Date 10/10/2017

Lecture 20

We denote the partial sums of the Fourier series as

$$S_1 = -\frac{4}{\pi}\sin x, \quad S_2 = -\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3}\right),$$

$$S_3 = -\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right) \ldots$$

The typical plots of S_1, S_2, S_3, S_{14} are given by

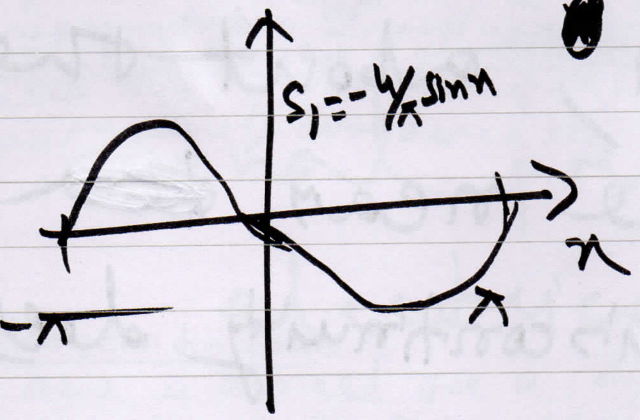

Fig 2:- (Plot of $S_1(x)$)

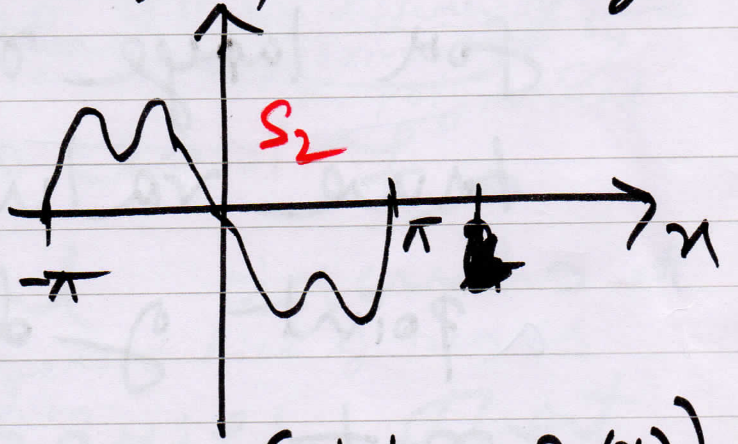

Fig 3:- (Plot of $S_2(x)$)

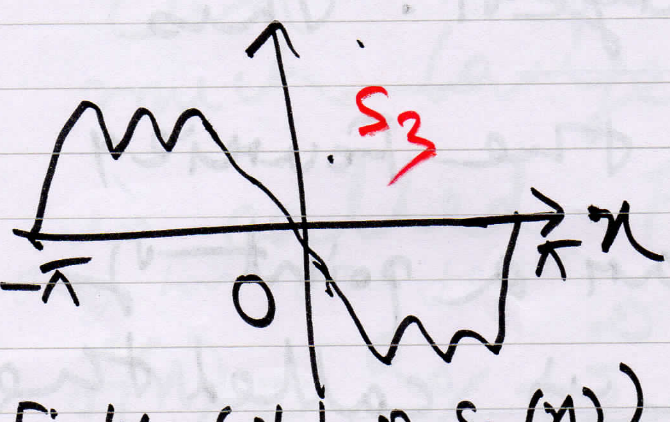

Fig 4:- (Plot of $S_3(x)$)

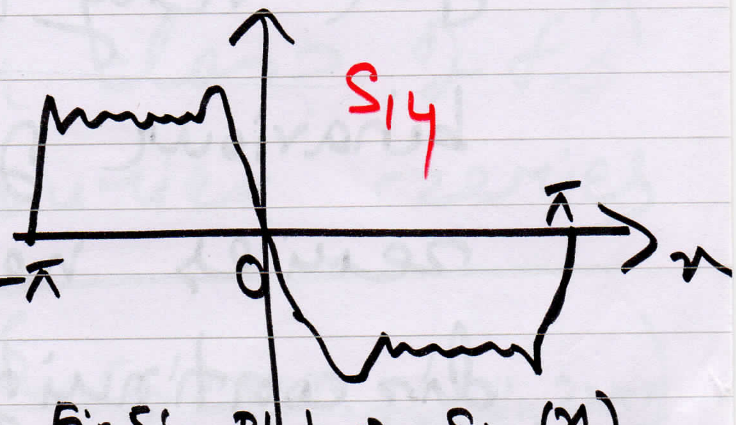

Fig 5:- Plot of $S_{14}(x)$.

It can be observed that the graph of $S_{14}(x)$ displays spikes near the discontinuity at $x = -\pi, 0, \pi$. (how?)

This oscillatory behaviour of the partial sums S_n for large n, about the true value near ~~the~~ a point of discontinuity does not smooth them out even for very large n. This behaviour of the Fourier series near a point of discontinuity is called the

$\boxed{\text{Gibbs phenomenon}}$.

Note:- $f(x)$ Recall that

a f^n, which is differentiable any no. of times at $x=a$ can be represented as a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

where, the co-efficients a_n are given by $a_n = \frac{1}{n!} f^n(a)$.

[So, just f^ns that are differentiable any no. of times have representation as a power series. This cond^n is pretty restrictive as any discontinuous f^n is not differentiable]

Thus the function $f(x)$ can be approximated by a polynomial.

The f^n frequently considered in signal processing, electrical circuits etc. are discontinuous. Thus there is a need for a different kind of series approximation of a given function.

– The type of series that can represent a much larger class of f^ns is called Fourier series.

They have the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right)$$

Non-periodic f^n.

A non-periodic f^n given on an interval can be extended to a periodic f^n outside that interval & then represented it by a Fourier series. We can sometimes extend it to an even or odd f^n & make the calculations simpler.

[EX Find the Fourier series for $f(x) = x^2$, $0 < x \le 2$ by extending the function to an even periodic function.]

$$f(x) = \sum f_n(\cancel{x})$$

(Concept of convergence of an infinite series of numbers)

Uniform Convergence

$$\sum x_n$$

$$a_1 + a_2 + \cdots \qquad \sum n, \quad \sum n^2, \ \ldots$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = l$$

Uniform Convergence

Suppose we have an infinite series $\sum_{n=1}^{\infty} u_n(x)$.

We define the R^{th} partial sum of the series i.e.

(to be the sum of the first R terms of the series)

$$S_R(x) = \sum_{n=1}^{R} u_n(x)$$

$$= u_1(x) + u_2(x) + \cdots + u_R(x) \longrightarrow (1)$$

Now, by defn this infinite series i.e. $\sum_{n=1}^{\infty} u_n(x)$ is said to converge to $f(x)$ in some interval if given any positive no.

ϵ, $\exists$ for each x in the interval a positive no. N such that
(number)

$$|S_R(x) - f(x)| < \epsilon,$$

whenever $R > N$. $\longrightarrow (2)$

$$|S_{R_1}(x) - f(x)| = d_1, \quad |S_{R_2}(x) - f(x)| = d_2, \quad - - \quad ;$$

$$|S_{N+1}(x) - f(x)| = 0.00001, \quad |S_{N+2} - f(x)| = 0.0000001 \ \text{etc.}$$

$$S_{R_1}(x), \ S_{R_2}(x), \ - - \ - - \ S_{N+1}(x), \ f(x), \ f(x) \Longrightarrow f(x)$$

$\longrightarrow f(x)$.

The no. N depends in general not only on ϵ but also on x.

we call $f(x)$ the sum of the series

(this is pointwise convergence)

An important case occurs when N depends on ϵ but not on the value of x in the interval.

In such case, we say that the series converges uniformly or, is uniformly convergent to $f(x)$.

** Two very important properties of uniformly convergent series are summarized in the following two theorems:

H.W

** Th-1/ If each term of an infinite series i.e. $\sum_{n=1}^{\infty} u_n(x)$ is continuous in an interval (a,b) & the series is uniformly convergent to the sum $f(x)$ in this interval, then

1. $f(x) = \sum_{n=1}^{\infty} u_n(x)$ is also continuous in the interval

2. The series can be integrated term by term, i.e.,

$$\int_a^b \left\{ \sum_{n=1}^{\infty} u_n(x) \right\} dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\, dx \rightarrow (3)$$

Th-2/ If each term of an infinite series i.e., $\sum_{n=1}^{\infty} u_n(x)$ has a derivative & the series of derivatives is uniformly convergent then the series can be differentiated term by term, i.e., Concept of limit:

$$\lim_{n\to\infty} a_n = l$$

ie, $a_n \to l$, as $n \to \infty$

$\in$ N

$|a_n - l| < \in$

as $n > N$

$N = 100$.

$|a_1 - l| = d_1$

$|a_2 - l| = d_2$

$\vdots$

$|a_{100} - l| = 0.0\ldots$

$|a_{100} - x| = 8.0\ldots$

i.e.,
$$\frac{d}{dx}\left[\sum_{n=1}^{\infty} u_n(x)\right] = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$$

$\rightarrow s(x)$

- There are various ways of proving the uniform convergence of a series. (Find out?)

Note: One most obvious way is to actually find the sum $S_R(x)$ in closed form & then apply the def^n directly. A second & most powerful way is to use a theorem called the Weierstrass M test

$S_R(x)$

(Weierstrass M test):

Th^{m}: If there exists a set of constants M_n, $n = 1, 2, \cdots$, such that for all x in an interval

i.e.,
$$\frac{d}{dx}\left[\sum_{n=1}^{\infty} u_n(x)\right] = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$$
$\to S(x)$

– There are various ways of proving the uniform convergence of a series. (Find out?)

Note: – One most obvious way is to actually find the sum $S_R(x)$ in closed form & then apply the def^n directly. A second & most powerful way is to use a theorem called the **Weierstrass M test**

$S_R(x)$

(Weierstrass M test):

Th^m

If there exists a set of constants M_n, $n = 1, 2, \cdots,$ such that for all x in an interval in an interval $\boxed{|u_n(x)| \leq M_n}$,

& if furthermore, $\sum_{n=1}^{\infty} M_n$ converges

then $\sum_{n=1}^{\infty} u_n(x)$ converges

uniformly in the interval.

Incidently, the series is also absolutely convergent, i.e., $\sum_{n=1}^{\infty} |u_n(x)|$

is convergent, under these conditions.

eg/ the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges

uniformly in the interval

$(-\pi, \pi)$ (or, in fact in any interval (why?)), since a set

of constants

$$M_n = \frac{1}{n^2}$$ (ie, $M_n = 1/n^2$) can be

found such that

$$\left| \frac{\sin(nx)}{n^2} \right| \le \frac{1}{n^2}$$

& $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by P-series Test)

Find out $\left(\sum_{n=1}^{\infty} \frac{1}{n^p}\right)$ converges for $P>1$ diverges for $P\le 1$

H.W. (EX) All uniformly convergent sequences are pointwise convergent but not vice versa