Tensor Analysis :

A point in N dimensional space is a set of N numbers $(x^1, x^2, \ldots\ldots, x^N)$.

Coordinate transformation -

Let $(x^1, x^2, \ldots\ldots, x^N)$ and $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ be coordinates of a point in two different frames of reference. Suppose there exists N independent relations between the coordinates of the two symbols having the form

$$\bar{x}^k = \bar{x}^k(x^1, x^2, \ldots, x^N) \quad ; \quad k = 1, 2, \ldots, N \qquad (1)$$

Then, conversely to each set of coordinates $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ there will correspond a unique set $(x^1, x^2, \ldots, x^N)$ given by $x^k = x^k(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N) \; : \; k = 1, 2, \ldots, N \qquad (2)$

The relations (1) or (2) define a transformation of coordinates from one frame to another.

The summation convention -

Whenever an index (subscript or superscript) is repeated in a given term: we are to sum over that index from 1 to N unless otherwise specified.

Thus, $a_1 x^1 + a_2 x^2 + \ldots\ldots + a_N x^N \equiv a_p x^p$, p is dummy index.

This is called summation convention.

Contravariant and covariant vectors -

If N quantities $A^1, A^2, \ldots, A^N$ in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N other quantities $\bar{A}^1, \bar{A}^2, \ldots, \bar{A}^N$ in another system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations-

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} A^q ,$$

they are called components of a contravariant vector or contravariant tensor of rank 1.

If N quantities $A_1, A_2, \ldots, A_N$ in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N other quantities $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_N$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations -

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} A_q ,$$

they are called components of a covariant vector or covariant tensor of rank 1.

<u>Contravariant</u>, <u>covariant and mixed tensors</u> -

If N^2 quantities A^{qs} in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N^2 other quantities $\bar{A}^{pr}$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations -

$$\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial \bar{x}^r}{\partial x^s} A^{qs} ,$$

they are called contravariant components of a tensor of rank 2.

The N^2 quantities A_{qs} are called covariant components of a tensor of rank 2 if -

$$\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \cdot \frac{\partial x^s}{\partial \bar{x}^r} \cdot A_{qs}$$

Similarly, the N^2 quantities A^q_s are called components of mixed tensor of rank 2. if -

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial x^s}{\partial \bar{x}^r} A^q_s$$

Tensors of higher rank may similarly be defined.
e.g. mixed term of rank 5 A^{qst}_{kl} (contravariant of order 3, covariant of order 2)

$$\bar{A}^{prm}_{ij} = \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial \bar{x}^r}{\partial x^s} \cdot \frac{\partial \bar{x}^m}{\partial x^t} \cdot \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A^{qst}_{kl}$$

The Kronecker Delta -

$$\delta^j_k = \begin{cases} 0 , & \text{if } j \neq k \\ 1 , & \text{if } j = k \end{cases} \Rightarrow \text{mixed tensor of rank 2 (prove)}$$

Scalars or invariants -

Suppose ϕ is a function of the coordinates x^k. and let $\bar{\phi}$ denote the functional value under a transformation to a new set of coordinates $\bar{x}^k$. Then ϕ is called a scalar or invariant w.r.t. the coordinate transformation if $\phi = \bar{\phi}$.

A scalar or invariant is a tensor of rank 0

Symmetric and skew-symmetric tensor -

A tensor is called symmetric w.r.t. two indices (contravariant or covariant) if its components remain unaltered upon interchange of the indices.

Thus if $A^{mpr}_{qs} = A^{pmr}_{qs}$, the tensor is symmetric in m and p.

A tensor is called skew-symmetric w.r.t. two indices if its components change sign upon interchange of indices.

Thus if $A^{mpr}_{qs} = -A^{pmr}_{qs}$, the tensor is skew-symmetric

Fundamental operations with tensors -

① **Addition** : The sum of two or more tensors of the same rank and type is also a tensor of the same rank and type.

Thus, if A^{mp}_q and B^{mp}_q are tensors, then $C^{mp}_q = A^{mp}_q + B^{mp}_q$ is also a tensor. Addition of tensors is commutative and associative.

(2) Subtraction : The difference of two tensors of same rank and type is also a tensor of same rank and type.

Thus, if A^{mp}_q and B^{mp}_q are tensors, then $C^{mp}_q = A^{mp}_q - B^{mp}_q$ is also a tensor.

(3) Outer multiplication : The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors.

e.g., $A^{pr}_q B^m_s = C^{prm}_{qs}$ is the outer product of A^{pr}_q and B^m_s.

Note that not every tensor can be written as a product of two tensors of lower rank. So, division of tensors is not always possible.

(4) Contraction : If one contravariant and one covariant index of a tensor are set equal, the result indicates that a summation over the equal indices is to be taken. The resulting sum is a tensor of rank 2 less than that of the original tensor. The process is called contraction.

e.g., in the tensor of rank 5, A^{mpr}_{qs}, set $r = s$ to obtain $A^{mpr}_{qr} = B^{mp}_q$, a tensor of rank 3. Further by setting $p = q$, we obtain $B^{mp}_p = B^m$, a tensor of rank 1.

(5) Inner multiplication : By the process of outer multiplication of two tensors followed by a contraction, we obtain a new tensor called an inner product of the given tensors. The process is called inner multiplication.

e.g., given the tensors A^{mp}_q and B^r_{st}, the outer product is $A^{mp}_q \cdot B^r_{st}$. Letting $q = r$, we obtain inner product $A^{mp}_r B^r_{st}$. Letting $q = r$ and $p = s$, another inner product $A^{mp}_r B^r_{pt}$ is obtained.

Inner and outer multiplication of tensors is commutative and associative.

⑥ <u>Quotient Law</u> : Suppose it is not known whether a quantity X is a tensor or not. If an inner product of X with an arbitrary tensor is itself a tensor, then X is also a tensor.

The line element and metric tensor -

In N-dimensional space with coordinates $(x^1, x^2, \dots, x^N)$ the line element ds is defined by quadratic form, called the metric form or metric.

$$ds^2 = g_{pq}\, dx^p dx^q$$

- The quantities g_{pq} are the components of a covariant tensor of rank two called the <u>metric tensor</u> or <u>fundamental tensor</u>. g_{pq} can be chosen symmetric.

<u>Proof</u> - $ds^2 = g_{pq}\, dx^p dx^q$ is a scalar or invariant.

Changing the dummy indices $p \to q$, $q \to p$,

$$ds^2 = g_{qp}\, dx^q dx^p = g_{qp}\, dx^p dx^q \quad \text{(ordinary product is commutative)}$$

$$\Rightarrow 2\, ds^2 = (g_{pq} + g_{qp})\, dx^p dx^q$$

$$2\, ds^2 = 2\, \tilde{g}_{pq}\, dx^p dx^q$$

$$\Rightarrow ds^2 = \tilde{g}_{pq}\, dx^p dx^q$$

where, $\tilde{g}_{pq} = \frac{1}{2}(g_{pq} + g_{qp})$ is symmetric.

Thus, <u>g_{pq} can be chosen symmetric.</u>

Now, since ds^2 is invariant,

$$\bar{g}_{pq}\, d\bar{x}^p d\bar{x}^q = g_{jk}\, dx^j dx^k = g_{jk} \frac{\partial x^j}{\partial \bar{x}^p} d\bar{x}^p \frac{\partial x^k}{\partial \bar{x}^q} d\bar{x}^q$$

$$= g_{jk} \frac{\partial x^j}{\partial \bar{x}^p} \cdot \frac{\partial x^k}{\partial \bar{x}^q}\, d\bar{x}^p d\bar{x}^q$$

$\Rightarrow g_{pq}$ is covariant tensor of rank two

The N-dimensional space where $ds^2 = g_{pq}\, dx^p dx^q$ is defined is, in general, called Riemannian space. In the special case where $\exists$ a coordinate transformation from x^j to $\bar{x}^k$ such that the metric form is transformed into diagnol form $(d\bar{x}^1)^2 + (d\bar{x}^2)^2 + \dots + (d\bar{x}^N)^2 \equiv d\bar{x}^k d\bar{x}^k$, then the space is called Euclidean space.

Conjugate or reciprocal tensor -

Let $g = |g_{pq}|$ denote the determinant with elements g_{pq} and suppose $g \neq 0$. Define g^{pq} by -

$$g^{pq} = \frac{G(p,q)}{g}, \quad G(p,q) \text{ is cofactor of } g_{pq}.$$

- g^{pq} is symmetric contravariant tensor of rank 2, called the conjugate or reciprocal tensor.

Proof - Since g_{jk} is symmetric, $G(j,k)$ is symmetric. and so $g^{jk} = G(j,k)/g$ is symmetric.

Let B^p be an arbitrary contravariant vector,

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$B_q = g_{pq} B^p$ is an arbitrary covariant vector.

Multiplying by g^{jq},

$$g^{jq} B_q = g^{jq} g_{pq} B^p = \delta^j_p B^p = B^j$$

$$\Rightarrow B^j = g^{jq} B_q$$

Since B^q is an arbitrary vector, g^{jq} is a contravariant tensor of rank two, by quotient law.

- $g_{jk}\, g^{pk} = \delta_j^p$

Proof - $g_{jk}\, G(j,k) = g$, where summation is over k only.

$$\Rightarrow g_{jk} \frac{G(j,k)}{g} = 1 \qquad \Rightarrow g_{jk}\, g^{jk} = 1$$

Again, $g_{jk}\, G(p,k) = 0 \;,\; p \neq j \;\Rightarrow\; g_{jk} \dfrac{G(p,k)}{g} = 0$

$$\Rightarrow g_{jk}\, g^{pk} = 0 \;,\; p \neq j$$

$$\therefore g_{jk}\, g^{pk} = \begin{cases} 0, & p \neq j \\ 1, & p = j \end{cases} \Rightarrow \underline{\underline{g_{jk}\, g^{pk} = \delta_j^{kp}}}$$

Associated Tensor -

Given a tensor, we can derive other tensors by raising or lowering indices. For example, given the tensor A_{pq} we obtain by raising the index p, the tensor $A^{p}_{\cdot q}$, the dot indicating the original position of moved index. Where no confusion can arise we shall often omit the dots.

These derived tensors can be obtained by forming inner products of the given tensor with the metric tensor g_{pq} or its conjugate g^{pq}.

$$A_{rq} \longrightarrow A^{p}_{\cdot q} = g^{rp} A_{rq}$$

$$A^{pq} = g^{rp} g^{sq} A_{rs}$$

$$A^{p}_{\cdot rs} = g_{rq} A^{pq}_{\cdot\cdot s} \;,\quad A^{qm\cdot tk}_{\cdot\cdot n} = g^{pk} g_{sn} g^{rm} A^{q\cdot st}_{\cdot r\cdot\cdot p}$$

All tensors obtained from a given tensor by forming inner products with the metric tensor and/or its conjugate are called associated tensors of the given tensor.

e.g., A^m and A_m are associated tensors and relation between them is $A_p = g_{pq} A^q$, $A^p = g^{pq} A_q$

In rectangular coordinates, $g_{pq} = 1$, if $p = q$ and 0 if $p \neq q$, so that $A_p = A^p$.

Length of a vector - A^p or A_p is $L^2 = A^p A_p = g^{pq} A_p A_q$
$= g_{pq} A^p A^q$

Angle between two vectors - A^p and B_p is $\cos\theta = \dfrac{A^p B_p}{\sqrt{(A^p A_p)(B^p B_p)}}$

The physical components of a vector A^p or A_p, denoted by A_u, A_v, A_w, are the projections of the vector on the tangents to the coordinate curves and are given in the case of orthogonal coordinates by -

$$A_u = \sqrt{g_{11}}\, A^1 = \frac{A_1}{\sqrt{g_{11}}} ; A_v = \sqrt{g_{22}}\, A^2 = \frac{A_2}{\sqrt{g_{22}}} ; A_w = \sqrt{g_{33}}\, A^3$$

Similarly, the physical components of a tensor A^{pq} or A_{pq} are -

$$A_{uu} = g_{11} A^{11} = \frac{A_{11}}{g_{11}} ; \quad A_{vv} = \sqrt{g_{11} g_{22}}\, A^{12} = \frac{A_{12}}{\sqrt{g_{11} g_{22}}} ;$$

$$A_{ww} = \sqrt{g_{11} g_{33}}\, A^{13} = \frac{A_{13}}{\sqrt{g_{11} g_{33}}} \quad \text{etc.}$$

Christoffel's symbol -

The symbols - $[pq, r] = \dfrac{1}{2}\left(\dfrac{\partial g_{pr}}{\partial x^q} + \dfrac{\partial g_{qr}}{\partial x^p} - \dfrac{\partial g_{pq}}{\partial x^r}\right)$

$\left\{ {s \atop pq} \right\} = g^{sr}[pq, r]$ are called the Christoffel symbols of the first and second kind respectively.

Transformation Laws for Christoffel symbols -

We denote by a bar a symbol in a coordinate system $\bar{x}^k$.

$$\overline{[jk, m]} = [pq, r] \frac{\partial x^p}{\partial \bar{x}^j} \cdot \frac{\partial x^q}{\partial \bar{x}^k} \cdot \frac{\partial x^r}{\partial \bar{x}^m} + g_{pq} \frac{\partial x^p}{\partial \bar{x}^m} \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}$$

$$\overline{\left\{ \begin{matrix} n \\ jk \end{matrix} \right\}} = \left\{ \begin{matrix} s \\ pq \end{matrix} \right\} \frac{\partial \bar{x}^n}{\partial x^s} \cdot \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial \bar{x}^n}{\partial x^q} \cdot \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}$$

These laws show that Christoffel symbols are not tensors unless the second terms on the right are zero.

Geodesics :

The distance S between two points t_1 and t_2 on a curve $x^r = x^r(t)$ in a Riemannian space is -

$$S = \int_{t_1}^{t_2} \sqrt{g_{pq} \frac{dx^p}{dt} \frac{dx^q}{dt}} \, dt$$

That curve in the space which makes the distance a minimum is called a geodesic of the space.

Particular Cases -

1. Geodesics on a plane are straight line.
2. Geodesics on a sphere are arcs of great circles.

- A necessary condition that $I = \int_{t_1}^{t_2} F(t, x, \dot{x})\, dt$ be an extremum is that -

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \quad \text{(Euler's or, Lagrange's eqn.)}$$

Proof: Let I be extremum on the curve $x = X(t)$, $t_1 \leq t \leq t_2$

Suppose $x = X(t) + \epsilon \eta(t)$, ($\epsilon$ independent of t) is a neighbouring curve through t_1 and t_2 so that $\eta(t_1) = \eta(t_2) = 0$. The value of I for the neighbouring curve is - $I(\epsilon) = \int_{t_1}^{t_2} F(t, x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta})\, dt$

A necessary condition that $I(\epsilon)$ be an extremum at $\epsilon = 0$ is that $\frac{dI}{d\epsilon} = 0$. Differentiating under integral sign,

$$\frac{dI}{d\epsilon}\bigg|_{\epsilon=0} = \int_{t_1}^{t_2}\left(\frac{\partial F}{\partial x}\eta + \frac{\partial F}{\partial \dot{x}}\dot{\eta}\right)dt = 0$$

Using by parts on second term,

$$\int_{t_1}^{t_2}\frac{\partial F}{\partial x}\eta\, dt + \underbrace{\frac{\partial F}{\partial \dot{x}}\eta\bigg|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2}\eta\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\eta\left[\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)\right]dt = 0$$

Since, η is arbitrary, $\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$

- Geodesics in a Riemannian space are found from the differential equation -

$$\frac{d^2x^r}{ds^2} + \left\{ {r \atop pq} \right\}\frac{dx^p}{ds}\frac{dx^q}{ds} = 0$$

Proof: $S = \int_{t_1}^{t_2}\sqrt{g_{pq}\dot{x}^p\dot{x}^q} \Rightarrow$ dist. between two points.

Geodesics is that curve for which dist. S is minimum.

Let $F = \sqrt{g_{pq}\dot{x}^p\dot{x}^q}$ so that $S = \int_{t_1}^{t_2} F\,dt$

For minimum, using Euler's eqn.,

$$\frac{\partial F}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right) = 0$$

$$\frac{\partial F}{\partial x^k} = \frac{1}{2\sqrt{g_{pq}\dot{x}^p\dot{x}^q}}\cdot\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q = \frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q$$

$$\frac{\partial F}{\partial \dot{x}^k} = \frac{1}{2\sqrt{g_{pq}\dot{x}^p\dot{x}^q}}\frac{\partial}{\partial \dot{x}^k}\left[g_{pk}\dot{x}^p\dot{x}^k + g_{kq}\dot{x}^k\dot{x}^q\right]$$

$$= \frac{1}{2\dot{s}}\left[g_{pk}\dot{x}^p + g_{kq}\dot{x}^q\right] = \frac{1}{2\dot{s}}\left[g_{pk}\dot{x}^p + g_{kp}\dot{x}^p\right]$$

$$= \frac{1}{\dot{s}}g_{pk}\dot{x}^p \quad [g_{pk} = g_{kp}]$$

By Euler's equation,

$$\frac{d}{dt}\left(\frac{g_{pk}\dot{x}^p}{\dot{s}}\right) - \frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q = 0$$

$$\Rightarrow g_{pk}\frac{\ddot{x}^p}{\dot{s}} + \frac{1}{\dot{s}}\frac{\partial g_{pk}}{\partial x^q}\dot{x}^p\dot{x}^q - \frac{1}{\dot{s}^2}g_{pk}\dot{x}^p\ddot{s} - \frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q = 0$$

$$\left[A = \frac{\partial g_{pk}}{\partial x^q}\dot{x}^p\dot{x}^q = \frac{\partial g_{qk}}{\partial x^p}\cdot\dot{x}^q\dot{x}^p \Rightarrow 2A = \left(\frac{\partial g_{pk}}{\partial x^q} + \frac{\partial g_{qk}}{\partial x^p}\right)\dot{x}^p\dot{x}^q\right.$$

$$\Rightarrow g_{pk}\ddot{x}^p + \frac{1}{2}\left[\frac{\partial g_{pk}}{\partial x^q} + \frac{\partial g_{qk}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^k}\right]\dot{x}^p\dot{x}^q = \frac{g_{pk}\dot{x}^p\ddot{s}}{\dot{s}}$$

Using arc length as parameter, we take $\dot{s} = 1$, $\ddot{s} = 0$.

Then above eqn. becomes -

$$g_{pk}\frac{d^2x^p}{ds^2} + [pq, k]\frac{dx^p}{ds}\cdot\frac{dx^q}{ds} = 0$$

Multiplying by g^{rk},

$$\delta^r_p\frac{d^2x^p}{ds^2} + \begin{Bmatrix} r \\ pq \end{Bmatrix}\frac{dx^p}{ds}\cdot\frac{dx^q}{ds} = 0$$

$$\Rightarrow \underline{\underline{\frac{d^2x^r}{ds^2} + \begin{Bmatrix} r \\ pq \end{Bmatrix}\frac{dx^p}{ds}\cdot\frac{dx^q}{ds} = 0}}$$

<u>The covariant derivative</u> - of a tensor A_p w.r.t. x^q is denoted by $A_{p,q}$ and is defined by

$$A_{p,q} = \frac{\partial A_p}{\partial x^q} - \begin{Bmatrix} s \\ pq \end{Bmatrix}A_s$$

The covariant derivative of A^p w.r.t. x^q is denoted by $A^p_{,q}$ and is defined by

$$A^p_{,q} = \frac{\partial A^p}{\partial x^q} + \begin{Bmatrix} p \\ qs \end{Bmatrix}A^s$$

<u>Note</u>: ① For rectangular system, the Christoffel's symbols are zero and covariant derivative reduces to usual partial derivative.

② The rules of covariant differentiation for sums and product of tensors are the same as those for ordinary differentiation.

③ Covariant derivatives express rates of change of physical quantities independent of any frames of reference.

Generalizing for high rank tensors, the covariant derivative of $A^{p_1 \dots p_n}_{r_1, \dots, r_n}$ w.r.t. x^q is $A^{p_1 \dots p_n}_{r_1 \dots r_n, q} = \frac{\partial A^{p_1 \dots p_n}_{r_1 \dots r_n}}{\partial x^q}$

$$- \left\{ {s \atop r_1 q} \right\} A^{p_1 \dots p_n}_{s r_2 \dots r_n} - \left\{ {s \atop r_2 q} \right\} A^{p_1 \dots p_n}_{r_1 s r_3 \dots r_n} - \dots - \left\{ {s \atop r_n q} \right\} A^{p_1 \dots p_n}_{r_1 r_2 \dots r_{n-1} s}$$

$$+ \left\{ {p_1 \atop q s} \right\} A^{s p_2 \dots p_n}_{r_1 \dots r_n} + \left\{ {p_2 \atop q s} \right\} A^{p_1 s p_3 \dots p_n}_{r_1 \dots r_n} + \dots + \left\{ {p_n \atop q s} \right\} A^{p_1 \dots p_{n-1} s}_{r_1 \dots r_n}$$

• Covariant Derivative of a tensor is also a tensor.

If A_p and A^p are tensors, then $A_{p,q}$ and $A^p_{,q}$ are tensors.

<u>Proof</u> – <u>To show $A_{p,q}$ is a tensor:</u>

$$A_{p,q} = \frac{\partial A_p}{\partial x^q} - \left\{ {s \atop pq} \right\} A_s$$

$$\bar{A}_j = \frac{\partial x^r}{\partial \bar{x}^j} A_r \Rightarrow \frac{\partial \bar{A}_j}{\partial \bar{x}^k} = \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial A_r}{\partial x^t} \frac{\partial x^t}{\partial \bar{x}^k} + \frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k} A_r \quad \text{—— (1)}$$

We now express $\frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k} A_r$ in terms of Christoffel symbol of 2nd kind. The transformation law for Christoffel symbol of 2nd kind is –

$$\overline{\left\{ {n \atop jk} \right\}} = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial \bar{x}^n}{\partial x^s} \left\{ {s \atop pq} \right\} + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial \bar{x}^n}{\partial x^p}$$

Multiplying by $\frac{\partial x^m}{\partial \bar{x}^n}$,

$$\frac{\partial x^m}{\partial \bar{x}^n} \overline{\left\{ {n \atop jk} \right\}} = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \delta^m_s \left\{ {s \atop pq} \right\} + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k} \delta^m_p \quad \left[\text{Using } \frac{\partial x^m}{\partial \bar{x}^n} \cdot \frac{\partial \bar{x}^n}{\partial x^s} = \delta^m_s \right]$$

$$= \frac{\partial x^p}{\partial \bar{x}^j} \cdot \frac{\partial x^q}{\partial \bar{x}^k} \left\{ {m \atop pq} \right\} + \frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k}$$

$$\Rightarrow \frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} = \overline{\left\{ {n \atop jk} \right\}} \frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \left\{ {m \atop pq} \right\} \quad \text{—— (2)}$$

Replacing m by r in (2) and substituting into (1),

$$\frac{\partial \bar{A}_j}{\partial \bar{x}^k} = \frac{\partial x^r}{\partial \bar{x}^j}\frac{\partial A_r}{\partial x^t}\frac{\partial x^t}{\partial \bar{x}^k} + \overline{\begin{Bmatrix} n \\ jk \end{Bmatrix}}\frac{\partial x^r}{\partial \bar{x}^n} A_r - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\begin{Bmatrix} r \\ pq \end{Bmatrix} A_r$$

$$\Rightarrow \frac{\partial \bar{A}_j}{\partial \bar{x}^k} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial A_p}{\partial x^q} + \overline{\begin{Bmatrix} n \\ jk \end{Bmatrix}}\bar{A}_n - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\begin{Bmatrix} r \\ pq \end{Bmatrix} A_r$$

$$\therefore \frac{\partial \bar{A}_j}{\partial \bar{x}^k} - \overline{\begin{Bmatrix} n \\ jk \end{Bmatrix}}\bar{A}_n = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\left[\frac{\partial A_p}{\partial x^q} - \begin{Bmatrix} r \\ pq \end{Bmatrix} A_r\right]$$

$$\Rightarrow \bar{A}_{j,k} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k} A_{p,q}$$

$\therefore$ $A_{p,q}$ is a covariant tensor of rank 2.

<u>To show :</u> <u>$A^p_{,q}$ is a tensor</u> –

$$A^p_{,q} \equiv \frac{\partial A^p}{\partial x^q} + \begin{Bmatrix} p \\ qs \end{Bmatrix} A^s$$

$$\bar{A}^j = \frac{\partial \bar{x}^j}{\partial x^r} A^r \Rightarrow \frac{\partial \bar{A}_j}{\partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial x^r}\frac{\partial A^r}{\partial x^t}\frac{\partial x^t}{\partial \bar{x}^k} + \frac{\partial^2 \bar{x}^j}{\partial x^r \partial x^t}\frac{\partial x^t}{\partial \bar{x}^k} A^r \quad \text{—— (1)}$$

We now express $\frac{\partial^2 \bar{x}^j}{\partial x^r \partial x^t}$ in terms of Christoffel symbol of 2nd kind. Transformation law for Christoffel symbol of 2nd kind is –

$$\begin{Bmatrix} n \\ jk \end{Bmatrix} = \frac{\partial \bar{x}^p}{\partial x^j}\cdot\frac{\partial \bar{x}^q}{\partial x^k}\cdot\frac{\partial x^n}{\partial \bar{x}^s}\overline{\begin{Bmatrix} s \\ pq \end{Bmatrix}} + \frac{\partial^2 \bar{x}^p}{\partial x^j \partial x^k}\frac{\partial x^n}{\partial \bar{x}^p}$$

Multiplying by $\frac{\partial \bar{x}^m}{\partial x^n}$,

$$\frac{\partial \bar{x}^m}{\partial x^n}\begin{Bmatrix} n \\ jk \end{Bmatrix} = \frac{\partial \bar{x}^p}{\partial x^j}\cdot\frac{\partial \bar{x}^q}{\partial x^k}\delta_s^m\overline{\begin{Bmatrix} s \\ pq \end{Bmatrix}} + \frac{\partial^2 \bar{x}^p}{\partial x^j \partial x^k}\cdot\delta_p^m$$

$$= \frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial \bar{x}^q}{\partial x^k}\overline{\begin{Bmatrix} m \\ pq \end{Bmatrix}} + \frac{\partial^2 \bar{x}^m}{\partial x^j \partial x^k}$$

$$\Rightarrow \frac{\partial^2 \bar{x}^m}{\partial x^j \partial x^k} = \begin{Bmatrix} n \\ jk \end{Bmatrix}\frac{\partial \bar{x}^m}{\partial x^n} - \frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial \bar{x}^q}{\partial x^k}\overline{\begin{Bmatrix} m \\ pq \end{Bmatrix}} \quad \text{—— (2)}$$

Replacing m, j, k by j, r, t in (2) and substituting into (1),

$$\frac{\partial \bar{A}^j}{\partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial x^r} \cdot \frac{\partial x^t}{\partial \bar{x}^k} \cdot \frac{\partial A^r}{\partial x^t} + \begin{Bmatrix} n \\ rt \end{Bmatrix} \frac{\partial \bar{x}^j}{\partial x^n} \cdot \frac{\partial x^t}{\partial \bar{x}^k} A^r - \frac{\partial \bar{x}^p}{\partial x^r} \frac{\partial \bar{x}^q}{\partial x^t} \overline{\begin{Bmatrix} j \\ pq \end{Bmatrix}} \frac{\partial x^t}{\partial \bar{x}^k} A^r$$

$$= \frac{\partial \bar{x}^j}{\partial x^r} \cdot \frac{\partial x^t}{\partial \bar{x}^k} \frac{\partial A^r}{\partial x^t} + \begin{Bmatrix} n \\ rt \end{Bmatrix} \frac{\partial \bar{x}^j}{\partial x^n} \cdot \frac{\partial x^t}{\partial \bar{x}^k} A^r - \frac{\partial \bar{x}^p}{\partial x^r} \delta^q_k \overline{\begin{Bmatrix} j \\ pq \end{Bmatrix}} A^r$$

$$= \frac{\partial \bar{x}^j}{\partial x^r} \cdot \frac{\partial x^t}{\partial \bar{x}^k} \frac{\partial A^r}{\partial x^t} + \begin{Bmatrix} r \\ st \end{Bmatrix} \frac{\partial \bar{x}^j}{\partial x^r} \cdot \frac{\partial x^t}{\partial \bar{x}^k} A^s - \overline{\begin{Bmatrix} j \\ pk \end{Bmatrix}} \bar{A}^p$$

$$\Rightarrow \frac{\partial \bar{A}^j}{\partial \bar{x}^k} + \overline{\begin{Bmatrix} j \\ pk \end{Bmatrix}} \bar{A}^p = \frac{\partial \bar{x}^j}{\partial x^r} \frac{\partial x^t}{\partial \bar{x}^k} \left[\frac{\partial A^r}{\partial x^t} + \begin{Bmatrix} r \\ st \end{Bmatrix} A^s \right]$$

$$\Rightarrow \overline{A^j_{,k}} = \frac{\partial \bar{x}^j}{\partial x^r} \frac{\partial x^t}{\partial \bar{x}^k} A^r_{,t}$$

$\therefore$ $A^p_{,q}$ is a mixed tensor of rank 2.

(i) $\dfrac{\partial x^p}{\partial x^q} = \delta^p_q$; (ii) $\dfrac{\partial x^p}{\partial \bar{x}^q} \dfrac{\partial \bar{x}^q}{\partial x^r} = \delta^p_r$

Proof - (i) for $p = q$, $\dfrac{\partial x^p}{\partial x^q} = 1$, since $x^p = x^q$.

for $p \neq q$, $\dfrac{\partial x^p}{\partial x^q} = 0$, since x^p and x^q are independent.

$$\therefore \frac{\partial x^p}{\partial x^q} = \delta^p_q$$

(ii) Coordinates x^p are functions of coordinates $\bar{x}^q$, which are in turn functions of coordinates x^r.

Then by chain rule,

$$\frac{\partial x^p}{\partial x^r} = \frac{\partial x^p}{\partial \bar{x}^q} \cdot \frac{\partial \bar{x}^q}{\partial x^r} = \delta^p_r \quad \left[\because \frac{\partial x^p}{\partial x^r} = \delta^p_r \right]$$

• The covariant derivatives of (a) g_{jk} ; (b) g^{jk} ; (c) δ^j_k are zero so that these quantities may be treated as constants in performing covariant differentiation.

Proof - (a) $g_{jk,q} = \dfrac{\partial g_{jk}}{\partial x^q} - \begin{Bmatrix} s \\ jq \end{Bmatrix} g_{sk} - \begin{Bmatrix} s \\ kq \end{Bmatrix} g_{sj}$

By def^n, $\begin{Bmatrix} s \\ pq \end{Bmatrix} = g^{sr} [pq, r] \Rightarrow g_{sk} \begin{Bmatrix} s \\ pq \end{Bmatrix} = \delta^r_k [pq, r]$

$$\Rightarrow [pq, k] = g_{sk} \begin{Bmatrix} s \\ pq \end{Bmatrix}$$

Using this result,

$$g_{jk,q} = \frac{\partial g_{jk}}{\partial x^q} - [jq,k] - [kq,j] \quad \text{———— (1)}$$

By def^n, $[jq,k] + [kq,j] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^q} + \cancel{\frac{\partial g_{qk}}{\partial x^j}} - \cancel{\frac{\partial g_{jq}}{\partial x^k}}\right) + \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^q} + \cancel{\frac{\partial g_{qj}}{\partial x^k}} - \cancel{\frac{\partial g_{kq}}{\partial x^j}}\right)$

$$= \frac{\partial g_{jk}}{\partial x^q} \quad [\because g_{ij} = g_{ji}] \quad \text{———— (2)}$$

Using (1) and (2), $g_{jk,q} = 0$

~~Using this result~~

(b) $$g^{jk}_{,q} = \frac{\partial g^{jk}}{\partial x^q} + \begin{Bmatrix} j \\ qs \end{Bmatrix} g^{sk} + \begin{Bmatrix} k \\ qs \end{Bmatrix} g^{js} \quad \text{———— (1)}$$

$$\frac{\partial}{\partial x^m}\left(g^{jk} g_{ij}\right) = \frac{\partial}{\partial x^m}\left(\delta_i^k\right) = 0$$

$$\Rightarrow g^{jk} \frac{\partial g_{ij}}{\partial x^m} + \frac{\partial g^{jk}}{\partial x^m} g_{ij} = 0 \quad \Rightarrow g_{ij} \frac{\partial g^{jk}}{\partial x^m} = - g^{jk} \frac{\partial g_{ij}}{\partial x^m}$$

Multiplying by g^{ir},

$$\delta_j^r \frac{\partial g^{jk}}{\partial x^m} = - g^{ir} g^{jk} \left([im,j] + [jm,i]\right)$$

$$\Rightarrow \frac{\partial g^{rk}}{\partial x^m} = - g^{ir} \begin{Bmatrix} k \\ im \end{Bmatrix} - g^{jk} \begin{Bmatrix} r \\ jm \end{Bmatrix}$$

Replacing r, m, i, j by j, q, s, s,

$$\frac{\partial g^{jk}}{\partial x^q} = - g^{sj} \begin{Bmatrix} k \\ sq \end{Bmatrix} - g^{sk} \begin{Bmatrix} j \\ sq \end{Bmatrix} \quad \text{———— (2)}$$

From (1) and (2), $g^{jk}_{,q} = 0$

(c) $$\delta^j_{k,q} = \frac{\partial \delta_k^j}{\partial x^q} - \begin{Bmatrix} s \\ kq \end{Bmatrix} \delta_s^j + \begin{Bmatrix} j \\ qs \end{Bmatrix} \delta_k^s$$

$$= 0 - \begin{Bmatrix} j \\ kq \end{Bmatrix} + \begin{Bmatrix} j \\ qk \end{Bmatrix} = 0$$

• Derive transformation laws for the Christoffel symbols of (a) the first kind, (b) the second kind.

Proof - (a) $\bar{g}_{jk} = \dfrac{\partial x^p}{\partial \bar{x}^j} \dfrac{\partial x^q}{\partial \bar{x}^k} g_{pq}$

$$\therefore \frac{\partial \bar{g}_{jk}}{\partial \bar{x}^m} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial g_{pq}}{\partial x^r}\frac{\partial x^r}{\partial \bar{x}^m} + \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial^2 x^q}{\partial \bar{x}^m \partial \bar{x}^k} g_{pq} + \frac{\partial^2 x^p}{\partial \bar{x}^m \partial \bar{x}^j}\cdot\frac{\partial x^q}{\partial \bar{x}^k} g_{pq} \quad \text{—— (1)}$$

By cyclic permutation of indices j, k, m by p, q, r,

$j \to k,\ k \to m,\ m \to j;\ p \to q,\ q \to r,\ r \to p$

$$\frac{\partial \bar{g}_{km}}{\partial \bar{x}^j} = \frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}\frac{\partial g_{qr}}{\partial x^p}\frac{\partial x^p}{\partial \bar{x}^j} + \frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^m} g_{qr} + \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m} g_{qr} \quad \text{—— (2)}$$

$j \to m,\ m \to k,\ k \to j;\ p \to r,\ r \to q,\ q \to p$

$$\frac{\partial \bar{g}_{mj}}{\partial \bar{x}^k} = \frac{\partial x^r}{\partial \bar{x}^m}\cdot\frac{\partial x^p}{\partial \bar{x}^j}\cdot\frac{\partial g_{rp}}{\partial x^q}\frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial x^r}{\partial \bar{x}^m}\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^j} g_{rp} + \frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^m}\frac{\partial x^p}{\partial \bar{x}^j} g_{rp} \quad \text{—— (3)}$$

(2) + (3) − (1), multiply by $1/2$ and using def.^{n} of Christoffel symbol of 1st kind,

$$[jk, m] = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}[pq, r] + \frac{1}{2}\left[\frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m} g_{qr} + \frac{\partial x^r}{\partial \bar{x}^m}\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^j} g_{rp}\right]$$
$$+ \frac{1}{2}\left[\frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^m} g_{qr}\frac{\partial x^q}{\partial \bar{x}^k} - \frac{\partial^2 x^p}{\partial \bar{x}^m \partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k} g_{pq}\right] + \frac{1}{2}\left[\frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^m}\frac{\partial x^p}{\partial \bar{x}^j} g_{rp} - \frac{\partial^2 x^q}{\partial \bar{x}^m \partial \bar{x}^k}\frac{\partial x^p}{\partial \bar{x}^j} g_{pq}\right]$$

We replace the indices as follows.

In 2nd term, $q \to p,\ r \to q$; in 3rd term, $r \to p$; in 4th term $r \to q$. Then we get -

$$[jk, m] = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}[pq, r] + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^m} g_{pq}$$

(b) The transformation law for Christoffel symbol of 1st kind is:

$$\overline{[jk, m]} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}[pq, r] + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^m} g_{pq} \quad \text{—— (1)}$$

Multiplying ① by $\bar{g}^{nm} = \dfrac{\partial \bar{x}^n}{\partial x^s} \dfrac{\partial \bar{x}^m}{\partial x^t} g^{st}$,

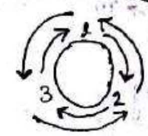
$$\bar{g}^{nm}[\overline{jk,m}] = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^m} \frac{\partial \bar{x}^n}{\partial x^s} \frac{\partial \bar{x}^m}{\partial x^t} g^{st}[pq,r] + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial x^q}{\partial \bar{x}^m} \frac{\partial \bar{x}^n}{\partial x^s} \frac{\partial \bar{x}^m}{\partial x^t} g^{st} g_{pq}$$

$$\Rightarrow \overline{\left\{ {n \atop jk} \right\}} = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial \bar{x}^n}{\partial x^s} \delta^r_t g^{st}[pq,r] + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial \bar{x}^n}{\partial x^s} \delta^q_t g^{st} g_{pq}$$

$$= \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial \bar{x}^n}{\partial x^s} \left\{ {s \atop pq} \right\} + \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial \bar{x}^n}{\partial x^p}$$

$$\left[\because \delta^r_t g^{st}[pq,r] = g^{sr}[pq,r] = \left\{ {s \atop pq} \right\} \text{ and, } \frac{\partial \bar{x}^n}{\partial x^s} \delta^q_t g^{st} g_{pq} = \frac{\partial \bar{x}^n}{\partial x^s} g^{sq} g_{pq} = \frac{\partial \bar{x}^n}{\partial x^s} \delta^s_p = \frac{\partial \bar{x}^n}{\partial x^p} \right]$$

Permutation Symbols -



We define e_{pqr} as follows

$e_{123} = e_{231} = e_{312} = +1$, $e_{213} = e_{132} = e_{321} = -1$.

$e_{pqr} = 0$ if two or more indices are equal.

e^{pqr} is defined in the same manner.

The symbols e_{pqr} and e^{pqr} are called permutation symbols in 3-dimensional space. Further we define,

$$\epsilon_{pqr} = \frac{1}{\sqrt{g}} e_{pqr} \quad ; \quad \epsilon^{pqr} = \sqrt{g}\, e^{pqr}$$

- ϵ_{pqr} and ϵ^{pqr} are covariant and contravariant tensor respectively. These are called <u>permutation tensors</u> in 3-D space.

<u>Note</u>: Generalization to higher dimensions are possible.

Tensor form of gradient, divergence and curl -

① **Gradient :** of an invariant (or scalar) ϕ is a covariant vector defined by $\vec{\nabla}\phi \equiv grad\ \phi = \phi_{,p} = \frac{\partial \phi}{\partial x^p}$

where, $\phi_{,p}$ is the covariant derivative of ϕ w.r.t. x^p.

② **Divergence :** of A^p is the contraction of its covariant derivative w.r.t. x^q, i.e., the contraction of $A^p_{,q}$ i.e.,

$$div\ A^p = A^p_{,p} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}\, A^k\right)$$

Note - $A^p_{,q} = \frac{\partial A^p}{\partial x^q} + \left\{ {p \atop qs} \right\} A^s$

$$\Rightarrow A^p_{,q} = \frac{\partial A^p}{\partial x^q} + g^{pr} A^s \frac{1}{2}\left(\frac{\partial g_{qr}}{\partial x^s} + \frac{\partial g_{sr}}{\partial x^q} - \frac{\partial g_{qs}}{\partial x^r}\right)$$

Setting $q = p$,

$$A^p_{,p} = \frac{\partial A^p}{\partial x^p} + g^{pr} A^s \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^s} + \frac{\partial g_{sr}}{\partial x^p} - \frac{\partial g_{ps}}{\partial x^r}\right)$$

$$A^s g^{pr}\frac{\partial g_{pr}}{\partial x^s} = \frac{1}{g} A^s \frac{\partial g}{\partial x^s} = \frac{1}{g} A^k \frac{\partial g}{\partial x^k} \quad \left[\because \frac{\partial g}{\partial x^m} = g g^{jr}\frac{\partial g_{jr}}{\partial x^m}\right]$$

$$g^{pr}\frac{\partial g_{sr}}{\partial x^p} - g^{pr}\frac{\partial g_{ps}}{\partial x^r} = g^{rp}\frac{\partial g_{sp}}{\partial x^r} - g^{pr}\frac{\partial g_{ps}}{\partial x^r} = 0$$

$$\therefore A^p_{,p} = \frac{\partial A^k}{\partial x^k} + \frac{1}{2g} A^k \frac{\partial g}{\partial x^k} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}\, A^k\right)$$

③ **Curl :** of A_p is a tensor of rank 2 and is defined by-

$$curl\ A_p = A_{p,q} - A_{q,p} = \frac{\partial A_p}{\partial x^q} - \frac{\partial A_q}{\partial x^p}$$

Note - $A_{p,q} = \frac{\partial A_p}{\partial x^q} - \left\{ {s \atop pq} \right\} A_s$, $A_{q,p} = \frac{\partial A_q}{\partial x^p} - \left\{ {s \atop qp} \right\} A_s$

$$\Rightarrow A_{p,q} - A_{q,p} = \frac{\partial A_p}{\partial x^q} - \frac{\partial A_q}{\partial x^p}$$

④ **Laplacian :** The laplacian of ϕ is the divergence of grad ϕ

i.e., $\nabla^2\phi = div\ \phi_{,p} = div\left(g^{pq}\phi_{,p}\right) = \left(g^{pq}\phi_{,p}\right)_{,q} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{jk}\frac{\partial \phi}{\partial x^k}\right)$

Note - $\phi_{,p} = \frac{\partial \phi}{\partial x^p}$, $div\left(g^{pq}\phi_{,p}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}\, g^{pk}\phi_{,p}\right)$

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{kj}\phi_{,k}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{jk}\frac{\partial \phi}{\partial x^k}\right)$$

The Intrinsic ~~and~~ or Absolute Derivative -

of A_p along a curve $x^q = x^q(t)$, denoted by $\frac{\delta A_p}{\delta t}$, is defined as the inner product of the covariant derivative of A_p and $\frac{\partial x^q}{\partial t}$ i.e., $A_{p,q}\frac{\partial x^q}{\partial t}$ and is given by-

$$\frac{\delta A_p}{\delta t} \equiv \frac{dA_p}{dt} - \left\{ {r \atop pq} \right\} A_r \frac{dx^q}{dt}$$

Similarly,
$$\frac{\delta A^p}{\delta t} \equiv \frac{dA^p}{dt} + \left\{ {p \atop qr} \right\} A^r \frac{dx^q}{dt}$$

The vectors A_p or A^p are said to move parallel, along a curve if their intrinsic derivatives along the curve are zero respectively.

Intrinsic derivatives of higher rank tensors are similarly defined.

Note :
$$\frac{\delta A_p}{\delta t} = A_{p,q}\frac{dx^q}{dt} = \left(\frac{\partial A_p}{\partial t} - \left\{ {s \atop pq} \right\} A_s\right)\frac{dx^q}{dt}$$
$$= \frac{\partial A_p}{\partial x^q}\frac{dx^q}{dt} - \left\{ {s \atop pq} \right\} A_s \frac{dx^q}{dt} = \frac{dA_p}{dt} - \left\{ {r \atop pq} \right\} A_r \frac{dx^q}{dt}$$

$$\frac{\delta A^p}{\delta t} = A^p_{,q}\frac{dx^q}{dt} = \left(\frac{\partial A^p}{\partial x^q} + \left\{ {p \atop qs} \right\} A^s\right)\frac{dx^q}{dt} = \frac{dA^p}{dt} + \left\{ {p \atop qr} \right\} A^r.$$

Relative and Absolute Tensors -

A tensor $A^{p_1 \dots p_m}_{r_1 \dots r_n}$ is called a relative tensor of weight w if its components transform according to the following equation -

$$\bar{A}^{q_1 \dots q_m}_{s_1 \dots s_n} = \left|\frac{\partial x}{\partial \bar{x}}\right|^w \frac{\partial \bar{x}^{q_1}}{\partial x^{p_1}} \cdots \frac{\partial \bar{x}^{q_m}}{\partial x^{p_m}} \frac{\partial x^{r_1}}{\partial \bar{x}^{s_1}} \cdots \frac{\partial x^{r_n}}{\partial \bar{x}^{s_n}} A^{p_1 \dots p_m}_{r_1 \dots r_n}$$

where, $J = \left|\frac{\partial x}{\partial \bar{x}}\right|$ is the Jacobian of the transformation.

If $w = 0$, the tensor is called absolute. If $w = 1$, the relative tensor is called a tensor density.

So far we have dealt with absolute tensors.

The operations of addition, multiplication, etc. of relative tensors are similar to those of absolute tensors.

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PROBLEMS :

① $\left\{ {p \atop pq} \right\} = \frac{\partial}{\partial x^q} \ln \sqrt{g}$

Sol) $g = g_{jk}\, G(j,k)$, sum over k only, where $G(j,k)$ is cofactor of g_{jk} in $g = |g_{jk}|$. Since, $G(j,k)$ does not contain g_{jr} explicitly, $\frac{\partial g}{\partial g_{jr}} = G(j,r)$.

g is a function of g_{jk}, where g_{jk} is function of x^q.

Then by Chain rule,

$$\frac{\partial g}{\partial x^m} = \frac{\partial g}{\partial g_{jr}} \cdot \frac{\partial g_{jr}}{\partial x^m} = G(j,r)\frac{\partial g_{jr}}{\partial x^m} = g\, g^{jr}\frac{\partial g_{jr}}{\partial x^m}$$

$$= g\, g^{jr}\left([jm, r] + [rm, j]\right) \qquad \left[g^{jk} = \frac{G(j,k)}{g}\right]$$

$$= g\left(\left\{ {j \atop jm} \right\} + \left\{ {r \atop rm} \right\}\right) = 2g \left\{ {j \atop jm} \right\}$$

$$\Rightarrow \left\{ {j \atop jm} \right\} = \frac{1}{2g}\frac{\partial g}{\partial x^m} = \frac{\partial}{\partial x^m} \ln \sqrt{g}$$

Homework :

② Write the law of transformation for the tensors :
ⓐ A^i_{jk} ; ⓑ B^{mn}_{ijk} ; ⓒ C^m .

③ A quantity $A(j,k,l,m)$ which is a function of coordinates x^i transforms to another coordinate system $\bar{x}^i$ according to the rule $\bar{A}(p,q,r,s) = \frac{\partial x^j}{\partial \bar{x}^p}\frac{\partial \bar{x}^q}{\partial x^k}\frac{\partial \bar{x}^r}{\partial x^l}\frac{\partial \bar{x}^s}{\partial x^m} A(j,k,l,m)$

Is the quantity a tensor? If so write the tensor in suitable notation. Give the contravariant and covariant order and rank.

④ Is dx^k a tensor? If so, state whether it is contravariant or covariant.

---

⑤ grad ϕ is a tensor.

<u>Proof</u>- ϕ is a function of x^k, where x^k is a function of $\bar{x}^j$. Therefore, $\frac{\partial \phi}{\partial \bar{x}^j} = \frac{\partial \phi}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^j}$

But ϕ is a scalar. Therefore $\phi(x^1, x^2, \ldots, x^n) = \bar{\phi}(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$

Therefore, $\dfrac{\partial \phi}{\partial \bar{x}^j} = \dfrac{\partial \bar{\phi}}{\partial \bar{x}^j}$

$\Rightarrow \dfrac{\partial \bar{\phi}}{\partial \bar{x}^j} = \dfrac{\partial x^k}{\partial \bar{x}^j} \cdot \dfrac{\partial \phi}{\partial x^k} \Rightarrow \dfrac{\partial \phi}{\partial x^k}$ is a covariant tensor of rank one. Therefore,

grad $\phi \equiv \dfrac{\partial \phi}{\partial x^k}$ is a covariant tensor of rank 1.

---

(6) A covariant tensor has components xy, $2y - z^2$, xz in rectangular coordinates. Find its covariant components in spherical coordinates.

Sol.) Let [redacted] A_j be covariant tensor in rectangular coordinates $x^1 = x$; $x^2 = y$, $x^3 = z$.

$A_1 = xy = x^1 x^2$; $A_2 = 2y - z^2 = 2x^2 - (x^3)^2$; $A_3 = xz = x^1 x^3$

Let $\bar{A}_k$ be transformed tensor in spherical coordinates $\bar{x}^1 = r$, $\bar{x}^2 = \theta$, $\bar{x}^3 = \phi$.

Now, $\bar{A}_k = \dfrac{\partial x^j}{\partial \bar{x}^k} A_j$

We know, $x^1 = \bar{x}^1 \sin \bar{x}^2 \cos \bar{x}^3$, $x^2 = \bar{x}^1 \sin \bar{x}^2 \sin \bar{x}^3$,
$x^3 = \bar{x}^1 \cos \bar{x}^2$

$\therefore \bar{A}_1 = \dfrac{\partial x^1}{\partial \bar{x}^1} A_1 + \dfrac{\partial x^2}{\partial \bar{x}^1} A_2 + \dfrac{\partial x^3}{\partial \bar{x}^1} A_3$

$= (\sin \bar{x}^2 \cos \bar{x}^3)\, x^1 x^2 + (\sin \bar{x}^2 \sin \bar{x}^3)\left[2x^2 - (x^3)^2\right] + \cos \bar{x}^2\, x^1 x^3$

$= \sin\theta \cos\phi\, r^2 \sin^2\theta \sin\phi \cos\phi + \sin\theta \sin\phi\,(2r\sin\theta\sin\phi - r^2\cos^2\theta)$
$+ \cos\theta\, r^2 \sin\theta \cos\theta \cos\phi$.

$\bar{A}_2 = \dfrac{\partial x^1}{\partial \bar{x}^2} A_1 + \dfrac{\partial x^2}{\partial \bar{x}^2} A_2 + \dfrac{\partial x^3}{\partial \bar{x}^2} A_3$

$= r\cos\theta\cos\phi \cdot r\sin^2\theta\cos\phi\sin\phi + r\cos\theta\sin\phi\,(2r\sin\theta\sin\phi - r^2\cos^2\theta)$
$+ (-r\sin\theta)\, r^2 \sin\theta\cos\theta\cos\phi$

$\bar{A}_3 = \dfrac{\partial x^1}{\partial \bar{x}^3} A_1 + \dfrac{\partial x^2}{\partial \bar{x}^3} A_2 + \dfrac{\partial x^3}{\partial \bar{x}^3} A_3$

$= (-r\sin\theta\sin\phi)\, r^2 \sin^2\theta \sin\phi\cos\phi + r\sin\theta\cos\phi\,(2r\sin\theta\sin\phi - r^2\cos^2\theta)$

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(7) (a) Show that $\frac{\partial A_p}{\partial x^q}$ is not a tensor even though A_p is a covariant tensor of rank one.

(b) Add suitable term to make it a tensor. What is it called ?

Sol.) (a) $\bar{A}_j = \frac{\partial x^p}{\partial \bar{x}^j} A_p \Rightarrow \frac{\partial \bar{A}_j}{\partial \bar{x}^j} \cdot \frac{\partial A_p}{\partial x^q} \frac{\partial x^q}{\partial x^k} + \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^j} A_p$

Due to 2^{nd} term on rhs, $\frac{\partial A_p}{\partial x^q}$ is not a tensor.

(b) If we add $-\left\{ {s \atop pq} \right\} A_s$, then it is called covariant derivative of A_p with respect to q, denoted by $A_{p,q}$ i.e.,

$$A_{p,q} = \frac{\partial A_p}{\partial x^q} - \left\{ {s \atop pq} \right\} A_s$$

We have already proved that $A_{p,q}$ is a covariant tensor of rank 2.

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(8) Show that the velocity of a fluid at any point is a contravariant tensor of rank one.

Sol.) The velocity of a fluid at any point has components $\frac{dx^k}{dt}$ in the coordinate system x^k. In the coordinate system $\bar{x}^j$ the velocity is $\frac{d\bar{x}^j}{dt}$. By chain rule,

$$\frac{d\bar{x}^j}{dt} = \frac{\partial \bar{x}^j}{\partial x^k} \frac{dx^k}{dt}$$

This shows that the $\frac{dx^k}{dt}$ is a contravariant tensor of rank one.

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(9) Show that δ^p_q is a mixed tensor of rank 2.

Sol.) $\bar{\delta}^j_k = \delta^j_k = 1$ if $j=k$ and 0 if $j \neq k$.

Now, $\delta^j_k = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^p}{\partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial \bar{x}^p} \frac{\partial x^q}{\partial \bar{x}^k} \delta^p_q$

$$\therefore \bar{\delta}^j_k = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^k} \delta^p_q$$

This shows that δ^p_q is a mixed tensor of rank 2.

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Homework :

(10) If A_r^{pq} and B_r^{pq} are tensors, prove that their sum and difference are tensors.

(11) If A_r^{pq} and B_t^{s} are tensors, prove that $C_{rt}^{pqs} = A_r^{pq} B_t^{s}$ is also a tensor.

(12) Let A_{rst}^{pq} be a tensor.

(a) Choose $p=t$ and show that A_{rsp}^{pq}, where the summation convention is [redacted] employed, is a tensor. What is its rank?

(b) Choose $p=t$ and $q=s$ and show similarly that A_{rqp}^{pq} is a tensor. What is its rank?

(13) Prove that the contraction of the tensor A_q^p is a scalar or invariant.

(14) Show that the contraction of the outer product of the tensors A^p and B_q is an invariant.

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(15) Show that an inner product of the tensors A_r^p and B_t^{qs} is a tensor of rank 3.

Sol) Outer product of A_r^p and $B_t^{qs} = A_r^p B_t^{qs}$.
Let us contract with respect to indices p and t. So, we set $t=p$. Now, $\bar{A}_k^j = \dfrac{\partial \bar{x}^j}{\partial x^p} \dfrac{\partial x^r}{\partial \bar{x}^k} A_r^p$.

$$\bar{B}_n^{lm} = \frac{\partial \bar{x}^l}{\partial x^q} \frac{\partial \bar{x}^m}{\partial x^s} \frac{\partial x^t}{\partial \bar{x}^n} B_t^{qs}$$

$$\therefore \bar{A}_k^j \bar{B}_n^{lm} = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial \bar{x}^l}{\partial x^q} \frac{\partial \bar{x}^m}{\partial x^s} \frac{\partial x^t}{\partial \bar{x}^n} A_r^p B_t^{qs}$$

Setting $j=n$,

$$\bar{A}_k^j \bar{B}_j^{lm} = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial \bar{x}^l}{\partial x^q} \frac{\partial \bar{x}^m}{\partial x^s} \cdot \frac{\partial x^t}{\partial \bar{x}^j} A_r^p B_t^{qs}$$

$$= \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial \bar{x}^l}{\partial x^q} \frac{\partial \bar{x}^m}{\partial x^s} \delta_p^t A_r^p B_t^{qs} = \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial \bar{x}^l}{\partial x^q} \frac{\partial \bar{x}^m}{\partial x^s} A_r^p B_p^{qs}$$

This shows that $A_r^p B_p^{qs} = C_r^{qs}$ is a tensor of rank three.

(16) A quantity $A(p,q,r)$ is such that in the coordinate system x^i, $A(p,q,r)\, B_r^{qs} = C_p^s$, where B_r^{qs} is an arbitrary tensor and C_p^s is a tensor. Prove that $A(p,q,r)$ is a tensor.

Sol.) In the transformed coordinate $\bar{x}^i$.

$$\bar{A}(j,k,l)\, \bar{B}_l^{km} = \bar{C}_j^m$$

Then, $$\bar{A}(j,k,l) \frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial \bar{x}^m}{\partial x^s} \frac{\partial x^r}{\partial \bar{x}^l} B_r^{qs} = \frac{\partial \bar{x}^m}{\partial x^s} \frac{\partial x^p}{\partial \bar{x}^j} C_p^s = \frac{\partial \bar{x}^m}{\partial x^s} \frac{\partial x^p}{\partial \bar{x}^j} A(p,q,r) B_r^{qs}$$

$$\Rightarrow \frac{\partial \bar{x}^m}{\partial x^s} \left[\frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} \bar{A}(j,k,l) - \frac{\partial x^p}{\partial \bar{x}^j} A(p,q,r) \right] B_r^{qs} = 0$$

Inner multiplication by $\frac{\partial x^n}{\partial \bar{x}^m}$,

$$\delta_s^n \left[\frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} \bar{A}(j,k,l) - \frac{\partial x^p}{\partial \bar{x}^j} A(p,q,r) \right] B_r^{qs} = 0$$

$$\Rightarrow \left[\frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} \bar{A}(j,k,l) - \frac{\partial x^p}{\partial \bar{x}^j} A(p,q,r) \right] B_r^{qn} = 0$$

Since, B_r^{qn} is arbitrary tensor, we have,

$$\frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} \bar{A}(j,k,l) - \frac{\partial x^p}{\partial \bar{x}^j} A(p,q,r) = 0$$

Inner multiplication by $\frac{\partial x^q}{\partial \bar{x}^m} \frac{\partial \bar{x}^n}{\partial x^r}$,

$$\delta_m^k \delta_l^n \bar{A}(j,k,l) - \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^m} \frac{\partial \bar{x}^n}{\partial x^r} A(p,q,r) = 0$$

$$\Rightarrow \bar{A}(j,m,n) = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^m} \frac{\partial \bar{x}^n}{\partial x^r} A(p,q,r)$$

This shows that $A(p,q,r) \equiv A_{pq}^r$ is a tensor.

Note- In the above problem, a special case of quotient law is established which states that if an inner product of a quantity X with an arbitrary tensor B is a tensor C, then X is a tensor.

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(17) If a tensor A_{st}^{pqr} is symmetric or skew-symmetric w.r.t. indices p and q in one coordinate system, show that it remains so w.r.t. p and q in any coordinate system.

(18) Show that every tensor can be expressed as the sum of two tensors of which one is symmetric and the other skew-symmetric in a pair of covariant or contravariant indices.

(19) If $\phi = a_{jk} A^j A^k$, show that we can always write $\phi = b_{jk} A^j A^k$, where b_{jk} is symmetric.

(20) Express in matrix notation the transformation equations for (a) a covariant vector, (b) a contravariant tensor of rank 2, assuming $N = 3$.

(21) Determine the metric tensor in (a) cylindrical and (b) spherical coordinates.

Note: In the above problem for both cases g is diagonal, because both coordinate systems are orthogonal.

For any orthogonal system, $g_{jk} = 0 \; ; \; j \neq k$

(22) Determine the conjugate metric tensor in (a) cylindrical and (b) spherical coordinates.

(23) Find (a) g and (b) g^{jk} corresponding to
$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6\,dx^1 dx^2 + 4\,dx^2 dx^3.$$

(24) If $A_j = g_{jk} A^k$, then show that $A^k = g^{jk} A_j$

(25) (a) Show that $L^2 = g_{pq} A^p A^q$ is an invariant.

(b) Show that $L^2 = g^{pq} A_p A_q$

Note: Length of vector A_p or A^p is $L = \sqrt{A_p A^p}$

(26) (a) If A^p and B^q are vectors, show that $g_{pq} A^p B^q$ is an invariant.

(b) Show that $\dfrac{g_{pq} A^p B^q}{\sqrt{(A^p A_p)(B^q B_q)}}$ is an invariant.

Note: Angle between 2 vectors A^p and B^q is $\cos\theta = \dfrac{g_{pq} A^p B^q}{\sqrt{(A^p A_p)(B^q B_q)}}$

if $\cos\theta = 0$ i.e., $g_{pq} A^p B^q = A^p B_p = 0$, vectors are orthogonal.

(27) Express
(a) A^{jkl} an

28) Prove
curves
given by

29) Prove
(a) $g_{12} =$

30) Prove

31) Evalu
(b) the

32) Dete
in (a)

33) Write
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(e) A^{jkl}_{mn}

34) Find
Note:
rules

35) Prov

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37) Prov

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compo

40) Expr

(27) Express the relationship between the associated tensors (a) A^{jkl} and A_{pqr} ; (b) $A^{\cdot k}_{j \cdot l}$ and A^{qkr} ; (c) $A^{p \cdot rs}_{\cdot q \cdot \cdot t}$ and $A^{\cdots sl}_{jqk}$

(28) Prove that the angles $\theta_{12}, \theta_{23}, \theta_{31}$ between the coordinate curves in a three dimensional coordinate system are given by - $\cos\theta_{12} = \dfrac{g_{12}}{\sqrt{g_{11} g_{22}}}$; $\cos\theta_{23} = \dfrac{g_{23}}{\sqrt{g_{22} g_{33}}}$; $\cos\theta_{31} = \dfrac{g_{31}}{\sqrt{g_{33} g_{11}}}$

(29) Prove that for an orthogonal coordinate system,
(a) $g_{12} = g_{23} = g_{31} = 0$; (b) $g_{11} = \dfrac{1}{g^{11}}$, $g_{22} = \dfrac{1}{g^{22}}$, $g_{33} = \dfrac{1}{g^{33}}$

(30) Prove (a) $[pq, r] = [qp, r]$; (b) $\left\{ {s \atop pq} \right\} = \left\{ {s \atop qp} \right\}$; (c) $[pq, r] = g_{rs} \left\{ {s \atop pq} \right\}$

(31) Evaluate the Christoffel symbols of (a) the first kind, (b) the second kind, for spaces where $g_{pq} = 0$ if $p \neq q$.

(32) Determine the Christoffel symbols of the second kind in (a) rectangular, (b) cylindrical, (c) spherical coordinates

(33) Write the covariant derivative with respect to x^q of each of the following tensors: (a) A_{jk} , (b) A^{jk}, (c) A^j_k , (d) A^j_{kl} , (e) A^{jkl}_{mn}.

(34) Find the covariant derivative of $A^j_k B^{lm}_n$ with respect to x^q.
Note: Thus covariant derivative of a product of tensors obey rules like those of ordinary derivatives of products.

(35) Prove: $(g_{jk} A^{km}_n)_{,q} = g_{jk} A^{km}_{n,q}$

(36) Prove that $\operatorname{div} A^p = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^k} (\sqrt{g} A^k)$

(37) Prove that $\nabla^2 \phi = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^k} \left(\sqrt{g}\, g^{kr} \dfrac{\partial \phi}{\partial x^r} \right)$

(38) Prove that $A_{p,q} - A_{q,p} = \dfrac{\partial A_p}{\partial x^q} - \dfrac{\partial A_q}{\partial x^p}$.

(39) Express the divergence of a vector A^p in terms of its physical components for: (a) Cylindrical, (b) Spherical coordinates.

(40) Express the Laplacian of ϕ, $\nabla^2\phi$ in: (a) Cylindrical, (b) Spherical coordinates.