Fatou's lemma :- E meas.

$(f_n) \geq 0$, (f_n) meas. Then

$$\int_E \liminf f_n \leq \liminf \int_E f_n$$

In addition $f_n \longrightarrow f$ then, "$\liminf f_n$" should be replaced by f,

$$\int_E f \leq \liminf \int_E f_n$$

Proof :- Let h be a bounded measurable function supported on E s.t. $h \leq \liminf f_n$. To show

(*) $\int h \leq \liminf \int_E f_n$

$\Downarrow$

(**) Taking supremum over all h s.t. $h \leq \liminf f_n$

$$\int \liminf f_n = \sup_{h \leq \liminf f_n} \int h$$

$$\Rightarrow \int_E \liminf f_n \leq \liminf \int_E f_n .$$

To establish (**),

$$h_n = \min \{ h, f_n \}$$

Since, $h \leq \liminf f_n$, therefore,

$h_n \longrightarrow h$ as $n \longrightarrow \infty$.

$\min \{ h, \liminf f_n \} = h$

We apply BCT,

$$\lim_{n \to \infty} \int_E h_n = \int_E h$$

$$\Rightarrow \int_E h = \lim_{n \to \infty} \int_E h_n$$

$$\leq \liminf \int_E f_n$$

$(h_n \leq f_n)$ ✓

From (**), we get the complete proof.

Corollary :- Suppose f is a nonnegative measurable function & $\{f_n\}$ a sequence of nonnegative functions with

$$f_n(x) \le f(x) \text{ for almost every } x.$$

Then, $\lim_{n \to \infty} \int f_n = \int f$.

Proof :- $f_n(x) \le f(x)$

Monotonicity, $\int f_n \le \int f \quad \forall n.$

$$\Rightarrow \quad \limsup \int f_n \le \int f. \quad \text{— (i)}$$

From Fatou's lemma,

$$\int f \le \liminf \int f_n \quad \text{— (ii)}$$

(i) & (ii) $\Rightarrow$

$$\limsup \int f_n \le \int f \le \liminf \int f_n \quad \text{— (iii)}$$

$$\liminf \int f_n \le \limsup \int f_n \quad \text{— (iv)}$$

$\Rightarrow$ (iii) & (iv) $\Rightarrow$ $\liminf \int f_n$

$= \limsup \int f_n$

$= \int f$

$\Rightarrow \lim \int f_n = \int f$ $\square$.

(A) $f_n \nearrow f$ if

$f_n(x) \leq f_{n+1}(x)$ a.e. x, $n \geq 1$,

$\lim f_n(x) = f(x)$ a.e. x.

(B) $f_n \searrow f$ if

$f_n(x) \geq f_{n+1}(x)$ a.e. x, $n \geq 1$

& $\lim\limits_{n \to \infty} f_n(x) = f(x)$ a.e. x.

Monotone Convergence theorem (MCT) :-

Suppose $\{f_n\}$ is a sequence of non-negative measurable functions, with $f_n \nearrow f$. Then,

$$\lim_{n\to\infty} \int f_n = f \quad , \quad \underline{f_n \le f} \checkmark$$

$\square$ $\quad f_n \searrow f \qquad f_1 \ge f_2 \ge f_3 \ge \cdots$

$$g_n = f_1 - f_n$$

$$g_n \geq 0, \quad g_n \nearrow f_1 - f$$

$$\lim_{n\to\infty} \int g_n = \int f_1 - f = \int f_1 - \int f$$

$$\Rightarrow \lim_{n\to\infty} \left(\int f_1 - \int f_n \right) = \int f_1 - \int f$$

$$\Rightarrow \lim_{n\to\infty} \int f_n = \int f. \qquad \square \checkmark$$

<u>Application</u>

① Let $\sum_{k=1}^{\infty} a_k(x)$ be a series of meas. functions, where $a_k(x) \geq 0 \quad \forall k \geq 1$.

<u>Then</u>, $\int \left(\sum_{k=1}^{\infty} a_k(x) \right) dx = \underline{\sum_{k=1}^{\infty} \int a_k(x)\, dx}$

If $\sum_{k=1}^{\infty} \int a_k(x)\, dx$ is finite,

then, $\sum_{k=1}^{\infty} a_k(x)$ is convergent for a.e. x.

$$\left(\sum_{k=1}^{\infty} a_k(x) < \infty \text{ for a.e. } x \right)$$

$$\Downarrow$$

$$m\left\{x : \sum_{k=1}^{\infty} a_k(x) = \infty\right\} = 0$$

Proof :- Let $f_n(x) = \sum_{k=1}^{n} a_k(x)$ &

$$f(x) = \sum_{k=1}^{\infty} a_k(x).$$

i) Each f_n is meas.

ii) $f_n \nearrow f$ & $f_n \to f$

$$\int f_n = \int \sum_{k=1}^{n} a_k(x) = \sum_{k=1}^{n} \int a_k(x).$$

By MCT,

$$\lim_{n\to\infty} \int f_n = \int f$$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} \int a_k(x)\,dx = \int \sum_{k=1}^{\infty} a_k(x)$$

$$\Rightarrow \underline{\sum_{k=1}^{\infty} \int a_k(x)\,dx} = \underline{\int \sum_{k=1}^{\infty} a_k(x)} \quad \boxed{***}$$

<u>Part II</u>,

If, $\sum \int a_k$ is finite, i.e. $< \infty$,

then, $\boxed{***} \Rightarrow$

$$\int \left(\sum_{k=1}^{\infty} a_k(x) \right) dx < \infty$$

$$\Rightarrow \sum_{k=1}^{\infty} a_k(x) < \infty \quad \text{a.e.}$$