https://github.com/Goobley/Lightspinner

But what is $\Lambda[\ldots]$?

And other radiative transfer questions.

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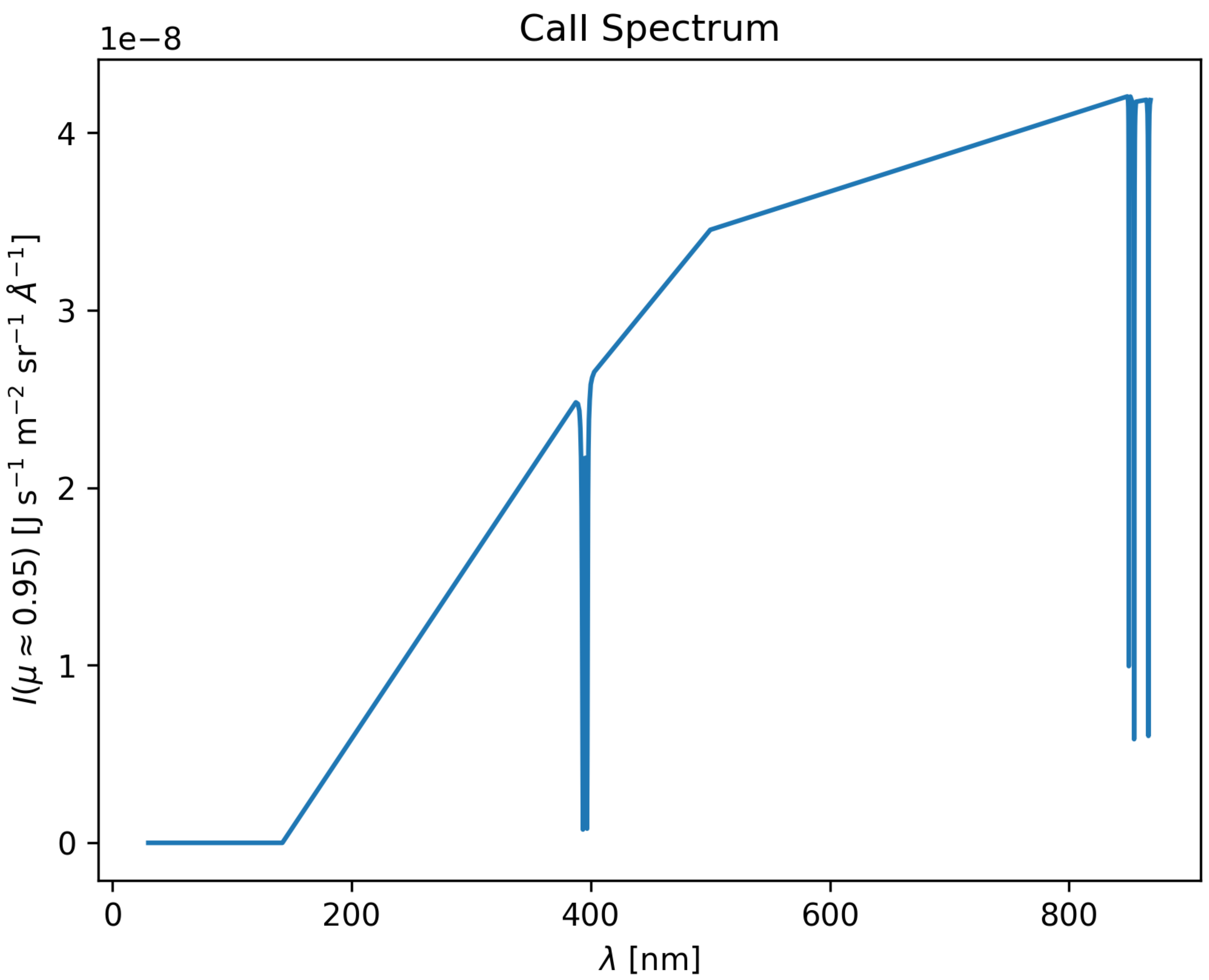
Outline

- Emissivity and Opacity
- Formal Solvers
- Operator “splitting”
- NLTE
- Evaluating the diagonal of Λ
- MALI
- Response Functions



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$$\sum_{l'} n_{l'} C_{l'l} + \sum_{l'} \oint d\Omega \int \frac{d\nu}{h\nu} \left\{ n_{l'} U_{l'l} + n_{l'} V_{l'l} I^{\mathrm{eff}}_{\nu,\mathbf{n}} + n_{l'} V_{l'l} \sum_{j} \sum_{i<j} \Psi^{*}_{\nu,\mathbf{n}} \left[n_j U_{ji} \right] \right\}$$

$$= \sum_{l''} n_{l} C_{ll'} + \sum_{l''} \oint d\Omega \int \frac{d\nu}{h\nu} \left\{ n_{l} U_{ll''} + n_{l} V_{ll''} I^{\mathrm{eff}}_{\nu,\mathbf{n}} + n_{l} V_{ll'} \sum_{j} \sum_{i<j} \Psi^{*}_{\nu,\mathbf{n}} \left[n_j U_{ji} \right] \right\}$$

Emissivity and Opacity

For a bound-bound transition (j > i):

$$\eta_\nu = n_j \frac{h\nu}{4\pi} A_{ji} \psi_{ij}$$

$$\chi_\nu = \frac{h\nu}{4\pi}(n_i B_{ij} \phi_{ij} - n_j B_{ij} \psi_{ij})$$

Emissivity: Addition of energy to path of light through emission processes.

Opacity: Removal of energy from path of light through absorption and scattering processes.

ϕ_{ij} - line absorption profile.

ψ_{ij} - line emission profile.

A, B - Einstein coefficients.

For a bound-free transition (j > i):

$$\eta_\nu = n_j n_e \Phi_{ij}(T) \left(\frac{2hv^3}{c^2} \right) e^{-hv/kT} \alpha_{ij}(v)$$

$$\chi_\nu = n_i \alpha_{ij}(v) - n_j n_e \Phi_{ij}(T) e^{-hv/kT} \alpha_{ij}(v)$$

$$\Phi_{ij}(T) = \frac{g_i}{2g_j} \left(\frac{h^2}{2\pi mkT} \right)^{3/2} \exp\left[\left(E_j - E_i \right) / kT \right]$$

- Saha-Boltzmann function.

α_{ij} - Bound-free cross section.

If multiple species are present, their emissivities and opacities simply sum.

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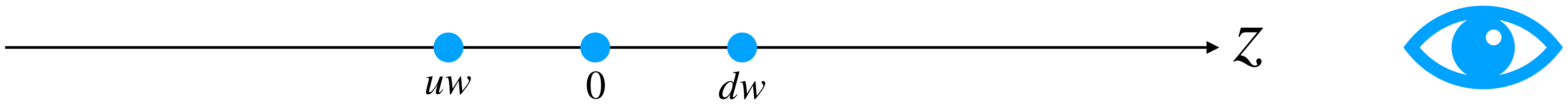
Time Independent Radiative Transfer Equation

$$\mu \frac{\partial I_{\mu\nu}}{\partial z} = \eta_{\mu\nu} - \chi_{\mu\nu} I_{\mu\nu}$$

$$\mu \frac{\partial I_{\mu\nu}}{\partial \tau_{\mu\nu}} = I_{\mu\nu} - S_{\mu\nu}$$

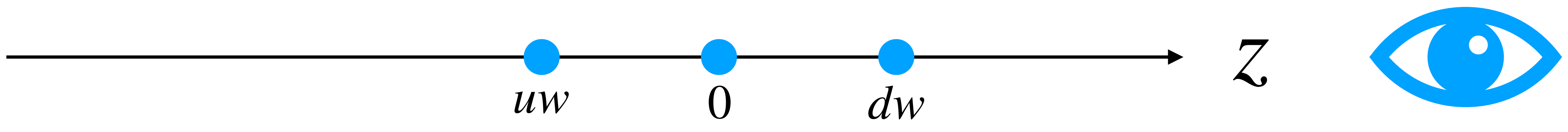


$$I_0 = I_{uw} e^{-\Delta\tau_{uw-0}} + \int_0^{\Delta\tau_{uw-0}} S(\tau) e^{-\tau} \mathrm{d}\tau$$

Optical depth: $\tau_{\mu\nu} = \int_z^{z_{\max}} \chi_\nu(z')\mathrm{d}z'/\mu$

Source Function (no scattering): $S_{\mu\nu} = \eta_{\mu\nu}/\chi_{\mu\nu}$

Short Characteristics Formal Solution of the RTE



$$I_0 = I_{uw} e^{-\Delta \tau_{uw-0}} + \int_0^{\Delta \tau_{uw-0}} S(\tau) e^{-\tau} \mathrm{d}\tau$$

- Assume piecewise linear variation of S between points i.e. $S(\tau) = S_0 + c_1 \tau$.

$$I_0 = I_{uw} e^{-\Delta \tau_{uw-0}} + S_0 w_0 + c_1 w_1$$

- This solver has issues, but the short characteristics works well with higher order interpolants.

Using:

$$\int_0^{\Delta \tau} e^{-\tau} \mathrm{d}\tau = 1 - e^{-\Delta \tau} =: w_0$$

$$\int_0^{\Delta \tau} \tau e^{-\tau} \mathrm{d}\tau = w_0 - \Delta \tau e^{-\Delta \tau} =: w_1$$

$$c_1 := \frac{S_{uw} - S_0}{\Delta \tau_{uw-0}}$$

(gradient along τ is opposite to traditional forward derivative as we want $S(0) = S_0$ and $S(\Delta \tau_{uw-0}) = S_{uw}$ or terms "closer" in optical depth to point 0 are damped more)

Outside of LTE

- If the atomic populations are non a-priori (for example in local thermodynamic equilibrium) then the method of the previous slide can be used to directly compute the outgoing intensity.

- LTE = every process in detailed balance i.e. for every $A \to B$ there is a $B \to A$.
  - Non-LTE (NLTE) everything for which this isn't true.
  - No radiation could escape from a star if it were in LTE!

- Expect departures from LTE if:
  - Radiative rates dominate collisional rates in important transition.
  - Radiation field is not blackbody-like.

Transition Rates

Total transition rate

$$P_{ij} = R_{ij} + C_{ij}$$

Collisional (de)excitation rate

Radiative (de)excitation rate

$$R_{ij} = \oint \mathrm{d}\Omega \int \frac{\mathrm{d}\nu}{h\nu} V_{ij}(\nu, \mathbf{n}) I(\nu, \mathbf{n}) \boxed{= B_{ij} J_{ij}}$$

$$R_{ji} = \oint \mathrm{d}\Omega \int \frac{\mathrm{d}\nu}{h\nu} \left\{ U_{ji}(\nu, \mathbf{n}) + V_{ji}(\nu, \mathbf{n}) I(\nu, \mathbf{n}) \right\} \boxed{= A_{ji} + B_{ji} J_{ij}}$$

In the case of complete redistribution i.e. $\phi_{ij} = \psi_{ij}$,
with $J_{ij} = \frac{1}{4\pi} \oint \mathrm{d}\Omega \int \mathrm{d}\nu \phi_{ij}(\nu, \mathbf{n}) I_{ij}(\nu, \mathbf{n})$, the absorption profile weighted integrated mean intensity.

Bound-bound:

$$V_{ij} = h\nu/4\pi B_{ij} \phi_{ij}(\nu, \mathbf{n}),$$

$$V_{ji} = h\nu/4\pi B_{ji} \psi_{ij}(\nu, \mathbf{n}),$$

$$U_{ji} = h\nu/4\pi A_{ji} \psi_{ij}(\nu, \mathbf{n})$$

Bound-free:

$$V_{ij} = \alpha_{ij}(\nu),$$

$$V_{ji} = n_e \Phi_{ij}(T) e^{-h\nu/kT} \alpha_{ij}(\nu)$$

$$U_{ji} = n_e \Phi_{ij}(T) \left(\frac{2h\nu^3}{c^2} \right) e^{-h\nu/kT} \alpha_{ij}(\nu)$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}) = \sum_{j \neq i} (n_j P_{ji}) - n_i \sum_{j \neq i} P_{ij}$$

Kinetic equilibrium equation (statistical equilibrium for LHS=0)

Lambda Iteration

- Treat the formal solver from before as an operator; $\Lambda_{\mu\nu}[S_{\mu\nu}] = I_{\mu\nu}$, then $J_\nu = \dfrac{1}{4\pi}\oint \mathrm{d}\Omega\, \Lambda_{\nu,\mathbf{n}}[S(\nu, \mathbf{n})]$
- The formal solution is needed to find the intensity throughout the atmosphere, to then find the rates, to update the populations, to find the new intensity, to update the rates, to update the populations…
  - Simply iterate this procedure?
  - This is pure lambda iteration and is extremely slow converging for optically thick problems.
- Accelerate this procedure with operator splitting (Cannon 1973):
  - $\Lambda = \Lambda^* + (\Lambda - \Lambda^*)$ i.e. $I_{\mu\nu} = \Lambda^*[S_{\mu\nu}] + (\Lambda_{\mu\nu} - \Lambda^*_{\mu\nu})[S^\dagger_{\mu\nu}]$
  - $\dagger$ denotes the value from the previous iteration.
  - This method still converges and can be shown to present an amplified version of the corrections applied in lambda iteration [HM pp.423]*

*[HM]: Hubeny & Mihalas (2015)

Approximate Diagonal Operator

- Olson, Auer & Buchler 1986 showed that the diagonal of the Λ operator is almost an optimum Λ^* whilst being trivial to invert.

- To evaluate the diagonal of Λ we set $S = \delta_{dd'}$, and knowing $J_d = \sum_{d'} \Lambda_{dd'} S_{d'}$ for each depth d in the atmosphere, we have $J_d = \Lambda_{dd}$.

- Applying our linear short characteristics method from before gives $\Lambda^*_{\mu\nu} = w_0 - \dfrac{w_1}{\Delta\tau_{uw-0}}$ at each depth.

MALI

- A direct application of accelerated lambda iteration leads to a nonlinear function in the atomic populations being solved for (whilst eliminating the mean intensity).

- Original approaches for solving the coupled system of statistical equilibrium and the radiative transfer equation was to linearise the equations of statistical equilibrium and solve the resulting system by Newton-Raphson iteration (e.g. Scharmer & Carlsson 1981, MULTI, Carlsson 1986)

- A more intuitive preconditioning approach, termed MALI, was first presented in Rybicki & Hummer (1991) and extended in Rybicki & Hummer (1992). It produces a linear scheme by analytically removing the scattering components of the radiative rates and making judicious choices of retaining the old populations in some components of the non-linear terms.

MALI 2

- Define a new operator $\Psi[\ldots] = \Lambda[(\chi^\dagger)^{-1}\ldots]$, where $\dagger$ again represents the value from the previous iteration.
- Perform the same operator splitting: $I(\nu, \mathbf{n}) = \Psi_{\nu,\mathbf{n}}[\eta^\dagger_{\text{tot}}] - \Psi^*_{\nu,\mathbf{n}}\left[\sum_{ij} n_j^\dagger U_{ji}^\dagger\right] + \Psi^*_{\nu,\mathbf{n}}\left[\sum_{ij} n_j U_{ji}^\dagger\right]$
- Why is U_{ji} written with a dagger when it seems constant? This is needed if we later modify the definitions as per [U01]* to support partial redistribution.
- Substitute these terms into the expressions for R_{ij}, R_{ji} and then substitute that into the statistical equilibrium equation to find…

[U01]: Uitenbroek (2001)

$$\sum_{l'} n_{l'} C_{l'l} + \sum_{l'} \oint d\Omega \int \frac{dv}{hv} \left\{ n_{l'} U_{l'l} + n_{l'} V_{l'l} I^{\text{eff}}_{v,\mathbf{n}} + n_{l'}^{\dagger} V_{l'l} \sum_{j} \sum_{i<j} \Psi^{*}_{v,\mathbf{n}} \left[n_j U_{ji} \right] \right\}$$

$$= \sum_{l''} n_l C_{ll'} + \sum_{l''} \oint d\Omega \int \frac{dv}{hv} \left\{ n_l U_{ll''} + n_l V_{ll''} I^{\text{eff}}_{v,\mathbf{n}} + n_l^{\dagger} V_{ll'} \sum_{j} \sum_{i<j} \Psi^{*}_{v,\mathbf{n}} \left[n_j U_{ji} \right] \right\}$$

With $I^{\text{eff}}_{\nu,\mathbf{n}} := \Psi_{\nu,\mathbf{n}}[\eta^{\dagger}_{\text{tot}}] - \Psi^{*}_{\nu,\mathbf{n}} \left[\sum_{j} \sum_{i<j} n_j^{\dagger} U_{ji}^{\dagger} \right] = I^{\dagger}(\nu, \mathbf{n}) - \Psi^{*}_{\nu,\mathbf{n}}[\eta_{\text{atom}}]$

MALI 4

- This can be used to write a linear set of equations $\sum_{l'} n_{l'} \Gamma_{l'l} = 0$

- With linear operator

$$\Gamma_{l'l} = C_{l'l} + \oint d\Omega \int \frac{d\nu}{h\nu} \left\{ U^{\dagger}_{l'l} + V^{\dagger}_{l'l} I^{\mathrm{eff}}_{\nu,\mathbf{n}} - \sum_{j} \left(n^{\dagger}_{l} V^{\dagger}_{lj} - n^{\dagger}_{j} V^{\dagger}_{jl} \right) \Psi^{*}_{\nu,\mathbf{n}} \left[\sum_{i<l'} U^{\dagger}_{l'i} \right] \right\}$$

$$-\delta_{l'l} \sum_{l''} \left\{ C_{ll''} + \oint d\Omega \int \frac{d\nu}{h\nu} \left[U^{\dagger}_{ll''} + V^{\dagger}_{ll''} I^{\mathrm{eff}}_{\nu,\mathbf{n}} \right] \right\}$$

- Drop term with δ_{ij} and only construct off-diagonal of matrix, using the fact that the δ_{ij} term gives $\sum_l \Gamma_{ll'} = 0$, allowing reconstruction of Γ_{ll}.
- $$\Gamma_{l'l} = C_{l'l} + \oint d\Omega \int \frac{d\nu}{h\nu} \left\{ U^{\dagger}_{l'l} + V^{\dagger}_{l'l} I^{\mathrm{eff}}_{\nu,\mathbf{n}} - \boxed{\sum_j \left(n^{\dagger}_l V^{\dagger}_{lj} - n^{\dagger}_j V^{\dagger}_{jl} \right)} \Psi^*_{\nu,\mathbf{n}} \left[\sum_{i<l'} U^{\dagger}_{l'i} \right] \right\}$$ **For off-diagonal**

Effective opacity at frequency (absorption subtracting stimulated emission)

MALI Implementation

- Replace integrals with appropriate frequency (fairly flexible choice) and angular (Gauss-Legendre) quadrature (i.e. these become discrete summations).

- Numerically integrate the terms over angle (and up/down), and frequency to construct Γ.

- Solve linear system of equations.

- Approach used in my codes:

    - Accumulate effective opacity, U for each (active) level, and total opacity of each atom whilst accumulating the total emissivity and opacity across all species.

    - Perform formal solution.

    - Implement almost directly the integrand of Γ using the previously stored values

Let’s have a go, shall we?

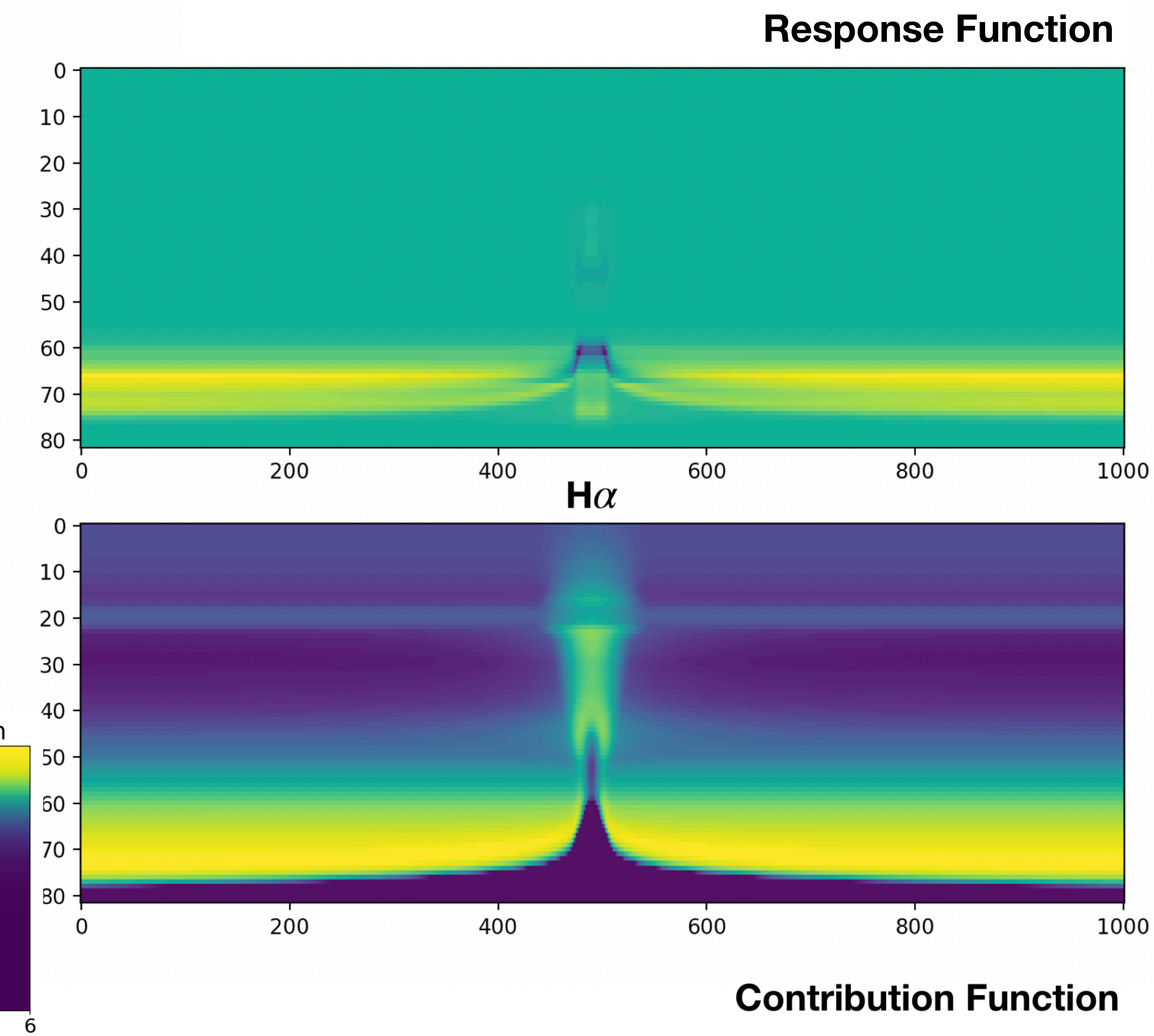
Response Functions

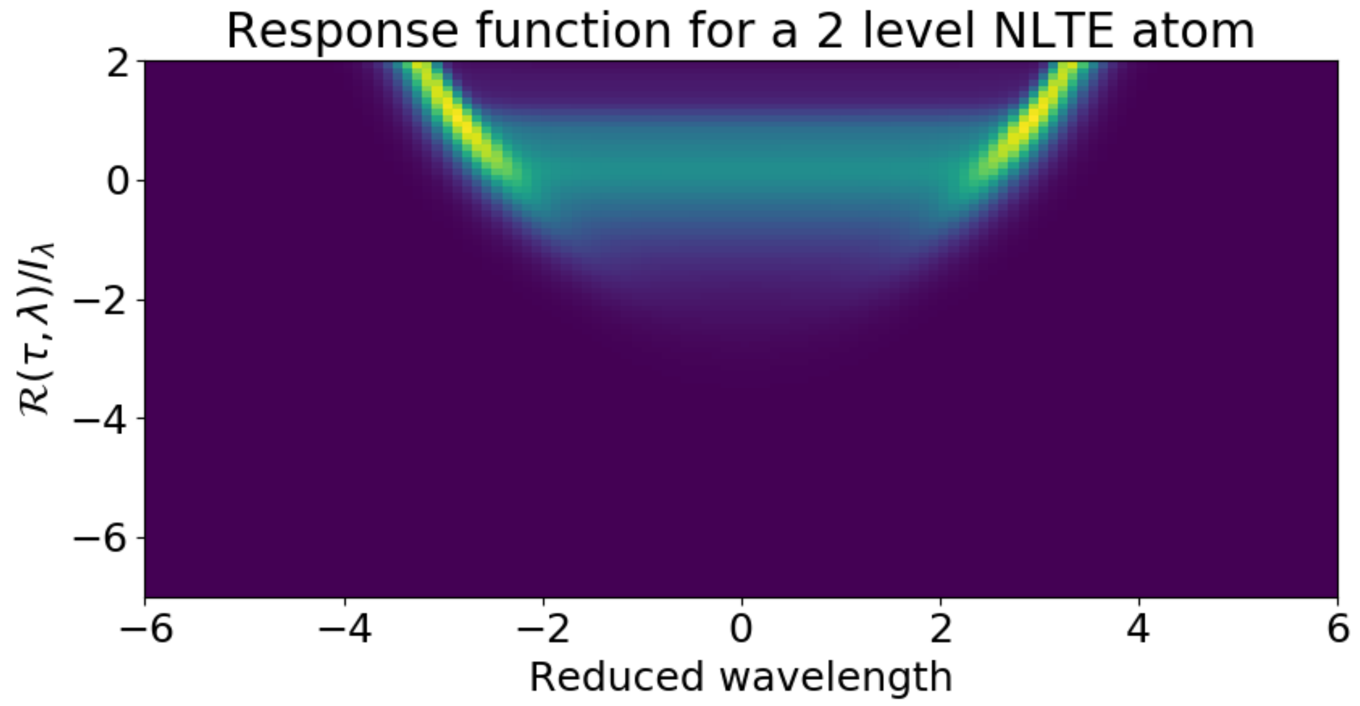
- A response function gives us the derivative of outgoing intensity wrt an atmospheric perturbation e.g. $\frac{\partial I}{\partial T}$.

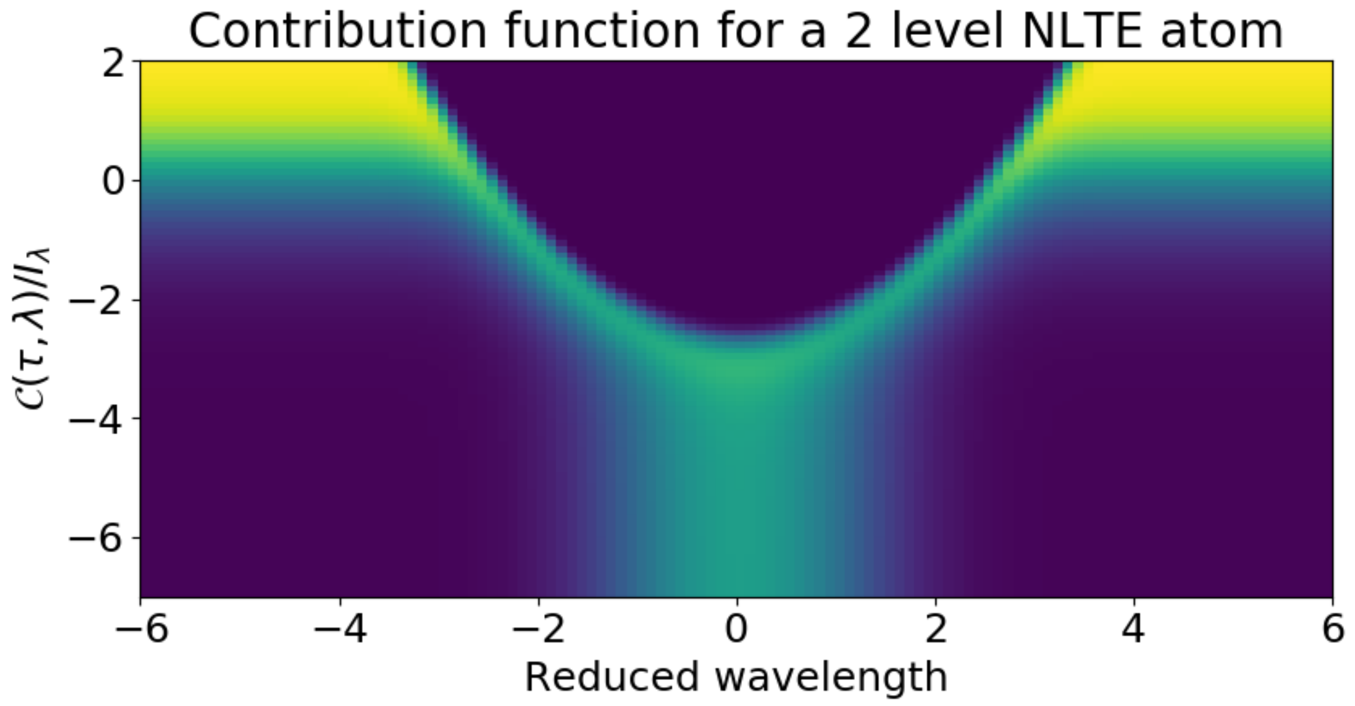
- Shows the contribution of non-local radiation from below the opaque $\tau = 1$ layer.

- Shows sensitivity to different parameters.

- Working on methods for constructing and interpreting time-dependent response functions.







Where do I start interpretation of my NLTE solutions?

- Define the departure coefficient $b_i = n_i / n_{i,\mathrm{LTE}}$

  - In multi-level systems these are determined by a competition between (for lines, general statements and often oversimplified for complex atmospheric structures):

    - Radiative effects (at lower photon destruction probability) causing an over population of the upper level of a line.

    - Collisional transitions that will tend to drive the two levels towards the same departure coefficient.

- Continua! They are important, but can be hard to analyse. Tend to form deeper than lines, so control the NLTE populations deep in the atmosphere which then influences the lines. These continua photoionise cooler layers of the atmosphere, that recombine more slowly due to local electrons being at temperature of cooler layer.

- To start thinking about time dependent interpretation I would recommend Carlsson & Stein (2002), and Judge (2005). A similar analysis to Carlsson & Stein (2002) (for H) was undertaken in Kerr et al (2019) for Mg II.

See [HM] for more details

Thank You!

- NLTE Radiative Transfer Processes are complex, but can be numerically solved with relative ease by applying well understood methods.
  - Important for people who use these codes to understand their contents — not just black boxes.
- We have only treated the case of complete redistribution, but following [U01] the extension to partial redistribution is relatively simple.
- Tools like response functions can yield a lot of insight into line formation.

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