Since $v = \frac{\pi x}{L}$ &

$f(v) = f(x)$,

formula (3) given eq^n (1) (how?).

In (4), we introduce,

$x = \frac{Lv}{\pi}$ as variable of integration

then the limits of integration

$v = \pm\pi$ become $x = \pm L$

Also, $v = \frac{\pi x}{L}$

$\Rightarrow dv = \frac{\pi}{L} dx$

Thus $\frac{dv}{2\pi} = \frac{dx}{2L}$ in a_0.

$\therefore \quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.$

S'lly, $\frac{dv}{\pi} = \frac{dx}{L}$ in a_n & b_n.

$$\therefore \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$n = 1, 2, \ldots$

$$\& \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$n = 1, 2, \ldots$

Thus (4) gives (2). //

Note

Interval of integration

In eqn (2), we may replace the interval of integration by any interval of length integration by any interval of length $p = 2L$, e.g., by the interval $0 \leq x \leq 2L$.

H.W

EX/ $a_0 = \frac{1}{2L} \int_{c}^{c+2L} f(x)\, dx$ etc.

Done in the next lecture

~~***~~

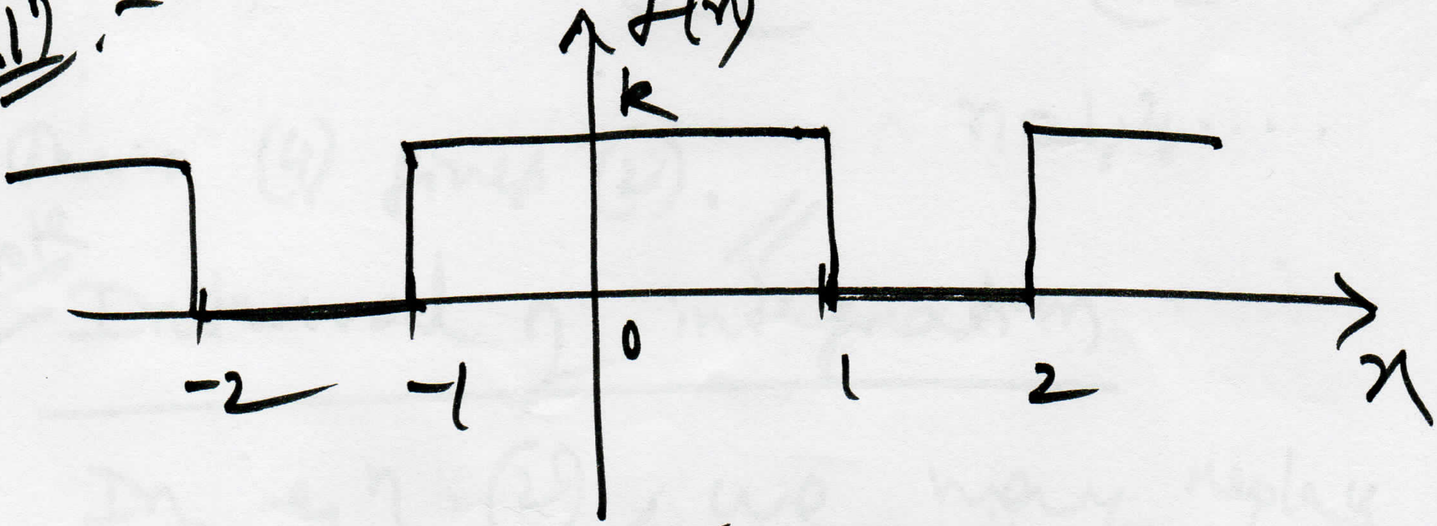
EX/ Periodic square wave

Find the Fourier Series of the function

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$

$\therefore P = 2L = 4, \; L = 2.$

Sol^n :-



From (2a) & (2b), we obtain

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{4}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{4}\int_{-1}^{1} k\,dx.$$

$$a_0 = \boxed{k/2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1} k\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{k}{2} \cdot \left[2 \frac{\sin(n\pi x/2)}{n\pi} \right]_{-1}^{1}$$

$$a_n = \frac{2k}{n\pi} \sin(n\pi/2).$$

Thus, $a_n = 0$, if n is even

$$a_n = \begin{cases} \frac{2k}{n\pi}, & \text{if } n = 1, 5, 9, \ldots \\ -\frac{2k}{n\pi}, & \text{if } n = 3, 7, 11, \ldots \end{cases}$$

From eqn (2c), we find that

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$\therefore\ b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right) dx$

$= \frac{1}{2}\int_{-1}^{1} k \sin\left(\frac{n\pi x}{2}\right) dx$

$= 0$ (as $\sin(n\pi x/2)$ is an odd f^n) (how?)

$\therefore$ the reqd. Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\| = \left(\frac{k}{2}\right) + \frac{2k}{\pi}\left[\cos\left(\frac{\pi}{2}x\right) - \frac{1}{3}\cos\left(\frac{3\pi}{2}x\right) + \frac{1}{5}\cos\left(\frac{5\pi}{2}x\right) - \cdots\right]$$

Mid-Sem Exam

Time: 2 hours

Marks: 30

1) Laplace Transform
Def^n & existence of L.T, convolution, Partial Fraction
Solving O.D. eq^{n}s with L.T (both I.V.P & B.V.P)
Integral eq^{n}s, Properties of L.T, periodic f^n, Special f^{n}s
Error f^{n}s.

2) Fourier Series.
of any length 2L.

—

—

H.W

Ex2

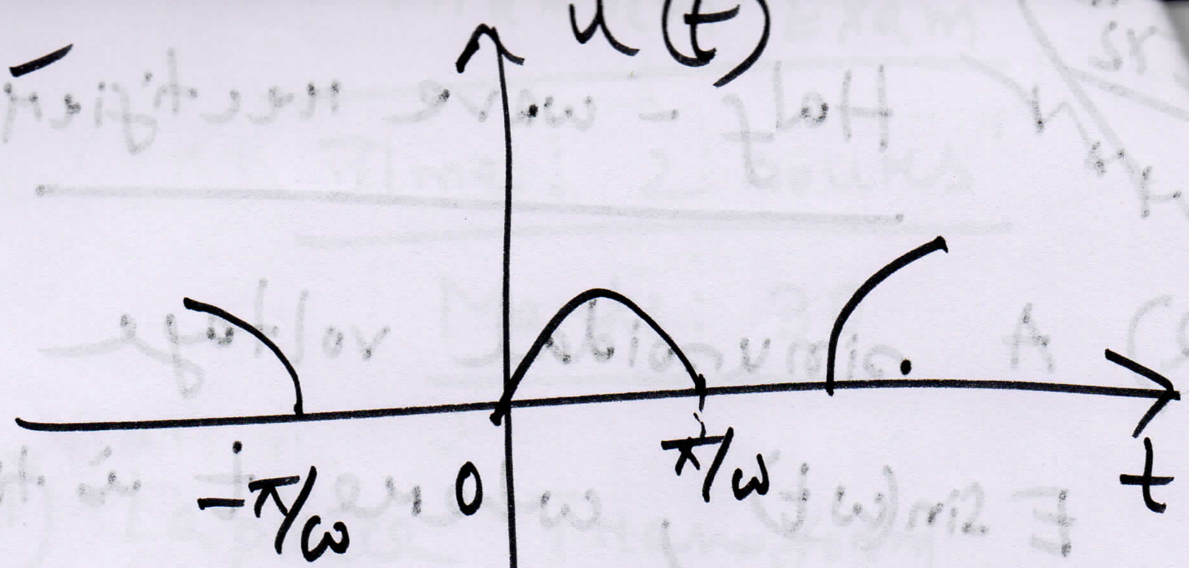
Half - wave rectifier

Q) A sinusoidal voltage $E \sin(\omega t)$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave. Find the Fourier series of the resulting periodic function.

$$u(t) = \begin{cases} 0, & \text{if } -L < t < 0 \\ E \sin \omega t, & \text{if } 0 < t < L \end{cases}$$

$P = 2L = 2\pi/\omega$, $L = \pi/\omega$

Sol^n.



since $u=0$, where $-L<t<0$

we obtain from (2a), with t instead x.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$= \frac{\omega}{2\pi}\int_0^{\pi/\omega} E\sin(\omega t)\,dt = \boxed{E/\pi}$$

& from (2b), with $x=\omega t$, $y=n\omega t$

$= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$a_n = \frac{1}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin\omega t\,\cos(n\omega t)\,dt.$$

[$x=\omega t$, $y=n\omega t$.

[illegible] =

$$a_n = \frac{\omega E}{2\pi}\int_0^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt$$

If $n=1$, the integral on the right is zero & if $n=2,3,\ldots$, we obtain

$$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$= \frac{E}{2\pi}\left[\frac{-\cos(1+n)\pi+1}{1+n} + \frac{-\cos(1-n)\pi+1}{(1-n)}\right]$$

If n is odd, this is equal to zero, & for even n, we have

$$a_n = \begin{cases} \dfrac{E}{2\pi}\left(\dfrac{2}{1+n}+\dfrac{2}{1-n}\right) = -\dfrac{2E}{(n+1)(n-1)\pi}, & (n=2,4,\ldots) \\ =0, \text{ otherwise}. \end{cases}$$

S/ly, from (2i), we find that $b_n = 0$, for $n=2,3,\ldots$

$$b_1 = \boxed{E/2}$$

& $\sin(n\pi)$ is odd.

Consequently, the reqd. F.S. (Fourier series)

is $$u(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$E/\pi + E/2 \sin \omega t$$

$$= \frac{2E}{\pi}\left[\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \dots\right)$$

Note :— The function in this example is even & have only cosine terms in its Fourier series, no sine terms. This is typical. Infact, unnecessary work (& corresponding sources of errors) in determining Fourier co-efficients can be avoided if a function is even or odd.

Even & Odd functions

↓

(Half - Range Expansions)

Even & Odd Functions

A function $y = g(x)$ is **even** if $g(-x) = g(x), \forall x$.

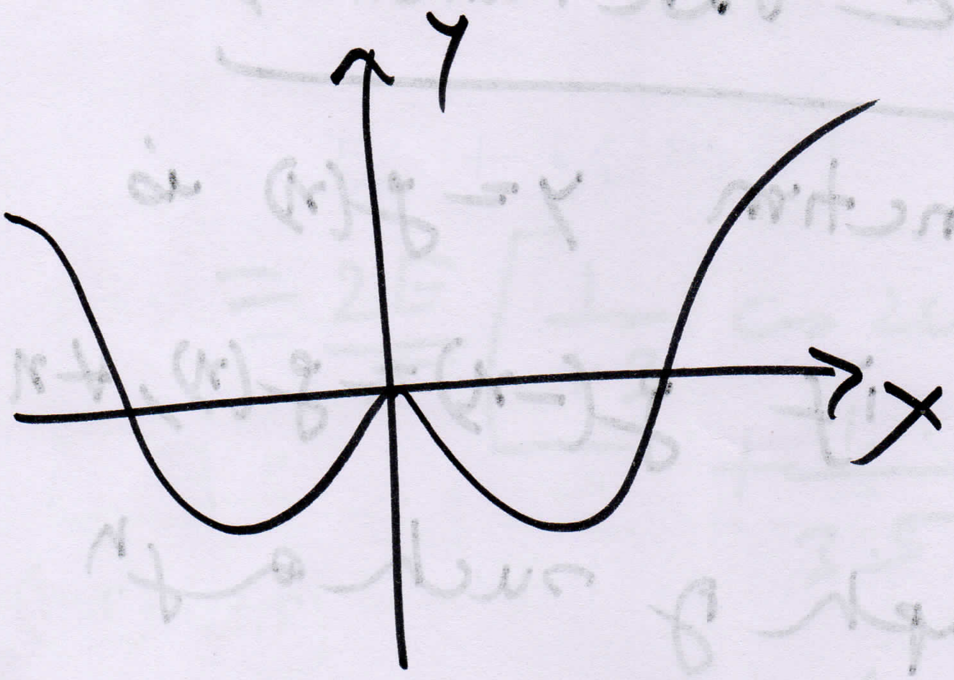
The graph of such a f^n is symmetric w.r.t. the Y-axis.

A function $h(x)$ is **odd** if $h(-x) = -h(x)$ $\forall x$
(i.e, $h(x)$ is anti-symmetric)

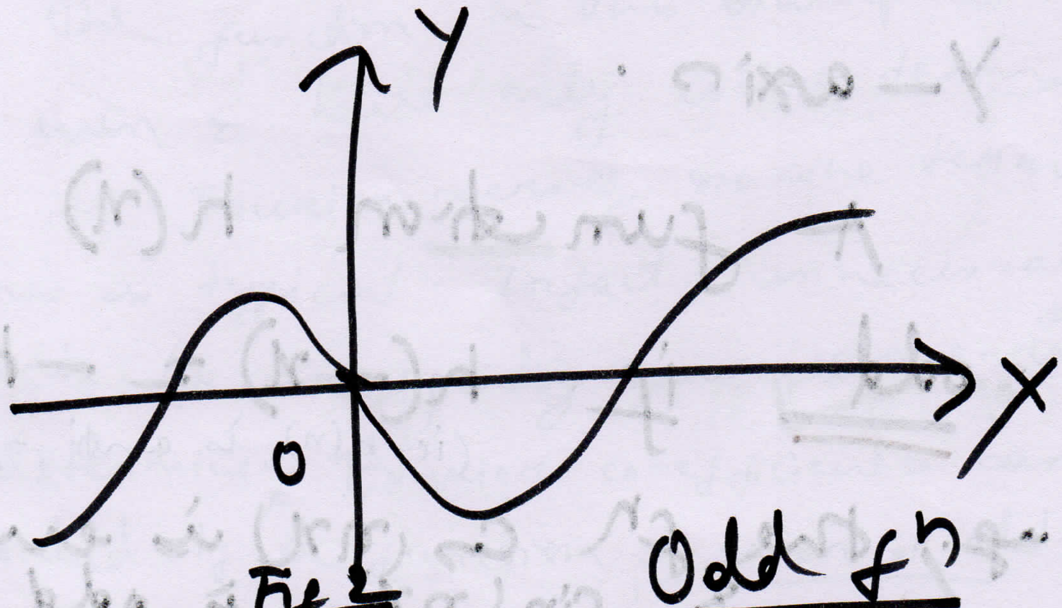
eg, the f^n $\cos(nx)$ is even & $\sin(nx)$ is odd.

ie., $h(x)$ is anti-symmetric.

It shows symmetry w.r.t the origin.



<u>Fig 1</u>:- <u>Even f^n</u>



<u>Fig 2</u> <u>Odd f^n</u>

Note :- 1) If $f(x)$ is an even f^n,

H.W

then $$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx,$$ (f even) $\rightarrow (1)$

H.W

2) If $h(x)$ is an odd f^n,

then $$\int_{-L}^{L} h(x)\,dx = 0$$ (h odd) $\rightarrow (2)$

***

3) The product of an even & an odd f^n is odd.

proof :- 3) Now, $g = fh$, even f & odd h, because

$$q(-x) = g(-x)\, h(-x)$$

$$= g(x)\,(-h(x))$$

$$= -\underbrace{g(x)\, h(x)}\ = -q(x)$$

Hence, $\therefore$ $q(x)$ is **odd**

if $f(x)$ is even, then

$f(x) \sin\left(\frac{n\pi x}{L}\right)$ is odd,

so (2) implies that $b_n = 0$

in eqn (2c).

Slly, if $f(x)$ is odd, so is

$f(x) \cos\left(\frac{n\pi x}{L}\right)$ & $a_n = 0$, $a_0 = 0$

in (2a) & (2b)

Fourier Cosine Series

Fourier sine series

Def^n:- The Fourier series of an even f^n of period $2L$ is a "Fourier Cosine series"

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) \rightarrow (3) \quad (f \text{ even})$$

with co-efficients

$$a_0 = \frac{1}{L}\int_0^L f(x)\, dx,$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx, \rightarrow (4) \quad n = 1, 2, \ldots$$

The Fourier series of an

<u>odd f^n of period $2L$</u>

is a "<u>Fourier sine series</u>"

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad \rightarrow (5) \quad (f \text{ odd})$$

with co-efficients

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx \quad \rightarrow (6).$$

The case of period 2π

In this case, Th-1 gives for an even f^n simply

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad (f \text{ even})$$

with co-efficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \rightarrow (3^*)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx, \quad n = 1, 2, \ldots \rightarrow (4^*)$$

S|||y, for an odd 2π-periodic f^n, we simply have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$(f \text{ odd})$

$\rightarrow (5^*)$

with co-efficients:

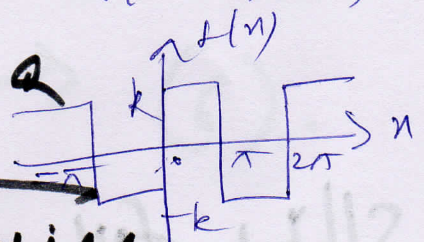
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx,$$

$$n = 1, 2, \dots$$

$\rightarrow (6^*)$

eg, $f(x)$ in $\underline{\text{EX1}}$ (Lec-14) (Rectangular wave), i.e., $f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$ & $f(x+2\pi) = f(x)$

is odd & therefore is represented by a Fourier sine series.



$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$$

** EX

Ch-2

Sum of functions

The Fourier co-efficients of a sum $\underline{f_1 + f_2}$ are the sums of the corresponding Fourier co-efficients of f_1 & f_2.

The Fourier co-efficients of cf are c times the corresponding Fourier co-efficient of f.

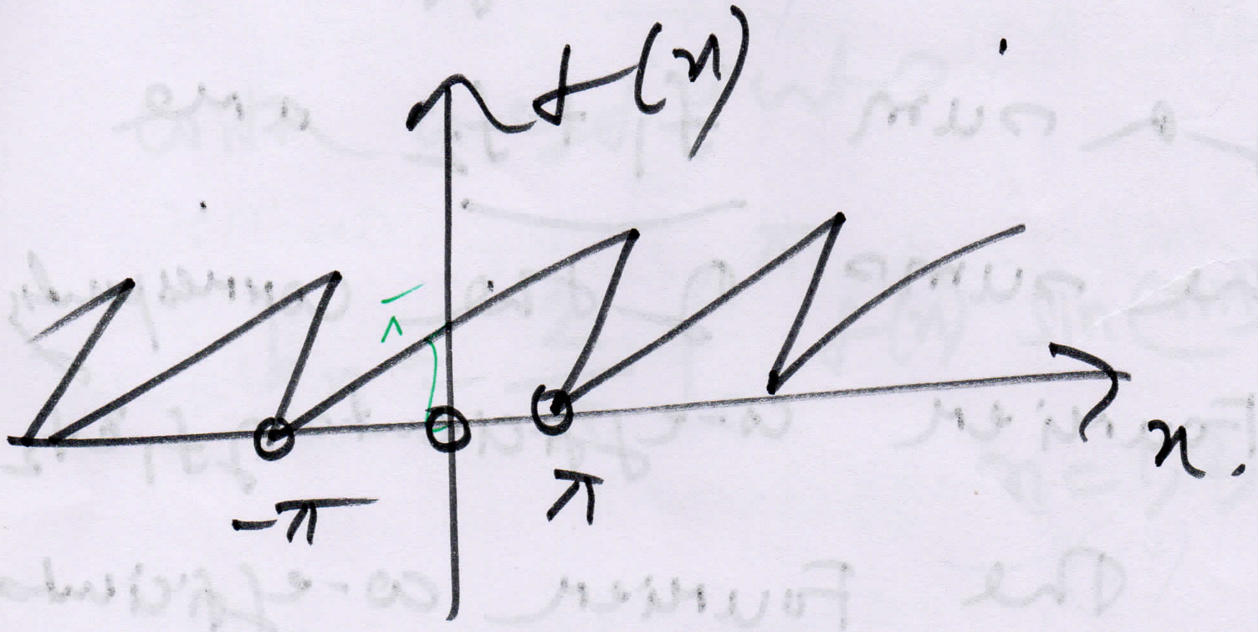
EX2

Sawtooth wave

Q) Find the Fourier series of the f^n

$f(x) = x + \pi$ if $-\pi < x < \pi$

& $f(x+2\pi) = f(x)$



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