(b) If $ab < 0$, then either (i) $a < 0, b > 0$ or
(ii) $a > 0, b < 0$.

(c) $a < b \Rightarrow a < \frac{a+b}{2} < b$.

Cor: (a) There is no least positive real number.

(b) If $a \in \mathbb{R}$ is such that $0 \le a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$

Proof: Let, if possible, a be the least possible positive number.

We know that $\frac{1}{2} > 0$ (as if $\frac{1}{2} \le 0$, then $\frac{1}{2} + \frac{1}{2} \le 0+0$
$\Rightarrow 1 \le 0$, a contradiction)

$\therefore$ $a > 0$ and $\frac{1}{2} > 0 \Rightarrow \frac{1}{2} \cdot a > 0$, but
$\frac{1}{2} < 1 \Rightarrow \frac{1}{2} \cdot a < 1 \cdot a = a$

$\therefore$ $0 < \frac{1}{2} \cdot a < a$, contradicting the fact that a is the least positive number.

(b) Suppose, $a > 0$. Then $0 < \frac{1}{2} \cdot a < a$. Take $\varepsilon_0 = \frac{1}{2} a$, which gives a contradiction. So, $a = 0$

Absolute value

Def: The absolute value of a real number, denoted by $|a|$, is defined by

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Theorem:- (a) $|a.b| = |a|.|b|$ for all $a, b \in \mathbb{R}$.

(b) $|a|^2 = a^2 \; \forall a \in \mathbb{R}$.

(c) If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$.

(d) $-|a| \leq a \leq |a|$.

Proof:- (c) Let $|a| \leq c$.

Let $a > 0$ $\therefore a \leq c$. Let $a = 0$, then $a \leq c$.

Let $a < 0$, then $-a \leq c \Rightarrow (-a) + a \leq c + a$

$\Rightarrow 0 \leq c + a$

$\Rightarrow (-c) + 0 \leq (-c) + (c + a)$

$\Rightarrow -c \leq a$

$\therefore -c \leq a \leq c$.

Let $a \leq c$. Let $a \geq 0$, then $|a| = a$

$\therefore |a| \leq c$.

Let $a < 0$, then ~~but~~ $|a| = -a$.

$\cancel{a \leq c \Rightarrow -c \leq -a \Rightarrow -c \leq |a|}$

$-c \leq a \Rightarrow -a \leq c \Rightarrow |a| \leq c$

$\therefore -c \leq a \leq c \Rightarrow |a| \leq c$.

(d) Take $c = |a|$ in (c). As, $|a| \leq c$. So,

$-c \leq a \leq c$

$\Rightarrow -|a| \leq a \leq |a|$.

Triangle Inequality:-

If $a, b \in \mathbb{R}$, then $|a+b| \leq |a| + |b|$.

Proof:- We have $-a \leq |a| \leq a$, $-b \leq |b| \leq b$.

Adding we get $-(a+b) \leq |a| + |b| \leq a+b$

$\therefore |a+b| \leq |a| + |b|$.

Cor.

(a) If $a, b \in \mathbb{R}$, then (i) $||a| - |b|| \leq |a-b|$

(ii) $|a-b| \leq |a| + |b|$.

(b) If $a_1, a_2, \ldots, a_n \in \mathbb{R}$, then

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|.$$

Remark: In the real line, $|a|$ is the distance of a from 0. More generally, the distance between a and b is $|a-b|$.

Completeness Property of $\mathbb{R}$:-

The previous two properties, i.e. algebraic and order property are also enjoyed by the set $\mathbb{Q}$ of rational numbers. But we know that $\sqrt{2}$ can't be represented as a rational number. So to define the set $\mathbb{R}$ of real numbers, we need to add one extra property. This additional property is named as the completeness (or the supremum) property and we say that $\mathbb{R}$ is a complete ordered field. This additional property permits us to develop various limiting procedures, which will be discussed in the following chapters.

Def:- Let $S \neq \Phi$ be a non-empty subset of $\mathbb{R}$.

(a) S is said to bounded above if there exists $u \in \mathbb{R}$ such that $s \leq u \ \forall s \in S$ and u is said to be an upper bound of S.

(b) S is said to be bounded below if there exists $l \in \mathbb{R}$ such that $l \leq s \ \forall s \in S$ and such l is said to be a lower bound of S.

Example:-

(a) Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. S is bounded below, 0 and all negative numbers are lower bounds. 1 is an upper bound of S. Here $0 \notin S$, but $1 \in S$.

(b) Let $S = \{x \mid 1 \leq x < 2\}$. S is bounded above, 2 being an upper bound. But $2 \notin S$.
S is bounded below as 1 is a lower bound of S. Here $1 \in S$.

(c) Let $S = \{n \mid n \in \mathbb{N}\}$. We shall show later than S is not bounded above. Though we know, S is bounded below by 0.

Let S be a non-empty set. Let S be bounded above, ~~then~~ and let u be an upper bound of S. Then $u+1, u+2, \ldots, u+n, \ldots$ are also upper bounds of S. So, S has infinitely many upper bounds.

Similarly, if l is a lower bound of S, then $l-n$ is again a lower bound of S. So, we can conclude that S has infinitely many lower bounds provided S is bounded below.

In the set of upper bounds (or lower bounds) of S, we are interested in the least upper bound (or greatest lower bound) of S.

Def:- Let S be a non-empty subset of $\mathbb{R}$.

(a) If S is bounded above, then a number u is said to be a supremum (sup S) (or a least upper bound) of S if

(1) u is an upper bound of S i.e. $s \leq u \; \forall s \in S$,

(2) if v is an upper bound of S, then $u \leq v$.

(b) If S is bounded below, then a number l is said to be an infimum (inf S) (or a greatest lower bound) of S if

(1) l is a lower bound of S i.e. $l \leq s \; \forall s \in S$,

(2) if m is a lower bound of S, then $m \leq l$.

Remark

(a) If S is bounded above, then there can be only one supremum, as if suppose u_1 and u_2 are two supremums of S with $u_1 < u_2$.
As u_1 is an upper bound of S and u_2 is a supremum of S, so $u_2 \leq u_1$. As $u_1 \neq u_2$, so $u_2 < u_1$ which is a contradiction.
Similarly, if S is bounded below, there can be only one infimum.

(b) If S is a non-empty finite set, then
sup S = maximum of S,
inf S = minimum of S.

We write the idea of the supremum in the following alternative ways:-

Theorem:-

(1) A number u is the supremum of a non-empty set S if and only if

(a) $s \le u \ \forall s \in S$,

(b) If $v < u$, then there exists $s' \in S$ such that $v < s' \le u$.

(2) A number u is the supremum of a non-empty set S if and only if

(a) $s \le u \ \forall s \in S$,

(b) for every $\varepsilon > 0$, there exists $s_\varepsilon \in S$ such that $\cancel{u - \varepsilon < s(\varepsilon)}$ $u - \varepsilon < s_\varepsilon \le u$.

Proof: We shall prove (2).

Suppose $u = \sup S$ and $v = u - \varepsilon$. Then, $v < u$. So, v is not an upper bound of S. So, $\exists\ s' \in S$ such that $v < s'$. We write s' as s_ε. Then, there is $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon \le u$.

Conversely, $u \in \mathbb{R}$ satisfies (a) and (b). Then, clearly u is an upper bound of S.

Let $v < u$. Then, take $\varepsilon := u - v$. $\therefore$ $\cancel{u = v + \varepsilon}$ $u - \varepsilon = v$.

$\therefore \exists$ an element $s'_\varepsilon \in S$ such that $v < s'_\varepsilon \le u$.

So, v is not an upper bound of S.

So, v is an upper bound of S, implies $u \le v$.

Example:-

(a) Let $S := \{a_1, \ldots, a_n\}$ be a finite set. Then, it can be proved that S_1 has a maximal elt a_i ($a_j \leq a_i \; \forall j$) and a minimum elt a_r ($a_r \leq a_j \; \forall j$), ~~And~~, (by Mathematical induction). Then, $a_i = \sup S$ and $a_r = \inf S$.

(b) Let $S := \{x : 0 \leq x < 1\}$. Then, clearly 1 is an upper bound of S. Let $v < 1$. If $v \leq 0$, then v is obviously not an upper bound. If $v > 0$, then $v < \frac{1+v}{2} < 1$ and $\frac{1+v}{2} \in S$. So, v is not an upper bound. So, $1 = \sup S$. Similarly, we can prove that $0 = \inf S$.

We have similar alternative definition for infimum of S.

Theorem:-

(1) A number l is the infimum of S if and only if

(a) $l \leq s \; \forall s \in S$,

(b) if $l < m$, then $\exists \; s' \in S$ such that $l \leq s' < m$.

(2) A number l is the infimum of S if and only if

(a) $l \leq s \; \forall s \in S$,

(b) for every $\epsilon > 0$, there exists an element $s(\epsilon) \in S$ such that $l \leq s(\epsilon) < l + \epsilon$.

Statement of Completeness Property:-

Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.

We can state the completeness property in term of infimum. Though these two approaches are equivalent i.e. one of these imply the other.

Theorem:- Let S be a non-empty subset of $\mathbb{R}$, which is bounded below. Then S has an infimum.

Proof:- Let $T := \{ y \mid y = -x \text{ where } x \in S \}$.

Let m be a lower bound of S, i.e. $m \le x$ ~~$\forall x \in S$~~ $\forall x \in S$.

$\therefore -x \le -m \quad \therefore y \le -m$

So, T is bounded above. So, by Completion property of $\mathbb{R}$, T has the supremum, say u.

Our claim is:- $-u$ is the infimum of S.

~~Take $-u+\varepsilon$~~ First, $-u$ is a lower bound of S.

~~So,~~ Take $-u+\varepsilon$. Then, there is $\underbrace{\cancel{x(\varepsilon)}}_{= -x(\varepsilon)}^{y(\varepsilon)}$ such that $u-\varepsilon < \bar{x}(\varepsilon) \le u$

$\therefore -u+\varepsilon > +x(\varepsilon) \geq -u$

$\therefore -u+\varepsilon > \overset{x(\varepsilon)}{\cancel{y(\varepsilon)}} \geq -u$. ~~Here $y(\varepsilon) \in T$.~~

So, $-u$ is the infimum of S.

Applications of the Supremum Property

~~Lemma~~

Proposition:- Let A and B be two non-empty subsets of $\mathbb{R}$, $a \le b$, $a \in A$ and $b \in B$. Then, $\sup A \le \inf B$.

Proof:-

$$a \le b \quad \forall a \in A, \forall b \in B$$

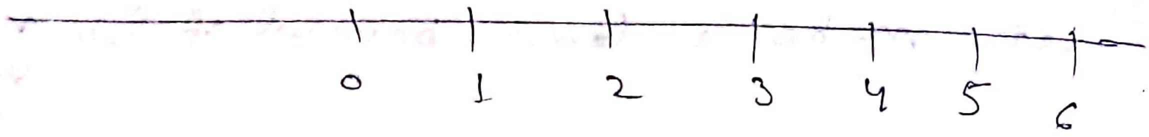
$\Rightarrow b$ is an upper bound of A.

$\Rightarrow \sup A \le b \quad \forall b \in B$

$\Rightarrow \sup A$ is a lower bound of B.

$\Rightarrow \sup A \le \inf B$

The Archimedean Property:-



It may seem obvious that the set $\mathbb{N}$ of natural numbers is not bounded in $\mathbb{R}$. We cannot prove this by algebraic and order properties of $\mathbb{R}$. Here we use the completeness property.

~~Theorem~~

Theorem:- If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \le n_x$

Proof: If possible, suppose $n \le x \ \forall n \in \mathbb{N}$.

Then, x is an upper bound of $\mathbb{N}$. So, $\mathbb{N}$ has the supremum u in $\mathbb{R}$. So, $u-1$ is not an upper bound of $\mathbb{N}$. So, $\exists\ n_1 \in \mathbb{N}$ s.t. $u-1 < n_1 < u < n_1+1$, contradicting the fact that u is an upper bound of $\mathbb{N}$. So, for $x \in \mathbb{R}$, there is a natural number n_x such that $x \le n_x$.

General Archimedean property of $\mathbb{R}$:-

If $x, y \in \mathbb{R}$ and $x > 0, y > 0$, then there exists a natural number n such that $ny > x$.

Proof:- By Archimedean property of $\mathbb{R}$, there exists a natural number n such that

$$\frac{x}{y} < n.$$

$$\therefore x < ny.$$

Cor:- Let $S := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Then $\inf S = 0$.

Proof:- We know that if n is a natural number, then $n > 0$. If $1/n \le 0$, then $n \cdot \frac{1}{n} \le n \cdot 0$

$\Rightarrow 1 \le 0$, a contradiction.

So, $\frac{1}{n} > 0$.

So, 0 is a lower bound of 0. Let $w := \inf S$ and $w \ne 0$. So, $w > 0$, So, $\exists$ a natural no. n_w such that $\frac{1}{w} < n_w$. $\therefore \frac{1}{n_w} < w$. So, w can't be a lower bound. So, $\inf S = 0$.

Cor:- If $y > 0$, there exists $n_y \in \mathbb{N}$ such that $n_y - 1 \le y < n_y$.

Proof:- Let $E_y := \{ m \in \mathbb{N} : y < m \}$.

By Archimedean property of $\mathbb{R}$, $E_y \ne \Phi$.

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So, by well-ordering property of $\mathbb{N}$, E_y has a least element, say n_y. Then, $n_y - 1 \notin E_y$, i.e. $n_y - 1 \leq y$.

So, $n_y - 1 \leq y < n_y$.

~~Note~~

~~Note:~~

The existence of positive square root:

The completeness property guarantees the existence of real numbers under certain hypothesis.

Theorem: There exist a positive real number x such that $x^2 = 2$.

Proof: Let $S := \{ s \in \mathbb{R} : s \geq 0 \text{ and } s^2 < 2 \}$

First, $1^2 = 1 < 2$ and $1 > 0$ imply that S is a non-empty subset of $\mathbb{R}$. Also, $2^2 = 4$, so $s \leq 2$. So, 2 is an upper bound of S. So, S has the supremum, say x. We shall show that $x^2 = 2$.

First, assume that, if possible, $x^2 < 2$.

We shall choose $n \in \mathbb{N}$ such that $x + \frac{1}{n} \in S$.

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2}{n} \cdot x + \frac{1}{n^2} \leq x^2 + \frac{2}{n} \cdot x + \frac{1}{n}$$

$$= x^2 + \frac{1}{n}(2x+1).$$

Now, as $x > 0$ (as $1 \in S$ and $1 > 0$, so $x \neq 0$).

By Archimedean property, we can choose a natural number n such that $\frac{1}{n} < \frac{2 - x^2}{2x+1}$.

$$\therefore \frac{2x+1}{n} < 2 - x^2.$$

$\therefore (x+\frac{1}{n})^2 < x^2+2-x^2 = 2$

So, $x+\frac{1}{n} \in S$. Therefore, x cannot be an upper bound. So, $x^2 \not< 2$.

Now, assume, if possible, that $x^2 > 2$. Then we shall choose a natural number m such that $x-\frac{1}{m}$ is an upper bound of S.

$$(x-\frac{1}{m})^2 = x^2 - \frac{2}{m}.x + \frac{1}{m^2} > x^2 - \frac{2}{m}x.$$

We choose $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{-2+x^2}{2x}$.

$$\therefore \frac{2x}{m} < x^2 - 2$$

$$\therefore (x-\frac{1}{m})^2 > x^2-(x^2-2) = 2.$$

$$\therefore s \in S \Rightarrow (x-\frac{1}{m})^2 > 2 > s^2$$

$$\Rightarrow x-\frac{1}{m} > s. \text{ (as } x-\frac{1}{m} > 0 \text{ and } s \geq 0 \Rightarrow x-\frac{1}{m}+s > 0)$$

So, $x-\frac{1}{m}$ is an upper bound contradicting the fact that x is the supremum.

Remark

(1) By modifying the preceding argument, one can show that if $a>0$, there is a unique $b>0$ such that $b^2=a$. We call b the positive square root of a and denote by $a^{\frac{1}{2}}$. We can also establish the existence of a unique positive nth root of a, denoted by $a^{1/n}$.

(2) Let $T := \{t \in \mathbb{Q} : t \geq 0 \text{ and } t^2 < 2\}$.

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If possible, suppose that $\sup T$ is a rational number, then one ~~can~~ can show similarly that $(\sup T)^2 = 2$. But, we have proved that if $x^2 = 2$, then x can't be a rational number. Therefore, T that consists of rational numbers does not have a supremum, which belongs to $\mathbb{Q}$.

As each rational number is a real number, so $\mathbb{Q}$ possesses the Archimedean property, but $\mathbb{Q}$ does not possess the completeness property.

Density Property in $\mathbb{R}$:-

We have shown the existence of at least one irrational number, namely $\sqrt{2}$. ~~Of~~ In fact, there are more irrational numbers than rational numbers in the sense that the set of rational numbers is countable, while the set of irrational numbers is uncountable (the second statement will be proved later). But, we shall show that the set of rationals is dense in $\mathbb{R}$.

The Density Theorem:-

If x and y are any real numbers with $x<y$, then there exists a rational number r such that $x<r<y$.

Proof: Without loss of generality, we assume that $x>0$. Since $y-x>0$, there exists a natural no n such that $\frac{1}{n} < y-x$ $\therefore 1 < ny - nx$.

$\therefore 1 + nx < ny$.

By the corallary fallowing Archimedean property, we get a natural number m such that $m-1 \leq nx < m$.

$\therefore m \leq nx+1 < ny$.

$\therefore nx < m < ny \Rightarrow x < \frac{m}{n} < y$.

Here $r := \frac{m}{n}$ and $x < r < y$.

<u>Cor</u>: If x and y are real numbers with $x<y$, then there exists an irrational number z such that $x < z < y$.

<u>Proof</u>: $x < y \Rightarrow \frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$. (as $\sqrt{2} > 0 \Rightarrow \frac{1}{\sqrt{2}} > 0$)

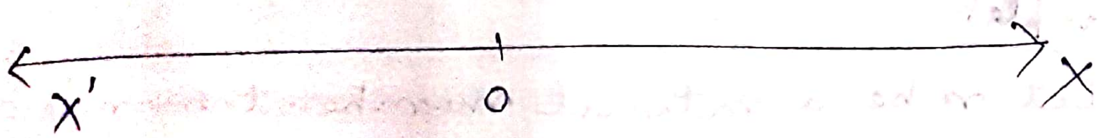
So, by the Density Theorem, there is a rational number r such that $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$.

$\therefore x < r\sqrt{2} < y$.

Let $z := r\sqrt{2}$. Then z is an irrational number and $x < z < y$.

<u>Geometric Representation of real numbers</u>:

We know that the rational numbers can be made to correspond to ~~to~~ points on a straight line. Let $X'X$ be a directed line.



Choose O on the line. O divides the line into two parts. The part to the right of O is called the positive part of O and the part to the left of O is called the negative side. Let's take a point A to the right of O, Let us take a point A to the right of O. Let O represent the real number 0 and A denote the real number one. We know how to represent the rational numbers on the line. These points are called rational points.

Q:- Is there any practical way of representing real numbers?

Answer:- Dedekind's idea was to think a real number as a partition of a line. The basic points behind his idea was — (i) the geometric intuition that any real number divides ~~the~~ the set of real numbers into two halves, & (ii) the set of rational numbers is dense in the set of real numbers.

Dedekind Cut:-

A Dedekind cut $x = (L, U)$ in $\mathbb{Q}$ is a pair of subsets L, U of $\mathbb{Q}$ satisfying the following:-

1. $L \cup U = \mathbb{Q}, L \cap U = \bar{\phi}, L \neq \bar{\phi}, U \neq \bar{\phi}$.

2. If $l \in L, u \in U$, then $l < u$.

3. L contains no largest element.

A real number is a Dedekind cut in $\mathbb{Q}$.

Example:-

1. Let r be a rational number, then r is a Dedekind cut of the form

$$\{q \in \mathbb{Q} : q < r\} \cup \{q \in \mathbb{Q} : q \geq r\}.$$

2. Suppose x is a real number. Then the corresponding Dedekind cut is

$$\{q \in \mathbb{Q} : q < x\} \cup \{q \in \mathbb{Q} : q > x\}.$$

For example, $\sqrt{2}$ can be represented as–

$$\{q \in \mathbb{Q} : q^2 < 2 \text{ or } q < 0\} \cup \{q \in \mathbb{Q} : q^2 > 2 \text{ and } q > 0\}.$$

Arithmetic Operations:-

1. For cuts $x = (A, B)$ and $y = (C, D)$, define

$$(x+y) = (E, \mathbb{Q} - E) \text{ where } E = \{q \in \mathbb{Q} : q = a + c \text{ for some } a \in A, c \in C\}.$$

2. For cut $x = (A, B)$, $(-x) := (C, \mathbb{Q} - C)$ with

$$C = \{q \in \mathbb{Q} : q = -b \text{ for } b \in B \text{ not the smallest element}\}.$$

3. We define the absolute value $|x|$ in the following way –

$$|x| := \begin{cases} x & \text{if } x = (A, B) \text{ \& } 0 \in A. \\ & \text{or if } x \in B, \text{ then } x \text{ is the least element of } B \\ -x, & \text{otherwise.} \end{cases}$$

4. We define multiplication in the following way

$$x \times y := \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0, \\ |x| \times |y| & \text{if } (x = (A,B) \text{ and } 0 \in A, \; y = (C,D) \text{ and } 0 \in C) \text{ or } \\ & (x = (A,B) \text{ and } 0 \in B, \; x \neq 0, \; y = (C,D) \text{ and } 0 \in D, \; y \neq 0), \\ -(|x| \times |y|) & \text{if } (x = (A,B) \text{ and } 0 \in A, \; y = (C,D) \text{ and } 0 \in D, \; y \neq 0) \\ & \text{or } (x = (A,B) \text{ and } 0 \in B, \; x \neq 0, \; y = (C,D) \text{ and } 0 \in C) \end{cases}$$

5. Given real numbers $x = (A, B)$ and $y = (C, D)$, x is less than or equal to y if $A \subseteq C$. We denote it by $x \leq y$.

(set of Dedekind cuts)

6. A set $S \subset \mathbb{R}$ is said to be bounded above if there is some $N \in \mathbb{R}$ such that $s \leq N \; \forall s \in S$. We define least upper bound or supremum for a bounded above set S as a real number (Dedekind cut) x which is an upper bound of S but no $y < x$ is an upper bound of S.

To give the existence of such supremum is not difficult, We define the supremum x of S as follows:-

$$x = (A, B) \text{ where}$$

$$A = \{ q \mid q \in \mathbb{Q} \text{ and } q < s \text{ for some } s \in S \}$$

$$B = \{ q \mid q \in \mathbb{Q} \text{ and } q = q_1 + q_2 \text{ where } s < q_1 \text{ for all } s \in S \text{ and } q_2 > 0 \}.$$

It can be checked that $x = (A, B)$ is a Dedekind cut and x is a least upper bound of S.