Pension management

Case of a single risky asset and a fixed pension coupon

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Abstract

At instant $k = 0$, a financial investor has a wealth $\mathcal{W}_0$ from which he wants to be able to receive a pension c at equally spaced instants $k = 0 \dots K-1$ (*e.g.* monthly). He asks himself what a smart investment strategy would be knowing that he can invest at each instant an amount ψ_k in a risky asset of returns

$$\mu_k = m + \sigma\epsilon_k \quad \epsilon_k \sim \mathcal{N}(0,1) \text{ i.i.d.}$$

and an amount ψ_k^0 a riskless asset of returns r. Figure 1 shows a diagram of the cash flows the investor is trying to optimize.

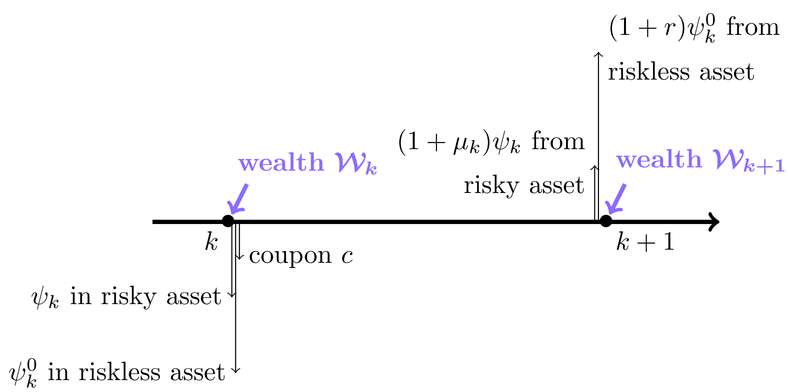

Figure 1: Cash flow diagram of a two period investment between instants k et $k+1$.

1 Model

1.1 Final actualized wealth $\tilde{\mathcal{W}}_K$.

1.1.1 Derivation of the formula

Between two instants k and $k+1$, the investor wants to invest ψ_k in the risky asset (without borrowing) and ψ_k^0 in the riskless asset after having received a coupon (*i.e.* the pension) c from the portfolio. In mathematical terms, the wealth $\mathcal{W}_k$ is the sum of these three terms positive terms

$$\mathcal{W}_k = c_k + \psi_k + \psi_k^0 \text{ with } c_k = \min(c, \mathcal{W}_k).$$

At instant $(k+1)^-$ (just before paying himself another coupon and reinvesting), he receives $(1+\mu_k)\psi_k$ from the risky asset and $(1+r)\psi_k^0$ from the riskless asset. In mathematical terms, this is written

$$\mathcal{W}_{k+1} = (1+\mu_k)\psi_k + (1+r)\psi_k^0.$$

Using these two expressions, we get to

$$\mathcal{W}_{k+1} - (1+r)\,\mathcal{W}_k = (\mu_k - r)\,\psi_k - (1+r)\,c_k$$

which calls for the definition of actualized to 0 quantities

$$\tilde{\mathcal{W}}_k = \frac{\mathcal{W}_k}{(1+r)^k} \qquad \tilde{\psi}_k = \frac{\psi_k}{(1+r)^k} \qquad \tilde{c}_k = \frac{c_k}{(1+r)^k}.$$

Using these new quantities, we have the following expression

$$\tilde{\mathcal{W}}_{k+1} - \tilde{\mathcal{W}}_k = \frac{\mu_k - r}{1+r}\tilde{\psi}_k - \tilde{c}_k \tag{1}$$

from which we finally have

$$\tilde{\mathcal{W}}_K = \mathcal{W}_0 + \sum_{k=0}^{K-1} \frac{\mu_k - r}{1+r}\tilde{\psi}_k - \sum_{k=0}^{K-1} \tilde{c}_k. \tag{2}$$

1.1.2 Case of no investment in the risky asset.

This case is interesting as it formalizes the naive strategy that would assume no investment in the risky asset and no interest rates. This naive strategy would simply divide the total available amount by the number of periods and the coupon would be

$$\gamma = \frac{\mathcal{W}_0}{K}$$

In the context of a riskless asset, assuming that $\tilde{W}_K = 0$ at the end of the period, the coupon c_l is obtained from the equality

$$\mathcal{W}_0 = \left(1 + \frac{1}{r}\right)\left(1 - \frac{1}{(1+r)^K}\right) c_l.$$

The naive strategy thus allows a coupon of

$$c_l = \frac{r}{1+r}\frac{(1+r)^K}{(1+r)^K - 1}\mathcal{W}_0. \tag{3}$$

In the following, this coupon is referred to as the linear benchmark. Note that the term linear comes from the limit $r \ll 1$ where

$$\frac{r}{1+r}\frac{(1+r)^K}{(1+r)^K - 1} \sim \frac{1}{K}$$

and gives a linear coupon of value $c_l \sim \gamma$ which is the expected coupon with interest rates.

1.2 Investment objectives.

1.2.1 Getting the highest coupon.

The objectives of the investment are to beat the linear benchmark c_l while controlling the investor's risks. At the end of the investment period, the wealth of the investor should be as close as possible to 0.

The previous statement suggests the average value $\left\langle \tilde{W}_K \right\rangle$ is equal to 0 and therefore the coupon value must follow the condition

$$\sum_{k=0}^{K-1} \frac{\langle \min(c, \mathcal{W}_k) \rangle}{(1+r)^k} = \mathcal{W}_0 + \frac{m-r}{1+r}\sum_{k=0}^{K-1}\left\langle \tilde{\psi}_k \right\rangle. \tag{4}$$

The right handside in the above formula depends then on the choice of investment strategy. This strategy is detailed below and consists in managing the risk of loss at every time step.

1.2.2 Controlling the risk taken by the investor at each period.

At each period, the investor wants to control the risk he is taking using probabilistic considerations. A simple risk measure is the probability of loss above a threshold. Say that the wealth $\mathcal{W}_k$ at instant k was not invested in the risky asset. Then at $k+1$, the investor would have $(1+r)(W_k - c)$.

The investor accepts to hold a risky position to beat this amount but refuses that his loss exceeds a fraction of this riskless amount with a probability level. Explicitly, say they do not wish to loose a fraction ρ of the riskless gain $(1+r)(W_k - c)$ with a probability superior to α. This is the same as writing

$$\text{Proba}\left[\tilde{W}_{k+1} \leq (1-\rho)\left(\tilde{W}_k - \tilde{c}_k\right)\right] \leq \alpha \tag{5}$$

Using (1) and assuming that the Sharpe ratio of the risky investment is such that[1]

$$\frac{m-r}{\sigma} \leq \mathcal{N}^{-1}(1-\alpha)$$

where $\mathcal{N}^{-1}$ is the inverse of the cumulative normal distribution function, we find after some algebra that the amount invested in risky asset must satisfy

$$\tilde{\psi}_k \leq \frac{\rho}{\rho_{\max}}\left(\tilde{\mathcal{W}}_k - \tilde{c}_k\right).$$

with

$$\rho_{\max} = \frac{1}{1+r}\left[\mathcal{N}^{-1}(1-\alpha) - \frac{m-r}{\sigma}\right]. \tag{6}$$

1.2.3 Investment strategy.

The previous formula was an inequality. However this inequality may be switched to an equal sign if the investor wishes to maximize its returns between two instants k and $k+1$.

As the investor cannot borrow money to conduct their strategy, they necessarily are capped in the amount they can invest in the risky asset. The maximum amount they can invest is $\tilde{\mathcal{W}}_k - \tilde{c}_k$. This results in the inequality

$$\tilde{\psi}_k \leq \tilde{W}_k - \tilde{c}_k. \tag{7}$$

The risk management inequality (6) and the no-borrowing condition (7) suggest to define the following strategy for the investor

$$\tilde{\psi}_k = f(\rho)\left(\tilde{\mathcal{W}}_k - \tilde{c}_k\right) \text{ with } f(\rho) = \min\left(1, \frac{\rho}{\rho_{\max}}\right) \tag{8}$$

with

$$\rho_{\max} = \frac{1}{1+r}\left[\mathcal{N}^{-1}(1-\alpha) - \frac{m-r}{\sigma}\right].$$

In layman's terms, the investor chooses an optimal Markowitz-like allocation at every time step. The risk aversion is measured in a probability of loss, and depending on this risk aversion, a coupon may be chosen to average the final wealth to zero.

1.3 Full statement of the problem.

The condition on an average final wealth to zero (4) with the strategy (8) leads to the inverse formula

$$\begin{aligned} \sum_{k=0}^{K-1} \frac{\langle \min\left[c(\rho), \mathcal{W}_k\right]\rangle}{(1+r)^k} &= \left\{\left[1-\omega(\rho)\right]\mathcal{W}_0 + \omega(\rho)\sum_{k=0}^{K-1}\frac{\langle \mathcal{W}_k\rangle}{(1+r)^k}\right\} \\ \omega(\rho) &= \frac{(m-r)\,f(\rho)}{1+r+(m-r)\,f(\rho)} \end{aligned} \tag{9}$$

[1] In the case where the Sharpe ratio is bigger than $\mathcal{N}^{-1}(1-\alpha)$, then all the remaining capital can be put in the risky asset.