Date
29/08/2017

$\boxed{\text{Lecture 13}}$

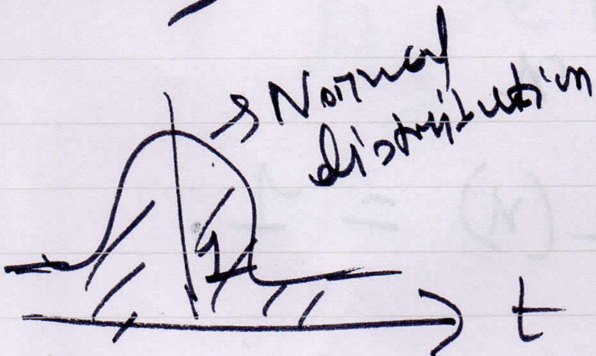
The Error function :-

The Error function $\operatorname{erf}(x)$

is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt .$$

$\longrightarrow$ the area under the Normal distribution curve in statistics.



$\left[\lim_{t \to \infty} \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx = 1. \right]$

$$\mathcal{L}^{-1}\left\{ \frac{1}{\sqrt{s}\,(s-1)} \right\} = e^t \operatorname{erf}\{\sqrt{t}\}$$

[by first shifting theorem

$\Rightarrow \boxed{\mathcal{L}\left[e^{-t} \cdot \{ e^t \operatorname{erf}\sqrt{t} \} \right] = F(s+1)}$

$\Rightarrow \mathcal{L}\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}$

$$\Rightarrow \mathcal{L}\underbrace{\left\{ \operatorname{erf}(\sqrt{t}) \right\}}_{= F(s+1)} = \frac{1}{s\sqrt{s+1}} \quad \text{(how?)}$$

The f^n $1 - \mathrm{erf}(x)$ is called the complementary error function & is written as $\mathrm{erfc}(x)$.

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt.$$

& $\mathrm{erf}(x) + \mathrm{erfc}(x) = 1.$

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Ex1/ Use a suitable double integral to evaluate the improper integral $\int_0^\infty e^{-t^2} dt$.

Sol^n: —

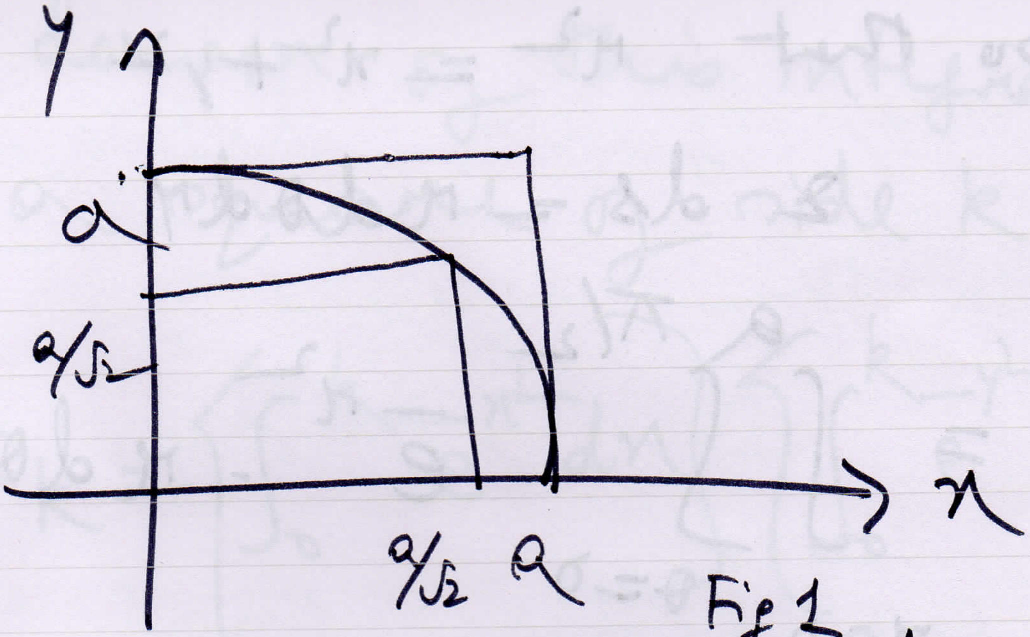


Fig 1

We consider the double integral

$$\iint_S e^{-(x^2+y^2)}\, dS,$$ where

S is the quarter disc
$x \geq 0,\ y \geq 0,\ x^2+y^2 \leq a^2.$

Converting to polar co-ordinates (r, θ), this integral becomes

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$|J| = r. \quad (\text{why?})$$

so that $r^2 = x^2 + y^2$

& $ds = r\,d\theta\,dr$

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} e^{-r^2} \cdot r\,d\theta\,dr$$

$$= \pi/4 \int_0^a e^{-r^2}\, d(r^2)$$

$$= -\pi/4 \left[e^{-r^2}\right]_0^a =$$

$$= \pi/4 \left[1 - e^{-a^2}\right]$$

As $a \to \infty$, $I \longrightarrow \pi/4$.

$\longrightarrow$ (*)

we now consider the double integral

$$I_k = \int_0^k \int_0^k e^{-(x^2+y^2)}\, dx\,dy$$

The domain of this integral is a square of side k.

Now $I_k = \left\{\int_0^k e^{-x^2} dx\right\}\left\{\int_0^k e^{-y^2} dy\right\}$

$$= \left\{\int_0^k e^{-x^2} dx\right\}^2$$

From the Fig 1, we can see that

$$I_{a/\sqrt{2}} < I < I_a$$

We also observe that

$$I_k \longrightarrow \left[\int_0^\infty e^{-x^2} dx\right]^2$$

as $k \to \infty$.

Hence, if we let $a \to \infty$ in the inequality

$$I_{a/\sqrt{2}} < I < I_a$$

we deduce that

$$I \to \left\{ \int_0^\infty e^{-x^2}\,dx \right\}^2$$

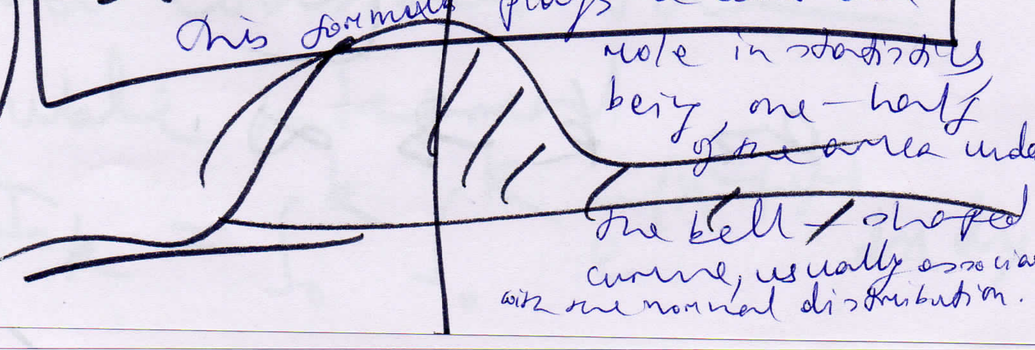
as $a \to \infty$

$$\therefore \left\{ \int_0^\infty e^{-x^2}\,dx \right\}^2 = \pi/4.$$

$$\boxed{\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2}$$

Find out more about them

This formula plays a central role in statistics being one-half of the area under the bell-shaped curve, usually associated with the normal distribution.



$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \mathcal{L}\left\{t^{-1/2}\right\}$$

$$= \sqrt{\pi/s}\,.$$

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty \frac{e^{-st}}{\sqrt{t}}\,dt$$

Sufficient cond^n satisfied at $t=0$?

$$= \int_{u=0}^\infty \frac{e^{-u^2}}{\cancel{s}\sqrt{s}} \cdot \frac{\sqrt{s}}{u}(2u\,du)$$

Let $st = u^2$

$\Rightarrow t = u^2/s$

$s\,dt = 2u\,du$

$$= \frac{2}{\sqrt{s}} \int_0^\infty e^{-u^2}\,du$$

$t=0,\ u=0$

$$= \frac{\cancel{2}}{\sqrt{s}} \cdot \frac{\sqrt{\pi}}{\cancel{2}} = \sqrt{\pi/s}$$

$t=\infty,\ u=\infty$

Hint: $\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\pi/s}$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}}$$

Q) Determine

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}\,(s-1)}\right\}.$$

Sol^n :- We know that

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}$$

& $$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}$$

Using the convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}\,(s-1)}\right\} = e^t * \frac{1}{\sqrt{\pi t}}$$

$$= \int_0^t \frac{1}{\sqrt{\pi \tau}}\, e^{(t-\tau)}\, d\tau$$

$$= \frac{e^t}{\sqrt{\pi}} \boxed{\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}}\, d\tau}$$

Let $\tau = u^2$

$\Rightarrow d\tau = 2u\,du$

Limits

$\tau=0,\ u=0$

$\tau=t,\ u=\sqrt{t}$

$$\therefore \int_{\tau=0}^t \frac{e^{-\tau}}{\sqrt{\tau}}\, d\tau = \boxed{2\int_{u=0}^{\sqrt{t}} e^{-u^2}\, du}$$

$$= e^t \cdot \left\{\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2}\, du\right\} = e^t \operatorname{erf}(\sqrt{t})$$

H.W ~~XXXXX~~

Q) Determine

$$\mathcal{L}\left[t^{-3/2}\exp\left\{-k^2/4t\right\}\right]$$

$$= \frac{2\sqrt{\pi}}{k} e^{-k\sqrt{s}}.$$

$$\Rightarrow \mathcal{L}^{-1}\left[e^{-k\sqrt{s}}\right] = \frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$$

Hint:- $= \int_0^\infty e^{-st} \cdot t^{-3/2} e^{-k^2/4t} \, dt$

$= \frac{4}{k}\int_0^\infty e^{-u^2} e^{-sk/4u^2} du$

$= \frac{4}{k} e^{-k\sqrt{s}} \int_0^\infty e^{-(u - k\sqrt{s}/2u)^2} du$

Let

$u = k/2\sqrt{t}$.

$du = -k/4 \, t^{-3/2} dt$

$u^2 + sk^2/4u^2$

$= \left\{u - \frac{k\sqrt{s}}{2u}\right\}^2$

a) Use convolution to find

$$\mathcal{L}^{-1}\left\{\frac{e^{-k\sqrt{s}}}{s}\right\}$$

$$= \operatorname{erfc}\left\{\frac{k}{2\sqrt{t}}\right\}$$

Hint:- we know,

$$\mathcal{L}^{-1}\left\{e^{-k\sqrt{s}}\right\} = \frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$$

$$\& \quad \mathcal{L}^{-1}\left\{1/s\right\} = 1$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{e^{-k\sqrt{s}}}{s}\right\} = \frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t} * 1$$

$$= \frac{k}{2\sqrt{\pi}} \int_0^t \tau^{-3/2} e^{-k^2/4\tau} d\tau$$

$$= \frac{2}{\sqrt{\pi}} \int_{k/2\sqrt{t}}^{\infty} e^{-u^2} du \quad \left[\text{Let } u^2 = k^2/4\tau\right.$$

$$= \operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$$

(Ex)

$$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$$

special case of convolution

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$$