Group Theory

Lecture 9

Quotient Group :

Let G be a gp and H is a subgp of G. Consider the collection of all left cosets of H. We will discuss when we can define a gp structure on the set of all left cosets.

Put $G/H = \{aH \mid a \in G\}$.

$$aH \cdot bH = \{ah_1 b h_2 \mid h_1, h_2 \in H\}.$$

$$\overset{?}{=}$$

$$abH = \{abh \mid h \in H\}.$$

Let us assume H is a normal subgp.

$$ah_1 bh_2 = a(h_1 b) h_2 = abh' h_2$$

$[b^{-1} h_1 b \in H \Rightarrow b^{-1} h_1 b = h' \Rightarrow h_1 b = bh']$

Lemma: Let $N \trianglelefteq G$. Then the product of two left cosets aN and bN is again a left coset in fact

$$(aN)\cdot(bN) = abN$$

Pf: I have already shown the definition make sense.

Now I will show that it is well-defined.

Suppose $aN = a'N$ and $bN = b'N$

we need to show $abN = a'b'N$.

WTS $(a'b')^{-1}ab \in N$.

Since $aN = a'N \Rightarrow a'^{-1}a \in N$

and $bN = b'N \Rightarrow b'^{-1}b \in N$.

$$(a'b')^{-1}ab = b'^{-1}\underbrace{a'^{-1}a}b$$
$$= b'^{-1}nb \in N \text{ as } N \trianglelefteq G.$$

$a'^{-1}a \in N \Rightarrow a'^{-1}a = n$ for some $n \in N$.

The multiplication of two left cosets is well-defined.

$G/N = \{$ Set of all left cosets of N where N is a normal subgp of G $\}$

with the binary operation
$$aN \cdot bN = abN$$ forms a gp. which is known as the quotient gp.

(1) Associativity you can check.

(2) Identity : N is the identity elt

$$gN \cdot N = gN \quad \text{and} \quad N \cdot gN = gN$$

(3) Inverse :

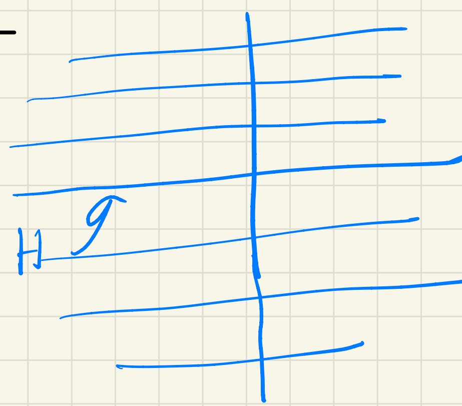
$$\boxed{(gN)^{-1} = g^{-1}N}$$

$$\overset{N}{\overset{\|}{gg^{-1}N}} = gN \cdot g^{-1}N = N.$$

Thus G/N is a group which is known as the quotient gp.

<u>First Isomorphism Thm</u> :

$$V/H \cong \mathbb{R}.$$



Let $\phi : G \to G'$ be a surjective gp homo. Then $G/\ker\phi \cong G'$.

<u>pf</u>: Let $N = \ker\phi \trianglelefteq G$.

$$\bar{\phi} : G/N \longrightarrow G' \quad \text{by}$$

$$\bar{\phi}(gN) = \phi(g)$$

First we show that $\bar{\phi}$ is well defined.

Let $gN = g_1N$ wTS $\phi(g) = \phi(g_1)$

$\hookrightarrow g_1^{-1} g \in N$.

$$\boxed{\phi(g_1^{-1}) = \phi(g_1)^{-1}}$$

$$\phi(g_1^{-1} g) = 1.$$

$$\Rightarrow \phi(g_1^{-1})\,\phi(g) = 1.$$

$$\Rightarrow \phi(g_1)^{-1}\phi(g) = 1 \Rightarrow \phi(g) = \phi(g_1).$$

Since ϕ is surjective so is $\bar{\phi}$.

Now want to show $\bar{\phi}$ is inj.

Let $\bar{\phi}(gN) = \bar{\phi}(g_1 N)$

NTS $gN = g_1 N$.

$\phi(g) = \phi(g_1)$

$\Rightarrow \phi(g_1^{-1} g) = 1$

$\Rightarrow g_1^{-1} g \in N$.

$\Rightarrow gN = g_1 N$.

$\therefore \bar{\phi}$ is injective.

Check $\bar{\phi}$ is a gp homo.

Hence $\bar{\phi}$ is an isomorphism.

$\therefore G/\ker\phi \cong G'$.

Example 1. $\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^{\times}$

det is a gp homo.

$\ker(\det) = SL_n(\mathbb{R})$

det is a surjective gp homo.

By 1st isomorphism Thm,

$$GL_n(\mathbb{R}) / SL_n(\mathbb{R}) \cong \mathbb{R}^{\times}$$

(2). $\phi : \mathbb{C}^{\times} \longrightarrow \mathbb{R}^{\times}_{>0} = \{$non-zero (+)ve real numbers wrt multiplic$\}$

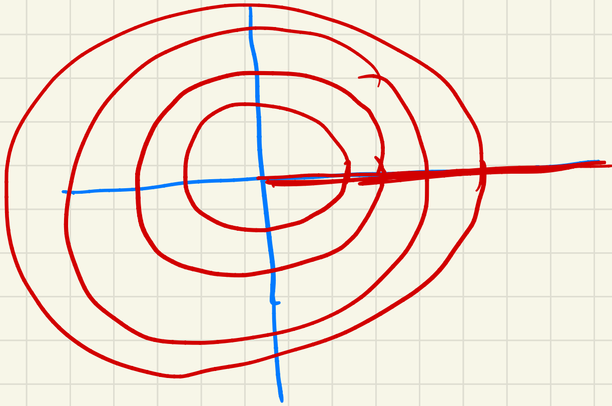
$\phi(z) = |z|$

$\ker\phi = \{ z \in \mathbb{C}^{\times} \mid |z| = 1 \} := U$

= unit circle with centre at $(0,0)$.

ϕ is surjective gp homo.

$$\mathbb{C}^{\times} / U \cong \mathbb{R}^{\times}_{>0}.$$



(3) $\det : O_n(\mathbb{R}) \longrightarrow \{1, -1\}$

$$\ker(\det) = SO_n$$

$$\therefore \quad O_n(\mathbb{R}) / SO_n \cong \mathbb{Z}/2\mathbb{Z}.$$

$[O_n(\mathbb{R}) : SO_n(\mathbb{R})]$ = no. of distinct left cosets

$= \left| O_n(\mathbb{R}) / SO_n(\mathbb{R}) \right|$

$= \left| \mathbb{Z}/2\mathbb{Z} \right|$

$= 2.$

Q24 $[m,n] = \mathrm{lcm}(m,n) = r$.

WTS $|xy| \mid [m,n]$

$\Rightarrow$ WTS $(xy)^r = 1$.

$(xy)^r = x^r y^r$ $[xy = yx]$.

$\hookrightarrow$ $\underbrace{xy\,xy \cdots xy}$

$\hookrightarrow = x^{mr'} y^{nr''}$

In S_3

$x = (1\,2)$ $y = (1\,2\,3)$.

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1/2 \\ 2 & 0 \end{pmatrix}$ $|AB| = \infty$

$|A| = 4$, $|B| = 2$, $AB = \begin{pmatrix} 2 & 0 \\ 0 & -1/2 \end{pmatrix}$

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

Q17. $G = S_3$.

$H = \langle (12) \rangle$ $\qquad$ $K = \langle (23) \rangle$.

$HK = \{ (1), (12), (23), (123) \}$.

$(12)(23)$

Q19 $\quad G = H_1 \cup H_2$.

$x \in H_1 \setminus H_2$. $\qquad y \in H_2 \setminus H_1$

$xy \in G$. let $xy \in H_1$

$x^{-1}(xy) \in H_1$

$\Rightarrow y \in H_1$

Tut 2

Q1 $\sigma(a_1 \cdots a_k)\sigma^{-1} = (\sigma(a_1) \cdots \sigma(a_k))$

$\sigma(a_1 \cdots a_k)\sigma^{-1}(\sigma(a_1))$

$= \sigma(a_1 \cdots a_k)(a_1)$

$= \sigma(a_2)$

Q3 $i_a : G \longrightarrow G$

$B(G)$ collection of all bijections

$\psi : G \longrightarrow B(G)$

$\psi(a) = i_a$

$\ker\psi = \{a \in G \mid \psi(a) = id\}$

$= \{a \in G \mid i_a = Id\}$

$= \{a \in G \mid i_a(g) = g \quad \forall g \in G\}$

$= \{a \in G \mid aga^{-1} = g \ \forall g \in G\}.$

$= Z(G).$

Q6. Show that Any cyclic gp is isomorphic with $\mathbb{Z}/n\mathbb{Z}$ for some $n = 0, 1, \ldots$?

Q7. $\phi : G \longrightarrow G$ be a gp automorph

$Aut(G) = \{$ Set of all automorphisms of $G \}.$

check: $Aut(G)$ is a gp wrt composition operation.

$f_1, f_2 \in Aut(G)$

$f_1 \circ f_2 \in Aut(G).$

Q What is $\mathrm{Aut}(\mathbb{Z})$?

$\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$.

$$\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}$$

$1 \longmapsto 1 \Rightarrow \phi(x) = x$

or $1 \longmapsto -1 \Rightarrow \phi(x) = -x$.

There are two automorphisms.

$|\mathrm{Aut}(\mathbb{Z})| = 2$.

$\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

$S_3 = \langle s, t \mid s^2 = 1,\ t^3 = 1,\ ts = st^2 \rangle$

$s \to$ 2-cycle

$t \to$ 3-cycle.

$\phi : S_3 \longrightarrow S_3$.

let $x \in S_3$ s.t $|x| = 2$.

$|\phi(x)| = ?$ if ϕ is an isomorphism?

Ex. Let $\phi : G \to G'$ be a gp isomorphism

~~Sol~~ If $x \in G$ s.t $|x| = n$ then show that $|\phi(x)| = n$.

$s \longmapsto s$ $t \longmapsto t$	$s \longmapsto s$ $t \longmapsto t^2$	$s \longmapsto st$ $t \longmapsto t$
$s \longmapsto st$ $t \longmapsto t^2$	$s \longmapsto st^2$ $t \longmapsto t$	$s \longmapsto st^2$ $t \longmapsto t^2$

$$\mathrm{Aut}(S_3) \cong S_3 .$$