Group Theory

Lecture 6

Recall A gp homo is map $f: G \to G'$ s.t $f(gh) = f(g) \cdot f(h)$

With f we have associated two subgps namely $\ker f$ which is a subgp of G and $\operatorname{im} f$ which is a subgp of G'.

ker f has a special property

Let $g \in \ker f$ and $h \in G$ be any elt.

Does $hgh^{-1} \in \ker f$?

$$f(hgh^{-1}) = f(h) f(g) f(h^{-1})$$

$$= f(h) \cdot 1_{G'} f(h)^{-1}$$

$$= 1_{G'}$$

<u>Defn</u> A subgp N of G is called a normal subgp of G ($N \trianglelefteq G$) if $hgh^{-1} \in N$ for all $g \in N$ and for all $h \in G$.

The kernel of a group homomorphism is always a normal subgp.

<u>Example</u>. (1) If $f: G_1 \longrightarrow G_2$ is a gp homo then $\ker f \trianglelefteq G_1$.

(2) $\det: GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^{\times}$ then

$$\ker(\det) = SL_n(\mathbb{R}) \trianglelefteq GL_n(\mathbb{R}).$$

(3) $\text{sign}: S_n \longrightarrow \{1, -1\}$.

$$\text{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

then $\ker(\text{sign}) = A_n \trianglelefteq S_n$.

(4) If G is an abelian group then any subgp H of G is a normal subgp.

(5) Consider U be the invertible upper triangular matrices in $GL_2(\mathbb{R})$. Is U a normal subgp of $GL_2(\mathbb{R})$?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = ?$$

U is not a normal subgp of $GL_2(\mathbb{R})$.

<u>Centre of a gp</u>

Let G be a gp. Then its centre is subgp $Z(G) = \{g \in G \mid gh = hg \ \forall h \in G\}$.

If G is abelian then $Z(G) = G$.

Check $Z(G)$ is a subgp of G.

$Z(G)$ is a normal subgp of G.

Ex 1. Show that $Z(GL_n(\mathbb{R})) = \{cI_n \mid c \in \mathbb{R}^*\}$.

Ex 2. Let $G = S_n$ then

$$Z(S_n) = \begin{cases} S_n & \text{if } n = 1, 2 \\ \{(1)\} & \text{if } n \geq 3, \end{cases}$$

Let $\phi : S \to T$ be a mapping of sets. This map defines an equivalance relation on the domain S by the rule $a \sim b$ if $\phi(a) = \phi(b)$

For any elt $t \in T$ the inverse image of t is defined as

$$\phi^{-1}(t) = \{ s \in S \mid \phi(s) = t \}.$$

The inverse images are called the fibers of ϕ.

The fibers are the equivalance classes and $S = \bigsqcup_{t \in T} \phi^{-1}(t)$.

For example consider the gp homo

$$\varphi : \mathbb{C}^{\times} \longrightarrow \mathbb{R}^{\times}_{>0} \quad \text{defined by}$$

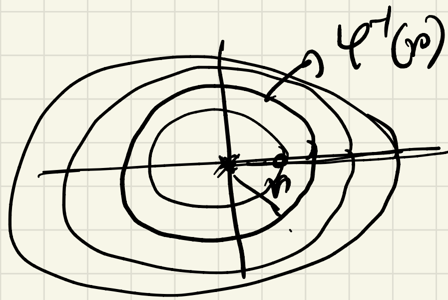
$$\varphi(z) = |z|.$$

Then the fibers of φ are concentric circles about origin.

and $\mathbb{C}^{\times} = \bigsqcup \varphi^{-1}(r)$

$$\varphi^{-1}(r) = \{ z \in \mathbb{C}^\times \mid |z| = r \}$$

$$\mathbb{C}^\times = \bigsqcup \varphi^{-1}(r)$$



$|z| = r_1$

$|z| = r_2$

<u>Propn</u> Let $\varphi : G \to G'$ be a gp homo with $\ker \varphi = N$ and let $a, b \in G$. Then $\varphi(a) = \varphi(b)$ iff $b = an$ for some $n \in N$ or equivalently if $a^{-1}b \in N$.

<u>Pf</u>: Let $\varphi(a) = \varphi(b)$

$$\Rightarrow \quad \varphi(a)^{-1}\varphi(a) = \varphi(a)^{-1}\varphi(b)$$

$$\Rightarrow \quad \varphi(a^{-1})\varphi(b) = 1 \Rightarrow \varphi(a^{-1}b) = 1$$

$\Rightarrow a^{-1}b \in N$

$\Rightarrow a^{-1}b = n$ for some $n \in N$.

$\Rightarrow b = an$.

Conversely, if $b = an$ for some $n \in N$

then $\varphi(b) = \varphi(an) = \varphi(a)\varphi(n)$
$= \varphi(a)$

<u>Cor</u>. A gp homo $\varphi : G \to G'$ is injective iff $\ker \varphi = \{1\}$.

<u>Defn</u>. A group homo $\varphi : G \to G'$ is called an isomorphism if φ is 1-1 and onto.

Moreover if $G' = G$ then an isomorphism is called an automorphism.

Example. (1) Any infinite cyclic gp is isomorphic to $(\mathbb{Z}, +)$.

Let G be any infinite cyclic gp s.t $G = \langle x \rangle$.

Define $\phi : \mathbb{Z} \longrightarrow G$.

$$\phi(1) = x.$$

$$\phi(n) = x^n.$$

Then check ϕ is an gp isomorphism.

(2) Any finite cyclic gp of order n is isomorphic with $(\mathbb{Z}/n\mathbb{Z}, +)$.

Let $G = \langle x \rangle$ be a cyclic gp of order n. i.e. $x^n = 1$.

Define $\phi : G \longrightarrow \mathbb{Z}/n\mathbb{Z}$ by $\phi(x) = \bar{1}$.

(3) Let G be any gp. and $g \in G$ be fixed.

Define $i_g : G \to G$ by

$$i_g(x) = gxg^{-1}$$

gp homo:

$$i_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1}$$
$$= (gxg^{-1})(gyg^{-1})$$
$$= i_g(x)\, i_g(y).$$

check i_g is 1-1 and onto.

$\therefore$ i_g gives an automorphism.

The elts gxg^{-1} are called conjugate of x $\forall g \in G$.

Tut - 1

Q9 In S_4.

$(12) \rightsquigarrow$ 2-cycles

$(12)(34) \rightsquigarrow$ Product of 2-cycles.

In S_n.

$\underline{n = 2k}$

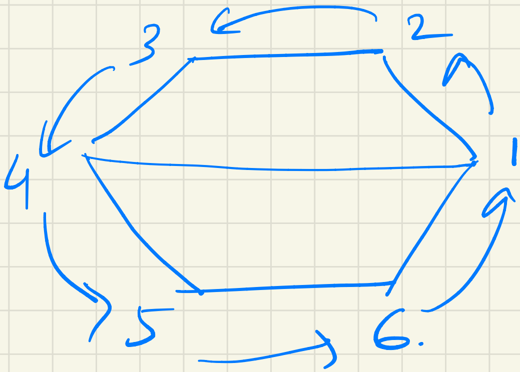
$$\underbrace{\binom{n}{2}}_{(12)} + \underbrace{\frac{\binom{n}{2}\binom{n-2}{2}}{2!}}_{(12)(34)} + \underbrace{\frac{\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}}{3!}}_{(12)(34)(56)} + \cdots + \frac{\binom{n}{2}\cdots\binom{2}{2}}{k!}$$

$$(12)(34) = (34)(12)$$

If $n = 2k+1$

$$\binom{n}{2} + \frac{\binom{n}{2}\binom{n-2}{2}}{2!} + \cdots + \frac{\binom{n}{2}\cdots\binom{3}{2}}{k!}$$

§16



$(1\ 2\ 3\ 4\ 5\ 6)$, $(1\ 3\ 5)(2\ 4\ 6)$,

§16 $GL_2(\mathbb{R})$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

$|A| = 4$, $|B| = 3$.

$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $|AB| = \infty$,

Q14 $2n = |G|$.

$$\{1, \underbrace{a_1, \ldots, a_{2n-1}}\}.$$

$$\underline{x^2 = 1}.$$

$$x = x^{-1}.$$

Q21

$$ah = h'$$

$$a = h'h^{-1} \in H.$$

Q27 $G = \langle a \rangle = \langle b \rangle$

$a^n = b$ and $b^m = a$

$$a = a^{mn}$$

$$\Rightarrow a^{mn-1} = 1.$$

$$\Rightarrow mn - 1 = 0. \Rightarrow mn = 1.$$

$$m, n = 1 \text{ or } -1.$$