$-2r\sin^2\theta/2 \geq -1 \Rightarrow 2r\sin^2\theta \geq \frac{1}{2}$

$r \geq \frac{1}{2\sin^2\theta}$

& $-2r\cdot\sin^2\theta/2 \leq 0$ → true for all r

$2r(1-\cos\theta) \leq 1$

$2r(1-\cos\theta) \leq 2$

$r \leq 1/2$

$1-\cos\theta \leq 2 \Rightarrow r \leq 1/2$

The explicit scheme is conditionally stable for $r = \frac{\nu\, dt}{(\delta x)^2} \leq \frac{1}{2}$

H.T → Stability for implicit & crank nicolson scheme.

16/3/2021

Stability for Linear PDE:

$\epsilon_j^n = A^n . e^{i\theta j}$, replace u_j^n by $\epsilon_j^n = A^n e^{i\theta j}$.

Find ξ = amplification factor = $\frac{A^{n+1}}{A^n}$

$$\left.\begin{array}{l} |\xi| \leq 1 \text{, stable} \\ |\xi| > 1 \text{, unstable} \end{array}\right\} \forall \theta$$

$r = \frac{\nu \delta t}{\delta x^2}$, $\nu > 0$, $r > 0$

Implicit scheme:-

$$u_j^{n+1} - u_j^n = r\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right).$$

$$\xi = \frac{1}{1 - 2r(\cos\theta - 1)} = \frac{1}{1 + 4r\sin^2\theta/2}$$

$$-1 \le \frac{1}{1 + 4r\sin^2\theta/2} \le 1 \quad \forall \theta,\ r > 0$$

~~$-1 \le 1 + 4r\sin^2\theta/2$~~

~~$4r\sin^2\theta/2$~~

$4r\sin^2\theta/2 \geq -2$

$1 + 4r\sin^2\theta/2 \geq -1$

$4r\sin^2\theta/2 \geq 0$

$1 + 4r\sin^2\theta/2 \geq 1$

$\frac{1}{1 + 4r\sin^2\theta/2} \le 1$

for any choice of r ⇒

Thus, the implicit scheme is unconditionally stable.

~~Q. Find the TE~~

Lax Equivalence Theorem:-

$u_j^n \cdot u(x,t)$. $u_j^n \to u(x_j, t_n)$ convergence as $\delta x, \delta t \to 0$

$|u_j^n - u(x_j, t_n)| \to$

A stable and consistent numerical scheme provides a converg.

Numerical solution of the linear PDE of parabolic or hyperbolic type (initial-boundary value problem) converge to the exact sol of the PDE iff the numerical scheme is consistent and stable.

TE: $O(\delta t^2, \delta x^2)$

→ merits of the numerical scheme

$t = 0$ → $t = t + \delta t$ $O(\delta t, \delta x^2)$

[HT] Show that Crank-Nicolson scheme is consistent & stable

$u_j^{n+1/2}$ (j, $n+1/2$) → expansion $u_t = \nu u_{xx}$

[HT2]

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\delta t} = \frac{u_{j-1}^n - 2\{\theta u_j^{n+1} + (1-\theta) u_j^{n-1}\} + u_{j+1}^n}{\delta x^2}$$

θ is a real no

Ⓐ Check for consistency when
(i) $\delta t = r\delta x$ (ii) $\delta t = r\delta x^2$

Find the values of θ for the consistent scheme

$\theta = 1/2$: Dufort-Frankel scheme

Ⓑ $u_j^{n+1} = u_j^{n-1} + \frac{2\delta t}{\delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).$

Leap frog scheme Central time central space

Prove that is unconditionally unstable

$u_t = \nu u_{xx}$ ~~Transport eq~~ Advection diffusion mechanisms.

The governing eq^n

$u_t + c u_x = \nu u_{xx}$

advection diffusion

$\boxed{u_t + u u_x = \nu u_{xx}}$ → non linear PDE

→ Transport eq^n

$u_t + c u_x = \nu u_{xx}$, $t > 0$, $0 < x < a$

I.C. $u(0, x) = f(x)$

B.C. $u(t, 0) = u_0$, $u(t, a) = u_a$, $t > 0$

Crank Nicolson Scheme

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{1}{2}\left[c\frac{\delta u}{\delta x}\Big|_j^{n+1} + c\frac{\delta u}{\delta x}\Big|_j^n\right] = \frac{\nu}{2}\left(\frac{\delta^2 u}{\delta x^2}\Big|_j^{n+1} + \frac{\delta^2 u}{\delta x^2}\Big|_j^n\right)$$

$\cos\theta + i\sin$

$-\omega t + a$

Central differences for space derivatives.

$$\frac{U_j^{n+1}-U_j^n}{\delta t}+\frac{1}{2}\left[C_j^{n+1}\frac{U_{j+1}^{n+1}-U_{j-1}^{n+1}}{2\delta x}+C_j^n\frac{U_{j+1}^n-U_{j-1}^n}{2\delta x}\right]$$

$$=\frac{\nu}{2}\left[\frac{U_{j-1}^{n+1}-2U_j^{n+1}+U_{j+1}^{n+1}}{\delta x^2}+\frac{U_{j-1}^n-2U_j^n+U_{j+1}^n}{\delta x^2}\right]$$

$\rightarrow$ $a_j U_{j-1}^{n+1}+b_j U_j^{n+1}+c_j U_{j+1}^{n+1}=d_j$, $n\geq 0$

$j=1,2,\dots,N-1$

Lab Task

Q $\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}=\alpha\frac{\partial^2 T}{\partial x^2}$, $0<x<1$

$u=0.1$, $\alpha=0.01$.

$T(0,t)=0$, $T(1,t)=100$, $T(x,0)=100x$

$\delta x\leq 0.2$ Solve by Crank-Nicolson Scheme.

$\boxed{U_t+cU_x=0}\rightarrow$ FTCS

$$\frac{U_j^{n+1}-U_j^n}{\delta t}+c\,\frac{U_{j+1}^n-U_{j-1}^n}{2\delta x}=0\rightarrow \text{unconditionally unstable}$$

$U_j^n=A^n e^{i\theta j}$, $\nu=c\,\delta t/\delta x$

$\xi=\frac{A^{n+1}}{A^n}$

$$\frac{A^{n+1}e^{i\theta j}-A^n e^{i\theta j}}{\delta t}+c\,\frac{A^n e^{i\theta(j+1)}-A^n e^{i\theta(j-1)}}{2\delta x}=0$$

$$\left(\frac{A^{n+1}}{A^n}\right)\frac{e^{i\theta j}-e^{i\theta j}}{\delta t}+c\,e^{i\theta j}\frac{(e^{i\theta}-e^{-i\theta})}{2\delta x}=0$$

A^{n+1} $\quad c\,e^{i\theta j}\frac{2i\sin\theta}{2\delta x}=0$

$$A^{n+1}-A^n=-\frac{\nu}{2}A^n\left(e^{i\theta}-e^{-i\theta}\right).$$

$\xi=1-i\nu\sin\theta$, $|\xi|^2=1+\nu^2\sin^2\theta\geq 1\ \forall\theta$

Unconditionally unstable.

22/3/21 $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}$, $t>0$, $0<x<a$

non-linear initial BVP

I.C: $u(x,0)=f(x)$: B.C.: $u(0,t)=U_0$

$u(a,t)=U_a$, $t>0$

Discretize by the Crank Nicolson scheme.

$$\frac{U_j^{n+1}-U_j^n}{\delta t}+\frac{1}{2}\left[u\frac{\partial u}{\partial x}\Big|_j^n+u\frac{\partial u}{\partial x}\Big|_j^{n+1}\right]=\frac{\nu}{2}\left[\frac{\partial^2 u}{\partial x^2}\Big|_j^n+\frac{\partial^2 u}{\partial x^2}\Big|_j^{n+1}\right]$$

$j=1,2,\dots,N-1$, $n\geq 0$

Use central difference scheme to discretize the space derivatives

$$\frac{U_j^{n+1}-U_j^n}{\delta t}+\frac{1}{2}\left[U_j^{n+1}\frac{U_{j+1}^{n+1}-U_{j-1}^{n+1}}{2\delta x}\right]-\frac{\nu}{2}\left[\frac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{\delta x^2}\right]$$

$$=\frac{\nu}{2}\left[\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{\delta x^2}-\frac{1}{2}U_j^n\frac{U_{j+1}^n-U_{j-1}^n}{2\delta x}\right].$$

Task is to obtain

$u^{n+1}=\left[U_1^{n+1}\ U_2^{n+1}\ \dots\ U_{N-1}^{n+1}\right]$

PAGE NO :

DATE :

Apply Newton's linearization technique to solve (*) iteratively

$$(u_j^{n+1})^{(k+1)} = (u_j^{n+1})^{(k)} + \Delta U_j^{n+1}$$

Obtain the ensuing tri-diagonal system which needs to be solved at any iteration

$\Rightarrow$ $a_j \Delta U_{j-1}^{n+1} + b_j \Delta U_j^{n+1} + c_j \Delta U_{j+1}^{n+1} = d_j$

with $\Delta U_0^{n+1} = \Delta U_N^{n+1} = 0$,

by solving (I), we get $\Delta U_j^{n+1}\ \forall j$

Find $(U_j^{n+1})^{(k+1)}$ and repeat the procedure

till $\max\limits_{1 \le j \le N-1} \left|\Delta U_j^{n+1}\right| < \epsilon$ — (**)

HT At every time level, $(n+1)$, the iteration process is continued till the convergence criteria $(**)$ is satisfied. To start the iteration $(U_j^{n+1})^{(0)}$, the initial guess is required, which are taken as the previous time step solu, i.e, $(U_j^{n+1})^{(0)} = U_j^n,\ \forall j$

NPTEL course $\rightarrow$ Boundary Value Problem

$u_t + u u_x = \nu u_{xx} \rightarrow$ nonlinear Burgers's equation

(*) $u_t + c u_x = \nu u_{xx} \rightarrow$ linear eqⁿ

(*) $u_t + c_1 u_x + c_2 u_y = \nu \nabla^2 u$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

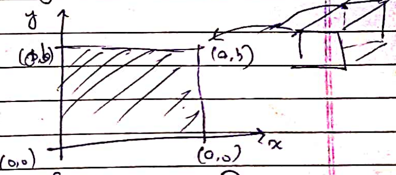
PAGE NO :

DATE :

(Q) $\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$, $t > 0$, $x, y \in \mathbb{R}$

$\mathbb{R}$: $0 < x < a$

$0 < y < b$



I.C.: $u(0, x, y) = f(x,y)$, $x, y \in \mathbb{R}$

B.C: $u(t, x, y)$ is prescribed on the boundary $\partial\Omega$ of $\mathbb{R}$.

$\partial\Omega$: $\left.\begin{array}{l} x = 0 \\ x = a \end{array}\right\}$ $0 < y < b$

and $\left.\begin{array}{l} y = 0 \\ y = b \end{array}\right\}$ $0 < x < a$

$\mathbb{R}$: $x^2 + y^2 \le a^2$

$\partial\Omega$: $x^2 + y^2 = a^2$

$\frac{\partial u}{\partial t} = \nu \nabla^2 u$, let the solution at t_n is known, we need to find the solution at t_{n+1}?

$u_{ij}^n = u(t_n, x_i, y_j)$, $n \ge 0$, $i = 0, 1, \dots, N$

$j = 0, 1, \dots, M$.

To find u_{ij}^{n+1} for $i = 1, 2, \dots, N-1$, $j = 1, 2, \dots, M-1$ when $n \ge 0$

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\delta t} = \frac{\nu}{2}\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\Big|_{ij}^{n+1} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\Big|_{ij}^{n}\right]$$

discretized by Crank-Nicolson scheme.

$u_{i+1,j}^{n+1}$, $u_{i-1,j}^{n+1}$, u_{ij}^{n+1}, $u_{i,j+1}^{n+1}$, $u_{i,j-1}^{n+1}$, $\rightarrow$ 5 unknowns in each eqⁿ

$(N-1)\times(M-1)$ linear algebraic eqⁿ involving $(N-1)\times(M-1)$ unknowns u_{ij}^{n+1}, which are not tri-diagonal system

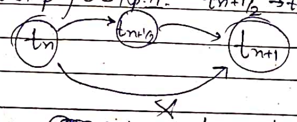
$$\frac{u_{ij}^{n+1}-u_{ij}^{n}}{\delta t}=\frac{\nu}{2}\left[\frac{u_{i+1j}^{n+1}-2u_{ij}^{n+1}+u_{i-1j}^{n+1}}{\delta x^2}+\frac{u_{ij+1}^{n+1}-2u_{ij}^{n+1}+u_{ij-1}^{n+1}}{\delta y^2}\right.$$

$$\left.+\frac{u_{i+1j}^{n}-2u_{ij}^{n}+u_{i-1j}^{n}}{\delta x^2}+\frac{u_{ij+1}^{n}-2u_{ij}^{n}+u_{ij-1}^{n}}{\delta y^2}\right]$$

let's say, $N, M = 99$. $\Rightarrow$ 98×98 equations.

Alternating direction Implicit Scheme (ADI) scheme

Solution at time level t_{n+1} is obtained by two-steps ① Step I, $t_n \to t_{n+1/2}$ (an intermediate time step) ② Step II. $t_{n+1/2} \to t_{n+1}$



Step I: ~~Discretize~~ $t_n \to t_{n+1/2}$, Discretize the derivatives w.r.t. x (or y) implicitly, and derivatives wrt y (or x) explicitly.

$$\frac{u_{ij}^{n+1/2}-u_{ij}^{n}}{\delta t/2}=\nu\left[\left.\frac{\partial^2 u}{\partial x^2}\right|_{ij}^{n+1/2}+\left.\frac{\partial^2 u}{\partial y^2}\right|_{ij}^{n}\right]$$

$$=\nu\left[\frac{u_{i+1j}^{n+1/2}-2u_{ij}^{n+1/2}+u_{i-1j}^{n+1/2}}{\delta x^2}+\frac{u_{ij+1}^{n}-2u_{ij}^{n}+u_{ij-1}^{n}}{\delta y^2}\right] \quad (*)$$

$i=1,2,\dots,N-1$

$j=1,2,\dots,M-1$

Here, variables are $u_{i-1j}^{n+1/2}, u_{ij}^{n+1/2}, u_{i+1j}^{n+1/2}$ $\forall j$

Eq^n (*) can be represented as

$$a_i\, u_{i-1j}^{n+1/2}+b_i\, u_{ij}^{n+1/2}+c_i\, u_{i+1j}^{n+1/2}=d_i \quad (**)$$

$i=1,2,\dots,N-1$

$j=1,2,\dots,M-1, \quad n\geq 0$

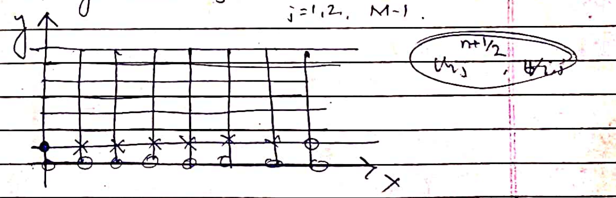
At a fixed j $(=1,2,\dots,M-1)$, eq^n (**) can be solved for all $i=1,2,\dots,N-1$, which leads to $A^j U_j^{n+1/2}=D_j$, $U_j^{n+1/2}=\begin{bmatrix} u_{1j}^{n+1/2}\\ u_{2j}^{n+1/2}\\ \vdots \\ u_{N-1j}^{n+1/2}\end{bmatrix}$

$A^j \to (N-1)\times(N-1)$ tri-diagonal system

Solve, $A^j U_j^{n+1/2}=D_j$, $j=1,2,\dots,M-1$

to get $u_{ij}^{n+1/2}$, $i=1,2,\dots,N-1$

$j=1,2,\dots,M-1$.



Step II: $t_{n+1/2}\to t_{n+1}$, $\to$ Discretize the derivatives w.r.t. x (or y) explicitly & derivatives wrt y (or x) implicitly.

$$\frac{u_{ij}^{n+1}-u_{ij}^{n+1/2}}{\delta t/2}=\nu\left[\left.\frac{\partial^2 u}{\partial x^2}\right|_{ij}^{n+1/2}+\left.\frac{\partial^2 u}{\partial y^2}\right|_{ij}^{n+1}\right]$$

$$\frac{u_{ij}^{n+1}-u_{ij}^{n+1/2}}{\delta t}=\frac{\nu}{2}\left[\frac{u_{i-1j}^{n+1/2}-2u_{ij}^{n+1/2}+u_{i+1j}^{n+1/2}}{\delta x^2}+\frac{u_{ij-1}^{n+1}-2u_{ij}^{n+1}+u_{ij+1}^{n+1}}{\delta y^2}\right]$$

$i=1,2,\dots,N-1$

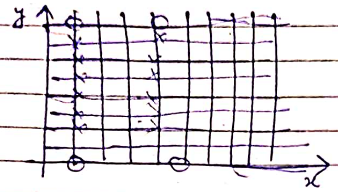
$j=1,2,\dots,M-1$

PAGE NO :
DATE :

$\Rightarrow a_j u_{i,j-1}^{n+1} + b_j u_{ij}^{n+1} + c_j u_{i,j+1}^{n+1} = d_i$

$j = 1, 2, \dots, M-1$

$A^i U^{n+1} = D^i$ For $i = 1, \dots, N-1$



implicit x

$n \xrightarrow{\text{explicit y}} n+\frac{1}{2} \xrightarrow[\text{implicit y}]{\text{explicit x}} n+1$

H.W. prove

ADI scheme, is $O(\delta t^2, \delta x^2, \delta y^2)$

Ex $\frac{\partial u}{\partial t} = \nabla^2 u, \quad -1 \le x, y \le 1, \quad t > 0$

$u(x, y, 0) = \cos\frac{\pi x}{2} \cdot \cos\frac{\pi y}{2}$

Lab Task

$\delta x, \delta y$

$u = 0, \quad x = \pm 1$

$u = 0, \quad y = \pm 1$

1) Crank-Nicolson scheme $\}$ $(N-1) \times (M-1)$

2) ADI scheme

$\delta x = \delta y = 1/2, \quad r = 1/6$

$\delta x = \delta y = 1/4, \quad r = 1/6$ $\}$

unconditionally stable

Q $u_t + u_x = \nu u_{xx}, \quad \nu > 0$

$u(x, 0) = \begin{cases} 1, & x < 0 \\ 0 & x > 0 \end{cases}$ Lab $(\nu = 1)$

$u(-5, t) = 1, \; u(5, t) = 0, \quad t \ge 0$

PAGE NO :
DATE :

H.W.

Q $u_t + u u_x = \nu u_{xx}$

(i) $u(x, 0) = \sin \pi x, \quad 0 < x < 1$ $\}$ H.W.

$u(0, t) = u(1, t) = 0, \; t > 0$

(Lab $\nu = 1$)

Also, $u(x, 0) = 4x(1-x) \quad 0 < x < 1$ $\}$ H.W. & Lab

(ii) $u(0, t) = u(1, t) = 0, \quad t > 0$