Ring Theory

lecture 27

In $k[x]$ the maximal ideals are of the form $(f(x))$ where $f(x)$ is irreducible over $k[x]$.

The maximal ideals of $\mathbb{C}[x]$ are of the form $\mathfrak{m}_a = (x-a)$ where $a \in \mathbb{C}$.

There is a 1-1 correspondence between the pts in $\mathbb{C}$ and the set of maximal ideals of $\mathbb{C}[x]$

$$\mathbb{C} \longleftrightarrow \mathbb{C}[x]$$

$$a \longmapsto \mathfrak{m}_a = (x-a).$$

$$\longleftarrow$$

Thm [Hilbert's Nullstellensatz]

The maximal ideals of the poly ring $R = \mathbb{C}[x_1, \ldots, x_n]$ are in 1-1 correspondence with the pts in $\mathbb{C}^n$. A pt $\underline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ corresponds to the kernel of the map

$$s_{\underline{a}} : \mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}$$

$$f(x_1, \ldots, x_n) \longmapsto f(a_1, \ldots, a_n).$$

The kernel of s_a is a maximal ideal and is gen by

$$(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n).$$

Remark. In order to prove the above thm we need to show that every maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]$ is of the form

$$m_{\underline{a}} = (x_1 - a_1, \ldots, x_n - a_n).$$

where $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$.

Defn. Let V be a subset of $\mathbb{C}^n$. If V can be defined as the set of common zeros of a finite number of polys in n-variables then V is called an alg variety.

$$V = Z(f_1, \ldots, f_r).$$

where $Z(f_1, \dots, f_r)$ denotes the set of common zeros of the polys $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_n]$

<u>Example</u>. (1) $f = ax + by + c = 0$ in $\mathbb{C}^2$. $Z(f)$ is a complex line in $\mathbb{C}^2$. Thus $Z(f)$ is a variety.

(2) Every pt $(a, b) \in \mathbb{C}^2$

$$(a, b) = Z(x - a, y - b).$$

is a variety..

$$f_1 = x - a = 0$$
$$f_2 = y - b = 0$$

Thm Let $f_1, \dots, f_r$ be polys in $\mathbb{C}[x_1, \dots, x_n]$ and $V = Z(f_1, \dots, f_r)$. Let $I = (f_1, \dots, f_r)$ be the ideal of $\mathbb{C}[x_1, \dots, x_n]$. The maximal ideals of the quotient ring $R = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{I}$ are in bijective correspondence with the pts of V.

Pf: The maximal ideal of R corresponds to those maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$ which contains I. Now a maximal ideal of $\mathbb{C}[x_1, \dots, x_n]$ is of the form

$$\mathfrak{m}_{\underline{a}} = (x_1 - a_1, \ldots, x_n - a_n)$$

where $\underline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$.

$$\mathfrak{m}_{\underline{a}} \supseteq \underline{I} = (f_1, \ldots, f_r).$$

$\Rightarrow$ $f_i \in \mathfrak{m}_{\underline{a}}$

iff $f_i(\underline{a}) = 0$. $\forall i$.

$\Rightarrow$ $\underline{a} \in V$.

$$f_i = g_1 (x_1 - a_1) + g_2 (x_2 - a_2) + \cdots + g_n (x_n - a_n)$$

where $g_i \in \mathbb{C}[x_1, \ldots, x_n]$.

$$\underline{f_i}(\underline{a}) = g_1(\underline{a})(a_1 - a_1) + g_2(\underline{a})(a_2 - a_2) + \cdots + g_n(\underline{a})(a_n - a_n) = 0.$$

Remark 1. Let R be a ring. Every ideal I of R which is not the unit ideal is contained in a maximal ideal.

Remark 2. The only ring R having no maximal ideals is the zero ring.

Cor. Let $f_1, \ldots, f_r$ be polys in $\mathbb{C}[x_1, \ldots, x_n]$. If the system eqn. $f_1 = 0, \ldots, f_r = 0$ has no soln in $\mathbb{C}^n$ then $1 = \sum g_i f_i$

i.e. $(f_1, \ldots, f_r) = \mathbb{C}[x_1, \ldots, x_n]$.

Pf: $V = Z(f_1, \dots, f_n) = \phi$

which means there is no maximal ideal containing $I = (f_1, \dots, f_n)$

Therefore I is the unit ideal.

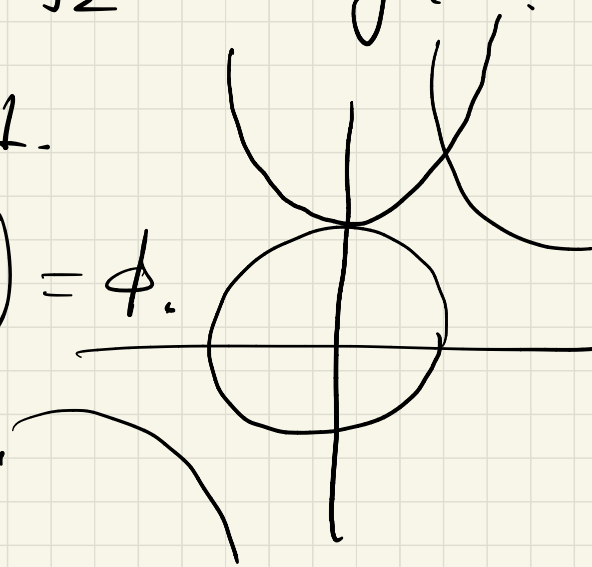
Hence $1 \in I$

$$\Rightarrow 1 = \sum g_i f_i$$

Example. Consider the ideal gen by

$f_1 = x^2 + y^2 - 1$, $f_2 = x^2 - y + 1$.

and $f_3 = xy - 1$.



$$Z(f_1, f_2, f_3) = \phi.$$

ie $(f_1, f_2, f_3) = 1$.