OVERLAND AND OPEN-CHANNEL FLOW
ESCI 4701: GEOMORPHOLOGY

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The speed of water flowing over the landscape or in a river channel is a major contributor to geomorphic change, and is crucial for fluvial and basin-scale processes. Here we will cover some commonly used equations, their inherient assumptions, and their pitfalls.

1. Setting the stage

Before we get started, let's visualize a river together, complete with its characteristics: its flow velocity, depth, slope, channel pattern, sediment grain size, etc.

Figure 1. Artist unknown.

2. Laminar vs. turbulent flow

In **Laminar flow**, parcels of fluid move smoothly against one another.
In **turbulent flow**, parcels of fluid swirl around and mix.

Figure 2. Draw schematic sketches of laminar and turbulent flow.

In *laminar flow*, viscosity dominates over inertia. Conceptually, this means that the dampening effect of how sticky or sluggish a fluid is will slow down and still the effects of perturbations. Think about moving your finger through a tray of water. The water might ripple and form vortices in response. This is because *inertia* – and turbulence – are dominant here. Now think about the same exercise in a tray of molasses. The molasses will smear and shear, but will not form vortices that spin away from your finger. This is because *viscosity* dominates over inertia.

The **Reynolds Number** is a dimensionless value that helps us to determine whether a flow will be laminar (viscous-dominated) or turbulent (inertia dominated)[1] It is the ratio of the inertial forces – those supplying momentum to maintain the motion of fluid parcels – to viscous forces – those dampening momentum-driven flow:

$$\mathrm{Re} = \frac{\rho u L}{\mu} = \frac{uL}{\nu} \tag{1}$$

ρ is the density of the fluid, u is the fluid velocity, and L is a length scale. μ is the *dynamic viscosity* of a fluid. $\nu = \mu/\rho$ is the *kinematic viscosity* of a fluid, which is simply defined as the dynamic viscosity divided by the density.

- Flow is **laminar** when $\mathrm{Re} \lessapprox 500$.
- Flow is **turbulent** when $\mathrm{Re} \gtrapprox 1000$.

Between these two Reynolds numbers, flow is classified as **transitional**: both inertial and viscous forces are important, and it is therefore more difficult to solve. You may be relieved, then, that we will not discuss transitional flows beyond this fair warning that they exist!

[1]The Reynolds Number appears in the *Navier–Stokes equation*, which is the master equation for fluid flow.

2.1. Intuition-building questions: laminar or turbulent?

2.1.1. *A 1-cm-thick layer of honey flowing down a ramp into the waiting mouth of a hungry bear.*

- $\rho_{\text{honey}} = 1450$ kg m^{-2}
- $u_{\text{honey}} = 0.01$ m s^{-1}
- $L_{\text{honey}} = 0.01$ m
- $\mu_{\text{honey}} = 10$ Pa s

2.1.2. *A 1-cm-thick layer of water flowing down a ramp into the waiting mouth of a thirsty hare.*

- $\rho_{\text{water}} = 1000$ kg m^{-2}
- $u_{\text{water}} = 0.1$ m s^{-1}
- $L_{\text{water}} = 0.01$ m
- $\mu_{\text{water}} = 10^{-3}$ Pa s

2.1.3. *A 1-m-deep river flowing next to the bear and the hare.*

- $\rho_{\text{water}} = 1000$ kg m^{-2}
- $u_{\text{water}} = 1$ m s^{-1}
- $L_{\text{water}} = 1$ m
- $\mu_{\text{water}} = 10^{-3}$ Pa s

3. Steady and/or uniform flow

Here, we will focus on the special cases of flows that are both *steady* and *uniform*. This helps to greatly simplify the mathematics and is applicable in many natural environments. What do these terms mean?

3.1. **Steady.** *Steady* flow is flow that is not changing in time. This means that all time derivatives are set to zero. Mathematically, for any quantity X,

$$\frac{\mathrm{d}X}{\mathrm{d}t} \rightarrow 0 \qquad \text{for steady flow.} \tag{2}$$

Most importantly, there are no accelerations (under the standard definition of accelerations being change in velocity over change in time).

Strictly speaking, we know that this is not true. Low flows in summer will be slower than high flows during snowmelt. But *compared to the amount of time that it takes for water to travel through a reach of a river, such changes are typically slow*. Exceptions exist, of course, such as flood waves after dam breaches. These situations require a more complete mathematical description of fluid flow.

Figure 3. Sketch the passage of a flood wave (e.g., from a dam break), and along with it, the expected thickness and velocity of the flow. Show how this changes with time. This is not a common situation!

3.2. **Uniform.** Uniform flows are not changing in space. Thus, for any quantity X and a spatial dimension x,

$$\cancelto{0}{\frac{\mathrm{d}X}{\mathrm{d}x}} \qquad \text{for uniform flow.} \tag{3}$$

Most flow in rivers can be approximated as uniform because the river channel changes only gradually downstream.

When considering natural flows, a common case of *nonuniform flow* occurs when a river approaches a lake, ocean, or waterfall. This is a zone of *spatial acceleration* as opposed to the more common temporal acceleration (which is most commonly called just "acceleration"). In this case, a single water molecule is speeding up or slowing down as it approaches the waterfall/lake/etc., but flow at a single location is steady.

FIGURE 4. Sketch the examples of a river entering a lake or ocean (so-called "M1 backwater") and a river flowing over an escarpment to form a waterfall (so-called "M2 backwater".

4. LAMINAR FLOW

Laminar flow follows the *Navier–Stokes equations* for the simplified case in which the viscosity dominates over the inertia. Assuming that the only force that acts on the flow is from the weight of the fluid itself, and that the flow is steady and uniform, this simplifies to:

$$\nu \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = -\mathbf{g} \tag{4}$$

Here, $\mathbf{g}$ is the gravitational **body stress** vector, which is that pulling the flow downslope. We will solve for it later using the "depth–slope product". For now, let's just integrate twice and solve for the shape of the velocity profile in a laminar flow being driven by gravity across an inclined bed:

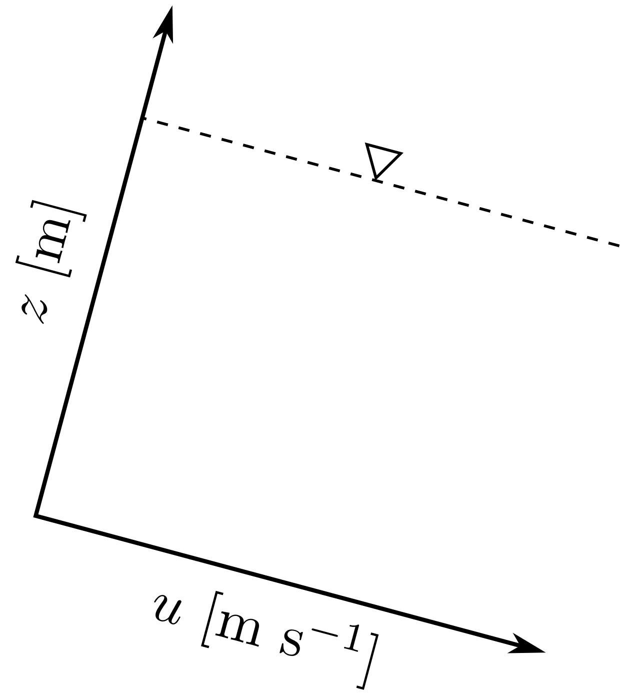

Figure 5. Draw a schematic laminar velocity profile.

5. Empirical or semi-empirical "laws" for turbulent overland and open-channel flow

What do we know in general about flow between the land surface and the air? Recall that the land surface (or channel bed) is a *no slip* boundary, and that the air is a *free slip* boundary.

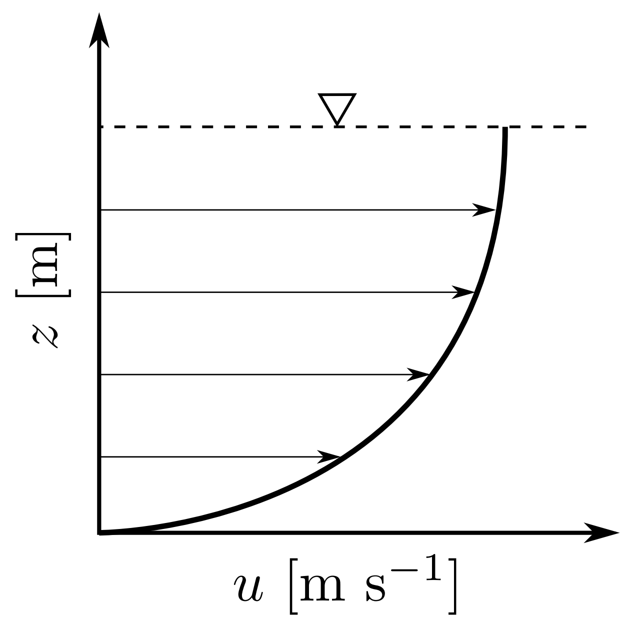

Figure 6. Schematic velocity profile.

Armed with this basic idea about what flow should look like, several engineers created equations to describe flow.

5.1. **Chézy's Equation.** Derived in the 1700's, originally for pipe flow and adopted to open channels.

$$\bar{u} = C_z u_\tau = C_z \sqrt{g R_h S} \quad (5)$$

$\bar{u}$ is mean (depth-averaged) flow velocity. The shear velocity, u_τ, is a velocity scale for the amount of shear stress imparted on the bed. C_z, the "Chézy coefficient" is analogous to an inverse friction coefficient: as it goes up, the flow goes faster. R_h is hydraulic radius (think pipes), and is comparable to h in channels, in which it is calculated as:

$$R_h = \frac{A}{L_{\text{wetted perimeter}}} \approx \frac{A}{b + 2h} \quad (6)$$

For channels that are much wider than they are deep (i.e. the majority in nature), this is approximately h, which is simply the flow depth:

$$\bar{u} \approx C_z \sqrt{ghS} \quad (7)$$

Does this behave properly at our limits (bottom and top of flow)?

Let's solve for the units of C_z.

$$C_z = 8.1 \left(\frac{h}{k_s} \right)^{1/6} \quad (8)$$

(*Parker*, 1991). k_s is a roughness height that will come up again for the Law of the Wall (below). It is variously given by (e.g., *Clifford et al.*, 1992):

$$k_s \approx (1.5 \text{ to } 3) D_{90} \approx 3.5 D_{84} \approx 5.9 D_{50} \quad (9)$$

where D stands for the particle (grain) size on the bed, and the subscript is the percentile (i.e. 90% of the grains smaller than the D_{90}). The D_{50} is the median grain size.

5.2. **Manning's Equation.** Manning in the 1840's observed that Chézy's C_z is a function of depth, so he came up with his own equation.

$$\bar{u} = \frac{1}{n} {R_h}^{2/3} S^{1/2} \tag{10}$$

(This version of the equation uses metric units.)

Manning's equation is still extremely widely used, from hydraulic engineering to hydrodynamic modeling to flood-hazard planning. There is a large literature devoted to determining "Manning's n" for different systems.

- Picture books to find what Manning's n should be! (*Harry and Barnes*, 1987; *Acrement and Schneider*, 1989)
- Tables for this same purpose (just search "Manning's n table" online)

These known coefficents are solved by back-calculating n in channels of known S, R_h and $\bar{u} = Q/A$.

$$n = \frac{{R_h}^{2/3} S^{1/2}}{\bar{u}} \tag{11}$$

Does this behave properly at our limits (bottom and top of flow)?

Let's solve for the units of n.

5.3. **Darcy-Weisbach equation.** This is the generalized form of the Darcy-Weisbach equation *for steady, uniform, open-channel flow*, in which the friction factor is denoted C_f.

$$\bar{u} = \sqrt{\frac{g R_h S}{C_f}} \tag{12}$$

Type of Channel and Description	Minimum	Normal	Maximum
A. *Natural Streams*			
1. Main Channels			
a. Clean, straight, full, no rifts or deep pools	0.025	0.030	0.033
b. Same as above, but more stones and weeds	0.030	0.035	0.040
c. Clean, winding, some pools and shoals	0.033	0.040	0.045
d. Same as above, but some weeds and stones	0.035	0.045	0.050
e. Same as above, lower stages, more ineffective slopes and sections	0.040	0.048	0.055
f. Same as "d" but more stones	0.045	0.050	0.060
g. Sluggish reaches, weedy. deep pools	0.050	0.070	0.080
h. Very weedy reaches, deep pools, or floodways with heavy stands of timber and brush	0.070	0.100	0.150
2. Flood Plains			
a. Pasture no brush			
1. Short grass	0.025	0.030	0.035
2. High grass	0.030	0.035	0.050
b. Cultivated areas			
1. No crop	0.020	0.030	0.040
2. Mature row crops	0.025	0.035	0.045
3. Mature field crops	0.030	0.040	0.050
c. Brush			
1. Scattered brush, heavy weeds	0.035	0.050	0.070
2. Light brush and trees, in winter	0.035	0.050	0.060
3. Light brush and trees, in summer	0.040	0.060	0.080
4. Medium to dense brush, in winter	0.045	0.070	0.110
5. Medium to dense brush, in summer	0.070	0.100	0.160
d. Trees			
1. Cleared land with tree stumps, no sprouts	0.030	0.040	0.050
2. Same as above, but heavy sprouts	0.050	0.060	0.080
3. Heavy stand of timber, few down trees, little undergrowth, flow below branches	0.080	0.100	0.120
4. Same as above, but with flow into branches	0.100	0.120	0.160
5. Dense willows, summer, straight	0.110	0.150	0.200

Figure 7. Tabulated Manning's n values.

Does this behave properly at our limits (bottom and top of flow)?

Let's solve for the units of C_f.

Empirically (*Leopold et al.*, 1964),

$$\frac{1}{\sqrt{C_f}} = \frac{1}{\sqrt{8}}\left[2.0\log\left(\frac{h}{D_{84}}\right)+1\right] \tag{13}$$

Hint: this equation is one to remember for later in the class, when we discuss shear stress and shear velocity. Note the logarithmic form of its empirical definition.

5.4. **Closing thoughts.** You should by algebra be able to derive relationships between each of these characterizations of flow roughness:

- C_z = Chézy coefficient
- n = Manning's n
- C_f Dimensionless bed resistance coefficient = C_z^{-2}

6. Shear stress

Thus far, we have developed equations that can provide mean velocity within a flow. They parameterize friction in some way, and generally behave as expected for a given velocity profile. Now, we want to build a more theoretically-based view of how open-channel flow works, to obtain a distribution of flow velocities with depth. Recall the constitutive relationship of water as a laminar fluid:

$$\tau = \mu \frac{\mathrm{d}u}{\mathrm{d}z} \tag{14}$$

This should tell you why it is important to know the velocity profile – especially considering how important shear stress can be in moving sediment and eroding bedrock (more on this soon in the course).

In order to understand how this equation, or any equation involving shear stresses (alternatively, shear tractions) on the bed, it helps to construct a diagram of a segment of flowing water.

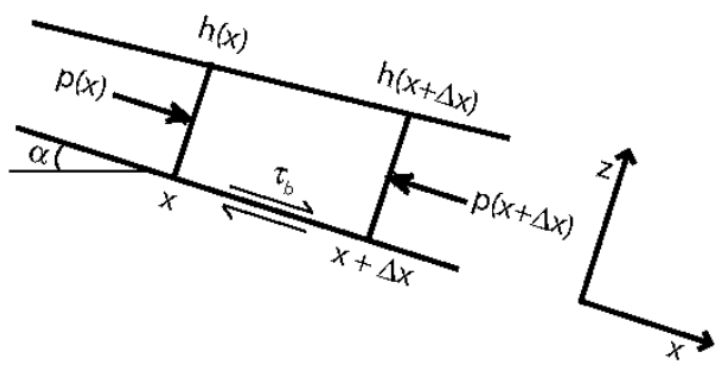

Figure 8. A segment of river, gratuitously lifted from Kelin Whipple's course notes on MIT OCW. Note that the coordinate system is aligned with the channel bed.

The solutions below all include an assumption of *steady flow*: that is, there are no accelerations in time. Of course, there will be some local accelerations that are related to eddies and turbulence, and therefore, geomorphologists often describe this as *quasi-steady* to indicate that – on average and on a whole, there is no acceleration beyond local-scale flickers.

6.1. **Pressure gradients.** Pressure gradients are produced through gradients in flow depth.[2]

$$\frac{\partial p}{\partial x} \approx \frac{p(x+\Delta x) - p(x)}{\Delta x} = \frac{\rho g h(x+\Delta x) - \rho g h(x)}{\Delta x} \tag{15}$$

[2] Changes in density could produce these as well, but rivers and overland flow comprise freshwater systems with sediment concentrations low enough that sediment is generally a small fraction of the total flow. Therefore, density can safely be assumed to be an approximate constant.

Therefore:

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x} \tag{16}$$

At the bed of the channel, this lateral pressure gradient produces a stress, which we denote as:

$$\tau_{b,\text{pressure}} = \frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x} \tag{17}$$

6.2. **Body stresses.** Body stresses (or forces) are those related to the weight of a parcel of material. The weight of the unit block of water is:

$$\text{Weight(water)} = \rho g h \tag{18}$$

We care about the shear stress on the bed, as this is capable of moving sediment, so therefore combine this weight of water with the trig function for its down-slope component to obtain the following equation

$$\tau_{b,\text{body}} = \rho g h \sin\alpha \approx \rho g h S \tag{19}$$

This is known as the "**depth–slope product**"[3], and is very important in fluvial geomorphology. It is analogous to solutions for a block on an inclined plane – with the exception that this solution is for a single column of water exerting pressure on the bed, and therefore is in terms of *stresses* instead of *forces*.

6.3. **Combining the pressure and body stresses.** Combining the body and pressure forces:

$$\tau_b = \tau_{b,\text{body}} + \tau_{b,\text{pressure}} \tag{20}$$

Solutions for when the pressure forces matter are called the **backwater equations**, and are important when flow is accelerating towards a waterfall or decelerating towards a body of standing water. These are a bit complicated, and we will not have time to cover them in this class. However, as a basic rule to see whether the backwater effect will be important, you may use a characteristic **backwater length** scale:

$$l_{bw} = h/S \tag{21}$$

where h is the depth *upstream of* the backwater reach.

This second right-hand term term disappears when flow depth is uniform, thereby leaving us with just the body forces. Therefore, for most places in most rivers, where we can assume steady, uniform flow. As a result, our most common equation for the applied bed shear stress becomes the depth–slope product. Although this is an approximation (S for $\sin\alpha$), I will use an equals sign from this point forward for convenience.

$$\tau_b = \rho g h S \tag{22}$$

6.4. **Basal shear stress: friction and definition.** Alternatively, one may represent the basal shear stress by using the roughness and flow-velocity relationships from earlier. Combining the Darcy–Weisbach friction factor with the shear velocity, one may obtain:

$$\tau_b = \rho C_f \bar{u}^2 \tag{23}$$

Finally, the definition of basal shear stress is:

$$\tau_b \equiv \rho u_\tau^2 \tag{24}$$

In the case of steady, uniform flow,

$$u_\tau = \sqrt{ghS} \tag{25}$$

[3]The depth–slope product holds true for both laminar and turbulent flow; it is merely a function of the flow geometry.

and as a result, we again obtain the depth–slope product for steady, uniform flow.

7. The Law of the Wall

We have an analogous equation to the laminar flow constitutive relationship for turbulent flow:

$$\tau = K \frac{\mathrm{d}u}{\mathrm{d}z} \tag{26}$$

K scales with the size of the eddies, and is therefore a property of the *flow* and not of the fluid (as true viscosity is). The eddies act to redistribute momentum within the flow, moving fast parcels down and slow parcels up. The efficiency of these eddies in doing so is related to the size of the eddies:

$$K \propto \rho l^2 \frac{\mathrm{d}u}{\mathrm{d}z} = \kappa^2 \rho l^2 \frac{\mathrm{d}u}{\mathrm{d}z} \tag{27}$$

where l is a characteristic eddy length-scale. κ describes the relationship between turbulent diffusivity K and the physical properties of eddies and the flow. It will become known as the von Kármán constant, and be very important for the rest of this discussion. Empirically, $\kappa = 0.407$.

Near the boundary of the flow, $l \approx z$, becuase the bed stops eddies. So...

$$K = \kappa^2 \rho z^2 \frac{\mathrm{d}u}{\mathrm{d}z} \tag{28}$$

Figure 9. Draw eddies near to and far from the wall.

Therefore, after doing a bit of equation combination:

$$\tau_b = \kappa^2 \rho z^2 \left(\frac{\mathrm{d}u}{\mathrm{d}z} \right)^2 \tag{29}$$

τ_b is bed shear stress. We're going to see a lot of it from here on out!

We can also further discuss a new term, defined originally by Prandtl, called the shear velocity:

$$u_\tau \equiv \frac{\sqrt{\tau_b}}{\rho} \tag{30}$$

This is a scaling that relates to the fact that shear stress, fundamentally, is about the near-wall gradient in velocity – hence, higher velocities in the channel produce higher shear as they approach the no-slip boundary.

This definition in turn leads us to

$$u_\tau = \kappa z \frac{\mathrm{d}u}{\mathrm{d}z} \tag{31}$$

Let's rearrange this Prandtl mixing theory equation to give us a velocity profile.

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{u_\tau}{\kappa}\frac{1}{z} \tag{32}$$

This is showing us how momentum mixes vertically by eddies to affect the velocity profile in a turbulent fluid.

Now let's integrate this expression to obtain something that gives us velocity instead of velocity gradient:

This final expression is the Law of the Wall.

$$u(z) = \frac{u_\tau}{\kappa}\ln\left(\frac{z}{z_0}\right) \tag{33}$$

z_0 is a roughness length scale, or roughness height. It sets the elevation above the bed at which the turbulent velocity profile reaches zero, though this is ficticious. In reality, there is a laminar sublayer beneath the turbulent profile, and z_0 is not particularly physically meaningful. Furthermore, this logarithmic profile is strictly only true for the first ∼20% of the distance from the bed to the water surface. Nevertheless, the logarithmic profile is a good approximation for flow in an open channel.

7.1. **Finding z_0.** The value of z_0 depends on whether the flow is hydraulically smooth or hydraulically rough.

Recall the Reynolds Number:

$$\mathrm{Re} = \frac{\rho u L}{\mu} = \frac{uL}{\nu} \tag{34}$$

L is a length scale and $\nu = 10^{-6}$ is the kinematic viscosity of water.

We can use it to define whether the flow will be hydraulically rough or hydraulically smooth. Based on Nikaradse's data/diagram:

- Hydraulically smooth: $\mathrm{Re} < 3$
- Hydraulically rough: $\mathrm{Re} > 100$

From the Equation 9, you can calculate a roughness length, k_s, as a function of grain size, D. This becomes the length scale relevant for determining the near-boundary shear Reynolds number:

$$\mathrm{Re}_b = \frac{u_\tau k_s}{\nu} \tag{35}$$

From this we can define z_0. For hydraulically smooth flow:

$$z_0 = \frac{\nu}{9 u_\tau} \tag{36}$$

Pretty much all natural rivers are hydraulically rough, however, so this case is more important to us:

$$z_0 = \frac{k_s}{30} \tag{37}$$

Based on the earlier relationships between grain size and k_s (Equation 9), one can find a straightforward linear relationship between z_0 and grain size. For those interested in further reading, these relationships are based on the experiments of *Nikuradse* (1933).

8. Summary and next steps

Here we have:

- Computed flow and shear stress within an open channel for both laminar and turbulent flows
- Gained insight into how viscosity and momentum affect the form of flows
- Become familiar with some standard equations and terminology used with open-channel flows

With this as a starting point, we will begin to study how these flows move sediments, form river channels, and evolve landscapes in ways that interact with the surrounding hillslopes and encompassing watersheds.

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