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MODERN ALGEBRA

1. (a) False.

We can consider the group.

G = the polynomial ring over F_p under addition. We can show that this is an abelian infinite group. & $a^p = 0$ $\forall a \in F_p$.

thus, every element has finite order.

(b) ~~True~~ False.

Consider S_3. In this we can construct all subgroup which is cyclic but S_3 is not cyclic

(c) True.

(d) False. For ex consider S_3 which is not abelian but all subgroups are normal ($\because$ we have seen in (b) that they are cyclic).

(e) True. Order of $G = p^3$. Now, for $Z(G)$, from Lagrange's theorem there can be 4 cases order $= 1, p, p^2, p^3$. ($\because p$ is a prime)

but we know that $|Z(G)| \geq p$.

& it cannot be p^3 otherwise $Z(G) = G$

So, we are left with p & p^2.

Now if $|Z(G)| = p^2 \Rightarrow |G/Z(G)| = p$.

$\Rightarrow G/Z(G) \Rightarrow$ abelian which is not true

thus order of $|Z(G)| = p$.

2. $|G| = 7^2 \cdot 11^2$. Then we have to check whether G is abelian or not.

consider J to be the normal sylow-7 subgroup

& K to be the " sylow-11 subgroup.

Now, since J & K are abelian groups.

we can construct a new group

$G = J \times K$ such that $|G| = 7^2 \cdot 11^2$.

Thus, G is abelian too ($\because J$ & K are abelian)

Given: The centre of Group G is of index n.

(3.) We have to prove that: Every conjugacy class of G has at most n elements.

Proof:

First of all, we can consider a conjugacy class C in G which contain an element $m \in G$.

Now, we know that the order of $C = |G : C_G(m)|$.

we can also say that $Z(G) \leq C_G(m)$. — ①

($\because$ if $g \in Z(G) \Rightarrow gmg^{-1} = mgg^{-1} = m$
$\therefore g \in C_G(m)$).

from ① we can say that

$Z(G) \leq C_G(m) \leq G$.

$\therefore |G : Z(G)| = |G : C_G(m)| \cdot |C_G(m) : Z(G)|$

$$\Rightarrow |G : C_G(m)| = \frac{|G : Z(G)|}{|C_G(m) : Z(G)|}$$

$$\Rightarrow |C| = \frac{n}{|C_G(m) : Z(G)|}$$

$\because |C_G(m) : Z(G)| \geq 1$.

$$\Rightarrow \frac{n}{|C_G(m) : Z(G)|} \leq n$$

$\Rightarrow \boxed{|C| \leq n}$ Thus conjugacy class has at most n elements.

(4.) We have to find all abelian groups which have order p^5 but elements have order at most p^3.

Now, in order to find these groups, we can find all groups

$$Z_{a_1} \times Z_{a_2} \times Z_{a_3} \times \cdots \times Z_{a_n}$$

such that $|a_1 a_2 a_3 \cdots a_n| = p^5$
& $p \leq a_i \leq p^3$

So, we can consider

for $n = 2$. (i) $a_1 = p^2$, $a_2 = p^3$
$\Rightarrow$ Group $\rightarrow Z_{p^2} \times Z_{p^3}$

for $n = 3$ (i) $a_1 = p^3$ $a_2 = p$ $a_3 = p$ $\rightarrow Z_{p^3} \times Z_p \times Z_p$
(ii) $a_1 = p^2$ $a_2 = p^2$ $a_3 = p$ $\rightarrow Z_{p^2} \times Z_{p^2} \times Z_p$

for $n = 4$ (i) $a_1 = p^2$, $a_2 = p$, $a_3 = p$ $a_4 = p \rightarrow Z_{p^2} \times Z_p \times Z_p \times Z_p$

for $n = 5$, $a_1 = a_2 = a_3 = a_4 = a_5 = p$
$\therefore$ Group $\rightarrow Z_p \times Z_p \times Z_p \times Z_p \times Z_p$.

So, in total we have got

$Z_{p^2} \times Z_{p^3}$
$Z_{p^3} \times Z_p \times Z_p \times Z_p$
$Z_{p^2} \times Z_{p^2} \times Z_p$
$Z_{p^2} \times Z_p \times Z_p \times Z_p$
$Z_p \times Z_p \times Z_p \times Z_p \times Z_p$