$y' = y$

$\frac{1}{y}dy = 1 \qquad \ln|y| = x + C$

$y = e^{x+C} \qquad y = Ce^{x}$

미분방정식을 보고 함수를 구하는 것이 공학수학의 목표 ① y 구하기
② 형태 의존성 ⇑
꼴을 보고 전략을 택해야 함

ODE 상미분방정식 $y' = y$ ← y가 1개의 변수에 의해 정의됨 $y(x)$

PDE 편미분방정식 $y(x,t)$, $y(x,t,z)$ 등

○ 일계 미분방정식 Separable ODE

$y' = \frac{dy}{dx}$ y 덩어리끼리 x 덩어리끼리 묶기

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$y' = y \qquad \frac{dy}{dx} = y \qquad \frac{1}{y}dy = dx \qquad \int \frac{1}{y}dy = \int 1 \cdot dx$

적분시 항상 적분 상수 고려

$\ln|y| = x + C$

$y = e^{x+C} \qquad y = C \cdot e^{x}$

○ $y' = ky \qquad \frac{1}{y}dy = k \qquad \ln|y| = kx + C$

$y = e^{kx+C} \qquad y = C \cdot e^{kx}$

○ $y' + 2xy = 0 \qquad y(0) = 3$

$y' = -2xy \qquad \frac{1}{y}y' = -2x \qquad \ln|y| = -2\frac{1}{2}x^2 + C$

$\ln|y| = -x^2 + C$

$y = 3e^{-x^2} \qquad y = e^{-x^2+C} \qquad y = Ce^{-x^2}$

○ $y' = 1 + y^2 \qquad \frac{dy}{dx} = 1 + y^2 \qquad \frac{1}{1+y^2}dy = dx$

$\tan^{-1} y = x + C \qquad y = \tan(x + C)$

○ $y^3 y' + x^3 = 0 \qquad y^3 \frac{dy}{dx} = -x^3$

$y^3 dy = -x^3 dx$

$\frac{1}{4}y^4 = -\frac{1}{4}x^4 + C \qquad x^4 + y^4 = C$ implicit

○ $y' = \sec^2 y \qquad \frac{1}{\sec^2 y}dy = dx \qquad \frac{1}{2}y + \frac{1}{4}\sin 2y = x + C$

$\sec y = \frac{1}{\cos y} \qquad \int \cos^2 y \, dy = \int dx \qquad 2y + \sin 2y = 4x + C$

$\sec^2 y = \frac{1}{\cos^2 y} \qquad \int \frac{1+\cos 2y}{2} dy = \int dx$

$1 + \tan^2 y = \sec^2 y \qquad 1 + \frac{\sin^2 y}{\cos^2 y} = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$

* 반각 공식

$\cos^2 y = \frac{1+\cos 2y}{2}$

$\sin^2 y = \frac{1-\cos 2y}{2}$

$\tan^2 y = \frac{1-\cos 2y}{1+\cos 2y}$

$y' = (y-1)^2 \qquad y(0) = 1$

$\frac{1}{(y-1)^2}dy = dx$

$(y-1)^{-2}dy = dx \qquad y = 1$

$-(y-1)^{-1} = x + C$

$-\frac{1}{y-1} = x + C \qquad -\frac{1}{1.01-1} = C = -\frac{1}{0.01} = -100$

$y(0) = 1.01$이면 $\qquad -\frac{1}{y-1} = x - 100$

$-\frac{1}{x-100} = y - 1 \qquad y = \frac{x-100}{x-100} - \frac{1}{x-100} = \frac{x-101}{x-100}$

○ $y + \sqrt{y'} = 1 \qquad \sqrt{y'} = 1 - y \qquad y' = (1-y)^2$

$\frac{1}{(1-y)^2}dy = dx \qquad (1-y)^{-2}dy = dx \qquad \frac{1}{1-y} = x + C$

$(1-y)^{-1} = x + C \qquad \frac{1}{x+C} = 1 - y$

속미분 고려

$y = 1 - \frac{1}{x+C}$

○ 치환변수 분리법

○ $2xyy' = y^2 - x^2$ 어떻게 묶지? → 치환 사용

$\frac{y}{x} = u$ → u와 x 변수분리 가능 90%는 $\frac{y}{x} = u$

치환 10%는 어떤 덩어리를 u로 치환

$y = ux$

$y' = u'x + u$ 대입

$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y} \qquad \frac{y}{x} = u$

$= \frac{1}{2}u - \frac{1}{2u}$

$u'x + u = \frac{1}{2}u - \frac{1}{2u} \qquad u'x = \frac{1}{2}u - \frac{1}{2u} - u = -\frac{1}{2}u - \frac{1}{2u}$

$= -\frac{1}{2}\left(u + \frac{1}{u}\right) = -\frac{1}{2}\left(\frac{u^2+1}{u}\right)$

u와 x에 관해 변수 분리

$\frac{du}{dx}x = -\frac{1}{2}\left(\frac{u^2+1}{u}\right)$

$\int \frac{2u}{u^2+1}du = \int -\frac{1}{x}dx \qquad u^2 + 1 = t \qquad \ln|x|^{-1}$

$2u\,du = dt \qquad \ln e^{c}$

$\int \frac{1}{t}dt = \ln|t| \quad \ln|u^2+1| = -\ln|x| + C$

$= \ln\frac{e^c}{x} = \ln\frac{C}{x}$

$u^2 + 1 = \frac{C}{x}$

$\frac{y^2}{x^2} + 1 = \frac{C}{x} \qquad y^2 + x^2 = Cx$

○ $xy' = y^2 + y$

$\frac{1}{y^2+y}y' = \frac{1}{x}$ 변수분리도 가능

$\frac{1}{y(y+1)}y' = \frac{1}{x}$

$\int\left(\frac{1}{y} - \frac{1}{y+1}\right)dy = \ln|y| - \ln|y+1|$ $\ln\frac{y}{y+1}$

$= \ln|x| + C = \ln Cx$ ($C \to \ln e^C$)

$\frac{y}{y+1} = Cx$ $\quad \frac{y+1}{y} = \frac{1}{Cx}$ $\quad 1 + \frac{1}{y} = \frac{1}{Cx}$ $\quad \frac{1}{y} = \frac{1}{Cx} - 1$

$\frac{1}{y} = \frac{1-Cx}{Cx}$

$y = \frac{Cx}{1-Cx}$

$= \frac{-Cx}{Cx-1} = \frac{-x}{x - \frac{1}{C}}$

$= \frac{x}{-x+C}$

치환 사용하면? $\quad y' = \frac{y^2}{x} + \frac{y}{x}$

$\frac{y}{x} = u \quad y = ux$

$y' = u'x + u$

$u'x + \cancel{u} = \frac{u^2x^2}{x} + u = u^2x + \cancel{u}$

$u' = u^2 \qquad \frac{1}{u^2}du = dx \qquad -\frac{1}{u} = x + C$

$-\frac{x}{y} = x + C$

$\frac{1}{y} = -\frac{x+C}{x}$

$y = -\frac{x}{x+C} = \frac{x}{-x+C}$

○ $xy' = y + 2x^3\sin^2\left(\frac{y}{x}\right) \qquad \frac{y}{x} = u \qquad y = u'x + u$

$y' = \frac{y}{x} + 2x^2\sin^2\left(\frac{y}{x}\right)$

$u'x + \cancel{u} = \cancel{u} + 2x^2\sin^2 u \qquad u'x = 2x^2\sin^2 u$

$u' = 2x\sin^2 u$

$\frac{1}{\sin^2 u}du = 2x\,dx$

$\int\frac{1}{\sin^2 u}du = -\frac{1}{\tan u} = x^2 + C \qquad \tan u = -\frac{1}{x^2+C}$

$\frac{y}{x} = u = \tan^{-1}\left(-\frac{1}{x^2+C}\right) = -\tan^{-1}\left(\frac{1}{x^2+C}\right)$

기함수

$\frac{1}{\cos u} = \sec u \qquad \frac{1}{\cos^2 u} = \sec^2 u \qquad \frac{1}{\tan u} = \cot u$

$\frac{1}{\sin u} = \csc u \qquad \frac{1}{\sin^2 u} = \csc^2 u$

$y = -x\tan^{-1}\left(\frac{1}{x^2+C}\right)$

$\tan' u = \sec^2 u = \frac{1}{\cos^2 u}$

$\cot' u = -\csc^2 u = -\frac{1}{\sin^2 u}$

$\left(\frac{\cos u}{\sin u}\right)' = \frac{-\sin^2 u - \cos^2 u}{\sin^2 u} = -\frac{1}{\sin^2 u}$

○ $xy' = y + 3x^4\cos^2\left(\frac{y}{x}\right) \qquad \frac{y}{x} = u$

$y' = \frac{y}{x} + 3x^3\cos^2\left(\frac{y}{x}\right) \qquad y' = u'x + u$

$u'x + \cancel{u} = \cancel{u} + 3x^3\cos^2 u \qquad u' = 3x^2\cos^2 u$

$\frac{1}{\cos^2 u}du = 3x^2dx$

$\int\sec^2 u\,du = \int 3x^2dx \to \tan u = x^3 + C$

$\frac{y}{x} = u = \tan^{-1}(x^3 + C) \qquad y = x\tan^{-1}(x^3+C)$

○ $xy' = x + y \qquad y' = 1 + \frac{y}{x} \qquad \frac{y}{x} = u$

$u'x + \cancel{u} = 1 + \cancel{u} \qquad u' = \frac{1}{x} \qquad u = \ln|x| + C$

$\frac{y}{x} = \ln|x| + \ln e^C = \ln e^C x = \ln Cx$

$y = x(\ln Cx) \qquad y = x(\ln x + C)$

○ $y' = (x+y-2)^2$

└ 이거 자체를 u로 보자

$u = x + y - 2$

$u - x + 2 = y$

$y' = u' - 1$

$u' - 1 = u^2 \qquad u' = 1 + u^2 \qquad \frac{1}{1+u^2}du = 1\,dx$

$\tan^{-1}(x)$ 미분 →

$\arctan'(x) = \frac{1}{1+x^2} \qquad \arctan(u) = x + C$

$u = \tan(x+C) = x + y - 2$

$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}} \qquad y = \tan(x+C) - x + 2$

$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$

○ $(x^2 - xy)y' + x^2 + y^2 = 0$

$y' = \frac{-x^2 - y^2}{x^2 - xy} = \frac{x^2+y^2}{xy - x^2} = \frac{1 + \frac{y^2}{x^2}}{\frac{y}{x} - 1}$

$\frac{y}{x} = u$ $\qquad$ 분자분모 모두 $\frac{1}{x^2}$로 나눠줌

$u'x + u = \frac{1+u^2}{u-1}$

$u'x = \frac{1 + \cancel{u^2} - \cancel{u^2} + u}{u-1} = \frac{u+1}{u-1}$

$\frac{u-1}{u+1}du = \frac{1}{x}dx$

$\int\frac{u+1-2}{u+1}du = \int\left(1 - \frac{2}{u+1}\right)du = u - 2\ln|u+1| = \ln|x| + C$

$\ln e^u + \ln|u+1|^{-2} = \ln Cx$

$\ln\frac{e^u}{(u+1)^2} = \ln Cx$

$e^u = Cx(u+1)^2$

$e^{\frac{y}{x}} = Cx\left(\frac{y}{x} + 1\right)^2$

$xe^{\frac{y}{x}} = Cx^2\left(\frac{y}{x} + 1\right)^2$

$= C(y + x)^2$

$\circ\ y' = \dfrac{y-x}{y+x} = \dfrac{\frac{y}{x}-1}{\frac{y}{x}+1}$ $\qquad \dfrac{y}{x} = u$

$u'x + u = \dfrac{u-1}{u+1}$

$u'x = \dfrac{u-1-u^2-u}{u+1} = -\dfrac{u^2+1}{u+1}$

$\displaystyle\int \frac{u+1}{u^2+1}du = -\int \frac{1}{x}dx$

$\displaystyle\int \frac{2u+2}{u^2+1}du = -\int \frac{2}{x}dx$

$\displaystyle\int \left(\frac{2u}{u^2+1} + \frac{2}{u^2+1}\right)du = -\int \frac{2}{x}dx$

$\ln|u^2+1| + 2\tan^{-1}u = -2\ln|x| + C$

$\ln|u^2+1| + \ln e^{2\tan^{-1}u} = \ln\dfrac{C}{x^2}$

$e^{2\tan^{-1}u}(u^2+1) = \dfrac{C}{x^2}$

$e^{2\tan^{-1}\frac{y}{x}}\left(\dfrac{y^2}{x^2}+1\right) = \dfrac{C}{x^2}$

$(x^2+y^2)e^{2\tan^{-1}\frac{y}{x}} = C$

$\sqrt{x^2+y^2}\,e^{\tan^{-1}\frac{y}{x}} = C$

$\circ\ y' = \dfrac{x+3y}{3x+y} = \dfrac{1+3\frac{y}{x}}{3+\frac{y}{x}}$ $\qquad \dfrac{y}{x} = u$

$u'x + u = \dfrac{1+3u}{3+u}$

$u'x = \dfrac{1+\cancel{3u}-\cancel{3u}-u^2}{3+u} = -\dfrac{u^2-1}{3+u}$

$\dfrac{3+u}{u^2-1}du = -\dfrac{1}{x}dx$

$\dfrac{3+u}{(u-1)(u+1)}$ $\qquad \displaystyle\int\left(\frac{2}{u-1} - \frac{1}{u+1}\right)du = -\int\frac{1}{x}dx$

$\to \dfrac{2}{u-1} - \dfrac{1}{u+1}$

부분분수 쉽게 분리하기?

$2\ln|u-1| - \ln|u+1| = -\ln|x| + C = \ln\dfrac{C}{x}$

$\ln|u-1|^2 + \ln\dfrac{1}{|u+1|} = \ln\dfrac{(u-1)^2}{(u+1)}$

$\dfrac{(u-1)^2}{u+1} = \dfrac{C}{x}$ $\qquad \dfrac{(\frac{y}{x}-1)^2}{(\frac{y}{x}+1)} = \dfrac{C}{x}$ $\qquad x^2\left(\dfrac{y}{x}-1\right)^2 = Cx\left(\dfrac{y}{x}+1\right)$

$(y-x)^2 = C(y+x)$

$\circ\ (x+\sqrt{xy})y' = y$ $\qquad \dfrac{y}{x} = u$

$y' = \dfrac{y}{x+\sqrt{xy}} = \dfrac{\frac{y}{x}}{1+\sqrt{\frac{y}{x}}} = \dfrac{u}{1+\sqrt{u}}$

$u'x + u$

$u'x = \dfrac{u}{1+\sqrt{u}} - \dfrac{1+\sqrt{u}}{1+\sqrt{u}}u = -\dfrac{u\sqrt{u}}{1+\sqrt{u}}$

$\dfrac{1+\sqrt{u}}{u\sqrt{u}}du = -\dfrac{1}{x}dx$

$\displaystyle\int\left(\frac{1}{u\sqrt{u}} + \frac{1}{u}\right)du = -2u^{-\frac{1}{2}} + \ln|u| = -\ln|x| + C$
$= \ln\dfrac{C}{x}$

$u^{-\frac{3}{2}}$ $\qquad -2u^{-\frac{1}{2}}$ $\qquad \ln e^{-2u^{-\frac{1}{2}}}u = \ln\dfrac{C}{x}$

$e^{-\frac{2}{\sqrt{u}}}u = \dfrac{C}{x}$

$e^{-2\sqrt{\frac{x}{y}}}\dfrac{y}{\cancel{x}} = \dfrac{C}{\cancel{x}}$ $\qquad -2\sqrt{\dfrac{x}{y}} + \ln y = C$

$(\ln y - c)^2 = 4\dfrac{x}{y}$

$y(\ln y - c)^2 = 4x$

함수 구하기 x와 y 관계 알기

$\frac{y}{x} = u$ 치환 $y' = u'x + u$ u와 x의 관계 → y, x의 관계

변수 분리 → 치환 변수 분리 → 완전 상미분 방정식(Exact)

$y' = y$

$\frac{dy}{dx} = y$ $y\,dx - dy = 0$

$\square\,dx + \square\,dy = 0$

x와 y의 관계식

$\boxed{P(x,y)\,dx + Q(x,y)\,dy} = 0$

$u(x,y)$ $du = \boxed{\underbrace{\frac{\partial u}{\partial x}}_{u_x} dx + \underbrace{\frac{\partial u}{\partial y}}_{u_y} dy}$ 전미분 관계

$du = 0$ 조건: $P = \frac{\partial u}{\partial x} = u_x$ $Q = \frac{\partial u}{\partial y} = u_y$

$u(x,y) = C$ $u_{xy} = u_{yx}$

x와 y의 관계식 → 목적: u 구하기 $P_y = Q_x$ 이게 만족되어야함

$P = \frac{\partial u}{\partial x}$ $Q = \frac{\partial u}{\partial y}$

$u = \int P\,dx + g(y)$

$u_y = \bigcirc\, y + g'(y) = Q$

예) $2xy\,dx + x^2 dy = 0$ $2xy + x^2 y' = 0$

$\underset{P(x,y)}{2xy}$ $\underset{Q(x,y)}{x^2}$ $y' = -\frac{2y}{x}$

① Exact 인지 확인

$\frac{\partial P}{\partial y} = 2x$ u_{xy}

$\parallel$

$\frac{\partial Q}{\partial x} = 2x$ u_{yx}

② u 구하기

$u_x = P(x,y) = 2xy$

$u = \int 2xy\,dx + g(y)$

$= yx^2 + g(y)$

$u_y = x^2 + g'(y) = Q(x,y) = x^2$

$g'(y) = 0$ $g(y) = C_1$

$u = x^2 y + C_1$

$u(x,y) = c = x^2 y$ $y = \frac{c}{x^2}$

$2xy\,dx + x^2 dy = 0$ 변수 분리로 풀면?

$x^2 dy = -2xy\,dx$

$\int \frac{1}{y} dy = -\int \frac{2}{x} dx$

$\ln|y| = -2\ln|x| + C$

$\ln y = \ln\frac{1}{x^2} + \ln e^c$

$y = \frac{c}{x^2}$

$\underset{P}{\cos(x+y)}\,dx + (\underset{Q}{3y^2 + 2y + \cos(x+y)})\,dy = 0$

$\frac{\partial P}{\partial y} = -\sin(x+y)$ ① exact인지 확인

$\parallel$ exact!

$\frac{\partial Q}{\partial x} = -\sin(x+y)$

같지 않을 때는 F라는 적분인자를 서로 곱해줘서 같게끔 만들어줌 (그러한 F를 찾기)

② u 구하기

$u = \int P\,dx + g(y)$

$= \int \cos(x+y)\,dx + g(y) = \sin(x+y) + g(y)$

$u_y = \cos(x+y) + g'(y) = 3y^2 + 2y + \cos(x+y) = Q$

$g'(y) = 3y^2 + 2y$ $g(y) = y^3 + y^2 + C_1$

$u = \sin(x+y) + y^3 + y^2 = C$

$P\,dx + Q\,dy = 0$ $du = u_x\,dx + u_y\,dy$

① $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ → $du = 0$ $P = u_x$ $Q = u_y$

② $u(x,y) = c$ 만족 못하면?

적분인자 곱함

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ 이면? $FP\,dx + FQ\,dy = 0$

$F(x) = e^{\int R(x)dx}$ $F(y) = e^{\int R(y)dy}$

$R(x) = \frac{P_y - Q_x}{Q}$ $R(y) = -\frac{P_y - Q_x}{P} = \frac{Q_x - P_y}{P}$ 부호 조심

이 공식을 사용하려면 F가 오직 하나의 변수로만 이루어져 있어야 한다

① R이 오직 x로만 이루어지는지 확인

② R이 오직 y로만 이루어지는지 확인

③ R이 x와 y로 이루어졌을 때 적분 인자로 푸는 문제가 아니거나 실수했거나

예제) $\underbrace{-y}_{P}\,dx + \underbrace{x}_{Q}\,dy = 0$ $x\,dy = y\,dx$

$\frac{1}{y}dy = \frac{1}{x}dx$

$\ln|y| = \ln|x| + C$

완전 상미분으로 풀면 $y = cx$

$$\frac{\partial P}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = 1$$

적분인자 구해줘야 함 $-Fy\,dx + Fx\,dy = 0$

① R이 오직 x로만 이루어져 있다고 가정 시

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{x}(-1-1) = -\frac{2}{x} \Rightarrow \text{됨}$$

② R이 오직 y로만 이루어져 있다고 가정 시

$$R = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = -\frac{1}{y}(1+1) = -\frac{2}{y} \Rightarrow \text{됨}$$

$$F = e^{-2\int \frac{1}{x}dx} = e^{-2\ln|x| + C}$$

$= C\frac{1}{x^2}$ C=1로 하자 (F 적분인자 간단하게 하는게 좋으므로)

$\underbrace{-\frac{y}{x^2}}_{M}dx + \underbrace{\frac{1}{x}}_{N}dy = 0$ $U_x = M$ $U_y = N$

$$\frac{\partial M}{\partial y} = -\frac{1}{x^2} = \frac{\partial N}{\partial x} = -\frac{1}{x^2}$$

$$U = \int M\,dx + g(y) = -y\int \frac{1}{x^2}dx + g(y)$$

$$= y\frac{1}{x} + g(y)$$

$U_y = \frac{1}{x} + g'(y) = N = \frac{1}{x}$ $g'(y) = 0$ $g(y) = C_1$

$U = \frac{y}{x} + C_1 = C$ $\frac{y}{x} = C$ $y = Cx$

예제) $\underbrace{(x^2+y^2)}_{P}dx \underbrace{-2xy}_{Q}\,dy = 0$ 변수 분리하기 어려움

완전 상미분 방정식으로 풀기

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ 되는지 확인

$\frac{\partial P}{\partial y} = 2y \neq \frac{\partial Q}{\partial x} = -2y$ 적분인자 구해보기

① $F(x) = e^{\int R(x)dx}$ $R(x) = \frac{P_y - Q_x}{Q}$

$\frac{2y+2y}{-2xy} = \frac{4y}{-2xy} = -\frac{2}{x}$ → x로 이루어짐

$F = e^{-2\int \frac{1}{x}dx} = e^{-2\ln|x| + C} = \frac{C}{x^2}$ $C=1$

$FP\,dx + FQ\,dy = 0$

$\underbrace{\frac{1}{x^2}(x^2+y^2)}_{M}dx + \underbrace{\frac{1}{x^2}(-2xy)}_{N}dy = 0$

$U_x = M$ $N = U_y$

$$U = \int \frac{1}{x^2}(x^2+y^2)\,dx + g(y) = \int\left(1 + \frac{y^2}{x^2}\right)dx + g(y)$$

$$= x - y^2\frac{1}{x} + C + g(y) = x - \frac{y^2}{x} = C$$

$U_y = -\frac{1}{x}2y + g(y) = -\frac{2y}{x}$ $g(y) = 0$ $g(y) = C$

$x = \frac{y^2}{x} + C$ $y^2 = x^2 + Cx$

예제) $\underbrace{(e^{x+y} + ye^y)}_{P}dx + \underbrace{(xe^y - 1)}_{Q}dy = 0$

$\frac{\partial P}{\partial y} = e^{x+y} + ye^y + e^y \neq \frac{\partial Q}{\partial x} = e^y$ 적분상수

① $F(x) = e^{\int R(x)dx}$ $R(x) = \frac{P_y - Q_x}{Q}$

$$= \frac{1}{xe^y - 1}\left(e^{x+y} + ye^y + \cancel{e^y} - \cancel{e^y}\right)$$

$= \frac{1}{xe^y - 1}e^y(e^x + y)$ → x로만 이루어져 있지 X

② $F(y) = e^{\int R(y)dy}$ $R(y) = \frac{Q_x - P_y}{P}$

$$= \frac{1}{e^{x+y} + ye^y}\left(-e^{x+y} - ye^y\right) = \frac{-1}{\cancel{e^{x+y} + ye^y}}\cancel{(e^{x+y} + ye^y)} = -1$$

$F(y) = e^{\int -1\,dy} = e^{-y+C} = Ce^{-y}$ $C=1$ → y로만 이루어짐

$FP\,dx + FQ\,dy = 0$ $e^{-y}(e^{x+y} + ye^y)dx + e^{-y}(xe^y - 1)\,dy = 0$

$\underbrace{(e^x + y)}_{U_x = M}dx + \underbrace{(x - e^{-y})}_{U_y = N}dy = 0$

$$U = \int (e^x + y)\,dx + g(y) = e^x + yx + C_1 + g(y) = e^x + yx + e^{-y} = C$$

$U_y = x + g'(y) = x - e^{-y} = N$ $g'(y) = -e^{-y}$

$g(y) = e^{-y}$

$\frac{dy}{dx}$ linear

$y'(x) + p(x)y = r(x)$ 이런 꼴이면 $y = \sim$ 구하는데 공식 사용 가능

$\underbrace{(p(x)y - r(x))}_{P}dx + \underbrace{1}_{Q} \cdot dy = 0$ 완전 미분방정식

$\frac{\partial P}{\partial y} = p(x) \neq \frac{\partial Q}{\partial x} = 0 \Rightarrow$ 적분 상수 구하기

$F(x) = e^{\int R(x)dx}$ $\quad R(x) = \frac{P_y - Q_x}{Q} = \frac{p(x)}{1} = p(x)$

$= e^{\int p(x)dx}$

$F(x)P(x,y)dx + F(x)Q(x,y)dy = 0$

$\underbrace{e^{\int p(x)dx}(p(x)y - r(x))}_{U_x = M}dx + \underbrace{e^{\int p(x)dx}}_{U_y = N}dy = 0$

U_x를 x에 관해 부분적분 해서 y에 관해 편미분하는 것보다

y에 관해 부분적분 해서 x에 관해 편미분하는게 더 편할 듯

$u = \int e^{\int p(x)dx}dy + g(x) = e^{\int p(x)dx} \cdot y + g(x) = C$

$U_x = p(x)e^{\int p(x)dx} \cdot y + g'(x) = e^{\int p(x)dx}(p(x)y - r(x))$

$g'(x) = -e^{\int p(x)dx}r(x)$

$g(x) = -\int r(x)e^{\int p(x)dx}dx$

$e^{\int p(x)dx} \cdot y - \int r(x)e^{\int p(x)dx}dx = C$

$y = e^{-\int p(x)dx}\left(\int r(x)e^{\int p(x)dx}dx + C\right)$

$y' + p(x)y = g(x)y^{\alpha}$ $\quad \alpha \neq 0, 1$ 선형으로 바꾸기

$u = y^{1-\alpha}$ $\quad u' = (1-\alpha)y^{-\alpha}y'$

$= (1-\alpha)y^{-\alpha}(g(x)y^{\alpha} - p(x)y)$

$= (1-\alpha)g(x) - p(x)(1-\alpha)\underbrace{y^{(1-\alpha)}}_{u}$

$u' + \underbrace{(1-\alpha)p(x)}_{p(x)}u = \underbrace{g(x)(1-\alpha)}_{r(x)}$

$u(x) = e^{-\int(1-\alpha)p(x)dx}\left(\int(1-\alpha)g(x)e^{\int(1-\alpha)p(x)dx}dx + C\right)$

$y(x) = \{u(x)\}^{\frac{1}{1-\alpha}}$

선형 일계 ODE

선형 상미분 방정식 $\quad y' + p(x)y = r(x)$

$y, y', \dots$에 관해 선형이어야 함

y'의 계수를 1로 만들 수 있어야 함

(예)

$y' + xy = x^2$ (o) $\qquad x^2y' + xy = x + 1$

$y' + (y+1)y = 1$ (x) $\qquad y' + \frac{1}{x}y = \frac{x+1}{x^2}$

$y' + xy = 1 + y^2$ (x)

$$y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}r(x)dx + C\right)$$

$$y = e^{-h}\left(\int e^{h}r\,dx + C\right), \quad h = \int p\,dx \text{ (단, 적분상수는 0)}$$

$(x^2y)'$

$x^2y' + 2xy = \cos x$

$e^xy + e^xy' = xe^x$

$(e^xy)' = xe^x$

$e^xy = \int xe^xdx$

$y' + py = r$

$e^{\int pdx}y' + e^{\int pdx}py = e^{\int pdx}r$ $\qquad (e^{\int pdx}y)' = e^{\int pdx}r$

$(\int pdx)'$

$e^{\int pdx}y = \int e^{\int pdx}rdx + C$

$y = e^{-\int pdx}\left(\int e^{\int pdx}rdx + C\right)$

$= e^{-h}\left(\int e^{h}rdx + C\right)$ $\quad h = \int pdx$

∘ $y' + \underbrace{(\tan x)}_{p}y = \underbrace{\sin 2x}_{r}$ $\quad y(0) = 1$

$y = e^{-h}\left(\int e^{h}rdx + C\right)$ $\quad h = \int pdx$

$h = \int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$ $\qquad \cos x = t \quad -\sin x dx = dt$

$= -\int \frac{1}{t}dt = -\ln|t| = -\ln|\cos x|$ 여기선 C를 고려 X

$y = e^{-h}\left(\int e^{h}rdx + C\right)$

$= e^{\ln\cos x}\left(\int e^{-\ln\cos x}\underbrace{\sin 2x}_{}dx + C\right)$

$= \cos x\left(\int \frac{1}{\cos x}2\sin x\cos x\,dx + C\right)$

$= \cos x\left(2\int \sin x\,dx + C\right)$

$= \cos x(-2\cos x + C) = -2\cos^2 x + C\cos x = -2\cos^2 x + 3\cos x$

$y(0) = 1$이므로 $\quad -2 + C = 1 \quad C = 3$

○ $y' = 2y - 4x \Rightarrow \underbrace{y' - 2y}_{p} = \underbrace{-4x}_{r}$

$h = \int -2\,dx = -2x$

$y = e^{2x}\left(\int e^{-2x}(-4x)\,dx + c\right)$

$= e^{2x}\left(\int (-4x)e^{-2x}\,dx + c\right)$

$\|$

$\left(-\frac{1}{2}\right)(-4x)e^{-2x} - \int \cancel{(-4)}^{2}\left(-\cancel{\tfrac{1}{2}}\right)e^{-2x}\,dx$

$2xe^{-2x} - 2\int e^{-2x}\,dx$

$-\cancel{2}\left(-\cancel{\tfrac{1}{2}}\right)e^{-2x}$

$2xe^{-2x} + e^{-2x}$ $\qquad y = 2x + 1 + C\cdot e^{2x}$

$(2x+1)e^{-2x}$

$= e^{2x}\cdot\left((2x+1)e^{-2x} + c\right) = 2x + 1 + C\cdot e^{2x}$

$2e^{-2x} + (2x+1)e^{-2x}(-2)$

○ 베르누이 방정식 $\quad y' + p(x)y = r(x)y^a$

비선형이므로 singular solution이 존재할 수 있는

$u = y^{1-a}$로 치환

상수항이 따로 있거나 y^a로 묶었을 때 x에 관한 식이 아니면 안됨

$y' + p(x)y = r(x)$

$y = e^{-h}\left(\int e^{h} r\,dx + C\right) \qquad h = \int p\,dx$

$y' + py = ry^a$ $\quad$ 선형 상미분 방정식 X
베르누이 방정식

(예) $y' + xy = x^2y^2 \qquad y' + xy = \frac{x^2}{y}$

$y' + py = ry^a \qquad u = y^{1-a}$로 치환 ☆

$u' + \boxed{p}\,u = \boxed{r}$

$a = 2$

○ $xy' + y = x^2y^2 \qquad y' + \frac{1}{x}y = xy^{2 = a}$ 베르누이 방정식임을 확인

$u = y^{1-a} = y^{1-2} = \frac{1}{y} = y^{-1}$

$u' = -y^{-2}y'$ 속미분 주의 $\qquad \times\ -y^{-2}$ 곱함

y' 포함 항이 이꼴이 되도록

$-y^{-2}y' + \frac{1}{x}y\left(-\frac{1}{y^2}\right) = xy^2(-y^{-2}) = -x$

$\underbrace{-y^{-2}y'}_{u'} + \frac{1}{x}\left(-\frac{1}{y}\right) = -x$

$u' - \frac{1}{x}\underbrace{y^{-1}}_{u} = -x \qquad u' - \underbrace{\frac{1}{x}}_{p}u = \underbrace{-x}_{r}$

$u = e^{-h}\left(\int e^{h} r\,dx + c\right) \qquad h = \int p\,dx = \int -\frac{1}{x}\,dx = -\ln|x|$

$= e^{\ln x}\left(\int e^{-\ln x}(-x)\,dx + c\right)$

$= x\left(\int \frac{1}{\cancel{x}}(-\cancel{x})\,dx + c\right) = x(-x + c)$

$= -x^2 + cx$

$y^{-1} = -x^2 + cx \qquad y = \frac{1}{-x^2 + cx}$

∘ $y' = Ay - By^2$ $\qquad$ $y' - Ay = -By^2$ $\quad \underset{\sim}{a=2}$

$u = y^{1-a} = y^{1-2} = y^{-1}$

$u' = -y^{-2}y'$ $\qquad$ $\times\ (-y^{-2})$ 곱하기

$$\underbrace{-y^{-2}y'}_{u'} + A\underbrace{y^{-1}}_{u} = B$$

$$u' + \underbrace{A}_{p}u = \underbrace{B}_{r}$$

$y^{-1} = u = e^{-h}\left(\int e^{h} r\,dx + c\right)$ $\qquad$ $h = \int p\,dx = \int A\,dx = Ax$

$= e^{-Ax}\left(\int e^{Ax} B\,dx + c\right)$

$= e^{-Ax}\left(\frac{B}{A}e^{Ax} + C\right) = \frac{B}{A} + Ce^{-Ax}$

$$y = \frac{1}{\frac{B}{A} + Ce^{-Ax}}$$