미분방정식 일계ODE

변수 분리형

○ $y' = ky$ (k는 상수)

$\frac{y'}{y} = k$ $\frac{1}{y}dy = kdx$

$\ln|y| = kx + C$ $y = e^{kx+C} \to y = Ce^{kx}$

○ $y' + 2xy = 0$ $y(0) = 3$

$y' = -2xy$ $\frac{1}{y}y' = -2x$

$\frac{1}{y}y' = -2x\,dx$ $\ln|y| = -x^2 + C$

$y = e^{-x^2+C} = Ce^{-x^2}$ $3 = Ce^0 = C$

$\therefore y = 3e^{-x^2}$

○ $y' = 1 + y^2$ $\frac{1}{1+y^2}dy = dx$

$\tan^{-1}y = x + C$ $y = \tan(x+c)$

○ $y^3y' + x^3 = 0$ $y^3y' = -x^3$

$y^3dy = -x^3dx$

$\frac{1}{4}y^4 = -\frac{1}{4}x^4 + C$ $y^4 = -x^4 + C$

$x^4 + y^4 = C$

○ $y' = \sec^2 y$ $y' = \frac{1}{\cos^2 y}$

$\cos^2 y\, y' = 1$ $\cos^2 y\, dy = dx$

$\frac{1+\cos 2y}{2}dy = dx$ $(1+\cos 2y)dy = 2dx$

$y + \frac{1}{2}\sin 2y = 2x + c$

$2y + \sin 2y = 4x + c$

○ 삼각함수의 역함수 미분

$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ $(\tan x)' = \sec^2 x$

$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ $(\sec x)' = \sec x \tan x$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ $(\csc x)' = -\csc x \cot x$

$(\cot x)' = -\csc^2 x$

$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

○ 반각 공식

$\cos^2 y = \frac{1+\cos 2y}{2}$

$\sin^2 y = \frac{1-\cos 2y}{2}$

$\tan^2 y = \frac{1-\cos 2y}{1+\cos 2y}$

○ $y' = e^{2x-1} y^2$ $\frac{1}{y^2} y' = e^{2x-1}$

$\frac{1}{y^2} dy = e^{2x-1} dx$

$-y^{-1} = \frac{1}{2} e^{2x-1} + C$

$y^{-2} \xrightarrow{적분} -y^{-1}$

$y^{-1} = \underbrace{-\frac{1}{2}e^{2x-1} - C}_{\frac{e^{2x-1}+C}{-2}}$

$y = \frac{-2}{e^{2x-1}+C}$

○ $(e^x+e^{-x}) y' = y^2$ $\frac{1}{y^2} y' = \frac{1}{e^x+e^{-x}}$

$\frac{e^x}{e^{2x}+1} = \frac{e^x}{(e^x)^2+1}$

$e^x = t$ $e^x dx = dt$

$-y^{-1} = \int \frac{1}{t^2+1} dt$

$= \tan^{-1}(t) + C$

$= \tan^{-1}(e^x) + C$

$y = -\frac{1}{\tan^{-1}(e^x)+C}$

○ $y' = (y-1)^2$ ① $y(0)=1$ ② $y(0)=1.01$

$\frac{1}{(y-1)^2} y' = 1$ $x=0$일 때 분모에 0이 올 수 X

$-(y-1)^{-1} = x+C$ ① $y(x)=1$

$(y-1)^{-1} = -x+C$

② $(y-1)^{-1} = -x+C$

$\frac{1}{0.01} = C$ $C=100$

$y-1 = \frac{-1}{x-C}$ $y = \frac{x-C-1}{x-C} = \frac{x-101}{x-100}$

○ Euler method

$y' = f(x,y)$ $y(x_0)=y_0$

$y(x_0+\Delta x) \approx \underbrace{f(x_0,y_0)}_{(x_0,y_0)에서의\ 미분계수}\Delta x + y_0$

$(x_0+\Delta x,\ f(x_0,y_0)\Delta x + y_0)$

예) $y' = f(x,y) = y+x$ $y(0)=0$

$x_0 = 0$ $y(x_0) = y_0$

$\frac{dy}{dx} = y+x$ $dy = (y+x)dx$

$\underbrace{(y+x)}_{P}dx + \underbrace{(-1)}_{Q}dy = 0$

$P = \frac{\partial u}{\partial x} = y+x$ $Q = \frac{\partial u}{\partial y} = -1$

$\frac{\partial P}{\partial y} = 1 \neq \frac{\partial Q}{\partial x} = 0$ $FP\,dx + FQ\,dy = 0$

$F = e^{\int R(x)dx} = e^{\int -1 dx} = e^{-x+\cancel{C}}$ $F = e^{\int R(y)dy}$

$R(x) = \frac{P_y - Q_x}{Q}$ $R(y) = \frac{Q_x - P_y}{P}$

$= \frac{1-0}{-1} = -1$

$\underbrace{e^{-x}(y+x)}_{=M=u_x}dx - \underbrace{e^{-x}}_{=N=u_y}dy = 0$

$u = \int e^{-x}(y+x)\,dx + g(y)$

$= y\int e^{-x} + \int xe^{-x}dx + g(y)$

$= -ye^{-x} + (-1)xe^{-x} - \int e^{-x}dx + g(y)$

$= -ye^{-x} - xe^{-x} + e^{-x} + g(y)$

$u_y = -e^{-x} + g'(y) = -e^{-x}$ $g'(y)=0$ $g(y)=c$

$u = -ye^{-x} - xe^{-x} + e^{-x} = C$

$y + x + 1 = ce^x$ $0 = C-0-1 = C-1$

$y = ce^x - x - 1$ $C=1$

$= e^x - x - 1$

- $y + \sqrt{y'} = 1$ $\qquad \sqrt{y'} = 1 - y$

$y' = (1-y)^2$ $\qquad \frac{1}{(1-y)^2} y' = 1$

$(1-y)^{-1} = x + C$

$1 - y = \frac{1}{x+C}$ $\qquad y = -\frac{1}{x+C} + 1$

$y(x_0 + \Delta x) \approx f(x_0, y_0)\Delta x + y_0$ $\qquad \Delta x = 0.2$

$y' = f(x, y) = y + x$

n	x_n	y_n	$y(x_n)$
0	$x_0 = 0$	$y_0 = 0$	$y_0 = 0$
1	$x_0 + \Delta x = 0.2$	$f(x_0, y_0)\Delta x + y_0 = 0 \cdot 0.2 + 0 = 0$	$e^{0.2} - 0.2 - 1$
2	$x_1 + \Delta x = 0.4$	$f(x_1, y_1)\Delta x + y_1 = 0.2 \cdot 0.2 + 0 = 0.04$	$e^{0.4} - 0.4 - 1$

치환 변수 분리형

○ $2xyy' = y^2 - x^2$ $\quad 2\left(\frac{y}{x}\right)y' = \left(\frac{y}{x}\right)^2 - 1$

$\frac{y}{x} = u \qquad y = ux \qquad y' = u'x + u$

$2u(u'x + u) = u^2 - 1 \qquad \frac{2u\,u'}{u^2+1} = -\frac{1}{x}$

$2uu'x + 2u^2 = u^2 - 1 \qquad u^2+1 = t \qquad \frac{2u}{u^2+1}du = -\frac{1}{x}dx$

$2uu'x = -u^2 - 1 \qquad 2u\,du = dt$

$\frac{1}{t}dt = -\frac{1}{x}dx$

$\ln|t| = -\ln|x| + C$

$\ln(u^2+1) = -\ln x + C \qquad u^2+1 = e^{-\ln x + C} = Ce^{-\ln x}$

$u^2 = Ce^{-\ln x} - 1 \qquad \left(\frac{y}{x}\right)^2 = Ce^{-\ln x} - 1 = C\cdot\frac{1}{x} - 1$

$y^2 = Cx^2\underbrace{e^{-\ln x}}_{\frac{1}{x}} - x^2 = Cx - x^2$

$\therefore x^2 + y^2 = Cx$

○ $xy' = y^2 + y \qquad \frac{1}{y^2+y}y' = \frac{1}{x}$

$\frac{1}{y(y+1)}y' = \frac{1}{x} \qquad \left(\frac{1}{y} - \frac{1}{y+1}\right)dy = \frac{1}{x}dx$

$\ln|y| - \ln|y+1| = \ln x + C$

$\ln\frac{y}{y+1} = \ln x + C \qquad \frac{y}{y+1} = e^{\ln x + C} = Ce^{\ln x} = Cx$

$\frac{y+1-1}{y+1} = 1 - \frac{1}{y+1} = Cx \qquad 1 - Cx = \frac{1}{y+1}$

$y+1 = \frac{1}{1-Cx} \qquad y = \frac{1-(1-Cx)}{1-Cx} = \frac{Cx}{1-Cx} = \frac{x}{-x+C}$

○ $xy' = y + 2x^3\sin^2\left(\frac{y}{x}\right) \qquad y' = \frac{y}{x} + 2x^2\sin^2\left(\frac{y}{x}\right)$

$\frac{y}{x} = u \qquad y = ux \qquad y' = u'x + u$

$u'\cancel{x} + \cancel{u} = \cancel{u} + 2x^{\cancel{2}}\sin^2 u \qquad \frac{1}{\sin^2 u}u' = 2x$

$\csc^2 u\ u' = 2x \qquad -\cot u = x^2 + C$

$\cot u = -x^2 + C \qquad \frac{y}{x} = \cot^{-1}(-x^2+C) \qquad y = x\cot^{-1}(-x^2+C)$

$\frac{1}{\tan u} = -x^2 + C \qquad \tan u = \frac{1}{-x^2+C}$

$\frac{y}{x} = \tan^{-1}\left(-\frac{1}{x^2+C}\right) \qquad y = x\tan^{-1}\left(-\frac{1}{x^2+C}\right)$

$= -x\tan^{-1}\left(\frac{1}{x^2+C}\right)$

완전 상미분 방정식

$u(x,y)=C$ $\quad du = u_x\,dx + u_y\,dy = 0$

$P\,dx + Q\,dy = 0$ $\quad P = u_x \quad Q = u_y$

$u_{xy} = u_{yx}$ $\quad P_y = Q_x$가 같아야함

같지 않으면 적분인자 구해서 곱함

$F(x) = e^{\int R(x)dx}$ $\quad R(x) = \dfrac{P_y - Q_x}{Q}$

$F(y) = e^{\int R(y)dy}$ $\quad R(y) = \dfrac{Q_x - P_y}{P}$

$\underbrace{FP}_{M = u_x}dx + \underbrace{FQ}_{N = u_y}dy = 0$

$u = \int M\,dx + g(y)$ $\quad u_y = \dfrac{\partial}{\partial y}\left(\int M\,dx\right) + g'(y)$

$= N$

○ $\underbrace{\cos(x+y)}_{P = u_x}dx + (\underbrace{3y^2+2y+\cos(x+y)}_{Q = u_y})\,dy = 0$

$P_y = -\sin(x+y)$ $\quad P_y = Q_x$

$Q_x = -\sin(x+y)$ $\quad$ exact!

$u = \int \cos(x+y)\,dx + g(y)$

$= \sin(x+y) + g(y)$

$u_y = \cos(x+y) + g'(y) = 3y^2+2y+\cos(x+y)$

$g'(y) = 3y^2+2y$

$g(y) = y^3 + y^2$

$u = \sin(x+y) + y^3 + y^2 = C$

$u = f(x,y)$

○ $(\underbrace{e^{x+y} + ye^y}_{P = u_x})dx + (\underbrace{xe^y - 1}_{Q = u_y})dy = 0$

$P_y = e^{x+y} + ye^y + e^y$ $\quad P_y \neq Q_x$

$Q_x = e^y$

$F(x) = e^{\int R(x)dx}$ $\quad F(y) = e^{\int R(y)dy}$

$R(x) = \dfrac{P_y - Q_x}{Q} = \dfrac{e^{x+y}+ye^y+\cancel{e^y} - \cancel{e^y}}{xe^y - 1}$

$R(y) = \dfrac{Q_x - P_y}{P} = \dfrac{\cancel{e^y} - e^{x+y} - ye^y - \cancel{e^y}}{e^{x+y}+ye^y} = -1$

$F(y) = e^{\int R(y)dy} = e^{\int -1\,dy} = e^{-y} \cancel{+c}$ $\quad 0$

$e^{-y}(e^{x+y}+ye^y)dx + e^{-y}(xe^{y}-1)dy = 0$

$(\underbrace{e^x + y}_{M = u_x})dx + (\underbrace{x - e^{-y}}_{N = u_y})dy = 0$

$u = \int M\,dx + g(y) = \int (e^x + y)\,dx + g(y)$

$= e^x + yx + g(y)$

$u_y = x + g'(y) = x - e^{-y} = N$

$g'(y) = -e^{-y}$ $\quad g(y) = e^{-y}$

$u = e^x + yx + e^{-y} = C$

○ $\underbrace{2x\tan y}_{P = u_x}\,dx + \underbrace{\sec^2 y}_{Q = u_y}\,dy = 0$ $\quad U = f(x,y) = C$

변수분리형 시도도 가능할 듯?

$P_y = 2x\sec^2 y$ $\quad P_y \neq Q_x$

$Q_x = 0$

$F(x) = e^{\int R(x)dx}$ $\quad R(x) = \dfrac{P_y - Q_x}{Q}$

$F(x) = e^{\int 2x\,dx} = e^{x^2}$ $\quad = \dfrac{2x\,\cancel{\sec^2 y}}{\cancel{\sec^2 y}} = 2x$

$\underbrace{e^{x^2}2x\tan y}_{M = u_x}\,dx + \underbrace{e^{x^2}\sec^2 y}_{N = u_y}\,dy = 0$

$u = e^{x^2}\tan y + g(y)$

$u_y = \cancel{e^{x^2}\sec^2 y} + \underbrace{g'(y)}_{0} = \cancel{e^{x^2}\sec^2 y}$

$U = e^{x^2}\tan y = C$ $\quad g(y) = c$

○ $\underbrace{(1+2x)\cos^2 y}_{P = u_x}\,dx + dy = 0$ $\quad u = f(x,y) = c$

$1 = Q = u_y$

$P_y = -(1+2x)\,2\cos y \sin y \qquad P_y \neq Q_x$

$Q_x = 0$

$F(x) = e^{\int R(x)dx}$

$R(x) = \dfrac{P_y - Q_x}{Q} = \dfrac{-2(1+2x)\cos y \sin y - 0}{1}$

$F(y) = e^{\int R(y)dy}$

$R(y) = \dfrac{Q_x - P_y}{P} = \dfrac{0 + \cancel{(1+2x)}\,2\cancel{\cos y}\sin y}{\cancel{(1+2x)}\cos^{2} y}$

$= 2\tan y$

$F(y) = e^{2\int \frac{\sin y}{\cos y}dy} = e^{-2\int \frac{1}{t}dt} = e^{-2\ln|t|}$

$= \dfrac{1}{t^2} = \dfrac{1}{\cos^2 y}$

$\underbrace{(1+2x)}_{M = u_x}\,dx + \underbrace{\dfrac{1}{\cos^2 y}}_{N = u_y}\,dy = 0$

$u = \int (1+2x)\,dx + g(y)$

$= x + x^2 + g(y)$

$u_y = g'(y) = \dfrac{1}{\cos^2 y} \qquad g(y) = \tan y$

$= \sec^2 y$

$u = x + x^2 + \tan y = C$

○ $\underbrace{xy}_{P = u_x}\,dx + \underbrace{(2x^2+3y^2-20)}_{Q = u_y}\,dy = 0$ $\quad u = f(x,y) = C$

$P_y = x \qquad P_y \neq Q_x$

$Q_x = 4x$

$F(x) = e^{\int R(x)dx} \qquad R(x) = \dfrac{P_y - Q_x}{Q}$

$F(y) = e^{\int R(y)dy} \qquad R(y) = \dfrac{Q_x - P_y}{P}$

$= e^{\int \frac{3}{y}dy} \qquad = \dfrac{4x - x}{xy} = \dfrac{3\cancel{x}}{\cancel{x}y}$

$= e^{3\ln|y|} = y^3$

$\underbrace{y^4 x}_{M = u_x}\,dx + \underbrace{y^3(2x^2+3y^2-20)}_{N = u_y} = 0$

$u = \int M\,dx + g(y) = \frac{1}{2}x^2y^4 + g(y)$

$u_y = 2x^2y^3 + g'(y) = 2x^2y^3 + 3y^5 - 20y^3$

$g'(y) = 3y^5 - 20y^3$

$g(y) = \frac{1}{2}y^6 - 5y^4$

$u = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C$

$x^2y^4 + y^6 - 10y^4 = C$

$\circ$ $(x+2y)y' + (x+y+1)y = 0$

$$\underbrace{(x+y+1)y}_{P=u_x}\,dx + \underbrace{(x+2y)}_{Q=u_y}\,dy = 0$$

$P = yx + y^2 + y \qquad Q = x + 2y$

$P_y = x + 2y + 1 \neq Q_x = 1$

$F(x) = e^{\int R(x)dx} = e^{\int 1 dx} = e^x$

$$R(x) = \frac{P_y - Q_x}{Q} = \frac{x+2y+\cancel{1}-\cancel{1}}{x+2y} = 1$$

$$\underbrace{e^x(xy+y^2+y)}_{M=u_x}\,dx + \underbrace{e^x(x+2y)}_{N=u_y}\,dy = 0$$

$u = g(x) + \int N dy = g(x) + e^x \int (x+2y)dy$

$= g(x) + e^x(xy+y^2) = g(x) + e^x xy + e^x y^2$

$u_x = g'(x) + \cancel{y}e^x(x+1)) + y^2 e^x$

$= \underbrace{g'(x)}_{0} + \underbrace{xe^x y + ye^x + y^2 e^x}_{e^x(xy+y+y^2)} \qquad g(x) = c$

$u = xe^x y + e^x y^2 = c$

$e^x(x+y)y$

$\circ$ $\underbrace{\cos x}_{P=u_x}\,dx + \underbrace{\left(1+\frac{2}{y}\right)\sin x}_{Q=u_y}\,dy = 0 \qquad u = f(x,y) = c$

$P_y = 0 \qquad P_y \neq Q_x$

$Q_x = \left(1+\frac{2}{y}\right)\cos x$

$$F(x) = e^{\int R(x)dx} = e^{\int \frac{\cos x}{\sin x}dx} = e^{-\int \frac{1}{t}dt} = e^{-\ln|\sin x|} = \frac{1}{\sin x}$$

$$R(x) = \frac{P_y - Q_x}{Q} = \frac{0 - \cancel{\left(1+\frac{2}{y}\right)}\cos x}{\cancel{\left(1+\frac{2}{y}\right)}\sin x} = -\frac{\cos x}{\sin x}$$

$$\underbrace{\frac{\cos x}{\sin x}}_{M=u_x}\,dx + \underbrace{\left(1+\frac{2}{y}\right)}_{N=u_y}\,dy = 0$$

$u = \int M dx + g(y) = \int \frac{\cos x}{\sin x}dx + g(y)$

$= \int \frac{1}{t}dt + g(y) = \ln \sin x + g(y)$

$u_y = g'(y) = \left(1+\frac{2}{y}\right)$ ☆ 꼭 잊지 말기!

$g(y) = y + 2\ln y$

$u = \ln \sin x + y + 2\ln y = C$

선형 상미분 방정식

$y' + p(x)y = r(x)$

$y = e^{-h}\left(\int e^{h} r\,dx + C\right) \qquad h = \int p\,dx$

○ $y' + \underbrace{(\tan x)}_{p}y = \underbrace{\sin 2x}_{r} \qquad y(0) = 1$

$y = e^{-h}\left(\int e^{h} r\,dx + C\right) \qquad h = \int p\,dx$

$= e^{\ln\cos x}\left(\int e^{-\ln\cos x}\sin 2x\,dx + C\right) \qquad = \int \tan x\,dx$

$\qquad = \int \frac{\sin x}{\cos x}\,dx$

$= \cos x\left(\int \frac{1}{\cos x}\sin 2x\,dx + C\right) \qquad = -\ln(\cos x)$

$= \cos x\left(\int \frac{2\sin x\cancel{\cos x}}{\cancel{\cos x}}dx + C\right)$

$= \cos x\left(2\int \sin x\,dx + C\right) = \cos x\left(-2\cos x + C\right)$

$= -2\cos^2 x + C\cos x \qquad = -2\cos^2 x + 3\cos x$

$y(0) = 1 \qquad -2 + C = 1 \qquad C = 3$

○ $y' = 2y - 4x$ ← linear $\qquad y' - 2y = -4x$

$p(x) = -2 \qquad r(x) = -4x$

$y = e^{-h}\left(\int e^{h} r\,dx + C\right) \qquad h = \int p(x)\,dx$

$h = \int (-2)\,dx = -2x$

$y = e^{2x}\left(\int e^{-2x}(-4x)\,dx + C\right)$

$= e^{2x}\left(-4\int x e^{-2x}\,dx + C\right)$

$= e^{2x}\left(-4\left[-\frac{1}{2}xe^{-2x} + \int +\frac{1}{2}e^{-2x}\,dx\right] + C\right)$

$= e^{2x}\left(2xe^{-2x} + \cancel{4}\cdot\frac{1}{\cancel{2}}\left(+\frac{1}{\cancel{2}}\right)e^{-2x} + C\right)$

$= e^{2x}\left(2xe^{-2x} + e^{-2x} + C\right)$

$= 2x + 1 + Ce^{2x} \qquad \therefore y = 2x + 1 + Ce^{2x}$

○ $y' + (\sin x)y = e^{\cos x} \qquad p(x) = \sin x \qquad r(x) = e^{\cos x}$

$y = e^{-h}\left(\int e^{h} r\,dx + C\right) \qquad h = \int p(x)\,dx$

$h = \int \sin x\,dx = -\cos x$

$y = e^{\cos x}\left(\int e^{-\cos x}e^{\cos x}\,dx + C\right)$

$= e^{\cos x}\left(\int 1\,dx + C\right) = e^{\cos x}(x + C)$

베르누이 방정식

- $xy' + y = x^2y^2 \rightarrow y' + \frac{1}{x}y = xy^2$

  $u = y^{1-2} = y^{-1}$

  $u' = -y^{-2}y'$

  $-y^{-2}$ 곱함

$$\underbrace{-y^{-2}y'}_{u'} - \frac{1}{x}y^{-1} = -x$$

$$u' - \underbrace{\frac{1}{x}}_{p(x)}u = \underbrace{-x}_{r(x)}$$

$$u = e^{-h}\left(\int e^h r\,dx + C\right) \qquad h = \int p(x)dx$$

$$h = -\int \frac{1}{x}dx = -\ln x$$

$$u = e^{\ln x}\left(\int e^{-\ln x}\cdot(-x)dx + C\right)$$

$$= x\left(-\int \frac{1}{\cancel{x}}\cancel{x}\,dx + C\right)$$

$$= x(-x + C) = -x^2 + Cx$$

$$y^{-1} = (-x^2 + Cx) \qquad y = \frac{1}{-x^2 + Cx}$$

$$y = e^{-h}\left(\int e^h r\,dx + C\right) \qquad h = \int p(x)dx$$

$$h = -\int \frac{2x}{1+x^2}dx = -\int \frac{1}{t}dt = -\ln|t|$$

$1 + x^2 = t$

$2x\,dx = dt$

$$= -\ln(1+x^2)$$

- $\sqrt{y}(y + y') = 1 \qquad y' + y = \frac{1}{\sqrt{y}} = y^{-\frac{1}{2}}$

  $u = y^{1+\frac{1}{2}} = y^{\frac{3}{2}}$

  $u' = \frac{3}{2}y^{\frac{1}{2}}y'$

  $\times \frac{3}{2}y^{\frac{1}{2}}$

$$\underbrace{\frac{3}{2}y^{\frac{1}{2}}y'}_{u'} + \frac{3}{2}\underbrace{y^{\frac{3}{2}}}_{u} = \frac{3}{2}$$

$$u' + \underbrace{\frac{3}{2}}_{p}u = \underbrace{\frac{3}{2}}_{r}$$

$$u = e^{-h}\left(\int e^h r\,dx + C\right) \qquad h = \int p\,dx = \int \frac{3}{2}dx = \frac{3}{2}x$$

$$= e^{-\frac{3}{2}x}\left(\int e^{\frac{3}{2}x}\cdot\frac{3}{2}dx + C\right) = e^{-\frac{3}{2}x}\left(e^{\frac{3}{2}x} + C\right)$$

$$\frac{3}{2}\int e^{\frac{3}{2}x}dx \qquad = 1 + C\cdot e^{-\frac{3}{2}x}$$

$$y^{\frac{3}{2}} = 1 + C\cdot e^{-\frac{3}{2}x}$$

$$y\sqrt{y} = 1 + C\cdot e^{-\frac{3}{2}x}$$

- $3(1+x^2)y' = 2xy(y^3 - 1)$

$$= 2xy^4 - 2xy$$

$$y' = \frac{2x}{3(1+x^2)}y^4 - \frac{2x}{3(1+x^2)}y$$

$$y' + \frac{2x}{3(1+x^2)}y = \frac{2x}{3(1+x^2)}y^4$$

$u = y^{1-4} = y^{-3}$

$u' = -3y^{-4}y'$

$\times(-3y^{-4})$

$$\underbrace{-3y^{-4}y'}_{u'} + \frac{2x}{\cancel{3}(1+x^2)}y(-\cancel{3}y^{-4}) = \frac{2x}{\cancel{3}(1+x^2)}y^4(-\cancel{3}y^{-4})$$

$$u' - \underbrace{\frac{2x}{1+x^2}}_{p}y^{-3} = \underbrace{-\frac{2x}{1+x^2}}_{r}$$

$$y = e^{\ln(1+x^2)}\left(\int e^{-\ln(1+x^2)}\cdot\left(-\frac{2x}{1+x^2}\right)dx + C\right)$$

$$= (1+x^2)\left(-\int \frac{2x}{(1+x^2)^2}dx + C\right)$$

$1+x^2 = t$

$2x\,dx = dt$

$$= (1+x^2)\left(-\int \frac{1}{t^2}dt + C\right)$$

$t^{-2} \xrightarrow[\text{적분}]{} -t^{-1}$

$$= (1+x^2)\left(+(+t^{-1}) + C\right)$$

$$= (1+x^2)\left(\frac{1}{1+x^2} + C\right)$$

$$= 1 + C(1+x^2) = y^{-3}$$

$$y = \left(1 + C(1+x^2)\right)^{-\frac{1}{3}}$$

∘ $y' = y(xy^3 - 1)$

$y' = xy^4 - y$ $\qquad$ $y' + y = xy^4$ $\qquad$ $\times(-3y^{-4})$

$u = y^{1-4} = y^{-3}$

$u' = -3y^{-4}y'$

$$\underbrace{-3y^{-4}y'}_{u'} + (-3y^{-4})y = (\cancel{-3y^{-4}})x\cancel{y^4}$$

$$u' \underbrace{-3}_{p}u = \underbrace{-3x}_{r}$$

$$u = e^{-h}\left(\int e^{h} r\,dx + C\right) \qquad h = \int p(x)dx$$

$$= e^{3x}\left(\int e^{-3x}\cdot(-3x)dx + C\right) \qquad = \int(-3)dx = -3x$$

$$= e^{3x}\left(\left(+\frac{1}{\cancel{3}}e^{-3x}\right)\cdot(\cancel{+3}x) + \int(\cancel{+3})\left(\cancel{-\frac{1}{3}}e^{-3x}\right)dx + C\right)$$

$$= e^{3x}\left(xe^{-3x} - \int e^{-3x}dx + C\right)$$

$$= e^{3x}\left(xe^{-3x} + \left(+\frac{1}{3}\right)e^{-3x} + C\right)$$

$$= x + \frac{1}{3} + Ce^{3x}$$

$y^{-3} = x + \frac{1}{3} + Ce^{3x}$ $\qquad$ $y = \left(x + \frac{1}{3} + Ce^{3x}\right)^{-\frac{1}{3}}$

○ $4y'' + 4y' + 17y = 0$ $\quad y(0) = -1 \quad y'(0) = 2$

$y = e^{\lambda x}$ $\qquad \hookrightarrow y'' + y' + \frac{17}{4}y = 0$

$e^{x}(4\lambda^2 + 4\lambda + 17) = 0$

$$\lambda = \frac{-4 \pm \sqrt{16 - 4\cdot 4\cdot 17}}{8} = \frac{-4 \pm \sqrt{16\cdot 16\cdot(-1)}}{8}$$

$$= \frac{-4 \pm 16i}{8} = -\frac{1}{2} \pm 2i$$

$y_1 = e^{-\frac{1}{2}x}(\cos(2x))$ $\qquad y_h = C_1 y_1 + C_2 y_2$

$y_2 = e^{-\frac{1}{2}x}(\sin(2x))$

$y_h = e^{-\frac{1}{2}x}(C_1\cos(2x) + C_2\sin(2x))$

$-1 = C_1\cos(0) + \cancel{C_2\sin(0)} = C_1$

$y'_h = -\frac{1}{2}e^{-\frac{1}{2}x}(-\cos(2x) + C_2\sin(2x))$

$\quad + e^{-\frac{1}{2}x}(2\sin(2x) + 2C_2\cos(2x))$

$2 = -\frac{1}{2}(-\cos(2\cdot 0) + \cancel{C_2\sin(2\cdot 0)})$

$\quad + (\cancel{2\sin(2\cdot 0)} + 2C_2\cos(2\cdot 0))$

$\quad = \frac{1}{2}\cdot 1 + 2C_2$ $\qquad \frac{3}{2} = 2C_2 \qquad C_2 = \frac{3}{4}$

$\therefore y_h = \left(-\cos(2x) + \frac{3}{4}\sin(2x)\right)e^{-\frac{1}{2}x}$

○ $y_1(x)$가 (H)의 solution으로 주어졌을 때

(H)의 두 번째 solution인 $y_2(x)$ 구하고

(N)의 해도 구하기

(1) $y'' - 4y = 2$ $\qquad y_1(x) = e^{-2x}$

$y'' - 4y = 0 \cdots$ (H) $\qquad p = 0 \quad q = -4$

$$y_2 = y_1\int \frac{1}{(y_1)^2} e^{-\int p\,dx} dx$$

$$= e^{-2x}\int \frac{1}{(e^{-2x})^2} e^{-\int 0\,dx} dx$$

$$= e^{-2x}\int e^{4x} \underbrace{e^{-0}}_{1} dx$$

$$= e^{-2x}\int e^{4x} dx = e^{-2x}\cdot \frac{1}{4}e^{4x} = \frac{1}{4}e^{2x}$$

$y_h = C_1 e^{-2x} + C_2 e^{2x}$

$y'' - 4y = 2 \cdots$ (N) $\qquad y_p = A = -\frac{1}{2}$

$0 - 4A = 2 \qquad A = -\frac{1}{2} \qquad y = y_h + y_p$

$y = C_1 e^{-2x} + C_2 e^{2x} - \frac{1}{2}$

(2) $y'' - 3y' + 2y = 5e^{3x}$

$y'' \underbrace{-3}_{p} y' + 2y = 0 \cdots$ (H) $\qquad y_1 = e^{x}$

$y = e^{\lambda x} \qquad e^{\lambda x}(\lambda^2 - 3\lambda + 2) = 0$

$(\lambda - 2)(\lambda - 1) = 0 \qquad \lambda = 2$ or $\lambda = 1$

$$y_2 = y_1\int \frac{1}{(y_1)^2} e^{-\int p\,dx} dx = y_1\int \frac{1}{(y_1)^2} e^{\int 3\,dx} dx$$

$$= e^{x}\int \frac{1}{e^{2x}} e^{3x} dx = e^{x}\int e^{x} dx = \underbrace{e^{2x}}$$

$y_h = C_1 e^{x} + C_2 e^{2x}$

$y_p = Ae^{3x} \qquad y'_p = 3Ae^{3x} \qquad y''_p = 9Ae^{3x}$

$\cancel{9Ae^{3x}} - \cancel{3\cdot 3Ae^{3x}} + 2Ae^{3x} = 5e^{3x}$

$\underbrace{A = \frac{5}{2}} \qquad y_p = \frac{5}{2}e^{3x}$

$y = C_1 e^{x} + C_2 e^{2x} + \frac{5}{2}e^{3x}$

○ $y''+4y'-2y=2x^2-3x+6$

(H) $y''+4y'-2y=0$ $\quad y=e^{\lambda x}$

$e^{\lambda x}(\lambda^2+4\lambda-2)=0$

$\lambda=\frac{-4\pm\sqrt{16-4\cdot1\cdot(-2)}}{2}$

$=\frac{-4\pm\sqrt{16+8}}{2}=\frac{-4\pm\sqrt{24}}{2}$

$=\frac{-4\pm2\sqrt{6}}{2}=-2\pm\sqrt{6}$

$y_1=e^{(-2+\sqrt{6})x}$

$y_2=e^{(-2-\sqrt{6})x}$ $\quad y_h=C_1e^{(-2+\sqrt{6})x}+C_2e^{(-2-\sqrt{6})x}$

$y_p=Ax^2+Bx+C$ $\quad y'_p=2Ax+B$

$y''_p=2A$

$2A+4(2Ax+B)-2(Ax^2+Bx+C)=2x^2-3x+6$

$-2Ax^2+(8A-2B)x+(2A+4B-2C)=2x^2-3x+6$

$-2A=2$ $\quad A=-1$ $\quad -2+4\cdot(-\frac{5}{2})-2C=6$

$-8-2B=-3$ $\quad -2-10-2C=6$

$-2B=5$ $\quad B=-\frac{5}{2}$ $\quad -12-2C=6$

$-18=2C$ $\quad C=-9$

$y_p=-x^2-\frac{5}{2}x-9$

$y=C_1e^{(-2+\sqrt{6})x}+C_2e^{(-2-\sqrt{6})x}-x^2-\frac{5}{2}x-9$

○ $y''-y'+y=2\sin 3x$

(H) $y=e^{\lambda x}$ $\quad e^{\lambda x}(\lambda^2-\lambda+1)=0$

$\lambda=\frac{1\pm\sqrt{1-4\cdot1\cdot1}}{2}=\frac{1\pm\sqrt{3}i}{2}$

$=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$y_1=e^{\frac{1}{2}x}\left(\cos\left(\frac{\sqrt{3}}{2}x\right)\right)$ $\quad y_h=e^{\frac{1}{2}x}\left(C_1\cos\left(\frac{\sqrt{3}}{2}x\right)+C_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$y_2=e^{\frac{1}{2}x}\left(\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$y_p=A\cos 3x+B\sin 3x=-\frac{24}{71}\cos 3x-\frac{2}{71}\sin 3x$

$y'_p=-3A\sin 3x+3B\cos 3x$

$y''_p=-9A\cos 3x-9B\sin 3x$

$-9A\cos 3x-9B\sin 3x+3A\sin 3x-3B\cos 3x+A\cos 3x+B\sin 3x=2\sin 3x$

$\sin 3x\cdot(\underbrace{-9B+3A+B}_{\parallel})+(\underbrace{-9A-3B+A}_{\parallel})\cos 3x=2\sin 3x$

$-6A+B=2$ $\quad -12B+A=0$ $\quad A=12B$

$-72B+B=2$ $\quad B=-\frac{2}{71}$ $\quad A=-\frac{24}{71}$

$-71B=2$

$y=e^{\frac{1}{2}x}\left(C_1\cos\left(\frac{\sqrt{3}}{2}x\right)+C_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right)-\frac{24}{71}\cos 3x-\frac{2}{71}\sin 3x$

$$\circ \quad y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

$$y = e^{\lambda x} \qquad e^{\lambda x}(\lambda^2 - 2\lambda - 3) = 0$$

$$(\lambda - 3)(\lambda + 1) = 0 \qquad \lambda = 3 \text{ or } \lambda = -1$$

$$y_1 = e^{3x} \qquad y_2 = e^{-x}$$

$$y_h = C_1 e^{3x} + C_2 e^{-x}$$

$$y_p = (Ax + B) + (Cx + D)e^{2x}$$

$$y_p' = A + (C)e^{2x} + (Cx + D)2 \cdot e^{2x}$$

$$= A + (2Cx + 2D + C)e^{2x}$$

$$y_p'' = 2C \cdot e^{2x} + (2Cx + 2D + C) \cdot 2e^{2x}$$

$$= e^{2x}(4Cx + 4D + 4C)$$

$$e^{2x}(\cancel{4Cx + 4D} + 4C) - 2A - 2(\cancel{2Cx + 2D} + C)e^{2x}$$

$$-3(Ax + B) - 3(Cx + D)e^{2x} = 4x - 5 + 6xe^{2x}$$

$$-3Ax - 3B - 2A - 3Cxe^{2x} + (2C - 3D)e^{2x} = 4x - 5 + 6xe^{2x}$$

$$-3A = 4$$

$$A = -\frac{4}{3} \qquad -3B - 2A = -5$$

$$3B + 2A = 5 \qquad B = \frac{23}{9}$$

$$3B - \frac{8}{3} = \frac{15}{3} \qquad 3B = \frac{23}{3}$$

$$C = -2 \qquad -4C - 3D = 0 \qquad D = -\frac{4}{3}$$

$$y_p = -\frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}$$

$$y = C_1 e^{3x} + C_2 e^{-x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}$$

$$\circ \quad y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

$$y'' - 6y' + 9y = 0 \qquad y = e^{\lambda x}$$

$$e^{\lambda x}(\lambda^2 - 6\lambda + 9) = 0 \qquad \lambda = 3 \ (\text{중근})$$

$$y_1 = e^{3x} \qquad y_2 = xe^{3x}$$

$$y_h = C_1 e^{3x} + C_2 x e^{3x}$$

$$y_p = (Ax^2 + Bx + C) + Dx^2(e^{3x})$$

$$y_p' = (2Ax + B) + 2Dxe^{3x} + Dx^2 \cdot 3e^{3x}$$

$$= (3Dx^2 + 2Dx)e^{3x} + (2Ax + B)$$

$$y_p'' = (6Dx + 2D)e^{3x} + (3Dx^2 + 2Dx) \cdot 3e^{3x} + 2A$$

$$= e^{3x}(9Dx^2 + 12Dx + 2D)e^{3x} + 2A$$

$$e^{3x}(\cancel{9Dx^2 + 12Dx} + 2D) + 2A - 6(\cancel{3Dx^2 + 2Dx})e^{3x}$$

$$-6(2Ax + B) + 9(Ax^2 + Bx + C) + \cancel{9Dx^2 e^{3x}}$$

$$= 6x^2 + 2 - 12e^{3x}$$

$$9Ax^2 = 6x^2 \qquad 2De^{3x} = -12e^{3x} \qquad D = -6$$

$$A = \frac{2}{3} \qquad (-12A + 9B) = 0 \qquad 9B = 8$$

$$-\cancel{12}^{4} \cdot \frac{2}{\cancel{3}} + 9B = 0 \qquad B = \frac{8}{9}$$

$$2A - 6B + 9C = 2$$

$$\frac{12}{9} - \frac{48}{9} + 9C = \frac{18}{9}$$

$$-\frac{36}{9} \qquad 9C - 4 = 2 \qquad 9C = 6 \qquad C = \frac{2}{3}$$

$$y_p = \left(\frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3}\right) - 6x^2 e^{3x}$$

$$y = C_1 e^{3x} + C_2 x e^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2 e^{3x}$$

○ $y'' \underbrace{-4}_{p(x)} y' + \underbrace{4}_{q(x)} y = \underbrace{(x+1)e^{2x}}_{r(x)}$

(H) $y'' - 4y + 4 = 0$ $\qquad y = e^{\lambda x}$

$e^{\lambda x}(\lambda^2 - 4\lambda + 4) = 0$ $\qquad \lambda = 2$ (중근)

$y_1 = e^{2x}$ $\qquad y_2 = xe^{2x}$

$y_h = C_1 e^{2x} + C_2 x e^{2x}$

$$y_p = y_1 \int \frac{W_1}{W} dx + y_2 \int \frac{W_2}{W} dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (2x+1)e^{2x} \end{vmatrix}$$

$$= e^{4x}(\cancel{2x}+1) - \cancel{2xe^{4x}} = e^{4x} \begin{matrix} \neq 0 \\ > 0 \end{matrix}$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & (2x+1)e^{2x} \end{vmatrix}$$

$$= -(x+1)xe^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix}$$

$$= (x+1)e^{4x}$$

$$y_p = e^{2x} \int \frac{-(x+1)x\cancel{e^{4x}}}{\cancel{e^{4x}}} dx + xe^{2x} \int \frac{(x+1)\cancel{e^{4x}}}{\cancel{e^{4x}}} dx$$

$$= -e^{2x} \int (x^2 + x) dx + xe^{2x} \int (x+1) dx$$

$$= -e^{2x}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right) + xe^{2x}\left(\frac{1}{2}x^2 + x\right)$$

$$= e^{2x}\left(-\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x^3 + x^2\right)$$

$$= e^{2x}\left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)$$

$$y = C_1 e^{2x} + C_2 x e^{2x} + e^{2x}\left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)$$

○ $4y'' + 36y = \csc 3x$ $\qquad y'' + 9y = \underbrace{\frac{1}{4}\frac{1}{\sin 3x}}_{r(x)}$

(H) $4y'' + 36y = 0$ $\qquad y = e^{\lambda x}$

$e^{\lambda x}(4\lambda^2 + 36) = 0$ $\qquad \lambda^2 = -9$ $\qquad \lambda = \pm 3i$

$y_1 = e^0(\cos(3x))$ $\qquad y_h = C_1\cos(3x) + C_2\sin(3x)$

$y_2 = e^0(\sin(3x))$

$$y_p = y_1 \int \frac{W_1}{W} dx + y_2 \int \frac{W_2}{W} dx$$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 x + 3\sin^2 x = 3$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\frac{1}{\sin 3x} & 3\cos 3x \end{vmatrix} = -\frac{1}{4}$$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4\sin 3x} \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

$$y_p = \cos(3x) \int \frac{1}{3}\left(-\frac{1}{4}\right) dx + \sin(3x) \int \frac{1}{3} \cdot \frac{1}{4} \frac{\cos 3x}{\sin 3x} dx$$

$$= -\frac{1}{12} x\cos(3x) + \frac{1}{36}\sin(3x) \int \frac{1}{t} dt$$

$\sin 3x = t \rightarrow 3\cos 3x\, dx = dt$

$\cos 3x\, dx = \frac{1}{3} dt$

$$= -\frac{1}{12} x\cos(3x) + \frac{1}{36}\sin(3x) \ln|\sin(3x)|$$

$$y = C_1\cos(3x) + C_2\sin(3x) - \frac{1}{12}x\cos(3x) + \frac{1}{36}\sin(3x)\ln|\sin(3x)|$$

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = \underbrace{2x^2e^x}_{r(x)}$$

○ $x^2y'' - 3xy' + 3y = 2x^4e^x$

(H) $x^2y'' - 3xy' + 3y = 0$ $\quad y = x^m$

$x^m\{m(m-1) - 3m + 3\} = 0$

$m^2 - 4m + 3 = 0 \qquad m=3$ or $m=1$

$(m-3)(m-1) = 0$

$y_1 = x^3 \qquad y_2 = x$

$y_h = c_1x^3 + c_2x$

$$y_p = x^3\int \frac{W_1}{W}dx + x\int \frac{W_2}{W}dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x \\ 3x^2 & 1 \end{vmatrix} = x^3 - 3x^3 = -2x^3 \neq 0$$

$$W_1 = \begin{vmatrix} 0 & x \\ 2x^2e^x & 1 \end{vmatrix} = -2x^3e^x$$

$$W_2 = \begin{vmatrix} x^3 & 0 \\ 3x^2 & 2x^2e^x \end{vmatrix} = 2x^5e^x$$

$$y_p = x^3\int \frac{\cancel{-2x^3}e^x}{\cancel{-2x^3}}dx + x\int \frac{\cancel{2x^5}\,x^2 e^x}{\cancel{-2x^3}}dx$$

$$= x^3\int e^x dx - x\int x^2e^x dx$$

$$= x^3e^x - x\left(x^2e^x - \int 2xe^x dx\right)$$

$$= x^3e^x - x\left(x^2e^x - \left(2xe^x - \int 2e^x dx\right)\right) \quad {}^{/\!/\,2e^x}$$

$$= \cancel{x^3e^x} - x\left(\cancel{x^2e^x} - 2xe^x + 2e^x\right)$$

$$= 2x^2e^x - 2xe^x$$

$$y = c_1x^3 + c_2x + 2x^2e^x - 2xe^x$$

○ $y''' + y'' = e^x\cos x = r(x)$

(H) $y''' + y'' = 0 \qquad y = e^{\lambda x}$

$e^{\lambda x}(\lambda^3 + \lambda^2) = 0 \qquad \lambda^2(\lambda+1) = 0$

$\lambda = -1 \quad \lambda = 0$ (중근)

$y_1 = e^{-x} \qquad y_2 = 1 \qquad y_3 = x$

$y_h = c_1e^{-x} + c_2 + c_3x$

(N) $y_p = (A\cos x + B\sin x)e^x$

$y_p' = (-A\sin x + B\cos x)e^x + (A\cos x + B\sin x)e^x$

$= ((A+B)\cos x + (-A+B)\sin x)e^x$

$y_p'' = (-(A+B)\sin x + (-A+B)\cos x)e^x$

$+ ((A+B)\cos x + (-A+B)\sin x)e^x$

$= (2B\cos x + (-2A)\sin x)e^x$

$y_p''' = (-2B\sin x - 2A\cos x)e^x$

$+ (2B\cos x + (-2A)\sin x)e^x$

$= ((2B-2A)\cos x + (-2A-2B)\sin x)e^x$

$((4B-2A)\cos x + (-4A-2B)\sin x)e^x = e^x\cos x$

$4B - 2A = 1 \qquad -4A - 2B = 0$

$-8A - 2A = 1 \qquad 2B = -4A \qquad B = -2A$

$A = -\frac{1}{10} \qquad B = \frac{1}{5}$

$$y_p = \left(-\frac{1}{10}\cos x + \frac{1}{5}\sin x\right)e^x$$

$$y = c_1e^{-x} + c_2 + c_3x + \left(-\frac{1}{10}\cos x + \frac{1}{5}\sin x\right)e^x$$

비선형 1계 미분 방정식

singular solution
존재 가능

○ $xy' = y^2 + y$

$y' - \frac{1}{x}y = \frac{1}{x}y^2$

$u = y^{1-2} = y^{-1}$

$u' = -y^{-2}y'$

$\times(-y^{-2})$

$-y^{-2}y' + \frac{1}{x}y^{-1} = -\frac{1}{x}$

$u' + \underbrace{\frac{1}{x}}_{p(x)} u = \underbrace{-\frac{1}{x}}_{r(x)}$

$u = e^{-h}\left(\int e^{h} r\,dx + C\right)$ $\quad h = \int p(x)\,dx$

$h = \int \frac{1}{x}\,dx = \ln|x|$

$u = e^{-\ln x}\left(\int e^{\ln x}\left(-\frac{1}{x}\right)dx + C\right)$

$= \frac{1}{x}\left(-\int \cancel{x}\,\frac{1}{\cancel{x}}\,dx + C\right)$

$= -\frac{1}{x}(x+c) = -\frac{x+C}{x} = y^{-1}$

$y = \frac{-x}{x+C}$ $\quad y=0$

○ $\frac{dy}{dx} = y^2-4$ $\quad y' = y^2-4$

$\frac{y'}{y^2-4} = 1$ $\quad \int \frac{1}{y^2-4}\,dy = \int 1\,dx$

$\frac{1}{(y-2)(y+2)} = \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right)$

$\frac{1}{4}\cdot\frac{y+2-(y-2)}{(y-2)(y+2)} = \frac{4}{(y-2)(y+2)}$ $\frac{1}{4}$

$\frac{1}{4}\int\left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy = \int 1\,dx$

$\frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = x + C$

$\ln\frac{|y-2|}{|y+2|} = 4x + c$

$\frac{|y-2|}{|y+2|} = e^{4x+c} = Ce^{4x}$

$y-2 = Ce^{4x}(y+2)$

$y - Ce^{4x}y = 2Ce^{4x} + 2$

$y(1-Ce^{4x}) = 2(Ce^{4x}+1)$

$y = \frac{2(Ce^{4x}+1)}{1-Ce^{4x}}$ $\quad$ or $\quad y = -2$

○ $\underbrace{\frac{dy}{dx}}_{y'} = (-2x+y)^2 - 11 \qquad y(0)=0$

y와 u 모두 x에 대한 함수

$-2x+y = u \qquad u' = -2+y'$

$u'+2 = y'$

$u'+2 = u^2 - 11$

$u' = u^2 - 9 \qquad \frac{1}{u^2-9}u' = 1$

$\frac{1}{(u-3)}\frac{1}{(u+3)} = \frac{1}{6}\left(\frac{1}{u-3} - \frac{1}{u+3}\right)$

$\cancel{u+3} - \cancel{(u-3)}$

$\frac{1}{6}\int\left(\frac{1}{u-3} - \frac{1}{u+3}\right)du = \int 1\,dx$

$\frac{1}{6}\left(\ln|u-3| - \ln|u+3|\right) = x + C$

$\ln\left|\frac{u-3}{u+3}\right| = 6x + C$

$\frac{u-3}{u+3} = C\cdot e^{6x}$

$u-3 = (u+3)C\cdot e^{6x}$

$-2x+y-3 = (-2x+y+3)\,c\cdot e^{6x}$

$-2x+y-3 = (-2cx+cy+3c)e^{6x}$

$y - c\cdot e^{6x}y = 2x - 2cxe^{6x} + 3 + 3ce^{6x}$

$y = \frac{2x-2cxe^{6x}+3+3ce^{6x}}{(1-ce^{6x})} \qquad y(0)=0$

$0 = \frac{0-0+3+3c}{1-c} \qquad 3c=-3 \quad c=-1$

$y = \frac{2x+2xe^{6x}+3-3e^{6x}}{1+e^{6x}} = \frac{2x(1+e^{6x})+3(1-e^{6x})}{1+e^{6x}}$

$= 2x + 3\frac{(1-e^{6x})}{1+e^{6x}}$

○ $\frac{dy}{dx} = \frac{xy^2 - \cos x\sin x}{y(1-x^2)} \qquad y(0)=2$

$y(1-x^2)\,dy = (xy^2 - \cos x\sin x)\,dx$

$\underbrace{(xy^2-\cos x\sin x)}_{P=u_x}dx + \underbrace{y(x^2-1)}_{Q=u_y}dy = 0$

$P_y = 2xy$

$Q_x = 2xy$

$u = g(x) + \int(x^2-1)y\,dy$

$= g(x) + (x^2-1)\frac{1}{2}y^2$

$u_x = g'(x) + \frac{1}{\cancel{2}}y^2(\cancel{2}x) = g'(x) + xy^2$

$g'(x) = -\cos x\sin x$

$t=\sin x$

$g(x) = -\int\cos x\sin x\,dx = -\int t\,dt = -\frac{1}{2}t^2 + C$

$= -\frac{1}{2}(\sin^2 x) + C$

$u = -\frac{1}{2}\sin^2 x \cancel{+C} + (x^2-1)\frac{1}{2}y^2 = C$

$-\frac{1}{2}\sin^2 x + (x^2-1)\frac{1}{2}y^2 = C$

$-\sin^2 x + (x^2-1)\frac{1}{2}y^2 = C = -2$

$y(0)=2 \qquad -0 + (-1)\frac{1}{\cancel{2}}\cancel{4}^{2} = -2$

$-(1-\cos^2 x) + (x^2-1)y^2 = -2$

$\cos^2 x - 1 + (x^2-1)y^2 = -2$

$(x^2-1)y^2 + \cos^2 x = -1$

$$U = f(x,y)$$

$$\circ \quad \underbrace{xy}_{P\,=\,U_x}dx + (\underbrace{2x^2+3y^2-20}_{Q\,=\,U_y})dy = 0$$

$$P_y = x \qquad\qquad P_y \neq Q_x$$

$$Q_x = 4x$$

$$F(y) = e^{\int R(y)dy} \qquad R(y) = \frac{Q_x - P_y}{P}$$

$$= e^{3\int \frac{1}{y}dy} \qquad = \frac{4x - x}{xy} = \frac{3\cancel{x}}{\cancel{x}y} = \frac{3}{y}$$

$$= e^{3\ln|y|} = y^3$$

$$FP\,dx + FQ\,dy = 0$$

$$\underbrace{y^3xy}_{\underbrace{xy^4}_{M\,=\,U_x}}dx + \underbrace{y^3(2x^2+3y^2-20)}_{N\,=\,U_y}dy = 0$$

$$U = \int M\,dx + g(y) = y^4\int x\,dx + g(y)$$

$$= y^4\frac{1}{2}x^2 + g(y) = \frac{1}{2}x^2y^4 + g(y)$$

$$U_y = \frac{1}{\cancel{2}}x^2\cdot \overset{2}{\cancel{4}}y^3 + g'(y) = 2x^2y^3 + 3y^5 - 20y^3$$

$$g'(y) = 3y^5 - 20y^3$$

$$g(y) = \frac{3}{6}y^6 - \frac{20}{4}y^4$$

$$= \frac{1}{2}y^6 - 5y^4$$

$$U = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C$$

$$\begin{cases} \dfrac{dx}{dt} = 6y \\ \dfrac{dy}{dt} = x+z \\ \dfrac{dz}{dt} = x+y \end{cases} \qquad X(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad X'(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$$

$$X'(t) = \underbrace{\begin{bmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \cdot X(t)$$

Let $X(t) = e^{\lambda t} \cdot K$

$X'(t) = A\,X(t) = A\,e^{\lambda t} \cdot K$

$X'(t) = \lambda e^{\lambda t} \cdot K$

$AK = \lambda K \qquad (\lambda I - A)K = 0$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -6 & 0 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = \lambda(\lambda^2 - 1) + (+6)(-\lambda - 1)$$

$$= (\lambda - 1)\lambda(\lambda + 1) - 6(\lambda + 1)$$

$$= (\lambda + 1)\{(\lambda - 1)\lambda - 6\}$$

$$= (\lambda + 1)(\lambda^2 - \lambda - 6) = (\lambda+1)(\lambda-3)(\lambda+2) = 0$$

$\lambda = -1$ or $\lambda = 3$ or $\lambda = -2$

$\lambda = -1$

$$\begin{bmatrix} -1 & -6 & 0 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & -6 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 6 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix} t$$

$k_1 + k_2 + k_3 = 0 \qquad 5k_2 - k_3 = 0 \qquad k_3 = t$

$k_1 = -\frac{6}{5}t \qquad k_2 = \frac{1}{5}t \qquad e^{-1 \cdot t}\begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix}$

$\lambda = 3$

$$\left[\begin{array}{ccc|c} 3 & -6 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right]$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$k_1 - 2k_2 = 0$

$k_2 - k_3 = 0 \qquad k_3 = t$

$k_2 = t \qquad k_1 - 2t = 0 \qquad k_1 = 2t$

$$e^{3t}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = -2$

$$\left[\begin{array}{ccc|c} -2 & -6 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 + 3k_2 = 0 \qquad k_2 - k_3 = 0 \qquad k_3 = t$

$k_2 = k_3 = t \qquad k_1 = -3t \qquad \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

$$e^{-2t}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore X_h(t) = c_1 e^{-t}\begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-2t}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{cases} \dfrac{dx}{dt} = x+y \\ \dfrac{dy}{dt} = 4x+y \end{cases} \qquad X(0)=0,\quad y(0)=1$$

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad X'(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} X(t)$$

$$= A \cdot X(t)$$

let $X(t) = e^{\lambda t}\cdot K$

$$X'(t) = \lambda e^{\lambda t}\cdot K = A e^{\lambda t}\cdot K$$

$(\lambda I - A)K = 0$

$$\det(\lambda I - A) = 0 \rightarrow \begin{vmatrix} \lambda-1 & -1 \\ -4 & \lambda-1 \end{vmatrix} = 0$$

$(\lambda-1)^2 - 4 = 0 \qquad \lambda^2 - 2\lambda + 1 - 4 = 0$

$(\lambda-3)(\lambda+1) = 0 \qquad \lambda^2 - 2\lambda - 3 = 0$

$\lambda = 3$ or $\lambda = -1$

$$\lambda = 3 \qquad \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -4 & 2 & 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$2k_1 - k_2 = 0 \qquad k_2 = s \qquad k_1 = \frac{1}{2}s$

$$X_1(t) = e^{3t}\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\lambda = -1 \qquad \left[\begin{array}{cc|c} -2 & -1 & 0 \\ -4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$2k_1 + k_2 = 0 \qquad k_2 = s \qquad k_1 = -\frac{s}{2}$

$$X_2(t) = e^{-1t}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \qquad X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X(t) = c_1 e^{3t}\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = \frac{1}{2}e^{3t}\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \frac{1}{2}e^{-t}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \qquad \begin{matrix} \frac{1}{2}c_1 - \frac{1}{2}c_2 = 0 & c_1 = c_2 \\ c_1 + c_2 = 1 & c_1 = \frac{1}{2} = c_2 \end{matrix}$$

$\{y = cx^2\} \perp \{y(x)\}$

$y' = 2cx \qquad -\frac{1}{2cx}$

$$\frac{y}{x^2} = c \Rightarrow \frac{y'x^2 - y\cdot 2x}{x^4} = 0$$

$y'x^2 = 2yx \qquad y' = \frac{2y}{x} \leftrightarrow y' = -\frac{x}{2y}$

$2yy' = -x$

$y^2 = -\frac{1}{2}x^2 + c \qquad \frac{1}{2}x^2 + y^2 = c$

IVP: $y' = f(x,y)$ $\quad y(x_0) = y_0$

$f(x,y)$가 어떤 사각 영역 R

R: $|x - x_0| < a$, $|y - y_0| < b$

내의 모든 점 (x,y)에서 연속이고

R에서 $|f(x,y)| \le K$로 bounded 되는 K가

존재하면

$f(x,y)$는 $\alpha = \min\{a, \frac{b}{K}\}$ 이고 $|x-x_0| < \alpha$인

정의역 구간에서 최소 한 개 이상의 solution을 지닌다

IVP가 $|x-x_0|<\alpha$인 정의역에서 solution $y(x)$를

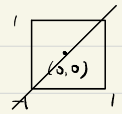
지닐 때, $\frac{\partial f}{\partial y}$가 어떤 특정 값 M에 대해

$\left|\frac{\partial f}{\partial y}\right| < M$로 bounded 되어야 하고

$\frac{\partial f}{\partial y}$가 사각 영역 R에서 연속이면

IVP는 유일한 해를 지닌다.

∘ $y' = f(x,y) = y^{\frac{1}{3}}$ $\quad y(0) = 0$



$y^{-\frac{1}{3}} y' = 1$

$y^{-\frac{1}{3}} dy = dx$

$\frac{3}{2} y^{\frac{2}{3}} = x + C$ $\quad y^{\frac{2}{3}} = \frac{2}{3}x + C$

$y = \{\frac{2}{3}x + C\}^{\frac{3}{2}}$

$y = \left(\frac{2}{3}x\right)^{\frac{3}{2}}$

IVP가 해를 지님

but $\frac{\partial f}{\partial y} = \frac{1}{3} y^{-\frac{2}{3}}$ is not bounded

∘ $y' = 1 + y^2$ $\quad y(0) = 0$

$y' = f(x,y) = 1 + y^2$

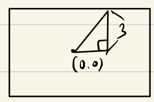
$\frac{1}{1+y^2} y' = 1$ $\quad \int \frac{1}{1+y^2} dy = \int 1 \cdot dx$

$\tan^{-1}(y) = x + C$

$y = \tan(x+C) \xrightarrow[y(0)=0]{} C=0$ $\quad (x_0, y_0) = (0,0)$

$a=5,\ b=3$

$-5 < x < 5$ $\quad -3 < y < 3$



$\alpha = \min\left(5, \frac{3}{10}\right)$

$|f(x,y)| = |1+y^2| = |1+9| = |10| \le K$

$y=3$일 때 $\quad K=10$

$\frac{\partial f}{\partial y} = 2y$ bounded on R

$\frac{\partial f}{\partial y}$가 R에서 연속이므로 Unique한 solution 지님