Linear Algebra

Lecture 13

Composition of linear transformation

Let $T: V \to W$ and $U: W \to Z$ be linear transformations and $\beta_1 = \{v_1, \dots, v_n\}$, $\beta_2 = \{w_1, \dots, w_m\}$ and $\beta_3 = \{z_1, \dots, z_p\}$ are ordered bases for V, W and Z respectively.

Let $A = [U]$

$B = [T]$

Composition of these linear fns:

$$UT : V \to Z$$

$$v_i \mapsto UT(v_i)$$

$$\begin{aligned}
UT(v_j) &= U(T(v_j)) \\
&= U\left(\sum_{k=1}^{m} B_{kj} w_k \right) \\
&= \sum_{k=1}^{m} B_{kj} U(w_k) \\
&= \sum_{k=1}^{m} B_{kj} \left(\sum_{i=1}^{p} A_{ik} z_i \right)
\end{aligned}$$

$$= \sum_{i=1}^{p} \left(\sum_{k=1}^{m} A_{ik} B_{kj} \right) z_i$$

$$(UT)(u_j) = \sum_{i=1}^{p} C_{ij} z_i$$

where $\underbrace{C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}}$

is the matrix representing linear tranformation UT.

<u>Matrix multiplication</u>: For $A \in \mathbb{F}^{m \times n}$
$B \in \mathbb{F}^{n \times p}$

$$C = AB$$

where $C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$, $\quad i = 1, \ldots, m$, $\quad j = 1, \ldots, p$

Invertibility :

Definition. Let V and W be vector spaces over a field $\mathbb{F}$ and let $T: V \to W$ be a linear map. A function $U: W \to V$ is said to be an iverse of T if

$$TU = I_W \quad \text{and} \quad UT = I_V$$

If T has an inverse, then T is said to be invertible and inverse of T is denoted as T^{-1}.

Lemma: If T and U are invertible functions, then

1) $(TU)^{-1} = U^{-1}T^{-1}$

2) $(T^{-1})^{-1} = T$

$$\left.\begin{aligned}(TU)U^{-1}T^{-1} &= T(UU^{-1})T^{-1} \\ &= TIT^{-1} \\ &= TT^{-1} \\ &= I\end{aligned}\right\} \begin{aligned}&(TU)^{-1} \\ &= U^{-1}T^{-1}\end{aligned}$$

$$U^{-1}T^{-1}(TU) = I$$

$T : V \to W$

T is invertible if and only if T is 1-1 and onto.

$$\text{Rank}(T) + \text{nullity}(T) = \dim(V)$$

$$\Rightarrow \text{Rank}(T) = \dim(V) \longrightarrow 1\text{-}1$$

$$\begin{array}{c} \| \\ \dim(W) \end{array} \longrightarrow \text{onto}$$

<u>Lemma:</u>

Let V & W be vector spaces over $\mathbb{F}$ and $T : V \to W$ be a linear invertible transformation. Then $T^{-1} : W \to V$ is also linear.

<u>Pf:</u> Let $y_1, y_2 \in W$ and $c \in F$.

Since T is 1-1 and onto, $\exists$ unique x_1 & x_2 such that

$$x_1 = T^{-1}(y_1), \quad x_2 = T^{-1}(y_2)$$

$$T^{-1}(cy_1 + y_2) = T^{-1}(c\,T(x_1) + T(x_2))$$

$$= T^{-1}(T(cx_1 + x_2))$$

$$= cx_1 + x_2$$

$$= cT^{-1}(y_1) + T^{-1}(y_2)$$

$\Rightarrow T^{-1}$ is a linear function. ∎

<u>Theorem</u>: Let T be invertible linear transformation from V to W. Then V is finite dimensional if and only if W is finite dimensional. Moreover $\dim(V) = \dim(W)$.

Matrix representation of T^{-1}

Let V and W be finite dimensional vector spaces over $\mathbb{F}$ with ordered basis $B_1 = \{v_1, \dots, v_n\}$ and $B_2 = \{w_1, \dots, w_m\}$

Let $T: V \to W$ be linear. Then T is invertible iff $m = n$.

Let $T^{-1} \in \mathcal{L}(W, V)$ and let B be a matrix representation of T^{-1}.

$$T^{-1}(w_j) = \sum_{i=1}^{n} B_{ij} v_i \quad \text{for } j = 1, 2, \dots, n$$

Let T have the matrix representation A with respect to the listed ordered basis.

$$TT^{-1} = AB \qquad\qquad T^{-1}T = BA$$

TT^{-1} = identity linear transformation from $W \to W$, so $AB = I_n$.

$T^{-1}T$ = identity linear transformation from $V \to V$, so $BA = I_n$.

This inspires the definition of inverse of a square matrix.

<u>Isomorphism</u>:

Let V and W be vector spaces over $\mathbb{F}$. We say V is isomorphic to W if there exists a linear transformation $T: V \to W$ which is invertible. Such a linear transformation is called as isomorphism.

Examples of isomorphic vector spaces.

<u>Ex</u>: Let $V = \mathbb{R}^{n+1}$ and $W = \mathbb{P}_n(\mathbb{R})$

Ex: Let V & W be finite dimensional vector spaces over $\mathbb{F}$.

$\mathcal{L}(V, W)$ and $\mathbb{R}^{m \times n}$ and $\mathbb{R}^{n \times m}$

where $m = \dim(W)$

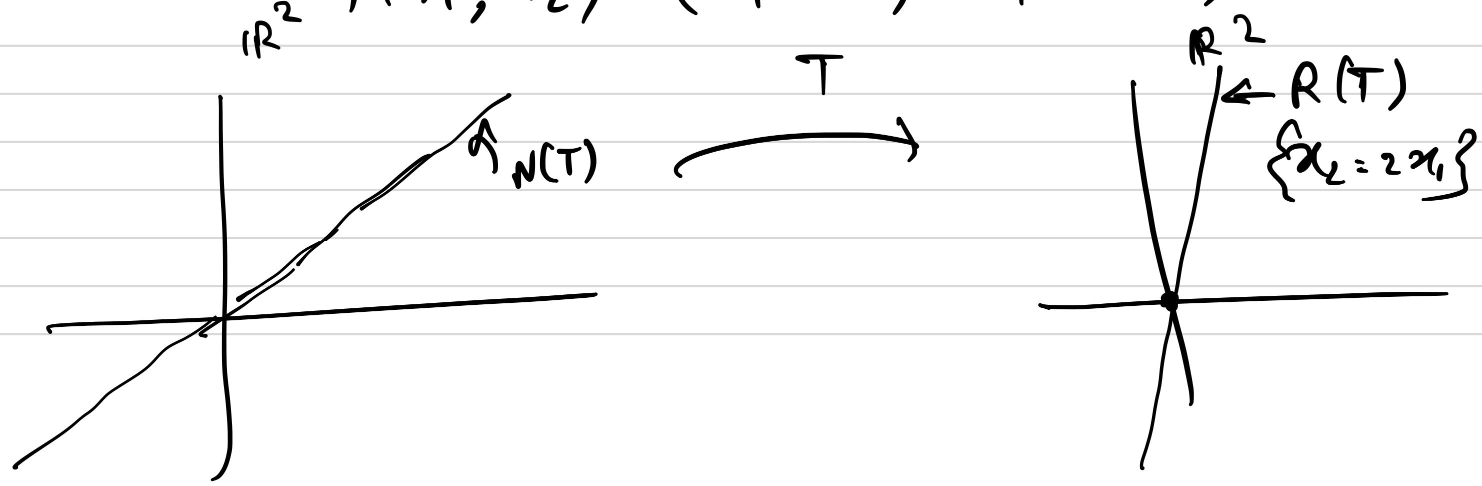
$n = \dim(V)$

<u>Ex</u>: Let V and W be finite dimensional vector spaces. Let T be a linear transformation from $V \to W$.

$V / N(T) \cong R(T)$ $\Big\}$ rank-nullity theorem.

<u>Example</u>: $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x_1, x_2) = (x_1 - x_2, 2x_1 - 2x_2)$$



Let $T : V \to Z$ be a linear transformation which is onto.

Define that map

$$\bar{T} : V/_{N(T)} \to Z$$

by $\bar{T}(\underbrace{v + N(T)}) = T(v)$

is any coset of $N(T)$

(a) Prove that $\bar{T}$ is well defined.

prove that for $v_1 + N(T) = v_2 + N(T)$

$$\bar{T}(v_1 + N(T)) = \bar{T}(v_2 + N(T))$$

$$T(v_1) = T(v_2)$$

(b) $\bar{T}$ is linear

$$\bar{T}\left[c(v_1 + N(T)) + (v_2 + N(T))\right]$$

$$= \bar{T}\left[cv_1 + v_2 + N(T)\right]$$

$$= T(cv_1 + v_2) = cT(v_1) + cT(v_2)$$

$$= c\bar{T}(v_1 + N(T)) + \bar{T}(v_2 + N(T))$$

(c) Is $\bar{T}$ an isomorphism??

$$\bar{T}(v_1 + N(T)) = \bar{T}(v_2 + N(T))$$

$$\Rightarrow \bar{T}((v_1 - v_2) + N(T)) = 0$$

$$\Rightarrow v_1 - v_2 \in N(T)$$

1-1

Take any element $z \in Z$.
Since T is onto, $\exists\, v \in V$ such that $T(v) = z$

$$\begin{array}{c} T(v) = z \\ \shortparallel \\ \bar{T}(v + N(T)) \\ \uparrow \\ V/N(T) \end{array}$$

onto

$$\begin{tikzcd} V \arrow[rr, "T"] \arrow[dr, "\eta"'] & & Z \\ & V/N(T) \arrow[ur, "\bar{T}"'] & \end{tikzcd}$$

$T = \bar{T}\eta$

Exercise.

onto linear

$$\begin{tikzcd} V \arrow[rr, "T"] \arrow[dr, "\eta"'] & & Z \\ & V/N(T) \arrow[ur, "\bar{T}"'] & \end{tikzcd}$$

$\bar{T}$ - isomorphism

Define $\eta : V \longrightarrow V/N(T)$

as $\eta(v) = v + N(T) \quad \forall v \in V.$

Prove: $T = \bar{T}\eta$ (The diagram commutes)

step 1: Show η is onto.

$R(\eta) = V/N(T)$

step: Composition $\bar{T}\eta$ is well-defined.

take $v \in V$

$\bar{T}\eta(v) = \bar{T}(\eta(v)) = \bar{T}(v + N(T))$

$= T(v)$