COVARIANCE DETERMINATION, PROPAGATION AND INTERPOLATION TECHNIQUES FOR SPACE SURVEILLANCE

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INTRODUCTION

The modern Space Surveillance systems require a precise knowledge of the position and velocity of the objects being analysed, but also the uncertainties of these orbital parameters in order to monitor their position with respect to a certain accuracy envelope.

As long as the mean error in the estimated orbit of the object is close to zero (which requires accurate orbit determination and propagation techniques), the orbital error follows a normal Gaussian distribution around the expected state vectors, and therefore, the covariance matrix will be representative of the position and velocity uncertainties [1].

The covariance matrix of the state vector contains the different covariance of every element (usually position and velocity of a Cartesian state vector in any reference frame) with respect to the others. This matrix must be symmetric and positive semi definite and, if the variance of every element is not null, positive-definite. Therefore, the covariance matrix defines a hyper-ellipsoid (or a proper ellipsoid in case of the position only covariance matrix) surface of equal probability density around the mean expected value of the state vector [2].
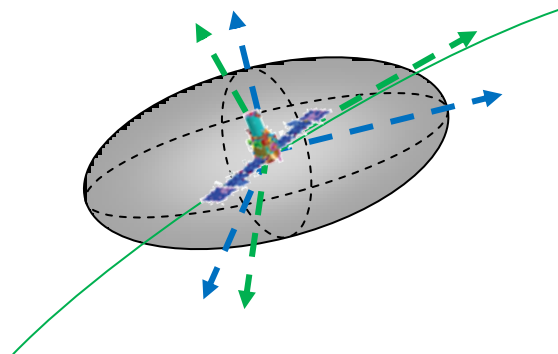

Fig. 1. Covariance ellipsoid representation in orbital frame

Given the importance of an accurate covariance computation in the Space Surveillance, this paper will review different methods for determination, propagation and interpolation of a set of covariance matrices in order to highlight several problems that may appear during the computation of orbital uncertainties.

COVARIANCE DETERMINATION AND PROPAGATION

The covariance matrix is a natural result of the orbit determination. Depending on the estimation method, the procedure to extract the covariance matrix can be completely different.

In case of a batch weighted least-squares estimator, the covariance matrix can be directly computed as the inverse of the normal matrix (N). Defining P_0 as the a-priori covariance matrix of the parameters, Q_0 as the observation a-priori covariance matrix and F the matrix of observation equations, which contains the partial derivatives of the computed observations with respect to the estimated parameters, the final covariance matrix at the estimated time can be computed as follows [3]:

$$\hat{P}_0 = N^{-1} = [P_0^{-1} + F^T \cdot Q_0^{-1} \cdot F]^{-1} \tag{1}$$

Usually, this covariance matrix is scaled with the root mean square of the determination process squared, or the equivalent loss function J divided by the number of observations m [4].

$$\hat{P}_0 = \frac{J}{m}[P_0^{-1} + F^T \cdot Q_0^{-1} \cdot F]^{-1} \tag{2}$$

The Square Root Information Filter (SRIF) is based on the estimation of the square root information matrix. This determination method is defined by the following equations [5]:

$$\tilde{z} = \tilde{R} \cdot x + \tilde{v}$$
$$z = A \cdot x + v \tag{3}$$

With x representing the parameters vector (dimension n), A the observations partials and z the new observations (dimension m), with an associated noise v. $\tilde{z}$ represents the fictitious observations with the information matrix from the previous solution $\tilde{R}$ and their associated noise, $\tilde{v}$.

The covariance matrix of the estimated parameters can be directly obtained from the information matrix with the following formula [6]:

$$\tilde{P} = \tilde{R}^{-1} \cdot \tilde{R}^T \tag{4}$$

Kalman filters are algebraically equivalent to the SRIF filter, but they are based on the direct estimation of the covariance matrix. The covariance matrix, however, can be computed with the equivalent equation to (1) in the batch estimator [7]:

$$P_0^+ = [I + K \cdot F] \cdot P_0^- \tag{5}$$

Being K the Kalman gain and F the same matrix defined in (1) and (2).

COVARIANCE PROPAGATION

When the covariance matrix is estimated at a reference epoch, it has to be propagated to any other time. With this covariance propagation it is possible to consider the increasing deviation of the orbital states during the orbit propagation.

As the accuracy of the propagation degrades with time, small position errors in radial direction accumulate a great uncertainty in the transverse (along-track) direction. The following figure shows the evolutions of the position sigma in the RTN frame, propagating a 1 metre spherical covariance of a Sun synchronous LEO satellite during 12 hours.
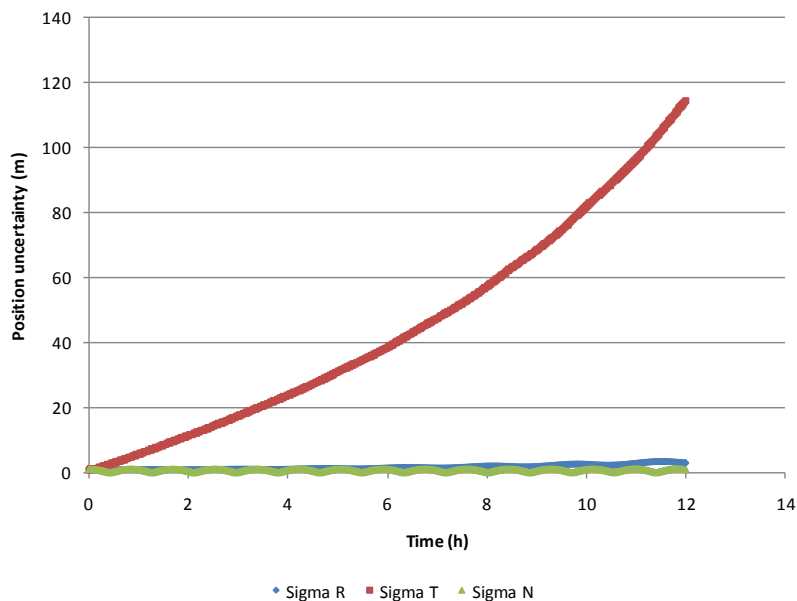

Fig. 2. Sigma evolution along the time

This propagation of the covariance can be performed through the state transition matrix, Φ [3][8]:

$$\hat{P} = \Phi(t) \cdot \hat{P}_0 \cdot \Phi^T(t) \tag{6}$$

The state transition matrix contains the variational partials of all the parameters with respect to their initial values. If the covariance matrix resulting from the orbit determination contains more elements than those required to be propagated (for instance, if the solar radiation pressure or the atmospheric drag are being estimated), a submatrix of this state transition matrix can be used to reduce the covariance at the reference epoch to the required state vector covariance, considering the influence of the other estimated (or considered) parameters in the position and velocity uncertainties.

Numerical Propagation

The state transition matrix can be integrated with numerical methods together with the orbit ephemeris, computing the relative change of the state vector components with respect to the initial state. This method can be highly configurable modifying the different perturbations to be considered in the propagation to adequate them to the orbit being considered. Fig. 3 shows the perturbing acceleration of the different dynamical effect on an orbit as a function of the distance from the Earth centre.
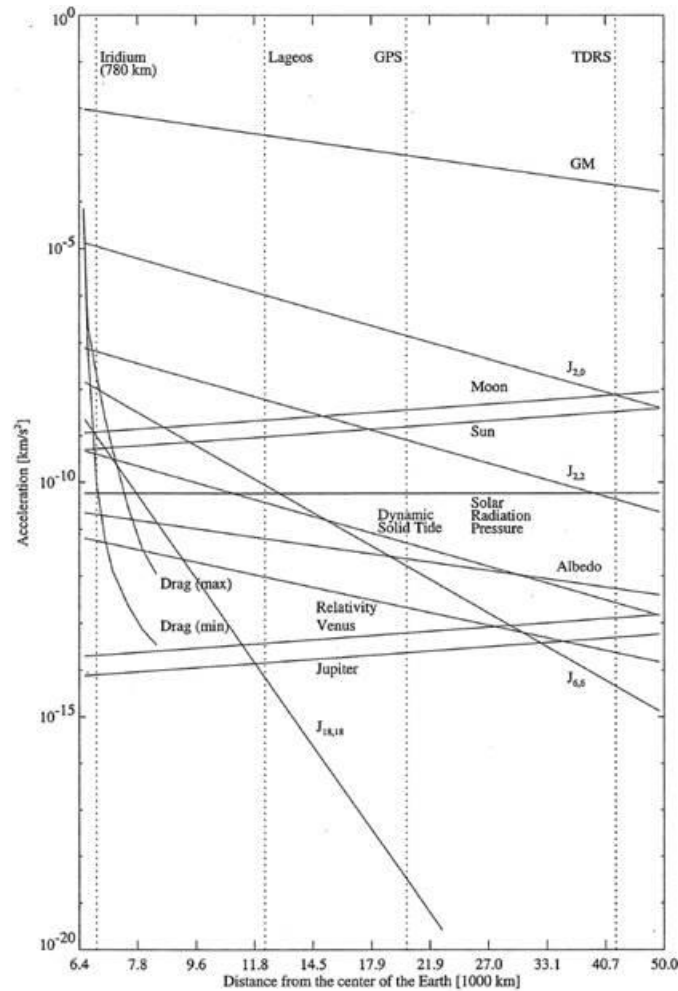

Fig. 3. Perturbing accelerations depending on height

The numerical propagation of the transition matrix also allows considering the effect of new parameters included in the propagation interval, such as the dispersion due to a manoeuvre depending on the performances or the degradation of the accuracy in the drag estimation due to the solar activity estimation. These parameters (called "consider parameters") should be already considered during the orbit determination, affecting the a-posteriori covariance matrix at the reference epoch [9], modifying the equations (1), (4) and (5).

However, the main disadvantage of the numerical methods is the high computational cost, even higher if the propagation has to be performed for a high number of objects, as is usually the case in Space Surveillance systems. If the performances of the numerical method can be improved by reducing the number of dynamical models to be used

during the propagation it will result in a loss of accuracy that could be enough to justify the usage of other analytical propagators.

Analytical Propagation

The analytical propagation methods for the covariance are used to obtain better performances with respect to the numerical integration. The results obtained with these methods are usually less accurate than the numerical ones, since they are often based in the linearization of the propagation problem (e.g. [10]) but the final error with respect to the actual values are usually enough for a coarse propagation that can be afterwards refined with a fine propagation if required.

The Clohessy-Wiltshire equations defined in [11] linearize the equations of motion and define the motion of two objects with very similar initial orbital states, one of them in a circular orbit around a central body. These equations provide a transition matrix that can be used to propagate the estimated covariance [12].

This propagation method has several limitations. The assumption of the circular orbit introduces errors that can be corrected re-evaluating the transition matrix with a small step size in an elliptical orbit. This method can only be applied when the covariance is quite low, since it was developed to analyse a rendezvous, so it only considers the movement of two orbiting objects very close to each other. As the covariance analysis becomes less and less important when the covariance increases, this constraint in the method is not a great limitation. Therefore, the main limitation of this method comes from the assumption of a central body, as it only takes into account the geocentric gravity. This limitation makes the method only useful for quite small time intervals.

The original Clohessy-Wiltshire equations were calculated for a cartesian frame. However the method can be applied with curvilinear coordinates [13]. This is an interesting approach as the main uncertainty of the position is usually along-track (see Fig. 2).

Since the Simplified General Perturbations (SGP) theory is often used for coarse orbit propagation, the analytical computation of a state transition matrix consistent with this theory has been analysed. Reference [14] defines the method for this computation obtaining the derivatives of the state transition matrix through the numerical derivation of the SGP equations. Defining x as the state vector position (in true-equator mean-equinox frame), p the orbital parameters considered in the SGP theory (orbital inclination, right ascension of the ascending node, eccentricity, argument of perigee, mean anomaly and mean motion) and q the additional parameters of the model (two first derivatives of the mean motion and the B* drag term), the SGP theory can be defined as the following equation:

$$x = SGP(\Delta t, p, q) \tag{7}$$

By numerical differentiation it is possible to obtain the Jacobean matrix of x with respect to p at any time. Inverting this matrix at the initial point the partials $\partial p/\partial x_0$ can be obtained, and so the state transition matrix can be obtained as:

$$\frac{\partial x}{\partial x_0} = \frac{\partial x}{\partial p} \cdot \frac{\partial p}{\partial x_0} \tag{8}$$

COVARIANCE INTERPOLATION

The covariance matrix interpolation has to be used when the covariance is known at other times than the one required. Due to the properties of the covariance matrix, this interpolation presents several numerical particularities that have to been dealt with.

Direct Covariance Matrix Interpolation

The first approach to interpolate the covariance is the interpolation of every single element in the matrix, using the Lagrange interpolating polynomials. This method is often used for state vector interpolation, since using polynomials of the same order than the numerical propagation model minimise the uncertainty during the interpolation. However, when applied to the correlation matrix, this method can introduce additional and artificial correlations between the different elements, as the entries are not completely independent of one another [15].

This effect can be mitigated with a high sampling of the covariance matrix and the usage of a reference frame close to the principal axes of the covariance ellipsoid (see Fig. 1). However, in some limit cases, this interpolation method cannot guarantee that the resultant matrix is positive-definite.

The computation of the eigenvalues and eigenvectors of the covariance matrix would allow the direct interpolation of the maximum deviations (eigenvalues) and the orientation of the principal covariance axes (eigenvectors). With this conversion no artificial correlation will be added during the interpolation.

The main problem of this method is the additional computational charge to obtain the eigenvalues and eigenvectors of every covariance matrix. Besides, most of the numerical methods optimised for this computation return the eigenvalues ordered by side, so some additional logic should have to be implemented to identify the principal axis associated to each value, especially when two of them are very similar to each other and their order change in the time.

Auxiliary Matrix Interpolation

To work around the problems of the covariance matrix interpolation it is possible to interpolate some auxiliary matrices that allow the reconstruction of the covariance.

During the covariance propagation, the state transition matrix is often interpolated with Lagrange polynomials. Once the transition matrix is known at the required time, the covariance is calculated applying (6) in the same way than during the propagation. Since the transition matrix is not symmetrical, there is an additional computational cost for the interpolation of all the single elements, but it solves most of the problems of the covariance interpolation. Unfortunately, the state transition matrix is usually only available during the propagation itself.

The sigma-correlation matrix can be obtained replacing the diagonal elements of the correlation matrix by the standard deviation of the position and velocity. Being $\hat{P}_{0ij}$ the element i,j of the covariance matrix, the same element of the correlation matrix C_{ij} can be computed with the following equation.

$$c_{ij} = \frac{\hat{P}_{0ij}}{\sqrt{\hat{P}_{0ii} \cdot \hat{P}_{0jj}}} \tag{9}$$

The diagonal elements in the correlation matrix are always equal to one, so they are replaced by the standard deviation of each element to allow the reconstruction of the covariance matrix. This sigma can be obtained directly from the diagonal element of the covariance:

$$\sigma_i = \sqrt{\hat{P}_{0ii}} \tag{10}$$

The interpolation of the sigma-correlation matrix obtained with (9) and (10) maintains the correlation of the elements from the covariance matrix, but depends on the frame used to represent the covariance, since the sigma is calculated directly from the diagonal element [15].

Analytical Interpolation Methods

If the sampling of the covariance matrix is very low, the previous interpolation methods can generate very coarse results compared with the actual covariance matrix at the required time. Some analytical interpolation methods can be considered as an alternative between the covariance interpolation and the analytical propagation of the last known covariance matrix.

If the two first derivatives of the covariance are known, it is possible to approximate the covariance evolution by a quintic spline to interpolate the covariance evolution. As they are usually not provided together with the covariance matrices, the actual derivatives can be approximated using a simple two-body Keplerian model [16]. This method provides good results with a covariance sampling of at least 20 matrices per orbit and, similarly to the direct covariance interpolation, the dispersion during the interpolation decreases in the TNW frame.

CONCLUSIONS

The covariance matrix provides information on the position and velocity uncertainty of a state vector. This positive definite matrix can be represented as an hyper-ellipsoid in a 6-dimensional space. The accurate knowledge of these uncertainties is critical for the Space Surveillance systems.

This matrix appears as a natural result of the orbit determination systems, but it has to be propagated to consider the increasing deviation of the state vector during the orbital evolution after the estimation time. Numerical propagation is very accurate, but it has low computational performances. Modifying the perturbation models applied during the propagation can improve the performance of this numerical method, reducing the accuracy at the same time. Analytical propagation methods can be used instead

The interpolation of the covariance matrix can introduce additional correlations depending on the sampling and the frame used for its representation. The interpolation of auxiliary matrices that can be used for reconstructing the covariance matrix can be used instead of a direct interpolation. Some analytical methods based on short propagation of the last known covariance state can be also considered when the sampling rate of the covariance matrix is very low.

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