이계 ODE 전략: 형태 → 풀이법

$y'' + py'(x) + qy(x) = r(x)$

ODE: 최종적인 함수가 하나의 변수에 대해서 이루어짐 $y(x)$

가장 많이 미분한 게 y''

선형 → $1 \cdot y'' + \boxed{p(x)}\, y' + \boxed{q(x)}\, y = \boxed{r(x)}$

오직 x에 대해서만 이루어진 식

우변을 보고 제차인지 비제차인지

제차: $r(x)=0$ ex) $y''+p(x)y'+q(x)y=0$

비제차: $r(x)\neq 0$ ex) $y''+p(x)y'+q(x)y = x+1$

기저(basis) $y''+p(x)y'+q(x)y=0$ 제차

해 근

$y''+y=0$ ← $y_1=\sin x$, $y_2=\cos x$ 기저

만족하는 y들이 2개

일차결합, 선형합

선형 2계 ODE라면 답을 $\boxed{y = C_1 y_1 + C_2 y_2}$로 나타내야 함

$y(0)=1$ $y'(0)=2$

$y = \square\, y_1 + \square\, y_2$ 특수해

기저를 대입했을 때 성립해야 함 $y_1''+py_1'+qy_1=0$

$y_2''+py_2'+qy_2=0$

if y_1이 주어졌을 때 y_2 구하는 방법 (하나의 기저가 주어진 경우)

가정

[공식] $\boxed{y_2 = uy_1 \quad u=\int U\,dx \quad U = \frac{1}{(y_1)^2}e^{-\int p\,dx}}$

$$\boxed{y_2 = y_1 \int \frac{1}{(y_1)^2} e^{-\int p\,dx}\, dx}$$

$y''+p(x)y'+q(x)y=0$ y_1과 y_2 대입할 때 성립해야 함

$y_2 = uy_1$ $y_2' = u'y_1 + uy_1'$ $y_2'' = u''y_1+u'y_1'+u'y_1'+uy_1''$

$= u''y_1+uy_1''+2u'y_1'$

$(u''y_1+uy_1''+2u'y_1') + p(u'y_1+uy_1')+q\,uy_1 = 0$

$y_1u''+u'(2y_1'+py_1)+u\underbrace{(y_1''+py_1'+qy_1)}_{0}=0$

$y_1u''+u'(2y_1'+py_1)=0$ u 구해야 함

$u'=U$

$y_1U'+U(2y_1'+py_1)=0$

$\frac{1}{U}U' = -\frac{2y_1'+py_1}{y_1}$ $U'=\frac{dU}{dx}$ 이므로

$\int \frac{1}{U}dU = \int -\left(2\frac{y_1'}{y_1}+p\right)dx$ $u'=U$ 이므로

$u=\int U\,dx$

$\ln|U| = -\ln|y_1^2| - \int p\,dx$

$U = e^{-2\ln y_1 - \int p\,dx} = y_1^{-2}e^{-\int p\,dx} = \frac{1}{y_1^2}e^{-\int p\,dx}$ $u=\int \frac{1}{y_1^2}e^{-\int p\,dx}$

○ $(x^2-x)y''-xy'+y=0$ $y_1=x$ $y_2=?$

$y'' - \underbrace{\frac{\cancel{x}}{\cancel{x}(x-1)}}_{p(x)}y' + \underbrace{\frac{1}{x(x-1)}}_{q(x)}y=0$ $y_2=uy_1=ux$

$=x\ln x+1$

$u=\int\frac{1}{(y_1)^2}e^{-\int p\,dx}dx = \int \frac{1}{x^2}e^{+\int \frac{1}{x-1}dx}dx = \int\frac{1}{x^2}e^{\ln|x-1|}dx$

$=\int \frac{1}{x^2}(x-1)dx = \int\frac{1}{x}dx - \int\frac{1}{x^2}dx$

$=\ln|x| + \frac{1}{x}$

∴ $y=C_1y_1+C_2y_2 = C_1x + C_2(x\ln x+1)$

○ $xy''+2y'+xy=0$ $y_1=\frac{\cos x}{x}$

$y''+\underbrace{\frac{2}{x}}_{p(x)}y'+y=0$ $y_2=uy_1 = u\frac{\cos x}{x} = \frac{\sin x}{\cancel{\cos x}}\cdot\frac{\cancel{\cos x}}{x} = \frac{\sin x}{x}$

$e^{-2\ln x} = x^{-2}$

$u=\int\frac{1}{(y_1)^2}e^{-\int p\,dx}dx = \int\frac{x^2}{\cos^2 x}e^{-\int\frac{2}{x}dx}dx$

$=\int\frac{\cancel{x^2}}{\cos^2x}\cdot\frac{1}{\cancel{x^2}}dx = \int\frac{1}{\cos^2x}dx = \int\sec^2 x\,dx = \tan x$

$y = C_1\frac{\cos x}{x} + C_2\frac{\sin x}{x}$

○ $x^2y''+xy'-4y=0$ $y''+\underbrace{\frac{1}{x}}_{p}y'-\frac{4}{x^2}y=0$ $y_1=x^2$

$y_2=uy_1=u\,x^2 = -\frac{1}{4}x^{-2}$

↳ 계수 필요없음 (C₂에 포함)

$u=\int\frac{1}{(y_1)^2}e^{-\int p\,dx}dx = \int\frac{1}{x^4}e^{-\int\frac{1}{x}dx}dx = \int\frac{1}{x^4}\cdot\frac{1}{x}dx$

$=\int\frac{1}{x^5}dx = -\frac{1}{4}x^{-4}$

$y=C_1x^2+C_2\cdot x^{-2}$

○ $y''-4y'+4y=0$ $y_1=e^{2x}$ $p(x)=-4$

$y_2=uy_1 = u\,e^{2x} = xe^{2x}$

$u=\int\frac{1}{(y_1)^2}e^{-\int p\,dx}dx = \int\frac{1}{e^{4x}}e^{\int 4dx} = \int\frac{1}{e^{4x}}e^{4x}dx = \int 1\,dx = x$

∴ $y=C_1e^{2x}+C_2xe^{2x} = e^{2x}(C_1+C_2x)$

- $(1-2x-x^2)y''+2(1+x)y'-2y=0 \qquad y_1=x+1$

$y''+\frac{2(1+x)}{1-2x-x^2}y'-\frac{2}{1-2x-x^2}y=0 \qquad \frac{2(1+x)}{1-2x-x^2}=p(x)$

$y_2=uy_1=u(x+1)=(-x-2\frac{1}{(x+1)})(x+1)=-x^2-x-2$
$=-(x^2+x+2)$

$u=\int\frac{1}{(y_1)^2}e^{-\int p dx}dx=\int\frac{1}{(x+1)^2}e^{-\int\frac{2x+2}{1-2x-x^2}dx}dx$

$t=1-2x-x^2 \qquad =\int\frac{1}{(x+1)^2}(1-2x-x^2)dx$

$\frac{dt}{dx}=-2x-2=-(2x+2)$

$e^{\int\frac{1}{t}dt}=e^{\ln|t|}=t=1-2x-x^2$

$\int\frac{1}{(x+1)^2}(1-2x-x^2)dx=\int\frac{-(x^2+2x+1)+2}{x^2+2x+1}dx$

$=\int -1\,dx+2\int\frac{1}{(x+1)^2}dx$

$=-x+2\int\frac{1}{k^2}dk=-x-2(x+1)^{-1}$

$y=C_1(x+1)+C_2(x^2+x+2)$

- $y=0$은 기저가 될 수 있을까? X
  기저의 조건에 위배 (independent 해야 함)

- 치환계수 내림 (단순 & 복합) 선형이 아닐 수 있음
  y'', y', x만 보이는 경우 $\boxed{y'=z\text{로 치환}}$

  y, y', y'' 모두 있는데 x가 안 보이는 경우 $\boxed{y'=z,\ y''=z\cdot\frac{dz}{dy}\text{로 치환}}$

- $y''-y'=0 \qquad y'=z$
  $z'-z=0 \qquad z'=z \qquad \int\frac{1}{z}dz=\int 1\,dx$

  $\ln|z|=x+C \qquad z=e^{x+C}=C_1e^x$

  $y'=C_1\cdot e^x \qquad y=C_1\int e^x dx=C_1e^x+C_2 \qquad \therefore y=C_1e^x+C_2$

- $2xy''=3y' \qquad y'=z$
  $2xz'=3z \qquad \frac{1}{z}z'=\frac{3}{2}\frac{1}{x} \qquad \ln|z|=\frac{3}{2}\int\frac{1}{x}dx$
  $=\frac{3}{2}\ln|x|+C_1$

  $y'=z=C_1x^{\frac{3}{2}} \qquad y=C_1\frac{2}{5}x^{\frac{5}{2}}+C_2=C_1x^{\frac{5}{2}}+C_2$

- $y''=1+(y')^2 \qquad y'=z=\tan(x+C_1)$
  $z'=1+z^2 \qquad \frac{1}{z^2+1}z'=1$

  $\tan^{-1}z=\int 1\cdot dx=x+C_1$

  $z=\tan(x+C_1)$

  $y=\int\tan(x+C_1)dx=\int\frac{\sin(x+C_1)}{\cos(x+C_1)}dx=-\ln|\cos(x+C_1)|+C_2$

- $xy''=y'+(y')^3 \qquad y'=z=(-1+\frac{C}{x^2})^{-\frac{1}{2}}$
  $xz'=z+z^3=z(1+z^2)$

  $\frac{1}{z(1+z^2)}z'=\frac{1}{x}$ 변수분리 안됨 → 베르누이 방정식으로

  $z'-\frac{1}{x}z=\frac{1}{x}z^3 \qquad u=z^{1-3}=z^{-2}$
  $u'=-2z^{-3}z'$

  $\times(-2z^{-3})$ 곱함

  $-2z^{-3}z'-\frac{1}{x}z(-2z^{-3})=\frac{1}{x}(-2)$

  $\underbrace{-2z^{-3}z'}_{u'}+\frac{2}{x}\underbrace{z^{-2}}_{u}=-\frac{2}{x} \qquad u'+\underbrace{\frac{2}{x}}_{p}u=\underbrace{-\frac{2}{x}}_{r}$

  $u=e^{-h}(\int e^h r dx+C) \qquad h=\int p dx=\int\frac{2}{x}dx=2\ln|x|$

  $=\frac{1}{x^2}(\int x^2(-\frac{2}{x})dx+C)=\frac{1}{x^2}(-\int 2x\,dx+C)$

  $=\frac{1}{x^2}(-x^2+C)=-1+\frac{C}{x^2}=z^{-2}$

  $z=(-1+\frac{C_1}{x^2})^{-\frac{1}{2}}$

  $y'=(-1+\frac{C_1}{x^2})^{-\frac{1}{2}}$

  $y=\int(-1+\frac{C_1}{x^2})^{-\frac{1}{2}}dx=\int(\frac{-x^2+C_1}{x^2})^{-\frac{1}{2}}dx=\int(\frac{x^2}{-x^2+C_1})^{\frac{1}{2}}dx$

  $=\int\frac{x}{\sqrt{-x^2+C_1}}dx=-\frac{1}{2}\int\frac{1}{\sqrt{t}}dt=-\frac{1}{2}\int t^{-\frac{1}{2}}dt$

  $t=-x^2+C_1 \qquad =-\frac{1}{2}2t^{\frac{1}{2}}+C_2=-t^{\frac{1}{2}}+C_2$
  $dt=-2x\,dx \qquad t^{-\frac{1}{2}}\xleftarrow{\text{미분}}2t^{\frac{1}{2}} \qquad =-(-x^2+C_1)^{\frac{1}{2}}+C_2$
  $=-\sqrt{C_1-x^2}+C_2$

○ 상수 계수를 가지는 이계 선형 ODE

[배경지식]

① 하나의 해 y_1이 주어졌을 때 $y_2 = u y_1$ $\quad u=\int \frac{1}{y_1^2} e^{-\int p dx} dx$

② $e^{ix} = \cos x + i\sin x$ 지수함수 e^x 삼각함수 $\sin x$ $\cos x$

$e^{-ix} = \cos x - i\sin x$

$e^{i\square} = \cos\square + i\sin\square$ $\quad e^{-i\square} = \cos\square - i\sin\square$

$\frac{e^{ix}+e^{-ix}}{2} = \cos x$ $\quad \frac{e^{ix}-e^{-ix}}{2i} = \sin x$

③ $y = C_1 y_1 + C_2 y_2$

$y'' + ay' + by = 0$ $\quad p(x)=a$ $\quad q(x)=b$

전략) $y=e^{\lambda x}$를 대입해서 λ를 구한다!

$y' = \lambda e^{\lambda x}$ $\quad y'' = \lambda^2 e^{\lambda x}$ $\quad y=e^{\lambda x}$

$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0$

$\underbrace{e^{\lambda x}}_{>0}(\lambda^2 + a\lambda + b) = 0$ $\quad \lambda^2 + a\lambda + b = 0$ 특성방정식

(i) λ가 2개의 실근을 가질 때

$y = C_1 e^{\lambda_1} + C_2 e^{\lambda_2}$ $\quad \lambda = \frac{-a \pm \sqrt{a^2-4b}}{2}$ $\quad a^2-4b>0$

(ii) λ가 중근을 가질 때

$\lambda_1 = -\frac{a}{2}$ $\quad a^2-4b=0$

(iii) λ가 복소근 2개를 가질 때

$\lambda_1 = -\frac{a}{2} + \omega i$ $\quad \lambda_2 = -\frac{a}{2} - \omega i$

(ii) $y_1 = e^{-\frac{a}{2}x}$ $\quad$ y_1에 x 곱한 결과 $\quad y = C_1 e^{-\frac{a}{2}x} + C_2 x e^{-\frac{a}{2}x}$

$y_2 = y_1 \int \frac{1}{(y_1)^2} e^{-\int a dx} dx$

$= e^{-\frac{a}{2}x} \int \cancel{e^{ax}}\, \cancel{e^{-ax}} dx = x e^{-\frac{a}{2}x}$

(iii) $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

$= C_1 e^{(-\frac{a}{2}+\omega i)x} + C_2 e^{(-\frac{a}{2}-\omega i)x}$

$= C_1 e^{-\frac{a}{2}x} e^{(\omega x)i} + C_2 e^{-\frac{a}{2}x} e^{-(\omega x)i}$

$= e^{-\frac{a}{2}x}(C_1 e^{(\omega x)i} + C_2 e^{-(\omega x)i})$

$= e^{-\frac{a}{2}x}\left(C_1(\cos(\omega x) + i\sin(\omega x)) + C_2(\cos(\omega x) - i\sin(\omega x))\right)$

$C_1\cos(\omega x) + C_1 i\sin(\omega x)$

$+ \ C_2\cos(\omega x) - C_2 i\sin(\omega x)$

$\cos(\omega x)(C_1+C_2) + i(C_1-C_2)\sin(\omega x)$

$C_1\cos(\omega x) + C_2\sin(\omega x)$

$\therefore y = e^{-\frac{a}{2}x}(C_1\cos(\omega x) + C_2\sin(\omega x))$

실수부 $\quad$ 허수부

○ $y'' + y' - 2y = 0$ $\quad y(0)=4$ $\quad y'(0)=-5$

$\downarrow y=e^{\lambda x}$

$\lambda^2 + \lambda - 2 = 0$

$\frac{-1\pm\sqrt{1-4(-2)}}{2}$

$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} = 3e^{-2x} + e^x$

$(\lambda+2)(\lambda-1)=0$

$\lambda_1 = -2$ $\quad \lambda_2 = 1$ $\quad y(0) = C_1 + C_2 = 4$

$y = C_1 e^{-2x} + C_2 e^x$ $\quad y'(0) = -2C_1 + C_2 = -5$

$y' = -2C_1 e^{-2x} + C_2 e^x$ $\quad 3C_1 = 9$ $\quad C_1 = 3$ $\quad C_2 = 1$

○ $2y'' - 5y' - 3y = 0$

$y'' - \frac{5}{2}y' - \frac{3}{2}y = 0$ $\quad \therefore y_1 = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

$\downarrow y=e^{\lambda x}$ $\quad = C_1 e^{-\frac{1}{2}x} + C_2 e^{3x}$

$\lambda^2 - \frac{5}{2}\lambda - \frac{3}{2} = 0$

$2\lambda^2 - 5\lambda - 3 = 0$ $\quad (2\lambda+1)(\lambda-3)=0$

$\lambda_1 = -\frac{1}{2}$ $\quad \lambda_2 = 3$

○ $y'' + 6y' + 9 = 0$ $\quad y_1 = e^{-3x}$ $\quad \therefore y = C_1 e^{-3x} + C_2 x e^{-3x}$

$\downarrow y=e^{\lambda x}$ $\quad y_2 = x e^{-3x}$

$\lambda^2 + 6\lambda + 9 = 0$ $\quad y_2 = e^{-3x}\int e^{6x} \cdot e^{-\int 6 dx} = e^{-3x}\int \cancel{e^{6x}}\cancel{e^{-6x}} dx$

$(\lambda+3)^2 = 0$ 중근 $\lambda = -3$ $\quad = x e^{-3x}$

○ $y'' - 10y' + 25 = 0$ $\quad y_1 = e^{5x}$

$\downarrow y=e^{\lambda x}$ $\quad y_2 = x e^{5x}$

$\lambda^2 - 10\lambda + 25 = 0$ $\quad y = C_1 e^{5x} + C_2 x e^{5x}$

$(\lambda-5)^2 = 0$ 중근 $\lambda = 5$

○ $y'' + 36y = 0$

$\downarrow y=e^{\lambda x}$ $\quad y = e^{0\cdot x}(C_1\cos 6x + C_2\sin 6x)$

$\lambda^2 + 36 = 0$ $\quad = C_1\cos 6x + C_2\sin 6x$

$\lambda_1 = 6i$ $\quad \lambda_2 = -6i$

○ $y'' + 4y' + 7y = 0$ $\quad y = e^{-2x}(C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x))$

$\downarrow y=e^{\lambda x}$

$\lambda^2 + 4\lambda + 7 = 0$

$\lambda = \frac{-4\pm\sqrt{16-4\cdot 7}}{2} = -2 \pm \frac{1}{2}\sqrt{16-28}$ $\quad (= 2\sqrt{3}i)$

$= -2 \pm \sqrt{3}i$

○ $3y'' + 2y' + y = 0$

$\downarrow y=e^{\lambda x}$ $\quad y = e^{-\frac{1}{3}x}\left(C_1\cos(\frac{\sqrt{2}}{3}x) + C_2\sin(\frac{\sqrt{2}}{3}x)\right)$

$3\lambda^2 + 2\lambda + 1 = 0$

$\lambda = \frac{-2\pm\sqrt{4-4\cdot 3\cdot 1}}{6}$

$= -\frac{1}{3} \pm \frac{1}{6}\sqrt{-8}$

$= -\frac{1}{3} \pm \frac{1}{\cancel{6}3}\cancel{2}\sqrt{2}i$

$= -\frac{1}{3} \pm \frac{1}{3}\sqrt{2}i$

° 오일러-코시 방정식

제차 2계 선형 방정식 $y''+p(x)y'+q(x)y=0$

	상수 계수	오일러 코시
식	$y''+ay'+by=0$	$y''+\frac{a}{x}y'+\frac{b}{x^2}y=0$ $x^2y''+axy'+by=0$
가정	$y=e^{\lambda x}$	$y=x^m$
특성방정식	$\lambda^2+a\lambda+b=0$	주의! $m^2+(a-1)m+b=0$
실근 2개	$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$	$y=C_1x^{m_1}+C_2x^{m_2}$
중근	$y=e^{\lambda x}(C_1+C_2x)$	$y=x^{m_1}(C_1+C_2\ln x)$
복소근 2개	$\lambda=\alpha\pm\beta i$ $y=e^{\alpha x}(C_1\cos(\beta x)+C_2\sin(\beta x))$	$\lambda=\alpha\pm\beta i$ $y=x^{\alpha}(C_1\cos(\beta\ln x)+C_2\sin(\beta\ln x))$

$$y''+\frac{a}{x}y'+\frac{b}{x^2}y=0 \qquad y=x^m \qquad y'=mx^{m-1}$$
$$x^2y''+axy'+by=0 \qquad y''=m(m-1)x^{m-2}$$

$$m(m-1)x^m+amx^m+bx^m=0$$
$$x^m\{m(m-1)+am+b\}=0 \qquad m^2+(a-1)m+b=0$$
$$x^m\{m^2+(a-1)m+b\}=0$$

$m^2+(a-1)m+b=0$ 이 중근을 가질 때

$$y_1=x^{m_1}$$
$$y_2=y_1\int\frac{1}{(y_1)^2}e^{-\int\frac{a}{x}dx}dx=x^{m_1}\int x^{-2m_1}e^{-a\ln x}dx$$
$$=x^{m_1}\int x^{-2m_1}x^{-a}dx=x^{m_1}\int x^{-2m_1-a}dx$$
$$m_1=\frac{-(a-1)}{2}=\frac{1-a}{2} \qquad =x^{m_1}\int x^{-1}dx=x^{m_1}\ln|x|$$
$$2m_1-1=-a \qquad a=1-2m_1$$

$m^2+(a-1)m+b=0$이 복소근 2개를 가질 때

$$m=\frac{-a+1\pm\sqrt{(a-1)^2-4b}}{2}=\frac{1-a}{2}\pm wi$$
$$y_1=x^m=x^{\frac{1-a}{2}+wi}=x^{\frac{1-a}{2}}x^{wi}$$
$$x^{wi}=e^{\ln x^{wi}}=e^{wi\ln x}=e^{i(w\ln x)}$$
$$=\cos(w\ln x)+i\sin(w\ln x)$$
$$x^{-wi}=e^{\ln x^{-wi}}=e^{-wi\ln x}=e^{-i(w\ln x)}$$
$$=\cos(w\ln x)-i\sin(w\ln x)$$

$$y=C_1y_1+C_2y_2=C_1x^{\frac{1-a}{2}}(\cos(w\ln x)+i\sin(w\ln x))$$
$$+C_2x^{\frac{1-a}{2}}(\cos(w\ln x)-i\sin(w\ln x))$$
$$=x^{\frac{1-a}{2}}\{C_1\cos(w\ln x)+C_1i\sin(w\ln x)+C_2\cos(w\ln x)-C_2i\sin(w\ln x)\}$$
$$=x^{\frac{1-a}{2}}\{\underbrace{(C_1+C_2)}_{C_1}\cos(w\ln x)+\underbrace{(C_1+C_2)}_{C_2}i\sin(w\ln x)\}$$
$$=x^{\underbrace{\frac{1-a}{2}}_{\text{실수부}}}\{C_1\cos(\underbrace{w}_{\text{허수부}}\ln x)+C_2\sin(\underbrace{w}_{\text{허수부}}\ln x)\}$$

(예) $x^2y''-\overset{a}{2}xy'-\overset{b}{4}y=0 \qquad y=x^m$
$$x^m\{m(m-1)-2m-4\}=0$$
$$m^2+(-2-1)m-4=0$$
$$m^2-3m-4=0$$
$$(m-4)(m+1)=0 \qquad m=4 \text{ or } m=-1$$
$$y=C_1x^4+C_2x^{-1}$$

(예) $x^2y''+\overset{a}{1.5}xy-\overset{b}{0.5}y=0 \qquad y=x^m$
$$x^m\{m(m-1)+1.5m-0.5\}=0$$
$$m^2+(1.5-1)m-0.5=m^2+0.5m-0.5=0$$
$$m^2+\frac{1}{2}m-\frac{1}{2}=0$$
$$2m^2+m-1=0 \qquad (2m-1)(m+1)=0 \qquad m=\frac{1}{2} \text{ or } m=-1$$
$$y=C_1x^{\frac{1}{2}}+C_2x^{-1}=C_1\sqrt{x}+\frac{C_2}{x}$$

(예) $x^2y''-2y=0 \qquad y=x^m$
$$x^m\{m(m-1)-2\}=0$$
$$m^2-m-2=(m-2)(m+1)=0 \qquad m=2 \text{ or } m=-1$$
$$y=C_1x^2+C_2x^{-1}$$

(예) $x^2y''-5xy'+9y=0 \qquad y=x^m$
$$x^m\{m(m-1)-5m+9\}=0$$
$$m^2-6m+9=(m-3)^2=0 \qquad m=3 \qquad y_1=x^3$$
$$\underbrace{y''-\frac{5}{x}}_{p}y'+\underbrace{\frac{9}{x^2}}_{q}y=0 \qquad y_2=y_1\int\frac{1}{(y_1)^2}e^{-\int p\,dx}dx$$
$$-\int p\,dx=\int\frac{5}{x}dx=5\ln x \qquad =x^3\int x^{-6}x^5dx=x^3\ln x$$
$$y=C_1x^3+C_2x^3\ln x=x^3(C_1+C_2\ln x)$$

☆ (예) $4x^2y''+8xy'+y=0 \qquad y=x^m$
$$4m(m-1)x^m+8mx^m+x^m=0$$
$$x^m\{4m(m-1)+8m+1\}=0$$
$$4m^2-4m+8m+1=4m^2+4m+1=(2m+1)^2=0 \qquad m=-\frac{1}{2}$$
$$y=x^{-\frac{1}{2}}(C_1+C_2\ln x)$$

좀더 빠르게 계산하려면 $x^2y''+2xy'+\frac{1}{4}y=0$ 로 먼저 고치고

$m^2+m+\frac{1}{4}=0$ 공식 적용

(예) $x^2y''-3xy'+10y=0 \qquad y=x^m$
$$x^m\{m(m-1)-3m+10\}$$
$$m^2-4m+10=0 \qquad m=\frac{4\pm\sqrt{16-4\cdot 10}}{2}=2\pm\frac{1}{2}\sqrt{-24}$$
$$=2\pm\frac{1}{2}2\sqrt{6}i=2\pm\sqrt{6}i$$
$$y=x^2(C_1\cos(\sqrt{6}\ln x)+C_2\sin(\sqrt{6}\ln x))$$

(예) $4x^2y'' + 5y = 0$ $\quad y = x^m$

$x^m\{4m(m-1) + 5\} = 0$

$4m^2 - 4m + 5 = 0 \quad m = \frac{-(-4) \pm \sqrt{16 - 4\cdot4\cdot5}}{2\cdot4} = \frac{4 \pm \sqrt{16-80}}{8}$

$= \frac{4 \pm \sqrt{-64}}{8} = \frac{4 \pm 8i}{8} = \frac{1}{2} \pm i$

$y = x^{\frac{1}{2}}\{C_1\cos(\ln x) + C_2\sin(\ln x)\}$

$y = x^{\frac{1}{2}\pm i} \quad e^{\ln x^{(\frac{1}{2}\pm i)}} = e^{(\frac{1}{2}\pm i)\ln x} = \underbrace{e^{\frac{1}{2}\ln x}}_{x^{\frac{1}{2}}} + \underbrace{e^{i\ln x}}_{\cos(\ln x) + i\sin(\ln x)}$

$C_1\cos(\ln x) + C_1 i\sin(\ln x)$

$C_2\cos(\ln x) - C_2 i\sin(\ln x)$ $\quad y = x^{\frac{1}{2}}(C_1\cos(\ln x) + C_2\sin(\ln x))$

$C_1\cos(\ln x) + C_2\sin(\ln x)$

Wronskian $\quad y = C_1y_1 + C_2y_2$

기저에서 해가 서로 독립

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ 0 이면 종속, 0 이 아니면 독립

$W = y_1y_2' - y_2y_1'$

주어진 함수들이 서로 독립인지 구하고

함수들이 기저가 되는 이계 제차 선형방정식 구하기

(예) $y_1 = \cos5x \quad y_2 = \sin5x \quad y(0) = 3 \quad y'(0) = 5$

$W = \begin{vmatrix} \cos5x & \sin5x \\ -5\sin5x & 5\cos5x \end{vmatrix} = 5\cos^2 5x + 5\sin^2 5x = 5 \neq 0$ 독립임!

$\lambda = \alpha \pm \beta i$

$y = e^0(C_1\cos5x + C_2\sin5x) \quad \alpha = 0 \quad \beta = 5 \quad \lambda = \pm5i$

$\lambda^2 + 0\cdot\lambda + 25 = 0 \quad y'' + 0y' + 25y = 0$

$y'' + 25y = 0$

$y = C_1\cos5x + C_2\sin5x \quad C_1 = 3 \quad C_2 = 1$

$y' = -5C_1\sin5x + 5C_2\cos5x \quad 5C_2 = 5$

$y = 3\cos5x + 5\sin5x$

(예) $y_1 = x^{m_1} \quad y_2 = x^{m_2}$ $(m_1 \neq m_2)$ $\quad y(1) = -2 \quad y'(1) = 2m_1 - 4m_2$

$y = x^m$

$y = C_1x^{m_1} + C_2x^{m_2}$ $\quad x^m\{m(m-1) + am + b\} = 0$

$m^2 + (a-1)m + b = 0$

$a - 1 = -(m_1 + m_2) \quad (m - m_1)(m - m_2) = 0$

$a = 1 - (m_1 + m_2) \quad m - (m_1 + m_2)m + m_1m_2 = 0$

$b = m_1m_2$

$x^2y'' + (1 - m_1 - m_2)xy' + m_1m_2y = 0$

$y(1) = C_1 + C_2 = -2 \quad C_1 = 2$

$y'(1) = m_1C_1x^{m_1-1} + m_2C_2x^{m_2-1} \quad C_2 = -4 \quad y = 2x^{m_1} - 4x^{m_2}$

$= m_1C_1 + m_2C_2 = 2m_1 - 4m_2$

비제차 선형 이계 ODE $y''+p(x)y'+q(x)y=r(x)$

일반해 $y=\underbrace{y_h}_{\text{제차해}}+\underbrace{y_p}_{\text{비제차해}}$

y_h는 여태까지 하던 방식으로 구하기

$r(x)=0$일 때의 해

$y''+p(x)y'+q(x)y=0$

y_p 구하기 ① 미정계수법 30%

② 매개변수법 70%

미정계수법

$r(x)$ 기반으로 y_p의 큰 틀을 잡고

$y''+p(x)y'+q(x)y=r(x)$에 대입해서 y_p를 구한다

$r(x)$	y_p
n차 다항함수	$ax^n+bx^{n-1}+cx^{n-2}+\cdots$
삼각함수 $\sin(wx)$ $\cos(wx)$	$A\cos(wx)+B\sin(wx)$
지수함수 e^{rx}	Ae^{rx}

(다항함수) × (지수함수) $(ax^2+bx+c)e^{rx}$

(예) $(3x^2+2x+1)e^{rx}$

(지수함수) × (삼각함수)

$e^{rx}\sin(wx)$ $Ae^{rx}\cos(wx)+Be^{rx}\sin(wx)$

$e^{rx}\cos(wx)$

(다항함수) × (삼각함수) $(ax^2+bx+c)\cos(wx)+(dx^2+ex+f)\sin(wx)$

$(3x^2+2x+1)e^{rx}$

초기값은 맨 마지막에 대입

만약에 y_p가 y_h에 속한다면 x 곱해서 속하지 않도록 하기

안 되면 또 x 곱해서 겹치지 않도록

합규칙 (예) $y''+py'+qy=\sin x+e^x$ r에 관해

$y_p=A\cos x+B\sin x$ $y_p=Ae^x$

(예) $y''+5y'+6y=2e^{-x}$

① y_h구하기: $y''+5y'+6y=0$ $y=e^{\lambda x}$

$\lambda^2+5\lambda+6=0$ $\lambda=-2$ or $\lambda=-3$

$y_h=c_1e^{-2x}+c_2e^{-3x}$

② y_p 구하기: $y_p=Ae^{-x}$ $y_p'=-Ae^{-x}$ $y_p''=Ae^{-x}$

$Ae^{-x}-5Ae^{-x}+6Ae^{-x}=2e^{-x}$

$2Ae^{-x}=2e^{-x} \Rightarrow A=1$ $y_p=e^{-x}$

$y=y_h+y_p=c_1e^{-2x}+c_2e^{-3x}+e^{-x}$

(예) $y''+3y'+2y=12x^2$

① y_h 구하기 $y''+3y'+2y=0$ $y=e^{\lambda x}$ $e^{\lambda x}(\lambda^2+3\lambda+2)=0$

$(\lambda+2)(\lambda+1)=0$ $\lambda=-2$ or $\lambda=-1$

$y_h=c_1e^{-2x}+c_2e^{-x}$

② y_p 구하기 $y_p=Ax^2+Bx+C$ $y_p'=2Ax+B$ $y_p''=2A$

$2A+3(2Ax+B)+2(Ax^2+Bx+C)=12x^2$ $A=6$ $B=-18$ $C=21$

$2Ax^2+(2B+6A)x+2A+3B+2C=12x^2$ $y_p=6x^2-18x+21$

$y_p=6x^2-18x+21$

(예) $y''-16y=2e^{4x}$

① y_h 구하기: $y''-16y=0$ $y=e^{\lambda x}$ $e^{\lambda x}(\lambda^2-16)=0$ $\lambda=\pm4$

$y_h=c_1e^{4}+c_2e^{-4}$ y_h에 포함됨

② y_p 구하기: $y_p=Ae^{4x}$ (X) $y_p=Axe^{4x}$ $y_p'=Ae^{4x}+\overbrace{Ax\,4e^{4x}}^{4Axe^{4x}}$

$y_p''=4Ae^{4x}+4Ae^{4x}+4Ax\cdot4e^{4x}$

$\cancel{16xe^{4x}}+8Ae^{4x}-\cancel{16Axe^{4x}}=2e^{4x}$ $A=\frac{1}{4}$ $y_p=\frac{1}{4}xe^{4x}$

$y=c_1e^{4}+c_2e^{-4}+\frac{1}{4}xe^{4x}$

(예) $y''-4y=8\cos(\pi x)$

① y_h 구하기: $y''-4y=0$ $y=e^{\lambda x}$ $e^{\lambda x}(\lambda^2-4)=0$ $\lambda=\pm2$

$y_h=c_1e^{2x}+c_2e^{-2x}$

② y_p 구하기: $y_p=A\cos(\pi x)+B\sin(\pi x)$

$y_p'=-\pi A\sin(\pi x)+\pi B\cos(\pi x)$

$y_p''=-\pi^2A\cos(\pi x)-\pi^2B\sin(\pi x)$

$-\pi^2A\cos(\pi x)-\pi^2B\sin(\pi x)-4A\cos(\pi x)-4B\sin(\pi x)=8\cos(\pi x)$

$(-\pi^2A-4A)\cos(\pi x)+(-\pi^2B-4B)\sin(\pi x)=8\cos(\pi x)$

$(-\pi^2-4)A=8$ $-\pi^2B-4B=0$ $(\pi^2+4)B=0$ $B=0$

$A=-\frac{8}{\pi^2+4}$ $y_p=-\frac{8}{\pi^2+4}\cos(\pi x)$

$y=c_1e^{2x}+c_2e^{-2x}-\frac{8}{\pi^2+4}\cos(\pi x)$

(예) $y'' + 4y = -12\sin(2x)$

① y_h 구하기: $y''+4y=0$ $\quad y=e^{\lambda x}$

$e^{\lambda x}(\lambda^2+4)=0 \quad \lambda=\pm 2i$

$y_1 = e^{0x}\cdot C_1\cos(2x) = C_1\cos(2x)$

$y_2 = e^{0x}\cdot C_2\sin(2x) = C_2\sin(2x)$

$y_h = C_1\cos(2x)+C_2\sin(2x)$

② y_p 구하기 $\quad y_p = A\cos(2x)+B\sin(2x)$ (X)

$y_p = x(A\cos(2x)+B\sin(2x))$

$y_p' = A\cos(2x)+B\sin(2x) + x(-2A\sin(2x)+2B\cos(2x))$

$y_p'' = -2A\sin(2x)+2B\cos(2x)+(-2A\sin(2x)+2B\cos(2x)) + x(-4A\cos(2x)-4B\sin(2x))$

$= -4A\sin(2x)+4B\cos(2x)-4Ax\cos(2x)-4Bx\sin(2x)$

$-4A\sin(2x)+4B\cos(2x)\cancel{-4Ax\cos(2x)-4Bx\sin(2x)}+\cancel{(4A\cos(2x)+4B\sin(2x))}x$

$= -12\sin(2x)$

$A=3 \quad B=0 \qquad y_p = 3x\cos(2x)$

$y = C_1\cos(2x)+C_2\sin(2x)+3x\cos(2x)$

(예) $y''-4y'+3y = e^x - \frac{9}{2}x$

① y_h 구하기 $\quad y''-4y'+3y=0 \quad y=e^{\lambda x} \quad \lambda^2-4\lambda+3=0$

$(\lambda-3)(\lambda-1)=0 \quad \lambda=3$ or 1

$y_h = C_1e^3 + C_2e^1$

② y_p 구하기 $\quad y_p = Ae^x + Bx + C$ (X) 합 규칙

$y_p = Axe^x + Bx + C \qquad y_p' = A(x+1)e^x + B$

$y_p'' = A(x+2)e^x \qquad y_p = -\frac{1}{2}xe^x - \frac{3}{2}x - 2$

$A(x+2)e^x - 4(A(x+1)e^x + B) + 3(Axe^x + Bx + C) = e^x - \frac{9}{2}x$

$\cancel{Axe^x+2Ae^x-4(Axe^x+Ae^x+B)+3Axe^x}+3Bx+3C$

$-2Ae^x + 3Bx - 4B + 3C = e^x - \frac{9}{2}x \qquad A = -\frac{1}{2}$

$3B = -\frac{9}{2} \qquad B = -\frac{3}{2} \qquad 6B+3C=0 \qquad C=-2$

$y = C_1e^3 + C_2e^1 - \frac{1}{2}xe^x - \frac{3}{2}x - 2$

매개변수법 $\quad y'' + p(x)y' + q(x)y = r(x)$

1) 먼저 y_h 구하고 $\quad y_1, y_2$ 구함

2) W 구하기 $\quad W = y_1 y_2' - y_1' y_2$

3) 아래 공식 사용해서 y_p 구하기

$$\boxed{y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx}$$

proof) Homogeneous case의 해와 독립이므로 $\underbrace{y_1, y_2}_{\text{Homogeneous case의 독립해}}$에 함수를 곱해서 y_p를 만들고 싶다

$y_p = u y_1 + v y_2 \quad (u, v : \text{not constant})$

$u' y_1 + v' y_2 = 0$이라고 가정

$q \quad y_p = (u y_1 + v y_2)\, q$

$p \quad y_p' = (u y_1' + \cancel{u' y_1 + v' y_2} + v y_2')\, p$

$+) \quad y_p'' = u' y_1' + u y_1'' + v' y_2' + v y_2''$

$r = u(\cancel{q y_1 + p y_1' + y_1''}) + v(\cancel{q y_2 + p y_2' + y_2''}) + u' y_1' + v' y_2'$

$$\underbrace{\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}}_{=W} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

Since $W(y_1(x), y_2(x)) \neq 0$, use cramer's rule

$$u'(x) = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{W} = \frac{-y_2 r}{W} \qquad v'(x) = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{W} = \frac{y_1 r}{W}$$

$$y_p = u y_1 + v y_2 = y_1 \int \left(-\frac{y_2 r}{W}\right) dx + y_2 \int \left(\frac{y_1 r}{W}\right) dx$$

$$= \boxed{-y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx}$$

단, r은 y''의 계수가 1일 때 공식 성립가능

(예) $y'' + y = \frac{1}{\cos x} \qquad p = 0 \quad q = 1 \quad r = \frac{1}{\cos x}$

y_h 구하기 $\quad y'' + y = 0 \qquad y = e^{\lambda x}$

$e^{\lambda x}(\lambda^2 + 1) = 0 \qquad \lambda = \pm i$

$y_h = e^0 (C_1 \cos(x) + C_2 \sin(x)) = C_1 \overset{y_1}{\cos x} + C_2 \overset{y_2}{\sin x}$

y_p 구하기 $\quad$ ① W 구하기 $\quad W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

$$= -\cos x \int \frac{\sin x}{\cos x} dx + \sin x \int \frac{\cos x}{\cos x} dx = \cos x \ln(\cos x) + x \sin x$$

$y = C_1 \cos x + C_2 \sin x + \cos x \ln(\cos x) + x \sin x$

(예) $x^2 y'' - x y' - 3y = x^2 \qquad y'' - \frac{1}{x} y' - \frac{3}{x^2} y = 1$

y_h 구하기 $\quad x^2 y'' - x y' - 3y = 0 \qquad y = x^m$

$x^m \{ m(m-1) - m - 3 \} = 0 \qquad m^2 - 2m - 3 = 0 \qquad (m-3)(m+1) = 0$

$m = 3$ or $m = -1$

$y_h = C_1 \overset{y_1}{x^3} + C_2 \overset{y_2}{x^{-1}}$

y_p 구하기 $\qquad W = \begin{vmatrix} x^3 & x^{-1} \\ 3x^2 & -x^{-2} \end{vmatrix} = -x - 3x = -4x \qquad r = 1$

$$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

$$= +x^3 \int \frac{x^{-1}}{4x} dx - x^{-1} \int \frac{x^3}{4x} dx = \frac{1}{4} x^3 \int x^{-2} dx - \frac{1}{4x} \int x^2 dx$$

$$= \frac{1}{4} x^3 (-x^{-1}) - \frac{1}{4} x^{-1} \cdot \frac{1}{3} x^3 = -\frac{1}{4} x^2 - \frac{1}{12} x^2$$

$$= -\frac{3}{12} x^2 - \frac{1}{12} x^2 = -\frac{4}{12} x^2 = -\frac{1}{3} x^2$$

$y = C_1 x^3 + C_2 x^{-1} - \frac{1}{3} x^2$