Lecture 24

<u>Lemma</u>:- Let $f: E \to \mathbb{R}$ be a bounded measurable function where $E \subseteq \mathbb{R}^d$ is a measurable set of finite measure. If $\{\varphi_k\}_{k=1}^{\infty}$ is any sequence of simple measurable functions, bounded by $M > 0$, i.e.,

$$|\varphi_k(x)| \leq M \quad \forall k \geq 1, \ \forall x,$$

& supported on E; with $\varphi_n(x) \to f(x)$ for almost everywhere $x \in E$, then

(i) $\lim_{k \to \infty} \int \varphi_k$ exists.

(ii) If $f = 0$ a.e, then $\lim_{n \to \infty} \int \varphi_n = 0$.

<u>proof</u>:- Given $\varphi_n(x) \to f(x)$ a.e on E.

$\therefore$ By Littlewood 3rd principle, given $\varepsilon > 0$, there exists a closed set $A_\varepsilon \subseteq E$ such that $m(E \setminus A_\varepsilon) \leq \varepsilon$ & $\varphi_n \to f$ uniformly on A_ε.

Set $I_n = \int_E \varphi_n$, $\forall n \geq 1$

For any $m, n \geq 1$,

Now $\left| I_n - I_m \right| = \left| \int \varphi_n - \int \varphi_m \right|$

$$\leq \int_E |\varphi_n - \varphi_m|$$

$$= \int_{A_\varepsilon} |\varphi_n - \varphi_m| + \int_{E \setminus A_\varepsilon} |\varphi_n - \varphi_m|.$$

$$\left(\because E = A_\varepsilon \,\dot{\cup}\, (E \setminus A_\varepsilon) \right)$$

$$\leq \int_{A_\varepsilon} |\varphi_n - \varphi_m| + \int_{E \setminus A_\varepsilon} \left(|\varphi_n| + |\varphi_m| \right)$$

$$\leq \int_{A_\varepsilon} |\varphi_n - \varphi_m| + 2M \int_{E \setminus A_\varepsilon} 1$$

$$\leq \int_{A_\varepsilon} |\varphi_n - \varphi_m| + 2M \, m(E \setminus A_\varepsilon)$$

$$\leq \int_{A_\varepsilon} |\varphi_n - \varphi_m| + 2M \varepsilon$$

But $\varphi_n \to f$ uniformly, on A_ε,

$\Rightarrow \quad |\varphi_n(x) - \varphi_m(x)| < \varepsilon \quad \forall m, n$ sufficiently large.

$\forall x \in A_\varepsilon$.

$\therefore \; |I_n - I_m| \leq \int_{A_\varepsilon} |\varphi_n - \varphi_m| + 2M\varepsilon.$

$\leq \int_{A_\varepsilon} \varepsilon + 2M\varepsilon.$

$\leq m(A_\varepsilon)\,\varepsilon + 2M\varepsilon.$

for m, n sufficiently large.

$\therefore \; \{I_n\}$ is a Cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is Complete, $\{I_n\}$ is a Convergent sequence.

i.e. $\lim_{n \to \infty} I_n$ exists, i.e. $\lim_{n \to \infty} \int \varphi_n$ exists.

This proves (i).

(ii) Suppose $f = 0$ a.e on E.

Consider

$$|I_n| = \left| \int_E \varphi_n \right|$$

$$\leq \int_E |\varphi_n|$$

$$= \int_{A_\varepsilon} |\varphi_n| + \int_{E \setminus A_\varepsilon} |\varphi_n| .$$

$$\leq \varepsilon\, m(A_\varepsilon) + M\, m(E \setminus A_\varepsilon),$$

for n sufficiently large.

$\boxed{\begin{array}{l} \varphi_n \to f = 0 \text{ a.e.} \\ \text{uniformly on } A_\varepsilon \\ |\varphi_n| \leq \varepsilon \text{ for } n \gg 0. \end{array}}$

$$\leq \left(m(A_\varepsilon) + M \right) \varepsilon$$

for n sufficiently large.

i.e. $\lim_{n \to \infty} I_n = 0$

$$\Rightarrow \lim_{n \to \infty} \int \varphi_n = 0.$$

<u>Def</u>:- Let f be a bounded function that is supported on a measurable set of finite measure. Then the <u>Lebesgue integral</u> of f is defined as

$$\int f := \lim_{n \to \infty} \int \varphi_n ,$$

where $\{\varphi_n\}$ is any sequence of simple functions satisfying: $|\varphi_n(x)| \leq M$ $\forall n \geq 1$ $\forall x$

& each φ_n is supported on $supp(f)$.

& $\varphi_n(x) \to f(x)$ a.e, as $n \to \infty$.

(By above Lemma, this limit exists).

<u>Claim:-</u> $\int f$ is independent of the limiting sequence $\{\varphi_n\}$ used, in order for the integral to be well-defined.

<u>proof:-</u>

Let $\{\psi_n\}$ be another sequence of simple funtions that is bounded by $M' > 0$; supported on $supp(f)$. & such that $\psi_n(x) \to f(x)$ a.e as $n \to \infty$.

Let $\eta_n = \varphi_n - \psi_n \quad \forall n \geq 1$.

Then $\{\eta_n\}$ is a sequence of simple funtions bounded by $M + M'$, supported on $supp(f)$; such that $\eta_n(x) \to 0$ a.e

$\therefore \{\eta_n\}$ satisfies all the assumptions of the above Lemma.

$\therefore$ By part (ii) in the Lemma, we get

$$\lim_{n\to\infty} \int \eta_n = 0.$$

$$\Rightarrow \lim_{n\to\infty} \int (\varphi_n - \psi_n) = 0.$$

$$\Rightarrow \lim_{n\to\infty} \left(\int \varphi_n - \int \psi_n \right) = 0.$$

$$\Rightarrow \lim_{n\to\infty} \int \varphi_n - \lim_{n\to\infty} \int \psi_n = 0.$$

$$\Rightarrow \lim_{n\to\infty} \int \varphi_n = \lim_{n\to\infty} \int \psi_n.$$

$\therefore$ The Lebesgue integral of f is well defined.

Def:- Let $E \subseteq \mathbb{R}^d$ be a measurable set of finite measure. Let f be a bounded measurable function with $m(\text{supp}(f)) < \infty$. Then

$$\int_E f := \int f \chi_E$$

<u>proposition</u>:- Suppose f, g are measurable & bounded functions supported on sets of finite measure. Then the following properties hold:

(i) Let $a, b \in \mathbb{R}$. Then

$$\int (af + bg) = a\int f + b\int g.$$

(ii) (Additive) Let $E, F \subseteq \mathbb{R}^d$ be disjoint measurable sets. Then

$$\int_{E \cup F} f = \int_E f + \int_F f.$$

(iii) (Monotonicity) If $f \leq g$, then $\int f \leq \int g$.

(iv) (triangular inequality)

If $|f|$ is bounded, supported on a set ~~of~~ finite measure, then

$$\left|\int f\right| \leq \int |f|.$$

<u>proof</u>:- Let $\{\varphi_n\}$, $\{\psi_n\}$ be sequences of simple functions, bounded & supported on $supp(f)$, $supp(g)$ respectively, such that

$\varphi_n \to f$ a.e
$\psi_n \to g$ a.e

$$\int f = \lim_{n\to\infty} \int \varphi_n \quad \&$$

$$\int g = \lim_{n\to\infty} \int \psi_n. \quad \text{Then}$$

$$\int (af+bg) = \lim_{n\to\infty} \int (a\varphi_n + b\psi_n)$$

$$= \lim_{n\to\infty} \left(a\int \varphi_n + b\int \psi_n\right)$$

$$= a\int f + b\int g.$$

Remaining properties: EXERCISE.