# Lecture 14

**Def:-** Let $E \subseteq \mathbb{R}$ be a measurable set. Then a function $f: E \to \mathbb{R} \cup \{\pm\infty\}$ is said to be a **Borel measurable function** if for each $\alpha \in \mathbb{R}$,

$$\{x \in E \,/\, f(x) > \alpha\} \text{ is a Borel set.}$$

**Remark:-** Every Borel measurable funtion is measurable.

**Example:-** Any Continuous funtion defined on a measurable set, is a Borel measurable funtion.

**proof:-**

$$\{x \in E \,/\, f(x) > \alpha\} = f^{-1}((\alpha, \infty))$$

is an open set $\in \mathcal{B}$.

**Theorem:-** Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$ be a funtion, where $E$ is a measurable set. Then the following statements are equivalent.

(1) $f$ is Borel measurable.

(2) For $\alpha \in \mathbb{R}$, $\{x \in E \,/\, f(x) \geq \alpha\}$ is a Borel set.

(3) For $\alpha \in \mathbb{R}$, $\{x \in E \,/\, f(x) < \alpha\}$ is a Borel set

(4)

(4) For $\alpha \in \mathbb{R}$, $\{x \in E \,/\, f(x) \leq \alpha\}$ is "

proof: EXERCISE.

Theorem:- Let $f, g$ be Borel measurable functions defined on a measurable set $E$. Then $f+c$, $f \pm g$, $fg$ are Borel measurable.

Theorem:- Let $\{f_n\}$ be a sequence of Borel measurable functions. Then

(i) $\sup\limits_{1 \leq i \leq n} (f_i)$ is Borel measurable. for any $n$.

(ii) $\inf\limits_{1 \leq i \leq n} (f_i)$ " " ".

(iii) $\sup(f_n)$ " "

(iv) $\inf(f_n)$ " "

(v) $\limsup f_n$ " "

(vi) $\liminf (f_n)$ " ".

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Def:- Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function. Then the <u>essential supremum</u> of $f$ is defined as

$$\operatorname{esssup}(f) := \inf \{ \alpha \in \mathbb{R} \mid f \leq \alpha \text{ a.e} \}.$$

$f \leq \alpha$ a.e means that $\{ x \in E \mid f(x) \nleq \alpha \}$ has measure 0.

<u>Examples</u> $f = \chi_{[0,1]} : \mathbb{R} \to \mathbb{R}$.

$$\begin{aligned} \operatorname{esssup}(f) &= \inf \{ \alpha \in \mathbb{R} \mid f \leq \alpha \text{ a.e} \} \\ &= \inf \{ \alpha \in \mathbb{R} \mid \chi_{[0,1]} \leq \alpha \text{ a.e} \} \\ &= 1 \end{aligned}$$

$\chi_{[0,1]} \leq 0$ a.e

$\boxed{\chi_{[0,1]} \leq 1 \text{ a.e}}$

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<u>proposition:</u> Let $f$ be a measurable function. Then $f \leq \operatorname{esssup}(f)$ a.e.

<u>proof:</u> To show: $\{ x \in E \mid f(x) \nleq \operatorname{esssup}(f) \}$ has measure 0.

i.e, $\{ x \in E \mid f(x) > \operatorname{esssup}(f) \}$ has measure zero.

Suppose $\operatorname{esssup}(f) = +\infty$. Then nothing to prove.

Suppose $\operatorname{esssup}(f) = -\infty$. Then by def,

$$f \leq n \text{ a.e } \forall n \in \mathbb{Z}.$$

$$\Rightarrow f \leq -\infty \text{ a.e.}$$

Suppose $\operatorname{esssup}(f)$ is a finite number.

Def $E_n = \left\{ x \in E \;\middle|\; f(x) > \frac{1}{n} + \operatorname{esssup}(f) \right\}$.

$$\operatorname{esssup}(f) = \inf \{ \alpha \in \mathbb{R} \mid f \leq \alpha \text{ a.e} \}.$$

$\Rightarrow E_n$ has measure $0$ $\forall n$. (by using inf. property)

Then we have

$$\bigcup_{n=1}^{\infty} \left\{ x \in E \;\middle|\; f(x) > \frac{1}{n} + \operatorname{esssup}(f) \right\} = \left\{ x \in E \;\middle|\; f(x) > \operatorname{esssup}(f) \right\} = F \text{ (say)}$$

$$\Rightarrow \underset{\substack{\parallel \\ m(F)}}{m^*(F)} \leq \sum_{n=1}^{\infty} m^*\left( \left\{ x \in E \;\middle|\; f(x) > \frac{1}{n} + \operatorname{esssup}(f) \right\} \right)$$

$$\leq \sum_{n=1}^{\infty} 0 = 0.$$

(we write $m$ for measurable sets only)

$$\Rightarrow m^*(F) = 0.$$

**Proposition:** Let $f, g$ be measurable functions defined on a measurable set $E$. Then

$$\operatorname{ess\,sup}(f+g) \leq \operatorname{ess\,sup}(f) + \operatorname{ess\,sup}(g).$$

**Proof:** We have

$$f+g \leq \underbrace{\operatorname{ess\,sup}(f) + \operatorname{ess\,sup}(g)}_{= \beta \text{ (say)}} \quad \text{a.e}$$

$$\Rightarrow \underbrace{\operatorname{ess\,sup}(f+g)}_{= \inf\{\alpha \in \mathbb{R} \,/\, f+g \leq \alpha \text{ a.e}\}} \leq \operatorname{ess\,sup}(f) + \operatorname{ess\,sup}(g).$$

$$\inf_{\beta \in}\{\alpha \in \mathbb{R} \,|\, f+g \leq \alpha \text{ a.e}\} \leq \beta.$$

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**Def:** Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function. Then the <u>essential infimum</u> of $f$ is defined as

$$\operatorname{ess\,inf}(f) := \sup\{\alpha \in \mathbb{R} \,/\, f \geq \alpha \text{ a.e}\}.$$

**Proposition:** $\operatorname{ess\,sup}(f) = -\operatorname{ess\,inf}(-f)$.

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**Def:** Let $f$ be a measurable function & $\operatorname{ess\,sup}(|f|) < \infty$. Then $f$ is said to be <u>essentially bounded</u>.

$$\sup(f_n)(x) = \sup\left(\{ f_n(x) \mid n \in \mathbb{N} \}\right).$$