2-Day Course – Spatial Modeling with Geostatistics

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“In two days, what a geoscientists needs to know about geostatistics, and workflows to get you started with applying geostatistics to impact your work.”

Spatial Modeling with Geostatistics

Probability Theory

Lecture outline . . .

- **Probability Definitions**
- **Venn Diagrams**
- **Frequentist Concepts**
- **Bayesian Concepts**

Prerequisites

Introduction

Probability Theory

Representative Sampling

Spatial Data Analysis

Spatial Estimation

Stochastic Simulation

Uncertainty Management

Machine Learning

What Will You Learn?

We will just cover some basic probability concepts.

- How to represent and calculate probability.
- Prerequisites for subsequent lectures.
- Tools you can apply just with probability concepts.

Probability
Helps in Making Decisions

For example:

- What is the probability that a well is a success? – *drill the well*
- What is the probability that a valve has a crack? – *replace the valve*
- What is the probability that a seismic survey finds a reservoir? – *acquire the seismic*
- What is the probability that a reservoir seal will fail? – *inject the CO2*

Most of our decisions involve uncertainty:

- By using rigorous probability concepts we can make better decisions.

Probability Definitions
What is Probability?

Measure of the likelihood that an event will occur. For random experiments and well-defined settings (such as coin tosses):

$$\mathrm{Prob}(A) = \mathrm{P}(A) = \lim_{n \to \infty} \left(\frac{n(A)}{n(\Omega)} \right)$$

frequentist approach to probability is the limit of relative frequency over a large number of trails.

Area(A) = area of A / total area = P(A)
Area(Ω) = total area / total area = probability of any possible outcome = P(Ω) = 1.0

where:

n(A) = number of times event A occurred
n(Ω) = number of trails

Example: Possibility of drilling a dry hole for the next well, encountering sandstone at a location ($\mathbf{u}_\alpha$), exceeding a rock porosity of 15% at a location ($\mathbf{u}_\alpha$).

Probability Concepts
Venn Diagrams

Venn Diagrams are a tool to communicate probability

Experiments (Sampling) (J): Establishment of conditions that produce an outcome.

Simple Event (x): A single outcome of an experiment.

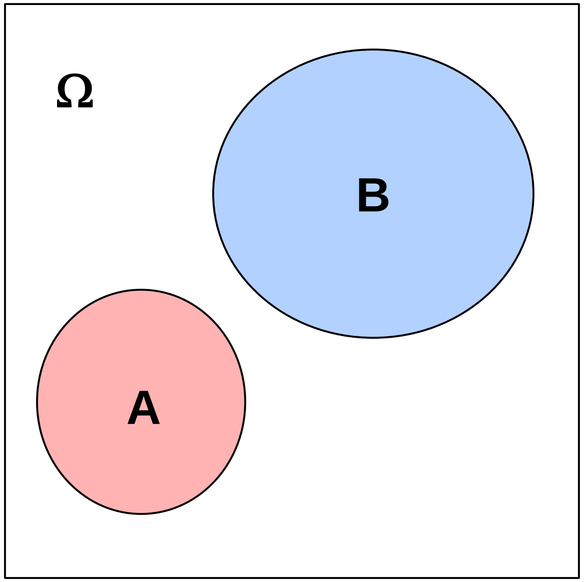
Event (A, B, …): Collection of simple events.

Occurrence of A: A has occurred if the outcome of experiment (sampling) belongs to it.

Sample Space (Ω): Collection of all possible events.

What do we learn from a Venn diagram?

- size of regions = probability of occurrence
- overlap = probability of joint occurrence
- excellent tool to visualize marginal, joint and conditional probability.



Venn Diagram – illustration of events and relations to each other.

Probability Definitions
Venn Diagram Example

Experiments (Sampling) (J):

- Facies determined from a set of well cores (N=3,000 measures at 1 foot increments)

Sample Space (Ω):

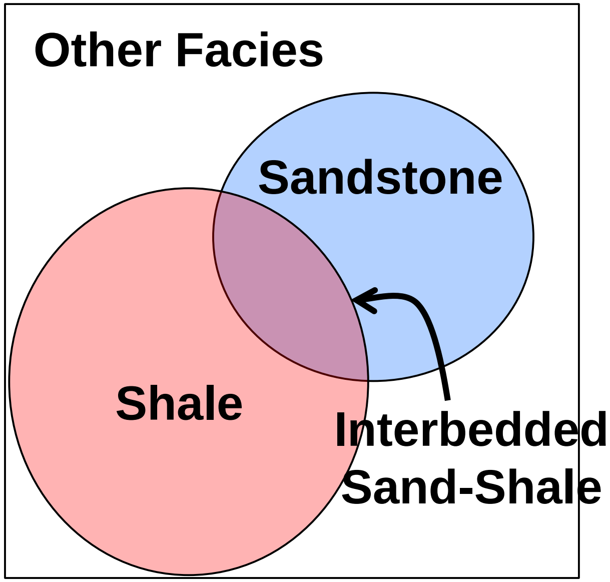
- Facies for the N=3,000 core measures

Event (A, B, …):

- Facies = {Sandstone, Interbedded Sandstone and Shale, Shale and other facies}

Venn Diagram Tells Us About Probability:

- Prob{Other Facies} > Prob{Shale} > Prob{Sandstone} > Prob{Interbedded} = Prob{Shale and Sandstone}
- Prob{Sandstone and Shale given Sandstone } < Prob{Sandstone}



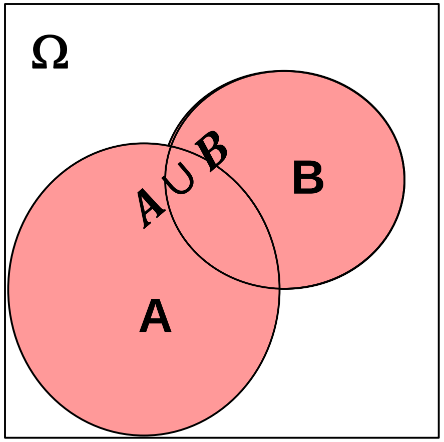
Venn Diagram – illustration of events and relations to each other.

Probability Definitions
Probability Operators

Union of Events:

- All outcomes in the sample space that belong to either event A or B

$$A \cup B = \{x: x \in A \; or \; x \; \in B\}$$

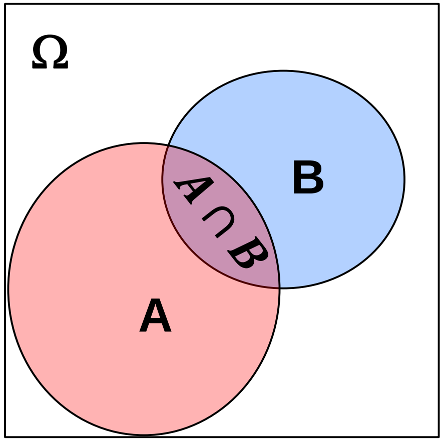


Venn Diagram – illustrating union.

Intersection of Events:

- All outcomes in the sample space that belong to both events A and B

$$A \cap B = \{x: x \in A \; and \; x \; \in B\}$$

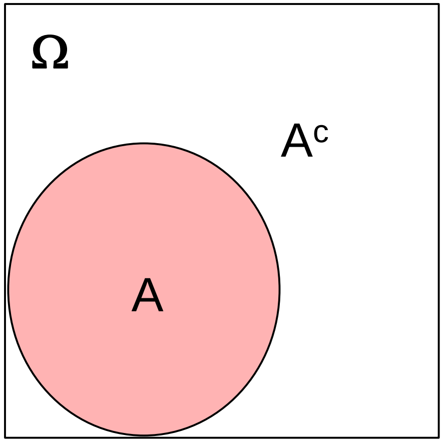


Venn Diagram – illustrating intersection.

Probability Definitions
Probability Operators

Complementary Events: A^c

- All outcomes in the sample space that do not belong to A

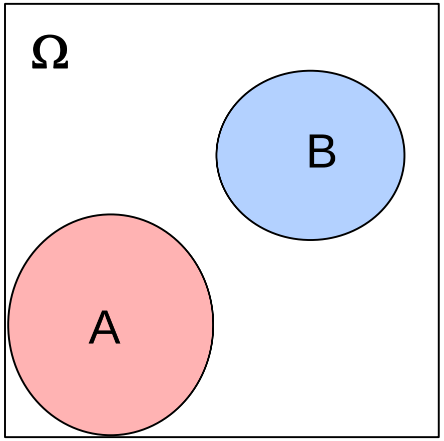


Venn Diagram – illustrating complementary events.

Mutually Exclusive Events:

- The events that do not intersect or do not have any common outcomes

$$A \cap B = \emptyset \rightarrow \text{Null Set}$$



Venn Diagram – illustrating mutually exclusive.

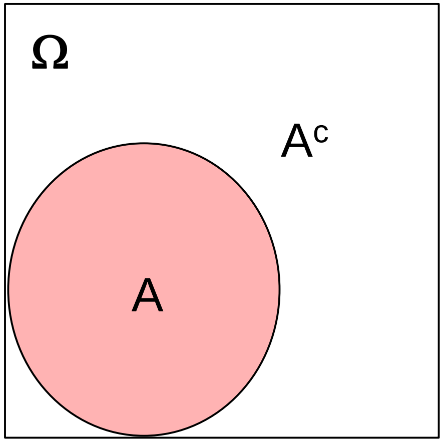
Probability Definitions
Probability Concepts

Non-negativity, Normalization:

- Fundamental probability constraints
  - Bounded $0 \leq P(A) \leq 1$
  - Closure $P(\Omega) = 1$
  - Null Sets $P(\phi) = 0$

Complimentary Events:

- Closure $P(A^c) + P(A) = 1$



Venn Diagram – illustrating complementary events.

Probability Definitions
Probability by Venn Diagram

The Addition Rule:

$$P(A \cup B) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P}(A \cap B)$$

Must account for the intersection!

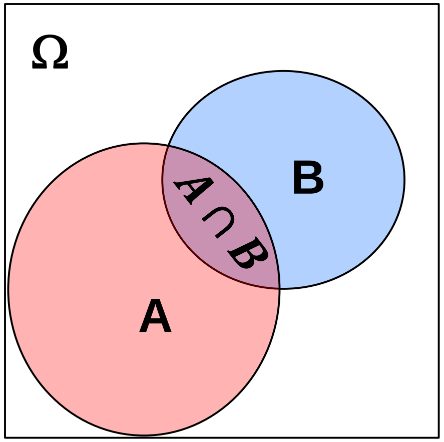
If mutually exclusive events:

$$A_i \cap A_j = \emptyset, \forall\, i \neq j$$

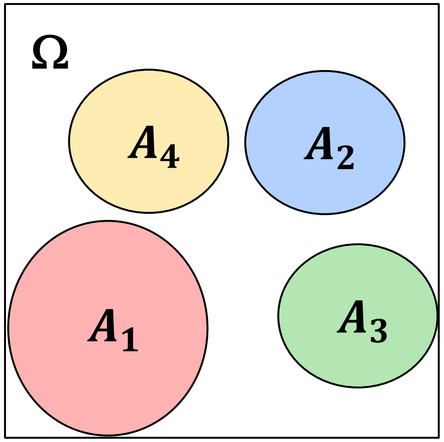
then,

$$\mathrm{P}\left(\bigcup_{i=1}^{k} \mathrm{A}_i\right) = \sum_{i=1}^{k} \mathrm{P}(\mathrm{A}_i)$$

no intersections to account for.



Venn Diagram – illustrating intersection.



Venn Diagram – illustrating intersection.

Probability Definitions Addition Rule Example

Calculate the following probabilities for event A and B:

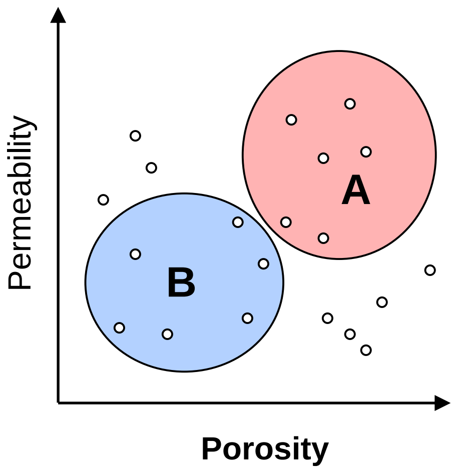
Note Event A: Sandstone and Event B: Shale

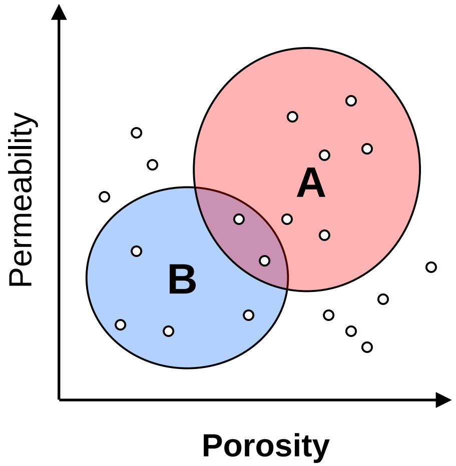
$$P(A) =$$

$$P(B) =$$

$$P(A \cap B) =$$

$$P(A \cup B) =$$





$$P(A) =$$

$$P(B) =$$

$$P(A \cap B) =$$

$$P(A \cup B) =$$

Hint: yes, it is just point counting!

Probability Definitions Addition Rule Example

Calculate the following probabilities for event A and B:

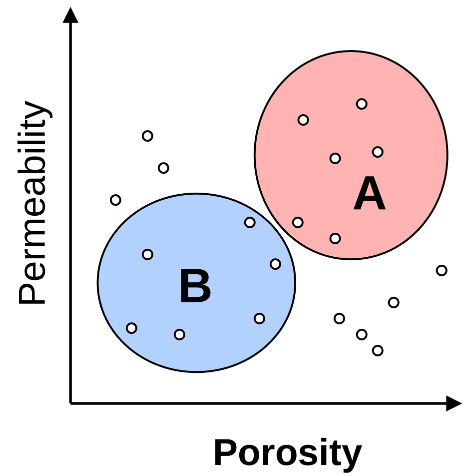
Note Event A: Sandstone and Event B: Shale

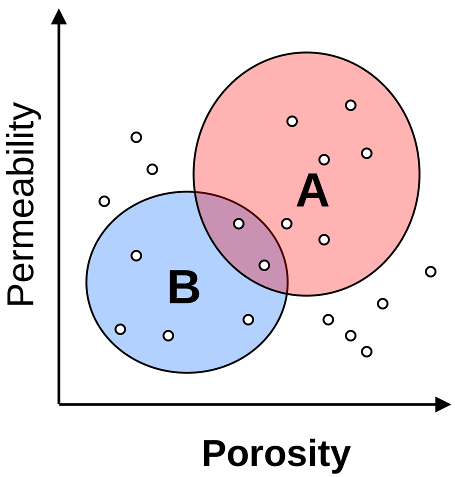
$$P(A) = {}^{6}/_{20} = 30\%$$

$$P(B) = {}^{6}/_{20} = 30\%$$

$$P(A \cap B) = {}^{0}/_{20} = 0\%$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 30\% + 30\% - 0\% = 60\%$$





$$P(A) = {}^{8}/_{20} = 40\%$$

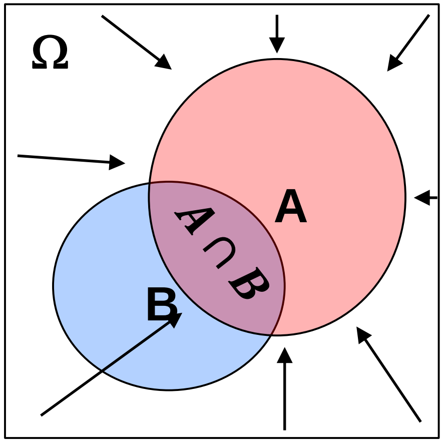
$$P(B) = {}^{6}/_{20} = 30\%$$

$$P(A \cap B) = {}^{2}/_{20} = 10\%$$

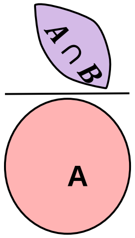
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 40\% + 30\% - 10\% = 60\%$$

Probability Definitions
Conditional Probability

Probability of B given A occurred? $P(B \mid A)$



$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

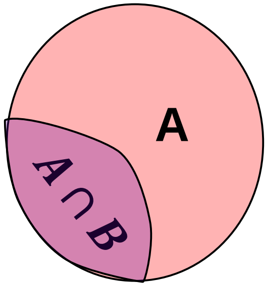


$P(A \cap B)$

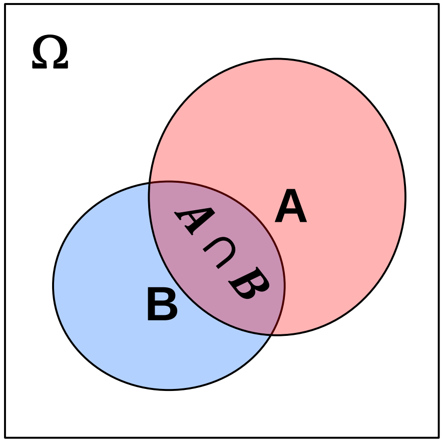
$P(A)$

Conceptually we shrink space of possible outcomes.

A occurred so we shrink our space to only event A.



Probability Definitions
Conditional, Marginal and Joint Probability



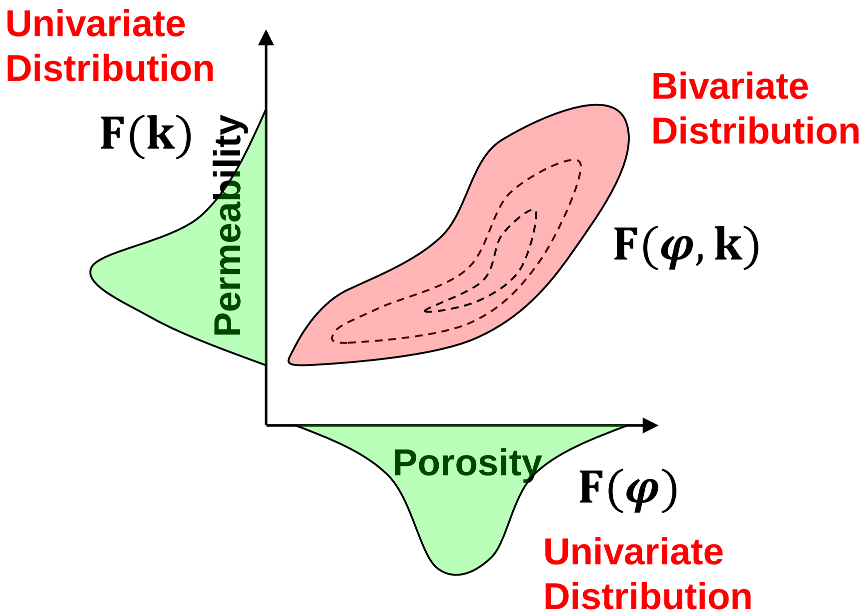
Probability of B given A occurred? $P(B \mid A)$

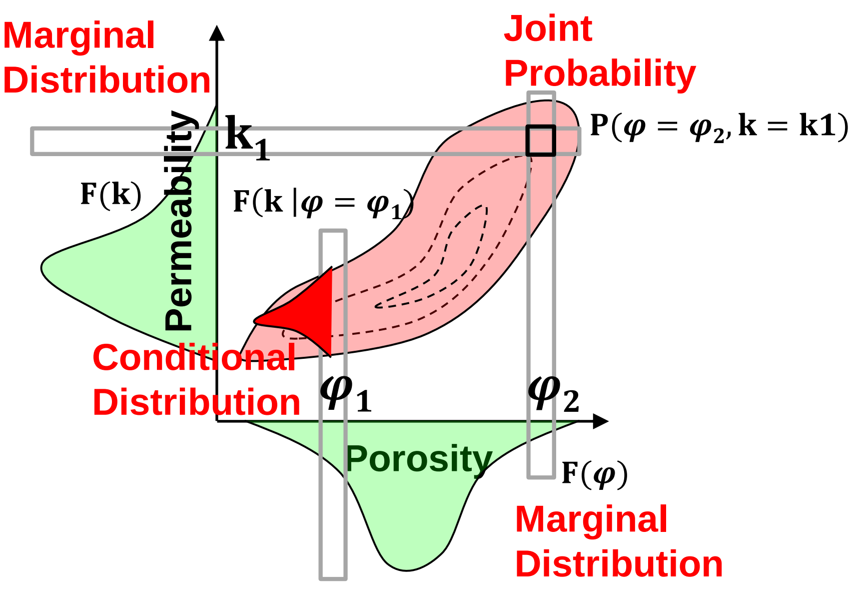
Conditional Probability

Joint Probability

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \text{ and } B)}{P(A)}$$

Marginal Probability





Probability Definitions
Conditional, Marginal and Joint Probability

Marginal Probability: Probability of an event, irrespective of any other event

$$P(X), P(Y)$$

Conditional Probability: Probability of an event, given another event is already true.

$$P(X \text{ } given \text{ } Y), P(Y \text{ } given \text{ } X)$$

$$P(X \mid Y), P(Y|X)$$

Joint Probability: Probability of multiple events occurring together.

$$P(X \text{ } and \text{ } Y), P(Y \text{ } and \text{ } X)$$

$$P(X \cap Y), P(Y \cap X)$$

$$P(X, Y), P(Y, X)$$

Probability Definitions
Conditional Probability

General Form for Conditional Probability?

$$P(C \mid B, A) = \frac{P(A \cap B \cap C)}{P(A \cap B)}$$

Recall:

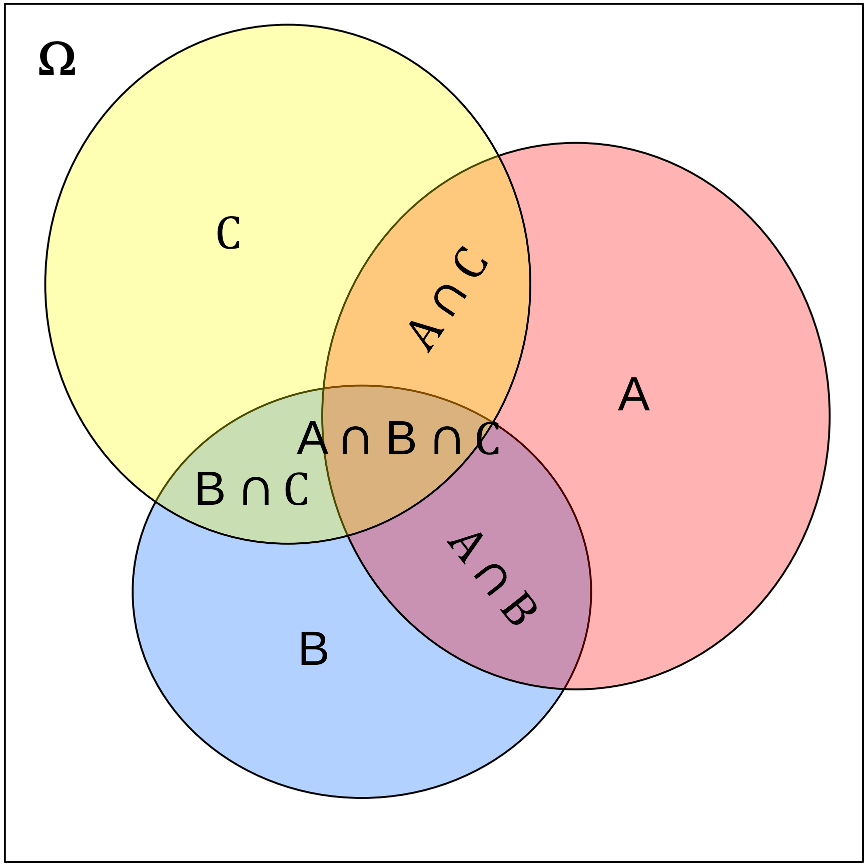
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Substitute:

$$P(C \mid B, A) = \frac{P(A \cap B \cap C)}{P(B|A)P(A)}$$

Reorganize:

$$P(A \cap B \cap C) = P(C \mid B, A)P(B|A)P(A)$$



General Form, Recursion of Conditionals

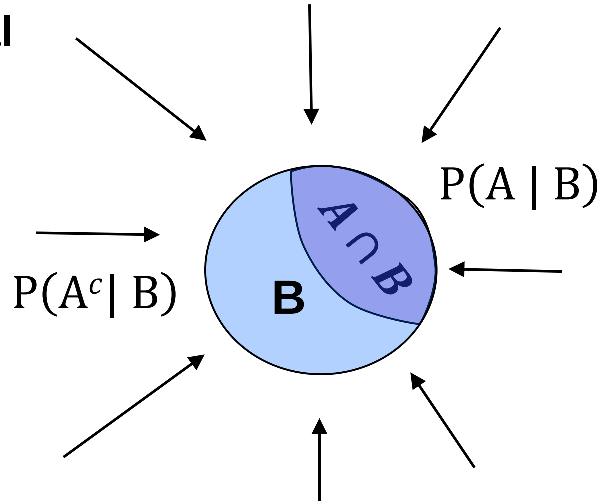
$$P(A_1 \cap \cdots \cap A_n) = P(A_n|A_{n-1}, \ldots, A_1)P(A_{n-1}|A_{n-2}, \ldots, A_1) \ \ldots \ P(A_1)$$

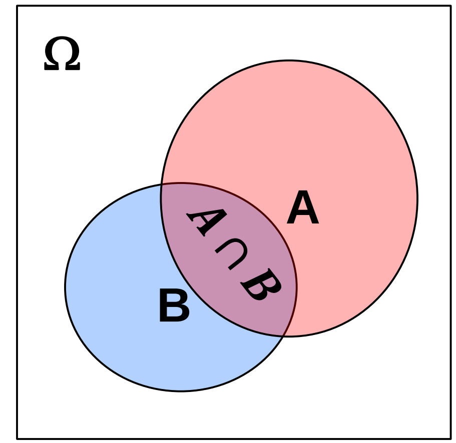
Probability Definitions
Conditional Probability

Other Relations with Conditional Probability

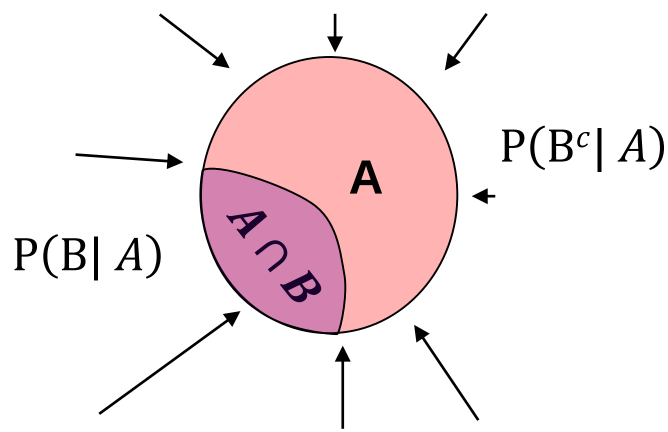
- Closure with conditional probabilities:

$$P(A \mid B) + P(A^c \mid B) = 1$$





$$P(B \mid A) + P(B^c \mid A) = 1$$



Probability Definitions
Conditional Probability Examples

Recall:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

For Case 1 calculate:

$P(A \mid B) =$

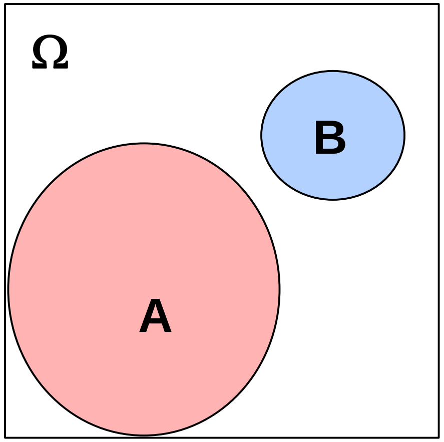
$P(B \mid A) =$

For Case 2 calculate:

$P(A \mid B) =$

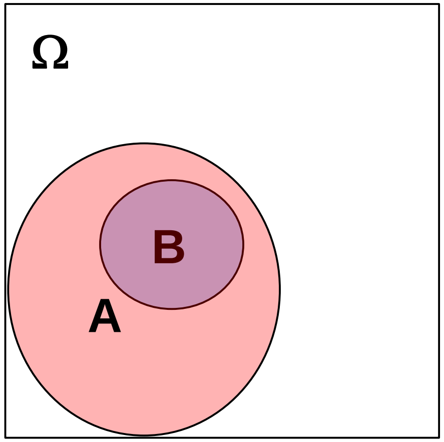
$P(B \mid A) =$

Case 1:



Venn Diagram – case 1.

Case 2:



Venn Diagram – case 2.

Probability Definitions
Conditional Probability Examples

Recall:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

For Case 1 calculate:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\emptyset}{P(B)} = \emptyset$$

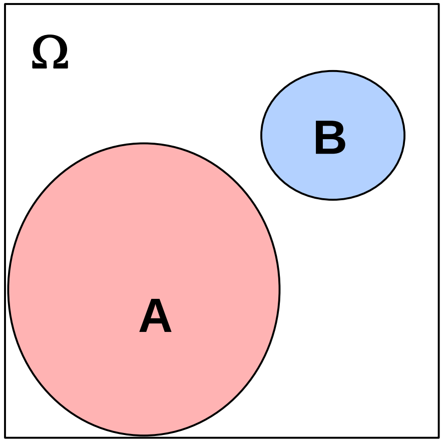
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\emptyset}{P(A)} = \emptyset$$

For Case 2 calculate:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)} = 1, \text{since } P(A \cap B) = P(B)$$

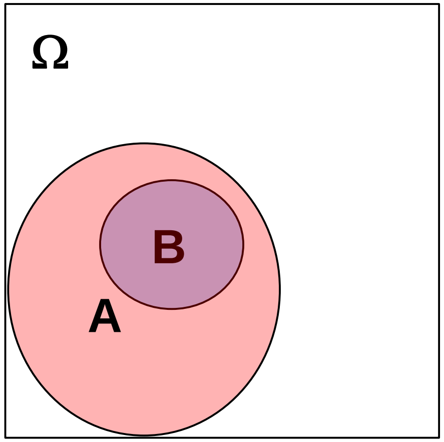
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)}, \text{since } P(A \cap B) = P(B)$$

Case 1:



Venn Diagram – case 1.

Case 2:



Venn Diagram – case 2.

Probability Definitions
Conditional Probability Examples

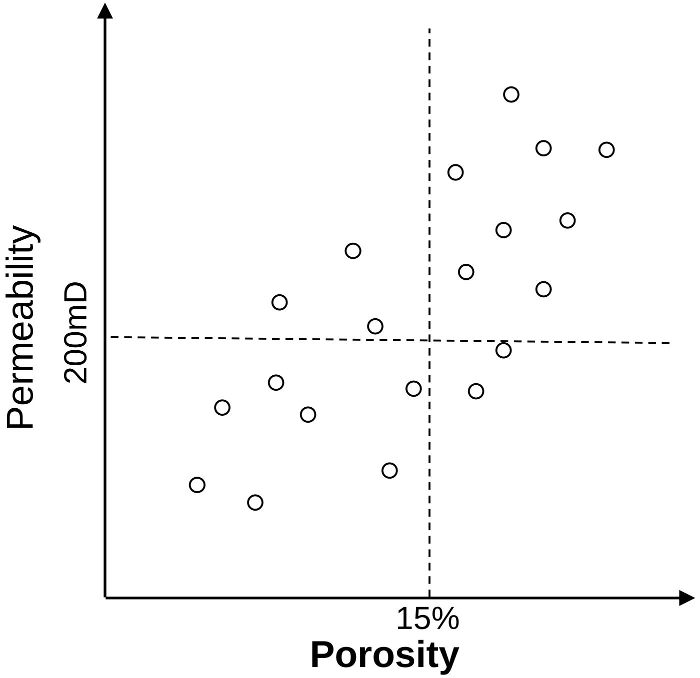
Event A: Porosity > 15%
Event B: Permeability > 200 mD

For Case 1 calculate:

$P(A \mid B) =$

$P(B \mid A) =$

Bonus Question: How much information does event B tell you about event A and visa versa?



Question: Calculate the following probabilities for events A and B:

Probability Definitions
Conditional Probability Examples

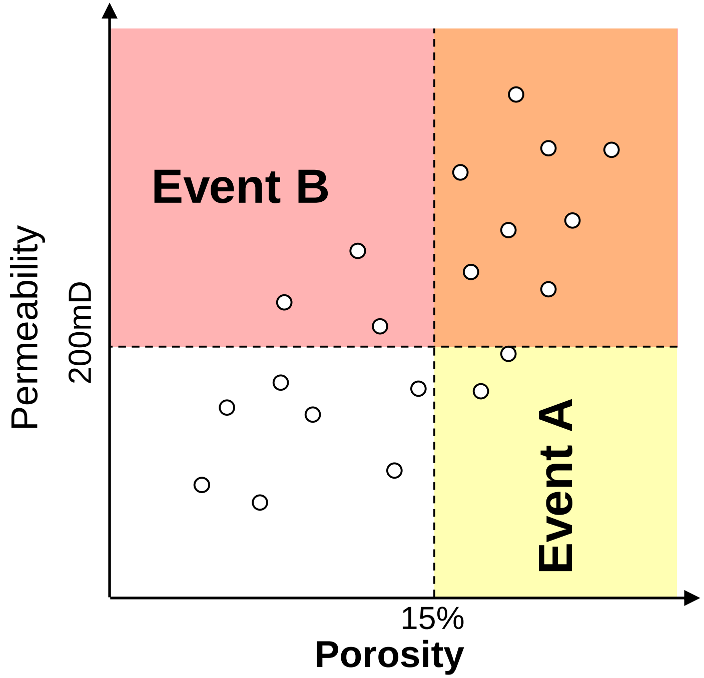
Question: Calculate the following probabilities for events A and B:

Event A: Porosity > 15%
Event B: Permeability > 200 mD

For Case 1 calculate:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{8/20}{11/20} = 8/11$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{8/20}{10/20} = 8/10$$



Bonus Question: How much information does B tell you about A and visa versa?
$P(A) = 10/20$, $P(A|B) = 8/11$ Probability from 50% → 73%
$P(B) = 11/20$, $P(B|A) = 8/10$ Probability from 55% → 80%
We cannot work with A and B independently, they provide information about each other.

Probability Definitions
Conditional, Marginal and Joint Probability

Joint Distribution:

$$f_{XY}(x,y)$$

Marginal Distribution:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)dy$$

Conditional Distribution:

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Table of Frequencies

Porosity (%) \ Fraction Shale (%)	10%	30%	50%	70%	90%
25%	1	1	0	0	0
20%	2	3	2	0	0
15%	1	2	2	1	0
10%	0	0	2	3	2
5%	0	0	1	1	1

Probability Definitions
Conditional, Marginal and Joint Probability

Joint Distribution:

$$f_{XY}(x,y)$$

Marginal Distribution:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)dy$$

Conditional Distribution:

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Table of Joint Probabilities

Porosity (%) \ Fraction Shale (%)	10%	30%	50%	70%	90%
25%	4%	4%	0	0	0
20%	8%	12%	8%	0	0
15%	4%	8%	8%	4%	0
10%	0	0	8%	12%	8%
5%	0	0	4%	4%	4%

Probability Definitions

Conditional, Marginal and Joint Probability

Given these joint probabilities calculate the:

Marginal Distributions:

Vsh	**10%**	**30%**	**50%**	**70%**	**90%**
$f_{Vsh}(v_{sh})$					

Porosity	**5%**	**10%**	**15%**	**20%**	**25%**
$f_{\varphi}(\varphi) =$					

Conditional Distribution:

Vsh	**10%**	**30%**	**50%**	**70%**	**90%**
$f_{Vsh\|\varphi}(v_{sh}\| \, \varphi = 15\%) =$					

Table of Joint Probabilities

Porosity (%)					
25%	4%	4%	0	0	0
20%	8%	12%	8%	0	0
15%	4%	8%	8%	4%	0
10%	0	0	8%	12%	8%
5%	0	0	4%	4%	4%
	10%	**30%**	**50%**	**70%**	**90%**
	Fraction Shale (%)				

Probability Definitions
Conditional, Marginal and Joint Probability

Given these joint probabilities calculate the:

Marginal Distributions:

Vsh	**10%**	**30%**	**50%**	**70%**	**90%**
$f_{Vsh}(v_{sh})$	16%	24%	28%	20%	12%

Porosity	**5%**	**10%**	**15%**	**20%**	**25%**
$f_{\varphi}(\varphi) =$	12%	28%	24%	28%	8%

Conditional Distribution:

Vsh	**10%**	**30%**	**50%**	**70%**	**90%**
	1/6	1/3	1/3	1/6	0

$$f_{Vsh|\varphi}(v_{sh}|\ \varphi = 15\%) = f_{Vsh,\varphi}(v_{sh}, \varphi = 15\%)/f_{\varphi}(\varphi = 15\%)$$

Table of Joint Probabilities

Porosity (%) \ Fraction Shale (%)	**10%**	**30%**	**50%**	**70%**	**90%**
25%	4%	4%	0	0	0
20%	8%	12%	8%	0	0
15%	4%	8%	8%	4%	0
10%	0	0	8%	12%	8%
5%	0	0	4%	4%	4%

Probability Definitions
Multiplication Rule

The Multiplication Rule:

$$P(A \cap B) = P(B|A)\,P(A)$$

If events A and B are independent**:**

$P(B|A) = P(B)$

Knowing something about A does nothing to help predict B. Then by substitution:

$$P(A \cap B) = P(B)\,P(A)$$

The general form given independence for all cases, $i = 1, \ldots, k$:

$$P\left(\cap_{i=1}^{k} A_i\right) = \prod_{i=1}^{k} P(A_i)$$

Probability Definitions Multiplication Rule Example

Given there is independence between fluid type and porosity:

Event A = Oil
Event B = Porosity > 10%

Given: $\mathrm{P(A)} = 30\%$ and $\mathrm{P(B)} = 50\%$

What is the P($A \cap B$)?

Given there is independence between fluid type, porosity and saturation:

Event A = Oil
Event B = Porosity > 10%
Event C = S_{oil} > 40%

Given: $\mathrm{P(A)} = 30\%$, $\mathrm{P(B)} = 50\%$, $\mathrm{P(C)} = 25\%$

What is the P($A \cap B \cap C$)?

Probability Definitions
Multiplication Rule Example

Given there is independence between fluid type and porosity:

Event A = Oil
Event B = Porosity > 10%

Given: $\mathrm{P(A)} = 30\%$ and $\mathrm{P(B)} = 50\%$

What is the P($A \cap B$)? 30% x 50% = 15%

Given there is independence between fluid type, porosity and saturation:

Event A = Oil
Event B = Porosity > 10%
Event C = S_{oil} > 40%

Given: $\mathrm{P(A)} = 30\%, \mathrm{P(B)} = 50\%, \mathrm{P(C)} = 10\%$

What is the P($A \cap B \cap C$)? 30% x 50% x 10% = 1.5%

Probability Definitions Evaluating Independence

Events A and B are independent if and only if:

$$\mathrm{P}(A \cap B) = \mathrm{P}(\mathrm{B})\mathrm{P}(A)$$

or

$$\mathrm{P}(\, A|B \,) = \mathrm{P}(A) \quad \text{and} \quad \mathrm{P}(\, B|A \,) = \mathrm{P}(\boldsymbol{B})$$

General Form:

Events $A_1, A_2, \ldots, A_n$ are independent if:

$$\mathrm{P}\left(\cap_{\boldsymbol{i=1}}^{\boldsymbol{k}} \mathrm{A_i}\right) = \prod_{\boldsymbol{i=1}}^{\boldsymbol{k}} \mathrm{P}(\mathrm{A_i})$$

Probability Definitions
Evaluating Independence Example

Example: Facies F1, F2 and F3 in 10 wells:

Position	Well 1	Well 2	Well 3	Well 4	Well 5	Well 6	Well 7	Well 8	Well 9	Well 10
Top	F3	F2	F2	F1	F1	F1	F2	F2	F1	F1
Middle	F1	F1	F1	F1	F2	F2	F1	F2	F2	F2
Bottom	F2	F2	F2	F3	F3	F3	F3	F3	F3	F2

Event A_1 = F1 is middle facies
Event A_2 = F3 is bottom facies

$P(\boldsymbol{A_1} \cap \boldsymbol{A_2}) = P(\boldsymbol{A_1})P(\boldsymbol{A_2})$ or

$P(\boldsymbol{A_1}|\boldsymbol{A_2}) = P(\boldsymbol{A_1})$ and $P(\boldsymbol{A_2}|\boldsymbol{A_1}) = P(\boldsymbol{A_2})$

Question: are events A1 and A2 independent?

Probability Definitions
Evaluating Independence Example

Example: Facies F1, F2 and F3 in 5 wells:

Position	Well 1	Well 2	Well 3	Well 4	Well 5	Well 6	Well 7	Well 8	Well 9	Well 10
Top	F3	F2	F2	F1	F1	F1	F2	F2	F1	F1
Middle	F1	F1	F1	F1	F2	F2	F1	F2	F2	F2
Bottom	F2	F2	F2	F3	F3	F3	F3	F3	F3	F2

Event A_1 = middle facies if F1
Event A_2 = bottom facies is F3

$P(A_1 \cap A_2) = P(A_1)P(A_2)$ or

$P(A_1|A_2) = P(A_1)$ and $P(A_2|A_1) = P(A_2)$

Question: are events A1 and A2 independent?

$$P(A_1) = {}^{5}/_{10} = 50\%, P(A_2) = {}^{6}/_{10} = 60\%, P(A_1 \cap A_2) = {}^{2}/_{10} = 20\%$$

$$P(A_1)P(A_2) = \mathbf{50\% \cdot 60\% = 30\%} \neq P(A_1 \cap A_2) = {}^{2}/_{10} = 20\% \text{ Not independent.}$$

Probability Definitions
Bayesian Statistics

Product Rule:

$$P(B \cap A) = P(A|B)\,P(B)$$

$$P(A \cap B) = P(B|A)\,P(A)$$

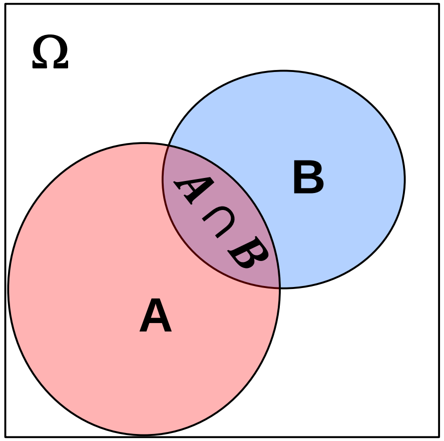
It follows that:

$$P(B \cap A) = P(A \cap B)$$

Therefore we combine two product rules, substitute:

$$P(A|B)\,P(B) = P(B|A)\,P(A)$$

We get Bayes' Theorem!



Venn Diagram – illustrating intersection.

Probability Definitions
Bayesian Theorem

Bayes' Theorem:

Make a easy adjustment and we get the popular form.

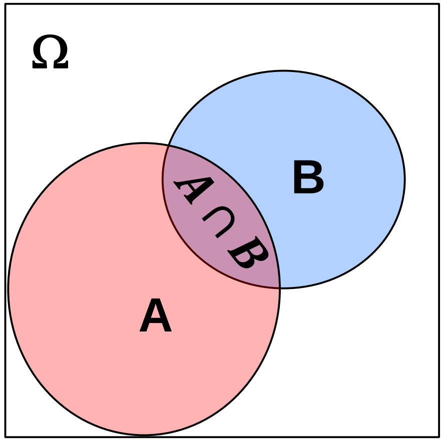
$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Observations:

1. We are able to get P(A | B) from P(B | A) as you will see this often comes in handy.
2. Each term is known as:

$$\text{Posterior} = \frac{\text{Likelihood x Prior}}{\text{Evidence}}$$

3. Prior should have no information from likelihood.
4. Evidence term is usually just a standardization to ensure closure.



Venn Diagram – illustrating intersection.

Note: we got to Bayes' Theorem by fundamental frequentist approaches.

Probability Definitions
Bayesian Theorem Example

Bayes Theorem:

A common example of Bayes' Theorem for modeling the subsurface.

Model Updating with a New Data Source:

$$P(\text{Model} \mid \text{New Data}) = \frac{\overbrace{P(\text{New Data} \mid \text{Model})}^{\textbf{Likelihood}}\ \overbrace{P(\text{Model})}^{\textbf{Prior}}}{\underbrace{P(\text{New Data})}_{\textbf{Evidence}}}$$

Probability Definitions
Bayesian Theorem Example

Bayes Theorem:

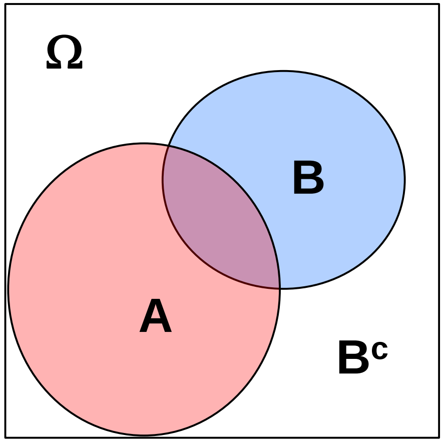
Alternative form, symmetry:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} \qquad P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}$$

Alternative form to calculate evidence term:

Given: $P(A) = \underbrace{P(A|B)\,P(B)}_{P(A \text{ and } B)} + \underbrace{P(A|B^c)\,P(B^c)}_{P(A \text{ and } B^c)}$

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B^c)\,P(B^c)}$$



Venn Diagram – illustrating intersection.

Probability and Statistics
Bayesian Methods

Bayesian approaches allow you to solve statistical problems that are not possible otherwise. Consider:

Event A	Event B
You have a disease	You test positive for the disease
There is fault compartmentalization	Geologist says there's a fault
Low permeability of a sample	The laboratory measure is low
A valve will fail	X-ray test is positive
You drill a dry well	Seismic AVO response looks poor

In all of these cases you need to calculate:

$$P\left(\begin{matrix}\text{Something is}\\ \text{Happening}\end{matrix} \,\middle|\, \begin{matrix}\text{Looks like}\\ \text{its happening}\end{matrix}\right) = \frac{P\left(\begin{matrix}\text{Looks like}\\ \text{its happening}\end{matrix} \,\middle|\, \begin{matrix}\text{Something is}\\ \text{Happening}\end{matrix}\right) P\left(\begin{matrix}\text{Something is}\\ \text{Happening}\end{matrix}\right)}{P\left(\begin{matrix}\text{Looks like}\\ \text{its happening}\end{matrix}\right)}$$

Probability and Statistics
Bayesian Methods

Bayesian approaches allow you to solve statistical problems that are not possible otherwise. Consider:

Correct Detection Rate x Occurrence Rate

$$P\left(\begin{matrix}\text{Something is}\\ \text{Happening}\end{matrix} \;\middle|\; \begin{matrix}\text{Looks like}\\ \text{its happening}\end{matrix}\right) = \frac{P\left(\begin{matrix}\text{Looks like}\\ \text{its happening}\end{matrix} \;\middle|\; \begin{matrix}\text{Something is}\\ \text{Happening}\end{matrix}\right) P\left(\begin{matrix}\text{Something is}\\ \text{Happening}\end{matrix}\right)}{P\left(\begin{matrix}\text{Looks like}\\ \text{its happening}\end{matrix}\right)}$$

All Detection Rate (included false positives)

Often these terms are much easier to collect:

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B^c)\,P(B^c)}$$

Let's try this out next.

Probability and Statistics
Bayesian Methods

Example: Prior information at a site suggests a channel feature exists at a given location with probability of 60%. We decide to further investigate this information using seismic data.

Seismic survey can show a feature
- is present with 90% probability if it really is present
- is not present with a probability 70% if it really is not

$P(A) = 0.6$

$P(B|A) = 0.9$

$P(B^c|A^c) = 0.7$

***A*=The feature is present**
***B* =Seismic shows the feature**
***A^c* =The feature not present**
***B^c* =Seismic does not show the feature**

Will seismic information be useful?

Probability and Statistics Bayesian Methods

Example: Prior information at a site suggests a channel feature exists at a given location with probability of 60%. We decide to further investigate this information using seismic data.

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- is present with 90% probability if it really is present
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***A*=The feature is present**
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***A^c^* =The feature not present**
***B^c^* =Seismic does not show the feature**

$P(A) = 0.6$
$P(B|A) = 0.9$
$P(B^c|A^c) = 0.7$
$P(B|A^c) = 1 - P(B^c|A^c) = 0.3$
$P(A^c) = 1- P(A) = 0.4$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)} = \frac{\overbrace{(0.9)(0.6)}^{\text{True Positive}}}{\underbrace{(0.9)(0.6)}_{\text{True Positive}} + \underbrace{(0.3)(0.4)}_{\text{False Positive}}} = 82\%$$

Probability and Statistics Bayesian Methods

Example: One in every thousand BOPs has a serious crack. X-ray analysis has a 99% chance of detecting the crack correctly. If the BOP does not have a crack, there is a 2% chance that the X-ray detects a crack. The rate of BOP cracks is 0.1%. A BOP has been X-rayed and the result is positive. What is the chance that the BOP actually does have a crack?

Solution:

***A* = BOP has cracks** **P(A|B) = ?**

***B* = BOP tests positive**

A^c = BOP does not have cracks

B^c = BOP did not test positive

P(A) = 0.001 – crack rate

P(B|A) = 0.99 – true positive

P(B|A^c) = 0.02 – false positive

Probability and Statistics Bayesian Methods

Example: One in every thousand BOPs has a serious crack. X-ray analysis has a 99% chance of detecting the crack correctly. If the BOP does not have a crack, there is a 2% chance that the X-ray detects a crack. The rate of BOP cracks is 0.1%. A BOP has been X-rayed and the result is positive. What is the chance that the BOP actually does have a crack?

Solution:

A **= BOP has cracks**   **P(A|B) = ?**

B **= BOP tests positive**

A^c = BOP does not have cracks

B^c = BOP did not test positive

$P(A) = 0.001$ – crack rate

$P(A^c) = 0.999$ – not cracked rate

$P(B|A) = 0.99$ – true positive

$P(B|A^c) = 0.02$ – false positive

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)} = \frac{\overbrace{(0.99)(0.001)}^{\text{True Positive}}}{\underbrace{(0.99)(0.001)}_{\text{True Positive}} + \underbrace{(0.02)(0.999)}_{\text{False Positive}}} = 4.7\%$$

Probability of a crack in the BOP given a positive crack test is only 4.7%! Why?
Cracks are very unlikely + high false positive rate (2%)!

Probability and Statistics
Excel Bayesian Inversion Demo

Bayesian Updating V2.0 - Inverting Conditional Probabilities

Michael Pyrcz, the University of Texas at Austin, Geostatistical Reservoir Modeling Class, @GeostatsGuy

With **Bayesian Updating** we can invert conditional probabilities (e.g. ***P(A|B)* → *P(B|A))*** . This ia very powerful, because often we can use an easier to calculate conditional probability to assess a more difficult to calculate, but more important conditional probability. For example, your doctor gives you a medical test that comes back positive for a disease. It would be important to know what is the probability that you have the disease given the positive test. This is a general category of problems that may be generalized as follows. **You have an positive indicator that something is happening, is the thing actually happening?** E.g. seismic interpretation indicates a fault, x-ray analysis indicates a crack etc.

$$P(\text{Actually Happening} \mid \text{Positive Indicator}) = \frac{P(\text{Positive Indicator} \mid \text{Actually Happening}) \ x \ P(\text{Actually Happening})}{P(\text{Positive Indicator})}$$

It turns out that the denominator (Evidence Term) is often hard to calculate so we may use probability logic to calculate it as follows:

$$P(\text{Positive Indicator}) = \underbrace{P(\text{Positive Indicator} \mid \text{Actually Happening}) x\, P(\text{Actually Happening})}_{\text{True Positive}} + \underbrace{P(\text{Positive Indicator} \mid \text{NOT Actually Happening}) \ x \ P(\text{NOT Actually Happening})}_{\text{False Positive}}$$

Returning to the doctor's office. Your doctor has just informed you that you have tested positive (Positive Indicator) for a disease. Don't panic, resort to probability math. What infromation do you have to work with?

Instructions:
Adjust the yellow probabilities (that would likely be available) and observe the resulting probability of having the disease given a positive test. Note intermediate calculated probabilities are in blue cells.

Probability of getting this disease	*P(Actually Happening)*	=	0.001%
By closure the compliment, probability of not getting this disease	*P(Not Actually Happening) = 1 - P(Actually Happening)*	=	99.999%
Probability of detecting the disease if you have it. This is the sensitivity of the test.	*P(Positive Indicator | Actually Happening)*	=	99.000%
Probability of detecting the disease if you don't have it. This is the false positive rate of the test.	*P(Positive Indicator | NOT Actually Happening)*	=	0.010%

P(Positive Indicator) = $P(\text{Positive Indicator} \mid \text{Actually Happening}) x\, P(\text{Actually Happening}) + P(\text{Positive Indicator} \mid \text{NOT Actually Happening}) \ x \ P(\text{NOT Actually Happening})$

P(Positive Indicat 0.99% x 0.00001% + 0.0001% x 0.99999% ⟶ *P(Positive Indicator)* = 0.011%

We now have everything we need to solve for the probability you have the disease given a postive test.

$$P(\text{Actually Happening} \mid \text{Positive Indicator}) = \frac{P(\text{Positive Indicator} \mid \text{Actually Happening}) \ x \ P(\text{Actually Happening})}{P(\text{Positive Indicator})}$$

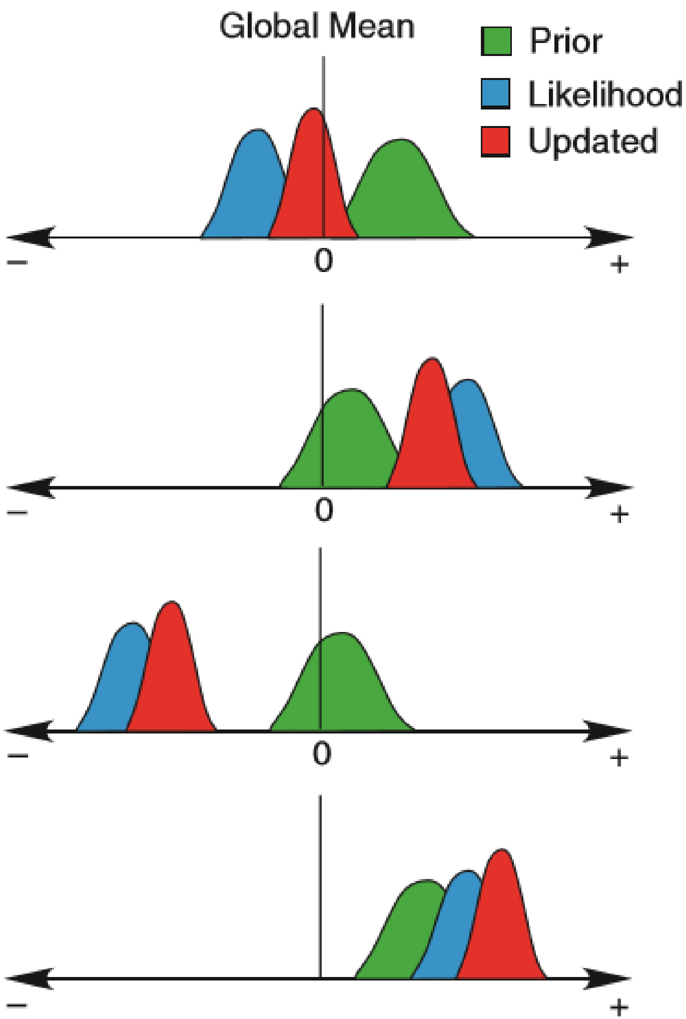
$$= \frac{99.000\% \quad x \quad 0.001\%}{0.011\%} \quad = \quad \boxed{P(\text{Actually Happening} \mid \text{Positive Indicator}) = 9.008\%}$$

Probability and Statistics Bayesian Methods

There is a simple analytical method for Bayesian Updating for the Case of Gaussian Distributions.

$$\bar{x}_{updated} = \frac{\bar{x}_{likelihood}(\mathbf{u}) \cdot \sigma^2_{prior}(\mathbf{u}) + \bar{x}_{prior}(\mathbf{u}) \cdot \sigma^2_{likelihood}(\mathbf{u})}{[1 - \sigma^2_{likelihood}(\mathbf{u})][\sigma^2_{prior}(\mathbf{u}) - 1] + 1}$$

$$\sigma^2_{updated}(\mathbf{u}) = \frac{\sigma^2_{prior}(\mathbf{u})\, \sigma^2_{likelihood}(\mathbf{u})}{[1 - \sigma^2_{likelihood}(\mathbf{u})][\sigma^2_{prior}(\mathbf{u}) - 1] + 1}$$



Probability and Statistics Bayesian Updating Demo

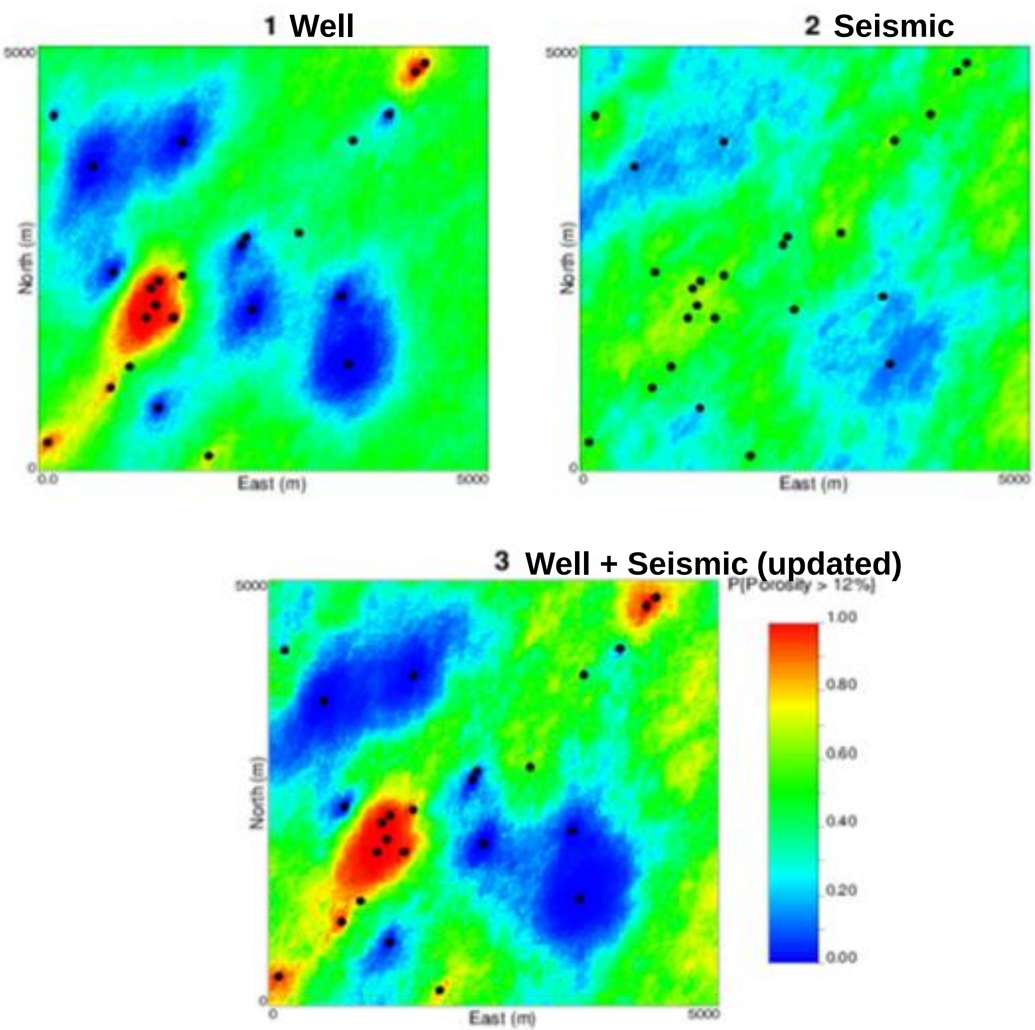
Example of Updating

1. Local prior probability of porosity exceeding 12% from well log data alone.

2. Local likelihood probability of porosity exceeding 12% from calibration with seismic data.

3. Local updated posteriori probability based on Bayesian updating.

4. Observations:
   - Strong priors (near data) are preserved.
   - When prior naïve, likelihood dominates.



This is the method of Mclennan et al., (2009) for fracture density modeling.

Probability and Statistics Excel Bayesian Updating Demo

Bayesian Gaussian Analytical Example Demo

Michael Pyrcz, the University of Texas at Austin, Geostatistical Reservoir Modeling Class

Formulation from Sivia, 1996.

$$\bar{x}_{\text{updated}} = \frac{\bar{x}_{\text{likelihood}}(\mathbf{u}) \cdot \sigma^2_{\text{prior}}(\mathbf{u}) + \bar{x}_{\text{prior}}(\mathbf{u}) \cdot \sigma^2_{\text{likelihood}}(\mathbf{u})}{[1 - \sigma^2_{\text{likelihood}}(\mathbf{u})][\sigma^2_{\text{prior}}(\mathbf{u}) - 1] + 1}$$

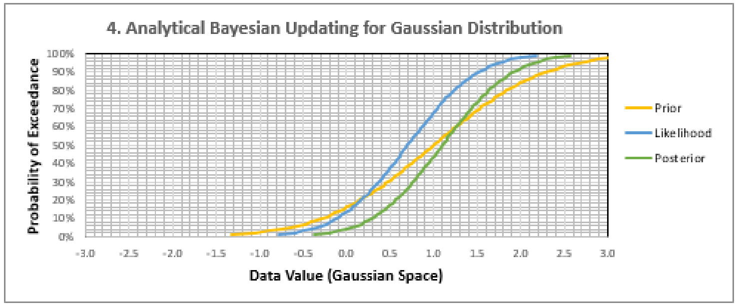
$$\sigma^2_{\text{updated}}(\mathbf{u}) = \frac{\sigma^2_{\text{prior}}(\mathbf{u})\,\sigma^2_{\text{likelihood}}(\mathbf{u})}{[1 - \sigma^2_{\text{likelihood}}(\mathbf{u})][\sigma^2_{\text{prior}}(\mathbf{u}) - 1] + 1}$$

1. Prior Distribution	
average	1.00
variance	1.00

2. Likelihood Distribution	
average	0.70
variance	0.40

3. Posterior Distribution	
average	1.10
variance	0.40

Percentile	Prior	Likelihood	Posterior
0.01	-1.326	-0.771	-0.371
0.02	-1.054	-0.599	-0.199
0.03	-0.881	-0.490	-0.090
0.04	-0.751	-0.407	-0.007
0.05	-0.645	-0.340	0.060
0.06	-0.555	-0.283	0.117
0.07	-0.476	-0.233	0.167
0.08	-0.405	-0.189	0.211
0.09	-0.341	-0.148	0.252
0.1	-0.282	-0.111	0.289
0.11	-0.227	-0.076	0.324
0.12	-0.175	-0.043	0.357
0.13	-0.126	-0.012	0.388
0.14	-0.080	0.017	0.417
0.15	-0.036	0.045	0.445
0.16	0.006	0.071	0.471
0.17	0.046	0.097	0.497
0.18	0.085	0.121	0.521
0.19	0.122	0.145	0.545
0.2	0.158	0.168	0.568
0.21	0.194	0.190	0.590
0.22	0.228	0.212	0.612
0.23	0.261	0.233	0.633
0.24	0.294	0.253	0.653
0.25	0.326	0.273	0.673
0.26	0.357	0.293	0.693
0.27	0.387	0.312	0.712
0.28	0.417	0.331	0.731
0.29	0.447	0.350	0.750
0.3	0.476	0.368	0.768
0.31	0.504	0.386	0.786
0.32	0.532	0.404	0.804
0.33	0.560	0.422	0.822
0.34	0.588	0.439	0.839
0.35	0.615	0.456	0.856
0.36	0.642	0.473	0.873



Instructions for Analytical Bayesian Updating for Gaussian Distributions

1. Set the average and the variance of the prior distribution (Gaussian parametric distribution).
2. Set the average and the variance of the likelihood distribution (Gaussian parametric distribution).
3. Observed the updated average and variance of the posterior distribution (Gaussian parametric distribution).
4. Observed the prior, likelihoof and posterior cumulative distribution functions (CDFs).

What did we learn?

1. The posterior variance is only a function of the prior and likelihood variances. The prior and likelihood means have no influence.
2. In general updating results in a reduction variance. Posterior variance is equal to or less than the greater of the prior and the likelihood variance.
3. High certainty in either prior or likelihood distribution (very low variance) causes either term to dominate the updated posterior.

Sivia, D.S., 1996, Data Analysis, A Bayesian Tutorial, Oxford Science Publications, 189 p.

Statistical Expectation

Statistical expectation is a probability weighted average. If all cases are equiprobable it is the same as arithmetic average.

For discrete variables:

$$\mathrm{E}[X] = \sum_{i=1}^{n} p_i \, x_i$$

$$\sum_{i=1}^{n} p_i = 1$$

probability of all outcomes sums to 1.0.

For continuous variables:

$$\mathrm{E}[X] = \int_{-\infty}^{+\infty} x \, f(x) dx$$

$$\int_{-\infty}^{+\infty} f(x) dx = 1$$

probability of all outcomes sums to 1.0.

The probability density function f(x) is the probability that the variate has a value equal to x.

Statistical Expectation Example

Example:

- The following grain sizes (mm) outcomes with probability in brackets

10 (10%), 20 (50%), 30 (10%), 40 (20%), 50 (10%)

Problem: Calculate the expected grain size.

Statistical Expectation Example

Example:

- The following grain sizes (mm) outcomes with probability in brackets

10 (10%), 20 (50%), 30 (10%), 40 (20%), 50 (10%)

Problem: Calculate the expected grain size.

$$E[X] = \sum_{i=1}^{N} p_i x_i = 10(0.1) + 20(0.5) + 30(0.1) + 40(0.2) + 50(0.1)$$

$E[X] =$ **27 mm**

Probability and Statistics New Tools

Topic	Application to Subsurface Modeling
Calculate Marginal, Conditional and Joint Probabilities	For univariate and bivariate settings calculate probabilities to support decision making. *Work out conditional statistics for permeability given facies and porosity. Move beyond simple correlations!*
Independence	Check for independence in sample data and use simplified workflows if present. *Check if facies proportions are independent of unit, if so consider combining the units to simplify the modeling workflow.*
Bayesian Methods	Solve for the probability of something given an indication of it occurring! *Calculate the probability of discontinuity in the subsurface given an indication in wells.*
Statistical Expectation	Report reservoir parameters in expected value. *Accounting for probability of each outcome with expected value.*

Probability and Statistics
What should you learn from this lecture?

Lecture outline . . .

- **Probability Definitions**
- **Venn Diagrams**
- **Frequentist Concepts**
- **Bayesian Concepts**

Prerequisites

Introduction

Probability Theory

Representative Sampling

Spatial Data Analysis

Spatial Estimation

Stochastic Simulation

Uncertainty Management

Machine Learning