Background

Before we can continue to the the `CNOT` gate from a quantum perspective we need to introduce some notation, namely the tensor product. The tensor product is used to be able to distinguish the different qubits when considering multiple qubits at once.

We denote the tensor product as follows $|0\rangle \otimes |1\rangle$, for example, means the first qubit is in state 0 and the second is in the state 1.

Now we can define the quantum `CNOT` gate as follows, if we start with both qubits in a general state,

$$\binom{\alpha}{\beta} \otimes \binom{\gamma}{\delta}$$

We can write this state in the following way:

$$(\alpha \binom{1}{0} + \beta \binom{0}{1}) \otimes (\gamma \binom{1}{0} + \delta \binom{0}{1}) .$$

The nice thing about the `CNOT` gate as well as the tensor products is that they are *linear*. What that means is that we can write the equation above as

$$\begin{gathered}\alpha\gamma \binom{1}{0} \otimes \binom{1}{0} + \\ \beta\gamma \binom{0}{0} \otimes \binom{1}{0} + \\ \alpha\delta \binom{1}{0} \otimes \binom{0}{1} + \\ \beta\delta \binom{0}{1} \otimes \binom{0}{1}\end{gathered}$$

When we apply a `CNOT` to these we can see the state $\binom{1}{0}$ as a 0 and $\binom{0}{1}$ as a 1 in the classical sense.

This means when we apply a `CNOT`, we get for example:

$$CNOT \left(\binom{0}{1} \otimes \binom{1}{0}\right) = CNOT(10) = 11 = \left(\binom{0}{1} \otimes \binom{0}{1}\right)$$

Now combining this information, let's look at one example,

$$\begin{aligned}&CNOT\left(\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) = \\ &CNOT\left(\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\right) \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) = \\ &\frac{1}{\sqrt{2}}\left(CNOT\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) + CNOT\left(\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right)\right) = \\ &\frac{1}{\sqrt{2}}\left(\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) + \left(\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}\right)\right)\end{aligned}$$