미분방정식 상계 이상 ODE

○ $y^{(4)}+2y^{(2)}+y=0 \qquad y=e^{\lambda x}$

$e^{\lambda x}(\lambda^4+2\lambda^2+1)=0 \qquad (\lambda^2+1)^2=0$

$\lambda=\pm i$

$y_1=\cos x \qquad y_2=\sin x$

$y_3=x\cos x \qquad y_4=x\sin x$

$y_h=C_1\cos x+C_2\sin x+C_3x\cos x+C_4x\sin x$

○ $y''+4y'-2y=2x^2-3x+6$

y_h 구하기 $\qquad y''+4y'-2y=0 \qquad y=e^{\lambda x}$

$e^{\lambda x}(\lambda^2+4\lambda-2)=0 \qquad \lambda=\frac{-4\pm\sqrt{16-4(-2)}}{2}$

$=\frac{-4\pm\sqrt{24}}{2}=\frac{-4\pm2\sqrt{6}}{2}=-2\pm\sqrt{6}$

$y_1=e^{(-2+\sqrt{6})x} \qquad y_2=e^{(-2-\sqrt{6})x}$

y_p 구하기 $\qquad y_p=Ax^2+Bx+C$

$y'_p=2Ax+B$

$y''_p=2A$

$2A+4(2Ax+B)-2(Ax^2+Bx+C)=2x^2-3x+6$

$=2A+8Ax+4B-2Ax^2-2Bx-2C=2x^2-3x+6$

$A=-1 \qquad -8-2B=-3 \qquad -5=2B \qquad B=-\frac{5}{2}$

$-2-10-2C=6 \qquad -12-2C=6$

$-18=2C \qquad C=-9 \qquad y_p=-x^2-\frac{5}{2}x-9$

$y=C_1e^{(-2+\sqrt{6})x}+C_2^{(-2-\sqrt{6})x}-x^2-\frac{5}{2}x-9$

○ $y'''+y''=e^x\cos x \qquad y=e^{\lambda x} \qquad e^{\lambda x}(\lambda^3+\lambda^2)=0$

$\lambda^2(\lambda+1)=0 \qquad \lambda^2=0$ or $\lambda=-1$

$y_1=e^{0x}=1 \qquad y_2=x \qquad y_3=e^{-x}$

$y_h=C_1+C_2x+C_3e^{-x}$

$y_p=(A\cos x+B\sin x)e^x$

$y'_p=(-A\sin x+B\cos x)e^x+(A\cos x+B\sin x)e^x$

$=\{(-A+B)\sin x+(A+B)\cos x\}e^x$

$y''_p=\{(-A+B)\cos x-(A+B)\sin x\}e^x+\{(-A+B)\sin x+(A+B)\cos x\}e^x$

$=(-2A\sin x+2B\cos x)e^x$

$y'''_p=(-2A\cos x-2B\sin x)e^x+(-2A\sin x+2B\cos x)e^x$

$=\{(-2A-2B)\sin x+(-2A+2B)\cos x\}e^x$

$\{(-2A-2B-2A)\sin x+(-2A+2B+2B)\cos x\}e^x=e^x\cos x$

$-4A-2B=0 \qquad -2A+4B=-2A-8A=-10A=1$

$B=-2A \qquad A=-\frac{1}{10} \qquad B=\frac{1}{5}$

$y_p=(-\frac{1}{10}\cos x+\frac{1}{5}\sin x)e^x$

$y=C_1+C_2x+C_3e^{-x}+(-\frac{1}{10}\cos x+\frac{1}{5}\sin x)e^x$

○ $y^{(4)} + y''' = 1 - x^2 e^{-x}$

y_h 구하기 $\quad y^{(4)} + y''' = 0 \qquad y = e^{\lambda x}$

$(\lambda^4 + \lambda^3) = \lambda^3(\lambda+1) = 0 \qquad \lambda^3 = 0$ or $\lambda = -1$

$y_1 = 1 \quad y_2 = x \quad y_3 = x^2 \quad y_4 = e^{-x}$

$y_h = C_1 + C_2 x + C_3 x^2 + C_4 e^{-x}$

y_p 구하기 $\quad Ax^3 + x(Bx^2 + Cx + D)e^{-x}$

○ $y'' - 4y' + 4y = (x+1)e^{2x}$

y_h 구하기 $\quad y'' - 4y' + 4y = 0 \qquad y = e^{\lambda x}$

$e^{\lambda x}(\lambda^2 - 4\lambda + 4) = 0 \qquad \lambda = 2$ (중근)

$y_1 = e^{2x} \qquad y_2 = xe^{2x}$

$y_h = C_1 e^{2x} + C_2 xe^{2x} \qquad r(x) = (x+1)e^{2x}$

매개변수법 $\quad y_p = y_1 \int \frac{W_1}{W} dx + y_2 \int \frac{W_2}{W} dx$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (2x+1)e^{2x} \end{vmatrix} = e^{4x}\cancel{(2x+1)} - \cancel{2xe^{4x}} = e^{4x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & (2x+1)e^{2x} \end{vmatrix} = -(x+1)xe^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = e^{4x}(x+1) - 0$$

$$y_p = e^{2x} \int \frac{-(x+1)x\cancel{e^{4x}}}{\cancel{e^{4x}}} dx + xe^{2x} \int \frac{\cancel{e^{4x}}(x+1)}{\cancel{e^{4x}}} dx =$$

$$= -e^{2x}\int (x^2+x)dx + xe^{2x}\int (x+1)dx$$

$$= -e^{2x}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right) + xe^{2x}\left(\frac{1}{2}x^2 + x\right)$$

$$= \left\{\left(\frac{1}{2} - \frac{1}{3}\right)x^3 + \left(1 - \frac{1}{2}\right)x^2\right\} e^{2x}$$

$$= \left(\frac{1}{6}x^3 + \frac{1}{2}\right)e^{2x}$$

$$y = C_1 e^{2x} + C_2 xe^{2x} + \left(\frac{1}{6}x^3 + \frac{1}{2}\right)e^{2x}$$

○ $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 3 + \ln x^3$

y_h 구하기 $\quad x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0 \qquad y = x^m$

$x^m\{m(m-1)(m-2) - 3m(m-1) + 6m - 6\} = 0$

$(m-1)\{m(m-2) - 3m + 6\}$

$= (m-1)(m-2)(m-3) = 0$

$y_1 = x \qquad y_2 = x^2 \qquad y_3 = x^3$

$y_h = C_1 x + C_2 x^2 + C_3 x^3$

$r(x) = \frac{3 + \ln x^3}{x^3}$

$$y_p = y_1 \int \frac{W_1}{W} dx + y_2 \int \frac{W_2}{W} dx + y_3 \int \frac{W_3}{W} dx$$

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = \begin{vmatrix} 1 & 2x & 3x^2 \\ 0 & -x^2 & -2x^3 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= 1 \cdot (-x^2 \cdot 6x + 4x^3) = -6x^3 + 4x^3 = -2x^3$$

$$W_1 = \begin{vmatrix} 0 & 2x & 3x^2 \\ 0 & -x^2 & -2x^3 \\ \frac{3+\ln x^3}{x^3} & 2 & 6x \end{vmatrix} = \frac{3+\ln x^3}{x^3}(-4x^4 + 3x^4)$$

$$= \frac{3+\ln x^3}{\cancel{x^3}}(-x^4) = -x(3+\ln x^3) = -3x(1+\ln x)$$

$$W_2 = \begin{vmatrix} 1 & 0 & 3x^2 \\ 0 & 0 & -2x^3 \\ 0 & \frac{3+\ln x^3}{x} & 6x \end{vmatrix} = 1 \cdot \left(0 + 2x^{\cancel{3}\,2} \frac{3\ln x^3}{\cancel{x}}\right)$$

$$= 6x^2 \ln x^3 = 18x^2 \ln x$$

$$W_3 = \begin{vmatrix} 1 & 2x & 0 \\ 0 & -x^2 & 0 \\ 0 & 2 & \frac{3+\ln x^3}{x} \end{vmatrix} = -x(3+\ln x^3)$$

$$= -3x(1+\ln x)$$

∘ $X'(t) = AX(t) \qquad A = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix}$

$x'(t) = -4x(t) + y(t) + z(t)$
$y'(t) = x(t) + 5y(t) - z(t)$
$z'(t) = y(t) - 3z(t)$

$X(t) = e^{\lambda t} \cdot k$ 라고 가정하자 $(\lambda I - A) = k$

$$\begin{vmatrix} \lambda+4 & -1 & -1 \\ -1 & \lambda-5 & 1 \\ 0 & -1 & \lambda+3 \end{vmatrix} = 0$$

$$(\lambda+4)\{(\lambda-5)(\lambda+3)+1\} + 1\{\underbrace{-(\lambda+3)-1}_{-\lambda-4 = -(\lambda+4)}\}$$

$$= (\lambda+4)(\lambda^2 - 2\lambda - 15 \cancel{+1} \cancel{-1})$$

$$= (\lambda+4)(\lambda-5)(\lambda+3) = 0$$

$\lambda = -4$일 때

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & -9 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ -1 & -9 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -10 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - 10k_3 = 0 \qquad k_3 = t \quad k_2 = -t$
$k_2 + k_3 = 0 \qquad k_1 = 10t$

$k = \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix} \qquad X_1(t) = e^{-4t} \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix}$

$\lambda = 5$일 때

$$\left[\begin{array}{ccc|c} 9 & -1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 8 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_3 = 0 \qquad k_3 = t \qquad k_1 = t$
$-k_2 + 8k_3 = 0 \qquad k_2 = 8t$

$k = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix} \qquad X_2(t) = e^{5t} \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$

$\lambda = -3$일 때

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -1 & -8 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_3 = 0 \qquad k_2 = 0 \qquad k_3 = t$
$k_1 = t$

$X_3(t) = e^{-3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$X(t) = c_1 e^{-4t} \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$