Lecture 22

Theorem (Lusin) (Littlewood 2nd principle) :-

Let E be a measurable set of finite measure. Let $f: E \to \mathbb{R}$ be a measurable function. Then for every $\varepsilon > 0$, there exists a closed set F_ε such that $F_\varepsilon \subseteq E$ & $m(E \setminus F_\varepsilon) \leq \varepsilon$ & such that

$f|_{F_\varepsilon}$ is continuous.

(every measurable function is nearly continuous).

proof:- Given that f is measurable.

$\Rightarrow$ there exists a sequence of simple functions $\{f_n\}$ such that

$f_n(x) \to f(x)$ a.e on E.

Claim:- For each f_n, there exists a set $E_n \subseteq E$ such that $m(E_n) < \frac{1}{2^n}$ & f_n is continuous on $E \setminus E_n$.

proof of the claim:-

f_n is a simple function on E

$x \in A_i$
$f_n(x) = a_i$

$\Rightarrow \quad f_n(x) = \sum_{i=1}^{s} a_i \chi_{A_i}$ where

A_i's are disjoint & a_i are constants.
(measurable) & $E = \bigcup_{i=1}^{n} A_i$.

For each A_i, there exists a closed set $B_i \subseteq A_i$ such that $m(A_i \setminus B_i) < \frac{1}{s 2^n}$.

Consider $E_n := E \setminus \bigcup_{i=1}^{s} B_i$

for the complement of B_i in $E \setminus E_n = \bigcup_{i=1}^{s} B_i$

is $\bigcup_{\substack{j \neq i \\ j=1}}^{s} B_j$ which is a closed set

$\Rightarrow \quad B_i$ is open in $E \setminus E_n$, $\forall i = 1, 2, \dots, s$

& $f_n\big|_{B_i}$ is the constant map a_i. $B_i \subseteq A_i$
which is continuous.

Thus $f_n\big|_{B_i}$ are continuous $\forall i = 1, \dots, s$
& B_i are open in $E \setminus E_n$.
$= \bigcup_{i=1}^{s} B_i$

$\Rightarrow \quad f_n\big|_{\bigcup_{i=1}^{s} B_i} = f_n\big|_{E \setminus E_n}$ is continuous

(by the patching Lemma)

& $m(E_n) = m\left(E \setminus \bigcup_{j=1}^{s} B_j\right)$

$$= m\left(\left(\bigcup_{i=1}^{s} A_i\right) \setminus \left(\bigcup_{j=1}^{s} B_j\right)\right)$$

$$\leq m\left(\left(\bigcup_{i=1}^{s} A_i\right) \setminus B_i\right)$$

$$= m\left(\bigcup_{i=1}^{s} (A_i \setminus B_i)\right)$$

$$\leq \sum_{i=1}^{s} m(A_i \setminus B_i)$$

$$< \sum_{i=1}^{s} \frac{1}{s2^n} = \frac{1}{2^n}.$$

$\therefore m(E_n) < \frac{1}{2^n}$.

This completes the proof the claim.

By Littlewood 3rd principle (Egorov's Thm), there exists a closed set $A_{\varepsilon/3} \subseteq E$ such that $\boxed{m(E \setminus A_{\varepsilon/3}) \leq \varepsilon/3}$ & $f_n \to f$ uniformly on $A_{\varepsilon/3}$.

Choose $N \in \mathbb{N}$ so that $\displaystyle\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{3}$.

Consider $F' = A_{\varepsilon/3} \setminus \bigcup_{n \geq N} E_n$.

Then $\underline{m(A_{\varepsilon/3} \setminus F')} = m\left(\bigcup_{n \geq N} E_n\right)$

$$\leq \sum_{n \geq N} m(E_n)$$

$$< \sum_{n \geq N} \frac{1}{2^n}$$

$$\underline{< \frac{\varepsilon}{3}} \quad \text{(by the chose of } N\text{)}$$

Now for every $n \geq N$, the funtions f_n is Continuous on $F' = A_{\varepsilon/3} \setminus \bigcup_{n' \geq N} E_{n'}$

$$= \bigcap_{n' \geq N} (A_{\varepsilon/3} \setminus E_{n'})$$

$$\subseteq \bigcap_{n' \geq N} (E \setminus E_{n'}) \subseteq E \setminus E_n$$

Thus f_n are Continuous for $n \geq N$, on F'

& $f_n \to f$ uniformly on $F' \subseteq A_{\varepsilon/3}$.

$\Rightarrow$ f is Continuous on F'

(because the uniform limit of a uniformly convergent seq. of continuous functions is continuous).

Since F' is measurable, there exists, a closed set $F_\varepsilon \subseteq F'$ such that

$m(F' \setminus F_\varepsilon) < \frac{\varepsilon}{3}$.

Now $E \setminus F_\varepsilon \subseteq (E \setminus A_{\varepsilon/3}) \cup (A_{\varepsilon/3} \setminus F') \cup (F' \setminus F_\varepsilon)$

$$\Rightarrow m(E \setminus F_\varepsilon) \le m(E \setminus A_{\varepsilon/3}) + m(A_{\varepsilon/3} \setminus F') + m(F' \setminus F_\varepsilon)$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus $m(E \setminus F_\varepsilon) < \varepsilon$ & $f|_{F_\varepsilon}$ is continuous.

This completes the proof.

Recall:- Let $\varphi = \sum_{k=1}^{n} a_k \chi_{E_k}$ be a simple function.

where E_k are measurable, a_k are constants.

The canonical representation of φ is the unique decomposition $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$, where a_k are distinct & the sets E_k are disjoint.

Remark:- Any simple function φ can be written uniquely in its canonical representation.

Pf:- Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$.

where a_k's need not be distinct & E_k's need not be disjoint.

Let $a_{i_1}, \ldots, a_{i_m}$ be the distinct values of $a_1, \ldots, a_N$.

Define $F_k = \{x \mid \varphi(x) = a_{i_k}\}$.

$1 \le k \le m$.

Then F_k are disjoint.

$\left(\because \text{ Supp } x \in F_k \cap F_\ell,\ k \neq \ell, \Rightarrow \varphi(x) = a_k = a_\ell. \ \Rightarrow\Leftarrow \right)$

Now $\varphi = \sum_{k=1}^{m} a_{i_k} \chi_{F_k}$,

which is the canonical repr. of φ.

<u>Def</u>:- Let φ be a simple function with canonical representation $\varphi(x) = \sum_{k=1}^{N} c_k \chi_{E_k}(x)$.

Then the <u>Lebesgue integral</u> of φ is defined as

$$\int \varphi \doteq \sum_{k=1}^{N} c_k\, m(E_k).$$

Another notation: $\int \varphi(x)\, dx$ <u>or</u> $\int_{\mathbb{R}^d} \varphi$

<u>or</u> $\int_{\mathbb{R}^d} \varphi(x)\, dx$

Def:- Let $E \subseteq \mathbb{R}^d$ be a measurable set with finite measure. Then $\varphi(x)\chi_E(x)$ is also a simple function

$$\left(\because \ \varphi\chi_E = \left(\sum_{k=1}^{N} a_k \chi_{E_k}\right)\chi_E \right.$$

$$= \sum_{k=1}^{N} a_k \chi_{E_k}\chi_E$$

$$= \sum_{k=1}^{N} a_k \chi_{E_k \cap E}$$

is a simple function on E.)

We define $\displaystyle\int_E \varphi := \int_{\mathbb{R}^d} \varphi(x)\chi_E(x)\, dx$

Another notation: $\displaystyle\int_E \varphi(x)\, dx$ or $\displaystyle\int_E \varphi(x)\, dm(x)$
