \infty$, weak convergence $\Longrightarrow$ strong convergence

Let $\dim X = K$, and $\{e_1, e_2, \dots, e_K\}$ be a basis for X.

Now for $(x_n) \in X$, $x \in X$

$$x_n = \alpha_1^{(n)} e_1 + \alpha_2^{(n)} e_2 + \dots + \alpha_K^{(n)} e_K, \quad \alpha_i^{(k)}, \; i \in \{1, \dots K\}$$

$$x = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_K e_K \qquad \alpha \in \mathbb{K}$$

$$x_n \rightharpoonup x \Rightarrow f(x_n) \longrightarrow f(x) \quad \forall f \in X'$$

$$\Downarrow$$

$$x_n \longrightarrow x$$

$\{f_1, f_2, \dots f_K\}$ is a dual basis for X'

$f_i \longrightarrow$ Definition for element in dual basis

$$f_j(x_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

$f_j(x_i) = \alpha_j^{(n)}$

$f_j(x) = \alpha_j$

$$f_j(x+y) = f_j(x) + f_j(y)$$

$$f_j(\alpha x) = \alpha f_j(x)$$

$$f_j(x_n) \longrightarrow f_j(x)$$

$$\alpha_j^{(n)} \longrightarrow \alpha_j$$

$$\|x_n - x\| = \left\|\sum_{j=1}^{k} \alpha_j^{(n)} e_j - \sum_{j=1}^{k} \alpha_j e_j\right\| = \left\|\left(\sum_{j=1}^{k} \alpha_j^{(n)} - \alpha_j\right) e_j\right\|$$

$$\leq \sum_{i=1}^{K} |\alpha_j^{(n)} - \alpha_j| \, \|e_i\|$$

$\downarrow$ as $n \longrightarrow \infty$

0

$$x_n \longrightarrow x \Longleftarrow$$

$\{T_n\}$ is a sequence of Operators in $B(X, Y)$

$$T_n \longrightarrow T, \quad T \in B(X, Y)$$

1. $T_n \longrightarrow T$ uniformly operator convergence

If $\|T_n - T\| \longrightarrow 0$ as $n \longrightarrow \infty$

(2) $T_n(x)$ converges strongly in Y iff
if $\lim_{n\to\infty} \|T_n(x) - y^*\| = 0$

Weak and strong convergences collapses for Bounded linear functionals as $\dim(Y = \mathbb{K}) = 1$

(3) $T_n(x)$ converges weakly in Y iff

$$g(T_n(x)) \longrightarrow g(Tx)$$

$$(f_n) \in X'$$

$(f_n) \in X'$ $\quad f_n \xrightarrow{w^*} f$ (weak* convergence)

$$f_n(x) \longrightarrow f(x) \;\; \forall x \in X.$$

$T: X \longrightarrow Y$ (continuous map)

(1) $\forall$ open subsets B of Y
$T^{-1}(B)$ is open in X.

Preimage of set B

$$T^{-1}(B) = \{x \in X \mid Tx \in B\}$$

Also, we can define image of a set

Let $A \subseteq X$, $\quad T(A) = \{Tx : x \in A\}$

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = \sin x$$

$A = (0, 2\pi) \longrightarrow$ open

$f(A) = [-1, 1] \longrightarrow$ closed or not-open.

In general, a continuous map can't take open set into open set.

Under what conditions, does a map takes open subsets to open subsets?

<u>Def</u>: $T: D(T) \subset X \longrightarrow Y$ is said to be an open map if $\forall$ open subset A of $D(T)$, $T(A)$ is open in Y.

$$f(0,1) = (\alpha, \beta)$$

$T: X \xrightarrow{\text{onto}} Y$. T is linear.

and T is bounded. Under what condition is it an open map?

X, Y are $\boxed{\text{Banach spaces.}}$

Page 262 No. 1

If $x_n \in C[a,b]$ and $x_n \xrightarrow{w} x \in C[a,b]$

Show that (x_n) is pointwise convergent on $[a,b]$

i.e. $(x_n(t))$ converges for all $t \in [a,b]$.

$$T_{t_0} : C[a,b] \longrightarrow \mathbb{K}$$
$$T_{t_0} \in C[a,b]'$$
$$T_{t_0}(x) = x(t_0).$$

$$x_n \xrightarrow{w} x$$
$$T_{t_0}(x_n) \longrightarrow T_{t_0}(x), \quad T_{t_0} \in C'[a,b]$$
$$x_n(t_0) \longrightarrow x(t_0).$$

Proof of open mapping thm:

$T : D(T) \subset X \longrightarrow Y$ is bounded linear map, X and Y are Banach spaces, T is onto. To show that $\forall$ open set A in X, $T(A)$: image set is open in Y.

i.e. $\forall y \in T(A), \exists r > 0$ s-t. $B(y, r) \subset T(A)$

Proof:

$\forall y \in T(A) \Rightarrow \exists x \in A,\ Tx = y$. Now consider the set $A - x$ which is open with centre at the origin. Take its radius to be r_1. (The open ball around the origin)

Define $k = \frac{1}{r_1}$. Then $k(A-x)$ contains a unit Ball $B(0;1)$

i.e. $B(0;1) \subset K(A-x) \subset X$.

Lemma: (Open Unit Ball) A Bounded linear operator T from a Banach space X onto a Banach space Y has the property that the image $T(B(0,1))$ contains an open unit ball about $0 \in Y$.

$$T(k(A-x)) = k\,T(A-x) = k\,(T(A) - Tx) \qquad (0)$$

$T(K(A-x))$ contains an open ball around origin, and so does $T(A) - Tx$.

Hence $T(A)$ contains an open ball about $Tx = y$

But since y is arbitrary, $T(A)$ is open.

Closed Graph Theorem

Let X and Y be Banach spaces.

$T: D(T) \subset X \longrightarrow Y$ is a closed linear map., where $D(T) \subset X$. Then if $D(T)$ is closed in X, then T is bounded.

Proof:

First, To show that $X \times Y$ is Banach with the norm $\|(x,y)\| = \|x\| + \|y\|$

Let $z_n = (x_n, y_n)$ be a cauchy sequence in $X \times Y$. Then for $\epsilon > 0$ $\exists$ N s.t.

$\|z_n - z_m\| = \|(x_n - x_m, y_n - y_m)\| = \|x_n - x_m\| + \|y_n - y_m\| < \epsilon$, $(m, n > N)$

which means

$$\|x_n - x_m\| < \epsilon$$
$$\|y_n - y_m\| < \epsilon \quad (m, n > N)$$

Hence (x_n) and (y_n) are cauchy sequence in X and Y

And X and Y are complete metric spaces.

$$\Rightarrow \quad x_n \longrightarrow x \in X$$
$$y_n \longrightarrow y \in Y$$

i.e. $z_n \longrightarrow z = (x, y)$

$$\|z_n - z_m\| = \|x_n - x_m\| + \|y_n - y_m\| < \epsilon$$

Let $m \to \infty$

$\Rightarrow$ we have $\|z_n - z\| < \epsilon$

$\Rightarrow$ $X \times Y$ is complete

(Projection map)

Let
$$P : X \times Y \longrightarrow D(T)$$
$$(x, y) \longmapsto x$$

$$P : G(T) \subset X \times Y \longrightarrow D(T)$$
$$(x, Tx) \longmapsto x$$

P is linear

$$P(\alpha(x, Tx) + \beta(y, Ty)) = P(\alpha x + \beta y, \alpha Tx + \beta Ty) = \alpha x + \beta y$$
$$= \alpha P(x, Tx) + \beta(y, Ty)$$

P is bijective

$$P(x, Tx) = P(x_1, Tx_1)$$

$$\Rightarrow x = x_1$$

i.e. $(x, Tx) = (x_1, Tx_1)$

one-one

Onto $\forall y \in D(T)$, $\exists (y, Ty)$ in $G(T)$
S.T. $P(y, Ty) = y$.

To show that P is unbounded
$$\|P(x, Tx)\| = \|x\| \leq \|Tx\| + \|x\| = \|(x, Tx)\|$$

$\|Tx\| \leq \alpha \|x\|$ → Definition of Boundedness.

$\Rightarrow$ P is Bdd, & $\|P\| = 1$

$P^{-1} : D(T) \longrightarrow G(T)$
is bounded by bounded inverse thm.

$$x \longmapsto (x, Tx)$$

$T : X \longrightarrow Y$
$\exists\, b$ S.T. $\|Tx\| \leq b\|x\|$

$\|(x, Tx)\| \leq b\|x\|$
for some b and $\forall x \in D(T)$.
Hence T is bdd because
$$\|Tx\| \leq \|x\| + \|Tx\| = \|(Tx, x)\| \leq b\|x\| \quad \forall x \in D(T)$$

Example: Give an example of a closed linear operator which is not bounded.

$$T : D(T) \longrightarrow C[0,1]$$
$$x \longmapsto x'$$

$D(T)$ is the subspace of the functions $x \in X$ which have continuous derivatives.

Let $(x_n) \in D(T)$ with $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$.
To show $y = Tx$.

Proof: $Tx_n = x_n'$
$x_n' = y$
$$\int_0^t y(\tau)\, d\tau = \int_0^t \lim_{n\to\infty} x_n'(\tau)\, d\tau$$

$$= \lim_{n\to\infty} \int_0^t x_n'(\tau)\, d\tau$$

$$= \lim_{n\to\infty} x_n(\tau)\Big|_0^t$$

$$x(t) - x(0) + \int_0^t y(\tau)\, d\tau = x(t) - x(0).$$

$x \in D(T)$ and $x' = y$

Banach-Steinhaus Theorem

Let X be a Banach Space. Y be a normed space and (F_n) is a sequence in $B(X,Y)$ such that, the sequences $(F_n(x))$ converges in Y $\forall x \in X$.

Space of bounded linear maps from X to Y

For $x \in X$, let $F(x) = \lim_{n\to\infty} F_n(x)$.

Then show that $F \in B(X,Y)$.

Proof: To show that F is linear.

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$$

By definition, $F(\alpha x + \beta y) = \lim_{n\to\infty} F_n(\alpha x + \beta y)$

As F_n is linear

$$= \lim_{n\to\infty} [\alpha F_n(x) + \beta F_n(y)]$$

$$= \lim_{n\to\infty} \alpha F_n(x) + \lim_{n\to\infty} \beta F_n(y)$$

$$= \alpha \lim_{n\to\infty} F_n(x) + \beta \lim_{n\to\infty} F_n(y)$$

$$= \alpha F(x) + \beta F(y)$$

Hence F is linear.

Now for each $x \in X$, the sequence $\{\|F_n(x)\|\}$ converges to $\|F(x)\|$
(Because $\|\cdot\|$ is a continuous function)

$$F_n(x) \longrightarrow F(x)$$

$$\|F_n(x)\| \longrightarrow \|F(x)\|$$

i.e. $\{\|F_n(x)\|\}$ converges to $F(x)$.

and hence bounded for every $x \in X$. So $\{\|F_n\|_{n=1,2,\dots}\}$ is bounded by uniform boundedness principle.

If $\|F_n\| \leq \alpha \quad \forall\, n$, then

$$\|F_n(x)\| \leq \|F_n\|\,\|x\| \leq \alpha\,\|x\|$$, letting $n \to \infty$

$$\|F(x)\| \leq \alpha\,\|x\| \quad \forall\, x \in X.$$

F is bdd.

Hahn Banach Theorem in Normed Spaces

Let f be a bounded linear functional on a a subspace Z of a normed space X. Then there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f to X and has the same norm,

$$\|\tilde{f}\|_X = \|f\|_Z$$

$$\|\tilde{f}\|_X = \sup_{\substack{x \in X \\ \|x\|=1}} |\tilde{f}(x)| , \quad \|f\|_Z = \sup_{\substack{x \in Z \\ \|x\|=1}} |f(x)|$$

and $\|f\|_Z = 0$ in the trivial case $Z = \{0\}$.

Proof:

Let $0 \neq a \in X$, then $\exists\, f \in X'$, such that

$f(x) = \|x\|$ and $\|f\| = 1$, $f|_Y = g$. , $f(a) = g(a) = \|a\|$

$\|f\| = 1$

$\|f\| = \sup |f(a)| = 1$

$\|a\| = 1$

Let $Y = \text{span}\{a\}$, which is a subspace of X.

let $g : Y \longrightarrow \mathbb{K}$ be given by

$g(ka) = k\|a\|$, g is linear IP

IP: $g(ka + k_1 a) = k\,g(a) + k_1\,g(a)$

$\Rightarrow g((k+k_1)a) = (k+k_1)\|a\|$

$= k\|a\| + k_1\|a\|$

g is bounded linear

$$\|g\| = \sup_{\|x\|=1} |g(x)|$$

For every $x \in X$ (normed space) we have

$$\|x\| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|}, \qquad \boxed{\text{Hence if } x_0 \text{ is such that } f(x_0) = 0 \ \forall f \in X', \text{ then } x_0 = 0.}$$

$$\|x_0\| = \sup_{f \in X'} \frac{|f(x)|}{\|f\|} = 0$$

$$\Rightarrow x_0 = 0$$

$$\sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|} \geq \frac{|\hat{f}(x)|}{\|\hat{f}\|} = \frac{\|x\|}{1} = \|x\|$$

1

$$\sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|} \leq \|x\| \quad \text{— } 2$$

7th Nov

For $0 \neq a \in X$ (normed spaces), $\exists f \in X'$ s.t. $f(a) = \|a\|$ and $\|f\| = 1$

Ex: Let X be a normed space and $E \subset X$. Then E is bounded in X iff $x'(E)$ is bounded in K $\forall x' \in X'$.

Proof: Suppose E is bounded i.e. $\forall x \in E$, $\|x\| \leq \alpha$, $\alpha \in \mathbb{R}$

To show that $x'(E)$ is bounded.

if $x' \in X'$, $|x'(x)| \leq \|x'\| \, \|x\| \leq \alpha \|x'\|$ $\forall x \in E$

$\Rightarrow \|x'(x)\| \leq \alpha \|x'\|$ $\forall x \in E$

Converse

($\Leftarrow$) Suppose $x'(E)$ is bounded in $\mathbb{K}$ $\forall x' \in X'$.

To show that E is bounded in X.

Now for a fixed $x \in E$, let $F_x : X' \xrightarrow{\text{evaluation map}} \mathbb{K}$ be given by

$$F_x(x') = x'(x)$$

Hence $F_x \in X''$.

$$F_x(\alpha x' + \beta y') = \alpha x'(x) + \beta y'(x)$$
$$= \alpha F_x(x') + \beta F_x(y')$$

$$|F_x(x')| = |x'(x)| \leq \|x'\| \, \|x\|$$

$\|F_x\| \leq \|x\|$ but $\exists x' \in X'$ s.t. $x'(x) = \|x\|$

i.e. $\|F_x\| = \|x\|$

Now, consider the subset $\mathcal{Y} = \{F_x : x \in E\} \subset B(X', \mathbb{K}) = X''$

$J : X \to X''$ $\rightarrow$ Canonical map

$x \longrightarrow F_x$. (Evaluation functional)

$F_x(x') = x'(x)$

When this map is onto, X is said to be Reflexive

This J is an isometry.

Also X' is Banach regardless of the fact whether X is Banach or Not.

For each $x' \in X'$,

$$\sup\{F_x(x') : F_x \in \mathcal{Y}\} = \sup\{x'(x) : x \in E\} < \infty$$

By assumption, Hence by uniform boundedness Principle.

$$\sup\{\|Fx\| : Fx \in \mathcal{F}\} = \sup\{\|x\| : x \in E\} < \infty$$

i.e. E is bounded.

For $x' \in X'$, we have

$$\|x'\| = \sup\{|x'(x)| \; \forall x \in X, \|x\| \leq 1\}.$$

And for all $x \in X$, we have

$$\|x\| = \sup\{x'(x) : x' \in X, \|x'\| \leq 1\}$$

Moreover

$$x_1' = x_2' \iff x_1'(x) = x_2'(x) \quad \forall x \in X.$$

$$x_1 = x_2 \iff x'(x_1) = x'(x_2) \quad \forall x' \in X'.$$

If X' is separable, then X is also separable.

$C(T)$, $C_0(T)$, $C_c(T)$ $\subsetneq C(T)$

Space of cont. bounded functionals

$$C_0(T) = \{x \in C(T) : \text{For every } \epsilon > 0, \text{ there is a compact set } E \subset T \text{ S.T. } |x(t)| < \epsilon \quad \forall t \notin E\}.$$

$$C_c(T) = \{x \in C(T) : \text{there is a compact set } E \subset T \text{ S.T. } x(t) = 0 \quad \forall t \notin E\}$$

8th November $\star$ Not in Exam.

1. Zorn's lemma
2. Axiom of Choice
3. $\prod_{\alpha} X_\alpha \neq \phi \quad \alpha \in I.$ → Cartesian Product.

Let R be a binary relation on X

$R \subset X \times X.$

R is said to be a Partial Ordered Relation

if

i) $(a,a) \in R \quad \forall a \in X$ Reflexivity $\quad a R a$

b) $(a,b) \in R$ & $(b,a) \in R \Rightarrow a = b$ Anti-symmetry $\quad aRb, bRa \Rightarrow a=b$

c) $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$ Transitivity. $\quad aRb, bRc \Rightarrow aRc.$

$(X, \leq)$ is a partial ordered set.

$Y \subseteq X, x \in X, y \leq x \quad \forall y \in Y.$

We say x is an upper bound of Y.

Maximal element: An element $m \in X$ is said to be a maximal element if $\forall x \in X, m \leq x.$

$\Rightarrow m = x$

$\mathbb{R}$ has no maximal element

for sets

X is a maximal elt if $X \subset A$ $\forall A \in P(X), A \subset X \; \forall A, A = X.$

$m \leq x \quad \forall x \in \mathbb{R}$

$\Rightarrow m = x \quad , x = m+1$

$m+1 \in \mathbb{R}.$

$a, b \in X$

if $a \leq b$ or $b \leq a$

$\Rightarrow a, b$ are comparable

A totally ordered set or a chain is a po-set X (partial ordered set.) if every two elements are comparable.

In other words, a chain is a partial ordered set that has no incomparable pair of elements.

Zorn's Lemma

Let $M \neq \phi$ be a partially ordered set suppose every chain $C \subset M$ has an upper bound in M, then M has atleast one maximal element.

Every non empty Vector Space has a Hamel Basis.

Proof: Let $\boxed{M}$ be the set of all linearly independent subsets of X.
Since $X \neq \{0\}$, it has an element $x \neq S$

$\{x\}$, $\{x\} \in M$, $M \neq \phi$, set $\leq = C$

Every chain $C \subset M$ has an upper bound i.e. the union of all subsets of X, which are elements of C.
By Zorn's Lemma M has a maximal element 'B'.

Let $Y = \text{span}\{B\}$. is a subspace of X.

TP: Then $Y = X$.

Suppose $Y \neq X$

Then $B \cup \{z\}$, $z \in X$, $z \notin Y$
would be linearly independent set containing B
as a proper subset, contrary to the maximality of B.