Problem 1: Integral (24)

1.1 (8)

$\int \sqrt{1 - x^2} \, \mathrm{d}x$

1.2 (8)

$\int \left(\sin(ax) e^{bx}\right) \mathrm{d}x$, where $a, b > 0$

1.3 (8)

$\int \left(\frac{1}{\left(1+x^2\right)^2}\right) \mathrm{d}x$

Note: No need to calculate the coefficient

Problem 2: Derivative (16)

2.1 (8)

$f(x) := \begin{cases} \frac{\sin x^2}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Try to get: $f^{(10)}(0), f^{(11)}(0)$

2.2 (8)

Check the convexity of the following curve:

$\begin{cases} x = \sin^3 t \\ y = \cos^3 t \end{cases}$, where $t \in \left(0, \frac{\pi}{2}\right)$

Problem 3 (20)

Prove Hölder's inequality: $\sum_1^n a_i b_i \leq \left(\sum_1^n a_i^p\right)^{\frac{1}{p}} \left(\sum_1^n b_i^q\right)^{\frac{1}{q}}$, where $a_i, b_i \geq 0, p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$

Problem 4 (10)

Find all possible function $x(t)$ such that: $x(t) \in C[a, b], x(t) \in D^2(a, b), x(a) = x(b) = 0$ and

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0, t \in (a, b)$$

where $p(t), q(t)$ are given function, and $\forall t \in (a, b), q(t) < 0$

Problem 5 (10)

$f(x) \in C^\infty[a, b]$, and $f^{(n)}(x) \geq 0$ for any n and x, prove that: $\exists M : f^{(n)}(x) \leq \frac{Mn!}{r^n}$ for any $n \geq 1$, where $x \in (a, b), r > 0, x + r \in (a, b)$

Problem 6 (20)

Find all possible function $f(x)$ such that: $f \in D[0, +\infty), f(0) = 0, |f(x) + \sqrt{x}f'(x)| \leq M\sqrt{x}\ |f(x)|$, where $M > 0$