

COORDINATE TRANSFORMATIONS FOR CADASTRAL SURVEYING

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ABSTRACT

A two-dimensional (2D) conformal transformation, (that preserves shape and hence angles), is a useful tool for practicing cadastral surveyors. It can be used as an aid to re-establishment where occupation boundary corners of allotments, surveyed in the field on an arbitrary survey coordinate system, can be transformed to the cadastral title coordinate system and occupation/title comparisons made. The transformation process consists of two parts. The first part is the determination of the transformation parameters; scale s , rotation θ and translations, t_x and t_y . This requires a minimum of two control points having coordinates in both the title and survey coordinate system. If there are three or more control points, then the transformation parameters are determined by a least squares process and a weighting scheme can be employed. The second part is to use the transformation parameters (determined from the control points) to transform the other surveyed points onto the title system. This paper sets out the necessary theory of 2D conformal transformations and the determination of transformation parameters using least squares. In addition, weighting schemes are discussed as well as transformations that preserve scale (i.e., a scale $s = 1$).

INTRODUCTION

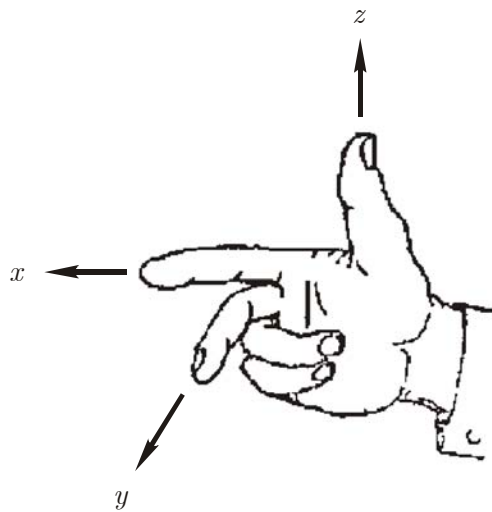
Conformal coordinate transformations, are widely used in the surveying profession. For instance, in geodesy, 3D conformal transformations can be used to convert coordinates related to the Australian Geodetic Datum (AGD66, AGD84) to the

Geocentric Datum of Australia (GDA94), in engineering surveying they form part of monitoring and control systems used in large projects such as the construction of elevated freeways and tunnels (Deakin 1998), and in photogrammetry they are used in the orientation (interior and exterior) of aerial digital images. In 2D form, transformations are used in cadastral survey re-establishments (Bebb 1981, Sprott 1983 and Bird 1984), matching digitized cadastral maps (Shmutter and Doytsher 1991) and "sewing together" the edges of strips of digital images (Bellman, Deakin and Rollings 1992).

In general, the effect of a 2D transformation on a polygon (a plane multi-sided figure) will vary from a simple change of location and orientation (with no change in shape or size) to a uniform change in scale (no change in shape) and finally to changes of shape and size of different degrees of nonlinearity (Mikhail 1976). The most common transformations in surveying applications, and the only type dealt with in this paper, are conformal, i.e., transformations that preserve angles and thus the shape of objects. Theory and applications of other coordinate transformations, such as affine, polynomial, projective etc. can be found in Mikhail (1976) and Moffitt and Mikhail (1980). In the theory that follows, transformations are expressed in the form of equations linking coordinates in one system with coordinates in another system and these equations contain rotation angles (usually denoted by θ), as well as scale s and translations t_x and t_y (or t_E and t_N) where the subscripts relate to the coordinate system axes labels, x,y ; E,N ; u,v ; etc. The idea of rotation is important and as we will see there are several different types of rotations, i.e., an object can have an actual rotation where it is rotated about a point; or an apparent rotation where its coordinates change because the coordinate axes are rotated; and these rotations can be clockwise or anticlockwise. In this paper we will only be considering apparent rotations caused by anticlockwise rotation of coordinate axes and to clarify these issues some rules and diagrams are helpful.

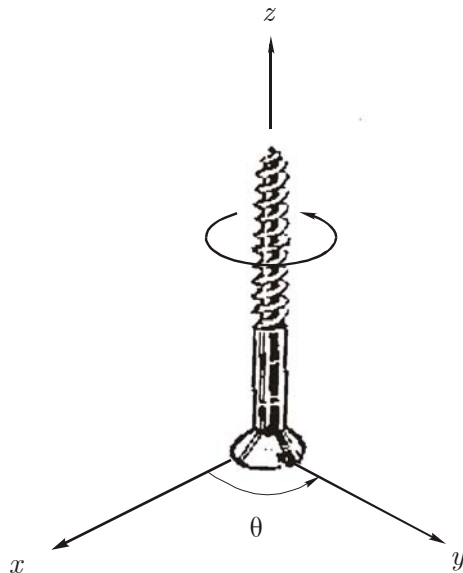
In general we consider points in space as being connected to the origin O of a 3D right-handed rectangular coordinate system x,y,z . Such a system can be visualised as the corner of a room where the intersection of two walls and the floor provide three reference lines Ox , Oy and Oz , known as the x -, y - and z -axes that are (usually) at

right angles to one another. The x - z and y - z planes are the walls and the x - y plane is the floor.



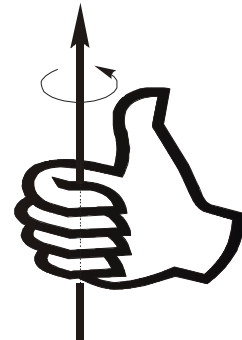
The three mutually perpendicular axes x , y and z are related by the right-hand rule as follows:

If the thumb, the forefinger and the second finger of the right hand are placed mutually at right angles then the thumb points in the z -direction, the forefinger points in the x -direction and the second finger points in the y -direction.



The axes x , y and z (in the cyclic order xyz) are a right-handed system (or dextral system) since a rotation from x towards y advances a right-handed screw in the direction of z . Similarly, a rotation from y towards z advances a right-handed screw in the direction of x and so on. The diagram on the left shows the right-hand-screw rule for the positive directions of rotations and axes of a right-handed rectangular coordinate system.

These rotations are considered positive anticlockwise when looking along the axis towards the origin; the positive sense of rotation being determined by the right-hand-grip rule where an imaginary right hand grips the axis with the thumb pointing in the positive direction of the axis and the natural curl of the fingers indicating the positive direction of rotation.



The right-handed coordinate system and positive anticlockwise rotations (given by the right-hand-grip rule) are consistent with conventions used in mathematics and physics and in mathematics, angles are measured positive anticlockwise from the x -axis; a convention we also use in these notes. As surveyors, we deal almost exclusively with angular quantities (bearing, azimuths, directions, etc) considered as positive clockwise and usually measured from north (or the y -axis in the x - y system or the v -axis in the u - v system) and this surveying convention of positive clockwise rotation from north could be described by a left-hand-grip rule but we do not usually do this.

CONFORMAL TRANSFORMATIONS IN TWO-DIMENSIONAL (2D) SPACE

In 2D conformal transformations all points lie in a plane and such points are considered to have only x,y (or u,v) coordinates, i.e., they lie in the x - y (or u - v) plane with a z -value = 0 (or w -value = 0). In these notes it is assumed that 2D conformal transformations are transformations from one rectangular coordinate system (u,v) which we could call the survey system to another rectangular system (x,y) that we could call the title system. Both of these coordinate systems could be thought of as arbitrary and it is immaterial where the origins of both systems lie. In addition, unless stated otherwise, a rotation is an angle considered to be positive in an anticlockwise direction as determined by the *right-hand-grip rule* and rotations of polygons (or objects) are apparent, since we are considering rotations of coordinate axes rather than actual rotations of polygons about a centre – more about this later. Also, transformation equations are conveniently expressed using matrix notation and a rotation matrix \mathbf{R} (whose elements are functions of the rotation angle θ) is a component of any conformal transformation equation. Rotation matrices are orthogonal, which is a very useful property, and there is an explanation of this property in the following sections.

The general conformal transformation formula are developed in a simple way. First, by considering transformations involving rotation only; then, involving both scale and rotation. And finally, the general case, involving scale, rotation and translations. We then show that this general conformal transformation (combining scale, rotation, translation) is the same as that obtained by using the mathematical principles of conformal mapping developed by C. F. Gauss.

Conformal Transformation involving Rotation only

u, v coordinates (survey system) are transformed to x, y (title system) coordinates by considering a rotation of the u, v coordinate axes through a positive anticlockwise angle θ . The transformation equations can be expressed in the following way

$$\begin{aligned} x &= u \cos \theta + v \sin \theta \\ y &= -u \sin \theta + v \cos \theta \end{aligned} \quad (1)$$

or in matrix notation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (2)$$

As an example consider the polygon $ABCD$ whose u, v coordinates are rotated by a positive anticlockwise angle $\theta = 30^\circ$. Figure 1 shows the initial location of the polygon in the u, v survey system and Figure 2 shows its transformed (rotated) location in the x, y title system.

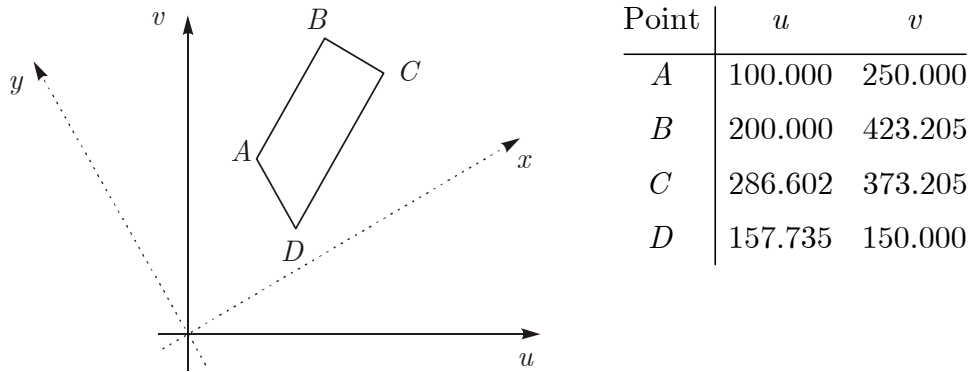


Figure 1 Polygon $ABCD$ with u, v coordinates in metres

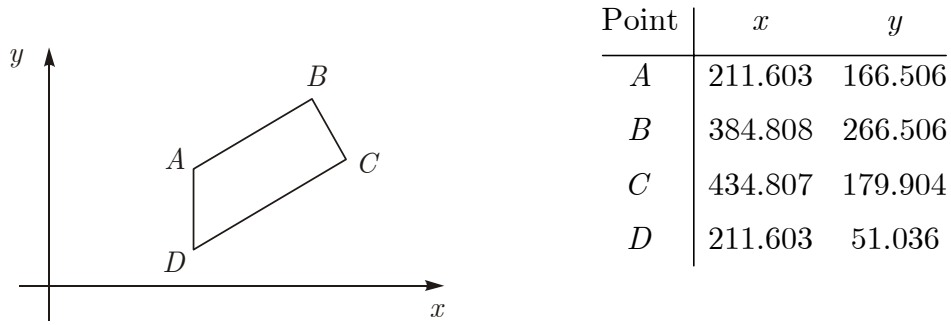


Figure 2 Rotated polygon $ABCD$ with x,y coordinates in metres

Comparing Figures 1 and 2 it appears that the size and shape of the polygon $ABCD$ has not changed but its orientation with respect to the coordinate axes has. This can be verified by considering the dimensions (bearings and distances) of the polygon $ABCD$ derived from the two coordinate sets.

Line	Bearing	Distance
AB	$30^{\circ} 00'$	200.000
BC	$120^{\circ} 00'$	100.000
CD	$210^{\circ} 00'$	257.735
DA	$330^{\circ} 00'$	115.470

Polygon dimensions in the u,v system

Line	Bearing	Distance
AB	$60^{\circ} 00'$	200.000
BC	$150^{\circ} 00'$	100.000
CD	$240^{\circ} 00'$	257.735
DA	$0^{\circ} 00'$	115.470

Polygon dimensions in the x,y system

This example demonstrates that a rotation of the coordinate axes causes an apparent rotation, in an opposite direction, of any polygon defined within the coordinate system. The size and shape of the polygon does not change.

Equation (1) and its matrix equivalent (2) can be obtained by considering Figure 3.

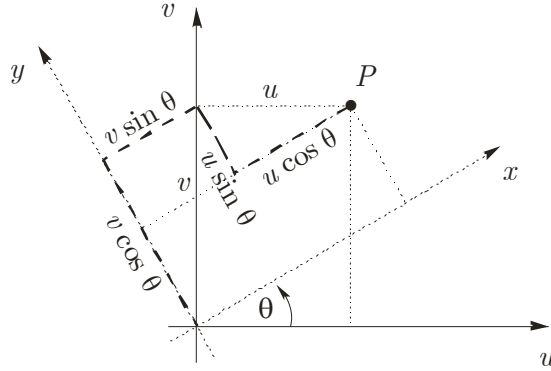


Figure 3 x, y coordinates of P as functions of u, v coordinates and rotation θ

Rotation matrices

Equation (2) can be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{R} \begin{bmatrix} u \\ v \end{bmatrix} \quad (3)$$

where $\mathbf{R} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ is known as a rotation matrix. Rotation matrices are orthogonal, i.e., the sum of squares of the elements of any row or column is equal to unity and an orthogonal matrix has the unique property that its inverse is equal to its transpose, i.e., $\mathbf{R}^{-1} = \mathbf{R}^T$. This useful property allows us to write the transformation from x, y coordinates to u, v coordinates as follows.

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \mathbf{R} \begin{bmatrix} u \\ v \end{bmatrix} \\ \mathbf{R}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} &= \mathbf{R}^{-1} \mathbf{R} \begin{bmatrix} u \\ v \end{bmatrix} \\ \mathbf{R}^T \begin{bmatrix} x \\ y \end{bmatrix} &= \mathbf{I} \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

and rearranging gives

$$\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{R}^T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (4)$$

We could write (4) as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{R}^* \begin{bmatrix} x \\ y \end{bmatrix}$$

which in words means: the x, y coordinates are transformed (rotated) to u, v coordinates. Equation (3) on the other hand means: the u, v coordinates are transformed (rotated) to x, y coordinates and it is interesting to note that \mathbf{R} and \mathbf{R}^* are in fact the same rotation matrix except in the former, θ is positive anticlockwise and in the latter θ is positive clockwise. Note that $\sin(-\theta) = -\sin \theta$ and $\cos(-\theta) = \cos \theta$.

Orthogonal Matrices

Orthogonal matrices are extremely useful since their inverse is equal to their transpose. Rotation matrices \mathbf{R} are orthogonal, hence $\mathbf{R}^{-1} = \mathbf{R}^T$. A proof of this can be found in Allan (1997) and is repeated here.

Consider the effect of a rotation on the coordinates \mathbf{x} of a point P , expressed as

$$\mathbf{X} = \mathbf{R}\mathbf{x}$$

\mathbf{X} is the transformed (or rotated) coordinates and \mathbf{R} is the rotation matrix.

Multiplying both sides of the equation by the inverse of \mathbf{R} gives

$$\mathbf{R}^{-1}\mathbf{X} = \mathbf{R}^{-1}\mathbf{R}\mathbf{x}$$

but from matrix algebra $\mathbf{R}^{-1}\mathbf{R} = \mathbf{I}$ and $\mathbf{I}\mathbf{x} = \mathbf{x}$ so

$$\mathbf{R}^{-1}\mathbf{X} = \mathbf{x}$$

or

$$\mathbf{x} = \mathbf{R}^{-1}\mathbf{X}$$

The length (actually squared length) of the line from the origin to the original position of point P is given by $\mathbf{x}^T\mathbf{x}$ and the length from the origin to the new (rotated) position is given by $\mathbf{X}^T\mathbf{X}$. This length does not change due to rotation, i.e., it is invariant under rotation. Hence

$$\mathbf{x}^T\mathbf{x} = \mathbf{X}^T\mathbf{X}$$

but

$$\mathbf{X} = \mathbf{R}\mathbf{x}$$

so

$$\begin{aligned} \mathbf{x}^T\mathbf{x} &= (\mathbf{R}\mathbf{x})^T \mathbf{R}\mathbf{x} \\ &= \mathbf{x}^T \mathbf{R}^T \mathbf{R} \mathbf{x} \end{aligned}$$

For this result to be possible

$$\mathbf{R}^T \mathbf{R} = \mathbf{I}$$

but

$$\mathbf{R}^{-1} \mathbf{R} = \mathbf{I}$$

Therefore

$$\mathbf{R}^T = \mathbf{R}^{-1}$$

Thus the *inverse* of a rotation matrix is equal to its *transpose*.

Rotation of Axes versus Rotation of Object

In these notes it is assumed that a rotation angle is a positive anticlockwise angle as determined by the right-hand-grip rule and that "apparent" rotations of objects (polygons) are caused by a rotation of the coordinate axes. This is not the only way that an object can be rotated.

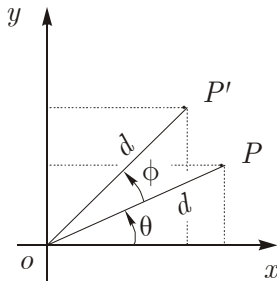


Figure 4

Consider Figure 4 where P with coordinates x, y moves to P' with coordinates x', y' by a positive anticlockwise rotation ϕ . The coordinates of P' are

$$\begin{aligned} x' &= d \cos(\theta + \phi) = d(\cos \theta \cos \phi - \sin \theta \sin \phi) \\ y' &= d \sin(\theta + \phi) = d(\sin \theta \cos \phi + \cos \theta \sin \phi) \end{aligned} \quad (5)$$

The coordinates of P are $x = d \cos \theta$ and $y = d \sin \theta$ which can be substituted into (5) to give

$$\begin{aligned} x' &= x \cos \phi - y \sin \phi \\ y' &= y \cos \phi + x \sin \phi \end{aligned} \quad \text{or in matrix form} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{\mathbf{R}} \begin{bmatrix} x \\ y \end{bmatrix} \quad (6)$$

Where $\bar{\mathbf{R}}$ is a rotation matrix and the rotation angle ϕ is a "right-handed" rotation. Inspection of equations (3) and (6) shows that $\bar{\mathbf{R}}$ is not the same form as \mathbf{R} , in fact it is identical in form to \mathbf{R}^T .

The rotation matrix \mathbf{R} causes an apparent rotation of the object by rotation of the coordinate axes whilst the rotation matrix $\bar{\mathbf{R}}$ rotates the object itself. Both \mathbf{R} and $\bar{\mathbf{R}}$ are "right-hand" rotation matrices (one is the transpose of the other) and there is often confusion amongst users of transformation software in defining the type of

rotation and the positive direction of rotation. You must be very careful in defining rotation, i.e., you must state what is being rotated, either axes or object and what is the positive direction of rotation. In these notes it is always assumed that the coordinate axes are being rotated and the rotations are always positive anticlockwise as defined by the right-hand-grip rule.

Conformal Transformation involving Rotation θ and a Scale change s

u, v coordinates (survey system) are transformed to x, y coordinates (title system) by considering a rotation of the u, v coordinate axes through a positive anticlockwise angle θ and a scaling of the u, v coordinates by a factor s . The transformation equations can be expressed in the following way

$$\begin{aligned} x &= (s \cos \theta) u + (s \sin \theta) v \\ y &= -(s \sin \theta) u + (s \cos \theta) v \end{aligned} \tag{7}$$

or in matrix notation

$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{8}$

Often, the coefficients of u and v in (7) are written as $a = s \cos \theta$ and $b = s \sin \theta$ giving

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{9}$$

and the scale factor s and the rotation angle θ are given by

$$\begin{aligned} s &= \sqrt{a^2 + b^2} \\ \theta &= \tan^{-1} \left(\frac{b}{a} \right) \end{aligned} \tag{10}$$

As an example consider the polygon $ABCD$ whose u,v coordinates (survey system) are rotated by a positive anticlockwise angle $\theta = 30^\circ$ and scaled by a factor $s = 0.6$. Figure 1 shows the initial location of the polygon in the u,v system and Figure 5 shows its transformed (rotated and scaled) location in the x,y system (title system).

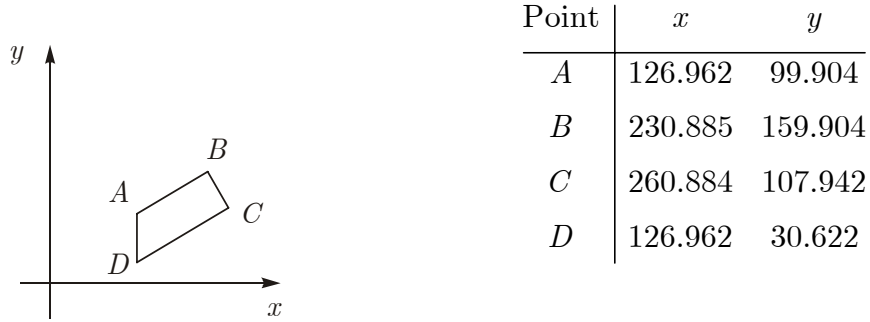


Figure 5 Rotated and scaled polygon $ABCD$ with x,y coordinates in metres

Comparing Figures 1 and 5 it appears that the shape of the polygon $ABCD$ has not changed but its size and orientation with respect to the coordinate axes has. This can be verified by considering the dimensions (bearings and distances) and area of the polygon $ABCD$ derived from the two coordinate sets.

Line	Bearing	Distance
AB	$30^\circ 00'$	200.000
BC	$120^\circ 00'$	100.000
CD	$210^\circ 00'$	257.735
DA	$330^\circ 00'$	115.470

Area= $22,886.75 \text{ m}^2$

Line	Bearing	Distance
AB	$60^\circ 00'$	120.000
BC	$150^\circ 00'$	60.000
CD	$240^\circ 00'$	154.641
DA	$0^\circ 00'$	69.282

Area= $8,239.23 \text{ m}^2$

Polygon dimensions in the u,v system

Polygon dimensions in the x,y system

Inspection of the two sets of dimensions reveals that bearings have been rotated by an angle $\theta = 30^\circ$ and distances scaled by a factor $s = 0.6$. Note that the shape of the polygon is unchanged but the area of the transformed figure has been reduced by a factor of s^2 .

Conformal Transformation with Rotation θ , Scale change s and Translations t_x, t_y

u, v coordinates (survey system) are first transformed to x', y' coordinates by considering a rotation of the u, v coordinate axes through a positive anticlockwise angle θ and a scaling of the u, v coordinates by a factor s . The x', y' coordinates are then transformed into x, y coordinates (title system) by the addition of translations t_x and t_y .

The transformation equations can be expressed in the following way

$$\begin{aligned} x &= (s \cos \theta) u + (s \sin \theta) v + t_x \\ y &= -(s \sin \theta) u + (s \cos \theta) v + t_y \end{aligned} \quad (11)$$

or in matrix notation

	$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad (12)$
or	$\begin{bmatrix} x \\ y \end{bmatrix} = s \mathbf{R} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

Similarly to before writing $a = s \cos \theta$ and $b = s \sin \theta$ gives

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad (13)$

This transformation is referred to by several names

- (i) *Four-parameter transformation*, the four parameters being a, b, t_x, t_y ,
- (ii) *2D Linear Conformal transformation*,
- (iii) *Similarity transformation* and
- (iv) *Helmert's transformation*, after the German geodesist F.R. Helmert (1843-1917).

Note that "linear" is sometimes used in the description of a conformal transformation to differentiate it from a polynomial conformal transformation. Polynomial conformal transformations are rarely used so the distinction will not be used hereafter.

The 2D (linear) conformal transformation equations may be derived by considering Figure 6. The x', y' coordinates are obtained by rotating and scaling the u, v coordinates; and then the x, y coordinates obtained by adding the translations t_x and t_y to the x', y' coordinates. This two-step process is given by the equations:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = s \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

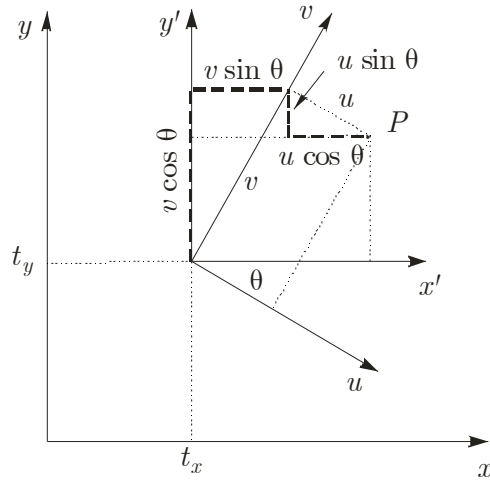


Figure 6. Schematic diagram of rotated and translated axes

Note that in Figure 6, t_x and t_y are both positive quantities, but in general, they may be positive or negative.

As an example of a 2D Conformal transformation, consider the polygon $ABCD$ whose u,v coordinates are rotated by a positive anticlockwise angle $\theta = 30^\circ$, scaled by a factor $s = 0.6$ and translated by $t_x = 50.000\text{ m}$ and $t_y = 150.000\text{ m}$. Figure 1 shows the initial location of the polygon in the u,v survey system and Figure 7 shows its transformed (rotated, scaled and translated) location in the x,y title system.

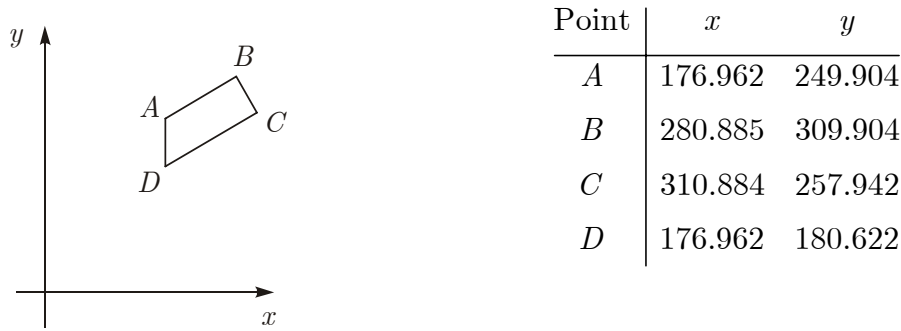


Figure 7 Rotated, scaled and translated polygon $ABCD$ with x,y coordinates in metres

Comparing Figures 1 and 7 it appears that the shape of the polygon $ABCD$ has not changed but its area and orientation with respect to the coordinate axes has. This can be verified by considering the dimensions (bearings and distances) and area of the polygon $ABCD$ derived from the two coordinate sets.

Line	Bearing	Distance	Line	Bearing	Distance
AB	$30^\circ 00'$	200.000	AB	$60^\circ 00'$	120.000
BC	$120^\circ 00'$	100.000	BC	$150^\circ 00'$	60.000
CD	$210^\circ 00'$	257.735	CD	$240^\circ 00'$	154.641
DA	$330^\circ 00'$	115.470	DA	$0^\circ 00'$	69.282
Area=22,886.75 m ²			Area=8,239.23 m ²		

Polygon dimensions in the u,v system

Polygon dimensions in the x,y system

Inspection of the two sets of dimensions reveals that bearings and distances of the polygon in the u,v system have been rotated by an angle $\theta = 30^\circ$ and scaled by a factor $s = 0.6$. Note that the shape of the polygon is unchanged but the area of the transformed figure has been reduced by a factor of s^2 . Comparison with the previous transformation demonstrates that translation has no effect on the area, shape and orientation of a polygon.

2D Conformal Transformation derived using conformal mapping theorems

C.F. Gauss (1777-1855) showed that the necessary and sufficient condition for a conformal transformation from the ellipsoid to the map plane is given by the complex expression (Lauf 1983)

$$y + ix = f(\chi + i\omega) \quad (14)$$

where the function $f(\chi + i\omega)$ is analytic, containing isometric parameters χ (isometric latitude) and ω (longitude) and in this equation the x -axis is east-west and the y -axis is north-south. i is the imaginary number ($i^2 = -1$). It should be noted here that isometric means *of equal measure*, and on the surface of the ellipsoid (or sphere) latitude and longitude are not equal measures of length. This is obvious if we consider a point near the pole where similar distances along a meridian and a parallel of latitude will correspond to vastly different angular values of latitude and longitude. Hence in conformal map projections, isometric latitude is determined to ensure that angular changes correspond to linear changes.

A necessary condition for an analytic function is that it must satisfy the Cauchy-Riemann equations

$$\frac{\partial y}{\partial \chi} = \frac{\partial x}{\partial \omega} \quad \text{and} \quad \frac{\partial y}{\partial \omega} = -\frac{\partial x}{\partial \chi} \quad (15)$$

Using this theorem, a conformal transformation from one plane rectangular coordinate system u, v (isometric parameters) to another plane rectangular system x, y (also isometric parameters) is given by the complex expression

$$y + ix = f(v + iu) \quad (16)$$

A function $f(v + iu)$ that satisfies the Cauchy-Riemann equations, is a complex polynomial, hence (16) can be given as

$$y + ix = \sum_{k=0}^n (a_k + ib_k)(v + iu)^k \quad (17)$$

Equation (17) can be expanded to the first power ($k = 1$) giving

$$\begin{aligned} y + ix &= (a_0 + ib_0)(v + iu)^0 + (a_1 + ib_1)(v + iu)^1 \\ &= a_0 + b_0i + a_1v + a_1ui + b_1vi + b_1ui^2 \end{aligned}$$

Equating real and imaginary parts (remembering that $i^2 = -1$) gives

$$\begin{aligned}x &= b_0 + a_1 u + b_1 v \\y &= a_0 - b_1 u + a_1 v\end{aligned}\tag{18}$$

or in matrix notation with translations a_0 and b_0 between the coordinate axes

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} b_0 \\ a_0 \end{bmatrix} \tag{19}$

These equations are of similar form to equations (13) in the section headed "Conformal Transformations with Rotation, Scale and Translations" and properly describe a *2D Conformal transformation*. Note that the elements of the leading diagonal of the coefficient matrix (a rotation matrix multiplied by a scale factor) are identical and the off-diagonal elements the same magnitude but opposite sign.

Equations (18) are essentially the same equations as in Jordan/Eggert/Kneissal (1963, pp. 70-73) in the section headed "*Das Helmerzsche Verfahren (Helmertsche Transformation)*" (Helmert's Transformation) although as noted by Bervoets (1992) in his bibliography, there is no reference to the original source. It is probable that F.R. Helmert developed this conformal transformation in his masterpiece on geodesy, *Die mathematischen und physikalischen Theorem der höheren Geodäsie*, (The mathematics and physical theorems of higher geodesy) on which he worked from 1877 and published in two parts: vol. 1, *Die mathematischen Theorem* (1880) and vol. 2, *Die physikalischen Theorem* (1884) [DSB 1972]. This probably accounts for the common usage of the term Helmert transformation when describing a 2D Conformal transformation.

The partial derivatives of (18) are

$$\frac{\partial x}{\partial u} = a_1, \quad \frac{\partial x}{\partial v} = b_1, \quad \frac{\partial y}{\partial u} = -b_1 \quad \text{and} \quad \frac{\partial y}{\partial v} = a_1$$

which satisfy the Cauchy-Riemann equations

$$\frac{\partial y}{\partial v} = \frac{\partial x}{\partial u} \quad \text{and} \quad \frac{\partial y}{\partial u} = -\frac{\partial x}{\partial v}$$

so verifying that the transformation is conformal.

SOLVING FOR CONFORMAL TRANSFORMATION PARAMETERS

Coordinate transformations, as used in practice, are models describing the assumed mathematical relationships between points in two rectangular coordinate systems; in these notes, the u,v (survey) and the x,y (title) systems. To determine the parameters of any transformation, coordinates of points common to both systems must be known. These points are known as control points or common points. The number of common points required for the solution of transformation parameters depends on the number of parameters in the transformation. In 2D transformations, each common point gives rise to two equations, thus n common points will give $2n$ equations. Therefore, if the four parameters of a *2D Conformal transformation* are to be determined, then a minimum of two common points are required to solve for the parameters.

It is good measurement practice to determine coordinate transformation parameters by using more than the minimum number of common points. This introduces redundant equations into the solution for the parameters and the theory of *least squares* is employed to calculate the best estimates. Parameters calculated in this manner are usually more reliable and the least squares process allows precision estimation of the parameters as well as an assessment (via residuals) of how well the transformation model fits the common points. By using least squares, several types of transformations can be "tested" on the common points to assess their suitability.

The solution for the transformation parameters involves the following steps

- (i) Select the common points ensuring that there are sufficient to allow a redundant set of equations.
- (ii) Select the appropriate weight matrix \mathbf{W} for the model.
- (iii) Solve for the parameters (contained in the vector \mathbf{x}) and residuals (contained in the vector \mathbf{v}).
- (iv) Assess the suitability of the model by analysis of the parameters and residuals.

Mathematical model for solution of 2D Conformal Transformation Parameters

The 2D Conformal transformation, or the *mathematical model*, consisting of rotation, scaling and translation is set out above [see equation (13)] and the transformation for the $k = 1, 2, 3, \dots, n$ common points is given in the form of *observation equations* (20)

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} v_{x_k} \\ v_{y_k} \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad (20)$$

where v_{x_k} and v_{y_k} are small unknown corrections or residuals simply added to the equations to account for the assumed inconsistency in the model. We could think of these residuals as consisting of two parts; one part associated with the u, v (survey) system and the other associated with the transformed x, y (title) system; the subscripts x and y attached to the residuals simply reflect the fact that they have been added to the "transformed" side of the model.

Re-arranging (20) so that all the "unknowns" are on to the left of the equals sign and the observations are to the right gives

$$\begin{aligned} v_{x_k} - a u_k - b v_k - t_x &= -x_k \\ v_{y_k} - a v_k + b u_k - t_y &= -y_k \end{aligned} \quad (21)$$

For n common points and $u = 4$ unknown parameters, the partitioned matrix representation of the $2n$ equations (21) is

$$\begin{bmatrix} v_{x_1} \\ v_{x_2} \\ v_{x_3} \\ \vdots \\ v_{x_n} \\ \hline v_{y_1} \\ v_{y_2} \\ v_{y_3} \\ \vdots \\ v_{y_n} \end{bmatrix} + \begin{bmatrix} -u_1 & -v_1 & -1 & 0 \\ -u_2 & -v_2 & -1 & 0 \\ -u_3 & -v_3 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -u_n & -v_n & -1 & 0 \\ \hline -v_1 & u_1 & 0 & -1 \\ -v_2 & u_2 & 0 & -1 \\ -v_3 & u_3 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -v_n & u_n & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \\ \vdots \\ -x_n \\ \hline -y_1 \\ -y_2 \\ -y_3 \\ \vdots \\ -y_n \end{bmatrix} \quad (22)$$

These equations are represented by the matrix equation

$$\mathbf{v} + \mathbf{B}\mathbf{x} = \mathbf{f} \quad (23)$$

where

- \mathbf{v} is a $(2n,1)$ column vector of residuals
- \mathbf{B} is a $(2n,u)$ matrix of coefficients
- \mathbf{x} is a $(u,1)$ vector of unknown parameters
- \mathbf{f} is a $(2n,1)$ column vector of numeric terms (coordinates)

The normal equations for the least squares solution of parameters \mathbf{x} and residuals \mathbf{v} are given in matrix form as

$$(\mathbf{B}^T \mathbf{W} \mathbf{B}) \mathbf{x} = \mathbf{B}^T \mathbf{W} \mathbf{f} \quad (24)$$

or

$$\mathbf{N}\mathbf{x} = \mathbf{t} \quad (25)$$

where

$\mathbf{N} = \mathbf{B}^T \mathbf{W} \mathbf{B}$ is the (u,u) symmetric coefficient matrix of the normal equations

$\mathbf{t} = \mathbf{B}^T \mathbf{W} \mathbf{f}$ is the $(u,1)$ vector of numeric terms of the normal equations

$$\mathbf{W} = \left[\begin{array}{cccc|cccc} w_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & w_2 & & & & & & 0 \\ \vdots & & \ddots & & & & & \vdots \\ 0 & & & w_n & & & & 0 \\ \hline & & & & w_1 & & & 0 \\ & & & & & w_2 & & \vdots \\ & & & & & & \ddots & 0 \\ 0 & & & & 0 & \cdots & 0 & w_n \end{array} \right]$$

is the $(2n,2n)$ diagonal weight matrix where the weights w_k on the upper-left diagonal are repeated on the lower-right diagonal. Weights are usually integer values and high weights are associated with "strong" points and low weights associated with "weak" points.

The general form of the normal equations $\mathbf{N}\mathbf{x} = \mathbf{t}$ are

$$\begin{bmatrix} \sum_{k=1}^n w_k (u_k^2 + v_k^2) & 0 & \sum_{k=1}^n w_k u_k & \sum_{k=1}^n w_k v_k \\ & \sum_{k=1}^n w_k (u_k^2 + v_k^2) & \sum_{k=1}^n w_k v_k & -\sum_{k=1}^n w_k u_k \\ \text{symmetric} & & \sum_{k=1}^n w_k & 0 \\ & & & \sum_{k=1}^n w_k \end{bmatrix} \begin{bmatrix} a \\ b \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n w_k (u_k x_k + v_k y_k) \\ \sum_{k=1}^n w_k (v_k x_k - u_k y_k) \\ \sum_{k=1}^n w_k x_k \\ \sum_{k=1}^n w_k y_k \end{bmatrix} \quad (26)$$

Centroidal coordinates

Computational savings can be made by reducing coordinates to a *weighted centroid*. For the n common points, the coordinates of the weighted centroid x_c, y_c in the x, y system are

$$\begin{aligned} x_c &= \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_n x_n}{w_1 + w_2 + w_3 + \cdots + w_n} = \frac{\sum_{k=1}^n w_k x_k}{\sum_{k=1}^n w_k} \\ y_c &= \frac{w_1 y_1 + w_2 y_2 + w_3 y_3 + \cdots + w_n y_n}{w_1 + w_2 + w_3 + \cdots + w_n} = \frac{\sum_{k=1}^n w_k y_k}{\sum_{k=1}^n w_k} \end{aligned} \quad (27)$$

Note here that coordinates of the weighted centroid x_c, y_c are just the weighted arithmetic means of the coordinates of the n common points. Also, note that if all points have the same weight then the coordinates of the centroid x_c, y_c are

$$x_c = \frac{\sum_{k=1}^n x_k}{n}, \quad y_c = \frac{\sum_{k=1}^n y_k}{n}$$

Now, the *centroidal* coordinates of the n common points in the x, y system are then

$$\begin{aligned} \bar{x}_1 &= x_1 - x_c & \bar{y}_1 &= y_1 - y_c \\ \bar{x}_2 &= x_2 - x_c & \bar{y}_2 &= y_2 - y_c \\ \bar{x}_3 &= x_3 - x_c & \bar{y}_3 &= y_3 - y_c \\ &\vdots & &\vdots \\ \bar{x}_n &= x_n - x_c & \bar{y}_n &= y_n - y_c \end{aligned} \quad (28)$$

Similar relationships can be written for centroidal coordinates in the u,v system. A useful property of the centroidal coordinates of the n common points is that their sums equal zero, i.e.,

$$\sum_{k=1}^n w_k \bar{x}_k = 0, \quad \sum_{k=1}^n w_k \bar{y}_k = 0, \quad \sum_{k=1}^n w_k \bar{u}_k = 0, \quad \sum_{k=1}^n w_k \bar{v}_k = 0 \quad (29)$$

Thus, replacing x,y and u,v coordinates with their centroidal counterparts \bar{x}, \bar{y} and \bar{u}, \bar{v} reduces the observation equations (20) to a centroidal form

$$\begin{bmatrix} \bar{x}_k \\ \bar{y}_k \end{bmatrix} + \begin{bmatrix} v_{x_k} \\ v_{y_k} \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} \bar{u}_k \\ \bar{v}_k \end{bmatrix} \quad (30)$$

It should be noted here that translations t_x and t_y are both zero when centroidal coordinates are used indicating that the centroids x_c, y_c and u_c, v_c are the same point.

For n common points and $u=2$ unknown parameters, the partitioned matrix representation of the $2n$ observation equations resulting from the centroidal model (30) is

$$\begin{bmatrix} v_{x_1} \\ v_{x_2} \\ v_{x_3} \\ \vdots \\ v_{x_n} \\ \hline v_{y_1} \\ v_{y_2} \\ v_{y_3} \\ \vdots \\ v_{y_n} \end{bmatrix} + \begin{bmatrix} -\bar{u}_1 & -\bar{v}_1 \\ -\bar{u}_2 & -\bar{v}_2 \\ -\bar{u}_3 & -\bar{v}_3 \\ \vdots & \vdots \\ -\bar{u}_n & -\bar{v}_n \\ \hline -\bar{v}_1 & \bar{u}_1 \\ -\bar{v}_2 & \bar{u}_2 \\ -\bar{v}_3 & \bar{u}_3 \\ \vdots & \vdots \\ -\bar{v}_n & \bar{u}_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\bar{x}_1 \\ -\bar{x}_2 \\ -\bar{x}_3 \\ \vdots \\ -\bar{x}_n \\ \hline -\bar{y}_1 \\ -\bar{y}_2 \\ -\bar{y}_3 \\ \vdots \\ -\bar{y}_n \end{bmatrix} \quad (31)$$

These equations are represented by the matrix equation (23) and the normal equations have the following simple form containing only three different numbers

$$\begin{bmatrix} \sum_{k=1}^n w_k (\bar{u}_k^2 + \bar{v}_k^2) & 0 \\ 0 & \sum_{k=1}^n w_k (\bar{u}_k^2 + \bar{v}_k^2) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n w_k (\bar{u}_k \bar{x}_k + \bar{v}_k \bar{y}_k) \\ \sum_{k=1}^n w_k (\bar{v}_k \bar{x}_k - \bar{u}_k \bar{y}_k) \end{bmatrix} \quad (32)$$

The solutions for the parameters a and b are

$$a = \frac{\sum_{k=1}^n w_k (\bar{u}_k \bar{x}_k + \bar{v}_k \bar{y}_k)}{\sum_{k=1}^n w_k (\bar{u}_k^2 + \bar{v}_k^2)} \quad (33)$$

$$b = \frac{\sum_{k=1}^n w_k (\bar{v}_k \bar{x}_k - \bar{u}_k \bar{y}_k)}{\sum_{k=1}^n w_k (\bar{u}_k^2 + \bar{v}_k^2)} \quad (34)$$

The translations t_x and t_y are obtained by re-arranging (13) and replacing x, y and u, v with the coordinates of the centroid x_c, y_c and u_c, v_c giving

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \end{bmatrix} - \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} u_c \\ v_c \end{bmatrix} \quad (35)$$

or

$$\begin{aligned} t_x &= x_c - au_c - bv_c \\ t_y &= y_c + bu_c - av_c \end{aligned} \quad (36)$$

After calculation of the parameters, $\mathbf{x} = \begin{bmatrix} a & b & t_x & t_y \end{bmatrix}^T$ the residuals are calculated using (31).

The least squares solution for the transformation parameters looks formidable, but it really is very simple. The parameters for any 2D Conformal transformation can be computed using a pocket calculator and this solution depends on forming only three numbers from a system of centroidal coordinates. Alternatively, a simple computer program spreadsheet (such as *Excel*) could be used.

In the following pages an example of a 2D Conformal transformation as an aid to cadastral re-establishment will be discussed.

CONFORMAL TRANSFORMATION EXAMPLE

Figure 8 shows a Plan of Subdivision (LP48556) with distances in links (1 chain = 100 links = 66 feet) and bearings related to True North. The plan shows two Reference Marks (RM's), one near the south-west corner of Lot 1 and the other near the south-east corner of the 100 link wide access to Lot 2. The subdivision was created and marked on the ground in the 1920's.

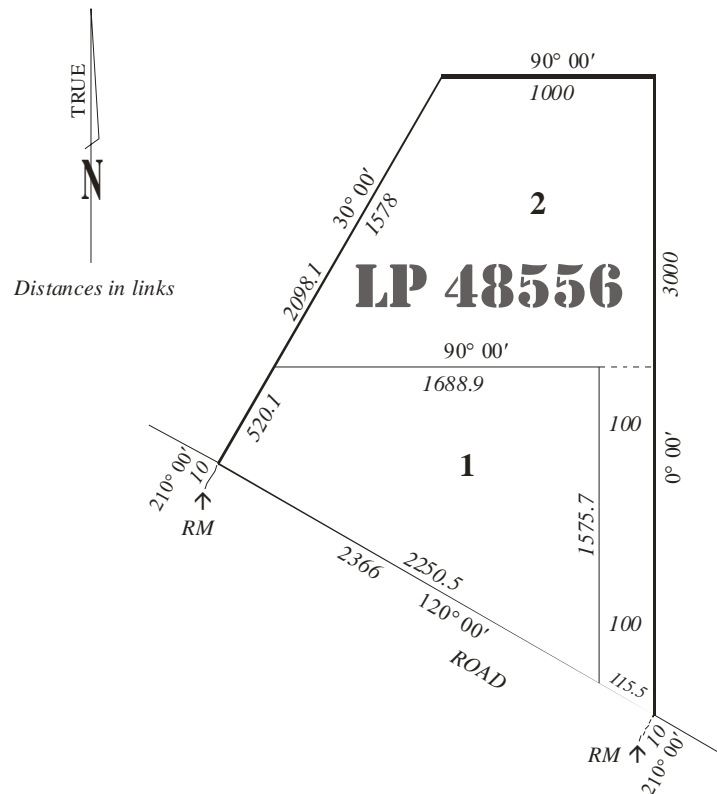


Figure 8 Plan of Subdivision LP48556

Figure 9 shows an Abstract of Fieldnotes of a recent survey conducted for the purposes of boundary re-establishment prior to purchase of Lot 2, LP 48556. At the time of survey only one of the RM's along the road was found and old pegs, thought to be original, were found at the south-west and north-east corners of Lot 2. Most of the fencing was fairly recent, probably replacing original fencing. The post at the north-east corner of Lot 2, which is new, is very close to the old peg which may have been disturbed when the new post was put in. The other old peg at the south-west corner of Lot 2 appeared to be original and undisturbed.

ABSTRACT OF FIELDNOTES

DATUM A-B

Distances in metres

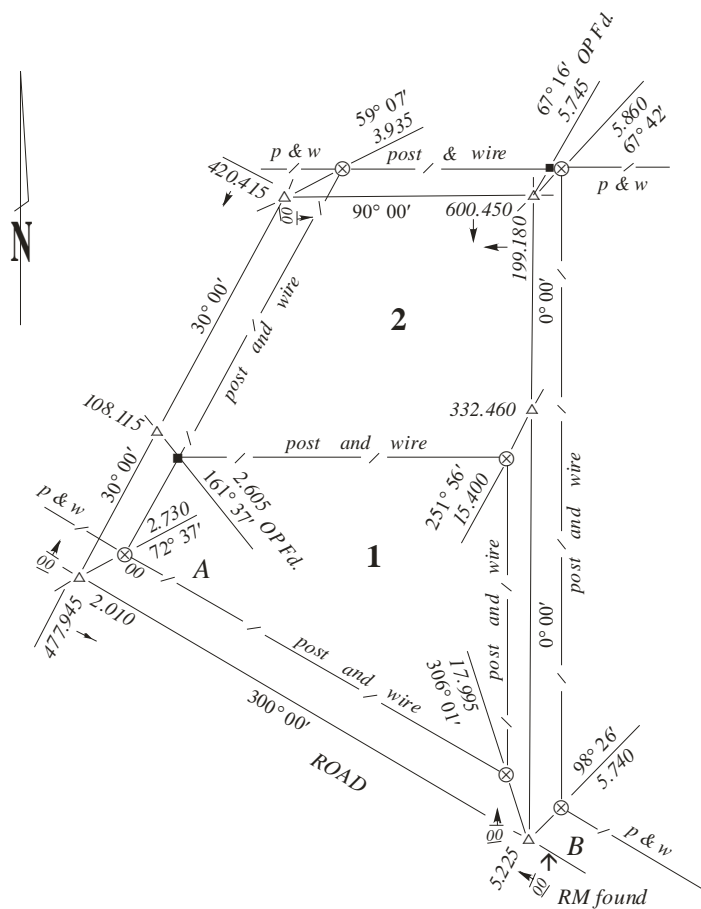


Figure 9 Abstract of Fieldnotes of survey of Lot 2, LP48556

The datum of the survey was the post A (south-west corner of Lot 1) and the RM B found near the south-east corner of the road access to Lot 2. A traverse line offset 2.010 m (10 links) from the post at A and passing through the RM was adopted for the bearing datum of $300^{\circ} 00'$

For this example we will perform a cadastral re-establishment using a 2D Conformal transformation (scale, rotation and translations) with weights based on the RM and the two old pegs of LP48556. In light of the information above, the RM will be given a weight of 10, the old peg at the south-west corner of Lot 2 will be given a weight of 5 and the other old peg (north-east corner of Lot 2) will be given a weight of 1.

The parameters of the transformation (scale, rotation and translations) will be determined and an inspection of residuals will give some indication as to the "correctness" of the re-establishment.

For the purposes of computing the transformation parameters, two arbitrary coordinate systems will be used. One system of coordinates, in metres, called *TITLE* will have values of 5000.000 E and 5000.000 N for the RM near the south-east corner of the road access to Lot 2. For the purpose of computing the *TITLE* coordinates the original dimensions in links will be converted to metres where 1 chain = 100 links = 66 feet, and 1 foot = 0.3048 metres (exactly) giving links $\times 0.201168$ = metres. The original dimensions of 1000 links, 3000 links and 1578 links will be converted to metres (3 decimal places) and the other dimensions "computed to close" and noted to 4 decimal places. The road access frontage will be derived by computation after converting the 100 link width to 20.117 metres. This computation process should ensure that coordinates are mathematically correct to 3 decimal places.

The other system of coordinates, also in metres, and called *SURVEY* will have values of 2000.000 E and 2000.000 N for the RM found. The traverse dimensions are mathematically correct (to a millimetre) and should yield *SURVEY* coordinates of traverse points and occupation correct to 3 decimal places.

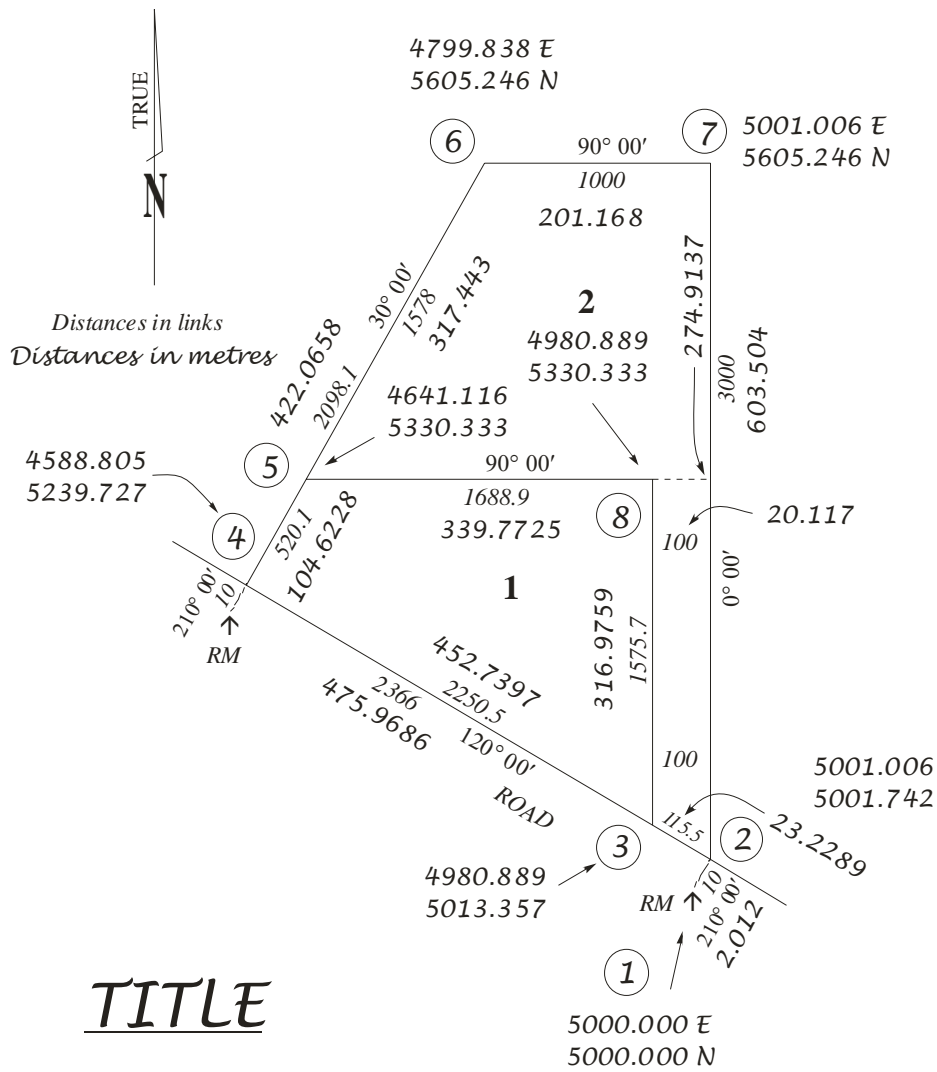


Figure 10 TITLE coordinates (metres)

POINT	Description	Weight	TITLE		CENTROIDAL TITLE	
			E	N	\bar{E}	\bar{N}
1	RM	10	5000.000	5000.000	112.088	-141.057
5	Old Peg	5	4641.116	5330.333	-246.796	189.276
7a	Old Peg	1	5001.006	5605.246	113.094	464.189
centroid			4887.9116	5141.0569		

Coordinates of the centroid computed using equation (27) and centroidal coordinates calculated using equation (28).

ABSTRACT OF FIELDNOTES

DATUM A-B

Distances in metres

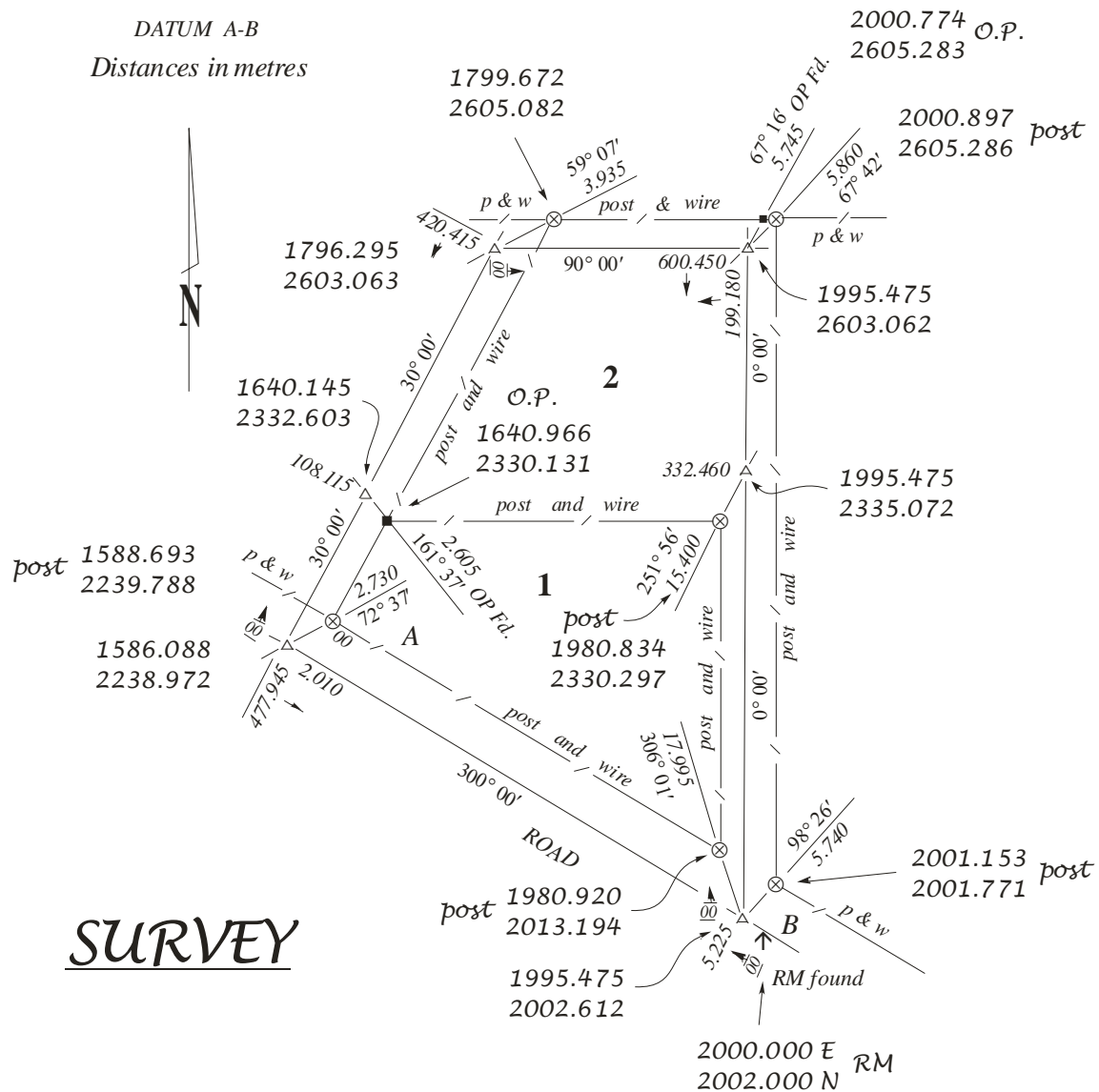


Figure 11 SURVEY coordinates (metres)

POINT	Description	Weight	SURVEY		CENTROIDAL SURVEY	
			U	V	\bar{U}	\bar{V}
1	RM	10	2000.000	2000.000	112.150	-140.996
5	Old Peg	5	1640.966	2330.131	-246.884	189.135
7a	Old Peg	1	2000.774	2605.283	112.924	464.287
centroid			1887.8503	2140.9961		

Coordinates of the centroid computed using equation (27) and centroidal coordinates calculated using equation (28).

Using the centroidal TITLE (\bar{E}, \bar{N}) and centroidal SURVEY (\bar{U}, \bar{V}) coordinates we can form the weighted centroidal coordinate products $\sum_{k=1}^n w_k (\bar{U}_k^2 + \bar{V}_k^2)$, $\sum_{k=1}^n w_k (\bar{U}_k \bar{E}_k + \bar{V}_k \bar{N}_k)$ and $\sum_{k=1}^n w_k (\bar{V}_k \bar{E}_k - \bar{U}_k \bar{N}_k)$ that are used in equations (33) and (34) to compute the parameters a and b . These products are

POINT	$\sum_{k=1}^n w_k (\bar{U}_k^2 + \bar{V}_k^2)$	$\sum_{k=1}^n w_k (\bar{U}_k \bar{E}_k + \bar{V}_k \bar{N}_k)$	$\sum_{k=1}^n w_k (\bar{V}_k \bar{E}_k - \bar{U}_k \bar{N}_k)$
1	324574.7369	324591.6483	154.7374
5	483619.1692	483643.2860	258.0952
7a	228314.0756	228287.9302	90.2643
sums	1036507.9817	1036522.8644	503.0970

The transformation parameters a and b are

$$a = \frac{\sum_{k=1}^n w_k (\bar{U}_k \bar{E}_k + \bar{V}_k \bar{N}_k)}{\sum_{k=1}^n w_k (\bar{U}_k^2 + \bar{V}_k^2)} = \frac{1036522.8644}{1036507.9817} = 1.000014359$$

$$b = \frac{\sum_{k=1}^n w_k (\bar{V}_k \bar{E}_k - \bar{U}_k \bar{N}_k)}{\sum_{k=1}^n w_k (\bar{U}_k^2 + \bar{V}_k^2)} = \frac{503.0970}{1036507.9817} = 0.000485377$$

The translations t_E and t_N are obtained from equations (36) as

$$t_E = E_c - aU_c - bV_c = 2998.995$$

$$t_N = N_c + bU_c - aV_c = 3000.946$$

where the coordinates of the centroid in both systems are given in the tables below Figures 9 and 10.

Having obtained a , b , t_E and t_N , residuals at the common points can be obtained from either equation (20) or (30).

These results, together with the transformed coordinates are shown on the *Excel* worksheet on the following page.

LEAST SQUARES SOLUTION OF PARAMETERS OF 2D LINEAR CONFORMAL TRANSFORMATION

Conformal Transformation Exercise

2D Linear Conformal Transformation (with weights)

E = a*U + b*V + t(E) E, N are coordinates in System 1.

N = -b*U + a*V + t(N) U, V are coordinates in System 2.

Rotations are considered positive anti-clockwise.

U, V coordinates (System 2) are transformed (scaled, rotated and translated) into E, N coordinates (System 1).

POINT	SYSTEM1 (TITLE)		Centroidal Coords	
	E	N	E(c)	N(c)
1	5000.000	5000.000	112.088	-141.057
5	4641.116	5330.333	-246.796	189.276
7a	5001.006	5605.246	113.094	464.189

Centroid 4887.9116 5141.0569

LEAST SQUARES SOLUTION

a = 1.000014359 t(E) = 2988.995

b = 0.000485377 t(N) = 3000.946

TRANSFORMED COORDINATES

E = a*U + b*V + t(E)

N = -b*U + a*V + t(N)

Point	SURVEY		TITLE	
	U	V	E	N
1	2000.000	2000.000	4999.995	5000.004
5	1640.966	2330.131	4641.116	5330.314
7a	2000.774	2605.283	5001.062	5605.296
2	2001.153	2001.771	5001.148	5001.775
3	1980.920	2013.194	4980.921	5013.208
4	1588.593	2239.798	4588.798	5239.995
6	1799.672	2605.082	4799.957	5605.192
7b	2000.897	2605.286	5001.185	5605.299
8	1980.634	2330.297	4980.989	5330.315
4,1	1586.088	2238.972	4586.193	5239.181

RM (Control Point)

OP (Control Point)

OP (Control Point)

post

post

post

post

post

RM South West corner

Weighted Centroidal Coordinate Products	
$W^*(U(c)^2+V(c)^2)$	$W^*(U(c)^*E(c)+V(c)^*N(c))$
324574.7369	324591.6483
483619.1692	483643.2860
228314.0756	228287.9302

Residuals	
E	N
-0.005	0.004
0.000	-0.019
0.056	0.050

SYSTEM2 (SURVEY)		Centroidal Coords	
U	V	U(c)	V(c)
2000.000	2000.000	112.150	-140.996
1640.966	2330.131	-246.884	189.135
2000.774	2605.283	112.924	464.287

Sums 1036507.9817 1036522.8644 503.0970

CONFORMAL TRANSFORMATION WITH SCALE FACTOR OF UNITY

For certain purposes it may be desirable to determine the transformation parameters from common points with the condition that the scale factor be equal to unity, i.e., $s = 1$. This can be achieved by the following:

Let's say that we are able to obtain a and b from equations (33) and (34) – a least squares solution with weights. The scale s and rotation angle θ are obtained from equations (10) which are re-stated here again as

$$s = \sqrt{a^2 + b^2} \quad (37)$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \quad (38)$$

If we divide both sides of equation (37) by s we have

$$\begin{aligned} \frac{s}{s} = 1 &= \frac{\sqrt{a^2 + b^2}}{s} \\ &= \sqrt{\frac{a^2 + b^2}{s^2}} \\ &= \sqrt{\left(\frac{a}{s}\right)^2 + \left(\frac{b}{s}\right)^2} \end{aligned}$$

If we define

$$a' = \frac{a}{s} \quad \text{and} \quad b' = \frac{b}{s} \quad (39)$$

the transformation, given by equation (13), becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a' & b' \\ -b' & a' \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} t'_x \\ t'_y \end{bmatrix} \quad (40)$$

and this transformation has a scale factor of unity, since

$$\sqrt{(a')^2 + (b')^2} = 1$$

Also, we note that $\frac{b}{a} = \frac{b'}{a'}$ since the scale s will cancel in the division so that the rotation angle θ computed from equation (38) is the same whether we use the parameters a, b or new parameters a', b' from equation (39).

It should be noted that the "new" transformation, with scale factor of unity, given by equation (40), has translations t'_x, t'_y and these will be different from the translations t_x, t_y of equation (13). The translations t'_x and t'_y are obtained by re-arranging equation (40) and replacing x, y and u, v with the coordinates of the centroid x_c, y_c and u_c, v_c giving

$$\begin{bmatrix} t'_x \\ t'_y \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \end{bmatrix} - \begin{bmatrix} a' & b' \\ -b' & a' \end{bmatrix} \begin{bmatrix} u_c \\ v_c \end{bmatrix} \quad (41)$$

or

$$\begin{aligned} t'_x &= x_c - a'u_c - b'v_c \\ t'_y &= y_c + b'u_c - a'v_c \end{aligned} \quad (42)$$

After calculation of the parameters, $\mathbf{x} = [a' \quad b' \quad t'_x \quad t'_y]^T$ the residuals are calculated using (31).

We define a', b', t'_x, t'_y as the parameters of a conformal transformation with a scale factor of unity.

Using the computed data from the example: $a = 1.000014359$, $b = 0.000485377$ giving $s = 1.000014476$ and from equations (39)

$$\begin{aligned} a' &= \frac{a}{s} = \frac{1.000014359}{1.000014476} = 0.999999882 \\ b' &= \frac{b}{s} = \frac{0.000485377}{1.000014476} = 0.000485370 \end{aligned}$$

The translations t'_E and t'_N are obtained from equations (42) as

$$\begin{aligned} t'_E &= E_c - a'U_c - b'V_c = 2999.022 \\ t'_N &= N_c + b'U_c - a'V_c = 3000.977 \end{aligned}$$

where the coordinates of the centroid in both systems are given in the tables below Figures 9 and 10.

Having obtained a' , b' , t'_E and t'_N , residuals at the common points can be obtained from either equation (20) or (30) by replacing a , b , t_x and t_y with a' , b' , t'_E and t'_N .

These results, together with the transformed coordinates (where the scale factor is unity) are shown on the *Excel* worksheet on the following page.

WEIGHTING SCHEMES

When solving for the transformation parameters, observation equations are formed – there are $2n$ equations, where n is the number of common points or control points – and the least squares principle leads to a set of normal equations [see equations (24) and (25)] that involve a (diagonal) weight matrix \mathbf{W} where the elements of the leading diagonal w_1, w_2, w_3, \dots etc. are known as weights and are usually integers (see page 19). High weights (large integers) are associated with "strong" points and low weights (small integers) associated with "weak" points.

This association may be best explained by reference to the example (see Figures 8 and 9) remembering that weights are only assigned to control points.

The three control points are the Reference Mark (RM found) near the S.E. corner of Lot 1, the old peg (OP) by the post at the N.E. corner of Lot 2 and the OP at the S.W. corner of Lot 2. Most surveyors would probably regard RM's and pegs (if they have not been disturbed) as very strong indicators of title corners (via title-connections in the case of RM's). Pegs might be slightly less well regarded as they could have been disturbed, and fence posts or fence intersections, would rank below that of pegs and RM's as important indicators of title corners. This would be a fairly normal hierarchy that a surveyor would gain from experience. Assigning weights is merely putting numbers into the transformation process that reflect that hierarchy.

In the exercise, the RM has been assigned a weight of 10, the OP at the S.W. corner of Lot 2 has been assigned a weight of 5 and the other OP at the N.E. corner of Lot 2 has been assigned a weight of 1. Perhaps here, the intention is to give less weight to the OP by the fence post, since there is a possibility that the peg could have been disturbed – by the fencing contractor perhaps. These are arbitrary numbers and are reflections of the surveyor's field experience. Control points of high weight will have smaller residuals than control points of low weight. You can adjust the magnitude of the weights to give a particular point (or points) lower residuals than other points.

It is interesting to note that in the *Excel* spreadsheet used to compute the transformation parameters, assigning a weight of zero effectively removes that point as a control point. This means that initially, all the occupation (RM's, OP's, posts, etc.) can be control points in an initial transformation and then removed from the process by assigning a weight of zero to points that have large residuals; indicating

that the occupation is not at, or near, a title corner. This adds flexibility to the re-establishment analysis.

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