Lecture 17

Proposition:- Let $E = E_1 \cup E_2$, where $E_1, E_2 \subseteq \mathbb{R}^d$.

Suppose $\text{dist}(E_1, E_2) > 0$. Then

$$m^*(E) = m^*(E_1) + m^*(E_2).$$

where $\text{dist}(E_1, E_2) = \inf\left\{ \|\underline{x} - \underline{y}\| \;\middle|\; \underline{x} \in E_1, \underline{y} \in E_2 \right\}$.

$$\|\underline{x}-\underline{y}\| = \sqrt{|x_1-y_1|^2 + \cdots + |x_d-y_d|^2}.$$

proof:- By subadditive property of m^*, we have

$$m^*(E) \leq m^*(E_1) + m^*(E_2).$$

To prove the reverse inequality, ~~let $\varepsilon > 0$~~.

Select a $\delta > 0$ such that $\text{dist}(E_1, E_2) > \delta > 0$.

Let $\varepsilon > 0$.

Choose a covering $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ by closed cubes Q_j

with $\sum_{j=1}^{\infty} |Q_j| \leq m^*(E) + \varepsilon$.

We may, after subdividing the cubes Q_j, assume that each Q_j has a diameter $< \delta$.

$\left(\text{diameter}(Q_j) = \sup\left\{ \|\underline{x} - \underline{y}\| \;/\; \underline{x}, \underline{y} \in Q_j \right\} \right)$

In this case, each Q_j can intersect at most one of the two sets E_1 or E_2.

Let J_1 = the set of those indices j for which Q_j intersects E_1.

& J_2 = " " " " " E_2.

Then $J_1 \cap J_2 = \emptyset$. &

$$E_1 \subseteq \bigcup_{j \in J_1} Q_j \quad , \quad E_2 \subseteq \bigcup_{j \in J_2} Q_j .$$

$\operatorname{dia} Q_j < \delta$.

$$\Rightarrow \quad m^*(E_1) \le m^*\Big(\bigcup_{j \in J_1} Q_j\Big) \le \sum_{j \in J_1} m^*(Q_j) = \sum_{j \in J_1} |Q_j| .$$

$$\& \quad m^*(E_2) \le \sum_{j \in J_2} |Q_j| .$$

$$\therefore m^*(E_1) + m^*(E_2) \le \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j|$$

$$= \sum_{j \in J_1 \cup J_2} |Q_j| \qquad J_1 \cup J_2 \subseteq \mathbb{N}$$

$$\le \sum_{j=1}^{\infty} |Q_j| .$$

$$\le m^*(E) + \varepsilon$$

true for any $\varepsilon > 0$.

$\therefore \quad m^*(E_1) + m^*(E_2) \leq m^*(E)$.

Hence $m^*(E) = m^*(E_1) + m^*(E_2)$.

Def:- (X, d) be a metric space. Then a subset $A \subseteq X$ is said to be <u>sequentially Compact</u>, if every sequence $\{x_n\}$ in A there exists a Convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to x$, for some $x \in A$, as $k \to \infty$

<u>Recall</u>

<u>Theorem:-</u> Let $K \subseteq \mathbb{R}^d$. Then K is Compact (Closed & bdd) if and only if K is sequentially Compact.

<u>Proposition:-</u> Let K be a compact set in $\mathbb{R}^d$ & F be a closed set in $\mathbb{R}^d$. Then $\operatorname{dist}(K, F) > 0$. & $K \cap F = \emptyset$.

<u>proof:-</u> Suppose $\operatorname{dist}(K, F) = 0$.

$$\Rightarrow \inf\left\{ \|\underline{x} - \underline{y}\| \,/\, \underline{x} \in K,\ \underline{y} \in F \right\} = 0.$$

Let $\underline{x}_n \in K$ & $\underline{y}_n \in F$ be sequences such that $\|\underline{x}_n - \underline{y}_n\| \longrightarrow 0$ as $n \to \infty$

$$0 = \operatorname{dist}(K, F).$$

Now K is compact & $\{\underline{x}_n\}$ is a sequence in K. Therefore there exists a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to \underline{x}$, for some $\underline{x} \in K$.

Now

$$0 \le \|\underline{x} - \underline{y}_{n_k}\| = \|(\underline{x} - \underline{x}_{n_k}) + (\underline{x}_{n_k} - \underline{y}_{n_k})\|$$

$$\le \underbrace{\|\underline{x} - \underline{x}_{n_k}\|}_{\downarrow \, 0} + \underbrace{\|\underline{x}_{n_k} - \underline{y}_{n_k}\|}_{\downarrow \, 0} \quad \text{as } k \to \infty$$

$$\Rightarrow \quad \|\underline{x} - \underline{y}_{n_k}\| \longrightarrow 0 \quad \text{as} \quad k \to \infty.$$

$$\Rightarrow \quad y_{n_k} \to \underline{x} \quad \text{as } k \to \infty. \qquad y_{n_k} \in F$$

Since F is closed & $\underline{x}$ is a limit point of F, therefore $\underline{x} \in F$.

Thus $\underline{x} \in K \cap F$.

$\Rightarrow \Leftarrow \quad \because K \cap F = \phi$.

Theorem:– The Complement of a measurable set in $\mathbb{R}^d$ is measurable.

proof:– Let $E \subseteq \mathbb{R}^d$ be measurable.

To show: $E^c = \mathbb{R}^d \setminus E$ is measurable.

For each $n \in \mathbb{N}$, there exists an open set U_n with $E \subseteq U_n$ & $m^*(U_n \setminus E) \leq \frac{1}{n}$.

Now U_n^c is closed & hence by above proposition, it is measurable.

$\Rightarrow S = \bigcup_{n=1}^{\infty} U_n^c$ is also measurable.

& $E^c \supseteq S$.

$E \subseteq U_n$
$\Rightarrow E^c \supseteq U_n^c \quad \forall n$
$\Rightarrow E^c \supseteq \bigcup_{n=1}^{\infty} U_n^c$

Now $E^c \setminus S \subseteq U_n \setminus E \quad \forall n$

$\Rightarrow m^*(E^c \setminus S) \leq m^*(U_n \setminus E) \leq \frac{1}{n}$

$\Rightarrow m^*(E^c \setminus S) \leq \frac{1}{n} \quad \forall n.$

$\Rightarrow m^*(E^c \setminus S) = 0.$

$\Rightarrow E^c \setminus S$ is measurable

$E^c \setminus S \subseteq U_n \setminus E$
Let $\underline{a} \in E^c \setminus S$
$\Rightarrow \underline{a} \in E^c$ & $\underline{a} \notin S$
$\Rightarrow \underline{a} \notin E$ & $\underline{a} \notin U_n^c \quad \forall n$
$\Rightarrow \underline{a} \notin E$ & $\underline{a} \in U_n$
$\Rightarrow \underline{a} \in U_n \setminus E$

Now $E^c = S \cup (E^c \setminus S)$ is also measurable.

measurable as required.

Theorem:- (Countably Additivity).

Let $E_1, E_2, \ldots$ be measurable disjoint sets in $\mathbb{R}^d$. Then

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j).$$

Thm:- Let $E \subseteq \mathbb{R}^d$ be a measurable set. Then $E + x$ is also measurable, for any $x \in \mathbb{R}^d$

& $m(E + x) = m(E)$.

- $m^*(E + x) = m^*(E) \quad \forall\, E \subseteq \mathbb{R}^d$.

Thus we have

$$\mathcal{M} = \{E \subseteq \mathbb{R}^d \,/\, E \text{ is measurable}\}.$$

$\mathcal{M}$ is a σ-algebra.

- $\mathbb{R}^d \in \mathcal{M}$
- $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$
- $E_1, E_2, \ldots \in \mathcal{M} \Rightarrow \bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$.

$\mathcal{B}$ = the smallest σ-algebra containing all open sets in $\mathbb{R}^n$.

- $\mathcal{B} \subseteq \mathcal{M}$ & the elements of $\mathcal{B}$ are called Borel sets.