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The generalized Cucconi test statistic for the two-sample problem

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When testing hypotheses in two-sample problem, the Lepage test statistic is often used to jointly test the location and scale parameters, and this test statistic has been discussed by many authors over the years. Since two-sample nonparametric testing plays an important role in biometry, the Cucconi test statistic is generalized to the location, scale, and location–scale parameters in two-sample problem. The limiting distribution of the suggested test statistic is derived under the hypotheses. Deriving the exact critical value of the test statistic is difficult when the sample sizes are increased. A gamma approximation is used to evaluate the upper tail probability for the proposed test statistic given finite sample sizes. The asymptotic efficiencies of the proposed test statistic are determined for various distributions. The consistency of the original Cucconi test statistic is shown on the specific cases. Finally, the original Cucconi statistic is discussed in the theory of ties.

1. Introduction

Hypothesis testing is one of the most important techniques used in nonparametric statistics. The two-sample problem is frequently considered in practice for testing hypothesis. However, in many applications, the underlying distribution is not adequately understood to assume normality or some other specific distributions, and nonparametric testing hypothesis must be used.

Let $X_i = \{X_{ij}; i = 1, 2, j = 1, 2, \ldots, n_i\}$ be two independent samples of size n_i from a distribution function $F_i(x) = F(\sigma_i x + \mu_i)$ with the location parameter μ_i and the scale parameter $\sigma_i > 0$. Additionally, let R_{ij} be the rank of the jth smallest observation in the ith sample in the pooled sample (X_1, X_2) with $N = n_1 + n_2$. The observations within each sample are assumed to be independent and identically distributed. Then we consider testing the location–scale hypotheses

$$H_0 : F_1 = F_2 \quad \text{against} \quad H_1 : F_1 \neq F_2.$$

In many cases, we have to test the location and scale parameters at the same time. If the scale parameters change, the test statistic for the location parameter is not useful. Similarly, if the location parameters change, the test statistic for the scale parameter is not useful. Therefore, it is preferable to jointly test for the location and scale differences to a continuous distribution functions. Then, Lepage (1971) developed the test statistic that combined the Wilcoxon (1945) and the Ansari and Bradley (1960) test statistics, namely $L_{1,1}$. Once Lepage test statistic was suggested, many researchers studied it as

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combination test statistics. Pettitt (1976) proposed a modified Lepage test statistic, namely $L_{1,2}$, which is a combination of the Wilcoxon (1945) test statistic and the Mood test statistic. In contrast, Cucconi (1968) suggested a test statistic for the location–scale parameter as follows:

$$T_C = \frac{U^2 + V^2 - 2\rho UV}{2(1-\rho^2)},$$

where

$$U = \frac{\displaystyle\sum_{j=1}^{n_1} R_{1j}^2 - \frac{n_1(N+1)(2N+1)}{6}}{\sqrt{\dfrac{n_1 n_2 (N+1)(2N+1)(8N+11)}{180}}},$$

$$V = \frac{\displaystyle\sum_{j=1}^{n_1} (N+1-R_{1j})^2 - \frac{n_1(N+1)(2N+1)}{6}}{\sqrt{\dfrac{n_1 n_2 (N+1)(2N+1)(8N+11)}{180}}},$$

$$\rho = \frac{2(N^2-4)}{(2N+1)(8N+11)} - 1.$$

If $T_C > -\log\alpha$, then we reject the null hypotheses H_0, where α is the significance level. In addition, Marozzi (2009) gave the exact critical values for various sample sizes. Recently, the Cucconi test statistic has been highly valued in hydrology (Rutkowska & Banasik, 2016), parasitology (Marozzi & Reiczigel, 2018) and many scientific fields. Additionally, a nonparametric one-way layout analysis of variance plays an important role in biometry. The extension of the T_C test statistic to multisample location–scale problems was proposed (Marozzi, 2014). More recently, Murakami (2016a) presented a test statistic based on the all-pair Cucconi test statistic for the multiple comparisons. However, in this paper, we focus on the two-sample Cucconi test statistic. Although Marozzi (2008) compared the power of the Cucconi test statistic with the Lepage test statistic by a simulation study, the result of the power comparison did not show a clear winner (Neuhäuser, 2012, p. 67). From the point of the asymptotic relative efficiency (ARE), we consider a critical winner for the hypotheses in this paper. In addition, various nonparametric test statistics were proposed to raise the ARE by using extra tuning parameters, e.g. Goria (1980), Pollicello and Hettmansperger (1976), Sen (1962, 1963) and Tamura (1963). As one of aims in this paper, we suggest a test statistic which is more powerful than the original Cucconi test statistic by raising the ARE. In Section 2, we propose the generalized two-sample Cucconi test statistic, namely $C_{p,q}$, to raise the ARE. In addition, we derive the limiting distribution of the $C_{p,q}$ test statistic under the hypotheses. Furthermore, we approximate the exact distribution of the $C_{p,q}$ test statistic by the gamma approximation. In Section 3, we discuss the existence and consistency of the $C_{p,q}$ test statistic for the specific p and q. Additionally, the asymptotic efficiency of the $C_{p,q}$ test statistic is derived based on the procedure of Murakami (2016b). In Section 4, we investigate the difference between the T_C test statistic and the $L_{1,2}$ test statistic. Finally, we conclude this paper in Section 5.

2. The generalized Cucconi test statistic

In this section, we propose a generalized two-sample Cucconi test statistic, namely $C_{p,q}$, with $p, q \in \mathbb{R}^+$ for the two-sample location–scale problem. At first, define that

$$\mathrm{S}_p \overset{\text{def.}}{=} \sum_{i=1}^{N} i^p, \quad \mathrm{T}_q \overset{\text{def.}}{=} \sum_{i=1}^{N} (N+1-i)^q, \quad \mathrm{ST}_{p,q} \overset{\text{def.}}{=} \sum_{i=1}^{N} i^p (N+1-i)^q.$$

Then, we suggest the generalized two-sample Cucconi test statistic as follows:

$$C_{p,q} = \sum_{i=1}^{2} \frac{N-n_i}{N} \frac{U_p^2 + V_q^2 - 2\rho^* U_p V_q}{1-\rho^{*2}} = \sum_{i=1}^{2} \frac{N-n_i}{N} \begin{bmatrix} U_p \\ V_q \end{bmatrix}' \begin{bmatrix} 1 & \rho^* \\ \rho^* & 1 \end{bmatrix}^{-1} \begin{bmatrix} U_p \\ V_q \end{bmatrix},$$

where

$$U_p = \frac{\mathrm{Q}_p - \mathrm{E}\left[\mathrm{Q}_p\right]}{\sqrt{\mathrm{var}\left[\mathrm{Q}_p\right]}}, \quad \mathrm{Q}_p = \sum_{t=1}^{n_1} R_{1t}^p,$$

$$\mathrm{E}\left[\mathrm{Q}_p\right] = \frac{n_1}{N}\mathrm{S}_p, \quad \mathrm{var}\left[\mathrm{Q}_p\right] = \frac{n_1 n_2}{N^2(N-1)}\left[N\mathrm{S}_{2p} - \mathrm{S}_p{}^2\right],$$

$$
\begin{aligned}
&V_q = \frac{Q_q - \mathrm{E}\left[Q_q\right]}{\sqrt{\mathrm{var}\left[Q_q\right]}}, \quad Q_q = \sum_{t=1}^{n_1}(N+1-R_{1t})^q, \\
&\mathrm{E}\left[Q_q\right] = \frac{n_1}{N}\mathrm{T}_q, \quad \mathrm{var}\left[Q_q\right] = \frac{n_1 n_2}{N^2(N-1)}\left[N\mathrm{T}_{2q} - {\mathrm{T}_q}^2\right], \\
&\rho^* = \mathrm{corr}(U_p, V_q) = \mathrm{cov}(U_p, V_q) = \frac{\mathrm{cov}(Q_p, Q_q)}{\sqrt{\mathrm{var}\left[Q_p\right]\mathrm{var}\left[Q_q\right]}}, \\
&\mathrm{cov}\left(Q_p, Q_q\right) = \frac{n_1 n_2}{N^2(N-1)}\left[N\mathrm{S}\mathrm{T}_{p,q} - \mathrm{S}_p\mathrm{T}_q\right].
\end{aligned}
$$

Note that the $C_{2,2}$ test statistic is equivalent to the $2T_C$ test statistic. Then, this relation indicates that the $C_{p,q}$ test statistic is the natural extension of the original Cucconi test statistic except for $|\rho^*| = 1$, see the detail in Section 3.1. In Section 2.1, we derive the limiting distribution of the $C_{p,q}$ test statistic under the hypotheses. In Section 2.2, we consider the approximation to the distribution of the $C_{p,q}$ test statistic.

2.1. The limiting distribution of the $C_{p,q}$ test statistic

In this section, we derive the limiting distribution of the $C_{p,q}$ test statistic. Define that

$$
\psi(u; p) = u^p, \quad \phi(u; q) = (1-u)^q, \quad u \in (0, 1) \tag{1}
$$

are the score functions of Q_p and Q_q, respectively. In addition, assume that these score functions satisfy the Chernoff–Savage condition, and suppose the following assumptions as same as Rublík (2005):

Assumption 1. Let $\psi : (0, 1) \to \mathbb{R}$ and there exists bounded function $g_\psi^{(d)} : (0, 1) \to \mathbb{R}$, $d = 1, 2$ with finitely many real numbers $0 = a_0 < \cdots < a_s = 1$ such that the first two derivatives of ψ exist as

$$
\psi'(u) = g_\psi^{(1)}(u), \quad \psi''(u) = g_\psi^{(2)}(u)
$$

for all $u \in (0, 1) - \{a_0, \ldots, a_s\}$. Note that $g_\psi^{(1)}$ is right-continuous and

$$
\psi(t_2) - \psi(t_1) = \int_{t_1}^{t_2} g_\psi^{(1)}(t)dt, \quad g_\psi^{(1)}(t_2) - g_\psi^{(1)}(t_1) = \int_{t_1}^{t_2} g_\psi^{(2)}(t)dt
$$

for all $0 < t_1 < t_2 < 1$. The second equation holds whenever $u_1 < u_2$ belongs to (a_m, a_{m+1}), $m = 0, \ldots, s-1$.

Assumption 2. There exist positive numbers $M, \delta > 0$ such that

$$
\left|\psi^{(d)}(t)\right| \le M\{t(1-t)\}^{\delta-d-1/2}, \quad d = 0, 1, 2
$$

for all $t \in (0, 1)$.

Remark that Assumptions 1 and 2 can adapt to the score function ϕ. Moreover, we add Assumptions 3 and 4 to derive the limiting distribution of the $C_{p,q}$ test statistic.

Assumption 3. For $i = 1, 2$, (2), (3), (4) are satisfied for the sample sizes:

$$
\min\{n_1, n_2\} \to \infty, \tag{2}
$$

$$
\hat{\xi}_i = \frac{n_i}{N}, \tag{3}
$$

$$
\lim_{N\to\infty} \hat{\xi}_i \to \xi_i > 0. \tag{4}
$$

Assumption 4. Let us assume that the sample $X_{i1}, \ldots, X_{in_i}$, $i = 1, 2$ is a random sample from the distribution function $F_i(x) = F(\sigma_i x + \mu_i)$, where μ_i and σ_i depend on the index N of the total sample size such that

$$
\sigma_i = 1 + \frac{a_i}{\sqrt{N}}, \quad \mu_i = \frac{b_i}{\sqrt{N}}, \quad \bar{a} = \sum_{i=1}^{2}\xi_i\sigma_i, \quad \bar{b} = \sum_{i=1}^{2}\xi_i\mu_i,
$$

where $a_i > 0$ and $a_i, b_i \in \mathbb{R}$ and there exist i_1, i_2 $(1 \le i_1, i_2 \le 2, i_1 \ne i_2)$ satisfying with $a_{i_1} \ne a_{i_2}$, $b_{i_1} \ne b_{i_2}$, or both.

Note that the given alternative hypothesis, i.e. μ_i and σ_i in Assumption 4, is Pitman's alternatives in which the local alternative values are changed by the sample sizes. Then, we obtain Theorems 1 and 2.

Theorem 1. *Suppose that Assumption 3 holds. Under H_0, the limiting distribution of the $C_{p,q}$ test statistic is the chi-square distribution with two degrees of freedom.*

The proof of Theorem 1. By using Theorem 3.1 in Rublík (2005) with $\psi(u;p)$ and $\phi(u;q)$ in the case of the two-sample problem, Theorem 1 is proved.

Therefore, we reject H_0 if $C_{p,q} > \chi_2^2(1-\alpha)$. As an another representation, we reject H_0 if $C_{p,q}/2 > -\log\alpha$.

Theorem 2. *Suppose that Assumptions 1–4 hold. In Assumption 4, there exist i_1, i_2 $(1 \le i_1, i_2 \le 2, i_1 \ne i_2)$ satisfying $a_{i_1} \ne a_{i_2}$ or $b_{i_1} \ne b_{i_2}$ or both. In addition, let $\ell = a_i/b_i$ for $i = 1, 2$. Under H_1, the limiting distribution of the $C_{p,q}$ test statistic is the chi-square distribution with two degrees of freedom and the non-central parameter is*

$$\frac{2\xi_2(\Sigma_\phi w_1^2 + \Sigma_\psi w_2^2 - 2\Sigma_{\psi,\phi} w_1 w_2)}{\xi_1(\Sigma_\psi \Sigma_\phi - \Sigma_{\psi,\phi}^2)} = \frac{2\xi_2}{\xi_1}\boldsymbol{w}'\Sigma^{-1}\boldsymbol{w},$$

where $\boldsymbol{w} = (w_1, w_2)'$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, $\Delta b = b_1 - b_2$ and

$$\begin{aligned}
w_1 &= \Delta b \int_{\overline{\mathbb{R}}} (\ell x + 1)\frac{d\psi(F(x);p)}{dx} dF(x),\\
w_2 &= \Delta b \int_{\overline{\mathbb{R}}} (\ell x + 1)\frac{d\phi(F(x);q)}{dx} dF(x),\\
\Sigma &= \begin{pmatrix} \Sigma_\psi & \Sigma_{\psi,\phi} \\ \Sigma_{\psi,\phi} & \Sigma_\phi \end{pmatrix},\\
\Sigma_\psi &= \int_0^1 \psi(u;p)^2 du - \left\{\int_0^1 \psi(u;p)du\right\}^2 = \frac{p^2}{(1+2p)(1+p)^2},\\
\Sigma_\phi &= \int_0^1 \phi(u;q)^2 du - \left\{\int_0^1 \phi(u;q)du\right\}^2 = \frac{q^2}{(1+2q)(1+q)^2},\\
\Sigma_{\psi,\phi} &= \int_0^1 \psi(u;p)\phi(u;q)du - \int_0^1 \psi(u;p)du \int_0^1 \phi(u;q)du\\
&= \frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(2+p+q)} - \frac{1}{(1+p)(1+q)}.
\end{aligned}$$

The proof of Theorem 2. By using Theorem 3.2 and Lemma 3.1 in Rublík (2005) with $\psi(u;p)$ and $\phi(u;q)$ in the case of the two-sample problem, Theorem 2 is shown.

Note that, by a simple calculation, ρ^* is represented as

$$\begin{aligned}
\rho^* &= \frac{\left[N\mathrm{ST}_{p,q} - \mathrm{S}_p\mathrm{T}_q\right]}{\sqrt{\left[N\mathrm{S}_{2p} - {\mathrm{S}_p}^2\right]\left[N\mathrm{T}_{2q} - {\mathrm{T}_q}^2\right]}}\\
&= \frac{\left[\frac{1}{N}\frac{\mathrm{ST}_{p,q}}{N^{p+q}} - \frac{1}{N}\frac{\mathrm{S}_p}{N^p}\frac{1}{N}\frac{\mathrm{T}_q}{N^q}\right]}{\sqrt{\left[\frac{1}{N}\frac{\mathrm{S}_{2p}}{N^{2p}} - \left(\frac{1}{N}\frac{\mathrm{S}_p}{N^p}\right)^2\right]\left[\frac{1}{N}\frac{\mathrm{T}_{2q}}{N^{2q}} - \left(\frac{1}{N}\frac{\mathrm{T}_q}{N^q}\right)^2\right]}}.
\end{aligned}$$

Herein, we apply the quadrature by parts for S_p, S_{2p}, T_q, T_{2q} and $\mathrm{ST}_{p,q}$. Then we obtain

$$\begin{aligned}
\mathrm{S}_p &= \sum_{i=1}^N i^p \iff \frac{1}{N}\frac{\mathrm{S}_p}{N^p} = \frac{1}{N}\sum_{i=1}^N \left(\frac{i}{N}\right)^p \longrightarrow \int_0^1 \psi(u;p)du,\\
\mathrm{S}_{2p} &= \sum_{i=1}^N i^{2p} \iff \frac{1}{N}\frac{\mathrm{S}_{2p}}{N^{2p}} = \frac{1}{N}\sum_{i=1}^N \left(\frac{i}{N}\right)^{2p} \longrightarrow \int_0^1 \psi(u;p)^2du,\\
\mathrm{T}_q &= \sum_{i=1}^N (N+1-i)^q \iff \frac{1}{N}\frac{\mathrm{T}_q}{N^q} = \frac{1}{N}\sum_{i=1}^N \left(\frac{N+1-i}{N}\right)^q\\
&\longrightarrow \int_0^1 \phi(u;q)du,
\end{aligned}$$

$$
\begin{aligned}
\mathrm{T}_{2q} &= \sum_{i=1}^{N}(N+1-i)^{2q} \iff \frac{1}{N}\frac{\mathrm{T}_{2q}}{N^{2q}} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{N+1-i}{N}\right)^{2q} \\
&\longrightarrow \int_0^1 \phi(u;q)^2 du, \\
\mathrm{ST}_{p,q} &= \sum_{i=1}^{N} i^p (N+1-i)^q \iff \frac{1}{N}\frac{\mathrm{ST}_{p,q}}{N^{p+q}} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{i}{N}\right)^p\left(\frac{N+1-i}{N}\right)^q \\
&\longrightarrow \int_0^1 \psi(u;p)\phi(u;q)du
\end{aligned}
$$

as $N \to \infty$. Therefore, we have

$$
\lim_{N\to\infty} \rho^* = \frac{\int_0^1 \psi(u;p)\phi(u;q)du - \int_0^1 \psi(u;p)du \int_0^1 \phi(u;q)du}{\sqrt{[\int_0^1 \psi(u;p)^2 du - \{\int_0^1 \psi(u;p)du\}^2][\int_0^1 \phi(u;q)^2 du - \{\int_0^1 \phi(u;q)du\}^2]}}.
$$

Furthermore, we transform the covariance matrix Σ in Theorem 2 to the correlation matrix as follows:

$$
\Sigma = \begin{bmatrix} \Sigma_\psi & \Sigma_{\psi,\phi} \\ \Sigma_{\psi,\phi} & \Sigma_\phi \end{bmatrix} \to \Sigma^* = \begin{bmatrix} 1 & \dfrac{\Sigma_{\psi,\phi}}{\sqrt{\Sigma_\psi \Sigma_\phi}} \\ \dfrac{\Sigma_{\psi,\phi}}{\sqrt{\Sigma_\psi \Sigma_\phi}} & 1 \end{bmatrix}.
$$

Therefore,

$$
\begin{bmatrix} 1 & \rho^* \\ \rho^* & 1 \end{bmatrix} \to \Sigma^*
$$

as $N \to \infty$.

2.2. Gamma approximation

In this section, we verify the accuracy of the approximation to the distribution of the generalized Cucconi test statistic. Calculation of the exact critical value of the test statistic is an important task in testing hypothesis. However, it is difficult to evaluate the exact critical value when the sample sizes are increased. Therefore, the approximation method must be used to estimate the exact critical values and the density or distribution functions under the finite sample sizes.

Herein, we give the exact first two moment of the $C_{p,q}$ test statistic for $(p,q) = (2,2), (2,3), (2,4)$. By applying the method of Murakami (2016b), we obtain that the first moment of the $C_{p,q}$ test statistic is $\mathrm{E}(C_{p,q}) \overset{\text{def.}}{=} m_1^{(p,q)} = 2$. The second moment of the $C_{p,q}$ test statistic, namely $m_2^{(p,q)}$, is given by

$$
\begin{aligned}
m_2^{(2,2)} &= \frac{4(A_1+A_2)}{35n_1n_2(N-2)^2(N+1)(N+2)}, \\
m_2^{(2,3)} &= \frac{A_3+A_4}{1001n_1n_2(N-2)^2(N+1)(N+2)A_7}, \\
m_2^{(2,4)} &= \frac{A_5+A_6}{2431n_1n_2(N-2)^2(N+1)(N+2)A_8},
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= -4N(N+1)(2N^3 - 29N^2 + 37N + 251), \\
A_2 &= 2n_1n_2(35N^4 - 127N^3 - 626N^2 + 1588N + 5184), \\
A_3 &= N(-102307872N^{12} + 1402710018N^{11} + 11822573691N^{10} \\
&\quad + 8591168250N^9 - 148533838507N^8 - 606668399746N^7 \\
&\quad - 1119446406012N^6 - 1131945712350N^5 - 587814682595N^4 \\
&\quad - 61035519372N^3 + 95824971371N^2 + 49101091600N \\
&\quad + 7695028324), \\
A_4 &= 2n_1n_2(666521856N^{12} + 3276866592N^{11} - 4906801026N^{10} \\
&\quad - 5224140287 1N^9 - 26228061195N^8 + 463409356908N^7 \\
&\quad + 1476400070874N^6 + 2055049916994N^5 + 1398163887404N^4
\end{aligned}
$$

$$
\begin{aligned}
&+278959229685N^3 - 208435305733N^2 - 137772290388N \\
&- 24678004300), \\
A_5 =\ & N(-11845215232N^{16} + 5089123681536N^{15} + 41178605383584N^{14} \\
&+ 43433988487776N^{13} - 610814898715968N^{12} \\
&- 2733317773427748N^{11} - 3862580301516772N^{10} \\
&+ 2891721977959632N^9 + 17234447180717076N^8 \\
&+ 22840265369448738N^7 + 8966015677871217N^6 \\
&- 8435333471062992N^5 - 10931287757338270N^4 \\
&- 3845935841261526N^3 + 358788297566721N^2 \\
&+ 515422424386584N + 85610839459644), \\
A_6 =\ & 2n_1n_2(1430287446016N^{16} + 10406063090688N^{15} \\
&+ 2874115166976N^{14} - 153124244927712N^{13} - 227095293444048N^{12} \\
&+ 1681419812524176N^{11} + 6704968911560572N^{10} \\
&+ 6110249211703776N^9 - 13434863071305552N^8 \\
&- 38200404133325916N^7 - 31420041807997422N^6 \\
&+ 3547370069160459N^5 + 21491472857910709N^4 \\
&+ 11370074025103677N^3 + 219459434004249N^2 \\
&- 1291066815110148N - 257166816196500), \\
A_7 =\ & (408N^4 + 1521N^3 + 2016N^2 + 1116N + 223)^2, \\
A_8 =\ & (12128N^6 + 62160N^5 + 116378N^4 + 91770N^3 + 20702N^2 - 6615N \\
&- 2073)^2.
\end{aligned}
$$

Note that, $m_2^{(2,2)}, m_2^{(2,3)}, m_2^{(2,4)} \to 8$ as $N \to \infty$ which is equivalent to the second moment of the chi-square distribution with two degrees of freedom.

Remark 1. Let Y be the random variable of a gamma distribution with a shape parameter κ_1 and a scale parameter κ_2. Note that the probability density function of Y is as

$$f_Y(y) = \frac{\kappa_2^{\kappa_1}}{\Gamma(\kappa_1)} y^{\kappa_1 - 1} \mathrm{e}^{-\kappa_2 y}, \qquad y > 0,$$

and $\mathrm{E}[Y] = \kappa_1/\kappa_2$, $\mathrm{var}[Y] = \kappa_1/\kappa_2^2$. Using $\mathrm{E}[Y]$ and $\mathrm{var}[Y]$, the parameters κ_1 and κ_2 are as follows:

$$\kappa_1 = \frac{\mathrm{E}[Y]^2}{\mathrm{var}[Y]}, \quad \kappa_2 = \frac{\mathrm{E}[Y]}{\mathrm{var}[Y]}.$$

Then we use the gamma approximation to the distribution of the $C_{p,q}$ test statistic by applying the exact moments. We report on the accuracy of the gamma approximation for $(p, q) = (2, 2), (2, 3), (2, 4)$. In Tables 1–3, we use as critical values i.e. 5.991, 7.378 and 9.210 in which 5%, 2.5% and 1% points of the upper tail probability are identified for a chi-square distribution with two degrees of freedom, respectively. The exact probabilities are estimated through simulations involving 10,000,000 Monte Carlo repetitions for the critical values. We list the type I error of the gamma approximation and the limiting distribution for each critical value of the $C_{p,q}$ test statistic and the relative errors (in parentheses) to the exact distribution.

The result of tables indicated that the gamma approximation is more suitable than the limiting distribution in all cases.

Remark 2. The proposed gamma approximation method is equivalent to $c\chi_\nu^2$-approximation where c is a constant scaled parameter and ν is degrees of freedom In general, it is more accurate to use a gamma approximation with two parameters based on matching moments than χ_2^2 approximation. One common example is that our simulation results are also obtained by the Welch–Satterthwaite approximation in Behrens–Fisher problem.

3. The property of the generalized Cucconi test statistic

In this section, we investigate some properties of the generalized Cucconi test statistic. At first, we define the Quasi-convex function and the Pseudo-convex function.

Definition 1 (*Quasi-convex Function*). Let $f : S \to \mathbb{R}$, where S is a nonempty convex set in $\mathbb{R}^n$. The function f is quasi-convex function if the following inequality is true for each $\boldsymbol{x}_1, \boldsymbol{x}_2 \in S$:

$$f(\lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2) \le \max\{f(\boldsymbol{x}_1), f(\boldsymbol{x}_2)\} \quad \text{for each } \lambda \in (0, 1).$$

Table 1
The case of the $C_{2,2}$ test statistic.

n_1	n_2	$1-\alpha$	Exact	Gamma approx.	Limiting dist.
10	20	0.950	5.746	5.721 (0.004)	5.991 (0.043)
		0.975	6.932	6.976 (0.006)	7.378 (0.064)
		0.990	8.414	8.627 (0.025)	9.210 (0.095)
10	25	0.950	5.770	5.757 (0.002)	5.991 (0.038)
		0.975	6.956	7.030 (0.011)	7.378 (0.061)
		0.990	8.541	8.704 (0.019)	9.210 (0.078)
10	30	0.950	5.783	5.810 (0.005)	5.991 (0.036)
		0.975	7.003	7.068 (0.009)	7.378 (0.053)
		0.990	8.643	8.760 (0.014)	9.210 (0.066)
15	15	0.950	5.796	5.726 (0.012)	5.991 (0.034)
		0.975	6.975	6.984 (0.001)	7.378 (0.058)
		0.990	8.435	8.637 (0.024)	9.210 (0.092)
15	20	0.950	5.805	5.765 (0.007)	5.991 (0.032)
		0.975	7.002	7.041 (0.006)	7.378 (0.054)
		0.990	8.515	8.721 (0.024)	9.210 (0.082)
15	25	0.950	5.833	5.793 (0.007)	5.991 (0.027)
		0.975	7.050	7.041 (0.006)	7.378 (0.054)
		0.990	8.627	8.780 (0.018)	9.210 (0.060)
15	30	0.950	5.833	5.814 (0.003)	5.991 (0.027)
		0.975	7.080	7.113 (0.005)	7.378 (0.042)
		0.990	8.685	8.825 (0.016)	9.210 (0.060)
20	20	0.950	5.850	5.795 (0.009)	5.991 (0.024)
		0.975	7.079	7.086 (0.001)	7.378 (0.042)
		0.990	8.685	8.785 (0.012)	9.210 (0.060)
20	25	0.950	5.858	5.818 (0.007)	5.991 (0.023)
		0.975	7.095	7.119 (0.003)	7.378 (0.040)
		0.990	8.685	8.833 (0.017)	9.210 (0.060)
20	30	0.950	5.855	5.835 (0.003)	5.991 (0.023)
		0.975	7.135	7.144 (0.001)	7.378 (0.034)
		0.990	8.785	8.870 (0.010)	9.210 (0.048)

Table 2
The case of the $C_{2,3}$ test statistic.

n_1	n_2	$1-\alpha$	Exact	Gamma approx.	Limiting dist.
10	20	0.950	5.726	5.713 (0.002)	5.991 (0.046)
		0.975	6.896	6.964 (0.010)	7.378 (0.070)
		0.990	8.406	8.609 (0.024)	9.210 (0.096)
10	25	0.950	5.726	5.753 (0.005)	5.991 (0.046)
		0.975	6.929	7.023 (0.013)	7.378 (0.065)
		0.990	8.535	8.694 (0.029)	9.210 (0.079)
10	30	0.950	5.729	5.781 (0.009)	5.991 (0.046)
		0.975	6.953	7.065 (0.016)	7.378 (0.061)
		0.990	8.613	8.755 (0.016)	9.210 (0.069)
15	15	0.950	5.786	5.715 (0.012)	5.991 (0.036)
		0.975	6.956	6.967 (0.002)	7.378 (0.061)
		0.990	8.370	8.614 (0.029)	9.210 (0.069)
15	20	0.950	5.815	5.756 (0.010)	5.991 (0.030)
		0.975	7.021	7.028 (0.001)	7.378 (0.051)
		0.990	8.527	8.701 (0.020)	9.210 (0.100)
15	25	0.950	5.826	5.786 (0.007)	5.991 (0.028)
		0.975	7.043	7.072 (0.004)	7.378 (0.047)
		0.990	8.589	8.765 (0.021)	9.210 (0.072)
15	30	0.950	5.841	5.809 (0.006)	5.991 (0.026)
		0.975	7.081	7.105 (0.003)	7.378 (0.042)
		0.990	8.680	8.814 (0.015)	9.210 (0.061)
20	20	0.950	5.849	5.787 (0.011)	5.991 (0.024)
		0.975	7.086	7.073 (0.002)	7.378 (0.041)
		0.990	8.630	8.767 (0.016)	9.210 (0.067)
20	25	0.950	5.866	5.810 (0.009)	5.991 (0.021)
		0.975	7.099	7.108 (0.001)	7.378 (0.039)
		0.990	8.669	8.818 (0.017)	9.210 (0.062)
20	30	0.950	5.869	5.829 (0.007)	5.991 (0.021)
		0.975	7.123	7.135 (0.002)	7.378 (0.036)
		0.990	8.732	8.857 (0.014)	9.210 (0.055)

Table 3
The case of the $C_{2,4}$ test statistic.

n_1	n_2	$1-\alpha$	Exact	Gamma approx.	Limiting dist.
10	20	0.950	5.699	5.697 (0.000)	5.991 (0.051)
		0.975	6.873	6.941 (0.010)	7.378 (0.073)
		0.990	8.412	8.576 (0.019)	9.210 (0.095)
10	25	0.950	5.770	5.757 (0.002)	5.991 (0.038)
		0.975	6.956	7.030 (0.011)	7.378 (0.061)
		0.990	8.541	8.704 (0.019)	9.210 (0.078)
10	30	0.950	5.783	5.810 (0.005)	5.991 (0.036)
		0.975	7.003	7.068 (0.009)	7.378 (0.053)
		0.990	8.643	8.760 (0.014)	9.210 (0.066)
15	15	0.950	5.796	5.726 (0.012)	5.991 (0.034)
		0.975	6.975	6.984 (0.001)	7.378 (0.058)
		0.990	8.435	8.637 (0.024)	9.210 (0.092)
15	20	0.950	5.805	5.765 (0.007)	5.991 (0.032)
		0.975	7.002	7.041 (0.006)	7.378 (0.054)
		0.990	8.515	8.721 (0.024)	9.210 (0.082)
15	25	0.950	5.833	5.793 (0.007)	5.991 (0.027)
		0.975	7.050	7.082 (0.005)	7.378 (0.047)
		0.990	8.627	8.780 (0.018)	9.210 (0.068)
15	30	0.950	5.833	5.814 (0.003)	5.991 (0.027)
		0.975	7.080	7.113 (0.005)	7.378 (0.042)
		0.990	8.685	8.825 (0.016)	9.210 (0.060)
20	20	0.950	5.850	5.795 (0.009)	5.991 (0.024)
		0.975	7.079	7.086 (0.001)	7.378 (0.042)
		0.990	8.685	8.785 (0.012)	9.210 (0.060)
20	25	0.950	5.858	5.818 (0.007)	5.991 (0.023)
		0.975	7.095	7.119 (0.003)	7.378 (0.040)
		0.990	8.689	8.833 (0.017)	9.210 (0.060)
20	30	0.950	5.855	5.835 (0.003)	5.991 (0.023)
		0.975	7.135	7.144 (0.001)	7.378 (0.034)
		0.990	8.785	8.870 (0.010)	9.210 (0.048)

Especially, if the above inequality is true as a strictly inequality for all $\boldsymbol{x}_1, \boldsymbol{x}_2$ which satisfy $f(\boldsymbol{x}_1) \neq f(\boldsymbol{x}_2)$, then f is a strictly quasi-convex function on S. Moreover, if we have strict inequality for each $\boldsymbol{x}_1, \boldsymbol{x}_2 \in S$ in the above definition, then the function f is said to be strongly quasi-convex function on S.

Definition 2 (*Pseudo-convex Function*)**.** Let $f : S \to \mathbb{R}$ be a differentiable function on S, where S is a nonempty open set in $\mathbb{R}^n$. If for each $\boldsymbol{x}_0, \boldsymbol{x} \in S$, $\boldsymbol{x}_0 \neq \boldsymbol{x}$,

$$\nabla f(\boldsymbol{x}_0)'(\boldsymbol{x} - \boldsymbol{x}_0) \geq 0 \text{ implies } f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0),$$

where ∇g is the gradient of g. Especially, if the above inequality can be rewritten as $f(\boldsymbol{x}) > f(\boldsymbol{x}_0)$ for all $\boldsymbol{x}_0, \boldsymbol{x} \in S$ and $\boldsymbol{x}_0 \neq \boldsymbol{x}$, then f is a pseudo-convex function on S.

In Section 3.1, we show the existence and correlation of the $C_{p,q}$ test statistic. In Section 3.2, we derive the asymptotic efficiency of the $C_{p,q}$ test statistic. Furthermore, we show that the original Cucconi test statistic $C_{2,2}$ is equivalent to the Lepage-type test statistic $L_{1,2}$ proposed by Pettitt (1976).

3.1. Existence and correlation

In this section, we consider the existence of the $C_{p,q}$ test statistic which depends on ρ^*. In addition, we investigate ρ^*'s behavior by p and q. According to the covariance of Q_p and Q_q, we obtain Theorem 3.

Theorem 3. *For p, q, we have two cases:*

(a) *The $C_{p,q}$ statistic exists except for $p = q = 1$.*
(b) *Since ρ^* is a two-variable function with p and q, ρ^* is a quasi-convex function on $(p, q) \in \mathbb{R}_+^2$, where $\mathbb{R}_+ = (0, \infty)$.*

Therefore, the $C_{p,q}$ test statistic exists for the case $\boldsymbol{d} \neq \mathbf{1}_2$. In addition, the correlation ρ^* becomes weak as the value $\boldsymbol{d}$ goes away from $\mathbf{1}_2$.

Proof of Theorem 3(a). First, we give Lemmas 1 and 2 to prove (a).

Lemma 1. *For $p, q > 0$, we have*

$$-\frac{1}{4} \le \Sigma_{\psi,\phi} < 0$$

by the Grüss's inequality and the equality of the left-side holds if and only if $(p, q) = (1, 1)$.

By Lemma 1, the correlation between Q_p and Q_q is negative. Thus, it is only necessary to consider the case of $\rho^* = -1$. To evaluate the correlation, we introduce Lemma 2.

Lemma 2 (Agarwal, Elezović, & Pecaric, 2005). *Let $p, q > 0$ and $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$. Then we have*

$$\left| B(1+p, 1+q) - \frac{1}{(1+p)(1+q)} \right| \le \frac{pq}{(1+p)(1+q)\sqrt{(1+2p)(1+2q)}}$$

and the equality occurs if and only if $p = q = 1$.

Then (a) immediately follows from Lemmas 1 and 2.

Proof of Theorem 3(b). Second, we give Lemma 3 to prove (b).

Lemma 3. *Let $f : S \to \mathbb{R}$ be a strictly pseudo-convex function on S, where S is a nonempty open set in $\mathbb{R}^n$. Then f is a strongly quasi-convex function on S.*

Then, it is enough to prove that $\tilde{\rho} := \rho^*(p, q)$ is the strictly pseudo-convex function at $\mathbf{1}_2 = (1, 1)'$ by Lemma 3. We prove that $\tilde{\rho}$ is a strictly pseudo-convex function at the specific point. Noticing that $\boldsymbol{d} = (p, q)'$ and $\nabla\tilde{\rho}(\boldsymbol{d}) = \mathbf{0}$, we obtain

$$\nabla\tilde{\rho}(\boldsymbol{d})'(\boldsymbol{d} - \mathbf{1}_2) = \mathbf{0}'(\boldsymbol{d} - \mathbf{1}_2) = 0$$

on $\boldsymbol{d} \in \mathbb{R}_+^2 - \{\mathbf{1}_2\} \overset{\text{def.}}{=} D$. Then $\tilde{\rho}(\boldsymbol{d}) > \tilde{\rho}(\mathbf{1}_2)$ on D by Lemma 2. In brief, $\tilde{\rho}(\boldsymbol{d})$ is the strictly pseudo-convex function at $\mathbf{1}_2$.

For the proof of Lemmas 1–3, see Appendix A.

Remark 3. When

$$\begin{bmatrix} 1 & \rho^* \\ \rho^* & 1 \end{bmatrix}$$

is singular, one possibility is that we define the inversed matrix by Moore–Penrose inverse. Then the limiting distribution of the $C_{p,q}$ test statistic is chi-square distribution with one degree of freedom for all $p, q > 0$ in this case.

3.2. Asymptotic efficiency

In this section, we derive the asymptotic efficiency of the $C_{p,q}$ test statistic. Marozzi (2008) compared the power of the original Cucconi test statistic with the original Lepage test statistic by a simulation study. However, the result of simulation study did not show a critical winner. We consider a critical winner for the hypotheses by the asymptotic efficiency of the $C_{p,q}$ test statistic relative to the original Lepage test statistic. Murakami (2016b) proposed the procedure to calculate the asymptotic efficiency for the combination of two linear rank test statistics. It is well known that the ARE of a test statistic with respect to another test statistic is equal to the ratio of two noncentral parameters (Andrews, 1954). Under Assumption 4, the asymptotic efficiency of the $C_{p,q}$ test statistic is given by

$$e(C_{p,q}) = B_3\{\text{var}\left[Q_q\right]B_1^2 + \text{var}\left[Q_p\right]B_2^2 - 2\text{cov}(Q_p, Q_q)B_1B_2\},$$

where

$$\begin{aligned}
B_1 &= \int_{-\infty}^{\infty}(\ell x + 1)\frac{d\psi(F(x); p)}{dx}dF(x) = p\int_{-\infty}^{\infty}(\ell x + 1)F(x)^{p-1}f(x)^2dx, \\
B_2 &= \int_{-\infty}^{\infty}(\ell x + 1)\frac{d\phi(F(x); q)}{dx}dF(x) \\
&= -q\int_{-\infty}^{\infty}(\ell x + 1)(1 - F(x))^{q-1}f(x)^2dx, \\
B_3 &= \left\{\text{var}(Q_p)\text{var}(Q_q) - \text{cov}(Q_p, Q_q)^2\right\}^{-1} \\
&= \left\{\frac{p^2q^2}{(1+p)^2(1+q)^2(1+2p)(1+2q)}\right. \\
&\quad \left. - \left(\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(2+p+q)} - \frac{1}{(1+p)(1+q)}\right)^2\right\}^{-1},
\end{aligned}$$

Table 4
Asymptotic efficiency of several test statistics under the symmetric distributions.

Distribution	$e(C_{2,2})$	$e(C_{2,3})$
Cauchy	$3/\pi^2 + 45\ell^2/\pi^4$	$\frac{5040-1680\pi^2+905\pi^4}{272\pi^6} + \frac{1512-99\pi^2}{136\pi^5}\ell + \frac{2421}{272\pi^4}\ell^2$
Laplace	$3/4 + 125\ell^2/144$	$25/34 + 75\ell/1088 + 33625\ell^2/39168$
Logistic	$1/3 + 5\ell^2/4$	$453/1360 + \ell/272 + 1345\ell^2/1088$
Normal	$3/\pi + 15\ell^2/\pi^2$	$0.964263 - 0.0402474\ell + 1.50306\ell^2$
t with 2 d.f.	$27\pi^2/512 + 4\ell^2/5$	$\frac{225\pi^2}{4352} + \frac{27\pi}{1088\sqrt{2}}\ell + \frac{269}{340}\ell^2$
t with 3 d.f.	$25/4\pi^2 + 6125\ell^2/64\pi^4$	$0.625938 + 0.0391485\ell + 0.97165\ell^2$
t with 4 d.f.	$297675\pi^2/4194304 + 6480\ell^2/5929$	$0.695615 + 0.0264742\ell + 1.08088\ell^2$
Uniform	$12 + 12\ell + 8\ell^2$	$905/68 + 185\ell/17 + 120\ell^2/17$
Distribution	$e(C_{2,4})$	$e(C_{2,0.50})$
Cauchy	$\frac{945(105-25\pi^2+6\pi^4)}{1516\pi^6} - \frac{210-210\pi^2+13\pi^4}{1516\pi^7}\ell + \frac{105-185\pi^2+86\pi^4}{1516\pi^8}\ell^2$	$0.318124 + 0.355626\ell + 0.257701\ell^2$
Laplace	$\frac{70245}{97024} + \frac{25949}{194048}\ell + \frac{295074689}{349286400}\ell^2$	$0.705125 + 0.430871\ell + 0.256721\ell^2$
Logistic	$126/379 + 49\ell/4548 + 33257\ell^2/27288$	$0.320833 + 0.214708\ell + 0.365023\ell^2$
Normal	$0.971641 - 0.0699408\ell + 1.49112\ell^2$	$0.982883 + 0.277788\ell + 0.361122\ell^2$
t with 2 d.f.	$\frac{632205\pi^2}{12419072} + \frac{2367\pi}{48512\sqrt{2}}\ell + \frac{1456}{1895}\ell^2$	$0.493869 + 0.39675\ell + 0.35\ell^2$
t with 3 d.f.	$0.61946 + 0.0792186\ell + 0.949608\ell^2$	$0.593681 + 0.380266\ell + 0.373625\ell^2$
t with 4 d.f.	$0.690793 + 0.0557193\ell + 1.06004\ell^2$	$0.659666 + 0.362552\ell + 0.380183\ell^2$
Uniform	$5670/379 + 3885\ell/379 + 2471\ell^2/379$	$315/4 + 105\ell + 35\ell^2$

Table 5
Asymptotic efficiency of several test statistics under the asymmetric distributions.

Distribution	$e(C_{2,2})$	$e(C_{2,3})$
Exponential	$6 + 4\ell/3 + 8\ell^2/9$	$645/68 + 95\ell/68 + 15\ell^2/17$
Gumbel	$0.88889 + 0.42039\ell + 1.28272\ell^2$	$0.882353 + 0.357172\ell + 1.21574\ell^2$
Rayleigh	$\frac{(76-30\sqrt{6})\pi}{3} + \frac{(72-20\sqrt{6})\sqrt{\pi}}{9}\ell + \frac{32}{9}\ell^2$	$-\frac{5(-651+252\sqrt{2}-112\sqrt{3}+180\sqrt{6})\pi}{272} - \frac{5(-99+7\sqrt{2}+22\sqrt{6})\sqrt{\pi}}{68}\ell + \frac{60}{17}\ell^2$
Levy	$1.63153 + 1.36914\ell + 0.413082\ell^2$	$1.96953 + 1.53626\ell + 0.429872\ell^2$
Maxwell	$2.26748 + 5.93179\ell + 5.46157\ell^2$	$2.33321 + 5.99697\ell + 5.44266\ell^2$
Pareto	$\frac{557056}{4563} + \frac{9728}{39}\ell + 128\ell^2$	$\frac{120913920}{830297} + \frac{85760}{289}\ell + \frac{2580}{17}\ell^2$
Distribution	$e(C_{2,4})$	$e(C_{2,0.50})$
Exponential	$\frac{4256}{379} + \frac{8246}{5685}\ell + \frac{74879}{85275}\ell^2$	$35/4 + 35\ell/12 + 385\ell^2/432$
Gumbel	$0.878089 + 0.309331\ell + 1.16042\ell^2$	$0.284379 + 0.255627\ell + 0.227833\ell^2$
Rayleigh	$\frac{53459-14700\sqrt{6}-8505\sqrt{10}+3780\sqrt{15}}{5685} + \frac{592200-131600\sqrt{6}-14742\sqrt{10}}{85275}\ell + \frac{299516}{85275}\ell^2$	$\frac{(50715+12320\sqrt{2}-18480\sqrt{3}-14700\sqrt{6})\pi}{144} - \frac{(315-280\sqrt{3}-70\sqrt{6})\sqrt{\pi}}{108}\ell + \frac{385}{108}\ell^2$
Levy	$2.34135 + 1.70251\ell + 0.443922\ell^2$	$1.88957 + 1.25305\ell + 0.348967\ell^2$
Maxwell	$2.39234 + 6.05338\ell + 5.43233\ell^2$	$2.28165 + 6.79899\ell + 5.59535\ell^2$
Pareto	$\frac{99921920}{576459} + \frac{400640}{1137}\ell + \frac{68096}{379}\ell^2$	$\frac{599680}{4563} + \frac{3520}{13}\ell + 140\ell^2$

and $\ell = a_i/b_i$. Note that $e(C_{p,q})$ can be expressed by the noncentral parameter $\boldsymbol{w}'\Sigma^{-1}\boldsymbol{w}$, see Section 2.1. We discuss the ARE for the Cauchy, Laplace, logistic, normal, Student's t with two, three and four degrees of freedom, exponential, Gumbel, Rayleigh, Levy, Maxwell, Pareto and uniform distributions. In this paper, we determine $(p, q) = (2, 0.5)$, $(2, 2)$, $(2, 3)$ and $(2, 4)$. Herein, we list the asymptotic efficiency of the $C_{p,q}$ test statistic for symmetric distributions and asymmetric distributions in Tables 4 and 5, respectively. The hypothesis is equivalent to the location problem when $\ell = 0$. Conversely, the hypothesis is equivalent to the scale problem when ℓ goes to infinity.

Furthermore, we obtain Theorem 4 for the symmetric distributions.

Theorem 4. *Suppose that $F(x)$ is the symmetric about 0. Then $e(C_{p,q}; \ell) = e(C_{q,p}; -\ell)$ is true.*

Proof of Theorem 4. By definition of B_1, B_2, B_3 and $\mathrm{cov}\left(Q_p, Q_q\right)$, it is enough to show the following equation:

$$h(\ell; p, q) = h(-\ell; q, p),$$

where

$$
\begin{aligned}
h(\ell; p, q) &\stackrel{\text{def.}}{=} \text{var}\left[Q_q\right] B_1^2 + \text{var}\left[Q_p\right] B_2^2 \\
&= \frac{q^2}{(1+q)^2(1+2q)} \left(p \int_{-\infty}^{\infty} (\ell x + 1) F(x)^{p-1} f(x)^2 dx \right)^2 \\
&\quad + \frac{p^2}{(1+p)^2(1+2p)} \left(-q \int_{-\infty}^{\infty} (\ell x + 1)(1 - F(x))^{q-1} f(x)^2 dx \right)^2 \\
&= \frac{p^2 q^2}{(1+q)^2(1+2q)} \left(\int_{-\infty}^{\infty} (\ell x + 1) F(x)^{p-1} f(x)^2 dx \right)^2 \\
&\quad + \frac{p^2 q^2}{(1+p)^2(1+2p)} \left(\int_{-\infty}^{\infty} (\ell x + 1)(1 - F(x))^{q-1} f(x)^2 dx \right)^2 .
\end{aligned}
$$

Note that $F(x) = 1 - F(-x)$ and $f(x) = f(-x)$ when F is symmetric about 0. Then, by transforming x into $-t$,

$$
\begin{aligned}
h(\ell; p, q) &= \frac{p^2 q^2}{(1+q)^2(1+2q)} \left(\int_{-\infty}^{\infty} (\ell(-t) + 1) F(-t)^{p-1} f(-t)^2 dx \right)^2 \\
&\quad + \frac{p^2 q^2}{(1+p)^2(1+2p)} \left(\int_{-\infty}^{\infty} (\ell(-t) + 1)(1 - F(-t))^{q-1} f(-t)^2 dx \right)^2 \\
&= \frac{q^2}{(1+q)^2(1+2q)} \left((-p) \int_{-\infty}^{\infty} ((-\ell)t + 1)(1 - F(t))^{p-1} f(t)^2 dt \right)^2 \\
&\quad + \frac{p^2}{(1+p)^2(1+2p)} \left(q \int_{-\infty}^{\infty} ((-\ell)t + 1) F(t)^{q-1} f(t)^2 dt \right)^2 \\
&= h(-\ell; q, p).
\end{aligned}
$$

It is indicated that we can only consider the positive or negative difference in the case of $p = q$ and F is symmetric about 0. To investigate the features of various cases, we plot the ARE, that is $e(C_{2,2})/e(C_{p,q})$ in Figs. 1 and 2.

As a variable q is shifted, ARE is nearly equivalent to 1 except for the specific cases. In the distributions defined on real line, the $C_{2,3}$ test statistic is more powerful than the other cases when $\ell \approx 0$. We obtain the same result in asymmetric cases except for the Levy and Pareto distributions. It is indicated that if the distributions of two populations X_1 and X_2 are unknown, but $X_1, X_2 \in \mathbb{R}$ and we investigate the stochastic larger $X_1 > X_2$ or $X_1 < X_2$, then we recommend to utilize the $C_{2,3}$ test statistic. Changing p, q is the meaningful to search the optimal statistic. Moreover it is revealed that the $C_{2,3}$ test statistic is more efficient than the $L_{1,1}$ statistic when we assume the light-tail distributions.

Focusing on $p = q = 2$, we obtain the noticeable result that the asymptotic efficiency of the $C_{2,2}$ test statistic is exactly equivalent to the Lepage-type test statistic $L_{1,2}$ proposed by Pettitt (1976), see Murakami (2016b), that is $e(C_{2,2}) = e(L_{1,2})$. To derive the connection between these two test statistics, we introduce Lemma 4.

Lemma 4. *Suppose that $p = q = 2$. The following identical equations are true.*

$$
\begin{aligned}
&Q_p^* = Q_p - \mathrm{E}[Q_p] = M - \mathrm{E}[M] + (N+1)(W - \mathrm{E}[W]), \\
&Q_q^* = Q_q - \mathrm{E}[Q_q] = M - \mathrm{E}[M] - (N+1)(W - \mathrm{E}[W]), \\
&1 + \rho = \frac{2\text{var}[M]}{\sigma^2}, \quad 1 - \rho = \frac{2(N+1)^2 \text{var}[W]}{\sigma^2}, \\
&W = \sum_{t=1}^{n_1} R_{1t}, \quad \mathrm{E}[W] = \frac{n_1(N+1)}{2}, \quad \text{var}[W] = \frac{n_1 n_2 (N+1)}{12}, \\
&M = \sum_{t=1}^{n_1} \left(R_{1t} - \frac{N+1}{2} \right)^2, \quad \mathrm{E}[M] = \frac{n_1(N^2-1)}{12}, \\
&\text{var}[M] = \frac{n_1 n_2 (N+1)(N^2-4)}{180}.
\end{aligned}
$$

By Lemma 4, we obtain Theorem 5.

Theorem 5. *The original Cucconi test statistic is equivalent to the Lepage-type test statistic proposed by Pettitt (1976).*

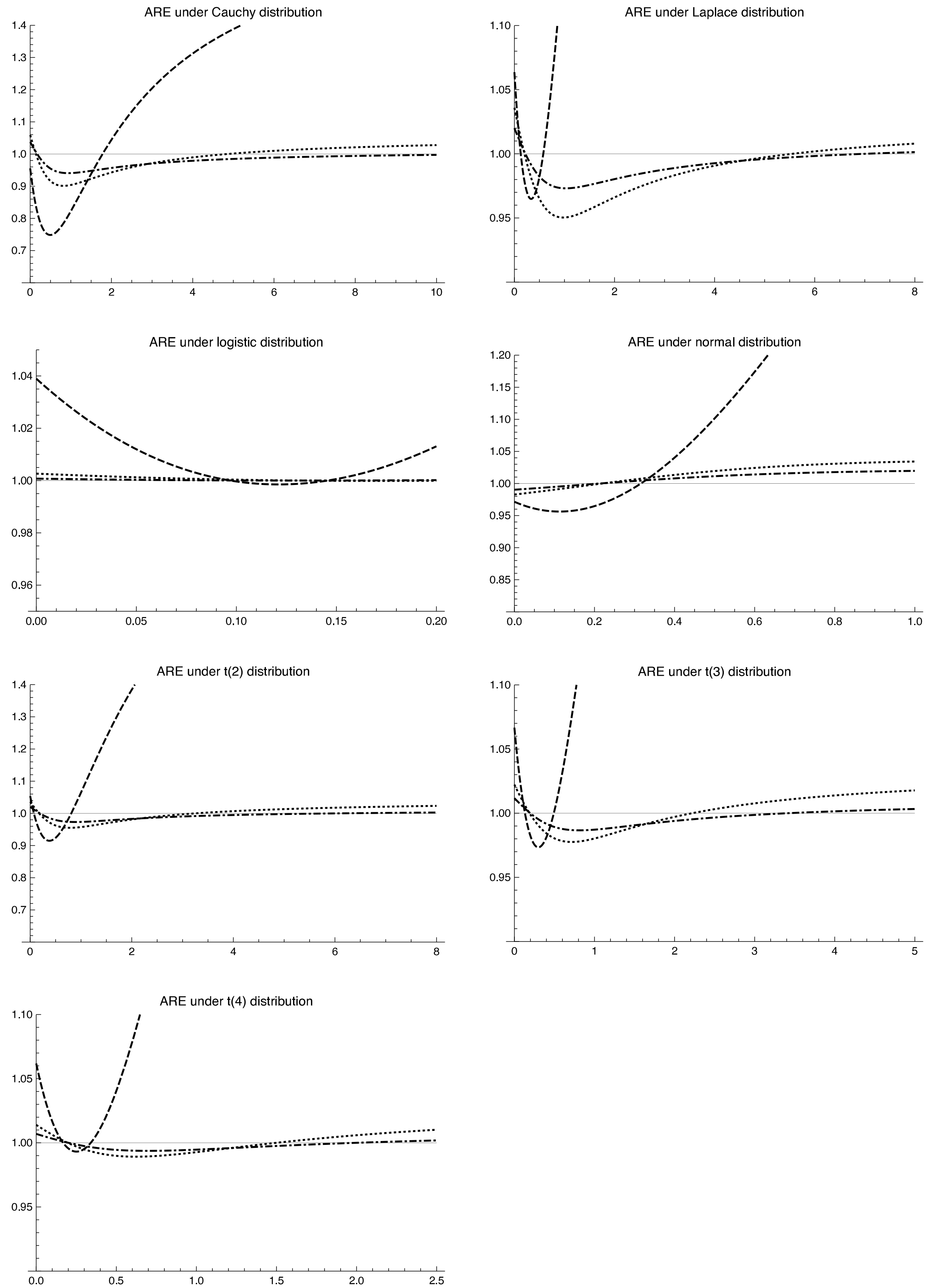

Fig. 1. ARE $e(C_{2,2})/e(C_{p,q})$ under the symmetric distributions. – – – : with $C_{2,3}$, $\cdots$: with $C_{2,4}$, – · – ·: with $C_{2,0.5}$.

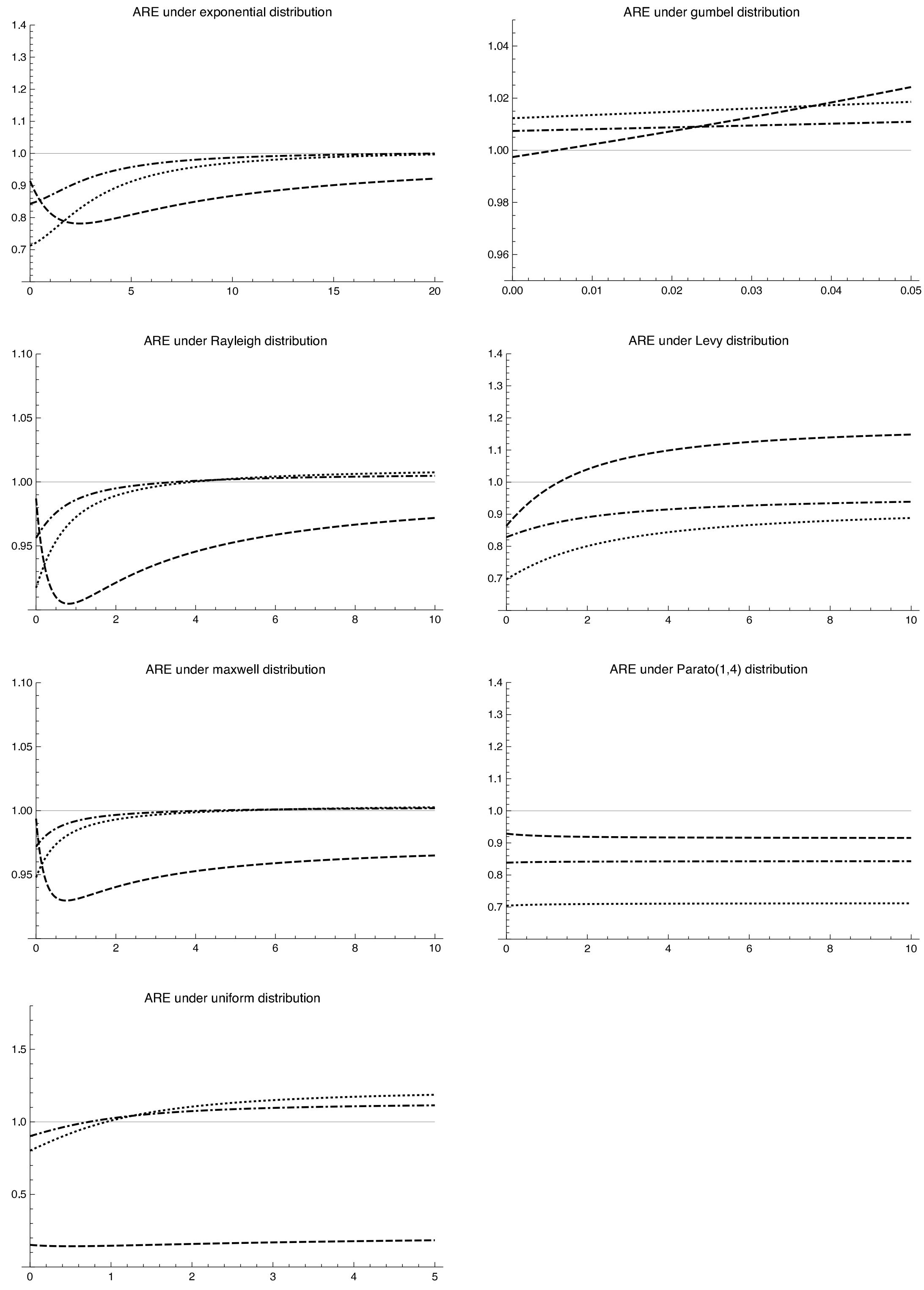

Fig. 2. ARE $e(C_{2,2})/e(C_{p,q})$ under various asymmetric distributions. – – – : with $C_{2,3}$, $\cdots$: with $C_{2,4}$, – · – ·: with $C_{2,0.5}$.

Proof of Theorem 5. We show the calculation process. Lemma 4 makes calculation easier.

$$
\begin{aligned}
C_{2,2} &= \frac{U^2+V^2-2\rho UV}{1-\rho^2} \\
&= \frac{(U-V)^2+2(1-\rho)UV}{1-\rho^2} \\
&= \frac{(Q_p-Q_q)^2+2(1-\rho)Q_p^*Q_q^*}{\sigma^2(1-\rho^2)} \\
&= \frac{2(1+\rho)(N+1)^2(W-\mathrm{E}[W])^2+2(1-\rho)(M-\mathrm{E}[M])^2}{\sigma^2(1-\rho^2)} \\
&= \frac{2(N+1)^2\mathrm{var}[W]}{\sigma^2(1-\rho)}\cdot\frac{(W-\mathrm{E}[W])^2}{\mathrm{var}[W]}+\frac{2\mathrm{var}[M]}{\sigma^2(1+\rho)}\cdot\frac{(M-\mathrm{E}[M])^2}{\mathrm{var}[M]} \\
&= \frac{(W-\mathrm{E}[W])^2}{\mathrm{var}[W]}+\frac{(M-\mathrm{E}[M])^2}{\mathrm{var}[M]} = L_{1,2}.
\end{aligned}
$$

3.3. Consistency of the original Cucconi test statistic

In previous section, we showed the equality between the original Cucconi test statistic and the Lepage-type test statistic proposed by Pettitt (1976). In Cucconi (1968), the consistency of the original Cucconi test statistic is only discussed for the shifted location parameter. In this section, we consider the conditions for the consistency of the original Cucconi test statistic. For a combination of two independent test statistics under the general class of alternatives H_A, we obtain Lemma 5. The consistency is defined by Definition 3.

Definition 3 (*Consistency*)**.** Suppose that n_1 and n_2 are functions of N, and consider the family of sequences of critical regions $T \geq t_\alpha(N)$, where the significance level $\alpha \in (0, 1)$ indexes the sequences of the family, N indexes the members of sequence, and $t_\alpha(N)$ is the least solution that satisfies $\Pr(T \geq t_\alpha(N)) \leq \alpha$ under the null hypothesis. If for all $\alpha \in (0, 1)$,

$$\lim_{N\to\infty} \Pr(T \geq t_\alpha(N)) = 1,$$

where the probabilities are taken under the alternative hypothesis. Then the test based on large value T is consistent.

Lemma 5. *For the null hypothesis H_0 versus a general class of alternatives H_A, the $L_{1,2}$ test statistic is consistent if the two-sided Wilcoxon test statistic W or the Mood test statistic M is consistent for H_A.*

Proof of Lemma 5. From the inequalities

$$
\begin{aligned}
&1 \geq \Pr(L_{1,2} \geq t|H_A) \geq \Pr(\{W-\mathrm{E}[W]\}^2/\mathrm{var}[W] \geq t|H_A), \\
&1 \geq \Pr(L_{1,2} \geq t|H_A) \geq \Pr(\{M-\mathrm{E}[M]\}^2/\mathrm{var}[M] \geq t|H_A),
\end{aligned}
$$

the result is immediate followed.

To investigate the consistency of the Cucconi test statistic, we introduce the following lemmas.

Lemma 6 (*Kruskal, 1952*)**.** *The Wilcoxon test statistic based on large value W is consistent if F is the continuous distribution.*

Lemma 7 (*van Eeden, 1964*)**.** *The Mood test statistic based on large value M is consistent if F is the continuous distribution and symmetric about 0.*

Taking into account Theorem 5, Lemmas 6 and 7, we have Corollary 1.

Corollary 1. *The original Cucconi test statistic based on large value is consistent for the cases (i) or (ii).*

(i) *F_1 and F_2 are continuous and the alternative hypothesis is the shifted location parameter, that is, "H_1: $F_2(x) = F_1(x-\mu)$".*
(ii) *F_1 and F_2 are continuous and symmetric about 0. In addition, the alternative is the changed scale parameter, that is, "H_1: $F_2(x) = F_1(\sigma x)$".*

4. The original Cucconi test statistic in the presence of ties

In this section, we mainly focus on the original Cucconi test statistic $C_{2,2}$ in the presence of ties. Hereafter, we suppose a general distribution function, which includes a discrete distribution. An original Cucconi test statistic is not suitable in the presence of ties. In order to correspond to the case of existing tied rank, firstly, we discuss the testing hypotheses similar to Paul and Mielke (1967). The observations in the pooled sample $(X_{11}, X_{12}, \ldots, X_{1n_1}, X_{21}, X_{22}, \ldots, X_{2n_2})$ are arranged in increasing order according to their real values. It is assumed that there are k classes of obtained values in the resulting ranking scheme. We assign the number $j \in \{1, \ldots, k\}$ to these classes in ascending order of the representative value. Note that tied observations with the same value from one class. Here let τ_j be the number of observations in the jth class, and $n_1 + n_2 = \sum_{j=1}^{k} \tau_j = N$ is the total number of observations in the pooled sample. Let α_j and β_j be the number of values related to the X_1's and X_2's in the jth class respectively, i.e. $\alpha_j + \beta_j = \tau_j$ for $j = 1, \ldots, k$. Define that $J_j = \sum_{h=1}^{j} \tau_h$ and $\alpha_0 = \beta_0 = \tau_0 = J_0 = 0$. In the presence of ties, the sums of squared rank Z_1 and Z_2 are respectively presented by

$$Z_1 = \sum_{j=1}^{k} \frac{\alpha_j}{\tau_j} \sum_{l=J_{j-1}+1}^{J_j} l^2, \quad Z_2 = \sum_{j=1}^{k} \frac{\alpha_j}{\tau_j} \sum_{l=J_{j-1}+1}^{J_j} (N+1-l)^2.$$

Herein we derive the moments of Z_s, $s = 1, 2$, under the null hypotheses with a similar procedure like that of Paul and Mielke (1967).

$$\begin{aligned}
\mathrm{E}[Z_s] &= \frac{n_1(N+1)(2N+1)}{6}, \\
\mathrm{var}[Z_1] &= \frac{n_1 n_2 (N+1)(2N+1)(8N+11)}{180} \\
&\quad - \frac{n_1 n_2}{180N(N-1)} \sum_{j=1}^{k} \tau_j(\tau_j^2 - 1)[(2\tau_j+1)(8\tau_j+11) + 60J_{j-1}^2 \\
&\quad + 60(\tau_j+1)J_{j-1}], \\
\mathrm{var}[Z_2] &= \frac{n_1 n_2 (N+1)(2N+1)(8N+11)}{180} \\
&\quad - \frac{n_1 n_2}{180N(N-1)} \sum_{j=1}^{k} \tau_j(\tau_j^2 - 1)[(2\tau_j+1)(8\tau_j+11) \\
&\quad + 60(N - J_{j-1} - \tau_j)(N - J_{j-1} + 1)], \\
\mathrm{cov}(Z_1, Z_2) &= -\frac{n_1 n_2 (N+1)(14N^2 + 30N + 19)}{180} \\
&\quad + \frac{n_1 n_2}{180N(N-1)} \sum_{j=1}^{k} \tau_j(\tau_j^2 - 1)[15(N+1)^2 \\
&\quad - 15(N - J_j - J_{j-1})^2 - \tau_j^2 + 4].
\end{aligned}$$

The detail of calculating these moments is provided by Corollary 2 in Appendix B. Let us denote $\mu^* = \mathrm{E}[Z_1] = \mathrm{E}[Z_2]$ and $\sigma_s^* = \sqrt{\mathrm{var}[Z_s]}$. Using these moments, we obtain the Cucconi test statistic in the presence of ties as follows:

$$C^* = \frac{U_*^2 + V_*^2 - 2\rho_*^2 U_* V_*}{2(1 - \rho_*^2)},$$

where

$$U_* = \frac{Z_1 - \mu^*}{\sigma_1^*}, \; V_* = \frac{Z_2 - \mu^*}{\sigma_2^*}, \; \rho_* = \frac{\mathrm{cov}(Z_1, Z_2)}{\sigma_1^* \sigma_2^*}.$$

In addition, we focus on the following two test statistics:

$$\begin{aligned}
W^* &= \sum_{j=1}^{k} \frac{\alpha_j}{\tau_j} \sum_{l=J_{j-1}+1}^{J_j} l, \\
M^* &= \sum_{j=1}^{k} \frac{\alpha_j}{\tau_j} \sum_{l=J_{j-1}+1}^{J_j} \left(l - \frac{N+1}{2} \right)^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
W' &= W^* - \mathrm{E}[W^*] = \frac{1}{2(N+1)} \{(Z_1 - \mu^*) - (Z_2 - \mu^*)\}, \\
M' &= M^* - \mathrm{E}[M^*] = \frac{1}{2} \{(Z_1 - \mu^*) + (Z_2 - \mu^*)\}.
\end{aligned}$$

From these equations, we represent $\mathrm{cov}(W^*, M^*)$ as follows:

$$\begin{aligned}\mathrm{cov}(W^*, M^*) &= \mathrm{E}[(W^* - \mathrm{E}[W^*])(M^* - \mathrm{E}[M^*])] \\ &= \frac{1}{4(N+1)}\mathrm{E}[(Z_1 - \mu^*)^2 - (Z_2 - \mu^*)^2] \\ &= \frac{n_1 n_2}{12N(N-1)} \sum_{j=1}^{k} \tau_j(\tau_j^2 - 1)(N - J_j - J_{j-1}).\end{aligned}$$

By Paul and Mielke (1967), we have

$$\begin{aligned}\mathrm{var}[W^*] &= \frac{n_1 n_2 (N+1)}{12} - \frac{n_1 n_2}{12N(N-1)} \sum_{j=1}^{k} \tau_j(\tau_j^2 - 1), \\ \mathrm{var}[M^*] &= \frac{n_1 n_2 (N+1)(N^2-4)}{180} - \frac{n_1 n_2}{180N(N-1)} \sum_{j=1}^{k} \tau_j(\tau_j^2 - 1) \\ &\quad \times \{\tau_j^2 - 4 - 15(N - J_i - J_{i-1})^2\}.\end{aligned}$$

Therefore, we obtain the Lepage-type test statistic in the presence of ties as follows:

$$L_{1,2}^* = \frac{W'^2 \mathrm{var}[M^*] - 2W'M'\mathrm{cov}(W^*, M^*) + M'^2 \mathrm{var}[W^*]}{\mathrm{var}[W^*]\mathrm{var}[M^*] - \mathrm{cov}(W^*, M^*)^2}.$$

Note that $2C^* \neq L_{1,2}$ in the presence of ties. For example, assume that $X_1 = \{1, 1, 2\}$, $X_2 = \{1, 3\}$. Then we obtain $2C^* = 8427/6808$ and $L_{1,2}^* = 16/9$. Therefore, it is meaningful to propose C^* and $L_{1,2}^*$ individually.

5. Conclusion and discussion

In this paper, we proposed the generalized Cucconi test statistic, named $C_{p,q}$, for the location, scale and location–scale parameters in the two-sample problem. The limiting distribution of the suggested test statistic was derived under the hypotheses. Under the null hypothesis, the limiting distribution of the $C_{p,q}$ test statistic is the chi-square distribution with two degrees of freedom. Additionally, we derived the non-central parameter of chi-square distribution under the alternative hypothesis. In addition, it is difficult to derive the exact critical value of the proposed test statistic when the sample sizes are increased. Therefore, we considered the gamma approximation to the distribution of the $C_{p,q}$ test statistic. The gamma approximation was more suitable than that of the limiting distribution of the $C_{p,q}$ test statistic for the moderate sample sizes with various p and q. Moreover, we calculate the asymptotic efficiency of the $C_{p,q}$ test statistic. In addition, the original Cucconi test statistic is exactly equivalent to the Lepage-type test statistic proposed by Pettitt (1976) for the continuous distributions but not for the discrete distributions.

Let $X_i = \{X_{ij}; i = 1, 2, \ldots, K, j = 1, 2, \ldots, n_i\}$ be K independent samples of size n_i from the continuous distribution function $F_i(x)$. Let R_{ij} be the rank of X_{ij} in the pooled sample $(X_1, \ldots, X_K)$. Recently, Nishino and Murakami (2018) considered the multisample Cucconi test statistic in following form:

$$MC_K^* = \sum_{i=1}^{K} \frac{N - n_i}{N} \frac{U_i^2 + V_i^2 - 2\rho_K U_i V_i}{1 - \rho_K^2},$$

where

$$\begin{aligned}U_i &= \frac{\displaystyle\sum_{j=1}^{n_i} R_{ij}^2 - \frac{n_i(N+1)(2N+1)}{6}}{\sqrt{\dfrac{n_i(N-n_i)(N+1)(2N+1)(8N+11)}{180}}}, \\ V_i &= \frac{\displaystyle\sum_{j=1}^{n_i} (N+1-R_{ij})^2 - \frac{n_i(N+1)(2N+1)}{6}}{\sqrt{\dfrac{n_i(N-n_i)(N+1)(2N+1)(8N+11)}{180}}}, \\ \rho_K &= \frac{2(N^2-4)}{(2N+1)(8N+11)} - 1, \quad N = n_1 + n_2 + \cdots + n_K.\end{aligned}$$

Then, the $C_{p,q}$ test statistic is easy to extend to the multisample version of test statistic as follows:

$$MC_{p,q,K} = \sum_{i=1}^{K} \frac{N-n_i}{N} \frac{U_{ip}^2 + V_{iq}^2 - 2\rho_{K*} U_{ip} V_{iq}}{1-\rho_{K*}^2},$$

where

$$U_{ip} = \frac{Q_{ip} - \mathrm{E}\left[Q_{ip}\right]}{\sqrt{\mathrm{var}\left[Q_{ip}\right]}}, \quad Q_{ip} = \sum_{j=1}^{n_i} R_{ij}^p,$$

$$V_{iq} = \frac{Q_{iq} - \mathrm{E}\left[Q_{iq}\right]}{\sqrt{\mathrm{var}\left[Q_{iq}\right]}}, \quad Q_{iq} = \sum_{j=1}^{n_i} (N+1-R_{ij})^q,$$

$$\rho_{K*} = \mathrm{corr}(U_{ip}, V_{iq}) = \mathrm{cov}(U_{ip}, V_{iq}).$$

In addition, the limiting distribution of the $MC_{p,q,K}$ test statistic under the null hypothesis is chi-square distribution with $2(K-1)$ degrees of freedom by a similar procedure of Nishino and Murakami (2018). However, calculation of the exact critical value of the test statistic is difficult when the sample sizes and the number of samples are increased. Then we have to approximate the exact distribution by an approximation method. Since the moment generating function of the MC_L^* test statistic, it is difficult to use a higher order approximation. In addition, the consistency of test statistic is important property in nonparametric statistics. It is not known whether the $MC_{p,q,K}$ test statistic is the Lepage-type test statistic or not for $p, q \neq 2$ and $K \geq 2$. Then, to derive the moment generating function and to show the consistency of proposed test statistic is one of the future works.

Various nonparametric test statistics were proposed to raise the ARE by using extra tuning parameters in many statistical journals such as Goria (1980), Pollicello and Hettmansperger (1976), Sen (1962, 1963) and Tamura (1963). The asymptotic efficiency of the $C_{p,q}$ test statistic depends on the parent distribution as same as previous references. If a density function can be assumed, we may determine p and q to maximize the asymptotic efficiency of the $C_{p,q}$ test statistic. However, if such a density function cannot fairly be guessed (or assumed), application of such a test seems to be dubious. Then, in practice, determination of optimal p and q is another future work.

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Appendix A. Proof of Lemmas

A.1. Proof of Lemma 1

$\Sigma_{\psi,\phi}$ is rewritten by

$$\Sigma_{\psi,\phi} = B(1+p, 1+q) - \frac{1}{(1+p)(1+q)},$$

where $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$. The inequality can be derived by the Grüss's inequality (see Grüss, 1935). In general, it is given by

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4}(A-\alpha)(B-\beta), \tag{A.1}$$

where f, g are integrable functions on $[a, b]$ and for all $x \in [a, b]$, $\alpha \leq f(x) \leq A$, $\beta \leq g(x) \leq B$ satisfies. In a similar manner of Dragomir, Agarwal, and Barnett (2000), we obtain the inequality by substituting the appropriate values which are $a = \alpha = \beta = 0$, $A = B = b = 1$, $f(x) = x^{1+p}$, $g(x) = x^{1+q}$. The sign of the left-hand side is derived by Theorem 3 of Dragomir et al. (2000). Herein we derive the condition to satisfy the equality because Dragomir et al. (2000) did not discuss about it. Set $\int_0^1 f(x)dx = F$ and $\int_0^1 g(x)dx = G$. $1-2F = 0$ and $1-2G = 0$ imply that the left-hand side and the right-hand side in (A.1) are equal. For our situation, $(p, q) = (1, 1)$ satisfies these conditions. Also, the left-hand side of (A.1) is equal to be 0 when $(p, q) = (0, 0)$. It is mentioned that we can omit the equalities on the right-hand side.

A.2. Proof of Lemma 2

The detail of proof is analogue to Dragomir (1998) and Agarwal et al. (2005). However, these papers did not discuss the case of equality. Herein, we derive the specific condition stated above. Since the equalities are induced by the Cauchy–Buniakowski–Schwarz integral inequality for double integral, we investigate the values $p, q > 0$ such that

$$\forall (u, v) \in [0, 1]^2, \quad u^p - v^p = k\{(1-u)^q - (1-v)^q\}.$$

Note that, the Cauchy–Buniakowski–Schwarz integral inequality for double integral is denoted by

$$\left(\iint_{[a,b]^2} f(x)g(y)h(x)h(y)dxdy \right)^2 \\ \leq \iint_{[a,b]^2} f(x)^2 h(x)h(y)dxdy \iint_{[a,b]^2} g(y)^2 h(x)h(y)dxdy,$$

where a, b are constant, two integrable functions f, g are defined on $[a, b]$, and $h : [a, b] \to \mathbb{R}^+$ is satisfied $\int_{[a,b]} h(x)dx > 0$. We choose three cases $(u, v) = (1, 0), (1, 1/2), (2/3, 1/3)$ and solve the simultaneous equations for p, q and k, then we obtain $p = q = 1$ and $k = -1$. The solution establishes the equation for all $(u, v) \in [0, 1]^2$.

A.3. Proof of Lemma 3

Assume that there exists $\hat{\boldsymbol{x}} = \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$ such that two distinct vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in S$ with $f(\hat{\boldsymbol{x}}) \geq \max\{f(\boldsymbol{x}_1), f(\boldsymbol{x}_2)\}$ and there exists $\lambda \in (0, 1)$. Taking account of the relation $f(\boldsymbol{x}_1) \leq f(\hat{\boldsymbol{x}})$ for $\hat{\boldsymbol{x}} \neq \boldsymbol{x}_1$ and pseudo-convexity of f, we obtain $\nabla f(\hat{\boldsymbol{x}})'(\boldsymbol{x}_1 - \hat{\boldsymbol{x}}) < 0$. Substituting $\hat{\boldsymbol{x}} = \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$ to it, we have $\nabla f(\boldsymbol{x}_0)'(\boldsymbol{x}_1 - \boldsymbol{x}_2) < 0$. Similarly, considering the other relation $f(\boldsymbol{x}_2) \leq f(\hat{\boldsymbol{x}})$ for $\hat{\boldsymbol{x}} \neq \boldsymbol{x}_2$, we derive the inequality $\nabla f(\hat{\boldsymbol{x}})'(\boldsymbol{x}_2 - \boldsymbol{x}_1) < 0$. Since two above inequalities have never incompatible, f is strongly quasi-convex function.

Appendix B. The moment of the $C_{2,2}$ test statistic in the presence of ties

Corollary 2. *For Z_1 and Z_2, we obtain the following moments.*

$$\begin{aligned}
\mathrm{E}[Z_s] &= \frac{n_1(N+1)(2N+1)}{6}, \\
\mathrm{var}[Z_1] &= \frac{n_1 n_2 (N+1)(2N+1)(8N+11)}{180} \\
&\quad - \frac{n_1 n_2}{180N(N-1)} \sum_{j=1}^{k} \tau_j(\tau_j^2 - 1)[(2\tau_j+1)(8\tau_j+11) \\
&\quad + 60J_{j-1}^2 + 60(\tau_j+1)J_{j-1}], \\
\mathrm{var}[Z_2] &= \frac{n_1 n_2 (N+1)(2N+1)(8N+11)}{180} \\
&\quad - \frac{n_1 n_2}{180N(N-1)} \sum_{j=1}^{k} \tau_j(\tau_j^2 - 1)[(2\tau_j+1)(8\tau_j+11) \\
&\quad + 60(N - J_{j-1} - \tau_j)(N - J_{j-1} + 1)], \\
\mathrm{cov}(Z_1, Z_2) &= -\frac{n_1 n_2 (N+1)(14N^2 + 30N + 19)}{180} \\
&\quad + \frac{n_1 n_2}{180N(N-1)} \sum_{j=1}^{k} \tau_j(\tau_j^2 - 1)[15(N+1)^2 - 15(N - J_j - J_{j-1})^2 \\
&\quad - \tau_j^2 + 4].
\end{aligned}$$

Proof of $\mathrm{E}[Z_1]$ **and** $\mathrm{var}[Z_1]$**.** The form of Z_1 is equivalent to the case of Taha (1964). Then its moment was already derived by Paul and Mielke (1967).

Proof of $\mathrm{E}[Z_2]$ **and** $\mathrm{var}[Z_2]$**.** As a similar procedure Paul and Mielke (1967), the conditional distribution of Z_2 for a specific configuration of ties is equivalent to a hypergeometric distribution as follows under H_0:

$$p(a_1, a_2, \ldots, a_k; \tau_1, \tau_2, \ldots, \tau_k, n_1) = \binom{N}{n_1}^{-1} \prod_{j=1}^{k} \binom{\tau_j}{a_j}.$$

Then the mean of Z_2 is derived in the following procedure:

$$\mathrm{E}[Z_2] = \frac{n_1}{N}\sum_{j=1}^{k}\tau_j \cdot \frac{1}{\tau_j}\sum_{I=J_{j-1}+1}^{J_j}(N+1-I)^2 = \frac{n_1(N+1)(2N+1)}{6}.$$

Herein, the score function of Z_2 is denoted by

$$\Psi_j = \frac{1}{\tau_j}\sum_{I=J_{j-1}+1}^{J_j}(N+1-I)^2.$$

Then, for the dispersion of Z_2, we have

$$\begin{aligned}
\mathrm{var}[Z_2] &= \frac{n_1 n_2}{N^2(N-1)}\left[N\sum_{j=1}^{k}\tau_j\Psi_j^2 - \left(\sum_{j=1}^{k}\tau_j\Psi_j^2\right)^2\right] \\
&= \frac{n_1 n_2}{N^2(N-1)}\left[N\sum_{I=1}^{N}(N+1-I)^4 - \left(\sum_{I=1}^{N}(N+1-I)^2\right)^2\right] \\
&\quad - \frac{n_1 n_2}{N(N-1)}\left[\sum_{j=1}^{k}\tau_j\left(\frac{1}{\tau_j}\sum_{I=J_{j-1}+1}^{J_j}(N+1-I)^2\right)^2\right. \\
&\quad \left. - \sum_{j=1}^{k}\sum_{I=J_{j-1}+1}^{J_j}(N+1-I)^4\right] \\
&= \frac{n_1 n_2(N+1)(2N+1)(8N+11)}{180} - \frac{n_1 n_2}{180N(N-1)}\sum_{j=1}^{k}\tau_j(\tau_j^2-1) \\
&\quad \times [(2\tau_j+1)(8\tau_j+11)+60(N-J_{j-1}-\tau_j)(N-J_{j-1}+1)].
\end{aligned}$$

Proof of the covariance of Z_1 and Z_2. Note that, we can obtain the relation in the presence of ties as

$$\begin{aligned}
Z_1 - \mu^* &= M' + (N+1)W', \\
Z_2 - \mu^* &= M' - (N+1)W'.
\end{aligned}$$

Using this relation, we obtain

$$\begin{aligned}
\mathrm{cov}(Z_1, Z_2) &= \mathrm{E}[(Z_1-\mu^*)(Z_2-\mu^*)] \\
&= \mathrm{E}[\{M^* - \mathrm{E}[M^*]\}^2] - (N+1)^2\mathrm{E}[\{W^* - \mathrm{E}[W^*]\}^2] \\
&= \mathrm{var}[M^*] - (N+1)^2\mathrm{var}[W^*].
\end{aligned}$$

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